Curve Meets Line

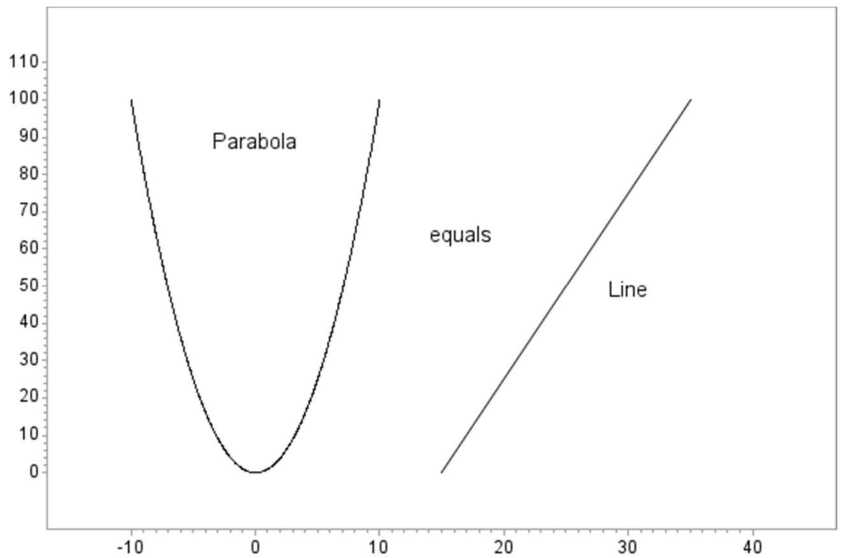
In high school algebra we learned to find the “roots” of a quadratic equation using the Quadratic Formula:

$$\text{Given}\quad ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The standard proof of this generally involves something called “completing the square,” followed by some algebra. But, looking at the polynomial equation one day, I realized that

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad ax^2 = -bx - c \quad \Rightarrow \quad x^2 = -\frac{b}{a}x - \frac{c}{a} = b'x + c'$$

or, equivalently



So, we have x^2 (a unit standard parabola) = $b'x + c'$ (a line). We find the 0, 1, or 2 points of coincidence between a standard parabola and a certain line. We compare a parabola to a line! I find this fascinating – what begins as an algebraic statement becomes geometric (of course, one interpretation of $ax^2 + bx + c = 0$ is as a graph, with roots at the x-coordinates where the graph intersects the x-axis - but the initial equation, at face value, is simply an algebraic polynomial). To find the roots, we first locate x_0, where $ax^2 + bx + c$ is minimized (when a is positive) or maximized (when a is negative) by recalling from calculus that a local min/max is found where the function’s first derivative is 0 (our function is continuous and has one inflection point giving it a single min/max). In our case, this is where $2ax_0 + b = 0 \Rightarrow x_0 = -\frac{b}{2a}$. Next, we note that the parabola is symmetric about x_0 since for every δ

$$a(x_0 - \delta)^2 + b(x_0 - \delta) + c = ax_0^2 - 2a\delta x_0 + a\delta^2 + bx_0 - b\delta + c$$

while

$$a(x_0 + \delta)^2 + b(x_0 + \delta) + c = ax_0^2 + 2a\delta x_0 + a\delta^2 + bx_0 + b\delta + c$$

and noting that the two right hand equations have a difference of

$$4a\delta x_0 - 2b\delta = 4a\delta\left(-\frac{b}{2a}\right) - 2b\delta = 0$$

we establish that the parabola has equal (elevation) values for x values equidistant above and below the point of symmetry, x_0, making it symmetric.

Back to finding roots and comparing parabolas and lines, having points common to both the parabola and line, we solve for δ such that

$$x = -\frac{b}{2a} \pm \delta \;\;\Rightarrow\;\; x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

which gives

$$\left(-\frac{b}{2a} \pm \delta\right)^2 = -\frac{b}{a}\left(-\frac{b}{2a} \pm \delta\right) - \frac{c}{a} \;\;\Rightarrow\;\; \frac{b^2}{4a^2} \pm \frac{b\delta}{a} + \delta^2 = \frac{b^2}{2a^2} \pm \frac{b\delta}{a} - \frac{c}{a}$$

$$\Rightarrow\;\; \delta^2 = \frac{-b^2 + 2b^2 - 4ac}{4a^2} \;\;\Rightarrow\;\; \delta = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \;\;\Rightarrow\;\; x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is the result we memorized in high school.