Taylor Pool
Math 290 Sec 2
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Assignment 11
Exercises: 11.1-11.6

Exercise 11.1a Prove there exist $a, b \in \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.

Proof. Fix $a = 1$, $b = 2$. Then $a^b = 1^2 \in \mathbb{Q}$. $\square$

Exercise 11.1b Prove there exist $a, b \in \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$.

Solution. Fix $a = 2$, $b = \frac{1}{2}$. Then $a^b = 2^{\frac{1}{2}} = \sqrt{2}$, which we have previously proved to be irrational. $\square$

Exercise 11.1c Prove there exist $a, b \in \mathbb{R} - \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$.

Proof. Let $a, b \in \mathbb{R} - \mathbb{Q}$.There are two cases.
Case 1: $\sqrt{2}^s qrt2$ is irrational.
Then $a, b = \sqrt{2}$, so we are done.
Case 2: $\sqrt{2}^{\sqrt{2}}$ is rational.
Then $\sqrt{2}^{\sqrt{2}}\sqrt{2}$ is irrational.
$(\sqrt{2}^{\sqrt{2}})\sqrt{2} = \sqrt{2}^{\sqrt{2}+1}$
So let $a = \sqrt{2}, b = \sqrt{2} + 1$. We are done. $\square$

Exercise 11.1d Prove there exist $a \in \mathbb{Q}$ and $b \in \mathbb{R} - \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.

Proof. Fix $a = 1, b = \sqrt{2}$. So $a^b = 1^{\sqrt{2}} = 1 \in \mathbb{Q}$. $\square$

Exercise 11.1e Prove there exist $a \in \mathbb{Q}$ and $b \in \mathbb{R} - \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$

Proof. Fix $a \in \mathbb{Q}, b \in \mathbb{R} - \mathbb{Q}$. There are two cases.
Case 1: $2^{\frac{1}{\sqrt{2}}}$ is irrational.
Then let $a = 2, b = \frac{1}{\sqrt{2}}$. We are done.
Case 2: $2^{\frac{1}{\sqrt{2}}}$ is rational.
Then let a $= 2^{\frac{1}{\sqrt{2}}}, b = \frac{1}{\sqrt{2}}$. So ${2^{\frac{1}{\sqrt{2}}}}^{\frac{1}{\sqrt{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$, which is irrational. $\square$

Exercise 11.1f Prove there exist $a \in \mathbb{R} - \mathbb{Q}$ and $b \in \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.

Proof. Fix $a = \frac{1}{\sqrt{2}}, b = 2$, so $a^b = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} \in \mathbb{Q}$. $\square$

Exercise 11.1g Prove there exist $a \in \mathbb{R} - \mathbb{Q}$ and $b \in \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$.

Proof. Fix $a = \frac{1}{\sqrt{2}}, b = 1$, so $a^b = (\frac{1}{\sqrt{2}})^1 = \frac{1}{\sqrt{2}} \in \mathbb{R} - \mathbb{Q}$. $\square$

Exercise 11.2a Let n be an integer larger than $\frac{1}{y-x}$. We then have $ny - nx = n(y-x) < \frac{1}{y-x}(y-x) = 1$. Explain why this proves that there is an integer strictly between nx and ny.

Proof. Since $n, x, y \in \mathbb{Z}$, and $n(y-x) < 1$, this means that $y - x \in \mathbb{Z}$.
Hence, $ny < nx + 1$. Let $z \in \mathbb{Z}$ be the largest integer smaller than nx. Then, $z < nx \leq z + 1 < nx + 1 < ny$. Hence, $z \in (nx, ny)$. So we are done. $\square$

Exercise 11.2b Let m be an integer between nx and ny. Show $\frac{m}{n} \in \mathbb{Q}$ is in the interval (x, y).

Proof. Let $m \in \mathbb{Z}, nx < m < ny$. Then $x < \frac{m}{n} < y$. So we are done. $\square$

Exercise 11.3 Prove or disprove: Given $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$, then $xy \in \mathbb{R} - \mathbb{Q}$.

Proof. Assume, by way of contradiction, that $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$ and $xy \in \mathbb{Q}$.
Thus, $x = \frac{p}{q}, xy = \frac{r}{s}$ for some $p, r \in \mathbb{Z} and q, s \in \mathbb{N}$.
Note that $p \neq 0$.
Then $y = \frac{q}{p}\frac{p}{q}y = \frac{q}{p}xy = \frac{qr}{ps}$, but this is a contradiction, since $\frac{qr}{ps} \in \mathbb{Q}$ but $y \in \mathbb{R} - \mathbb{Q}$. So we are done. $\square$

Exercise 11.4 Prove or disprove: Let $s \in \mathbb{Z}$. If $6s - 3$ is odd, then s is odd.

Proof. Assume, by way of the contrapositive that s is even, so $s = 2k$ for some $k \in \mathbb{Z}$.
Thus, $6s - 3 = 6(2k) - 3 = 12k - 3 = 12k - 4 + 1 = 2(6k - 2) + 1$, which is odd. this violates the contrapositive property for truthfulness.
So this statement is disproved. $\square$

Exercise 11.5 Prove or disprove: There exists an integer x such that $x^2 + x$ is odd.

Proof. Let x be an arbitrary integer. There are two cases.
Case 1: Suppose x is odd.
Thus, $x = 2k + 1$ for some $k \in \mathbb{Z}$.
$x^2 + x = (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6K + 2 = 2(2k^2 + 3k + 1)$, which is even.
Case 2: Suppose x is even.
Thus, $x = 2k$ for some $k \in \mathbb{Z}$.
$x^2 + x = (2k)^2 + 2k = 4k^2 + 2k + 2(2k^2 + k)$, which is even.
Thus, in all cases, $x^2 + x$ is even for any integer x. $\square$

Exercise 11.6 Prove or disprove: Given any positive rational number a, there is an irrational number $x \in (0, a)$.

Proof. Let $a \in \mathbb{Q}, a > 0$.
Assume $\frac{1}{\sqrt{2}}$ is irrational.
Since $1 < 2, \sqrt{1} < \sqrt{2}$. Thus, $\frac{1}{\sqrt{1}} = 1 > \frac{1}{\sqrt{2}}$.
$0 < a\frac{1}{\sqrt{2}} < a$. Since a rational number multiplied by an irrational number is always irrational, and since $\frac{1}{\sqrt{2}} < 1$, we are done. $\square$