Wednesday, 2 November 2016

Time allowed: 2 hours

For candidates applying to Physics, Physics and Philosophy, Engineering, or Materials

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There are two Sections (A and B) to this test, each carrying equal weight.

Section A: Mathematics for Physics Q1-Q11 [50 Marks]

Section B: Physics Q12-Q21 [50 Marks]

Answers should be written on the question sheet in the spaces provided, and you should attempt as many questions as you can from each Section.

The numbers in the margin indicate the marks expected to be assigned to each question. You are advised to divide your time according to the marks available, and to spend equal effort on Sections A and B.

No calculators, tables, or formula sheets may be used.

Answers in Section A should be given exactly and in simplest terms unless indicated otherwise.

Numeric answers in Section B should be calculated to 2 significant figures unless indicated otherwise.

Do NOT turn over until told that you may do so.

Section A

1. Differentiate the expression $x \sin x^2$ with respect to x. [2]

$$f'(x) = \sin(x^2) + x\cos(x^2)\cdot 2x$$
$$= \sin(x^2) + 2x^2\cos(x^2)$$

product and chain rules

2. Find all values of θ between 0 and 2π which satisfy the equation

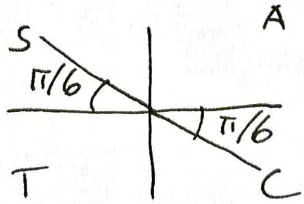
$$\sqrt{3}\tan^2\theta - 2\tan\theta - \sqrt{3} = 0.$$

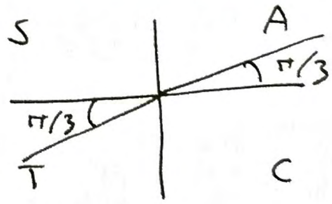
[5]

$$(\sqrt{3}\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$$

$$\tan\theta = -\frac{1}{\sqrt{3}} \quad \text{or} \quad \tan\theta = \sqrt{3}$$

$$\theta = \frac{5\pi}{6}, \frac{11\pi}{6} \qquad \theta = \frac{\pi}{3}, \frac{4\pi}{3}$$





θ	0	30	45	60	90
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
$\cos\theta$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

3. Given the two equations

$$\log_4\left(\frac{64^x}{16^y}\right) = 13 \quad (1)$$
$$\log_{10} 10^x + \log_3 3^y = 1, \quad (2)$$

find x and y. [5]

$(1): \log_4 64^x - \log_4 16^y = 13$

$\log_4 4^{3x} - \log_4 4^{2y} = 13$

$3x \log_4 4 - 2y \log_4 4 = 13$

$3x - 2y = 13 \quad (3)$

$(2): x \log_{10} 10 + y \log_3 3 = 1$

$x + y = 1$

$2x + 2y = 2 \quad (4)$

$(3)+(4): 5x = 15$

$x = \underline{3}$

In $(4): 2(3) + 2y = 2$

$y = \underline{-2}$

$\log_a a = 1$

$\log_c ab = \log_c a + \log_c b$

$\log_c\left(\frac{a}{b}\right) = \log_c a - \log_c b$

$\log_c a^b = b \log_c a$

4. What is the term independent of x in the expansion

$$\left(x - \frac{1}{x^2}\right)^{12}?$$

[3]

$${}^{12}C_0\, x^{12} \left(-\frac{1}{x^2}\right)^0$$

$$+ \; {}^{12}C_1\, x^{11} \left(-\frac{1}{x^2}\right)^1$$

$$+ \; \ldots$$

$$+ \; {}^{12}C_4\, x^8 \left(-\frac{1}{x^2}\right)^4 \longrightarrow {}^{12}C_4$$

$$+ \; \ldots$$

$$\boxed{{}^nC_r = \frac{n!}{r!\,(n-r)!}}$$

$${}^{12}C_4 = \frac{12!}{4!\;8!} = \frac{\overset{3}{\cancel{12}} \times 11 \times \overset{5}{\cancel{10}} \times \cancel{9}^{3}}{\cancel{4} \times \cancel{3} \times \cancel{2}}$$

$$= \underline{495}$$

5. How many numbers greater than 5000 may be formed by using some or all of the digits 3, 4, 5, 6, and 7 (but only those) without repetition? [3]

$4 \times 3 \times 2 = 24$

4 digits: 5 _ _ _
6 _ _ _
7 _ _ _

$\left.\right\}$ total $= 24 \times 3 = 72$

5 digits: $\underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 120$

$+$

192

6. A seed is planted. After one month, there is one twig containing two leaves. In the following month, the twig grows two further twigs, each new twig again containing two leaves. If in successive months each new twig produces two further twigs with their leaves, how many leaves will be on the plant after 10 months? (You may assume that no leaves fall off the plant, and once a new twig has produced two twigs it does not produce further twigs in subsequent months.) [4]

month:	0	1	2	3	...
Twig:	0	1	2	4	
Leaves:	0	2	4	8	

G.P. with $a = 2$, $r = 2$

$$\boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$$

$$S_{10} = \frac{2(2^{10} - 1)}{2 - 1} = 2 \times (1024 - 1)$$

$$= \underline{2046}$$

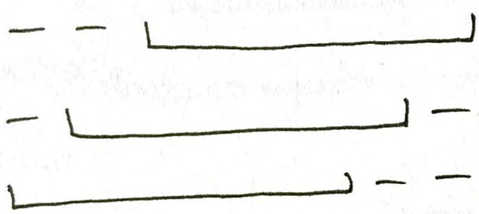
7. In a dice game, a player throws a fair 6-sided die n times. To win the game, the player needs to throw the following exact sequence within the n throws: 6, 5, 4, four 3's, two 2's, and 1. What is the probability that the player will win if the die is thrown (a) 8 times, (b) 10 times, and (c) 12 times? (You may leave your answer in terms of powers.) [4]

Need 6, 5, 4, 3, 3, 3, 3, 2, 2, 1 (10 terms)

a) 0 since $n < 10$

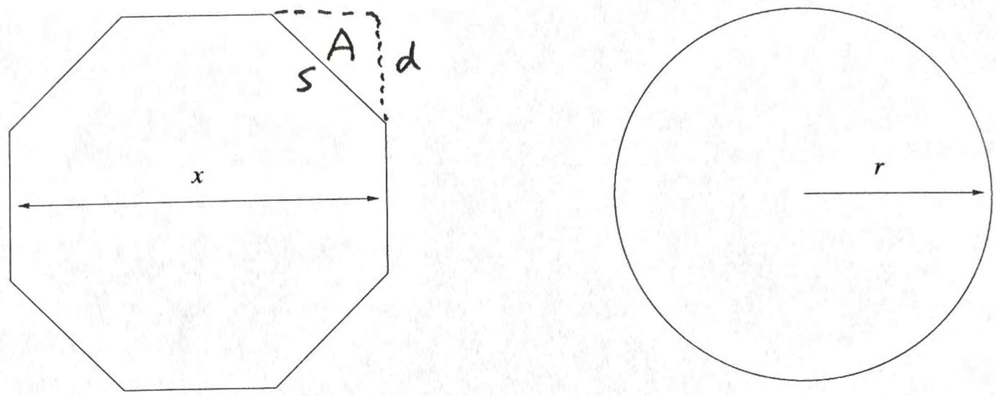
b) $\left(\frac{1}{6}\right)^{10}$

c)



$3 \times \left(\frac{1}{6}\right)^{10} = \frac{3}{6^{10}}$

8. Compare a regular octagon with a circle of radius r. The distance between two parallel sides of the octagon is x.



What should x be (in terms of r only) if the two shapes are to have the same area? [6]

$$\text{Area} = x^2 - 4A$$

$$A = \frac{1}{2}d^2$$

$$s^2 = d^2 + d^2$$

$$d^2 = \frac{s^2}{2}$$

$$\Rightarrow A = \frac{1}{2}\cdot\frac{s^2}{2}$$

$$= \frac{s^2}{4}$$

$$\Rightarrow \text{Area} = x^2 - 4\frac{s^2}{4}$$

$$= x^2 - s^2$$

$$x = s + 2d$$

$$= s + 2\times\frac{s}{\sqrt{2}}$$

$$= s(1+\sqrt{2})$$

$$\Rightarrow \text{Area} = s^2(1+\sqrt{2})^2 - s^2$$

$$= s^2 + 2\sqrt{2}s^2 + 2s^2 - s^2$$

$$= 2s^2(1+\sqrt{2})$$

$$= 2s^2\frac{(1+\sqrt{2})^2}{1+\sqrt{2}}$$

$$= \frac{2x^2}{1+\sqrt{2}}$$

For area $= \dfrac{2x^2}{1+\sqrt{2}} = \pi r^2$,

$$x = \sqrt{\frac{\pi r^2(1+\sqrt{2})}{2}}$$

9. For what range(s) of x is the following inequality satisfied?

$$5 - 3x < \frac{2}{x}$$

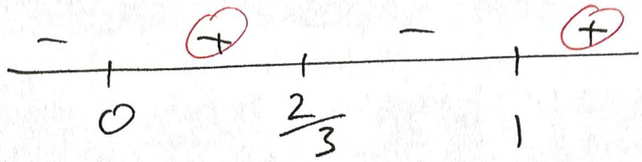
[6]

multiply by x^2:
always positive

$$5x^2 - 3x^3 < 2x$$

$$0 < x(3x^2 - 5x + 2)$$

$$0 < x(3x - 2)(x - 1)$$

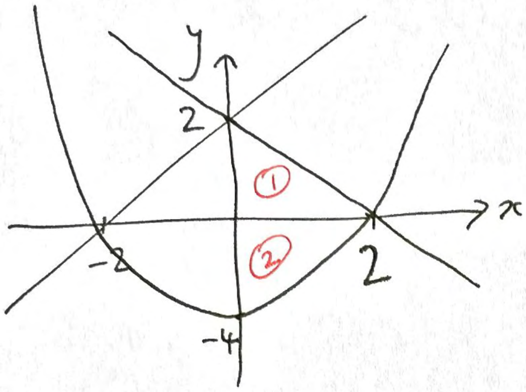


$$\{0 < x < \tfrac{2}{3}\} \cup \{x > 1\}$$

10. Find the area of the region enclosed between the following curves and including the origin:

$$y = 2 - x$$
$$y = 2 + x$$
$$y = x^2 - 4.$$

[6]



$① : \frac{1}{2} \times 2 \times 2 = 2$

$② : \int_0^2 (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_0^2$

$= \left(\frac{8}{3} - 8\right) - 0$

$= -\frac{16}{3}$

$\Rightarrow \text{Area} = 2 \times \left(2 + \frac{16}{3}\right)$

$= \frac{44}{3}$

11. A cylinder of dough is squashed such that its height h decreases linearly with time t as

$$h(t) = h_0 - \alpha t$$

for $t < h_0/\alpha$. Assume that the volume V of the dough remains constant, and it retains a cylindrical shape. Find an expression for the *rate of change* of the radius of the cylinder as a function of time and the parameters h_0, α, and V.

Does the rate of change increase or decrease with time? [6]

Find $\frac{dr}{dt} = \frac{dh}{dt} \cdot \frac{dr}{dh}$ (chain rule)

(cons.) $V = \pi r^2 h$

$\frac{V}{\pi} \cdot r^{-2} = h$

$\frac{dh}{dr} = -\frac{2V}{\pi} \cdot r^{-3}$

$\frac{dr}{dh} = -\frac{\pi r^3}{2V}$

$h = h_0 - \alpha t$

$\frac{dh}{dt} = -\alpha$

$\Rightarrow \frac{dr}{dt} = \frac{\alpha \pi r^3}{2V}$

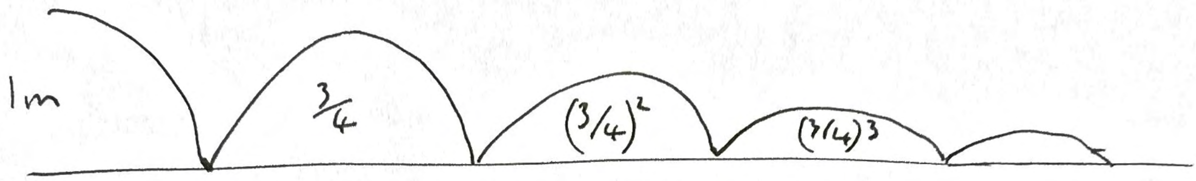
$= \alpha \pi \left(\frac{V}{\pi h}\right)^{3/2} \cdot \frac{1}{2V}$

$= \frac{\alpha}{2}\sqrt{\frac{V}{\pi} \cdot \frac{1}{(h_0 - \alpha t)^3}}$

Rate increases with time

Section B

12. A ball of mass 100 g bounces on a hard surface. Every time it hits the floor, it loses a quarter of its kinetic energy. If the ball is released from a height of 1 m, after how many bounces will the ball bounce no higher than 0.25 m? [3]



$\frac{3}{4} > \frac{1}{4}$

energy $\propto$ height

$\left(\frac{3}{4}\right)^2 = \frac{9}{16} > \frac{1}{4}$

$\left(\frac{3}{4}\right)^3 = \frac{27}{64} > \frac{1}{4}$

$\left(\frac{3}{4}\right)^4 = \frac{81}{256} > \frac{1}{4}$

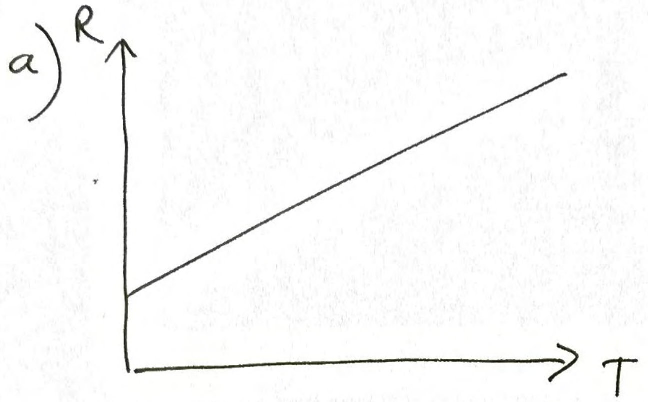
$\left(\frac{3}{4}\right)^5 = \frac{243}{1024} < \frac{1}{4} \Rightarrow$ Bounces 5 times

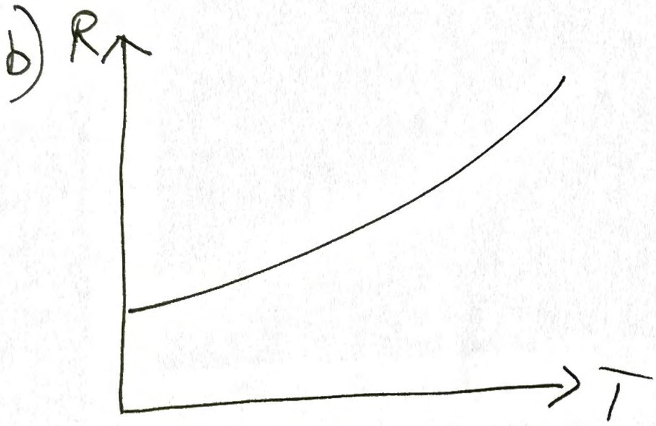
13. For a range of temperatures around room temperature and above, sketch how the resistance of the following components vary with temperature:

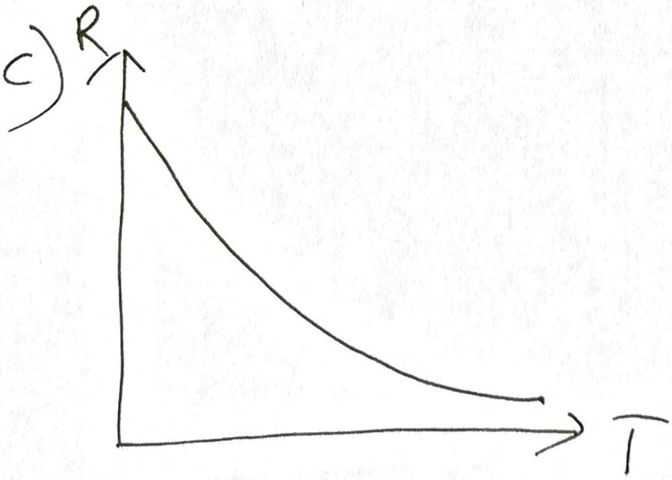
(a) an ideal wire,

(b) a filament light bulb, and

(c) a thermistor. [3]







14. Assume that the moons Io and Europa travel in circular orbits around Jupiter. The radius of Europa's orbit is 1.6 times that of Io's. In what ratio are the moons' orbital periods? Give the answer to 2 significant figures. [4]

$T^2 \propto r^3$

$kT^2 = r^3$

(Kepler's 3rd Law)

Given $\frac{r_E}{r_I} = 1.6$, find $\frac{T_E}{T_I}$

$$\frac{r_E^3}{r_I^3} = 1.6^3$$

$$\frac{kT_E^2}{kT_I^2} = 1.6^3$$

$$\frac{T_E}{T_I} = (1.6)^{3/2}$$

$$= \left(\frac{16}{10}\right)^{3/2}$$

$$= \frac{64}{10\sqrt{10}}$$

$$= \frac{6.4}{\sqrt{10}}$$

$3^2 = 9$

$4^2 = 16$

$3.2^2 = (3+0.2)^2 = 9 + 1.2 + 0.04 > 10$

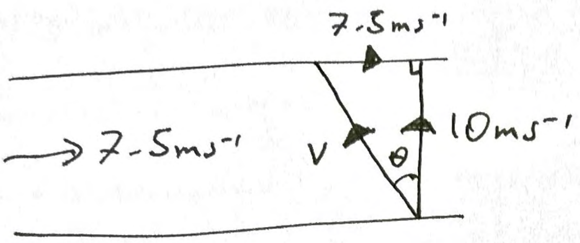
$3.1^2 = (3+0.1)^2 = 9 + 0.6 + 0.01 < 10$

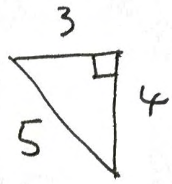
$3.15^2 = (3+0.15)^2 = 9 + 0.9 + 0.0225 < 10$

$\Rightarrow \sqrt{10} = 3.2$ (1dp)

$\Rightarrow \frac{T_E}{T_I} = \frac{6.4}{3.2} = \underline{2.0}$ (2sf)

15. A rower wants to cross a river to a point on the opposite river bank directly opposite from where she will start. The distance to cover is 100 m, and she wants to cross in 10 seconds. The river flows uniformly at 7.5 m/s. How fast must she row her boat, and at what angle, relative to the flowing water? Give your answers as exact expressions in simplest terms. [3]





$V = 12.5\,ms^{-1}$

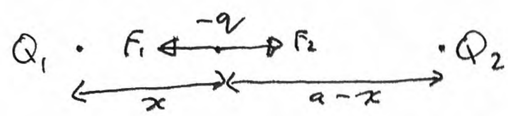
$$\tan\theta = \frac{7.5}{10} = \frac{3}{4}$$

$$\theta = \arctan\left(\frac{3}{4}\right)$$

Angle relative to water is $\frac{\pi}{2} + \arctan\left(\frac{3}{4}\right)$

16. A negative point charge, with charge $-q$ (with $q > 0$ C), lies on a line between two fixed positive point charges with charges Q_1 and Q_2. Assume that charge Q_1 lies at $x = 0$, and charge Q_2 lies at $x = a$. Find all positions where the charge $-q$ experiences no net force from the positive charges, for cases $Q_1 = Q_2$ and $Q_1 \neq Q_2$.

In the latter case, what is the physical significance of your unused solution for the position of the charge $-q$? [6]



$$F_1 = k\frac{Q_1 q}{x^2} \qquad F_2 = k\frac{Q_2 q}{(a-x)^2}$$

Let $Q_2 = nQ_1$. For no net force,

$$F_1 = F_2$$

$$\frac{Q_1}{x^2} = \frac{nQ_1}{(a-x)^2}$$

$$a^2 - 2ax + x^2 = nx^2$$

$$x^2(1-n) - 2ax + a^2 = 0$$

$$x = \frac{2a \pm \sqrt{4a^2 - 4a^2(1-n)}}{2(1-n)}$$

$$= \frac{2a \pm 2a\sqrt{1-(1-n)}}{2(1-n)}$$

$$= \frac{a \pm a\sqrt{n}}{1-n}$$

$$= \frac{a(1 \pm \sqrt{n})}{(1+\sqrt{n})(1-\sqrt{n})}$$

$$= \frac{a}{1+\sqrt{n}} \quad \text{or} \quad \frac{a}{1-\sqrt{n}}$$

Since x is positive for like charges Q_1 and Q_2,

$$x = \frac{a}{1+\sqrt{n}}$$

For $Q_1 = Q_2$, $n = 1$

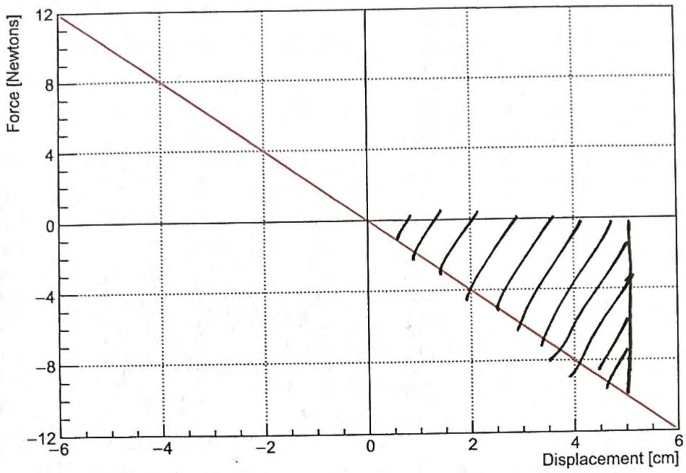
$$\Rightarrow x = \frac{a}{2}$$

For $Q_1 \neq Q_2$, $n = Q_2/Q_1$

$$\Rightarrow x = \frac{a}{1+\sqrt{\frac{Q_2}{Q_1}}}$$

The unused solution is for the case where Q_1 and Q_2 are unlike charges

17. The diagonal line in the graph below shows the relationship between force and displacement for a certain frictionless physical system:



(a) Initially, a mass of 20 grams is placed at a displacement of 5 cm and released. What work is done in moving the mass to a displacement of 0 cm?

(b) What is the speed of the mass when it reaches a displacement of 0 cm?

(c) Describe the mass's motion after it reaches 0 cm. [7]

a) $W = \frac{1}{2} \times 0.05 \times 10 = \underline{0.25\,J}$

b) $\frac{1}{2}mv^2 = 0.25$

$v^2 = \frac{0.5}{0.02} = 25$

$v = \underline{5\,ms^{-1}}$

c) $F = -kx \Rightarrow SHM \Rightarrow a = -w^2 x$

$w^2 = \frac{-a}{x} = \frac{-F/m}{x} = \frac{-(-12/0.02)}{0.06} = 10000$

$w = 100$

$T = \frac{2\pi}{w} = \frac{\pi}{50} = 0.06\,s$

Moves with <u>SHM</u>, period <u>0.06s</u>, amplitude <u>0.05m</u>

18. The drag force F on a sphere is related to the radius of the sphere (r), the velocity of the sphere (v), and the coefficient of viscosity of the fluid the drop is falling through (η) by the formula

$$F = kr^x \eta^y v^z$$

where k is a dimensionless constant, and x, y, and z are integers. By considering the units of the equation, work out the values of x, y, and z. (The coefficient of viscosity has units of kg m^{-1}s^{-1}.) [4]

LHS: $N \equiv kg\,m\,s^{-2}$ (using $F = ma$)

RHS: $m^x (kg\,m^{-1}s^{-1})^y (m\,s^{-1})^z \equiv m^x kg^y m^{-y} s^{-y} m^z s^{-z}$

$\equiv m^{x+z-y} s^{-y-z} kg^y$

kg: $\underline{y = 1}$

s: $-y - z = -2$

$z = -1 + 2$

$\underline{z = 1}$

m: $x + z - y = 1$

$x = 1 - 1 + 1$

$\underline{x = 1}$

19. Light of wavelength 625 nm shines onto a metal surface with a work function of 1.0 eV. Electrons are emitted from the metal.

(a) What is the maximum speed of the emitted electrons?

(b) The electrons are then accelerated through a potential difference of 5.0 keV towards a fluorescent screen. What is the final speed of the electrons when they hit the screen? [6]

[You may assume that Planck's constant is 6.0×10^{-34} J s, the speed of light is 3.0×10^{8} m/s, the mass of the electron is 1.0×10^{-30} kg, and the charge on the electron is 1.6×10^{-19} C.]

a) $\frac{hc}{\lambda} = \phi + \frac{1}{2} m v_{max}^2$

$\frac{6\times10^{-34} \times 3\times10^{8}}{625\times10^{-9}} = 1\times1.6\times10^{-19} + \frac{1}{2}\times1\times10^{-30} v^2$

$\frac{36\times10^{-26}}{6.25\times10^{-7}} = 3.2\times10^{-19} + 10^{-30} v^2$

$5.7\times10^{-19} - 3.2\times10^{-19} = 10^{-30} v^2$

$v^2 = \frac{2.5\times10^{-19}}{10^{-30}} = 25\times10^{10}$

$v = \underline{5\times10^{5}} \text{ ms}^{-1}$

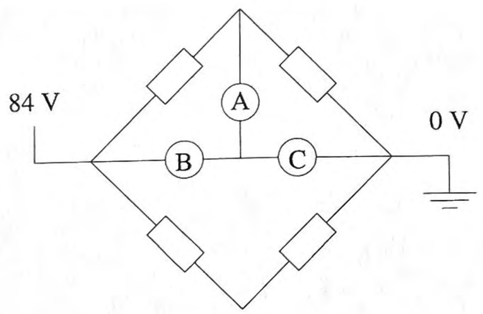
b) $5\times10^{3} \times 1.6\times10^{-19} = \frac{1}{2} \times 10^{-30} v^2$ (ignore initial v)

$10^{-30} v^2 = 1.6\times10^{-15}$

$v^2 = 16\times10^{14}$

$v = \underline{4\times10^{7}} \text{ ms}^{-1}$

20. In the resistor network shown below, each rectangle represents a resistor with fixed resistance 12 Ω, and each circle a water heater with fixed resistance 6 Ω. Each water heater contains 1 kg of water, initially at 20°C, and you may assume that all the power dissipated by the water heater heats the water.



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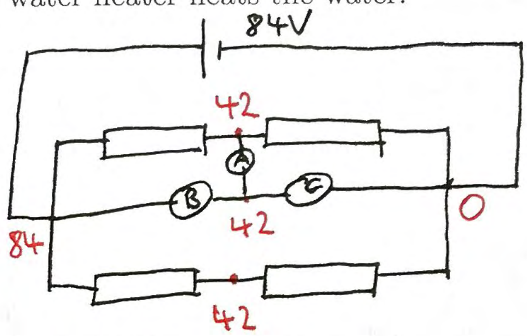


A voltage 84 V is applied as indicated in the figure. How long does it take for each individual water heater to raise their waters' temperature to 27°C?

[The specific heat capacity of water is approximately 4200 J/kg °C.] [8]

A does not heat since there's no p.d. across it.

B and C have 42 V across each

$$P = \frac{V^2}{R} = \frac{42^2}{6} = \frac{42 \times 6 \times 7}{6} = 294 = 3 \times 10^2 \text{ W}$$

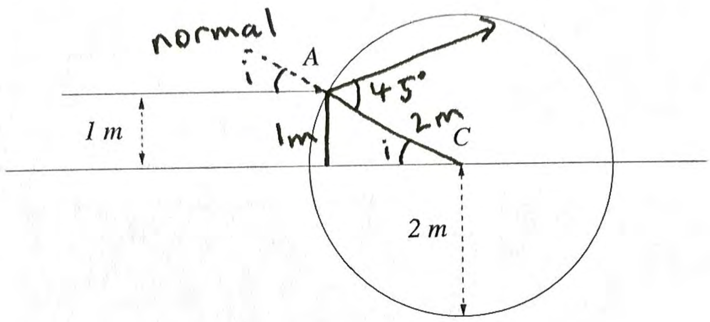
$$Q = mc\Delta T$$

$$300t = 1 \times 4200 \times (27-20)$$

$$t = \frac{4200 \times 7}{300} = 14 \times 7$$

$$t = 1 \times 10^2 \text{ s}$$

21. A sphere of radius $R = 2$ m is made of a material with the same refractive index as vacuum, and is supported in a fluid with refractive index n by a straight wire going through its centre C. A laser is directed toward the sphere with its beam parallel to the wire but offset by 1 m, and strikes the sphere at point A. What refractive index n would be needed for the refracted beam to travel at 45° relative to the line AC (which includes points A and C)?



Another beam, not necessarily parallel with the wire, strikes the sphere at point A. At what angle, relative to the line AC, should the beam be directed such that it would be completely reflected by the sphere? [6]

a) $\sin i = \frac{1}{2}$ $\quad n = \frac{\sin r}{\sin i} = \frac{\sin 45}{1/2} = \frac{\sqrt{2}/2}{1/2} = \sqrt{2} = 1.4$

b) Total internal reflection, find the critical angle.

Assume $n = \sqrt{2}$

$$\sqrt{2} = \frac{\sin 90}{\sin \theta_c}$$

$$\sin \theta_c = \frac{1}{\sqrt{2}}$$

$$\theta_c = 45°$$

$\therefore$ Beam should be directed at an angle greater than 45°