PHYSICS ADMISSIONS TEST
October 2023

Time allowed: 2 hours

For candidates applying to Physics, Physics and Philosophy, Engineering, or Materials Science

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Total 26 questions [100 Marks]

Answers should be written on the question sheet in the spaces provided, and you are encouraged to show your working.
You should attempt as many questions as you can.

No tables, or formula sheets may be used.

Answers should be given exactly and in simplest terms unless indicated otherwise.

Indicate multiple-choice answers by circling the best answer.
Partial credit may be given for correct workings in multiple choice questions.

The numbers in the margin indicate the marks expected to be assigned to each question. You are advised to divide your time according to the marks available.

You may take the gravitational field strength on the surface of Earth to be $g \approx 10\,\mathrm{m\,s^{-2}}$

Do NOT turn over until told that you may do so.

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1. What speed does a bull elephant (mass 4900 kg) have to move at to have the same kinetic energy as a cyclist (mass 100 kg) moving at 30 km h^{-1}? [2]

A	B	C	D	E
0.6 m s^{-1}	1.2 m s^{-1}	4.2 m s^{-1}	8.3 m s^{-1}	16.6 m s^{-1}

E : Elephant
C : Cyclist

$$\frac{1}{2} m_E v_E^2 = \frac{1}{2} m_C v_C^2$$

$$\therefore \quad v_E = \sqrt{\frac{m_C v_C^2}{m_E}}$$

$$v_E = \sqrt{\frac{m_C}{m_E}}\, v_C$$

$$= \sqrt{\frac{10\cancel{0}}{490\cancel{0}}} \times \frac{1}{12} \times 10^2$$

$$= \frac{1}{\sqrt{49}} \times \frac{1}{12} \times 100$$

$$= \frac{1}{84} \times 100$$

$$= \frac{25}{21} \approx 1\tfrac{4}{21} \quad (\text{"1 and a bit"} \therefore B)$$

$$30\,\text{km h}^{-1}$$

$$= 30\times10^3\,\text{m h}^{-1}$$

$$= \frac{30\times10^3}{3600}\,\text{m s}^{-1}$$

$$= \frac{3\times10^4}{36\times10^2}\,\text{m s}^{-1}$$

$$v_C = \frac{1}{12}\times10^2\,\text{m s}^{-1}$$

2. A seed packet contains 100 seeds. When planted, 75 will successfully become plants, but of these only a third will have flowers, and of these only one fifth will produce fruit. How many seeds produce fruiting plants? [2]

A	B	C	D	E
5	10	15	20	25

100

75

$\div 3$

$\frac{75}{3}$

$\div 5$

$\frac{75}{15} = 5$

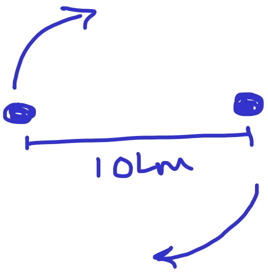
One orbit creates one wave cycle

3. Two black holes orbit each other and emit gravitational waves arising from the periodic nature of the orbit. The orbital separation is around 10 km, the relative speeds of the black holes are close to the speed of light, and gravitational waves travel at the speed of light. Which of the following would best describe the frequency of the emitted radiation? [2]

A	**B**	**C**	**D**	**E**
10^{-2} Hz	10 Hz	10^{4} Hz	10^{7} Hz	10^{10} Hz

$v \approx c$

$r \approx 5\text{km}$



$$v = \frac{2\pi r}{T}$$

$$v = 2\pi r f$$

$$c = 2\pi \times 5 \times 10^3 f$$

$$\therefore f = \frac{3 \times 10^8}{2\pi \times 5 \times 10^3}$$

$\pi \approx 3$:

$$f \approx \frac{\cancel{3} \times 10^8}{2 \times \cancel{3} \times 5 \times 10^3}$$

$$= \frac{10^8}{10 \times 10^3}$$

$$= \frac{10^8}{10^4}$$

$$= 10^4 \text{ Hz}$$

4. What is the next number in the sequence $\frac{1}{5}, \frac{3}{25}, \frac{7}{125}, \frac{3}{125}, \frac{31}{3125}$? [2]

A	B	C	D	E
$\frac{7}{125}$	$\frac{27}{3125}$	$\frac{59}{3125}$	$\frac{59}{15625}$	$\frac{63}{15625}$

$n=1$ $\frac{2^1-1}{5^1}$ $n=2$ $\frac{2^2-1}{5^2}$ $n=3$ $\frac{2^3-1}{5^3}$ $n=4$ $\frac{2^4-1}{5^4}$ $n=5$ $\frac{2^5-1}{5^5}$

$$\downarrow$$

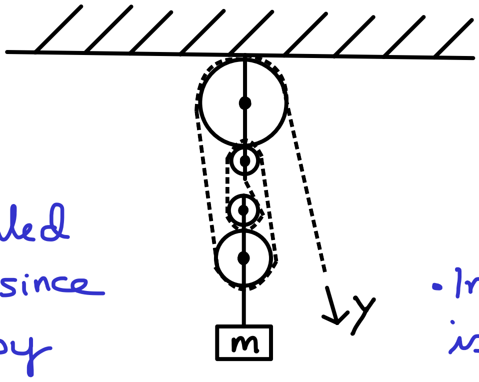
$$\frac{15}{625} = \frac{15}{5^4} = \frac{3\times\cancel{5}}{\cancel{5}\times 5^3} = \frac{3}{125}$$

$\therefore$ Next term : $\frac{2^6-1}{5^6} = \frac{63}{5^6}$ $\therefore$ E

5. Consider the pulley system in the diagram, containing 4 wheels. If you pull the free end a distance y, how far will m rise by? [2]

A type of "Block & Tackle" system

[Key idea is mechanical advantage: $\frac{F_{output}}{F_{input}}$]



If the free end is pulled by a force, T, then since the mass is held by four sections of rope the upwards force will be 4T.

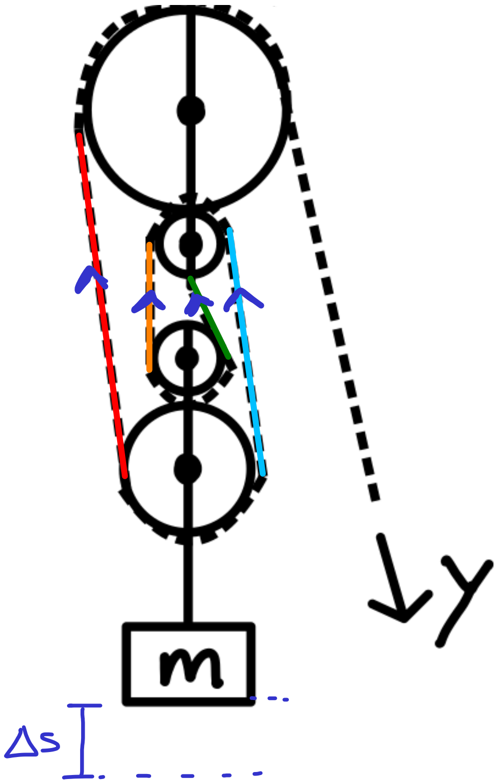
• In terms of length, the rope is folded "in four" (coloured on left). Since it is all the same rope, the total distance moved by all sections must add to y:

Same tension throughout ∴ distance moved in each section is the same.

$$\Delta s + \Delta s + \Delta s + \Delta s = y$$

$$4\Delta s = y$$

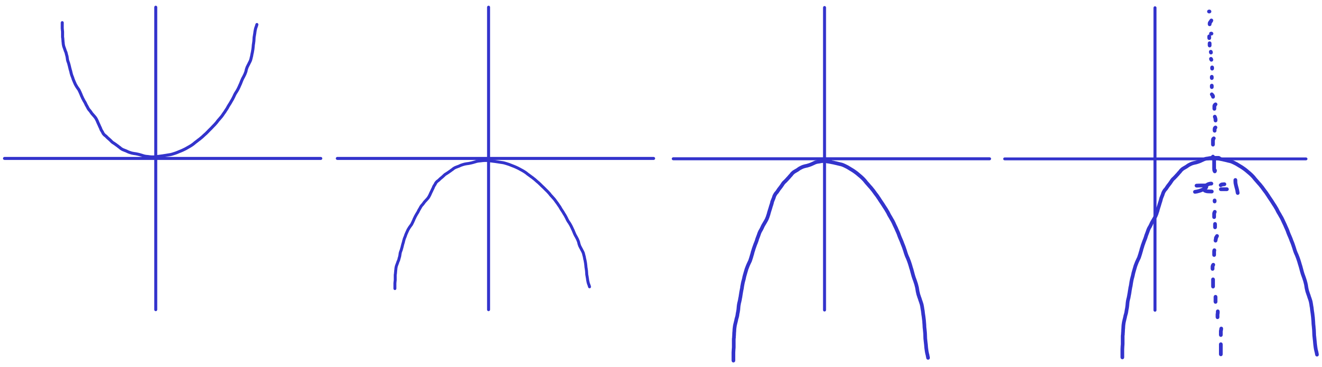
$$\Delta s = \frac{y}{4}$$



A	B	C	D	E
$y/16$	$y/4$	$y/2$	$2y$	$4y$

6. Consider $f(x) = x^2$. You want to transform the function so you get a new function $g(x)$ stretched by a vertical scale factor of 2, with a line of symmetry about $x = 1$ and which is never positive. $g(x)$ would be equal to which of the following functions? [2]

A	**B**	**C**	**D**	**E**
$-2f(x-1)$	$-f(x-1)$	$-2f(x+1)$	$-f(x+1)$	$-f(2x-2)$



x^2 $\quad$ $-x^2$ $\quad$ $-2x^2$ $\quad$ $-2(x-1)^2$

$f(x)$ $\quad$ $-f(x)$ $\quad$ $-2f(x)$ $\quad$ $-2f(x-1)$

7. If $y = \left(2 + \frac{x}{2}\right)^4$, which of the following is $\frac{dy}{dx}$? [2]

- **A** $4 + 2x + \frac{3x^2}{4} + \frac{x^3}{4}$
- **B** $8 + 6x + \frac{3x^2}{2} + \frac{x^3}{8}$
- **C** $32 + 24x + 6x^2 + \frac{x^3}{2}$
- **D** $16 + 12x + 3x^2 + \frac{x^3}{4}$
- **E** $2 + x + \frac{3x^2}{8} + \frac{x^3}{8}$

$$\frac{dy}{dx} = 4\left(2+\frac{x}{2}\right)^3 \times \frac{1}{2}$$

$$= 2\left(2+\frac{x}{2}\right)^3$$

$$= 2\left(2+\frac{x}{2}\right)\left(2+\frac{x}{2}\right)\left(2+\frac{x}{2}\right)$$

const term:

$2(2)^3$

$= 16$

x^3 term:

$2\left(\frac{x}{2}\right)^3$

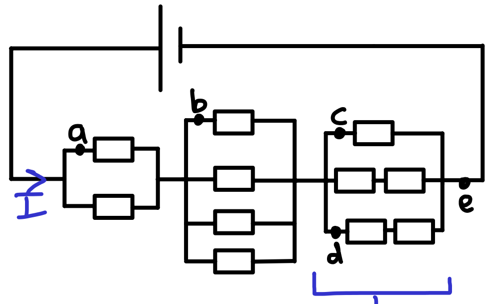
$= \frac{x^3}{4}$

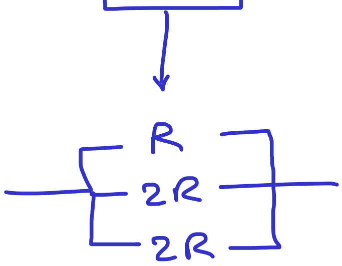
$\therefore$ D

8. All resistors in the circuit below have the same value. If an ammeter is placed in the circuit in turn at points (a) through to (e), which of the following sets of points will give the same reading? [2]

KCL:

$\sum I = 0$ at a node.





$I_a = \frac{I}{2}$

$I_b = \frac{I}{4}$

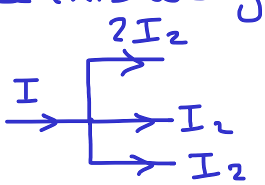
$I_c = \frac{I}{2}$

$I_d = \frac{I}{4}$

$I_e = I$

Resistance splits $1:2:2$

So the same current along the middle and bottom branches, I_2, but twice this along the top



$\therefore I = 4I_2$

$I_2 = \frac{1}{4} I$

A	B	C	D	E
a, b	a, c	b, e	c, d	a, b, c

c measures $2I_2 = \frac{1}{2} I$

d measures $I_2 = \frac{1}{4} I$

$\{a, c\}$

$\{b, d\}$

will give the same reading

9. If $\frac{dy}{dx} = x^2 + \frac{1}{x^3}$ and $y = 0$ when $x = 1$, what is $\int_1^3 y\,dx$? [2]

A	B	C	D	E
$\frac{4}{3}$	$\frac{8}{3}$	$\frac{20}{3}$	8	$\frac{22}{3}$

$$y = \int \frac{dy}{dx}\,dx = \int x^2 + x^{-3}\,dx$$

$$y = \frac{x^3}{3} - \frac{x^{-2}}{2} + C$$

$$\boxed{x = 1,\ y = 0}$$

$$0 = \frac{1}{3} - \frac{1}{2} + C$$

$$\therefore C = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$\therefore\ y = \frac{x^3}{3} - \frac{x^{-2}}{2} + \frac{1}{6}$$

$$\int_1^3 \frac{x^3}{3} - \frac{x^{-2}}{2} + \frac{1}{6}\,dx = \left[\frac{x^4}{12} + \frac{x^{-1}}{2} + \frac{x}{6}\right]_1^3$$

$$= \frac{3^4}{12} + \cancel{\frac{1}{2(3)}} + \cancel{\frac{3}{6}} - \left(\frac{1}{12} + \cancel{\frac{1}{2}} + \cancel{\frac{1}{6}}\right)$$

$$= \frac{81}{12} - \frac{1}{12}$$

$$= \frac{80}{12} = \frac{20}{3}$$

10. A particle of mass m, travelling freely at an initial speed v, can be stopped in a distance d by a constant retarding force F. What magnitude of force (applied in a direction perpendicular to the motion) would be needed to change the trajectory of the same particle (at the same speed v) into a circular arc of radius d? [2]

A	**B**	**C**	**D**	**E**
$F/2$	$F/\sqrt{2}$	F	$\sqrt{2}F$	$2F$

Work Done = KE lost

$$Fd = \frac{1}{2}mv^2$$

$$2Fd = mv^2$$

- Centripetal force :

$$F_c = \frac{mv^2}{r} \qquad \therefore F_c = \frac{mv^2}{d} = \frac{2F\cancel{d}}{\cancel{d}} = 2F$$

11. What is the (integer) m such that $\sum_{n=1}^{m}(3+2n) = 140$? [2]

A	B	C	D	E
6	8	10	12	14

$$\sum_{r=1}^{k} r = \frac{k}{2}(k+1)$$

$$\sum_{r=1}^{k} \overset{\text{const.}}{a} = \underbrace{a + a + \dots + a}_{k \text{ times}}$$

$$= ka$$

$$\sum_{n=1}^{m}(3+2n) = \sum_{n=1}^{m} 3 + 2\sum_{n=1}^{m} n = 140$$

$$= 3m + \cancel{2}\frac{m}{\cancel{2}}(m+1) = 140$$

$$3m + m^2 + m = 140$$

$$m^2 + 4m - 140 = 0$$

$$(m+14)(m-10) = 0$$

$$\therefore m = -14 \quad \text{or} \quad \underline{\underline{10}}$$

12. A device uses 3 kW of power at a voltage of 60 V. It is connected to a power supply via an ideal transformer. The transformer has N turns on the winding connected to the device and $20N$ turns on the winding connected to the power supply. What current flows in the winding connected to the power supply? [2]

A	B	C	D	E
1 mA	0.4 A	2.5 A	50 A	1 kA

i : initial
s : secondary

Ideal transformer means 100% power efficiency: $I_i V_i = I_s V_s$

$N_i = 20N$

$N_s = N$

$V_s = 60V$

$P_s = 3kW$

$I_i = ?$

$P_s = I_s V_s$

$3000 = I_s \times 60$

$\therefore I_s = \frac{300}{6} = 50\text{ A}$

$$\frac{N_i}{N_s} = \frac{V_i}{V_s} = \frac{I_s}{I_i}$$

$$\frac{N_i}{N_s} = \frac{I_s}{I_i}$$

$$20 = \frac{50}{I_i}$$

$$\therefore I_i = \frac{5}{2} = 2.5A$$

13. You use some measuring scales to discover the following relationships between masses of apples (each of mass m_A), bananas (m_B) and carrots (m_C):

$$2m_A + 3m_B + 4m_C = 4m_A + 3m_B + 3m_C \quad (1)$$

$$m_A + 4m_B + m_C = 8m_B \quad (2)$$

Find all combinations of apples and/or bananas that have the same mass as 5 carrots (note that only whole numbers of apples and bananas are allowed). [6]

(1) $m_C = 2m_A$ (2) $m_C = 4m_B - m_A$

Eliminate m_A:

$(1) \rightarrow (2)$

$m_C = 4m_B - \frac{m_C}{2}$

$2m_C = 8m_B - m_C$

$3m_C = 8m_B$ (3)

Eliminate m_C:

$(1) \rightarrow (2)$

$2m_A = 4m_B - m_A$

$3m_A = 4m_B$ (4)

- From (1), $\underline{5m_C = 10m_A}$ (5)
- From (3), $5m_C = \frac{40}{3}m_B$ ∴ No integer number of "just" bananas.
- From (5), 10 apples = 5 carrots. But from (4), 3 apples = 4 bananas.

Combinations of apples & bananas equivalent to 10 apples:

7 apples + 4 bananas.

4 apples + 8 bananas

1 apple + 12 bananas

Solution:

5 carrots is equivalent to:

- 10 apples
- 7 apples + 4 bananas
- 4 apples + 8 bananas
- 1 apple + 12 bananas

14. If $2^{x+2y} = 16$ and $xy = 2$, find x and y. [3]

$$2^{x+2y} = 16$$

$$xy = 2 \quad (2)$$

$$\log_2\left(2^{x+2y}\right) = \log_2(16)$$

$$(x+2y)\log_2(2) = \log_2(16)$$

$$\log_2(2) = 1 :$$

$$x+2y = 4 \quad (1)$$

Substitute (1) into (2) :

$$(4-2y)y = 2$$

$$4y - 2y^2 = 2$$

$$2y^2 - 4y + 2 = 0$$

$$y^2 - 2y + 1 = 0$$

$$(y-1)(y-1) = 0$$

$$\therefore \underline{\underline{y = 1}} \longrightarrow$$

substitute into (2) :

$$\underline{\underline{x = 2}}$$

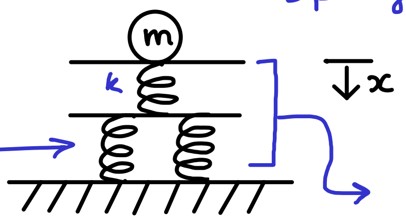
15. A ball of mass m sits in equilibrium on top of a set of three identical springs of spring constant k as in the diagram (you can assume that the springs are stiff and that the ball is light). The ball is pressed down by a distance x and then released. Assuming that 90% of the stored energy is transferred to the ball, how high will the ball go above its point of release (in terms of m, k, x and g, where g is the acceleration due to gravity)?

[6]

Springs in series : $\frac{1}{k_{TOT}} = \frac{1}{k_1} + \frac{1}{k_2} + \dots$

Springs in parallel : $k_{TOT} = k_1 + k_2 + \dots$

$k_{TOT} = 2k$



$$\frac{1}{k_{TOT}} = \frac{1}{k} + \frac{1}{2k}$$

$$\frac{1}{k_{TOT}} = \frac{3}{2k}$$

$$\therefore k_{TOT} = \frac{2k}{3} \quad (1)$$

- Stored Energy = Elastic Potential Energy

$$= \frac{1}{2} k_{TOT} x^2 = \frac{1}{3} k x^2$$

Sub in (1)

- 90% of $\frac{1}{3}kx^2$ gets transferred to the ball :

$$\frac{9}{10} \times \frac{1}{3} k x^2 = \frac{3}{10} k x^2$$

- This energy gets converted to GPE. At its highest point, h :

$$\frac{3}{10} k x^2 = mgh$$

$$\therefore h = \frac{3kx^2}{10mg}$$

$[L]$: length
$[T]$: time
$[M]$: mass

16. In astrophysics, the Jeans length λ_J is a measure of the size of a cloud of gas in which internal pressure just supports the cloud against collapse under gravity. It depends on the speed of sound in the gas, c_s, the gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}}$ and the mass density of the cloud ρ. The dependences may be expressed in the form $\lambda_J = c_s^\alpha G^\beta \rho^\gamma$. What values of α, β and γ are required for λ_J to have the correct units (or dimensions) of length? [4]

c_s units : $m s^{-1}$ — c_s dimensions : $[L][T]^{-1}$

G units : $kg^{-1} m^3 s^{-2}$ — G dimensions : $[M]^{-1}[L]^3[T]^{-2}$

ρ units : $kg m^{-3}$ — ρ dimensions : $[M][L]^{-3}$

λ_J units : m — λ_J dimensions : $[L]$

$$\lambda_J = c_s^\alpha G^\beta \rho^\gamma$$

Restate the equation in terms of dimensions:

$$[L] = \left([L][T]^{-1}\right)^\alpha \left([M]^{-1}[L]^3[T]^{-2}\right)^\beta \left([M][L]^{-3}\right)^\gamma$$

$$[L] = [L]^\alpha [T]^{-\alpha} [M]^{-\beta} [L]^{3\beta} [T]^{-2\beta} [M]^\gamma [L]^{-3\gamma}$$

Group terms:

$$[L] = [L]^{\alpha+3\beta-3\gamma} \cdot [T]^{-\alpha-2\beta} \cdot [M]^{-\beta+\gamma}$$

Equate powers:

$[L]$: $1 = \alpha + 3\beta - 3\gamma$ ①

$[T]$: $0 = -\alpha - 2\beta \Rightarrow \alpha = -2\beta$ ②

$[M]$: $0 = -\beta + \gamma \Rightarrow \beta = \gamma$ ③

Substitute ② and ③ into ① :

$1 = -2\beta \cancel{+3\beta} \cancel{-3\beta}$

$\therefore \beta = -\frac{1}{2}$

Substitute β into ② & ③ :

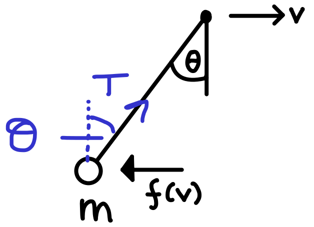
$\alpha = 1$ $\qquad \gamma = -\frac{1}{2}$

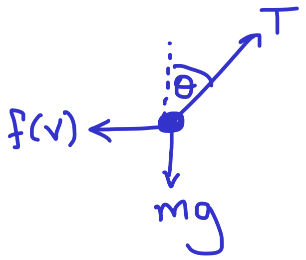
$$\lambda_J = c_s G^{-\frac{1}{2}} \rho^{-\frac{1}{2}}$$

17. A mass m on the end of a rigid rod of negligible mass hangs from a (pivot) point. The pivot point moves horizontally at speed v and the mass experiences a drag force $f(v)$ in the direction opposite to its velocity. At speed v, the rod makes a constant angle θ to the vertical.

(a) Find an expression for $f(v)$ in terms of the angle θ that the pendulum makes to the vertical.

(b) Sketch θ as a function of v in the case that $f(v)$ is proportional to v. [4]





a.)

$(\xrightarrow{+})$ $\sum F = ma$, $a = 0$

$T\sin\theta - f(v) = 0$

$f(v) = T\sin\theta$ ①

$(\uparrow +)$ $\sum F = ma$

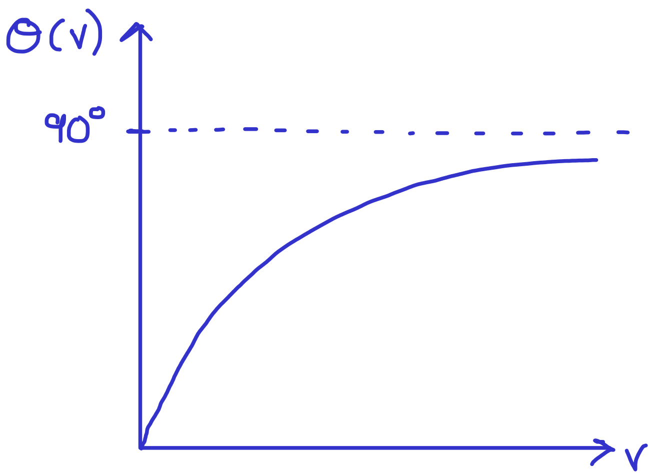
$T\cos\theta - mg = 0$

$T\cos\theta = mg$ ②

$$\frac{\cancel{T}\sin\theta}{\cancel{T}\cos\theta} = \frac{f(v)}{mg}$$

$$\therefore \underline{\underline{f(v) = mg\tan(\theta)}}$$

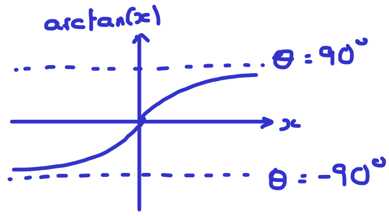
b.)



- $f(v) \propto v$. ③
- From a.), $f(v) \propto \tan(\theta)$ ④
- Combine ③ & ④ :

$v \propto \tan(\theta)$

$\therefore \theta \propto \arctan(v)$



18. A quartic polynomial function $f(x)$ has the following properties:

$$\frac{d^2f}{dx^2} = 0 \quad \text{at } x = 1 \text{ and } x = 3 \quad \textbf{only}$$

$$\frac{df}{dx} = 0 \quad \text{at } x = 2$$

$$f(0) = 0,$$

$$f(1) = 3.$$

Find $f(x)$. [7]

- Since $f(x)$ is a quartic, $\frac{df}{dx}$ is a cubic and $\frac{d^2f}{dx^2}$ is a quadratic.
- $\frac{d^2f}{dx^2}$ has roots at $x=1$, $x=3$. Then since $\frac{d^2f}{dx^2}$ is a quadratic:

$$\frac{d^2f}{dx^2} = A(x-1)(x-3)$$

$$= Ax^2 - 4Ax + 3A \quad (1)$$

A is some constant (vertical stretch factor)

- Integrating we get $\frac{df}{dx} = \frac{A}{3}x^3 - 2Ax^2 + 3Ax + B \quad (2)$

Substituting $x=2$, $\frac{df}{dx} = 0$ into (2):

$$0 = \frac{8A}{3} - 8A + 6A + B$$

$$0 = 8A - 24A + 18A + 3B$$

$$0 = 2A + 3B \quad (3)$$

- Integrating (2) we get $f(x) = \frac{A}{12}x^4 - \frac{2A}{3}x^3 + \frac{3A}{2}x^2 + Bx + \cancel{C}^{\,0} \quad (4)$

$f(0) = 0$ means $C = 0$

Substituting $x=1$, $f(x) = 3$ into (4):

$$3 = \frac{A}{12} - \frac{2A}{3} + \frac{3A}{2} + B$$

$$36 = A - 8A + 18A + 12B$$

$$36 = 11A + 12B \quad (5)$$

- Substitute (3) into (5):

$$36 = 11A - 8A$$

$$36 = 3A$$

$$\therefore A = 12$$

$\therefore$ From (3): $0 = 24 + 3B$

$$B = -8$$

- Substituting A & B into (4) to get $f(x)$:

$$f(x) = \underline{\underline{x^4 - 8x^3 + 18x^2 - 8x}}$$

$e^{2x} = e^{x+x} = e^{x} \cdot e^{x} = (e^{x})^2$

19. Solve the following equation for real x, [3]

$$6e^{2x} + e^x = 15$$

$6(e^x)^2 + e^x - 15 = 0$

$6(e^x)^2 - 9e^x + 10e^x - 15 = 0$

$3e^x(2e^x - 3) + 5(2e^x - 3) = 0$

$(3e^x + 5)(2e^x - 3) = 0$

$\therefore e^x = -\frac{5}{3}$

no solutions as e^x only has values > 0

$e^x = \frac{3}{2}$

$x = \ln\left(\frac{3}{2}\right)$

$+$ $\times$

1 -90

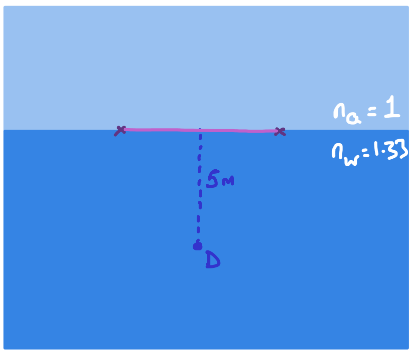
$\therefore 10 \,\&\, -9$

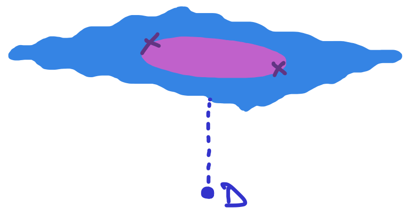
[Snell's Window]

20. A diver 5 metres under the surface of the sea looks up. They see a circle of light directly above them, where they can see what is on the surface, but outside of this circle the diver only sees a reflection of what is under the water. Explain why there is such a circle and calculate its radius. You may assume $n_{air} = 1$ and $n_{water} = 1.33$. [4]

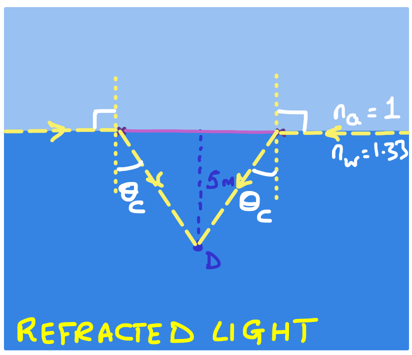
D : Diver

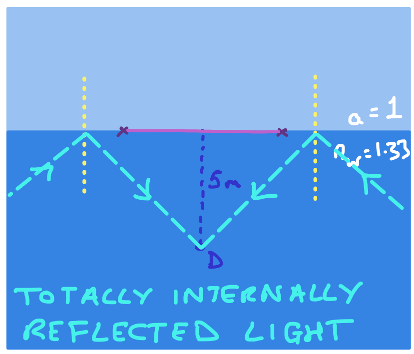
x : Outer edges of the circle of light





- When the diver looks up towards the surface of the water, light from above is refracted. It is refracted towards the normal since $n_{water} > n_{air}$.





- The largest possible angle of incidence for light hitting the water is 90°. This is geometrically equivalent to light exiting the water at the critical angle of the water → air boundary (angle of refraction = 90°)

- If the diver looks further out, the angle of incidence of light from above the water can't increase further (beyond 90°) – the only light visible past this point will be that which has reflected from objects below the water. This light will TIR off the surface of the water.

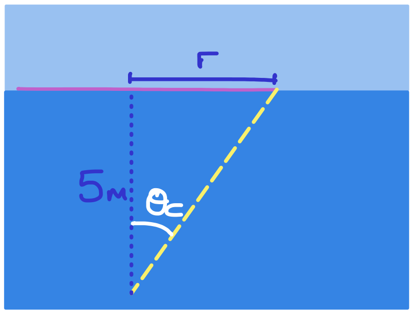
- Critical angle of the water + air boundary:

$$\theta_c = \arcsin\left(\frac{1}{1.33}\right) = 48.8° \ (3sf)$$

- $\tan(48.8°) = \frac{r}{5}$

$$\therefore r = 5\tan(48.8°)$$

$$= \underline{\underline{5.71m}} \ (3sf)$$



21. Find all values of x that satisfy the equation. [5]

$$4\sin x\left(\sin x+\cos^2 x\right)=3+\sin x$$

- Substitute $\cos^2 x = 1-\sin^2 x$:

$$4\sin x\left(\sin x + 1 - \sin^2 x\right) = 3+\sin x$$
$$4\sin^2 x + 4\sin x - 4\sin^3 x = 3 + \sin x$$
$$4\sin^3 x - 4\sin^2 x - 3\sin x + 3 = 0$$

- $\sin x$ is defined for $-1 \le x \le 1$.
- Try $\sin x = 1$:

$$4-4-3+3=0 \checkmark$$

So $\sin x = 1$ is a solution, hence $\underset{(u-1)}{(\sin x - 1)}$ is a factor.

- Let $u = \sin x$:

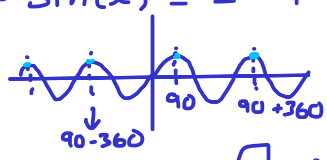
$$4u^3 - 4u^2 - 3u + 3 = 0$$

$$
\begin{array}{r|l}
 & 4u^2 + 0 - 3 \\
\hline
u-1 & 4u^3 - 4u^2 - 3u + 3 \\
 & 4u^3 - 4u^2 \\
\hline
 & 0 \; -3u \\
 & -\; 0 \cdot 0 \\
\hline
 & -3u + 3 \\
 & -3u + 3 \\
\hline
 & 0
\end{array}
$$

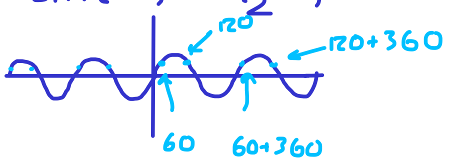
$$\therefore 4u^3 - 4u^2 - 3u + 3 = 0$$
$$(u-1)(4u^2-3) = 0$$
$$\therefore u = 1, \quad u = \pm\sqrt{\frac{3}{4}}$$
$$u = \pm\frac{\sqrt{3}}{2}$$

$$\therefore \sin(x) = 1 \qquad \sin(x) = \pm\frac{\sqrt{3}}{2}$$

- $\sin(x) = 1$ for $x = (90 \pm 360n)^\circ$



- $\sin(x) = \frac{\sqrt{3}}{2}$ for $x = (60 \pm 360n)^\circ$ and $(120 \pm 360n)^\circ$



• $\sin(x) = -\dfrac{\sqrt{3}}{2}$ for $-60^\circ, 300^\circ \ldots\ldots$ $x = \underline{\underline{(-60^\circ \pm 360n)}}^\circ$

$-120^\circ, 240^\circ \ldots\ldots$ $x = \underline{\underline{(-120^\circ \pm 360n)}}^2$

22. Two identical spacecraft of mass m are in stable circular orbits around the Earth – one at height R_E and the other at height $2R_E$ above the surface of the Earth. What is the difference in the total energy between the two spacecraft? The radius of the Earth is R_E. [6]

$$E_k = \frac{1}{2}mv^2 \quad (1)$$

$$E_p = -\frac{GMm}{r}$$

Equate centripetal with gravitational force to find v:

$$\frac{\cancel{m}v^2}{\cancel{r}} = \frac{GM\cancel{m}}{r^{\cancel{2}}}$$

$$\therefore v^2 = \frac{GM}{r}$$

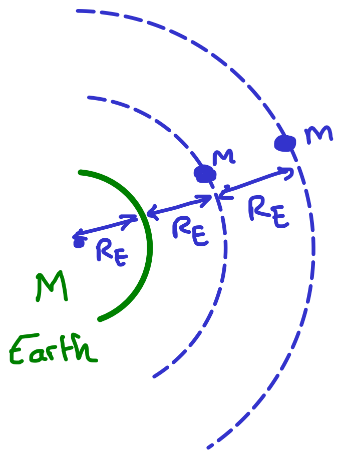
Into (1):

$$E_k = \frac{GMm}{2r}$$

$\therefore$ Total energy of an object a distance, r, from the centre of the orbit is:

$$E_{TOT} = E_k + E_p = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$E_{TOT} = -\frac{GMm}{2r}$$



Inner orbit, $r = 2R_E$:

$$E_{TOT} = -\frac{GMm}{4R_E}$$

Outer orbit, $r = 3R_E$:

$$E_{TOT} = -\frac{GMm}{6R_E}$$

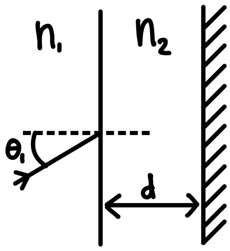
Difference in Total Energy $= -\dfrac{GMm}{6R_E} - \left(-\dfrac{GMm}{4R_E}\right)$

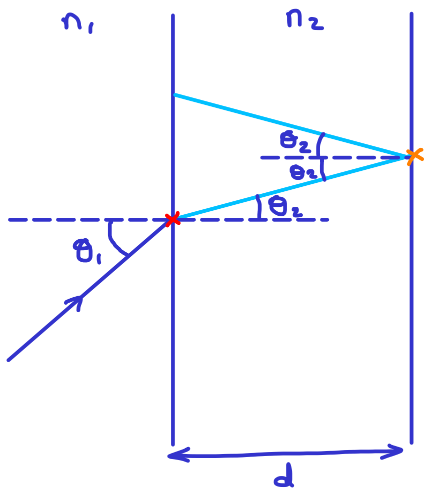
$$= \frac{GMm}{R_E}\left(-\frac{1}{6} + \frac{1}{4}\right)$$

$$= \frac{GMm}{12R_E}$$

23. A beam of light in a medium with refractive index n_1 is incident at an angle θ_1 on a slab of material of thickness d with refractive index $n_2 > n_1$ as shown in the figure. The rear surface of the slab is mirrored and perfectly reflective.

$n_2 > n_1$ ∴ slows down in n_2 (bends toward the normal)

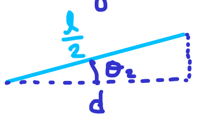




a) • Apply Snell at the boundary × :

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad (1)$$

• Triangle from × to × :



$$\cos\theta_2 = \frac{2d}{l} \quad (2)$$

• To get rid of θ_2 use (1) and (2) in the identity:

$$\sin^2\theta_2 + \cos^2\theta_2 = 1$$

$$\frac{n_1^2 \sin^2\theta_1}{n_2^2} + \frac{4d^2}{l^2} = 1$$

$$\frac{4d^2}{l^2} = 1 - \frac{n_1^2 \sin^2\theta_1}{n_2^2}$$

$$\frac{l^2}{4d^2} = \frac{1}{1 - \frac{n_1^2 \sin^2\theta_1}{n_2^2}} \quad \begin{matrix} \times n_2^2 \\ \times n_2^2 \end{matrix}$$

$$l^2 = \frac{4d^2 n_2^2}{n_2^2 - n_1^2 \sin^2\theta_1}$$

$$\underline{\underline{l = \frac{2dn_2}{\sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}}}$$

b) $0^\circ \leq \theta_1 \leq 90^\circ$

As $\theta \to 0^\circ$, $\sin\theta \to 0$

$\theta \to 90^\circ$, $\sin 90 \to 1$

• At small θ_1:

$$l \approx \frac{2dn_2}{\sqrt{n_2^2 - 0}} = \frac{2dn_2}{n_2} = \underline{\underline{2d}}$$

• A large θ_1:

$$l \approx \underline{\underline{\frac{2dn_2}{\sqrt{n_2^2 - n_1^2}}}}$$

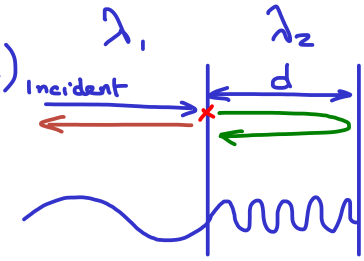
(a) What distance, l, does the beam transmitted into the slab travel before re-emerging from it? Express your answer in terms of n_1, n_2, θ_1 and d. [3]

(b) What are the limiting values of l at large and small θ_1? [3]

(c) Now consider the case in which light of wavelength (in medium 1) λ_1 is incident normally. For what value(s) of d would light reflected from the two surfaces interfere constructively? Ignore any phase changes that might occur at reflections. [3]

$$n = \frac{c}{v} = \frac{\cancel{f}\lambda_0}{\cancel{f}\lambda} = \frac{\lambda_0}{\lambda} \qquad n_1 = \frac{\lambda_0}{\lambda_1} \qquad n_2 = \frac{\lambda_0}{\lambda_2}$$

$$\therefore n_1\lambda_1 = n_2\lambda_2$$

c.) 

- Some light gets transmitted, some is reflected at ×.
- Path difference between the transmitted & reflected waves is $2d$.
- For constructive interference, each cycle of the reflected wave must meet in phase with each cycle of the transmitted wave.
- This occurs if the path difference, $2d$, is an integer number of wavelengths:

$$2d = m\lambda_2 \qquad m \in \mathbb{Z}^+$$

- Substituting in $n_1\lambda_1 = n_2\lambda_2$:

$$2d = \frac{mn_1\lambda_1}{n_2}$$

$$d = m\frac{n_1\lambda_1}{2n_2}$$

24. A ship floating at anchor moves vertically only, as waves on the surface of the sea cause the surface height to vary with position x and time t as

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right),$$

General form:
$A\sin(kx - \omega t)$
k: wavenumber $\frac{2\pi}{\lambda}$
ω: angular frequency $\frac{2\pi}{T}$

where A, λ and v are positive constants.

(a) What is the period P of the ship's vertical oscillations? [1]

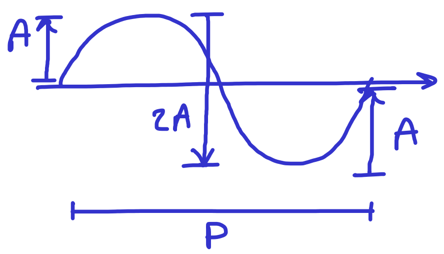
(b) What total vertical distance does the ship move through during a time interval equal to P? [1]

(c) Sketch curves for the ship's kinetic and potential energies as functions of time on the same plot, from $t = 0$ to $t = 2P$. [3]

a.) Period, $P = \frac{1}{f}$ and $v = f\lambda$

$$P = \frac{\lambda}{v}$$

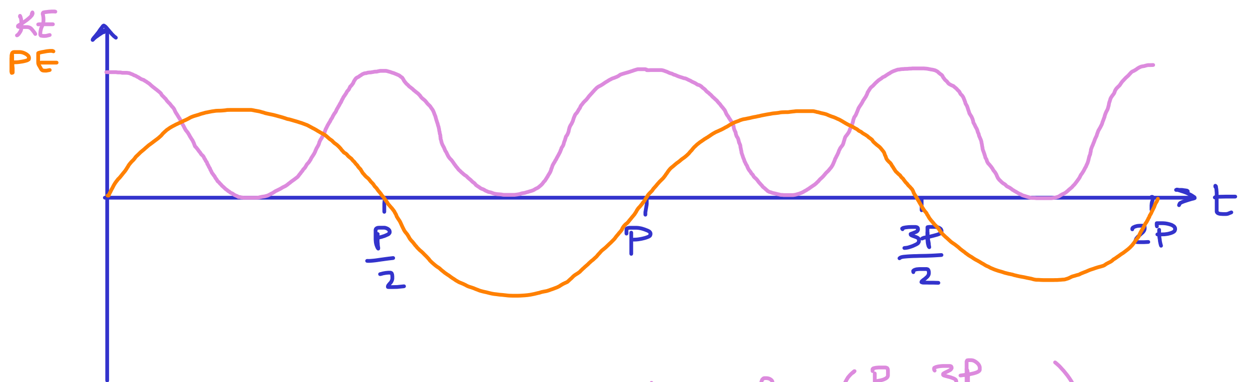
b.)



During one period, P, the ship moves vertically a distance of $A + 2A + A = 4A$

Note: Its vertical displacement after one period is zero.

c.) Let zero potential energy be at the equilibrium position



KE will be zero at crests and troughs $\left(\frac{P}{4}, \frac{3P}{4} \ldots\right)$ and maxima when the curve of y is steepest $(0, \frac{P}{2}, P \ldots)$

PE will be zero when vertical displacement is zero, negative at troughs and positive at crests.

25. Sketch $y = x^4 - 2x^3$ and $y = 2x - x^2$ on the same axes, showing clearly the natures of the stationary points and labelling their coordinates. Write down an integral expression for the finite area enclosed between the two curves (you do not need to evaluate the integral). [7]

$y = x^4 - 2x^3 = x^3(x-2)$

roots: $x = 0, x = 2$

x^4 positive

Triple root at $x=0$ ∴ inflection ("cubic" shape)

Single root at $x=2$ ∴ crosses

$\frac{dy}{dx} = 4x^3 - 6x^2$ ∴ stationary points when:

$4x^3 - 6x^2 = 0$

$2x^2(2x-3) = 0$

$x = 0, \ x = \frac{3}{2}$

$\boxed{x = \frac{3}{2}}$

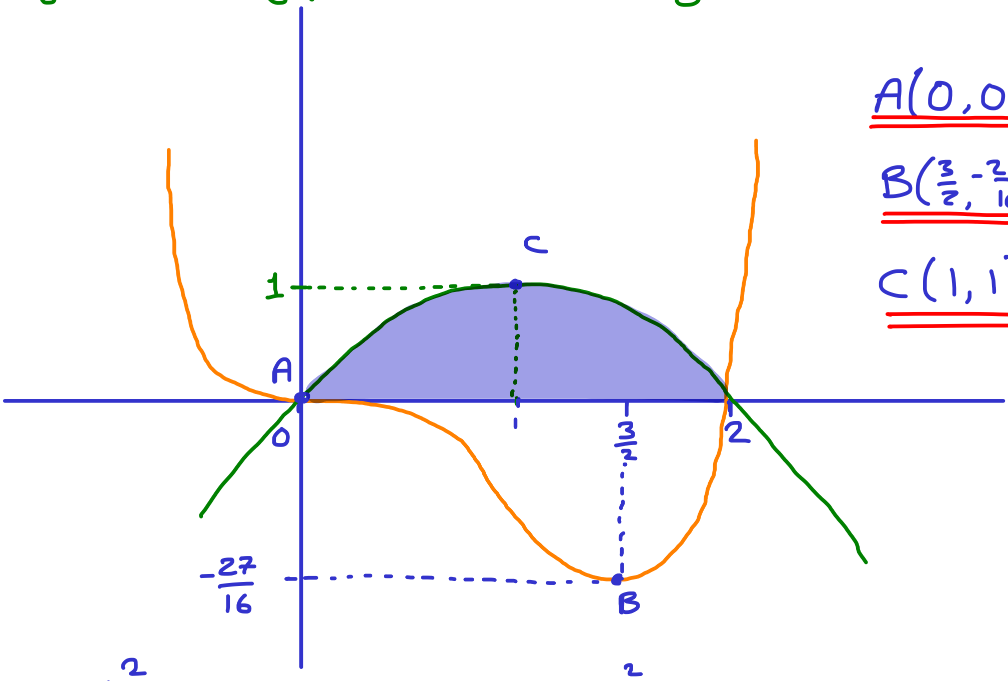
$y = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = \frac{81}{16} - \frac{27}{4} = \frac{-27}{16}$

$y = 2x - x^2 = x(2-x)$

x^2 negative

roots: $x = 0, x = 2$

By symmetry, stationary point at $x = 1$, $y = 2(1) - (1)^2 = 1$



$\underline{A(0,0)}$ inflection

$\underline{B\left(\frac{3}{2}, -\frac{27}{16}\right)}$ minimum

$\underline{C(1,1)}$ maximum

$$\text{Area} = \int_0^2 x^4 - 2x^3 - (2x - x^2)\, dx = \underline{\int_0^2 x^4 - 2x^3 + x^2 - 2x \, dx}$$

$\{8,8\}$: 64 possible outcomes. $\{10,6\}$: 60 possible outcomes.

26. Two separate pairs of unbiased dice are rolled. One pair consists of two eight-sided dice (with faces numbered 1-8). The other pair consists of one six-sided die (with faces numbered 1-6) and one ten-sided die (with faces numbered 1-10).

(a) Which pair is most likely to show a total of 16? [1]

(b) Are any totals equally likely to be rolled using the two pairs of dice? [1]

(c) What is the smallest total that is more probable when using the pair consisting of 8-sided dice? [1]

(d) Which pair is more likely to give a total that is divisible by 3? [2]

(e) Given that at least one of the eight-sided dice has landed 5, is a total of 11 or 10 more likely? Give a reason for your answer. [2]

a.) For each pair, there is only one way to roll a 16 : 8+8 or 10+6. However, the 10+6 pair will be more likely as there is a lower number of possible outcomes :

$P(8+8=16) = \frac{1}{64}$ $\quad$ $P(10+6=16) = \frac{1}{60}$ $\quad$ $\frac{6}{60} = \frac{1}{10}$

b.) The smallest no. of outcomes for $\frac{x}{64}$ and $\frac{y}{60}$ to be equal is if $x=16$, $y=15$ (to give probabilities of $\frac{1}{4}$). However, there are no totals for which there are 15 or 16 appearances (Max for $\{8,8\}$ is eight 9s, and for $\{6,10\}$ there are six 7s, 8s, ..., 11s). So, not possible

Factors of 60 : 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60

Factors of 64 : 1, 2, 4, 8, 16, 32, 64

$\frac{15}{60} = \frac{1}{4}$ $\quad$ $\frac{16}{64} = \frac{1}{4}$

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c.) Upto 7, the {8,8} and {6,10} have the same number of totals. Since 64ths are smaller than 60ths, the probability of rolling a 1,2,3....,7 will always be smaller for the {8,8} than the {6,10}.

	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16

For a total of 8, the {8,8} has a probability of $\frac{7}{64} ? \frac{1}{10} \left(\frac{6.4}{64}\right)$

the {10,6} has a probability of $\frac{6}{60} = \frac{1}{10}$

$\frac{7}{64} > \frac{6}{60}$ $\therefore$ the smallest total for which {8,8} has a higher probability than for {6,10} is $\underline{8}$

d.) Divisible by 3

	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

$P(\div 3) = \frac{22}{64} > \frac{22}{66} \left(\frac{1}{3}\right)$

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16

$P(\div 3) = \frac{20}{60} = \frac{1}{3}$

$\therefore$ $\underline{\{8,8\} \text{ is more likely}}$.

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e.)

Where at least one die is a 5

10

[5,5]

11

[5,6], [6,5]

2 ways to roll a total of 11 using the {8,8}, only 1 way to roll a 10. Hence 11 is more likely