THE COLLEGES OF OXFORD UNIVERSITY

PHYSICS

Wednesday 6 November 2013

Time allowed: 2 hours

For candidates applying for Physics, Physics and Philosophy, Engineering or Materials

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There are two parts (A and B) to this test, carrying equal weight.

Part A: Mathematics for Physics Q1-Q10 [50 Marks]

Part B: Physics Q11-Q21 [50 Marks]

Answers should be written on the question sheet in the spaces provided and you should attempt as many questions as you can from each part.

Marks for each question are indicated in the right hand margin. There are a total of 100 marks available and total marks for each section are indicated at the start of a section. You are advised to divide your time according to the marks available, and to spend equal effort on parts A and B.

No calculators, tables or formula sheets may be used.

Answers in Part A should be given exactly unless indicated otherwise. Numeric answers in Part B should be calculated to 2 significant figures.

Use $g = 10\,\mathrm{m\,s^{-2}}$.

Do NOT turn over until told that you may do so.

Part A: Mathematics for Physics [50 Marks]

1. What is the sum of the series $\dfrac{2}{3} - \dfrac{2}{9} + \dfrac{2}{27} - \ldots$? [4]

Geometric progression with:

$r = -\dfrac{1}{3}$, $a = \dfrac{2}{3}$

$$S_\infty = \frac{a}{1-r} = \frac{2/3}{1-(-\frac{1}{3})} = \frac{2}{3} \div \frac{4}{3}$$

$$= \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$$

2. If $y = \sqrt{x\nu}$ and $(y - \sqrt{x})^2 = u$,
   find an expression for x in terms of u and ν. [3]

$$u = \left(\sqrt{x\nu} - \sqrt{x}\right)^2 = x\nu + x - 2\sqrt{x\nu}\sqrt{x}$$

$$= x\nu + x - 2x\sqrt{\nu}$$

$$u = x\left(\nu + 1 - 2\sqrt{\nu}\right)$$

$$x = \frac{u}{1 + \nu - 2\sqrt{\nu}}$$

3. 50 people are in a room. 8 people in the room have red hair, 3 people have black hair and 20 people are male. You may assume that hair colour and gender are independent.

(a) If a person is selected at random from the room, what is the probability that they will be a female with red hair?

(b) If a person is selected at random from the room, what is the probability that they will be a male who does not have red or black hair?

[3]

Events are independent

a) $\Rightarrow$ $P(FR) = P(F) \times P(R)$

$$= \frac{30}{50} \times \frac{8^4}{50_{25}} = \frac{12}{125}$$

b) $P(\text{male, other colour}) = \frac{20}{50} \times \left[\frac{50 - (8+3)}{50} \right]$

$$= \frac{2}{5} \times \frac{39}{50}$$

$$= \frac{39}{125}$$

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4. Consider the function $f(x) = x^3 - x^2 - 4x + 4$

(a) Show that $x = 1$ is a root of $f(x) = 0$ and hence factorise $f(x)$ to find the remaining roots.

(b) Having found the roots of $f(x) = 0$, find the area bounded by the curve $f(x)$ and the x-axis between the two smallest roots.

[5]

a) $f(1) = (1)^3 - (1)^2 - 4(1) + 4 = 1 - 1 - 4 + 4$

$$= 0$$

$f(x) = x^3 - x^2 - 4x + 4 = (x-1)(x^2 + 0x - 4)$
$$= (x-1)(x^2 - 4) = (x-1)(x-2)(x+2)$$

$$\therefore x = 1, \ 2 \ or \ -2$$

b) $\int_{-2}^{1} (x^3 - x^2 - 4x + 4) dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 4x \right]_{-2}^{1}$

$$= \left(\frac{1}{4} - \frac{1}{3} - 2 + 4 \right) - \left(\frac{16}{4} + \frac{8}{3} - 8 - 8 \right)$$

$$= \left(\frac{3 - 4 + 24}{12} \right) - \left(\frac{48 + 32 - 192}{12} \right)$$

$$= \frac{23 + 112}{12} = \frac{45}{4}$$

5. If $x = \log_{10} 100 + \log_5 \sqrt{25} - \log_3 y^2$ and
$\frac{x}{2} = \log_2 8 - 9 \log_{10} \sqrt{10} + 2 \log_3 y$

find x and y. [4]

$$\frac{1}{2}\left[\log_{10} 100 + \log_5 \sqrt{25} - \log_3 y^2\right] = \log_2 8 - 9\log_{10}\sqrt{10} + 2\log_3 y$$

$$2 + 1 - 2\log_3 y = 2\left[3 - \frac{9}{2}\log_{10} 10 + 2\log_3 y\right]$$

$$3 - 2\log_3 y = -3 + 4\log_3 y$$

$$6 = 6\log_3 y$$

$$1 = \log_3 y$$

$$y = 3$$

$$x = \log_{10} 100 + \log_5 \sqrt{25} - \log_3 (3)^2$$

$$= 2 + 1 - 2 = 1$$

6. Find the equation of the straight line that passes through the centres of the two circles: $x^2 + 4x + y^2 - 2y = -1$ and $x^2 - 4x + y^2 - 6y = 3$ [5]

$x^2 + 4x + y^2 - 2y = -1$

$(x + 2)^2 - 4 + (y - 1)^2 - 1 = -1$

$(x + 2)^2 + (y - 1)^2 = 4$

$\therefore C_1 (-2, 1)$

$x^2 - 4x + y^2 - 6y = 3$

$(x - 2)^2 - 4 + (y - 3)^2 - 9 = 3$

$(x - 2)^2 + (y - 3)^2 = 16$

$\therefore C_2 (2, 3)$

$\text{Gradient} = \dfrac{3 - 1}{2 + 2} = \dfrac{2}{4} = \dfrac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - 3 = \dfrac{1}{2}(x - 2)$

$y = \dfrac{x}{2} + 2$

7. How many terms in the binomial expansion would be needed to determine $(3.12)^5$ to one decimal place? [4]

$$(3.12)^5 = (3 + 0.12)^5 = [3(1 + 0.04)]^5 = 3^5(1 + 0.04)^5$$

$$= 3^5\left[1 + 5(0.04) + \frac{5(4)(0.04)^2}{2} + \frac{5(4)3(0.04)^3}{6 \cdot 1} + \ldots\right]$$

$$= 3^5\left[1 + 0.20 + 0.016 + 0.00064\right]$$

$$3^5 \times 0.00064 = 0.16 \implies \text{term is required}$$

$$\therefore \ 4 \quad \text{terms in total}$$

8. Sketch the regions in the xy plane defined by the inequalities: $1 < xy < 2$ and $\frac{1}{2} < \frac{y}{x} < 2$ [7]

$y = 2x$

$y = \frac{x}{2}$

$y = \frac{2}{x}$

$y = 1/x$
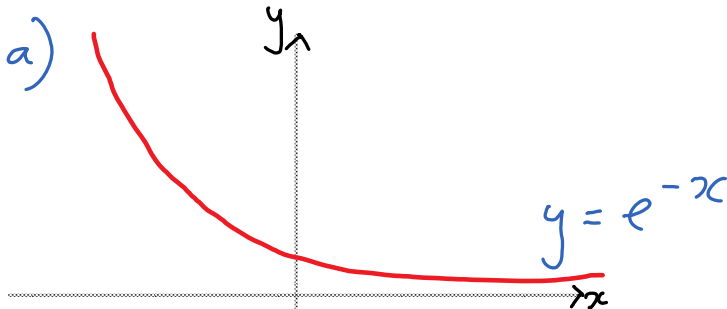
$y = \frac{2}{x}$ $y = 1/x$
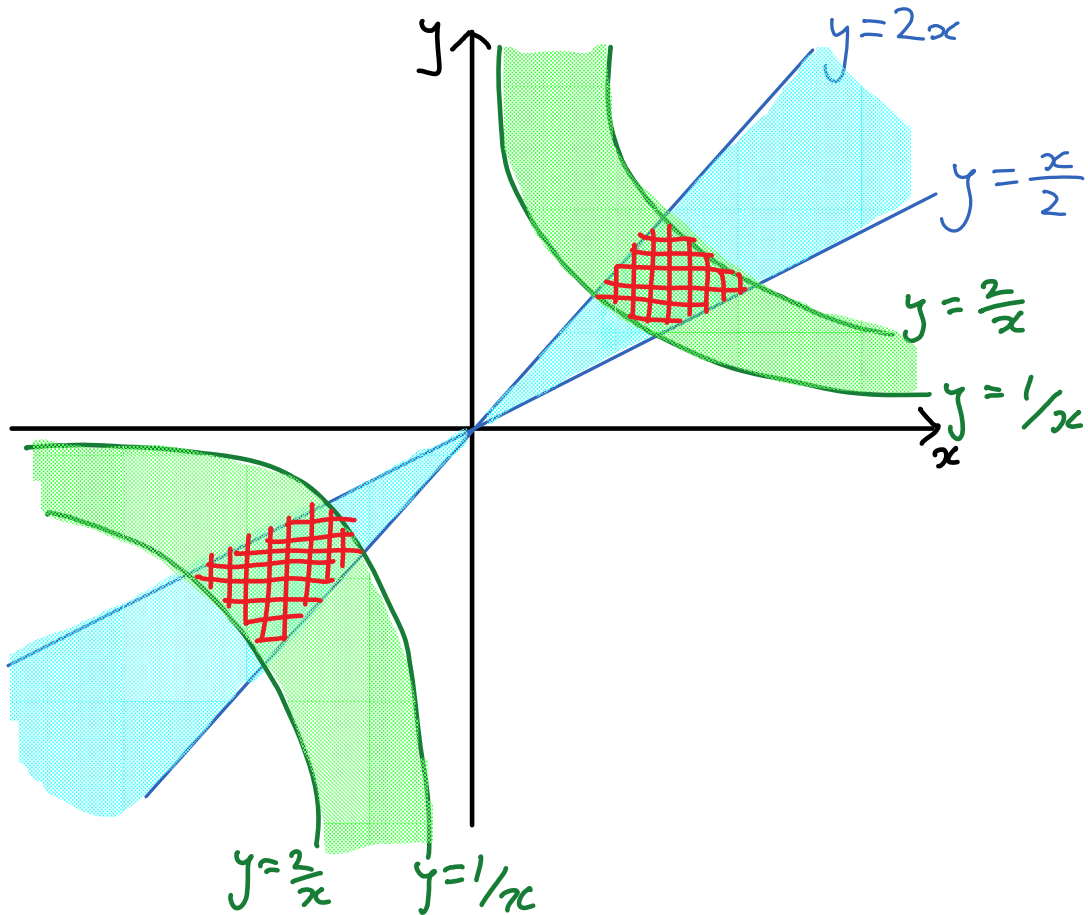
9. (a) Sketch $y = \exp(-x)$

(b) Sketch $y = 3\{\exp[-2(x - 1)] - 2\exp[-(x - 1)]\}$ for $x > 0$

[$\exp(x)$ is defined as $\exp(x) \equiv e^x$]   [8]

a)

$y = e^{-x}$

b) $y = 3\left[e^{-2(x-1)} - 2e^{-(x-1)} \right]$

As $x \to +\infty$, $y \to 0$ (from $-ve$)

$x \to -\infty$, $y \to \infty$

When $y = 0$, $3\left[e^{-2(x-1)} - 2e^{-(x-1)} \right] = 0$

$e^2 \cdot e^{-2x} - 2e \cdot e^{-x} = 0$

$e^2 \cdot (e^{-x})^2 - 2e \cdot e^{-x} = 0$

$e^{-x}\left(e^2 e^{-x} - 2e \right) = 0$

$e^{-x} = 0$ or $e^{-x} = \dfrac{2}{e}$

No soln. $+ve$ solution

$\Rightarrow$ Cuts the x-axis once on positive x

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When $x = 0$, $y = 3\left[e^2 - 2e\right] > 0$

$\Rightarrow$ Cuts the y-axis once on positive y

Turning point:

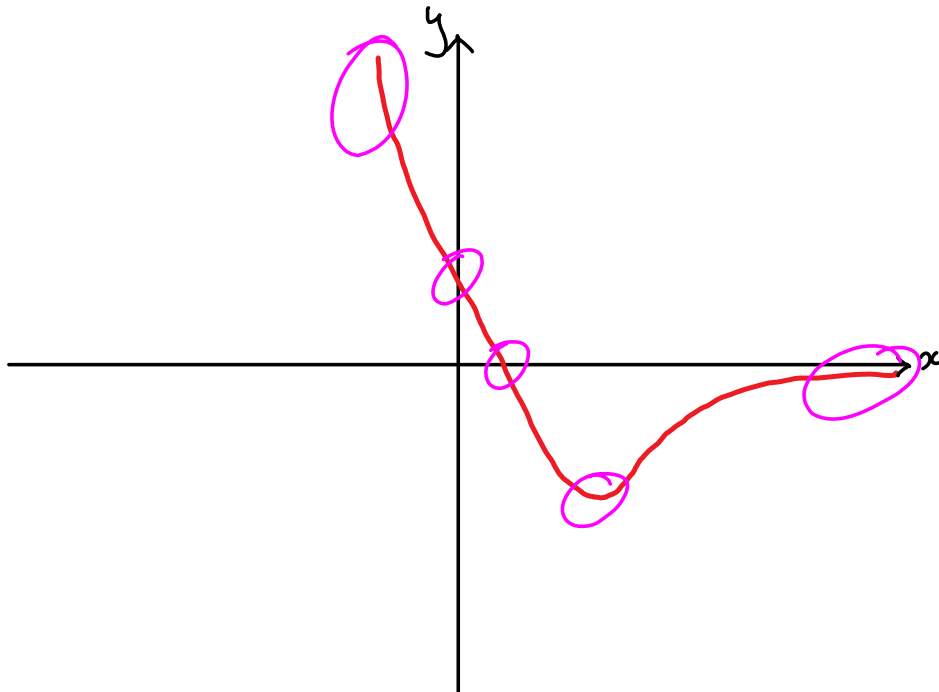$\dfrac{dy}{dx} = 3\left[-2e^{-2(x-1)} + 2e^{-(x-1)}\right] = 0$

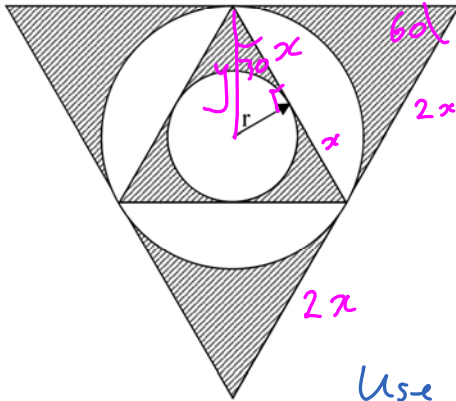$-e^2 \cdot e^{-2x} + e \cdot e^{-x} = 0$

$e^2(e^{-x})^2 - e \cdot e^{-x} = 0$

$e^{-x}\left(e^2 e^{-x} - e\right) = 0$

$e^{-x} = 0$ or $e^{-x} = \dfrac{1}{e} > 0$

$\Rightarrow$ One turning point on positive x. This is more than where the curve meets x-axis

10. In the figure below, all triangles are equilateral. Find the shaded area in terms of r. [7]

$$\tan 30 = \frac{r}{x}$$

$$x = \frac{r}{1/\sqrt{3}} = \sqrt{3}\, r$$

$$\sin 30 = \frac{r}{y}$$

$$y = \frac{r}{1/2} = 2r$$

Use area of triangle $= \frac{1}{2} ab \sin C$

and circle πr^2

Area of small circle $= \pi r^2$

" " big " $= \pi (2r)^2 = 4\pi r^2$

" " small triangle $= \frac{1}{2}(2\sqrt{3}r)(2\sqrt{3}r)\sin 60$

$$= 3\sqrt{3}\, r^2$$

Area of big triangle $= \frac{1}{2}(4\sqrt{3}r)(4\sqrt{3}r)\sin 60$

$$= 12\sqrt{3}\, r^2$$

$\therefore$ Shaded area $= (12\sqrt{3} - 4\pi + 3\sqrt{3} - \pi)r^2$

$$= (15\sqrt{3} - 5\pi)r^2$$

$$= 5r^2(3\sqrt{3} - \pi)$$

[Turn over] 13

Part B: Physics [50 Marks]

Multiple choice (10 marks)

Please circle **one** answer to each question only.

11. An ideal transformer has 100 turns on the primary coil. It is connected to an alternating supply of 100 V, 2.4 A. How many turns are required on the secondary coil to supply 4.8 A?

    A   25 turns
    B   50 turns
    C   75 turns
    D   200 turns

    [2]

$$P = I_p V_p = I_s V_s$$

$$2.4 \times 100 = 4.8 V_s$$

$$V_s = 50 V$$

$$\frac{N_p}{N_s} = \frac{V_p}{V_s}$$

$$\frac{100}{N_s} = \frac{100}{50}$$

$$N_s = 50$$

12. A radioactive sample contains two different isotopes, A and B. A has a half-life of 3 days, B has a half-life of 6 days. Initially in the sample there are twice as many atoms of A as of B. At what time will the ratio of the number of atoms of A to B be **reversed**?

    A   3 days
    B   6 days
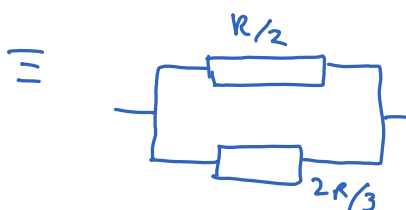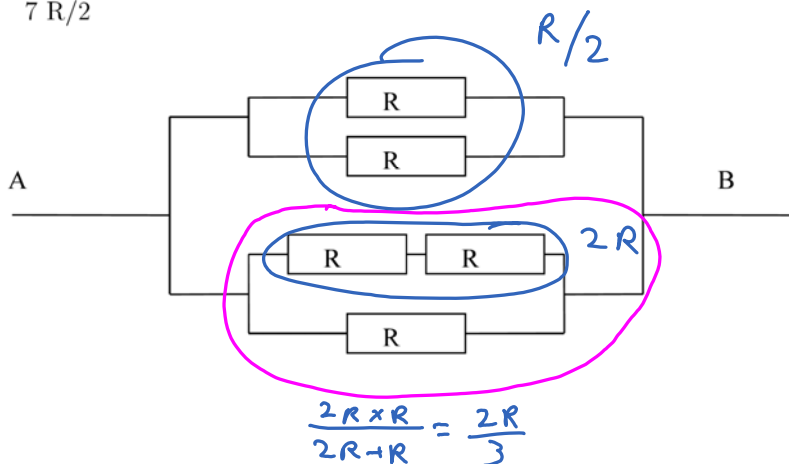   C   12 days
    D   ratio will never be reversed

    [2]

A: 80   40   20   10   5

B: 40   20   10

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+----+----+----+----+
0    3    6    9    12
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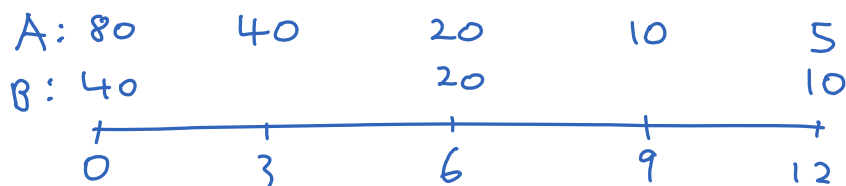
13. Consider the resistor network shown below. What is the overall resistance between A and B?

A 2 R/7

B R/2

C 3 R/2

D 7 R/2

[2]

R/2

2R

A ──────── B

R   R

R   R

R

$\dfrac{2R \times R}{2R + R} = \dfrac{2R}{3}$

≡

R/2

2R/3

$R_T = \dfrac{R/2 \times 2R/3}{R/2 + 2R/3}$

$= \dfrac{R^2/3}{7R/6}$

$= \dfrac{2R}{7}$

For 2 resistors in parallel,

$\dfrac{1}{R_T} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{R_2 + R_1}{R_1 R_2}$

$R_T = \dfrac{R_1 R_2}{R_1 + R_2}$

[Turn over] 15

Written answers (20 marks)

16. A four wheeled car, of mass 1000 kg, rests on the ground. If each tyre is inflated to 2 bar (where 1 bar = 100 kPa), what area of **each** tyre is in contact with the ground? (Assume a uniform distribution of mass across the car). [4]

Weight of car $= mg = 1 \times 10^4 N$

On each tyre, there will be $\frac{W}{4} = 2.5 \times 10^3 N$

$P = \frac{F}{A}$

$A = \frac{2.5 \times 10^3}{200 \times 10^3} = 1.25 \times 10^{-2} m^2$

[Turn over] 17

17. Two masses, m and M, are connected by a massless string of fixed length on a slope inclined at an angle α as sketched in the figure below. The pulley P is massless. Ignoring friction, calculate the acceleration of mass m and the tension of the string. What is the condition for the masses to be stationary?

[5]
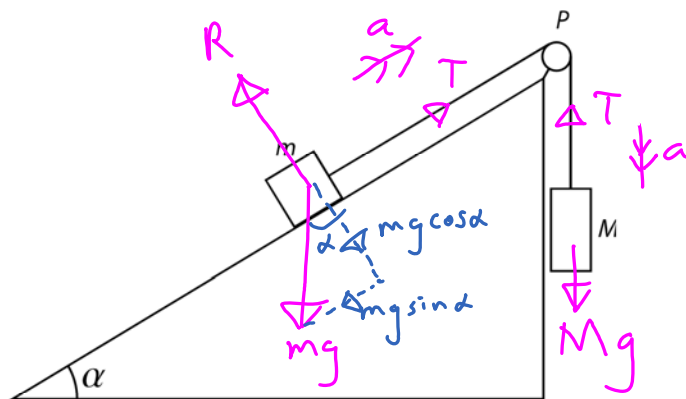
Find $a, T.$

For $M \downarrow [F = ma]$

$$Mg - T = Ma \quad ①$$

For $m \nearrow [F = ma]$

$$T - mg\sin\alpha = ma$$
$$T = ma + mg\sin\alpha \quad ②$$

② in ①: $Mg - ma - mg\sin\alpha = Ma$

$$g(M - m\sin\alpha) = a(M + m)$$

$$a = \frac{g(M - m\sin\alpha)}{M + m}$$

In ②: $T = \frac{mg(M - m\sin\alpha)}{M + m} + \frac{mg\sin\alpha}{(M + m)} \times (M + m)$

$$= \frac{mg(M - m\sin\alpha + M\sin\alpha + m\sin\alpha)}{M + m}$$

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$$\frac{mgM(1 - \sin\alpha)}{M + m}$$

18. A projectile of mass 0.2 kg and speed 122 m s^{-1} hits a ball of mass 12 kg hanging on a massless string of fixed length, as sketched in the figure below. The projectile was moving at the height of the centre of the ball and after hitting the ball it stops inside the ball, i.e. it becomes stationary with respect to the ball. What is the maximum height that the ball (with the projectile inside) will reach above its original position?

[5]

Find h.

$\uparrow$ h

$\leftarrow$ 122 n/s

m = 0.2

M = 12 kg
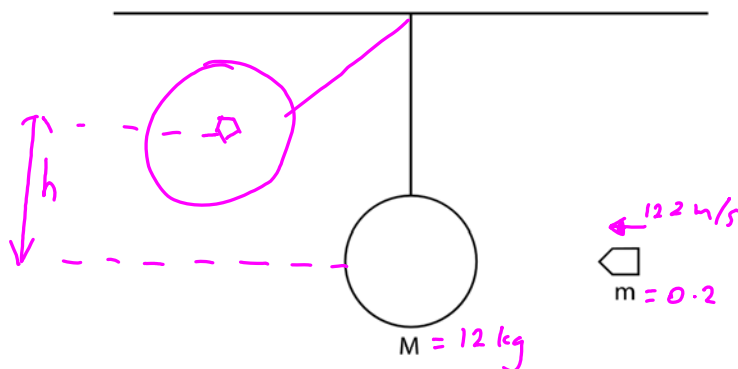
Conservation of momentum:

$$0.2 (122) = (12 + 0.2) v$$

$$v = 2 \, ms^{-1}$$

Conservation of energy:

$$\frac{1}{2} (M+m)(2)^2 = (M+m) gh$$

$$h = \frac{4}{2g}$$

$$= 0.2 \, m$$

19. A monochromatic point light source of wavelength λ is shining through two narrow slits separated by a distance d (d is of the order of λ) on a screen which is a distance D away ($D \gg d$) from the slits. Sketch the pattern of light intensity observed on the screen. Explain why there are minima and maxima. If λ corresponds to red light, what would the pattern look like for green light; make a sketch on the same scale. [6]

fringe separation ∝ wavelength
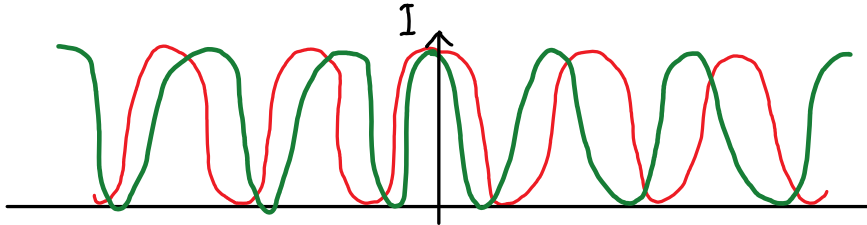
The narrow slits act like coherent sources, which means they have a constant phase difference between them. Light from those sources arrive at a particular point on the screen having travelled different distances. This is called path difference. Since the sources are coherent, the path difference at a particular point on the screen does not change.
When the path difference is an integer multiple of the wavelength, light from the two sources arrive in phase so there is constructive interference. This is where the bright fringes are. At points that the path difference is a half-integer multiple of the wavelength, light arrives in anti-phase so there is destructive interference. This gives rise to the dark fringes.

Long questions (20 marks)

20. An explorer tests her gas fuelled cooking stove before setting off on an expedition. She has a pot which has a square base of side 10 cm and height 15 cm. Starting from 20 °C, how much energy is required to heat the water in a totally full pot to boiling point? (You may assume the specific heat capacity of water is 4.2 kJ kg^{-1} K^{-1} and the density of water is 1 g cm^{-3}. You may also neglect the specific heat capacity of the pot.) [4]

$$\Delta Q = mc\Delta T ; \quad m = \rho V$$

$$\Delta Q = (10 \times 10 \times 15) \times 10^{-3} \times 4.2 \times 10^{3} \times (100 - 20)$$

$$\simeq 5.0 \times 10^{5} \text{ J}$$

The explorer now goes up Mount Everest. She discovers that the boiling point of water decreases by 1 °C every 300 m. What physical effect causes this reduction in boiling point? [1]

Decrease in pressure

[Turn over] 21

When she reaches 6000 m she uses her stove to make a cup of tea. Her mug only requires 100 g of water. How much energy will it take to boil the water and make the cup of tea (assuming it is 10 °C in her tent at 6000 m)? [2]

New boiling point: $100 - (6000 \div 300) = 80°C$

$\Delta Q = mc\Delta T = 100 \times 10^{-3} \times 4.2 \times 10^{3} \times (80 - 10)$

$$= 3 \times 10^4 \text{ J}$$

She discovers there is a problem with her stove and it now only produces 50 % of the power it did at sea level. If a full pot took 15 minutes to reach boiling point at sea level, how long will it take to boil the water for the cup of tea? [3]

Assuming the stove was working properly at sea level,

$$\text{time} = \left(\frac{3 \times 10^4}{5 \times 10^5} \times 15 \times 60\right) \times 2$$

$$= 108\,s$$

21. A particle of mass m moves with a velocity v_0 along the positive x direction before entering a region of length d where it is accelerated by a constant force f acting along the direction x.

(a) What is the velocity of the particle as it leaves the region of the acceleration?

$$\left[v^2 = u^2 + 2as\right] \qquad a = \frac{f}{m}$$

[1]

$$v^2 = v_0^2 + \frac{2f}{m}d$$

$$v = \sqrt{v_0^2 + \frac{2fd}{m}}$$

After being accelerated, the particle travels a distance D and then enters a region where a force of a magnitude F proportional to its speed v acts on it, $F = \alpha v$ and $\alpha > 0$ is constant. As sketched below, the force acts in the plane of the figure, and is perpendicular to the velocity at every point on the particle trajectory. A detector is placed as shown below, extending downwards, from the point of the entry to the region were the force is acting, along the direction y. (There is no gravitational force involved.)
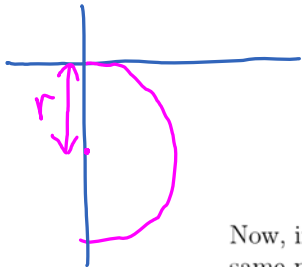

[Turn over] 23
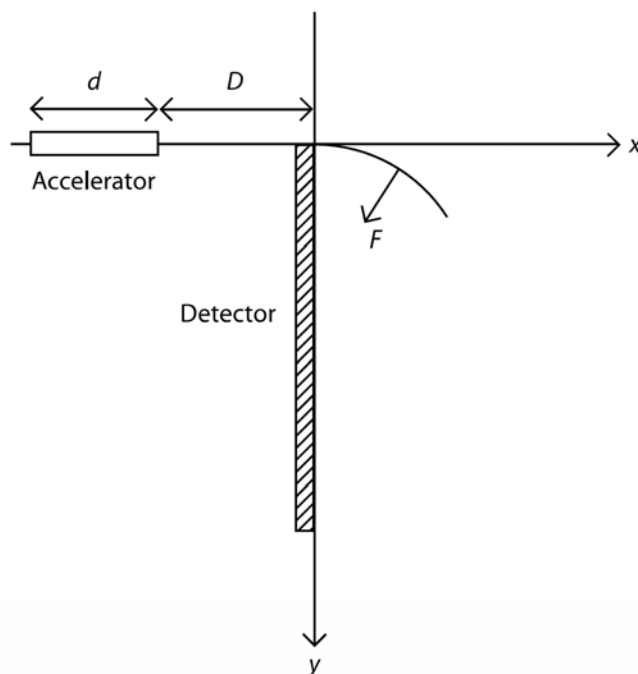
(b) Derive an expression for the y coordinate where the particle is detected.

Force is perpendicular to direction [4] of travel so motion is circular. Speed is constant.

$$F = \frac{mv^2}{r}$$

$$\alpha v = \frac{mv^2}{r}$$

$$r = \frac{mv}{\alpha}$$

$$\Rightarrow y = \frac{2mv}{\alpha} \text{, where } v \text{ was found in part (a)}$$

Now, instead of one particle, there are many particles, initially following the same path as the first particle with speeds v, ranging between v_1 and v_2, at the point of entry to the region where the force F is acting. The detector has a CCD like structure, meaning it is segmented into pixels of size $\triangle y$, the same for all y.

(c) What is the minimal spread of the speeds $\triangle v > 0$ such that v and $v + \triangle v < v_2$ are resolved by the detector? [2]

$$\triangle y < \frac{2m(v + \triangle v)}{\alpha} - \frac{2mv}{\alpha}$$

$$\triangle y < \frac{2m}{\alpha}(v + \triangle v - v)$$

$$\triangle y < \frac{2m}{\alpha} \triangle v$$

$$\triangle v > \frac{\alpha \triangle y}{2m}$$

(d) For a given particle, how much work is done by both the forces involved from when the particle enters the accelerator until it strikes the detector?

[3]

In the accelerator, work done = f × d

Outside, there is no force, hence no work done

In the region of force F (circular motion), force is perpendicular to velocity, so no work is done

∴ Total work done is fd