Expected Value of the Flow of Knowledge

```
flow = μ - α + γ + 1/n (1 - μ - γ + α) - (μ + 1/n (1 - μ))
FullSimplify[flow]
Limit[flow, n → ∞]
```

$$-\alpha+\gamma-\frac{1-\mu}{n}+\frac{1+\alpha-\gamma-\mu}{n}$$

$$-\frac{(-1+n)\ (\alpha-\gamma)}{n}$$

$$-\alpha+\gamma$$

Expected Value of Positive Learning

```
pl = (n - 1)/n γ + (1 - μ - γ) 1/n (n - 1)/n
FullSimplify[pl]
Limit[pl, n → ∞]
```

$$\frac{(-1+n)\ \gamma}{n}+\frac{(-1+n)\ (1-\gamma-\mu)}{n^2}$$

$$\frac{(-1+n)\ (1+(-1+n)\ \gamma-\mu)}{n^2}$$

$$\gamma$$

Expected Value of Negative Learning

```
nl = (n - 1)/n α + 1/n (n - 1)/n (1 - γ - μ)
FullSimplify[nl]
Limit[nl, n → ∞]
```

$$\frac{(-1+n)\ \alpha}{n}+\frac{(-1+n)\ (1-\gamma-\mu)}{n^2}$$

$$\frac{(-1+n)\ (1+n\,\alpha-\gamma-\mu)}{n^2}$$

$$\alpha$$

Expected Value of Retained Learning

```
rl = μ - α + (1 - γ - μ) 1/n 1/n + γ 1/n + α 1/n
Limit[rl, n → ∞]
```

$$-\alpha + \frac{\alpha}{n} + \frac{\gamma}{n} + \frac{1-\gamma-\mu}{n^2} + \mu$$

$$-\alpha + \mu$$

Expected Value of Zero Learning

```
zl = (1 - μ - γ) (n - 1)/n (n - 1)/n
FullSimplify[zl]
Limit[zl, n → ∞]
```

$$\frac{(-1+n)^2\,(1-\gamma-\mu)}{n^2}$$

$$-\frac{(-1+n)^2\,(-1+\gamma+\mu)}{n^2}$$

$$1 - \gamma - \mu$$

Solve for Estimators

```
FullSimplify[Solve[{epl == pl, enl == nl, erl == rl}, {μ, γ, α}] ]
```

$$\left\{\left\{\mu \to \text{enl} + \text{erl} + \frac{-1+\text{enl}+\text{erl}}{-1+n},\ \gamma \to \frac{n\,(-1+\text{enl}+\text{erl}+\text{epl}\,n)}{(-1+n)^2},\ \alpha \to \frac{n\,(-1+\text{epl}+\text{erl}+\text{enl}\,n)}{(-1+n)^2}\right\}\right\}$$

Comparative Statics

```
FullSimplify[D[pl, μ]]
FullSimplify[D[nl, μ]]
FullSimplify[D[zl, μ]]
FullSimplify[D[rl, μ]]
```

$$\frac{1-n}{n^2}$$

$$\frac{1-n}{n^2}$$

$$-\frac{(-1+n)^2}{n^2}$$

$$1 - \frac{1}{n^2}$$

```
FullSimplify[D[pl, n]]
FullSimplify[D[nl, n]]
FullSimplify[D[zl, n]]
FullSimplify[D[rl, n]]
```

$$\frac{-2\ (-1+\gamma+\mu)\ +n\ (-1+2\ \gamma+\mu)}{n^3}$$

$$\frac{-2\ (-1+\gamma+\mu)\ +n\ (-1+\alpha+\gamma+\mu)}{n^3}$$

$$-\frac{2\ (-1+n)\ (-1+\gamma+\mu)}{n^3}$$

$$\frac{-n\ (\alpha+\gamma)\ +2\ (-1+\gamma+\mu)}{n^3}$$

Comparing Bias

```
FullSimplify[pl - γ]
FullSimplify[nl - α]
FullSimplify[zl - (1 - μ - γ)]
FullSimplify[rl - (μ - α)]
```

$$\frac{-1+\gamma+\mu-n\ (-1+2\ \gamma+\mu)}{n^2}$$

$$\frac{-1+\gamma+\mu-n\ (-1+\alpha+\gamma+\mu)}{n^2}$$

$$\frac{(-1+2\ n)\ (-1+\gamma+\mu)}{n^2}$$

$$-\frac{-1+\gamma-n\ (\alpha+\gamma)\ +\mu}{n^2}$$

Observed zl is smaller than the true value, while observed rl is greater than the true value. There is no simple relationship for pl and nl.