Rotation Between Two Vectors in $\mathbb{R}^3$

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July 25, 2024

1 Notation

- $\circ$ is the group operator between two rotations.
- $\odot$ is the rotation operater acting between a rotation and a vector.
- $\cdot$ is the inner product operater acting between two vectors.
- $\times$ is the cross product operator acting betwen two vectors.

2 Problem Statement

Let $a, b \in \mathbb{R}^3$ be unit vectors. Find a rotation ${}^aR_b$ such that

$$b = {}^bR_a \odot a \tag{1}$$

3 Solution

First, compute the inner product between a and b:

$$\cos\theta = a \cdot b \tag{2}$$

3.1 Non-opposite Case

Now, if for some $\epsilon > 0$,

$$\cos\theta \geq \epsilon - 1 \tag{3}$$

Then this means that the two vectors are sufficiently non-opposite. We can then compute the cross product from a to b as

$$\omega \sin\theta = a \times b \tag{4}$$

Now, from the double angle identities, we have

$$\cos\frac{\theta}{2} = \sqrt{\frac{\cos\theta+1}{2}} \tag{5}$$

$$\omega\sin\frac{\theta}{2} = \omega\sin\theta\frac{1}{2\cos\frac{\theta}{2}} \tag{6}$$

Then we may express ${}^{b}R_{a}$ as a quaternion:

$${}^{b}R_{a} = \begin{bmatrix}\cos\frac{\theta}{2}\\ \omega\sin\frac{\theta}{2}\end{bmatrix} \tag{7}$$

3.2 Opposite Case

Suppose now that for $\epsilon > 0$,

$$\cos\theta < \epsilon - 1 \tag{8}$$

In this case, a and b are pointing in opposite directions, which has direct numerical stability issues, since the cross product between two parallel vectors is the zero vector. To solve this issue, we compute an intermediate rotation ${}^{c}R_{a}$ of π radians about an axis perpendicular to a.

3.2.1 Finding the axis

We select the index, i of a with the minimum absolute value.

$$i = \arg\min_{j} |a_j| \tag{9}$$

Then we compute our axis of rotation, μ as

$$\mu = \text{normalized}(e_i - a a_i) \tag{10}$$

where e_i is the unit vector with 1 as the i-th element.

3.2.2 Defining the Intermediate Rotation

Then we may define ${}^{c}R_{a}$ in terms of a quaternion as:

$${}^{c}R_{a} = \begin{bmatrix}0\\ \mu\end{bmatrix} \tag{11}$$

3.2.3 Final Solution

Now, let $c = {}^{c}R_{a} \odot a$, and we may find ${}^{b}R_{c}$ via the method proposed in Section 3.1. Finally

$${}^{b}R_{a} = {}^{b}R_{c} \circ {}^{c}R_{a} \tag{12}$$

4 Remarks

4.1 Efficiency

The solution presented Section 3 is very efficient. Even though sin and cos terms appear, these quantities are calculated via the inner and cross products. The most expensive operation of the procedure is the square root in Equation 5.

4.2 Inspiration

We take inspiration from Ethan Eade's approach, however Eade does not specify how to choose the axis of intermediate rotation, which is a nontrivial task. Furthermore, Eade's approach is in terms of rotation matrices. For more information, see `https://ethaneade.com/rot_between_vectors.pdf`.