PHYSICS ADMISSIONS TEST
November 2021

Time allowed: 2 hours

For candidates applying to Physics, Physics and Philosophy, Engineering, or Materials Science

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Total 24 questions [100 Marks]

Answers should be written on the question sheet in the spaces provided, and you are encouraged to show your working. You should attempt as many questions as you can.

No tables, or formula sheets may be used.

Answers should be given exactly and in simplest terms unless indicated otherwise.

Indicate multiple-choice answers by circling the best answer. Partial credit may be given for correct workings in multiple choice questions.

The numbers in the margin indicate the marks expected to be assigned to each question. You are advised to divide your time according to the marks available.

You may take the gravitational field strength on the surface of Earth to be $g \approx 10\,\mathrm{m\,s^{-2}}$

Do NOT turn over until told that you may do so.

These solutions are provided by Luke, an experienced PAT tutor. You can find more information and contact him through his profile below:

DBS

5.0 (35)

Luke G. PMT Courses Tutor

University of York - MSc Physics

Physics, Maths and Further Maths tutor | Specific experience as a mentor and tutor for students with ASC and ADHD

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View

2^5 $2^{10} = (2^2)^5 = 4^5$ Try successive powers of 5

1. What is the next number in the sequence? 1, 32, 243, 1024, 3125 [2]

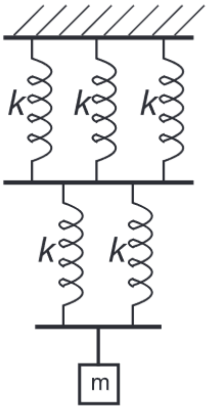
A	B	C	D	E
5040	6225	7164	7776	8192

$1 = 1^5$
$32 = 2^5$
$243 = 3^5$
$1024 = 4^5$
$3125 = 5^5$

$\therefore\ 6^5 = 7776$

k_{eff} = Effective / "overall" spring const.

2. What is the effective spring constant of the combination of springs shown in the diagram, if each spring has spring constant k? [2]



A	B	C	D	E
$\frac{5}{6}k$	k	$\frac{6}{5}k$	$2k$	$5k$

Springs in Series:

k_1, k_2

Hooke:

$F_1 = k_1 x_1, \quad F_2 = k_2 x_2$

Tensions are equal:

$F_1 = F_2 = F$.

Extensions add:

$x = x_1 + x_2$

$x = \frac{F}{k_1} + \frac{F}{k_2}$

$x = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$

$\frac{x}{F} = \frac{1}{k_1} + \frac{1}{k_2}$

$\frac{1}{k} \rightarrow \frac{x}{F}$

$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$

Springs in Parallel:

k_1, k_2

$F_1 = k_1 x_1$

$F_2 = k_2 x_2$

Extensions are equal:

$x_1 = x_2 = x$

Tensions add:

$F = F_1 + F_2$

$F = k_1 x + k_2 x$

$\therefore F = (k_1 + k_2) x$

$k_{eff} = k_1 + k_2$

$$\frac{1}{k_{TOT}} = \frac{1}{3k} + \frac{1}{2k}$$

$$= \frac{2+3}{6k}$$

$$\therefore k_{TOT} = \frac{6k}{5}$$

$$\sum_{k=1}^{N} 1 = \underbrace{1+1+\dots+1}_{N \text{ times}} = N$$

$$\sum_{k=1}^{N} k = 1+2+\dots+N \quad \text{(Arithmetic Progression)}$$

$$= \frac{N}{2}\left(2\times 1 + (N-1)1\right) \qquad S_N = \frac{N}{2}\left(2a+(n-1)d\right)$$

5

$$= \frac{N}{2}(N+1)$$

3. Evaluate $\sum_{n=1}^{10}\left(2 - \frac{n}{2} + 2^n\right)$. [2]

A	B	C	D	E
$2^{10} - \frac{11}{2}$	$2^{12} - \frac{19}{2}$	$2^{11} - \frac{19}{2}$	$2^{10} - \frac{11}{4}$	$2^{11} - \frac{11}{2}$

Geometric:

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\sum_{n=1}^{10} 2 - \frac{n}{2} + 2^n$$

GEO. $a=2$, $r=2$

$$= \sum_{1}^{10} 2 - \sum_{1}^{10} \frac{n}{2} + \sum_{1}^{10} 2^n$$

ARITHM. $a=1$, $d=1$

$$= \underbrace{(2+2\dots+2)}_{10 \text{ times}} - \frac{1}{2}\sum_{1}^{10} n + \frac{2(1-2^{10})}{1-2}$$

$$= 20 - \frac{1}{2}\left[\frac{10}{2}(10+1)\right] + 2(2^{10}-1)$$

$$= 20 - \left(\frac{10}{4}\times 11\right) + 2^{11} - 2$$

$$= 18 - \frac{110}{4} + 2^{11}$$

$$= \frac{36}{2} - \frac{55}{2} + 2^{11}$$

$$= 2^{11} - \frac{19}{2}$$

Work Done = K.E. Gained

$QV = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{2QV}{m}}$

4. Five different ions are accelerated from rest by the same potential difference. Which will have the smallest final velocity? [2]

	A	B	C	D	E
	$^{6}_{3}Li^{2+}$	$^{7}_{3}Li^{2+}$	$^{7}_{3}Li^{3+}$	$^{9}_{4}Be^{3+}$	$^{9}_{4}Be^{4+}$
$\frac{Q}{m}$:	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{3}{9}$	$\frac{4}{9}$ ×14
	×21	×18	×18	×14	
	$=\frac{42}{126}$	$=\frac{36}{126}$	$=\frac{54}{126}$	$=\frac{42}{126}$	$=\frac{56}{126}$

$v \propto \sqrt{\frac{Q}{m}}$. Smallest final velocity = smallest Q to m ratio.

5. Gravity on the Moon satisfies

$$g_{Moon} = \frac{1}{6} g_{Earth}.$$

A ball dropped on Earth from a height h takes a time t to reach the ground. From which height should it be dropped on the Moon so that it takes the same time t to reach the surface? You can neglect all effects of air resistance. [2]

A	B	C	D	E
$\frac{1}{36}h$	$\frac{1}{6}h$	$\frac{1}{\sqrt{6}}h$	h	$6h$

Earth:

$(\downarrow +)$

s	h
u	0
$\cancel{v}$	
a	g
t	t

$$h = \frac{1}{2} g t^2$$

Moon:

$(\downarrow +)$

s	$?$
u	0
$\cancel{v}$	
a	$\frac{1}{6}g$
t	t

$$s = \frac{1}{2}\left(\frac{g}{6}\right)t^2$$

$$s = \frac{1}{6}\left(\frac{1}{2} g t^2\right)$$

$$s = \frac{h}{6}$$

6. Two unbiased dice are rolled and the numbers obtained are added. If the probability of getting the sum S is $P(S)$, which of the following statements are true? [2]

1. $P(10) + P(11) = P(6)$
2. $P(6) > P(8)$
3. $P(2) + P(3) + P(4) > P(7)$
4. $P(7) = \frac{3}{2}P(5)$
5. $P(11) = P(3)$

A	B	C	D	E
1,2,4	3,4,5	2,3,4	1,3,5	1,4,5

Outcome Space m+n :

m \ n	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

36 possible outcomes.

- $P(2) = \frac{1}{36}$
- $P(3) = \frac{2}{36}$
- $P(4) = \frac{3}{36}$
- $P(5) = \frac{4}{36}$
- $P(6) = \frac{5}{36}$
- $P(7) = \frac{6}{36}$
- $P(8) = \frac{5}{36}$
-
- $P(10) = \frac{3}{36}$
- $P(11) = \frac{2}{36}$
-

1. $\frac{3}{36} + \frac{2}{36} = \frac{5}{36}$ True.

2. $\frac{5}{36} > \frac{5}{36}$. False.

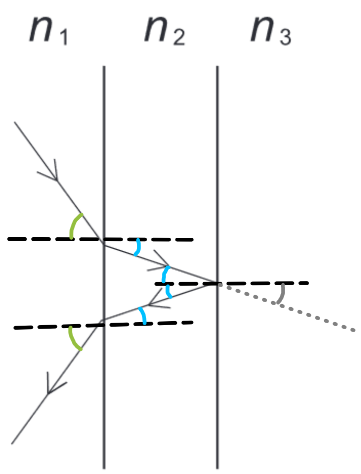
3. $\frac{1}{36} + \frac{2}{36} + \frac{3}{36} > \frac{6}{36}$ False.

4. $\frac{6}{36} = \frac{3}{2} \times \frac{4}{36}$ True.

5. $\frac{2}{36} = \frac{2}{36}$ True

$\therefore 1, 4, 5.$

7. A light ray follows a path through three media separated by plane boundaries as shown in the diagram, with refractive indices n_1, n_2 and n_3.



Snell:
$n_1 \sin\theta_1 = n_2 \sin\theta_2$

TIR:
$n_1 \sin\theta_c = n_2 \sin(90°)$

$\therefore \sin\theta_c = \frac{n_2}{n_1}$

So, TIR only occurs if
$\frac{n_2}{n_1} < 1$
$n_2 < n_1$

$\left.\begin{array}{l} n_2 > n_1 \\ n_2 > n_3 \\ n_1 > n_3 \end{array}\right\} \therefore n_3 < n_1 < n_2$

Which of the following sequences puts the refractive indices in order of increasing value? [2]

A	B	C	D	E
n_1, n_2, n_3	n_2, n_1, n_3	n_1, n_3, n_2	n_3, n_1, n_2	n_3, n_2, n_1

$1 \to 2$:
$n_1 \sin\theta_1 = n_2 \sin\theta_2$ (A)

Light bends towards normal
so 2 is more optically dense than 1.
$n_2 > n_1$

$2 \to 3$:
θ_2 must be $> \theta_c$.
$\therefore \sin\theta_2 > \frac{n_3}{n_2}$ (B)
and $n_3 < n_2$ since TIR occurs.

Combine (A) & (B)
$$\frac{n_1 \sin\theta_1}{n_2} > \frac{n_3}{n_2}$$
$$\therefore n_1 \sin\theta_1 > n_3$$
$$\sin\theta_1 > \frac{n_3}{n_1} \quad (C)$$

If n_1 and n_3 were to share a boundary, from $n_1 \to n_3$ the critical angle would be:
$$\sin\theta_c = \frac{n_3}{n_1}$$
So from (C), this means $\sin\theta_1 > \sin\theta_c$. So TIR would occur for angle of incidence θ_1. $\therefore n_1 > n_3$

8. If $f(x) = x^2$ and $g(x) = x + 3$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y = f(g(x)) - g(f(x))$. [2]

A	B	C	D	E
6	$2x + 5$	$2x - 1$	$6x + 6$	2

$f(g(x)) = f(x+3) = (x+3)^2$

$g(f(x)) = g(x^2) = x^2 + 3$

$y = (x+3)^2 - (x^2+3)$

$= x^2 + 6x + 9 - x^2 - 3$

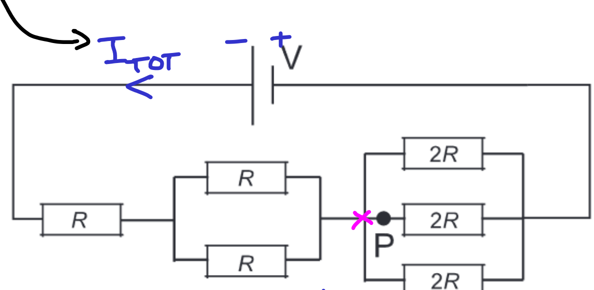
$y = 6x + 6$

$\therefore \frac{dy}{dx} = 6$

9. What is the current at the point P in the diagram? [2]

A	B	C	D	E
$\frac{2V}{13R}$	$\frac{2V}{11R}$	$\frac{V}{9R}$	$\frac{6V}{13R}$	$\frac{6V}{11R}$

Note: Electron flow.



$R \quad \underbrace{\tfrac{R}{2}} \quad \underbrace{\tfrac{2R}{3}}$

$$\therefore \quad R_{TOT} = R + \frac{R}{2} + \frac{2R}{3} = \frac{6R+3R+4R}{6} = \frac{13R}{6}$$

- At point ×, total current splits evenly three ways.

$$\therefore \quad I_P = \frac{I_{TOT}}{3}$$

- $V_{TOT} = I_{TOT} R_{TOT} \implies I_{TOT} = \dfrac{V}{\frac{13R}{6}} = \dfrac{6V}{13R}$

$$\therefore \quad I_P = \frac{1}{3} \times \frac{6V}{13R} = \frac{2V}{13R}$$

10. Which of these represents a simpler form for $\cos\left(\sin^{-1}(x)\right)$? [2]

A	B	C	D	E
$\sqrt{1-x^2}$	$\sqrt{1+x^2}$	$\sqrt{1-x}$	$1-x^2$	$\sqrt{x-1}$

$$\cos^2(x)+\sin^2(x)=1$$

$$\therefore \cos(x)=\sqrt{1-\sin^2(x)}$$

$$\cos(x)=\sqrt{1-[\sin(x)]^2} \quad (1)$$

Then replace $x \to \sin^{-1}(x)$ in (1) :

$$\cos\left(\sin^{-1}(x)\right)=\sqrt{1-[\sin(\sin^{-1}(x))]^2}$$

$$=\sqrt{1-x^2}$$

11. Consider the following five graphs.

1. Force (y-axis) against distance (x-axis) $Fd =$ Work
2. Force (y-axis) against time (x-axis) $Ft =$ Impulse (Change in Momentum)
3. Velocity (y-axis) against time (x-axis) $vt =$ Displacement
4. Mass (y-axis) against velocity squared (x-axis) $mv^2 = 2 \times K \cdot E$.
5. Voltage (y-axis) against charge (x-axis) $VQ =$ Work ← Definition of Voltage: $V = \frac{W}{Q}$.

For which graphs could the area under the graph potentially be a measurement of energy? [2]

A	B	C	D	E
1, 4, 5	1, 5	1, 4	1, 3, 4	All of them.

The areas under 1, 4 & 5 all have units of energy - Joules (kgm^2s^{-2})

Odd Function : Changes sign across the origin.

Even Function : Same sign across the origin.

Integrals with symmetric limits → odd integrand → = zero

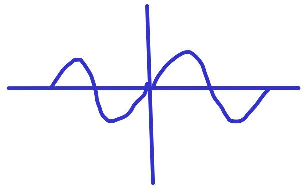
→ even integrand → ≠ zero in general

12. Which of the following integrals are equal to zero (you do **not** need to evaluate the integrals explicitly)? [2]

1. $\int_{-\pi/2}^{\pi/2} \sin(3x)\ \mathrm{d}x$ odd ∴ zero
2. $\int_{-\sqrt{7}}^{\sqrt{7}} \frac{1}{9}x^5 - \frac{1}{7}x^3 + \frac{1}{21}x\ \mathrm{d}x$ odd ∴ zero
3. $\int_0^1 \cos x\ \mathrm{d}x$ non-zero (asymmetric limits) $\approx \int_0^{\frac{\pi}{3}} \cos x\, dx$
4. $\int_{-2}^{2} \frac{1}{4}x^4 + \frac{1}{12}x^2\ \mathrm{d}x$ even ∴ non-zero
5. $\int_{-90}^{90} \sin(5x) - \frac{1}{2}\sin x\ \mathrm{d}x$ odd ∴ zero

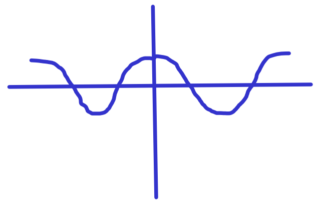
A	B	C	D	E
1,2	3,4	1	1,2,5	1,5

sin(nx)



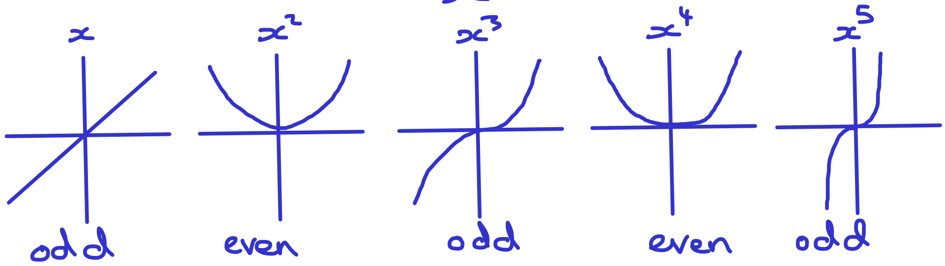
odd

cos(nx)



even

x^n



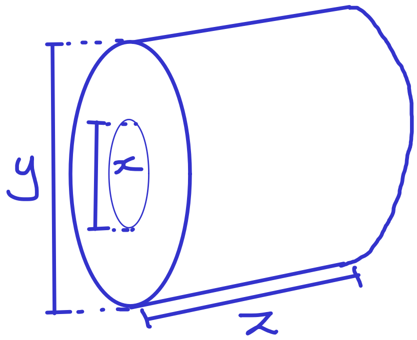
x^n → odd n = odd graph.

→ even n = even graph.

$$V_{cylinder} = \pi r^2 h = \frac{\pi d^2 h}{4}$$

13. A toilet roll is made up of an inner cardboard tube, diameter x cm, with toilet paper wrapped around it to give an overall diameter of y cm. The length of the cardboard tube is z cm. Suppose the diameter of the inner tube is reduced from x cm to $(x-1)$ cm, but the volume of toilet *paper* is kept the same. What is the difference in the *total* volume of the roll? [3]

$$V_{TOT}^{before} = \frac{\pi y^2 z}{4}$$



$$V_{paper}^{before} = \frac{\pi y^2 z}{4} - \frac{\pi x^2 z}{4}$$

$$= (y^2 - x^2)\frac{\pi z}{4}$$

$x \to x-1 \quad y \to y' \quad z \to z$ (assume length is the same)

$$V_{TOT}^{after} = \frac{\pi y'^2 z}{4} \quad A$$

$$V_{paper}^{after} = \frac{\pi y'^2 z}{4} - \frac{\pi (x-1)^2 z}{4}$$

$$= \left(y'^2 - (x-1)^2\right)\frac{\pi z}{4} = V_{paper}^{before}$$

$$\left(y'^2 - (x-1)^2\right)\frac{\pi z}{4} = (y^2 - x^2)\frac{\pi z}{4}$$

$$y'^2 - \cancel{x^2} + 2x - 1 = y^2 - \cancel{x^2}$$

$$y'^2 = y^2 - 2x + 1 \quad B$$

$B \to A$

$$V_{TOT}^{after} = \frac{\pi (y^2 - 2x + 1) z}{4}$$

$$V_{TOT}^{after} = \frac{\pi y^2 z}{4} + \frac{\pi (1-2x) z}{4}$$

$$\therefore V_{TOT}^{after} - V_{TOT}^{before} = \frac{\pi (1-2x) z}{4}$$

14. A machine requires 10kW of electrical power when operating at 100V.

(a) If power is delivered through a cable with resistance 100Ω, how much power is lost to the cable's resistance? [2]

(b) In order to reduce power loss, a transformer is used between the cable and the machine. What ratio of turns is required on the transformer in order to reduce power loss on the cable by a factor of 10^4? [2]

a.) $P = IV$

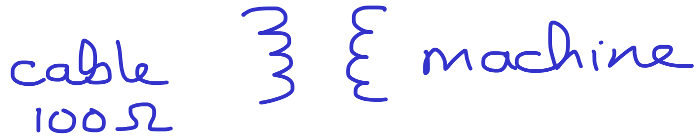
$\Rightarrow I = \frac{100\cancel{00}}{1\cancel{00}} = 100A$ drawn by the machine.

$\therefore P_{\text{dissipated in wire}} = I^2 R = (100)^2 (100)$

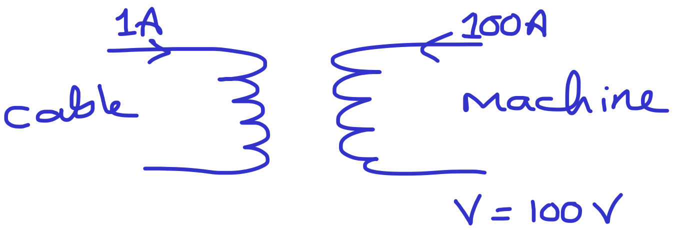
$= (10^2)^2 \times 10^2$

$= 10^6 W = \underline{\underline{1 MW}}$

b.)



To reduce $P_{disp} = I^2 R$ by a factor of 10^4 (since R is constant for the wire), I^2 must be reduced by a factor of 10^4 i.e. I must be reduced by 10^2 in the cable. $I = 100A \rightarrow I = 1A$



$\frac{V_P}{V_S} = \frac{N_P}{N_S}$

$I_S V_S = I_P V_P$

$\frac{V_P}{V_S} = \frac{I_S}{I_P} = \frac{100}{1} = \frac{N_P}{N_S}$ $\therefore$ $N_P : N_S$ $\underline{\underline{100 : 1}}$

15. The observed brightness of a Sun-like star shows periodic dips. The time period between these dips $T = 225$ days.

(a) If these dips are interpreted as being due to the transits of a planet in a circular orbit around the star, estimate the radius R of that orbit.

You may assume that the star has the same mass as the Sun and that the planet's mass is much smaller than the mass of the star. You can take the mean radius of the Earth's orbit around the Sun to be approximately $R_E = 1.5 \times 10^{11}$ m. [5]

a.) Circular orbit $\therefore F_G = F_c$

$$\frac{GM\cancel{m}}{R^2} = \cancel{m}\omega^2 r$$

$$\omega^2 = \frac{GM}{R^3}$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{GM}{R^3}$$

$$\therefore \frac{T^2 M}{R^3} = \text{const.} \qquad \frac{T_E^2 M_S}{r_E^3} = \text{const}$$

$M \approx M_S$

$$\frac{T^2}{R^3} = \frac{T_E^2}{r_E^3}$$

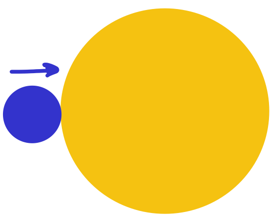
$$R^3 = \left(\frac{T}{T_E}\right)^2 r_E^3$$

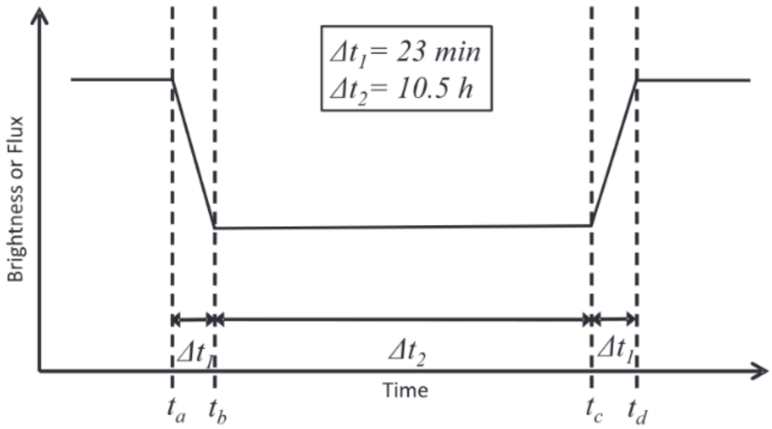
$$R = \left(\frac{225}{365}\right)^{\frac{2}{3}} \times (1.5 \times 10^{11})$$

$$= \underline{1.09 \times 10^{11}\,\text{m (3sf)}}$$

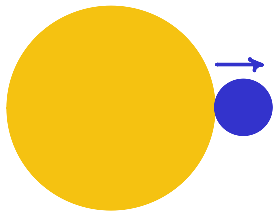
(b) A model of a transit and its effect on the star's brightness is shown in the figure below. Assuming that we observe the system in the plane of the planet's orbit, draw the relative positions of the planet and the star at the times t_a to t_d and calculate the radii of the planet, R_P and the star, R_S.

At t_a:





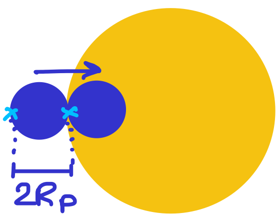
At t_d:



[5]

$t_a \rightarrow t_b$

From when the star begins to dim, to when the planet is completely in front of the star.

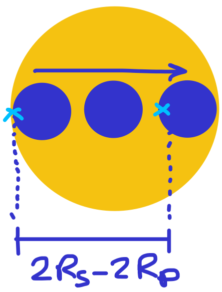


A point on the planet moves a distance $2R_p$ in a time Δt_1.

$$v = \frac{2R_p}{\Delta t_1} \quad (1)$$

$t_b \rightarrow t_c$

When the planet transits across the star.

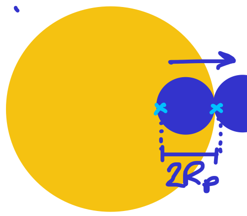


A point on the planet moves a distance $2R_s - 2R_p$ in a time Δt_2.

$$v = \frac{2(R_s - R_p)}{\Delta t_2} \quad (2)$$

$t_c \rightarrow t_d$

From when the star begins to brighten, to when the planet stops obscuring the star.



A point on the planet moves a distance $2R_p$ in a time Δt_1.

$$v = \frac{2R_p}{\Delta t_1}$$

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Orbit radius from a.)

Orbital speed of the planet, $v = \frac{2\pi R}{T} = \frac{2\pi\,(1.09 \times 10^{11})}{225 \times 24 \times 3600}$

$= 35100\ ms^{-1}$ (3sf)

Sub v into ①:

$R_P = \frac{1}{2} \times 35\,115\ldots \times (23\,min \times 60) = \underline{2.42 \times 10^{7}\,m}$ (3sf)

Sub v & R_P into ②:

$R_s = R_P + \frac{v\Delta t_2}{2} = (2.42\ldots \times 10^{7}) + \frac{(35100\ldots \times 10.5\,hrs \times 3600)}{2}$

$R_s = \underline{6.88 \times 10^{8}\,m}$ (3sf)

16. A vehicle travels in a fixed direction at a velocity $v(t)$ that varies with time as follows:

$$\begin{array}{rll} t < t_1 & v(t) = At^2 & \xrightarrow{\frac{d}{dt}} a(t) = 2At \\ t_1 < t < t_2 & v(t) = C - B(t-t_2)^2 & \longrightarrow a(t) = -2B(t-t_2) \\ t > t_2 & v(t) = v_2. & \longrightarrow a(t) = 0 \end{array}$$

Here A, B, C and v_2 are constants.

(a) Find A, B and C such that both the velocity and acceleration are continuous at $t = t_1$ and $t = t_2$. [3]

(b) Sketch the velocity and acceleration as a function of time. [3]

(c) Find the total distance travelled between $t = 0$ and $t = t_3$, where $t_3 > t_2$. [4]

a.) For velocities to be continuous:

$\boxed{t = t_1}$

$$At_1^2 = C - B(t_1 - t_2)^2$$

$\boxed{t = t_2}$

$$C - B(t_2 - t_2)^2 = v_2$$

$$C = v_2 \quad (1)$$

Substitute (1):

$$At_1^2 = v_2 - B(t_1 - t_2)^2 \quad (2)$$

For accelerations to be continuous:

$\boxed{t = t_1}$

$$2At_1 = 2B(t_1 - t_2)$$

$$At_1 = B(t_1 - t_2) \quad (3)$$

$\times(t_1 - t_2)$

$$At_1(t_1 - t_2) = -B(t_1 - t_2)^2 \quad (4)$$

$\boxed{t = t_2}$

$$-2B(t_2 - t_2) = 0$$

$$0 = 0$$

Consistent, but no further information.

(4) into (2):

$$At_1^2 = v_2 + At_1(t_1 - t_2)$$

$$\cancel{At_1^2} = v_2 + \cancel{At_1^2} - At_1t_2$$

$$\therefore A = \frac{v_2}{t_1 t_2} \quad (5)$$

(5) into (3):

$$\frac{v_2}{\cancel{t_1} t_2} \times \cancel{t_1} = B(t_1 - t_2)$$

$$\therefore B = \frac{v_2}{t_2(t_1 - t_2)}$$

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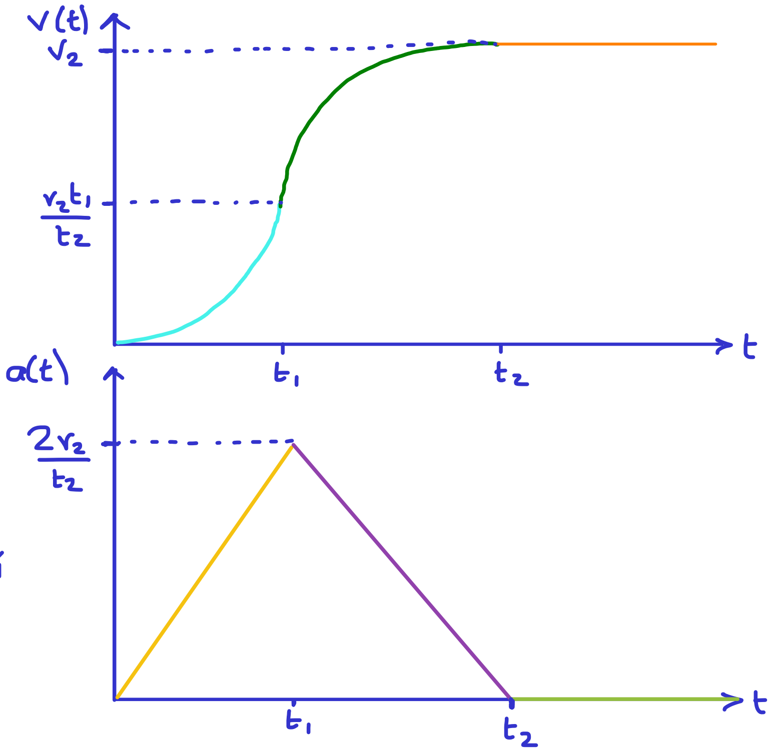
b.)

At $t = t_1$:

$v(t_1) = At_1^2$

$= \frac{v_2}{\cancel{t_1} t_2} \times t_1^{\cancel{2}}$

$= \frac{v_2 t_1}{t_2}$



$t<t_1 : At^2$ Quadratic ∪ from the origin

$t_1<t<t_2 : C-B(t-t_2)^2$ Quadratic ∩ which connects on (continuous $v(t)$)

$t>t_2 : v_2$ Horizontal line

At t_1,

$a(t_1) = 2At_1$

$= 2\frac{v_2}{\cancel{t_1} t_2}\cancel{t_1}$

$= \frac{2v_2}{t_2}$

$t<t_1 : 2At$ $y=mx$ graph

$t_1<t<t_2 : -2B(t-t_2)$ $y=mx$ graph with negative gradient which connects on (continuous $a(t)$)

$t>t_2 : 0$

Area under v-t graph = displacement.

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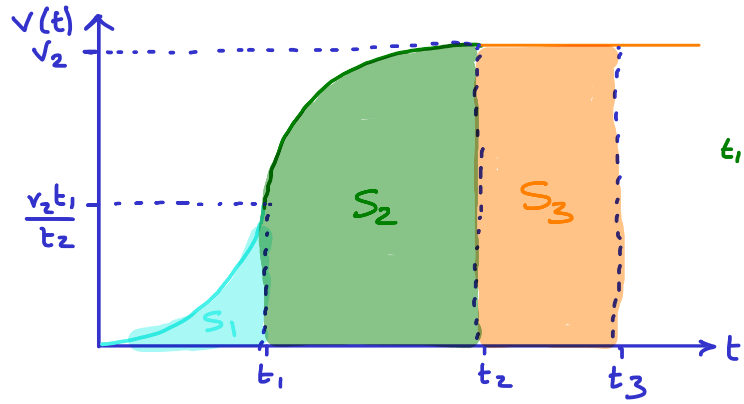
c.)



$t < t_1 : At^2 = \frac{v_2}{t_1 t_2} t$

$t_1 < t < t_2 : C - B(t-t_2)^2$
$= v_2 - \frac{v_2 (t-t_2)^2}{t_2(t_2-t_1)}$

$$S_1 = \int_0^{t_1} At^2 \, dt = \frac{At^3}{3}\Big|_0^{t_1} = \frac{At_1^3}{3}$$

$$S_2 = \int_{t_1}^{t_2} C - B(t-t_2)^2 \, dt = \left[Ct - \frac{B(t-t_2)^3}{3} \right]_{t_1}^{t_2}$$
$$= Ct_2 - 0 - \left(Ct_1 - \frac{B(t_1-t_2)^3}{3} \right)$$
$$= Ct_2 - Ct_1 + \frac{B(t_1-t_2)^3}{3}$$

$$S_3 = v_2(t_3 - t_2)$$

$$S_T = \frac{v_2 t_1^{\cancel{3}\,2}}{3\cancel{t_1} t_2} + \cancel{v_2 t_2} - v_2 t_1 \cancel{+}- \frac{v_2 (t_1 - t_2)^{\cancel{3}\,2}}{3t_2\cancel{(t_2 - t_1)}} + v_2 t_3 - \cancel{v_2 t_2}$$

$$= \frac{v_2 t_1^2}{3t_2} - v_2 t_1 - \frac{v_2 (t_1-t_2)^2}{3t_2} + v_2 t_3$$

$$= \frac{v_2}{3t_2} \left\{ t_1^2 - (t_1 - t_2)^2 \right\} - v_2 t_1 + v_2 t_3$$

$$= \frac{v_2}{3t_2} \left\{ 2t_1 t_2 - t_2^2 \right\} - v_2 t_1 + v_2 t_3$$

$$= \frac{2}{3} v_2 t_1 - \frac{1}{3} v_2 t_2 - v_2 t_1 + v_2 t_3$$

$$= \frac{v_2}{3} \left(2t_1 - t_2 - 3t_1 + 3t_3 \right) = \frac{v_2}{3} \left(-t_1 - t_2 + 3t_3 \right)$$

Dimensions :

[M] mass

[L] length

[T] time

1 dimensionless quantity

17. In Einstein's theory of gravity, light passing a star of mass M, at a distance R from the centre of the star, is bent by an angle θ (measured in radians, 2π radians $= 360°$) as shown in the diagram.

(a) Assume that θ depends on the gravitational constant G and also depends on the mass of the star M, the distance R and the speed of light $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$, as

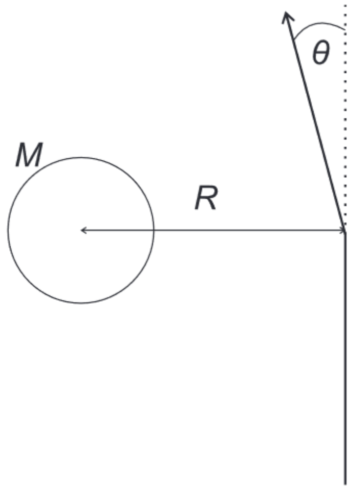
$$\theta = \lambda G M^{\alpha} R^{\beta} c^{\gamma},$$

where λ is an undetermined dimensionless constant.

By considering the dimensions of these quantities determine α, β and γ. [4]

(b) In SI units the numerical value of G is 6.67×10^{-11}.

Neglecting any spatial extent to the star, and taking $\lambda = 1$ and $M = 2 \times 10^{30}$ kg, determine the distance R such that light is bent so much that it falls directly into the star. [3]



To find units of G, consider gravitational force:

$$F = \frac{GMm}{r^2} \Rightarrow G = \frac{Fr^2}{Mm} = \frac{mar^2}{Mm} = \frac{ar^2}{M}$$

a) $\theta = \lambda G M^{\alpha} R^{\beta} c^{\gamma}$

$$[G] = \frac{[L][T]^{-2}[L]^2}{[M]}$$

$$[G] = [L]^3[M]^{-1}[T]^{-2}$$

Dimensions :

θ : 1

λ : 1

G : $[L]^3[M]^{-1}[T]^{-2}$

M : $[M]$

R : $[L]$

c : $[L][T]^{-1}$

$$1 = 1\,[L]^3[M]^{-1}[T]^{-2}[M]^{\alpha}[L]^{\beta}[L]^{\gamma}[T]^{-\gamma}$$

$$1 = [L]^{\beta+\gamma+3}[M]^{\alpha-1}[T]^{-\gamma-2}$$

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$$1 = [L]^{\beta+\gamma+3}[M]^{\alpha-1}[T]^{-\gamma-2}$$

All exponents on RHS must be zero as LHS is dimensionless.

$\beta+\gamma+3=0$ (1) $\quad \alpha-1=0 \quad -\gamma-2=0$

$\underline{\underline{\alpha=1}} \quad \underline{\underline{\gamma=-2}}$ (2)

(2) into (1):

$\underline{\underline{\beta=-1}}$

b) $\therefore \theta = \frac{\lambda GM}{Rc^2} \implies R = \frac{\lambda GM}{\theta c^2}$

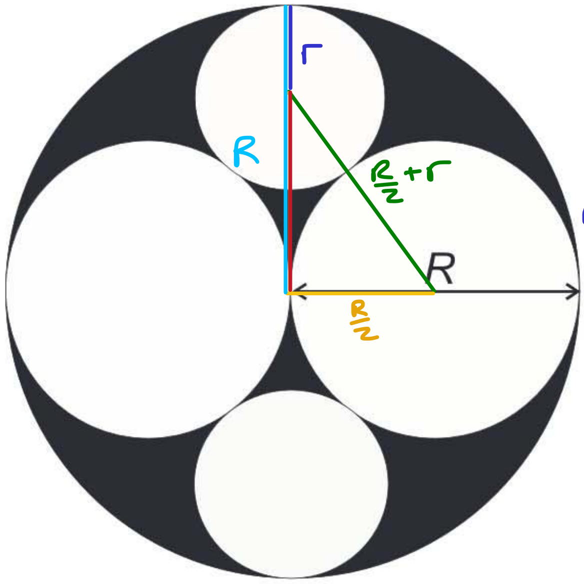
For the light to fall directly into the star, $\theta = \frac{\pi}{2}$:

$$R = \frac{1 \times 6.67\times10^{-11} \times 2\times10^{30}}{\frac{\pi}{2} \times (3\times10^8)^2}$$

$\underline{\underline{R = 944\text{m}}}$ (3sf)

18. Find the area of the dark regions of this figure in terms of the overall radius of the outer circle (denoted R). [5]

R | r | $R-r$



$$\left(R-r\right)^2 + \left(\frac{R}{2}\right)^2 = \left(\frac{R}{2}+r\right)^2$$

$$R^2 - 2Rr + \cancel{r^2} + \cancel{\frac{R^2}{4}} = \cancel{\frac{R^2}{4}} + Rr + \cancel{r^2}$$

$$R^2 = 3Rr$$

$$R = 3r$$

$$\therefore\ r = \frac{R}{3}$$

Dark Region $= \pi R^2 - 2$ large circles $- 2$ small circles

$$= \pi R^2 - 2 \times \pi \left(\frac{R}{2}\right)^2 - 2\pi\left(\frac{R}{3}\right)^2$$

$$= \pi R^2 - \frac{1}{2}\pi R^2 - \frac{2}{9}\pi R^2$$

$$= \left(\frac{1}{2} - \frac{2}{9}\right)\pi R^2$$

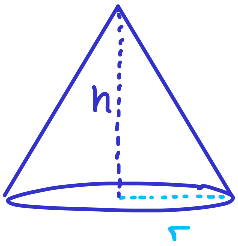
$$= \underline{\underline{\frac{5}{18}\pi R^2}}$$

19. Sand is poured at a constant rate onto a flat horizontal surface. It forms a pile in the shape of a cone with a constant slope.

(a) If r is the radius of the base of the cone, find how r varies with t, given that $r(0) = 0$. [3]

(b) If it takes a time t_1 for r to reach r_1, how long does it take for the radius to reach $2r_1$ (in terms of t_1)? [2]

a.)



Constant rate $\therefore \frac{dV}{dt} = \text{const.}$

$$\int dV = \int k \, dt$$

$$V = kt + C$$

$$\boxed{V=0, t=0}$$

$$C = 0$$

$$\therefore V = kt$$

$V_{cone} = \pi r^2 h$

$$\therefore \pi r^2 h = kt$$

Constant slope $\therefore$ gradient $= \frac{h}{r}$ is a constant, say θ.

$$\pi r^3 \frac{h}{r} = kt$$

$$\pi r^3 \theta = kt$$

$$r^3 = \frac{k}{\pi\theta} t$$

$$\therefore \underline{\underline{r(t) \propto \sqrt[3]{t}}}$$

b.) Constant rate $\therefore$ ratio of volume to time will remain the same.

$$\frac{r_1 \times \pi r_1^2 h_1}{r_1 \times t_1} = \frac{\pi r_2^2 h_2 \times r_2}{t_2 \times r_2}$$

As above, $\frac{h_1}{r_1} = \frac{h_2}{r_2} = \text{const.}$

$$\frac{\cancel{\pi} r_1^3}{t_1} \times \cancel{\frac{h_1}{r_1}} = \frac{\cancel{\pi} r_2^3}{t_2} \times \cancel{\frac{h_2}{r_2}}$$

$$\frac{r_1^3}{t_1} = \frac{(2r_1)^3}{t_2}$$

$$\frac{1}{t_1} = \frac{8}{t_2} \implies \underline{\underline{t_2 = 8t_1}}$$

$$^nC_r = \frac{n!}{r!(n-r)!}$$

$$^cC_2 = \frac{c!}{2!(c-2)!} = \frac{c(c-1)\cancel{(c-2)}\cancel{(c-3)}\ldots}{2\cancel{(c-2)}\cancel{(c-3)}\ldots} = \frac{1}{2}(c^2-c)$$

28

20. Expand $(a+bx)^c$, where c is a positive integer, as a polynomial in x up to and including terms involving x^2.

If the coefficient of the x^0 term is equal to $\frac{1}{16}$, $b=\frac{1}{4}$ and the coefficient of the x^2 term is $\frac{3}{32}$, find a and c. [4]

$$(a+bx)^c = {}^cC_0\, a^c (bx)^0 + {}^cC_1\, a^{c-1}(bx)^1 + {}^cC_2\, a^{c-2} b^2 x^2$$

$$= a^c + ca^{c-1}bx + \frac{1}{2}(c^2-c)a^{c-2}b^2x^2$$

$$a^c = \frac{1}{16} \quad (1)$$

c has to be a positive integer
∴ the only possible sets are:

I $a=\frac{1}{16}$, $c=1$

II $a=\frac{1}{4}$, $c=2$

III $a=\frac{1}{2}$, $c=4$

$$\frac{1}{2}(c^2-c)a^{c-2}b^2 = \frac{3}{32}$$

$$\frac{1}{2}(c^2-c)a^{c-2}\left(\frac{1}{4}\right)^2 = \frac{3}{32}$$

$$(c^2-c)a^{c-2} = 3$$

$$(c^2-c)\frac{a^c}{a^2} = 3$$

Sub in (1)

$$(c^2-c)\frac{1}{16} = 3a^2$$

$$c^2-c = 48a^2$$

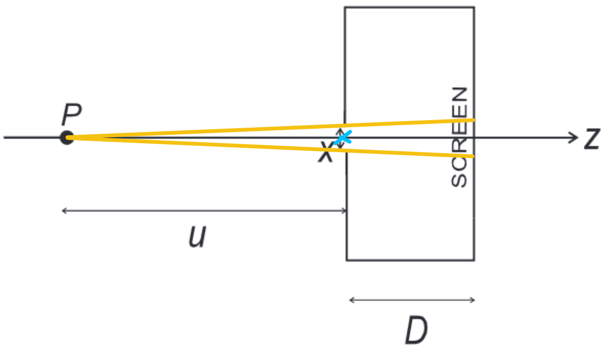
$$c(c-1) = 3\times16a^2$$

$$c(c-1) = 3\times(4a)^2 \quad (2)$$

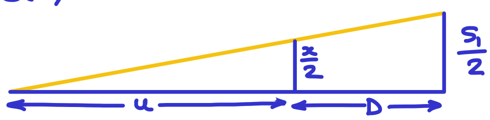
The only set that satisfies (2) is III

$$\therefore\ a=\frac{1}{2},\quad c=4$$

21. A pinhole camera consists of a box with a small aperture (of diameter x). A point light source P is at a distance u from the aperture outside the box. It projects an image on the screen, which is inside the box at a distance D from the aperture.



a.)



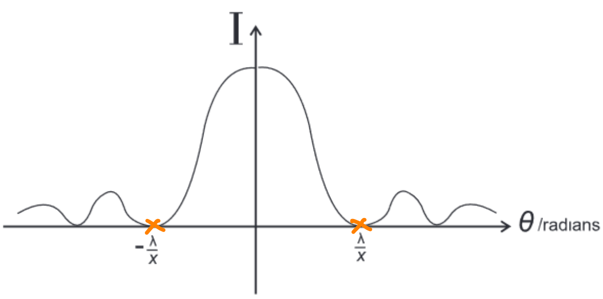
Similar triangles:

$$\frac{u}{\frac{x}{\cancel{2}}} = \frac{u+D}{\frac{s_1}{\cancel{2}}}$$

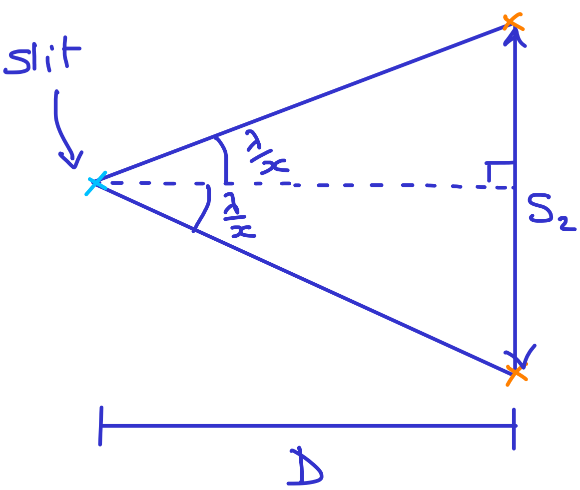
$$s_1 = \frac{(u+D)x}{u} = \left(1+\frac{D}{u}\right)x$$

(a) If diffraction is ignored, what is the diameter s_1 of the spot projected on the screen? [2]

(b) If $D \gg x$, diffraction of light of wavelength λ by the aperture produces a distribution of intensity that varies with angle θ (measured in radians from the z axis marked on the diagram; 2π radians $= 360$ degrees) approximately as shown below:



What is the approximate diameter, s_2, of the central peak of the diffraction pattern projected on the screen? [2]



$$\tan\left(\frac{\lambda}{x}\right) = \frac{\frac{s_2}{2}}{D}$$

$$\tan\left(\frac{\lambda}{x}\right) = \frac{s_2}{2D}$$

If $D \gg x$, small angle approx. $\tan(\theta) \approx \theta$

$$\frac{\lambda}{x} \approx \frac{s_2}{2D} \implies s_2 \approx \frac{2\lambda D}{x}$$

(c) Assume that the effects in part (a) and (b) can be combined to give a total diameter s_3, where

$$s_3^2 = s_1^2 + s_2^2.$$

Find the pinhole size that produces the sharpest image of the light source on the screen. [4]

x

$$S_3^2 = \left[\left(1+\frac{D}{u}\right)x\right]^2 + \left[\frac{2\lambda D}{x}\right]^2$$

$$S_3 = \left[\left(1+\frac{D}{u}\right)^2 x^2 + \frac{4\lambda^2 D^2}{x^2}\right]^{\frac{1}{2}}$$

Sharpest image = Minimum S_3. i.e. when $\frac{dS_3}{dx} = 0$.

$$\frac{dS_3}{dx} = \frac{1}{2}\left(\left(1+\frac{D}{u}\right)^2 x^2 + \frac{4\lambda^2 D^2}{x^2}\right)^{-\frac{1}{2}} \times \left[2\left(1+\frac{D}{u}\right)^2 x - \frac{8\lambda^2 D^2}{x^3}\right]$$

$\frac{dS_3}{dx} = 0$ when the numerator $= 0$:

$$2\left(1+\frac{D}{u}\right)^2 x - \frac{8\lambda^2 D^2}{x^3} = 0$$

$$\left(1+\frac{D}{u}\right)^2 x^4 = 4\lambda^2 D^2$$

$$(u+D)^2 x^4 = 4\lambda^2 D^2 u^2$$

$$x^4 = \frac{4\lambda^2 D^2 u^2}{(u+D)^2}$$

$$x^2 = \frac{2\lambda D u}{u+D}$$

$$x = \sqrt{\frac{2\lambda D u}{u+D}}$$

22.

(a) Sketch the functions $f(x) = x^2 + 2x - 7$ and $g(x) = \frac{2}{x}$ on the same graph. [2]

(b) Determine the coordinates of the points of intersection of the two curves and label their exact values. You are **not** required to determine coordinates of intersections with x or y axes. [4]

(c) Evaluate the finite area enclosed between the two curves. [4]

You are expected to show clear working for all parts.

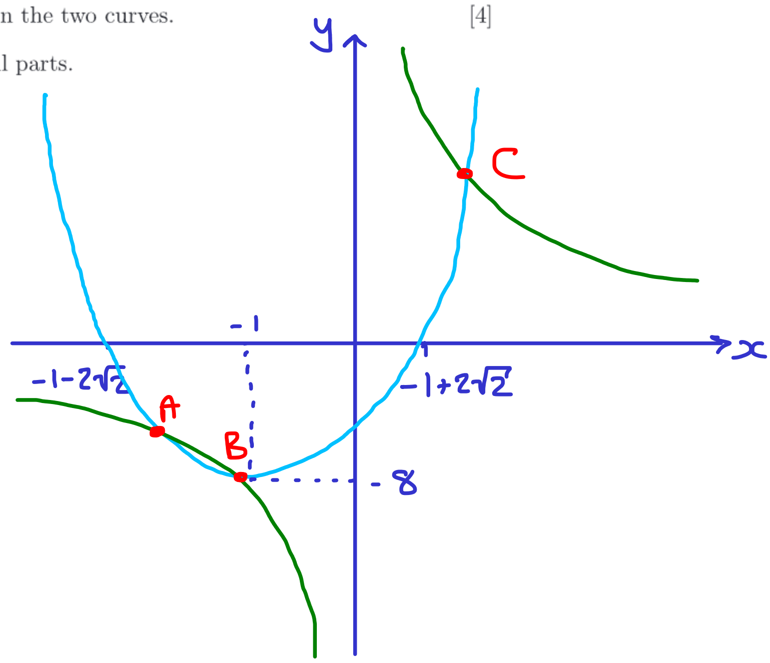
a.)

$$f(x) = x^2 + 2x - 7$$
$$= (x+1)^2 - 8$$

Minimum: $(-1, -8)$

Roots: $x = \dfrac{-2 \pm \sqrt{4 - 4(1)(-7)}}{2}$

$$= \frac{-2 \pm \sqrt{32}}{2}$$
$$= -1 \pm 2\sqrt{2}$$



b.) $x^2 + 2x - 7 = \dfrac{2}{x}$

$$x^3 + 2x^2 - 7x - 2 = 0$$

Try $x = 2$:

$$8 + 8 - 14 - 2 = 0$$

$\therefore x = 2$ is a root and $(x-2)$ is a factor

$$\begin{array}{r|l} & x^2 + 4x + 1 \\ x-2 & x^3 + 2x^2 - 7x - 2 \\ & -(x^3 - 2x^2) \\ & \quad 4x^2 - 7x \\ & \quad -(4x^2 - 8x) \\ & \qquad x - 2 \\ & \qquad -(x-2) \\ & \qquad 0 \end{array}$$

$\therefore x^3 + 2x^2 - 7x - 2 = (x-2)(x^2 + 4x + 1)$

$x = 2$ $\qquad$ $x = \dfrac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$

Then find the y-values.

$\boxed{x = 2}$

$$g(2) = \frac{2}{2} = 1$$

$\boxed{x = -2 \pm \sqrt{3}}$

$$g(-2 \pm \sqrt{3}) = \frac{2}{-2 \pm \sqrt{3}} \times \frac{(-2 \mp \sqrt{3})}{(-2 \mp \sqrt{3})} = \frac{-4 \mp 2\sqrt{3}}{4 - 3}$$
$$= -4 \mp 2\sqrt{3}$$

A $(-2 - \sqrt{3}, -4 + 2\sqrt{3})$ $\qquad$ B $(-2 + \sqrt{3}, -4 - 2\sqrt{3})$ $\qquad$ C $(2, 1)$

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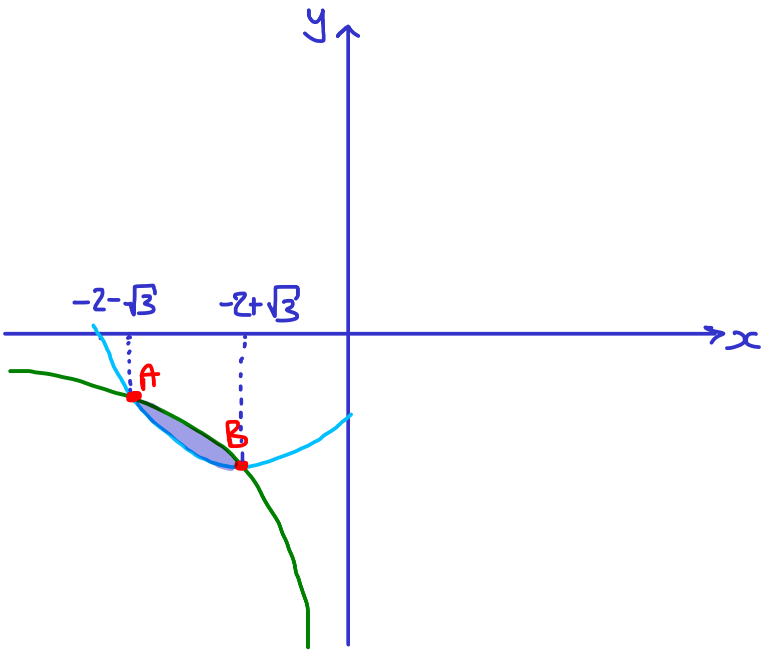
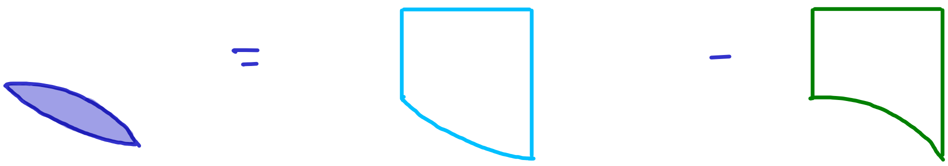
c.)





$$\text{Enclosed Area} = \int_{-2-\sqrt{3}}^{-2+\sqrt{3}} x^2+2x-7\,dx - \int_{-2-\sqrt{3}}^{-2+\sqrt{3}} \frac{2}{x}\,dx$$

$$= \int_{-2-\sqrt{3}}^{-2+\sqrt{3}} x^2+2x-7-\frac{2}{x}\,dx$$

$$= \left[\frac{x^3}{3}+x^2-7x-2\ln x\right]_{-2-\sqrt{3}}^{-2+\sqrt{3}}$$

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$$= \frac{(-2+\sqrt{3})^3}{3} + (-2+\sqrt{3})^2 - 7(\cancel{-2}+\sqrt{3}) - 2\ln(-2+\sqrt{3}) \ldots\ldots$$

$$\ldots\ldots - \frac{(-2-\sqrt{3})^3}{3} - (-2-\sqrt{3})^2 + 7(\cancel{-2}-\sqrt{3}) + 2\ln(-2-\sqrt{3})$$

$$= \frac{\cancel{-26}+15\sqrt{3}}{3} - \frac{(\cancel{-26}-15\sqrt{3})}{3} + \cancel{7} - 4\sqrt{3} - (\cancel{7}+4\sqrt{3}) - 14\sqrt{3} + 2\ln\left(\frac{-2-\sqrt{3}}{-2+\sqrt{3}}\right)$$

$$= 10\sqrt{3} - 8\sqrt{3} - 14\sqrt{3} + 2\ln\left(\frac{-2-\sqrt{3}}{-2+\sqrt{3}} \times \frac{-2-\sqrt{3}}{-2-\sqrt{3}}\right)$$

$$= -12\sqrt{3} + 2\ln\left(\frac{4+4\sqrt{3}+3}{4-3}\right)$$

$$= -12\sqrt{3} + 2\ln(7+4\sqrt{3})$$

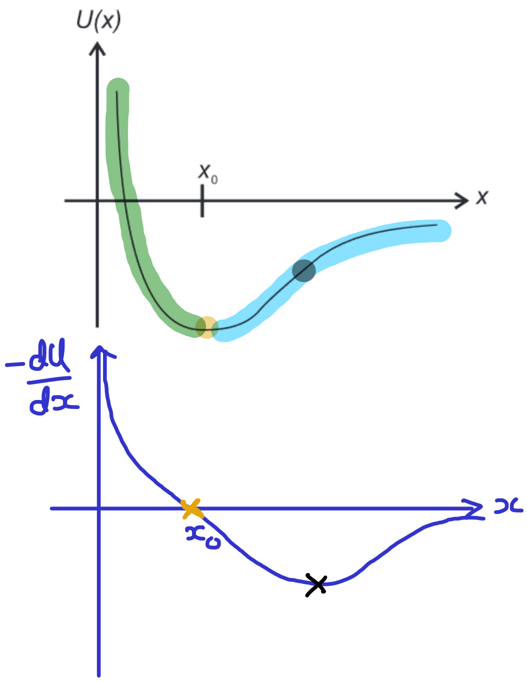
Area is below the x-axis $\therefore$ integral is negative, so multiply by -1 to get the magnitude of the area.

$$\text{Area} = \underline{\underline{12\sqrt{3} - 2\ln(7+4\sqrt{3})}} \approx 15.52$$

23. The potential energy for a system of two atoms separated by a distance x is $U(x)$ (shown below).

Sketch $-\frac{dU}{dx}$. [2]

What does this new curve represent physically and what does the point x_0 represent? [2]



$\frac{dU}{dx}$ is negative

$\frac{dU}{dx} = 0$

$\frac{dU}{dx}$ is positive, reaches a maximum, then tends to zero.

- $-\frac{dU}{dx}$ is force*. The x_0 point represents the equilibrium position, where the attractive and repulsive forces balance.

* Work done is equal to the negative change in the potential energy. (to justify the negative sign, consider an object being raised from the floor. It moves up $\therefore$ potential energy increases, but the force opposes the direction of motion so the work done is negative).

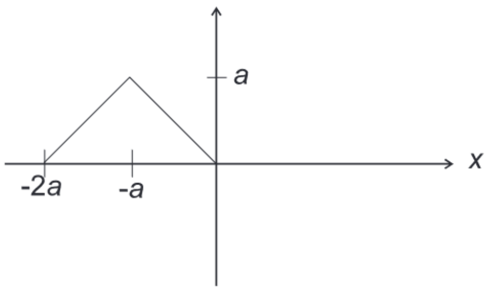
$$W = -dU$$

$$\int F\,dx = -dU$$

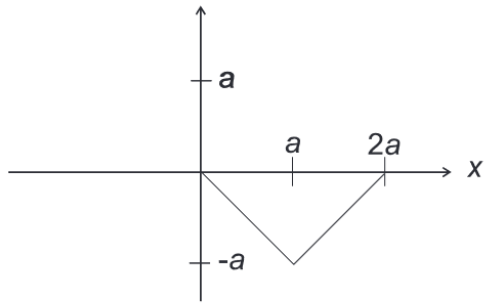
Diff both sides w.r.t. x.

$$F = -\frac{dU}{dx}$$

24. If $f(x)$ is



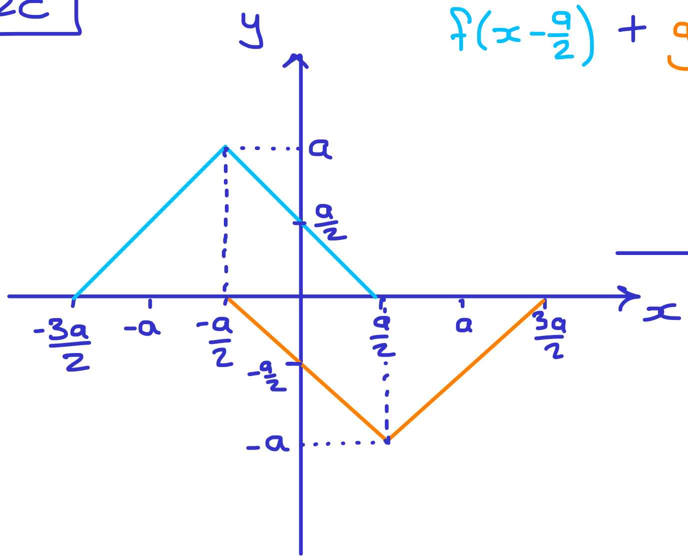
and $g(x)$ is

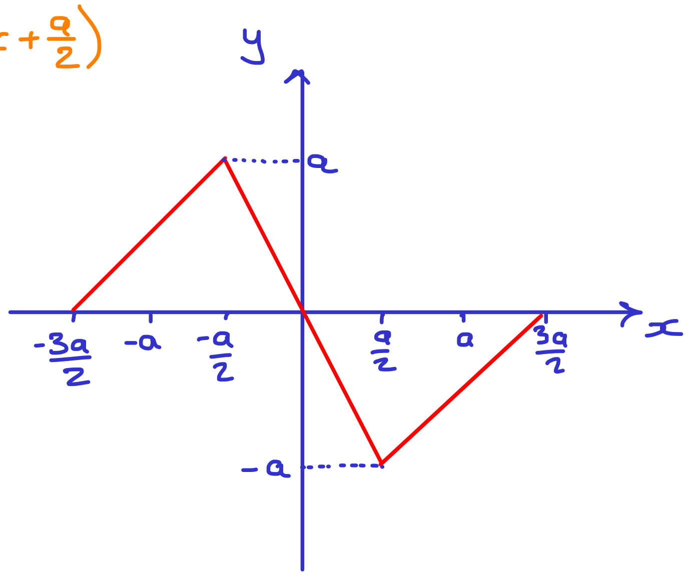


then sketch $f(x-ct)+g(x+ct)$ for $t = a/2c$, $3a/4c$ and a/c. Each sketch should be done separately with its own set of axes. [6]

$\boxed{t = \frac{a}{2c}}$

$f(x - \frac{a}{2}) + g(x + \frac{a}{2})$





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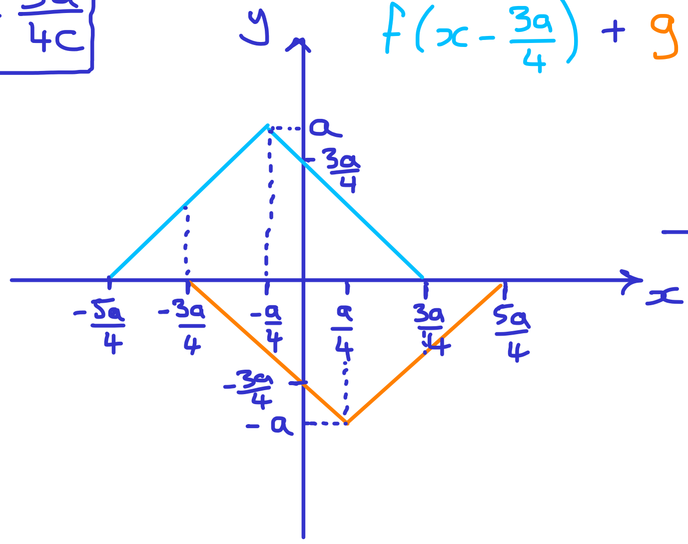
$\boxed{t = \frac{3a}{4c}}$

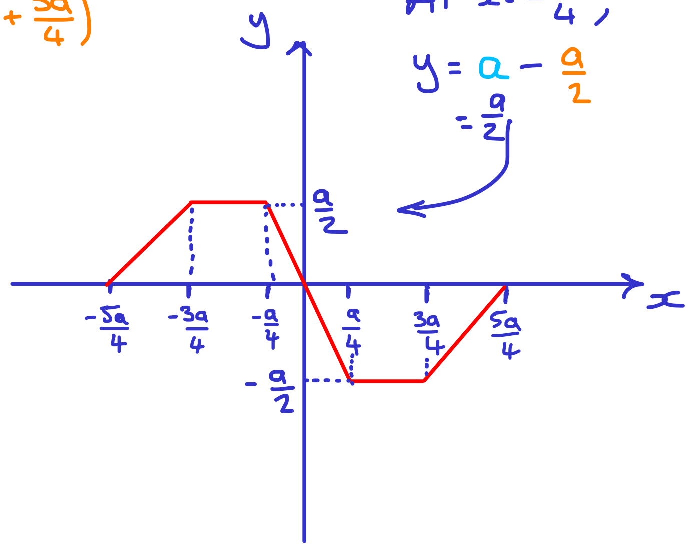
$f\left(x - \frac{3a}{4}\right) + g\left(x + \frac{3a}{4}\right)$



At $x = -\frac{a}{4}$,

$y = a - \frac{a}{2}$

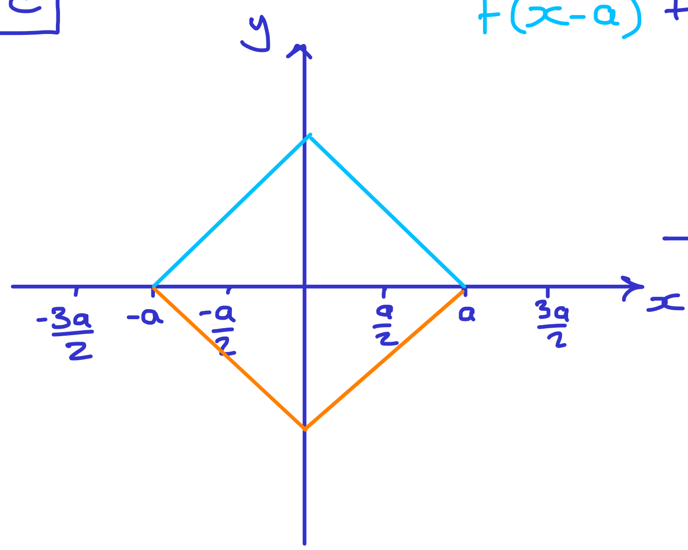
$= \frac{a}{2}$

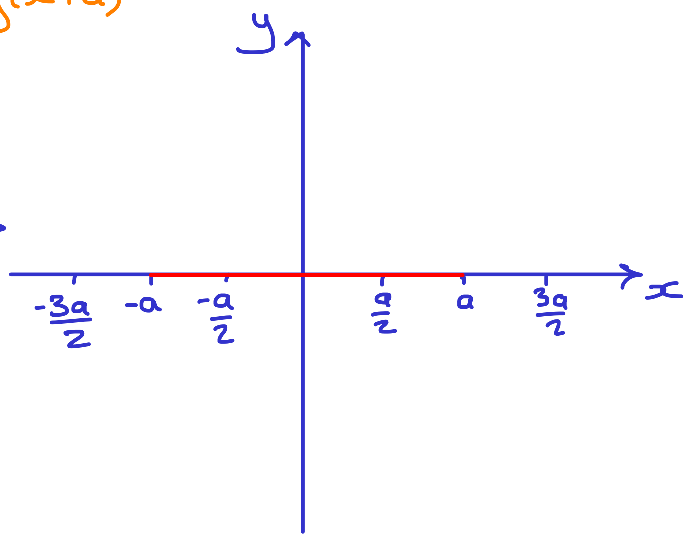


Note: For $-\frac{3a}{4} \le x \le -\frac{a}{4}$ and $\frac{a}{4} \le x \le \frac{3a}{4}$, the graph flat lines as f and g are symmetric across the lines $y = \frac{a}{2}$ and $y = -\frac{a}{2}$ respectively.

$\boxed{t = \frac{a}{c}}$

$f(x-a) + g(x+a)$





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