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# Reactive Construction of Planar Euclidean Spanners with Constant Node Degree

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## Kurzfassung

## Abstract

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## 1 Introduction

Wireless ad-hoc sensor networks are very useful. You can create warning systems for emergency purposes. For instance, deploying many sensor nodes into the sea or a forest to check and caution for tsunamis or fires, respectively.

If a node detects an event and sends a message, it is obvious that this message needs to *arrive* at a certain station. Possibly, this message needs to travel a long distance which one node cannot cover. The solution is to send the message to a neighbour of this node and this node forwards the message to another, and so on, until the message arrives at its destination. While sending from one node to another it may happen that the message gets lost or stuck in a loop, thus, never arriving at its destination. This must be prohibited. To achieve guaranteed message delivery in a multi-hop network a specific graph-property called planarity must be satisfied.

Explaining planarity imagine a graph setup watched from above. It creates a 2d-view of this graph. Planarity says that from this view no two edges are allowed to cross each other except in the endpoints.

One approach to planarize a graph is removing edges. If edges are arbitrarily removed from this graph it may result in a disconnected graph or at least randomly long paths. This needs to be prohibited and can be achieved if a so called *euclidian t-spanner* property is satisfied. With this property satisfied a path in a subgraph

## 2 Preamble

In this part of this work we define some notations and declare definitions which we will use. In addition, some former mentioned aspects are being formalized.

Nodes which are contained in a graph are denoted in lower case arabic letters and upper case arabic letters are complete graphs which consist of a set of nodes and a set of edges which connect nodes. Let  $\bigcirc abc$  be a circle with  $a, b, c$  on its border. The circle with center  $o$  is denoted with  $(o)$ .  $\triangle abc$  is the triangle with corners  $a, b$  and  $c$ .

Furthermore, we assume that there are no four points in any graph which are cocircular since the Delaunay Triangulation is then not unique anymore. This leads to unnecessary case differentiation.

The Unit Disk Graph of a node set  $s$  is denoted as  $U(s)$ . This graph contains all nodes of node set  $s$  and connects two nodes if and only if their distance between each other is at most 1. In addition, we will make use of the so called *Gabriel Graph*, denoted as  $GG$ . It is the graph which contains all nodes of a supergraph  $U$  and it contains an edge  $uv \in U$  if the Gabriel circle of  $uv$  contains no other node. The Gabriel circle of an edge  $uv$  is denoted as  $disk(u, v)$ . It is the circle with  $u$  and  $v$  on its border and with its center on line  $uv$ . In this work  $U$  denotes the unit disk graph with unit disk radius  $R = 1$ .

Another important graph in order to follow this work is the Partial Delaunay Triangulation (PDT) [1]. It is a planar, t-spanner of the Unit Disk Graph.

The following abbreviations are used throughout this work and introduced here. *Ready To Send (RTS)* and *Clear To Send (CTS)* describe the process of a node pair to interchange messages while starting the

### 2.1 Partial Delaunay Triangulation

The Partial Delaunay Triangulation produces a connected, planar, t-spanner of any connected graph. In this part we will see an example of the reactive construction of *PDT*.

### 3 Related work

In the past years several topology controls were invented and further developed. We are interested in local algorithms only, and hence, centralized algorithms are ignored in this related work. There are a lot of different approaches with different results. The following is an extract of these approaches and can be divided into two main groups:

1. reactive algorithms
2. algorithms which produce a planar t-spanner with constant node degree

Reactive algorithms generally need less messages as only localized algorithms due to the lack of beaconing. They do not need the whole  $k$ -neighbourhood of every node to function, but only a fractional amount of their direct neighbours. As time of writing there are three reactive algorithms:

1. Beaconless Forwarder Planarization (BFP)
2. Guaranteed delivery beaconless forwarding (GDBF) with extension
3. reactive Partial Delaunay Triangulation

First, we describe an algorithm briefly and in the following there is a short section about properties of the produced graph. The BFP-algorithm ([2]) is divided into two phases. First, in the Selection Phase the executing node  $F$  starts the algorithm by sending a RTS message. In the following every node, which receives this message, starts a timer corresponding to a specific delay function. The closer a node resides to the executing node, the earlier it answers with a CTS. If a node  $W$  overhears a CTS of a node  $W'$  it checks whether or not it is contained in a certain area corresponding to node  $W'$  and  $F$ . This area is defined by geometric regions, in the following denoted as  $Reg(A, B)$ , with  $A$  and  $B$  being two nodes specifying this region. The minimum region  $Reg(F, W')$  is the Gabriel circle  $disk(F, W')$  and the maximum region  $Reg(F, W')$  is the Relative Neighbourhood Graph lune over  $F$  and  $W'$ . The latter describes the area of the intersection of two circles around two neighbouring nodes  $UV$  with radii equal to  $|UV|$  and with middlepoints  $U$  and  $V$ , respectively. Different regions cause the algorithm to use different amounts of messages. This will be discussed later.

Suppose  $W$  is contained in such an area it cancels its timer and is, henceforth, called a *hidden node*. Hidden nodes further participate in the algorithm. If a hidden node  $H$  receives a message from another node  $T$ , it memorizes this node if  $H$  lies in the former defined region.

The Protest Phase lets hidden nodes protest against violating edges. An edge  $UV$  is called a violating edge if there is a node in  $Reg(U, V)$ . If hidden nodes

have nodes they memorize they restart the above timer. As soon as a message from another hidden node  $W'$  arrives at hidden node  $W$ , the latter checks its memorized nodes: A node  $X$  can be removed from the set of memorized nodes if  $W' \in \text{Reg}(F, X)$ . If the timer of a node expires and there are still nodes which are memorized, the node sends a protest message consisting of the violating node. The forwarder node  $F$  removes violating edges when it receives protests.

This algorithm performed on each node of a graph  $G$  produces a planar subgraph  $G'$ . However,  $G'$  is not a t-spanner of  $G$  and has no constant node degree despite the underlying region  $(GG, RNG, CNG)$  (refer to ... for an example of these three regions).

*GDBF* is a scheme to forward messages in a network. All messages will be greedy forwarded to the node which lies closest to the destination until a node which has no neighbours closer to the destination, called a local minimum, is reached. From that point a recovery mode is used until the local minimum is exited and the algorithm can switch back to greedy mode. In greedy mode the message holder broadcasts a RTS-message to all neighbours. Every neighbour instantiates a timer with length depending on how far the neighbour is away from the destination. Nodes closer to the destination answer earlier. A CTS-message is sent as soon as the timer expires and the message holder forwards the message to its sender. Every other node cancels its timer and remains silent. In recovery mode a RTS message from the message holder is sent as well. Now, all neighbours instantiate a timer corresponding to the distance to the message holder  $M$  (closer nodes respond first). If a neighbour  $N$  overhears another nodes  $N'$  message, it cancels its timer if  $N' \in \text{disk}(M, N)$ . For more detailed information, refer to [3].

*GDBF* can be extended to reactively produce a planar subgraph of a given input graph. Since this graph is equal to the Gabriel graph, this is not a t-spanner of the input graph and also has no constant node degree.

The understanding of the Partial Delaunay Triangulation is crucial to follow this work and, thus, it is already explained in the preamble. PDT has a constant spanning ratio of at most  $\frac{1+\sqrt{5}}{4}\pi^2 \approx 7.98$ . In addition, the output is a planar graph, but it has no constant bounded degree.

The second group consists of the following algorithms:

1.  $H_{PLOS}$
2.  $\Delta_{11-\text{Spanner}}$
3.  $PuDel$

$H_{PLOS}$  (Planar Localized Optimum Spanner)[4] produces a planar Euclidean spanner with stretch-factor  $1 + \epsilon$  with  $\epsilon > 0$  arbitrarily small and constant node degree. However, it needs a node to be aware of its complete 2-hop-neighbourhood.

$\Delta_{11}\text{-Spanner}$  [5] constructs a spanner with an upper bound of 7 and a constant node degree of at most 11. The obtained graph is not planar and the algorithm needs a node to know its 4-hop-neighbourhood.

*PuDel* [6] produces a subgraph which is equal to the subgraph produced by *PDT* [7] and hence, it has an Euclidean stretch-factor of  $\approx 7.98$ , is planar, but has no constant node degree.



## 4 Algorithm

This chapter introduces the *reactive Modified Yao Step (RMYS)* and explains its functionality. For the sake of completeness follows a scheme of the Modified Yao Step taken from [8] and how this can be changed to a reactive approach. In addition, there is an explanation of how RMYS operates. Then there is a proof of correctness, followed by a brief analysis of the message complexity and message size of RMYS. Afterwards, we see which properties the graph produced by RMYS obtains and which not. At last, there is an improved version of RMYS which needs less messages.

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**Algorithm 1** Modified Yao Step
 

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**Input:** planar, connected graph  $G$ ; integer  $k \geq 14$

**Output:** planar, connected graph  $G'$  with constant node degree of at most  $k$

- 1: **for** each node  $p \in G$  **do**
- 2:   Define  $k$  disjoint cones of size  $2\pi/k$  around  $p$ .
- 3:   Select for each non empty cone the shortest edge.
- 4:   **for** each maximal sequence  $s$  of empty cones **do**
- 5:     **if**  $|s| == 1$  **then**
- 6:       Let  $nx$  and  $ny$  be the incident edges on  $p$  clockwise and
- 7:       counterclockwise, respectively, from the empty cone.
- 8:       **if** either  $nx$  or  $ny$  has already been selected **then**
- 9:         select the other edge
- 10:      **else**
- 11:       Select the shorter edge
- 12:      **else**
- 13:       select the first  $\lfloor \frac{|s|}{2} \rfloor$  unselected edges incident on  $n$  clockwise from  $s$
- 14:       select the first  $\lceil \frac{|s|}{2} \rceil$  unselected edges incident on  $n$  counterclockwise from  $s$

$G'$  is the subgraph of  $G$  consisting of all nodes which are in  $G$  and all edges which fulfil that both endpoints of this edge have selected it.

---

This scheme does not tell, in particular, how this can be computed on a node. However, my reactive approach, called rMYS, is the following: Since every node knows its PDT-neighborhood (refer to algorithm 2) which is used by the Modified Yao Step, it can execute everything from line 1 to 14 without further knowledge about its neighborhood and hence, does not need to send any messages at all. The basic approach of RMYS needs to send one message to each possible neighbor which all send a message back whether or not they did select this edge. This ensures that only bidirectional edges are used.

The following algorithm is the definition of RMYs. For clarity, notice that both acronyms RMYs and rMYs mean “reactive Modified Yao Step“, but former is the algorithm which consists of rPDT, the reactive version of PDT, and rMYs, the reactive way of applying the Modified Yao Step to a planar and connected graph described above.

---

**Algorithm 2** Reactive Modified Yao Step

---

**Input:** any connected graph  $G$ ; integer  $k \geq 14$

**Output:** planar, connected graph  $G'$  with constant node degree of at most  $k$

---

**for** each node  $p \in G$  **do**

    create the PDT-Neighborhood of  $p$  using rPDT

    apply rMYs to  $p$  using PDT-graph

    to ensure bi-directional edges let each neighbor of  $p$  create its RMYs-neighbors and

    send a protest message if  $p$  is not among them causing  $p$  to remove this edge

---

#### 4.1 Proof of correctness

*Proof.*

$$\begin{aligned}
 &MYS(PDT) \leftrightarrow RMYs \\
 &MYS(PDT(v)) \stackrel{a)}{\leftrightarrow} rMYS(rPDT(v)) \\
 &MYS(PDT(v)) \stackrel{b)}{\leftrightarrow} rMYS(PDT(v)) \\
 &MYS(PDT(v)) \stackrel{c)}{\leftrightarrow} MYs(PDT(v))
 \end{aligned}$$

We need to proof that the proposed reactive version of this algorithm is equal to a simple concatenation of first, the Partial Delaunay Triangulation and secondly, the Modified Yao Step on any node  $v \in G$ . a) is the fragmentation of the proposition applied to a node  $v$ . It is well known that  $rPDT$  produces the same graph as the simple local approach, so b) holds true.  $rMYS$  does the same calculation as  $MYS$  right before the broadcast in the end. Therefore, we need only to look at this broadcast. The executing node  $v$  sends a broadcast which must be overheard by all  $PDT$ -Neighbors of  $v$ . Because of the assumptions that every message arrives and arrives instantaneously, the message cannot get lost. Every informed node checks whether or not it selects  $v$  to be in its RMYs-Neighborhood. If yes, it remains silent and otherwise it sends a protest message causing  $v$  to remove this edge. Hence,  $v$  can check whether or not each node in it's neighborhood accepts this edge. This leads to the same behavior  $MYS$  does and therefore, c) is true and completing this proof. ■

## 4.2 Message Complexity

Let  $N_{PDT}(u)$  be the message complexity of  $PDT$  creating the neighborhood of Node  $u \in G$ . First,  $rPDT$  needs at most  $n$  messages to create the  $PDT$ -neighborhood. Next, the executing node sends at most  $k$  messages to its neighbors to ask whether they accept their connection.  $k$  answers come back and therefore  $k*2$ . Every one of this  $k$  neighbors needs to calculate it's  $PDT$ -neighborhood and hence,  $k*N_{PDT}(u)$ . The following equation put these reflections into one formula.

$$N_{RMYS}(u) = \underbrace{N_{PDT}(u)}_{\theta(n)} + k * \underbrace{2}_{\theta(1)} + k * \underbrace{N_{PDT}(v)}_{\theta(n)}$$

$$\theta(N_{RMYS}(u)) = \theta(n)$$

Since  $k$  is a constant it can be omitted in  $O$ -Notation. The sum of the same complexity remains in the same complexity and hence, the message complexity of this algorithm is  $\theta(n)$ .

## 4.3 Message Size

If the assumption that every node has a unique position holds this position can be used to identify each node uniquely. Hence, two floats can be used to save this position resulting in a constant number of bits.

## 4.4 Properties of the RMYS-graph

This section is devoted to the graph-properties the RMYS algorithms inherits. First, it is important to know whether RMYS produces from any connected Unit Disk Graph a connected subgraph. The first part of RMYS, the Partial Delaunay Triangulation, creates from a connected graph a connected subgraph. Since rMYS removes edges it may be possible that the graph will be disconnected. To analyze this we need to recognize that rMYS finds at least one edge per non empty cone. Since these cones around a node  $p$  cover the whole area around  $p$  and if there is a node in it, there will be an edge for that cone. Hence, the graph cannot become disconnected.

Following this, there is planarity. The reactive approach of the Partial Delaunay Triangulation produces a planar Graph and since the rMYS step of the RMYS-algorithm does not add any edges, the planarity property cannot be violated. Hence, RMYS produces a planar graph.

Every node has a constant node degree of at most  $k$ . Since for each cone one edge

## 5 Proof

Let  $U$  be the Unit Disk Graph of the Euclidean Graph  $E$  with a set of nodes  $S$  in the plane as vertex-set and containing edge  $AB$  if  $|AB| \leq R$  with unit disk radius  $R = 1$ . The authors of [8] use  $LDel^{(2)}(U)$  as the underlying subgraph of the Modified Yao Step.  $LDel^{(2)}(U)$  is defined as the union of the Gabriel-graph and the subgraph of  $U$  in which the circumcircle of every triangle does not contain a 2-hop-neighbor of the nodes which create the triangle. However, it is not known whether  $LDel^{(2)}(U)$  can be constructed reactively. At this point I want to introduce the *Partial Delaunay Triangulation (PDT)* [1] which might be a valid replacement. The following part of this work will examine the possibility of this replacement and, thus, proving the correctness of the following proposition:

**Proposition 5.1.** *Let  $G$  be the PDT-subgraph of  $U$ .*

*For every integer  $k \geq 14$ , there exists a subgraph  $G'$  of  $G$  such that  $G'$  has maximum degree  $k$  and stretch factor  $1 + 2\pi(k * \cos \frac{\pi}{k})^{-1}$ .*

With  $GG$  being the Gabriel Graph, we define the Partial Delaunay Triangulation as follows:

**Definition 5.1.** *An edge  $UV \in U$  is in  $G$  if either*

(i)  $UV \in GG$

(ii) *or  $\exists W \in U$  : maximizes  $\angle UWV$ ,  $\bigcirc UVW \setminus \{U, V, W\} = \emptyset$  and  $\sin \angle UWV \geq \frac{|CA|}{R}$ , with  $R > 0$  being the unit disk radius.*

Additionally, the following Delaunay graph property is being used:

**Lemma 5.1.** *If  $CA$  and  $CB$  are edges of the PDT graph then the region  $R_1$  of  $(O) = \bigcirc ABC$  subtended by chord  $CA$  and away from  $B$  and the region  $R_2$  of  $(O)$  subtended by chord  $CB$  and away from  $A$  contain no points that are two hop neighbours of  $A$ ,  $B$  and  $C$ .*

Refer to Figure 1 for a graphical illustration of the above lemma. This property also holds true for PDT.

*Proof.* Let  $disk(A, C)$  be the circle with  $C$  and  $A$  on it's border and the midpoint on Line  $CA$ . Since  $CA \in G$  either:

(i)  $CA \in GG$ :

$B$  cannot lie inside  $disk(A, C)$ . Since  $disk(A, C)$  can overlap circle  $\bigcirc ABC$  on one side of  $AC$  only (where  $B$  is),  $R_1$  must be completely inside  $disk(A, C)$ .

(ii) or  $CA \in G \setminus GG$  is satisfied.

Since  $CA \in G$  and  $CA \notin GG$ ,  $\exists W \in U$  :  $W$  maximizes the interior angle  $\angle CWA$ , more specifically,  $W$  is the closest node to  $CA$ . There are two cases, where  $W$  can be located:

(a)  $W$  lies in the halfplane subtended by line  $CA$  away from  $B$ .

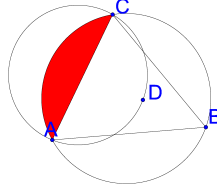
$W$  cannot reside in  $R_1$ , since the circumcircle  $\odot ACW$  would contain  $B$ , which is not allowed by precondition. Thus,  $W \notin R_1$  is true. Therefore,  $\odot ACW$  does certainly contain  $R_1$ .

(b)  $W$  lies in the halfplane subtended by line  $CA$  towards  $B$ .

Since  $W$  is the angle maximizing node with respect to  $CA$ , the following is true:  $\angle CWA \geq \angle BCA$ . Therefore,  $W \in \odot ABC$  and since  $\odot ACW$  does not overlap  $\odot ABC$  on the side subtended by line  $CA$  where  $W$  is, it must overlap  $\odot ABC$  on the other side. Thus, it must contain  $R_1$  completely.

These deductions work for  $R_2$  analogously. Therefore,  $R_1$  and  $R_2$  cannot contain one or two hop neighbours of  $A$ ,  $B$  and  $C$ . ■

In order to proof proposition 5.1 we need to show that there is a path from  $A$  to  $B$ . First, we divide the proof into two cases: when  $\triangle ABC$  contains nodes of  $G$  and when this triangle is devoid of any nodes of  $G$ .



**Figure 1:** The red marked region contains no Points of  $G$  because it is always contained in  $\odot ACD$  which must be empty by definition.

Keil and Gutwin [9] proved the existence of a path between the points  $A$  and  $B$  and showed that the length of this path is delimited by the length of the arc from  $A$  to  $B$  on the circle  $\odot ABC$ . This path connects  $A$  and  $B$  when no other points of  $G$  are inside  $\triangle ABC$ . The only precondition is that lemma 5.1 holds (which it does). This path is called the *outward path*.

First, notice the recursive definition of this path taken from [8]:

1. **Base case:** If  $AB \in G$ , the path consists of edge  $AB$ .

2. **Recursive step:** Otherwise, a point must reside in the region  $R_3$  of  $(O)$  subtended by chord  $AB$  and away from  $C$ . Let  $T$  be such a point with the property that the region of  $\odot ATB$  subtended by chord  $AB$  closer to  $T$  is empty. We call  $T$  an *intermediate point* with respect to the pair of points  $(A, B)$ . Let  $(O_1)$  be the circle passing through  $A$  and  $T$  whose center  $O_1$  lies on segment  $AO$  and let  $(O_2)$  be the circle passing through  $B$  and  $T$  whose center  $O_2$  lies on segment  $BO$ . Then both  $(O_1)$  and  $(O_2)$  lie inside  $(O)$ , and  $\angle AO_1T$  and  $\angle TO_2B$  are both less than  $\angle AOB \leq \frac{4\pi}{k}$ . Moreover, the region of  $(O_1)$  subtended by chord  $BT$  and containing  $O_2$  is empty. Therefore, we can recursively construct a path from  $A$  to  $T$  and a path from  $T$  to  $B$ , and then concatenate them to obtain a path from  $A$  to  $B$ .

Figure 2 contains an example for an intermediate point.

The recursive steps assumes  $AB \notin G$  and concludes that there must be a point in  $R_3$ . For  $G = PDT$  the following lemma proofs the correctness of this assumption:

**Lemma 5.2.** *For three points  $A, B, C \in G$  and  $\gamma = \angle ACB \leq \frac{2\pi}{k}$  with  $k \geq 14$ ,  $|AB| \leq R$  is satisfied.*

*Proof.*

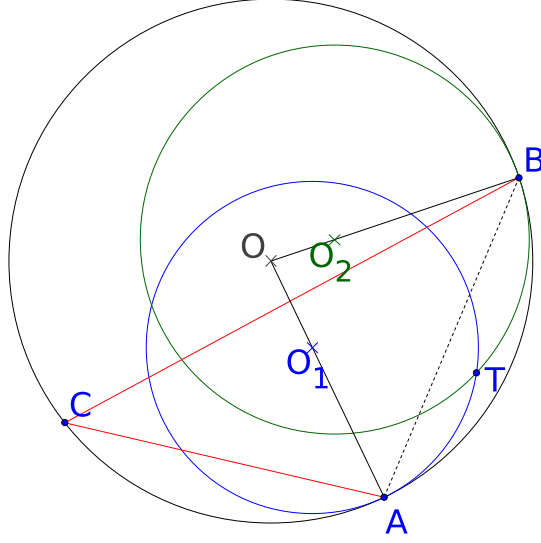
$$\begin{aligned}
 |AB|^2 &= |BC|^2 + |AC|^2 - 2|BC||AC|\cos\gamma \\
 &\leq R^2 + R^2 - 2R^2\cos\gamma \\
 &\leq 2R^2 - 2R^2\cos\frac{2\pi}{k} \\
 &\stackrel{a)}{\leq} 2R^2 - 2R^2\cos\frac{\pi}{7} \\
 &\stackrel{b)}{\leq} 2R^2 - 2R^2 \cdot 0.9 = 0.2R^2 = \\
 |AB| &\leq \sqrt{0.2}R \leq R
 \end{aligned}$$

In order to minimize  $2R^2\cos\frac{2\pi}{k}$ ,  $\gamma$  must be maximized and hence, it is  $\frac{2\pi}{k}$  obtaining a). Then adjust  $\cos\frac{\pi}{7}$  downward to 0.9 and receive b). ■

Lemma 5.1 and lemma 5.2 proof that there must be a node in  $R_3$ , if  $AB \notin G$ .

In order to proof proposition 5.1 we need the following proposition (which is from [8]):

**Proposition 5.2.** *In every recursive step of the outward path construction described above, if  $M_p$  is an intermediate point with respect to a pair of points  $(M_i, M_j)$ , then:*



**Figure 2:** The intermediate point  $T$  with respect to pair  $(A, B)$ , and the circles  $O_1$  and  $O_2$ , which are completely within  $O$ .

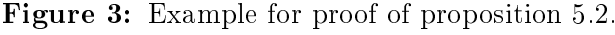
- a) *there is a circle passing through  $C$  and  $M_p$  that contains no point of  $G$ , and*
- b) *circles  $\odot CM_i M_p$  and  $\odot CM_j M_p$  contain no points of  $G$ , except, possibly, in the region subtended by chords  $M_i M_p$  and  $M_p M_j$ , respectively, away from  $C$ .*

Note that every point  $p = 1, \dots, r-1$ , is an intermediate point with respect to a pair  $(M_i, M_j)$ , where  $0 \leq i < p < j \leq r$ . Furthermore, Keil and Gutwin [9] showed that the length of the path  $A = M_0, M_1, \dots, M_r = B$  is bounded by the length of arc  $AB$ . For completeness I copy the proof for proposition 5.2 from [8] with adapted notation.

*Proof.* We assume, by induction, that there are circles  $\odot CM_i$  and  $\odot CM_j$  passing through  $C$  and  $M_i$ , and  $C$  and  $M_j$ , respectively, containing no points of  $G$ , and that the circle  $\odot CM_i M_j$  contains no point of  $G$  in the interior of the region  $R'$  subtended by chord  $M_i M_j$  closer to  $C$ . (This is certainly true in the base case because  $CA, CB \in G$ , by lemma 5.1 and by our initial assumptions).

Since  $M_i M_j$  is not an edge in  $G$ , the point  $M_p$  chosen in the construction is the point with the property that the region  $R$  of  $\odot M_i M_p M_j$  subtended by chord  $M_i M_j$  away from  $C$ , contains no point of  $G$ . Then the circle passing through  $C$  and  $M_p$  and tangent to  $\odot M_i M_p M_j$  at  $M_p$  is completely inside  $\odot CM_i \cup \odot CM_j \cup R \cup R'$ , and therefore devoid of points of  $G$ . This proves part a).

The region of  $\odot CM_i M_p$  subtended by chord  $M_i M_p$  and containing  $C$  is inside  $\odot M_i \cup R \cup R'$ , and therefore contains no point of  $G$  in its interior. The same is



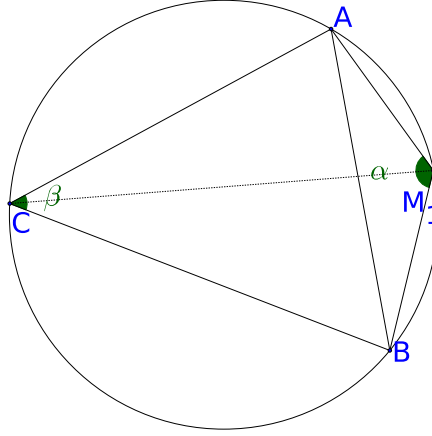
**Fact 5.1.** *If four points  $A$ ,  $B$ ,  $C$  and  $M_1$  are on one circle and  $C$  and  $M_1$  are on different halfplanes of chord  $AB$ , then  $\angle AM_1B + \angle ACB = \pi$  is true (please refer to figure 4 for an graphical illustration of this fact).<sup>1</sup>*

**Lemma 5.3.** *Let  $k \geq 14$  be an integer, and let  $CA$  and  $CB$  be edges in  $G$  such that  $\angle BCA \leq \frac{2\pi}{k}$  and  $CA$  is the shortest edge in the angular sector  $\angle BCA$ . There exists a path  $p : A = M_0, M_1, \dots, M_r = B$  in  $G$  such that:*

$$(i) \quad |CA| + \sum_{i=0}^{r-1} |M_i M_{i+1}| \leq (1 + 2\pi(k \cos(\frac{\pi}{k}))^{-1})|CB|$$

<sup>1</sup>see Euklid, book 3, proposition 22, for proof





**Figure 4:** Example of fact 5.1

(ii) There is no edge in  $G$  between any pair  $M_i$  and  $M_j$  lying in the closed region delimited by  $CA, CB$  and the edges of  $p$ , for any  $i$  and  $j$  satisfying  $0 \leq i < j - 1 \leq r$ .

(iii)  $\angle M_{i-1}M_iM_{i+1} > \pi - \frac{2\pi}{k}$ , for  $i = 1, \dots, r - 1$ .

(iv)  $\angle CAM_1 \geq \frac{\pi}{2} - \frac{\pi}{k}$ .

*Proof.* This proof is performed almost equal to [8], but covering more details.

(i)

$$\begin{aligned}
 |CA| + |\widehat{AB}| &= |CB| + 2\theta \cdot |OA| \\
 &\stackrel{a)}{=} |CB| + \left(\frac{\theta}{\sin \theta}\right) \cdot |AB| \\
 &\stackrel{b)}{=} |CB| + \left(\frac{\theta}{\cos \frac{\theta}{2}}\right) \cdot |CB| \\
 &\stackrel{c)}{\leq} (1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|
 \end{aligned}$$

In [9] Keil and Gutwin proved that the length of the path between  $A$  and  $B$  is bounded by  $|\widehat{AB}|$  and thus, it suffices to show that  $|CA| + |\widehat{AB}| \leq (1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$ . Since  $|CA| \leq |CB|$ ,  $|CA| + |\widehat{AB}|$  is largest, when  $CA$  and  $CB$  are symmetrical to the diameter of  $\odot ABC$ , we can assume  $|CA| = |CB|$ .  $|\widehat{AB}|$  can be replaced with  $2\theta \cdot |OA|$  (angle times radius). For every chord  $s$  of a circle ( $c$ ) it is true, that  $s = 2r \sin \frac{\alpha}{2}$ , with  $r$  being the radius of ( $c$ ) and  $\alpha$  being the angle between the endpoints of  $s$  in midpoint  $c$  facing  $s$ . Note that  $\alpha = 2\theta$ . These equations proof a).

Next, substitute  $|AB|$  with  $|AB| = \sin \frac{\theta}{2} \cdot 2|CB|$  and replace  $\sin \theta$  with the trigonometry identity  $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ . You receive equation b).

At last, substitute  $\theta$  using inequality  $\theta \leq \frac{2\pi}{k}$  with  $k > 2$ , obtaining c).

- (ii) Suppose,  $M_i$  and  $M_j$  is an edge in  $G$ , then there exists a circle with these two points on it's border which does not contain any other node of  $G$ . So,  $M_p$  must lie outside of this circle. By proposition 5.2 part a) there is a circle  $\odot CM_p$  through  $C$  and  $M_p$  which is empty. These two last observations contradict each other, since  $\odot CM_p$  would always contain  $M_i$  or  $M_j$ .
- (iii) Since the angles  $\alpha$  and  $\beta$  between opposite points of a chord in a rectangle which corners lie on a circle are supplementary, this is a fact:  $\angle AM_1B = \pi - \angle ACB$  (see lemma 5.1 for more details). The angle  $\angle M_{i-1}CM_{i+1}$  is smallest, if  $M_{i-1}$  and  $M_{i+1}$  lie on the circle. Note, by precondition we assume  $\angle BCA \leq \frac{2\pi}{k}$ . These facts proof following inequalities:

$$\begin{aligned} \angle M_{i-1}M_iM_{i+1} &\geq \pi - \angle M_{i-1}CM_{i+1} \\ &\geq \pi - \angle BCA \\ &\geq \pi - \frac{2\pi}{k} \end{aligned}$$

- (iv) Since  $M_1$  is inside the area subtended by chord  $AB$  from  $\odot ABC$  away from  $C$ , it is true that  $\angle CAM_1 \geq \angle CAB \geq \frac{\pi}{2} - \frac{\pi}{k}$ . The last inequality is true because:

$$\begin{aligned} \angle CAB + \angle ABC + \underbrace{\angle BCA}_{\leq \frac{2\pi}{k}} &= \pi \\ \angle CAB + \angle ABC &\geq \pi - \frac{2\pi}{k} \\ \angle CAB &\geq \frac{\pi - \frac{2\pi}{k}}{2} = \frac{\pi}{2} - \frac{\pi}{k} \end{aligned}$$

Since  $CA \leq CB$ ,  $\angle CAB$  can be at most the half of  $\pi - \frac{2\pi}{k}$ , proving the last inequality. ■

## 5.1 inward Path

Now, we perform the proof for the case when  $\triangle ABC$  contains other nodes.

Let  $S$  be the set of points which contains points  $A$  and  $B$ , and all the points interior to  $\triangle ABC$  excluding  $C$ . Then  $CH(S)$  are all the points which are on the

convex hull of  $S$ . Let these points be called  $N_0 = A$  and  $N_t = B$  and points  $N_1, \dots, N_{t-1}$  are the points on  $CH(S)$  which lie inside  $\triangle ABC$ . The following two propositions are taken from [8]:

**Proposition 5.3.** *The following are true:*

- a) for every  $i = 0, \dots, s-1$  :  $|CN_i| \leq |CN_{i+1}|$ , and
- b) for every  $i = 0, \dots, s-2$  :  $\angle N_i N_{i+1} N_{i+2} \geq \pi$ , where  $\angle N_i N_{i+1} N_{i+2}$  is the angle facing point  $C$ .

*Proof.* Since  $CA$  is the shortest edge in the angular sector  $\angle BCA$ ,  $|CA| \leq |CN_i|$ , for  $i = 1, \dots, t-1$  and since  $N_1, \dots, N_t$  are on  $CH(S)$ , a) is true.

Part b) follows from the convexity of  $CH(S)$ . All interior angles to  $CH(S)$  measure at most  $\pi$ , so all the exterior angles fulfil  $\angle N_{i-1} N_i N_{i+1} \geq \pi$  ■

**Proposition 5.4.** *The following are true:*

- a) for every  $i = 0, \dots, s-1$ , the interior of  $\triangle CN_i N_{i+1}$  is devoid of points of  $G$ ,
- b) for every  $i = 0, \dots, s$ , there exists a circle passing through  $CN_i$  whose interior is devoid of points of  $G$ .

*Proof.* Since  $N_0, \dots, N_s$  are on  $CH(S)$  and, hence, no other point can reside closer to  $C$ , part a) is true. ■



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