Fachbereich 4: Informatik

Reactive Construction of Planar Euclidean Spanners with Constant Node Degree

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Tim Budweg

Betreuer: M. Sc. Florentin Neumann, Institut für Informatik, Fachbereich Informatik, Universität Koblenz-Landau
Erstgutachter: M. Sc. Florentin Neumann, Institut für Informatik, Fachbereich Informatik, Universität Koblenz-Landau
Zweitgutachter: Prof. Dr. Hannes Frey, Institut für Informatik, Fachbereich Informatik, Universität Koblenz-Landau

Koblenz, im 11 2015

Kurzfassung

Abstract

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1 Introduction

Wireless ad-hoc sensor networks are very useful. You can create warning systems for emergency purposes. For instance, deploying many sensor nodes into the sea or forest to check and caution for tsunamis or fire, respectively.

If a node detects something and sends a message, it is obvious that this message needs to *arrive* at a certain station. Possibly, this message needs to travel a long distance which one node cannot cover. The solution is to send the message to a neighbour of this node and this node forwards the message to another, and so on, until the message arrives at it's destination. While sending from one node to another it may be that the message gets lost or stuck in a loop, thus, never arriving at its destination. This must be prohibited. To achieve this guaranteed message delivery in a multi-hop network a specific graph-property called planarity must be satisfied.

Explaining planarity imagine a graph setup watched from above. It creates a 2d-view of this graph. Planarity says that from this view no two edges are allowed to cross each other except in the endpoints.

To planarize a graph some edges must be removed. If edges are arbitrarily removed from this graph it may result in a disconnected graph or at least randomly long paths. This needs to be prohibited and can be achieved if a so called *euclidian t-spanner* property is satisfied. With this property satisfied a path in a subgraph

2 Proof

Let U be the Unit Disk Graph of the Euclidean Graph of a Node Set S. The authors of [1] use $LDel^{(2)}(U)$ as the underlying subgraph of the Modified Yao Step. $LDel^{(2)}(U)$ is defined as the union of the Gabriel-graph and the subgraph of U in which the circumcircle of every triangle does not contain a 2-hop-neighbor of the nodes which create the triangle. However, it is not known whether $LDel^{(2)}(U)$ can be constructed reactively. At this point I want to introduce the *Partial Delaunay Triangulation (PDT)* [2] which might be a valid replacement. The following part of this work will examine the possibility of this replacement and, thus, proving the correctness of the following proposition:

Proposition 2.1. *Let G be the PDT-subgraph of U. For every integer $k \geq 14$, there exists a subgraph G' of G such that G' has maximum degree k and stretch factor $1 + 2\pi(k * \cos\frac{\pi}{k})^{-1}$.*

With GG being the Gabriel Graph, we define the Partial Delaunay Triangulation as follows:

Definition 2.1. *An edge UV is in G if either*

(i) $UV \in U$ and $UV \in GG$

(ii) or $\exists W \in U$: maximizes $\angle UWV$ and $\bigcirc UVW \backslash \{U, V, W\} = \emptyset$

Additionally, the following Delaunay Graph property is being used:

Lemma 2.1. *If CA and CB are edges of the PDT graph then the region R_1 of $(O) = \bigcirc ABC$ subtended by chord CA and away from B and the region R_2 of (O) subtended by chord CB and away from A contain no points that are two hop neighbours of A, B and C.*

See Figure 1 for a graphical illustration of the above lemma. Let $disk(A, C)$ be the circle with C and A on it's border and the middlepoint on Line CA. This property also holds true for PDT.

Proof. Since $CA \in G$ either:

(i) $CA \in GG$:

Since $CA \in GG$, B cannot lie inside $disk(A, C)$. Therofore, R_1 must be completely inside $disk(A, C)$.

(ii) or $CA \in G \backslash GG$ is satisfied.

Since $CA \in G$ and $CA \notin GG$, $\exists W \in U : W$ maximizes the interior $\angle CWA$, more specifically, W is the closest node to CA. There are two cases, where W can be located:

(a) W lies in the halfplane subtended by line CA away from B.

(b) W lies in the halfplane subtended by line CA towards B.

■

We need to show that there is a path from A to B. First, we divide the proof into two cases: when $\triangle ABC$ contains nodes of G and when this triangle is devoid of any nodes of G.

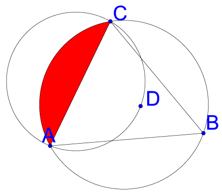

Figure 1: The red marked region contains no Points of G because it is always contained in $\bigcirc ACD$ which must be empty by definition.

Keil and Gutwin [3] proved the existence of a path between the points A and B and showed that the length of this path is delimited by the length of the arc from A to B on the circle $\bigcirc ABC$. This path connects A and B when no other points of G are inside $\triangle ABC$. The only precondition is that lemma 2.1 holds (which it does). This path is called the *outward path*.

The recursive definition of this path taken from [1] is as follows:

1. **Base case:** If $AB \in G$, the path consists of edge AB.

2. **Recursive step:** Otherwise, a point must reside in the region of (O) subtended by chord AB and away from C. Let T be such a point with the property that the region of $\bigcirc ATB$ subtended by chord AB closer to T is empty. We call T an *intermediate point* with respect to the pair of points (A, B). Let (O_1) be the circle passing through A and T whose center O_1 lies on segment AO and let (O_2) be the circle passing through B and T whose center O_2 lies on segment BO. Then both (O_1) and (O_2) lie inside (O), and $\angle AO_1T$ and $\angle TO_2B$ are both less than $\angle AOB \leq \frac{4\pi}{k}$. Moreover, the region of (O_1) subtended by chord BT and containing O_2 is empty. Therefore, we can recursively construct a path from A to T and a path from T to B, and then concatenate them to obtain a path from A to B.

Figure 2 contains an example for an intermediate point.

We must proof the following proposition (which is from [1]).

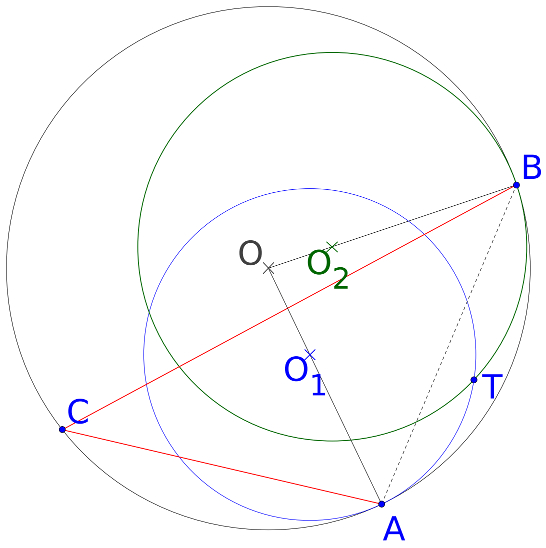

Figure 2: The intermediate point T with respect to pair (A, B), and the circles O_1 and O_2, which are completely within O.

Proposition 2.2. *In every recursive step of the outward path construction described above, if M_p is an intermediate point with respect to a pair of points (M_i, M_j), then:*

1. *there is a circle passing through C and M_p that contains no point of G, and*
2. *circles $\bigcirc CM_iM_p$ and $\bigcirc CM_jM_p$ contain no points of G, except, possibly, in the region subtended by chords M_iM_p and M_pM_j, respectively, away from C.*

Proof. Let G be the set of nodes which is created by PDT. Since CA and CB are edges in G there are circles $\bigcirc CM_i$ and $\bigcirc CM_j$ which have C and M_i, and C and M_j, respectively, on it's border and do not contain any other nodes of G. At this point I assume, without loss of generality, that M_i and M_j lie on the y-axis of the coordinate system and C lies to the left of these points. First, notice that $\triangle CM_iM_j$ is empty by precondition and the area $R_{\bigcirc M_iM_pM_j}$ of $\bigcirc M_iM_pM_j$ subtended by chord M_iM_j away from C contains no other points either. This proof is divided into three cases.

1. $\bigcirc CM_p$ is tangential to $\bigcirc CM_iM_j$ at C.
2. $\bigcirc CM_p$ overlaps $\bigcirc CM_iM_j$ in the upper halfplane subtended by chord CM_i (away from M_j).

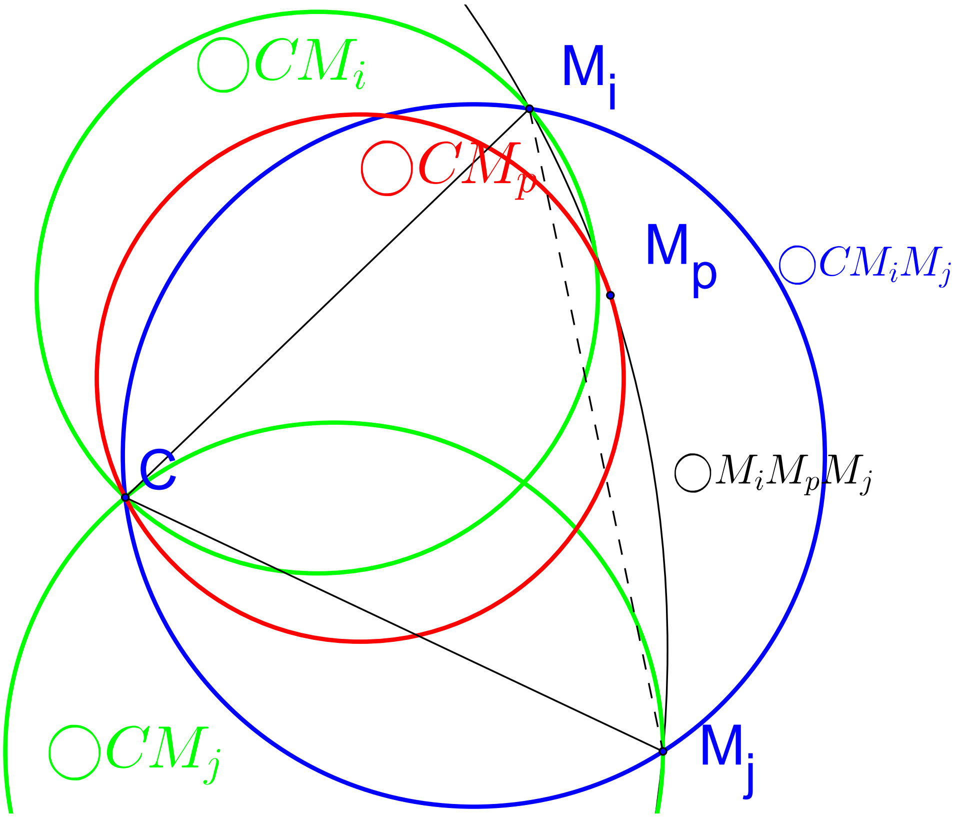

Figure 3: Example for proof of proposition 2.2.

3. $\bigcirc CM_p$ overlaps $\bigcirc CM_iM_j$ in the lower halfplane subtended by chord CM_j (away from M_i).

Since the two circles $\bigcirc CM_p$ and $\bigcirc CM_iM_j$ share the point C, $\bigcirc CM_p$ cannot overlap $\bigcirc CM_iM_j$ on both sides of the edge CM_p. For case 1 $\bigcirc CM_p$ is completely inside $\bigcirc CM_iM_j$ and therefore devoid of any points. For case 2 $\bigcirc CM_p$ is completely inside the area $R_{\bigcirc M_iM_pM_j} \cup \bigcirc CM_iM_j \bigcirc CM_i$ and, therefore, empty. $\bigcirc CM_p$ cannot overlap $\bigcirc CM_i$ because of the following lemma.

Lemma 2.2. *If A and B are points in the plane and are located in different halfplanes of a line CM_i, and A and B are not allowed to reside inside a circle $\bigcirc CM_i$, there is no circle $\bigcirc ABC$ which does not contain M_i and overlaps circle $\bigcirc CM_i$.*

Proof. Let, without loss of generality, C and M_i be on a line CM_i which is parallel to the y-axis. A and B are then located to the left and right of this line. You can see this construction in figure 4.

This proof uses a contradiction. A cannot reside inside $\bigcirc CM_i$, since CM_i is a PDT-edge. The circle c_1 must not contain M_i and, therefore, must cross the circle $\bigcirc CM_i$ in front of M_i. Notice that the next part of the circle c_1 must be inside of circle $\bigcirc CM_i$, since we started outside. Because c_1 must cross B and B lies outside of $\bigcirc CM_i$, c_1 crosses a second time $\bigcirc CM_i$. And the last conclusion is that C is a common point of $\bigcirc CM_i$ and c_1. Hence, we have got three intersections of $\bigcirc CM_i$ and c_1 at least. Since A and B lie outside of $\bigcirc CM_i$, these two circles are not equal. But two circles which intersect at least three times and are not equal, do not exist. ∎

The proof of case 3 works analogously. These conclusions proof part a) of proposition 2.2.

The following part of this work proofs part b) of the same proposition. The main argument of this proof is that $\bigcirc CM_iM_p$ is contained completely in the area $\bigcirc CM_i \cup R_{\bigcirc M_iM_pM_j} \cup R2_{\bigcirc CM_iM_j}$ with $R2_{\bigcirc CM_iM_j}$ being the area of $\bigcirc CM_iM_j$ subtended by chord M_iM_j closer to C. I show that every possible position of $\bigcirc CM_i$ includes the area $R3_{\bigcirc CM_iM_p}$ of $\bigcirc CM_iM_p$ subtended by chord CM_i away from M_j. The proof is divided into three cases of how $\bigcirc CM_i$ can be located:

1. $\bigcirc CM_i$ is equal to $\bigcirc CM_iM_p$.

2. $\bigcirc CM_i$ overlaps $\bigcirc CM_iM_p$ in the halfplane subtended from line CM_i away from M_j.

3. $\bigcirc CM_i$ overlaps $\bigcirc CM_iM_p$ in the halfplane subtended from line CM_i closer to M_j.

First, assume, without loss of generality, that C and M_i lie on a horizontal line and M_j is located below this line. $\bigcirc CM_iM_p$ can only overlap $\bigcirc CM_i$ on one side of line CM_i, since they share these two points.

For case 1, since $\bigcirc CM_i$ is empty, $\bigcirc CM_iM_p$ must be empty, too.

For case 2, $\bigcirc CM_i$ moves up, away from M_j, expanding the area in the same direction of which $R3_{\bigcirc CM_iM_p}$ is located. So, $R3_{\bigcirc CM_iM_p}$ cannot overlap $\bigcirc CM_i$ in this case.

Case 3, the only case where $R3_{\bigcirc CM_iM_p}$ would overlap $\bigcirc CM_i$, cannot occur, since $M_p \in G$ and M_p is, obviously, located on $\bigcirc CM_iM_p$. So, if $\bigcirc CM_i$ moves down, towards M_j, it would contain M_p, which cannot happen, since $\bigcirc CM_i$ is a PDT-circle.

Notice that the proof for $\bigcirc CM_jM_p$ works analogously substituting M_i with M_j and vice versa.

∎

Another lemma we need in order to proof proposition 2.1 is the following:

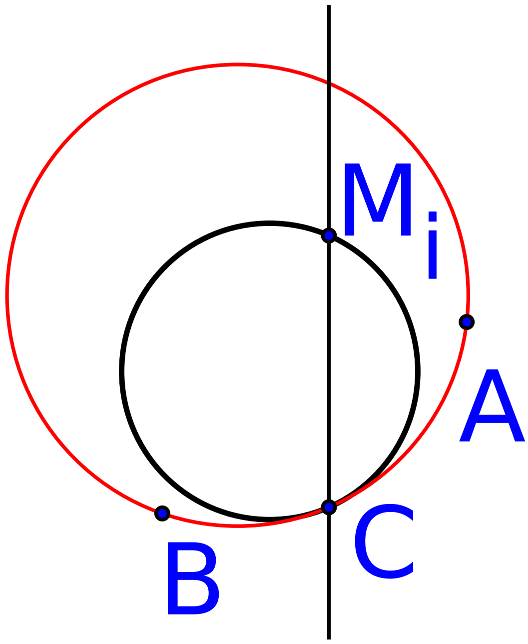

Figure 4: Example of the construction of lemma 2.2.

Lemma 2.3. *If four points A, B, C and M_1 are on one circle and C and M_1 are on different halfplanes of chord AB, then $\angle AM_1B + \angle ACB = \pi$ is true (see figure 5 for an graphical illustration of this lemma).*

Proof. see Euklid, book 3, Proposition 22. (Nochmal besprechen)

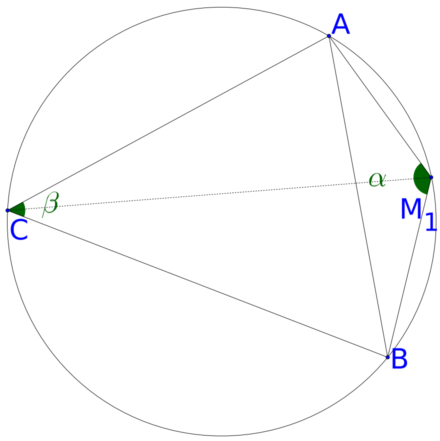

Figure 5: Example for lemma 2.3

∎

Now, we can proof the following lemma from [1], which shows, that for the case of the outward path, proposition 2.1 is satisfied:

Lemma 2.4. *Let $k \geq 14$ be an integer, and let CA and CB be edges in G such that $\angle BCA \leq \frac{2\pi}{k}$ and CA is the shortest edge in the angular sector $\angle BCA$. There exists a path $p : A = M_0, M_1, \dots, M_r = B$ in G such that:*

(i) $|CA| + \sum_{i=0}^{r-1} |M_i M_{i+1}| \leq (1 + 2\pi(k\cos(\frac{\pi}{k}))^{-1})|CB|$

(ii) There is no edge in G between any pair M_i and M_j lying in the closed region delimited by CA, CB and the edges of p, for any i and j satisfying $0 \leq i < j - 1 \leq r$.

(iii) $\angle M_{i-1} M_i M_{i+1} > \pi - \frac{2\pi}{k}$, *for* $i = 1, \dots, r-1$.

(iv) $\angle CAM_1 \geq \frac{\pi}{2} - \frac{pi}{k}$.

Proof. This proof is performed almost equal to [1], but covering more details.

(i)

$$\begin{aligned} |CA| + |\widehat{\mathrm{AB}}| &= |CB| + 2\theta \cdot |OA| \\ &\overset{a)}{=} |CB| + (\frac{\theta}{\sin\theta}) \cdot |AB| \\ &\overset{b)}{=} |CB| + (\frac{\theta}{\cos\frac{\theta}{2}}) \cdot |CB| \\ &\overset{c)}{\leq} (1 + 2\pi(k\cos\frac{\pi}{k})^{-1})|CB| \end{aligned}$$

Since $|CA| \leq |CB|$, $|CA| + |\widehat{\mathrm{AB}}|$ is largest, when CA and CB are symmetrical to the diameter of $\bigcirc ABC$, we can assume $|CA| = |CB|$. $|\widehat{\mathrm{AB}}|$ can be replaced with $2\theta \cdot |OA|$ (angle times radius). For every chord s of a circle (c) it is true, that $s = 2r\sin\frac{\alpha}{2}$, with r being the radius of (c) and α being the angle between the endpoints of s in middlepoint c facing s. Note that $\alpha = 2\theta$. These equations proof a).

Next, substitute $|AB|$ with $|AB| = \sin\frac{\theta}{2} \cdot 2|CB|$ and replace $\sin\theta$ with the trigonometry identity $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. You receive equation b).

At last, substitute θ using inequality $\theta \leq \frac{2\pi}{k}$ with $k > 2$, obtaining c).

(ii) Suppose, M_i and M_j is an edge in G, then there exists a circle with these two points on it's border which does not contain any other node of G. So, M_p must lie outside of this circle. By proposition 2.2 part a) there is a circle $\bigcirc CM_p$ trough C and M_p which is empty. These two last observations contradict each other, since $\bigcirc M_iM_j$ would always contain M_p. If M_p does not reside in the circle $\bigcirc M_iM_j$, this circle and the circle $\bigcirc CM_p$ would cross at least three times (and are not equal), which cannot exist.

(iii) Since the angles α and β between opposite points of a chord in a rectangle which corners lie on a circle are supplementary, this is a fact: $\angle AM_1B = \pi - \angle ACB$ (see lemma 2.3 for more details). The angle $\angle M_{i-1}CM_{i+1}$ is smallest, if M_{i-1} and M_{i+1} lie on the circle. Note, by precondition we assume $\angle BCA \leq \frac{2\pi}{k}$. These facts proof following inequalities:

$$\begin{aligned}\angle M_{i-1}M_iM_{i+1} &\geq \pi - \angle M_{i-1}CM_{i+1}\\ &\geq \pi - \angle BCA\\ &\geq \pi - \frac{2\pi}{k}\end{aligned}$$

(iv) Since M_1 is inside the area subtended by chord AB from $\bigcirc ABC$ *away* from C, it is true that $\angle CAM_1 \geq \angle CAB \geq \frac{\pi}{2} - \frac{\pi}{k}$. The last inequality is true because:

$$\begin{aligned}\angle CAB + \angle ABC + \underbrace{\angle BCA}_{\leq \frac{2\pi}{k}} &= \pi\\ \angle CAB + \angle ABC &\geq \pi - \frac{2\pi}{k}\\ \angle CAB &\geq \frac{\pi - \frac{2\pi}{k}}{2} = \frac{\pi}{2} - \frac{\pi}{k}\end{aligned}$$

Since $CA \leq CB$, $\angle CAB$ can be at most the half of $\pi - \frac{2\pi}{k}$, proving the last inequality.

■

2.1 inward Path

Now, we perform the proof for the case when $\triangle ABC$ contains other nodes.

Let S be the set of points which contains points A and B, and all the points interior to $\triangle ABC$ excluding C. Then $CH(S)$ are all the points which are on the convex hull of S. Let these points be called $N_0 = A$ and $N_t = B$ and points $N_1, \cdots, N_{t-1}$ are the points on $CH(S)$ which lie inside $\triangle ABC$. The following proposition is taken from [1]:

Proposition 2.3. *For every* $i = 1, \cdots, t-1$:

a) $CN_i \in G$*,*

b) $|CN_i| \leq |CN_{i+1}|$*, and*

c) $\angle N_{i-1}N_iN_{i+1} \geq \pi$*, where* $\angle N_{i-1}N_iN_{i+1}$ *is the angle facing point* C*.*

Proof. Since CA is the shortest edge in the angular sector $\angle BCA$, $|CA| \leq CN_i, for i = 1, \cdots, t-1$ and since $N_1, \cdots, N_t$ are on $CH(S)$, b) is true.

Part c) follows from the convexity of $CH(S)$. All interior angles to $CH(S)$ measure at most π, so all the exterior angles fulfil $\angle N_{i-1}N_iN_{i+1} \geq \pi$ ∎

Since $CN_i \leq CN_{i+1}$ and no other point of G lies inside $\triangle N_iCN_{i+1}$, CN_i is the shortest edge in the angular sector $\angle N_iCN_{i+1}$.

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