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Reactive Construction of Planar Euclidean Spanners with Constant Node Degree

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Koblenz, im 11 2015

Kurzfassung

Abstract

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1 Introduction

Wireless ad-hoc sensor networks are very useful. You can create warning systems for emergency purposes. For instance, deploying many sensor nodes into the sea or forest to check and caution for tsunamis or fire, respectively.

If a node detects something and sends a message, it is obvious that this message needs to *arrive* at a certain station. Possibly, this message needs to travel a long distance which one node cannot cover. The solution is to send the message to a neighbour of this node and this node forwards the message to another, and so on, until the message arrives at it's destination. While sending from one node to another it may be that the message gets lost or stuck in a loop, thus, never arriving at its destination. This must be prohibited. To achieve this guaranteed message delivery in a multi-hop network a specific graph-property called planarity must be satisfied.

Explaining planarity imagine a graph setup watched from above. It creates a 2d-view of this graph. Planarity says that from this view no two edges are allowed to cross each other except in the endpoints.

To planarize a graph some edges must be removed. If edges are arbitrarily removed from this graph it may result in a disconnected graph or at least randomly long paths. This needs to be prohibited and can be achieved if a so called *euclidian t-spanner* property is satisfied. With this property satisfied a path in a subgraph

2 Preamble

In this part of this work we define some notations and declare definitions which we will make use of. In addition, some former mentioned aspects are being formalized.

Let $\bigcirc ABC$ be a circle with A, B, C on its border. The circle with center O is denoted with (O). $\triangle ABC$ is the triangle with corners A, B and C.

Furthermore, we assume that there are no four points in any graph which are cocircular.

2.1 Gabriel Graph

The Gabriel Graph, denoted as GG, is the graph which contains all nodes of a supergraph U and it contains an edge $UV \in U$ if the Gabriel circle of UV contains no other node. The Gabriel circle of an edge UV is denoted as $disk(U, V)$. It is the circle with U and V on its border and with its center on line UV. In this work U is the unit disk graph with unit disk radius $R = 1$.

2.2 Partial Delaunay Triangulation

The Partial Delaunay Triangulation produces a connected, planar, t-spanner of any connected graph. In this part we will see an example of the reactive construction of PDT.

3 Related work

In the past years several topology controls were invented and further developed. We are interested in local algorithms only, and hence, centralized algorithms are ignored in this related work. There are a lot of different approaches with different results. The following is an extract of these approaches and can be divided into two main groups:

1. reactive algorithms
2. algorithms which produce a planar t-spanner with constant node degree

Reactive algorithms generally need less messages as only localized algorithms due to the lack of beaconing. They do not need the whole k-neighbourhood of every node to function, but only a fractional amount of their direct neighbours. As time of writing there are three reactive algorithms:

1. Beaconless Forwarder Planarization (BFP)
2. Guaranteed delivery beaconless forwarding (GDBF) with extension
3. reactive Partial Delaunay Triangulation

First, we describe an algorithm briefly and in the following there is a short section about properties of the produced graph. The BFP-algorithm ([1]) is divided into two phases. First, in the Selection Phase the executing node F starts the algorithm by sending a RTS message. In the following every node, which receives this message, starts a timer corresponding to a specific delay function. The closer a node resides to the executing node, the earlier it answers with a CTS. If a node W overhears a CTS of a node W' it checks whether or not it is contained in a certain area corresponding to node W' and F. This area is defined by geometric regions, in the following denoted as $Reg(A, B)$, with A and B being two nodes specifying this region. The minimum region $Reg(F, W')$ is the Gabriel circle $disk(F, W')$ and the maximum region $Reg(F, W')$ is the Relative Neighbourhood Graph lune over F and W'. The latter describes the area of the intersection of two circles around two neighbouring nodes UV with radii equal to $|UV|$ and with middlepoints U and V, respectively. Different regions cause the algorithm to use different amounts of messages. This will be discussed later.

Suppose W is contained in such an area it cancels its timer and is, henceforth, called a *hidden node*. Hidden nodes further participate in the algorithm. If a hidden node H receives a message from another node T, it memorizes this node if H lies in the former defined region.

The Protest Phase lets hidden nodes protest against violating edges. An edge UV is called a violating edge if there is a node in $Reg(U, V)$. If hidden nodes have nodes they memorize they restart the above timer. As soon as a message from another hidden node W' arrives at hidden node W, the latter checks its memorized nodes: A node X can be removed from the set of memorized nodes if $W' \in Reg(F, X)$. If the timer of a node expires and there are still nodes which are memorized, the node sends a protest message consisting of the violating node. The forwarder node F removes violating edges when it receives protests.

This algorithm performed on each node of a graph G produces a planar subgraph G'. However, G' is not a t-spanner of G and has no constant node degree despite the underlying region (GG, RNG, CNG) (refer to ... for an example of these three regions).

$GDBF$ is a scheme to forward messages in a network. All messages will be greedy forwarded to the node which lies closest to the destination until a node which has no neighbours closer to the destination, called a local minimum, is reached. From that point a recovery mode is used until the local minimum is exited and the algorithm can switch back to greedy mode. In greedy mode the message holder broadcasts a RTS-message to all neighbours. Every neighbour instantiates a timer with length depending on how far the neighbour is away from the destination. Nodes closer to the destination answer earlier. A CTS-message is sent as soon as the timer expires and the message holder forwards the message to its sender. Every other node cancels its timer and remains silent. In recovery mode a RTS message from the message holder is sent as well. Now, all neighbours instantiate a timer corresponding to the distance to the message holder M (closer nodes respond first). If a neighbour N overhears another nodes N' message, it cancels its timer if $N' \in disk(M, N)$. For more detailed information, refer to [2].

GDBF can be extended to reactively produce a planar subgraph of a given input graph. Since this graph is equal to the Gabriel graph, this is not a t-spanner of the input graph and also has no constant node degree.

The understanding of the Partial Delaunay Triangulation is crucial to follow this work and, thus, it is already explained in the preamble. PDT has a constant spanning ratio of at most $\frac{1+\sqrt{5}}{4}\pi^2 \approx 7.98$. In addition, the output is a planar graph, but it has no constant bounded degree.

The second group consists of the following algorithms:

1. H_{PLOS}
2. $\Delta_{11-Spanner}$
3. $PuDel$

H_{PLOS} (Planar Localized Optimum Spanner)[3] produces a planar Euclidean spanner with stretch-factor $1 + \epsilon$ with $\epsilon > 0$ arbitrarily small and constant node degree. However, it needs a node to be aware of its complete 2-hop-neighbourhood.

$\Delta_{11-Spanner}$ [4] constructs a spanner with an upper bound of 7 and a constant node degree of at most 11. The obtained graph is not planar and the algorithm needs a node to know its 4-hop-neighbourhood.

PuDel [5] produces a subgraph which is equal to the subgraph produced by *PDT* [6] and hence, it has an Euclidean stretch-factor of ≈ 7.98, is planar, but has no constant node degree.

4 Algorithm

This chapter introduces the $reactiveModifiedYaoStep$ and explains it's functionality. First, there is the basic version of this algorithm, followed by an improved version which needs less messages to construct a node's neighborhood.

Algorithm 1 Modified Yao Step

Input: planar, t-spanner G; integer $k \geq 14$
Output: planar, t-spanner G' with constant node degree

for each node $p \in G$ **do**
    Define k disjoint cones of size $2\pi/k$ around p.
    Select for each non empty cone the shortest edge.
    for each maximal sequence s of empty cones **do**
        if $|s| == 1$ **then**
            Let nx and ny be the incident edges on p clockwise and counterclockwise, respectively, from the emtpy cone.
            if either nx or ny has already been selected **then**
                select the other edge
            else
                Select the shorter edge
        else
            select the first $\lfloor \frac{|s|}{2} \rfloor$ unselected edges incident on n clockwise from s
            select the first $\lceil \frac{|s|}{2} \rceil$ unselected edges incident on n counterclockwise from s
G' is the subgraph of G consisting of all nodes which are in G and all edges which fulfil that both endpoints of this edge have selected it.

4.1 Proof of correctness

Proof.

$$MYS(PDT) \leftrightarrow RMYS$$

$$MYS(PDT(v)) \overset{a)}{\leftrightarrow} rMYS(rPDT(v))$$

$$MYS(PDT(v)) \overset{b)}{\leftrightarrow} rMYS(PDT(v))$$

$$MYS(PDT(v)) \overset{c)}{\leftrightarrow} MYS(PDT(v))$$

We need to proof that the proposed reactive version of this algorithm is equal to a simple concatenation of first, the Partial Delaunay Triangulation and secondly, the Modified Yao Step on any node $v \in G$. a) is the fragmentation of the proposition applied to a node v. It is well known that $rPDT$ produces the same graph as the simple local approach, so b) holds true. $rMYS$ does the same calculation as MYS until the broadcast in the end. Therefore, we need only to look at this broadcast. The executing node v sends a broadcast which must be overheard by all PDT -Neighbors of v. Because of the assumptions that every message arrives and arrives instantaneously, the message cannot get lost. Every informed node sends an answer back which must arrive. Hence, v can check whether or not each node in it's neighborhood accepts this edge. This leads to the same behavior MYS does and therefore, c) is true completing this proof. ■

4.2 Message Complexity

Let $N_{PDT}(u)$ be the message complexity of PDT creating the neighborhood of Node $u \in G$. First, $rPDT$ needs at most n messages to create the PDT-neighborhood. Next, the executing node sends at most k messages to its neighbors to ask whether they accept their connection. k answers come back and therefore $k*2$. Every one of this k neighbors needs to calculate it's PDT-neighborhood and hence, $k*N_{PDT}(u)$. The following equation put these reflections into one formula.

$$\begin{aligned} N_{RMYS}(u) &= \underbrace{N_{PDT}(u)}_{\theta(n)} + k * \underbrace{2}_{\theta(1)} + k * \underbrace{N_{PDT}(v)}_{\theta(n)} \\ \theta(N_{RMYS}(u)) &= \theta(n) \end{aligned}$$

Since k is a constant it can be omitted in O-Notation. The sum of the same complexity remains in the same complexity and hence, the complexity of this algorithm is $\theta(n)$.

5 Proof

Let U be the Unit Disk Graph of the Euclidean Graph E with a set of nodes S in the plane as vertex-set and containing edge AB if $|AB| \leq R$ with unit disk radius $R = 1$. The authors of [7] use $LDel^{(2)}(U)$ as the underlying subgraph of the Modified Yao Step. $LDel^{(2)}(U)$ is defined as the union of the Gabriel-graph and the subgraph of U in which the circumcircle of every triangle does not contain a 2-hop-neighbor of the nodes which create the triangle. However, it is not known whether $LDel^{(2)}(U)$ can be constructed reactively. At this point I want to introduce the *Partial Delaunay Triangulation (PDT)* [8] which might be a valid replacement. The following part of this work will examine the possibility of this replacement and, thus, proving the correctness of the following proposition:

Proposition 5.1. *Let G be the PDT-subgraph of U.*
For every integer $k \geq 14$, there exists a subgraph G' of G such that G' has maximum degree k and stretch factor $1 + 2\pi(k * \cos\frac{\pi}{k})^{-1}$.

With GG being the Gabriel Graph, we define the Partial Delaunay Triangulation as follows:

Definition 5.1. *An edge $UV \in U$ is in G if either*

(i) $UV \in GG$

(ii) or $\exists W \in U$: maximizes $\angle UWV$, $\bigcirc UVW \backslash \{U, V, W\} = \emptyset$ and $\sin \angle UWV \geq \frac{|CA|}{R}$, with $R > 0$ being the unit disk radius.

Additionally, the following Delaunay graph property is being used:

Lemma 5.1. *If CA and CB are edges of the PDT graph then the region R_1 of $(O) = \bigcirc ABC$ subtended by chord CA and away from B and the region R_2 of (O) subtended by chord CB and away from A contain no points that are two hop neighbours of A, B and C.*

Refer to Figure 1 for a graphical illustration of the above lemma. This property also holds true for PDT.

Proof. Let $disk(A, C)$ be the circle with C and A on it‘s border and the middlepoint on Line CA. Since $CA \in G$ either:

(i) $CA \in GG$:

B cannot lie inside $disk(A, C)$. Since $disk(A, C)$ can overlap circle $\bigcirc ABC$ on one side of AC only (where B is), R_1 must be completely inside $disk(A, C)$.

(ii) or $CA \in G \backslash GG$ is satisfied.

Since $CA \in G$ and $CA \notin GG$, $\exists W \in U : W$ maximizes the interior angle $\angle CWA$, more specifically, W is the closest node to CA. There are two cases, where W can be located:

(a) W lies in the halfplane subtended by line CA away from B.

W cannot reside in R_1, since the circumcircle $\bigcirc ACW$ would contain B, which is not allowed by precondition. Thus, $W \notin R_1$ is true. Therefore, $\bigcirc ACW$ does certainly contain R_1.

(b) W lies in the halfplane subtended by line CA towards B.

Since W is the angle maximizing node with respect to CA, the following is true: $\angle CWA \geq \angle BCA$. Therefore, $W \in \bigcirc ABC$ and since $\bigcirc ACW$ does not overlap $\bigcirc ABC$ on the side subtended by line CA where W is, it must overlap $\bigcirc ABC$ on the other side. Thus, it must contain R_1 completely.

These deductions work for R_2 analogously. Therefore, R_1 and R_2 cannot contain one or two hop neighbours of A, B and C.

■

In order to proof proposition 5.1 we need to show that there is a path from A to B. First, we divide the proof into two cases: when $\triangle ABC$ contains nodes of G and when this triangle is devoid of any nodes of G.

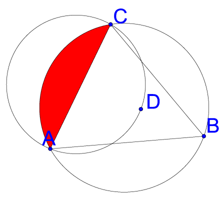

Figure 1: The red marked region contains no Points of G because it is always contained in $\bigcirc ACD$ which must be empty by definition.

Keil and Gutwin [9] proved the existence of a path between the points A and B and showed that the length of this path is delimited by the length of the arc from A to B on the circle $\bigcirc ABC$. This path connects A and B when no other points of G are inside $\triangle ABC$. The only precondition is that lemma 5.1 holds (which it does). This path is called the *outward path*.

First, notice the recursive definition of this path taken from [7]:

1. **Base case:** If $AB \in G$, the path consists of edge AB.

2. **Recursive step:** Otherwise, a point must reside in the region R_3 of (O) subtended by chord AB and away from C. Let T be such a point with the property that the region of $\bigcirc ATB$ subtended by chord AB closer to T is empty. We call T an *intermediate point* with respect to the pair of points (A, B). Let (O_1) be the circle passing through A and T whose center O_1 lies on segment AO and let (O_2) be the circle passing through B and T whose center O_2 lies on segment BO. Then both (O_1) and (O_2) lie inside (O), and $\angle AO_1T$ and $\angle TO_2B$ are both less than $\angle AOB \leq \frac{4\pi}{k}$. Moreover, the region of (O_1) subtended by chord BT and containing O_2 is empty. Therefore, we can recursively construct a path from A to T and a path from T to B, and then concatenate them to obtain a path from A to B.

Figure 2 contains an example for an intermediate point.

The recursive steps assumes $AB \notin G$ and concludes that there must be a point in R_3. For $G = PDT$ the following lemma proofs the correctness of this assumption:

Lemma 5.2. *For three points $A, B, C \in G$ and $\gamma = \angle ACB \leq \frac{2\pi}{k}$ with $k \geq 14$, $|AB| \leq R$ is satisfied.*

Proof.

$$
\begin{aligned}
|AB|^2 &= |BC|^2 + |AC|^2 - 2|BC||AC|\cos\gamma \\
&\leq R^2 + R^2 - 2R^2\cos\gamma \\
&\leq 2R^2 - 2R^2\cos\frac{2\pi}{k} \\
&\overset{a)}{\leq} 2R^2 - 2R^2\cos\frac{\pi}{7} \\
&\overset{b)}{\leq} 2R^2 - 2R^2 \cdot 0.9 = 0.2R^2 = \\
|AB| &\leq \sqrt{0.2}R \leq R
\end{aligned}
$$

In order to minimize $2R^2\cos\frac{2\pi}{k}$, γ must be maximized and hence, it is $\frac{2\pi}{k}$ obtaining a). Then adjust $\cos\frac{\pi}{7}$ downward to 0.9 and receive b).

∎

Lemma 5.1 and lemma 5.2 proof that there must be a node in R_3, if $AB \notin G$.

In order to proof proposition 5.1 we need the following proposition (which is from [7]):

Proposition 5.2. *In every recursive step of the outward path construction described above, if M_p is an intermediate point with respect to a pair of points (M_i, M_j), then:*

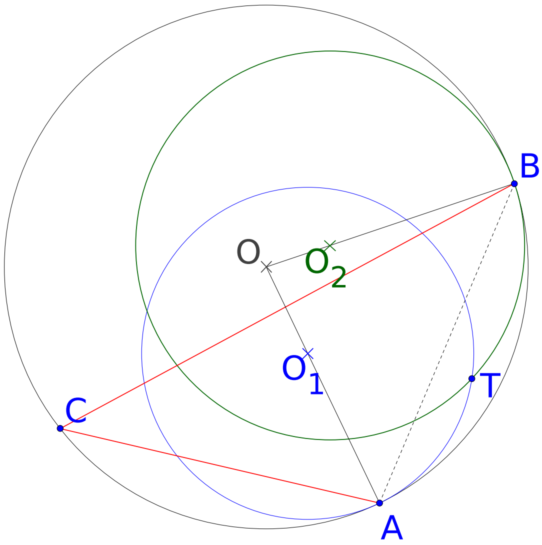

Figure 2: The intermediate point T with respect to pair (A, B), and the circles O_1 and O_2, which are completely within O.

a) there is a circle passing through C *and* M_p *that contains no point of* G*, and*

b) circles $\bigcirc CM_iM_p$ *and* $\bigcirc CM_jM_p$ *contain no points of* G*, except, possibly, in the region subtended by chords* M_iM_p *and* M_pM_j*, respectively, away from* C*.*

Note that every point $p = 1, \cdots, r-1$, is an intermediate point with respect to a pair (M_i, M_j), where $0 \leq i < p < j \leq r$. Furthermore, Keil and Gutwin [9] showed that the length of the path $A = M_0, M_1, \cdots, M_r = B$ is bounded by the length of arc $\overset{\frown}{AB}$. For completeness I copy the proof for proposition 5.2 from [7] with adapted notation.

Proof. We assume, by induction, that there are circles $\bigcirc CM_i$ and $\bigcirc CM_j$ passing through C and M_i, and C and M_j, respectively, containing no points of G, and that the circle $\bigcirc CM_iM_j$ contains no point of G in the interior of the region R' subtended by chord M_iM_j closer to C. (This is certainly true in the base case because $CA, CB \in G$, by lemma 5.1 and by our initial assumptions).

Since M_iM_j is not an edge in G, the point M_p chosen in the construction is the point with the property that the region R of $\bigcirc M_iM_pM_j$ subtended by chord M_iM_j away from C, contains no point of G. Then the circle passing through C and M_p and tangent to $\bigcirc M_iM_pM_j$ at M_p is completely inside $\bigcirc CM_i \cup \bigcirc CM_j \cup R \cup R'$, and therefore devoid of points of G. This proves part a).

The region of $\bigcirc CM_iM_p$ subtended by chord M_iM_p and containing C is inside $\bigcirc M_i \cup R \cup R'$, and therefore contains no point of G in its interior. The same is

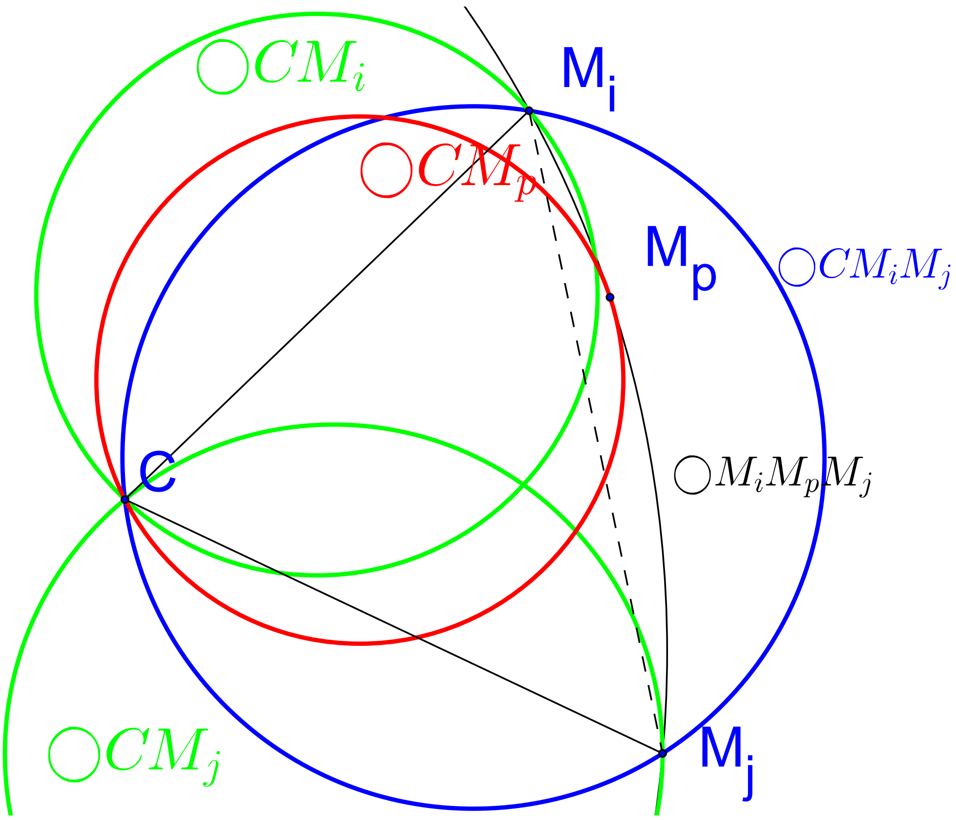

Figure 3: Example for proof of proposition 5.2.

true for the region of $\bigcirc CM_jM_p$ subtended by chord M_jM_p and containing C, and part b) holds as well. ■

Another fact we need in order to proof proposition 5.1 is the following:

Fact 5.1. *If four points A, B, C and M_1 are on one circle and C and M_1 are on different halfplanes of chord AB, then $\angle AM_1B + \angle ACB = \pi$ is true (please refer to figure 4 for an graphical illustration of this fact).*[1]

Now, we can proof the following lemma from [7], which shows, that for the case of the outward path, proposition 5.1 is satisfied:

Lemma 5.3. *Let $k \geq 14$ be an integer, and let CA and CB be edges in G such that $\angle BCA \leq \frac{2\pi}{k}$ and CA is the shortest edge in the angular sector $\angle BCA$. There exists a path $p : A = M_0, M_1, \ldots, M_r = B$ in G such that:*

(i) $|CA| + \sum_{i=0}^{r-1} |M_iM_{i+1}| \leq (1 + 2\pi(k\cos(\frac{\pi}{k}))^{-1})|CB|$

[1]see Euklid, book 3, proposition 22, for proof

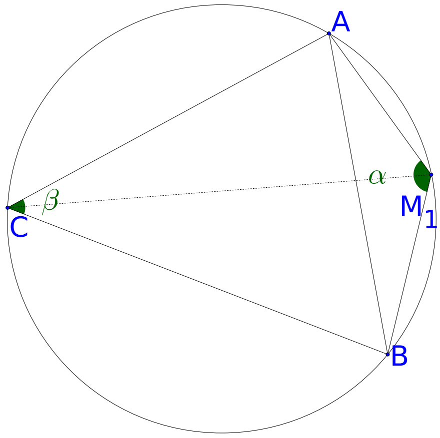

Figure 4: Example of fact 5.1

(ii) *There is no edge in G between any pair M_i and M_j lying in the closed region delimited by CA, CB and the edges of p, for any i and j satisfying $0 \leq i < j-1 \leq r$.*

(iii) $\angle M_{i-1}M_iM_{i+1} > \pi - \frac{2\pi}{k}$, *for* $i = 1, \ldots, r-1$.

(iv) $\angle CAM_1 \geq \frac{\pi}{2} - \frac{pi}{k}$.

Proof. This proof is performed almost equal to [7], but covering more details.

(i)

$$\begin{aligned}|CA| + |\overset{\frown}{\mathrm{AB}}| &= |CB| + 2\theta \cdot |OA| \\ &\overset{a)}{=} |CB| + (\frac{\theta}{\sin\theta}) \cdot |AB| \\ &\overset{b)}{=} |CB| + (\frac{\theta}{\cos\frac{\theta}{2}}) \cdot |CB| \\ &\overset{c)}{\leq} (1 + 2\pi(k\cos\frac{\pi}{k})^{-1})|CB|\end{aligned}$$

In [9] Keil and Gutwin proved that the length of the path between A and B is bounded by $|\overset{\frown}{\mathrm{AB}}|$ and thus, it suffices to show that $|CA| + |\overset{\frown}{\mathrm{AB}}| \leq (1+2\pi(k\cos\frac{\pi}{k})^{-1})$. Since $|CA| \leq |CB|$, $|CA|+|\overset{\frown}{\mathrm{AB}}|$ is largest, when CA and CB are symmetrical to the diameter of $\bigcirc ABC$, we can assume $|CA| = |CB|$. $|\overset{\frown}{\mathrm{AB}}|$ can be replaced with $2\theta \cdot |OA|$ (angle times radius). For every chord s of a circle (c) it is true, that $s = 2r\sin\frac{\alpha}{2}$, with r being the radius of (c) and α being the angle between the endpoints of s in middlepoint c facing s. Note that $\alpha = 2\theta$. These equations proof a).

Next, substitute $|AB|$ with $|AB| = \sin\frac{\theta}{2} \cdot 2|CB|$ and replace $\sin\theta$ with the trigonometry identity $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. You receive equation b).

At last, substitute θ using inequality $\theta \leq \frac{2\pi}{k}$ with $k > 2$, obtaining c).

(ii) Suppose, M_i and M_j is an edge in G, then there exists a circle with these two points on it's border which does not contain any other node of G. So, M_p must lie outside of this circle. By proposition 5.2 part a) there is a circle $\bigcirc CM_p$ through C and M_p which is empty. These two last observations contradict each other, since $\bigcirc CM_p$ would always contain M_i or M_j.

(iii) Since the angles α and β between opposite points of a chord in a rectangle which corners lie on a circle are supplementary, this is a fact: $\angle AM_1B = \pi - \angle ACB$ (see lemma 5.1 for more details). The angle $\angle M_{i-1}CM_{i+1}$ is smallest, if M_{i-1} and M_{i+1} lie on the circle. Note, by precondition we assume $\angle BCA \leq \frac{2\pi}{k}$. These facts proof following inequalities:

$$\begin{aligned} \angle M_{i-1}M_iM_{i+1} &\geq \pi - \angle M_{i-1}CM_{i+1} \\ &\geq \pi - \angle BCA \\ &\geq \pi - \frac{2\pi}{k} \end{aligned}$$

(iv) Since M_1 is inside the area subtended by chord AB from $\bigcirc ABC$ *away* from C, it is true that $\angle CAM_1 \geq \angle CAB \geq \frac{\pi}{2} - \frac{\pi}{k}$. The last inequality is true because:

$$\begin{aligned} \angle CAB + \angle ABC + \underbrace{\angle BCA}_{\leq \frac{2\pi}{k}} &= \pi \\ \angle CAB + \angle ABC &\geq \pi - \frac{2\pi}{k} \\ \angle CAB &\geq \frac{\pi - \frac{2\pi}{k}}{2} = \frac{\pi}{2} - \frac{\pi}{k} \end{aligned}$$

Since $CA \leq CB$, $\angle CAB$ can be at most the half of $\pi - \frac{2\pi}{k}$, proving the last inequality.

■

5.1 inward Path

Now, we perform the proof for the case when $\triangle ABC$ contains other nodes.

Let S be the set of points which contains points A and B, and all the points interior to $\triangle ABC$ excluding C. Then $CH(S)$ are all the points which are on the convex hull of S. Let these points be called $N_0 = A$ and $N_t = B$ and points $N_1, \cdots, N_{t-1}$ are the points on $CH(S)$ which lie inside $\triangle ABC$. The following two propositions are taken from [7]:

Proposition 5.3. *The following are true:*

a) for every $i = 0, \cdots, s-1 : |CN_i| \leq |CN_{i+1}|$, and

b) for every $i = 0, \cdots, s-2 : \angle N_i N_{i+1} N_{i+2} \geq \pi$, where $\angle N_i N_{i+1} N_{i+2}$ is the angle facing point C.

Proof. Since CA is the shortest edge in the angular sector $\angle BCA$, $|CA| \leq CN_i$, for $i = 1, \cdots, t-1$ and since $N_1, \cdots, N_t$ are on $CH(S)$, a) is true.

Part b) follows from the convexity of $CH(S)$. All interior angles to $CH(S)$ measure at most π, so all the exterior angles fulfil $\angle N_{i-1} N_i N_{i+1} \geq \pi$ ■

Proposition 5.4. *The following are true:*

a) for every $i = 0, \cdots, s-1$, the interior of $\triangle CN_i N_{i+1}$ is devoid of points of G,

b) for every $i = 0, \cdots, s$, there exists a circle passing trough CN_i whose interior is devoid of points of G.

Proof. Since $N_0, \cdots, N_s$ are on $CH(S)$ and, hence, no other point can reside closer to C, part a) is true.

■

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