Taylor Pool
Math 290 Sec 2
Date Feb 14, 2017

Assignment 1
Exercises: 1

Exercise 13.1 Prove that for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Proof. Let $P(n) : \sum_{i=1}^{n}(2i-1) = n^2$.

Base Case: Prove $P(1) : \sum_{i=1}^{1}(2i-1) = 1^2$.
$2(1) - 1 = 1$ so $1 = 1$, which is true.

Inductive Step: Assume $P(k) : \sum_{i=1}^{k}(2i-1) = k^2$.
Then we need to prove $P(k+1) : \sum_{i=1}^{k+1}(2i-1) = (k+1)^2$.
We begin on the left-hand side.

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + 2(k+1) - 1 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

But $(k+1)^2$ is the right-hand side, so we have proved $P(k+1)$.
Thus, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Exercise 13.2 Prove that for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

Proof.

$$\text{Let } P(n) : \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

Base Case: Prove $P(1) : \sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{2(1)+1}$.
$\frac{1}{(2-1)(2+1)} = \frac{1}{3}$ so $\frac{1}{3} = \frac{1}{3}$, which is true.

Inductive Step: Assume $P(k) : \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$.
Then we need to prove

$$P(k+1) : \sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \frac{k+1}{2(k+1)+1} \text{ is true.}$$

$$\begin{aligned} LHS = \sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\ &= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\ &= \frac{k+1}{2k+3} \\ &= \frac{k+1}{2(k+1)+1} = RHS \end{aligned}$$

Thus, since $P(k+1)$ is true, we have proved the proposition by mathematical induction.

□

Exercise 13.3 Prove that for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

$$\text{Let } P(n) : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base Case: Prove $P(1) : \sum_{i=1}^{1} i^2 = \frac{(1)(1+1)(2(1)+1)}{6}$.
$1^2 = \frac{2(3)}{6}$, so $P(1)$ is true.

Inductive Step: Assume $P(K) : \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.
Then we need to prove

$$P(k+1) : \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \text{ is true.}$$

$$\begin{aligned}
LHS = \sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = RHS
\end{aligned}$$

Thus, since $P(k+1)$ is true when $P(k)$ is true, and since $P(1)$ is true, we have proved the proposition by mathematical induction. □

Exercise 13.4 Prove that for every $n \in \mathbb{N}$, $n < 3^n$.

Proof. Let $P(n) : n < 3^n$.

Base Case: Prove $P(1) : 1 < 3^1$.
$1 < 3$, so $P(1)$ is true.

Inductive Step: Assume $P(k) : k < 3^k$ is true.
Then we need to prove $P(k+1) : k+1 < 3^{k+1}$ is true.

$$RHS = k + 1 < 3^k + 1$$

□

Exercise 13.5 Let $a, x \in \mathbb{R}$, with $x \neq 1$. Prove that $\forall n \in \mathbb{N}$,

$$\sum_{i=0}^{n} x_i = \frac{1 - x^{n+1}}{1 - x}$$

Proof.

$$\text{Let } P(n) : \sum_{i=0}^{n} x_i = \frac{1 - x^{n+1}}{1 - x}$$

Base Case:

$$\text{Prove } P(1) : \sum_{i=0}^{1} x_i = \frac{1 - x^{1+1}}{1 - x}$$

$$\begin{aligned}
\sum_{i=0}^{1} x_i &= \frac{1 - x^{1+1}}{1 - x} \\
x^0 + x^1 &= \frac{1 - x^2}{1 - x} \\
1 + x &= \frac{1 - x^2}{1 - x} \\
\frac{(1 + x)(1 - x)}{1 - x} &= \frac{1 - x^2}{1 - x} \\
\frac{1 - x^2}{1 - x} &= \frac{1 - x^2}{1 - x}
\end{aligned}$$

Thus, we have proved the base case.

Inductive Step:

$$\text{Assume } P(k) : \sum_{i=0}^{k} x_i = \frac{1 - x^{k+1}}{1 - x} \text{ is true.}$$

$$\text{Prove } P(k+1) : \sum_{i=0}^{k+1} x_i = \frac{1 - x^{k+1+1}}{1 - x} \text{ is true.}$$

$$\begin{aligned}
LHS = \sum_{i=0}^{k+1} x_i &= \sum_{i=0}^{k} x^i + x^{k+1} \\
&= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \\
&= \frac{1 - x^{k+1} + (1 - x)x^{k+1}}{1 - x} \\
&= \frac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1 - x} \\
&= \frac{1 - x^{k+2}}{1 - x} = RHS
\end{aligned}$$

Thus, since $P(1)$ is true and $P(k+1)$ is true if $P(k)$ is true, we have proved the proposition.

$\square$