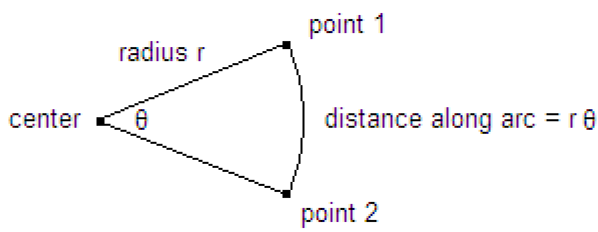
***Following is a letter to my daughter, Ginger, who is finishing her master's degree in Geospatial Information Science (GIS) at NC State University*.**

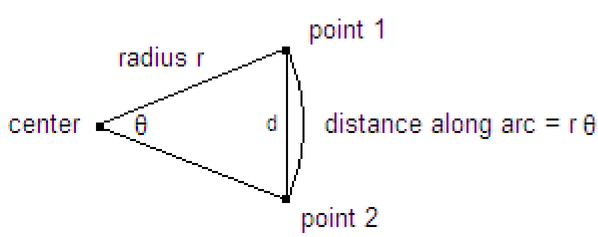
Hi Ginger,

Here is the small problem I mentioned that you might try writing a procedure for. The objective is to calculate the distance between two points on a sphere (with obvious application to distances between points on the earth's surface). I have used this method in determining vicinity of scheduled delivery locations for potential load combination and in determining whether supplier locations are within a maximum radius for LEED (Leadership in Energy and Efficiency Design) consideration.

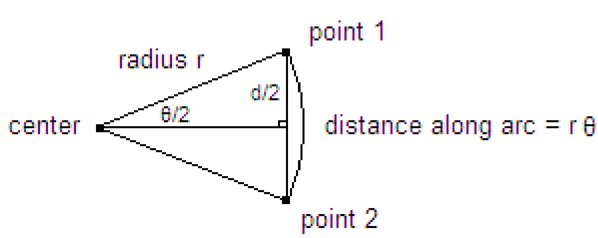
Basically, the idea is to connect the two points by a minimal arc along the surface of the sphere and calculate the length of that arc. The shortest distance between two points on a sphere is along an arc lying in the plane containing the two points and the center point of the sphere. Recalling from geometry, the length of this arc equals the angle between the lines connecting the points to the center of the sphere times the radius of the sphere, as shown here:



The radius is easy in our case (average earth radius is about 3,959 miles) reducing the challenge to finding θ, the angle between the arc endpoints. Notice that both lines above have equal length (the sphere radius), so if we form an isosceles triangle by connecting the two points by a straight line of length *d*, as seen here:



then bisect that line and connect another one perpendicular to it to the center point as such:



we have a nice right triangle with angle $\frac{\theta}{2}$ and hypotenuse r. Recalling from trigonometry that $\sin\left(\frac{\theta}{2}\right) = \frac{\mathrm{d}/2}{\mathrm{r}}$, our problem becomes that of finding d and for that we will use the Pythagorean theorem. Recall, given two points (x_1, y_1) and (x_2, y_2) in a plane, the straight line distance between them is

$$\mathrm{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which extends to three dimensions (as in a sphere), where the points are (x_1, y_1, z_1) and (x_2, y_2, z_2):

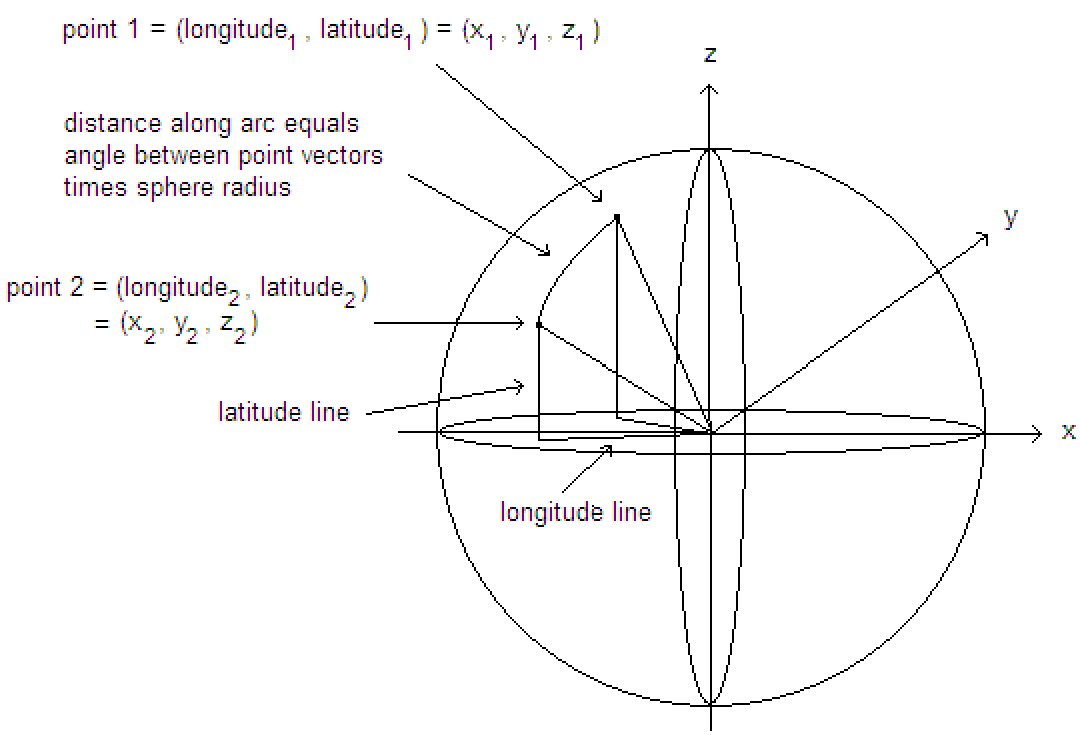
$$\mathrm{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Once d is known, θ can be solved for by taking the arcsine of $\frac{\mathrm{d}/2}{\mathrm{r}}$ and multiplying by 2. This follows from

$$\sin\left(\frac{\theta}{2}\right) = \frac{\mathrm{d}/2}{\mathrm{r}} \;\Rightarrow\; \frac{\theta}{2} = \arcsin\left(\frac{\mathrm{d}/2}{r}\right) \;\Rightarrow\; \theta = 2\arcsin\left(\frac{\mathrm{d}/2}{r}\right)$$

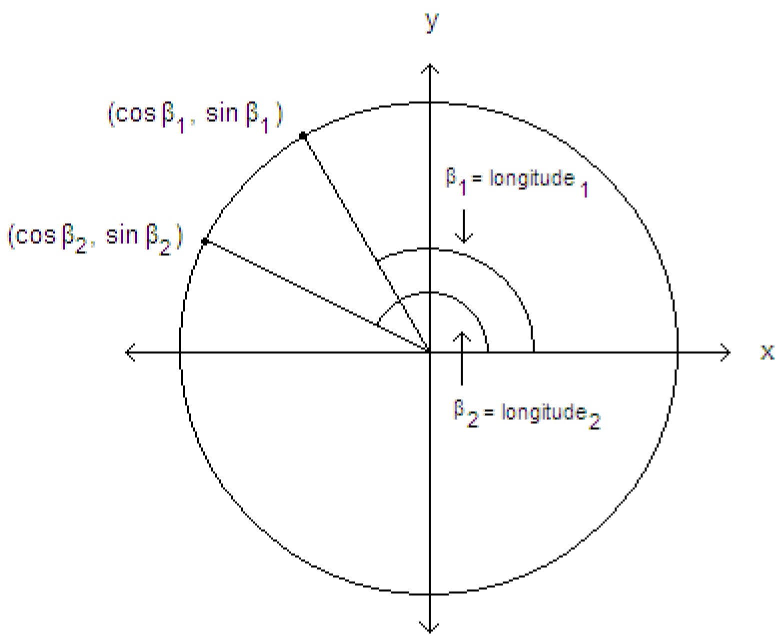
Finally, multiplying θ by 3,959 gives us the minimum distance between the points along an arc on the earth's surface.

Now, geographic points are given in longitude and latitude, so in order to use the above method we must convert given longitude and latitude coordinates into *x*, *y*, and *z* coordinates. From the following diagram, we see that longitude sweeps a point around the equator and determines *x* and *y* coordinates, while

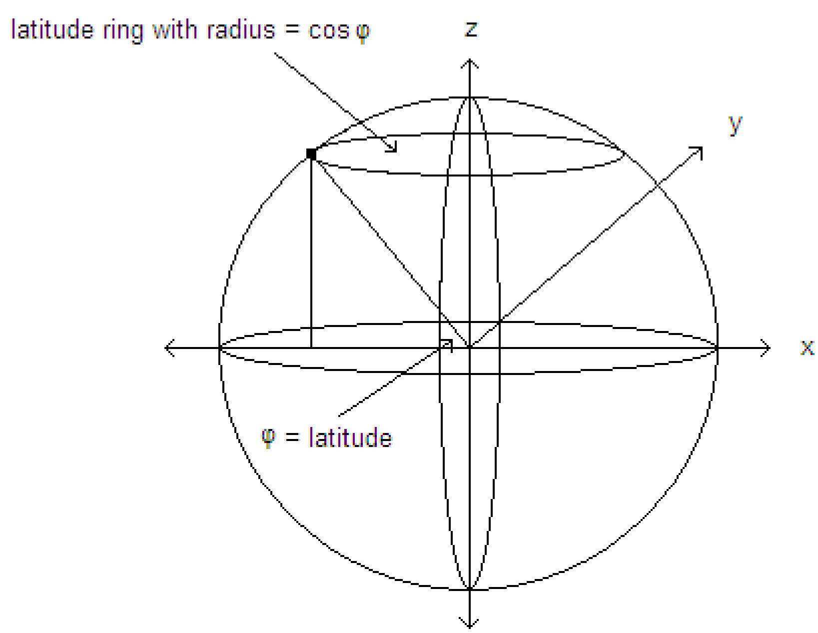


latitude raises a point toward a pole, giving its *z* coordinate.

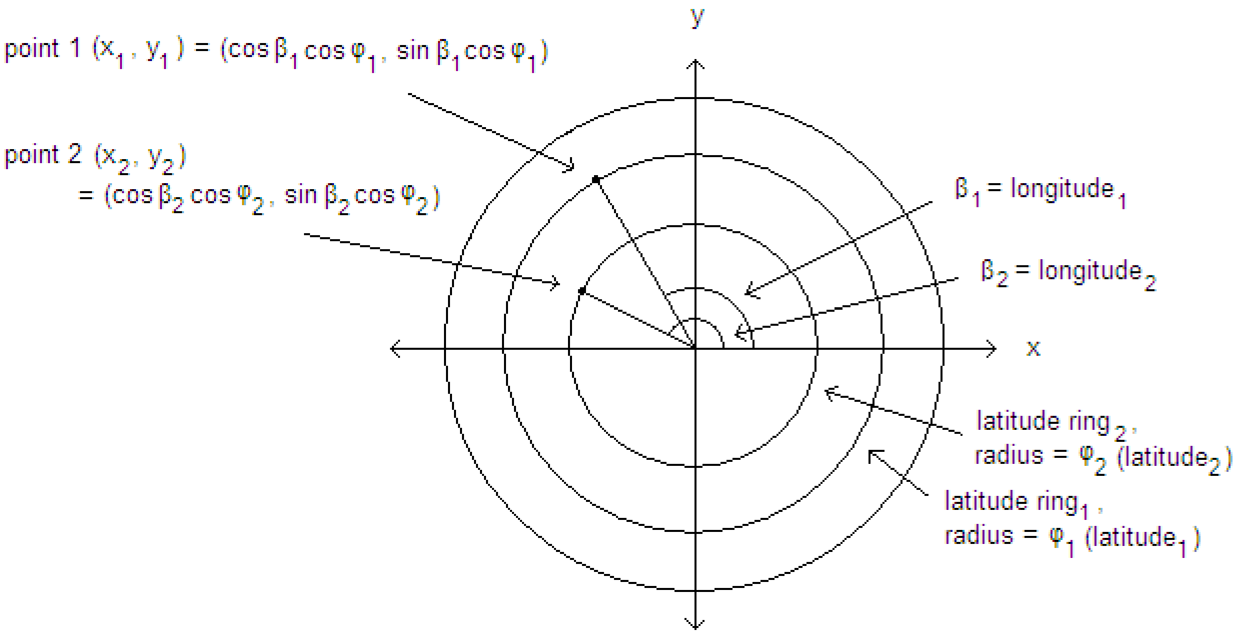
Looking down from the north pole, the longitude lines for points 1 and 2 lie in the equatorial plane as follows:



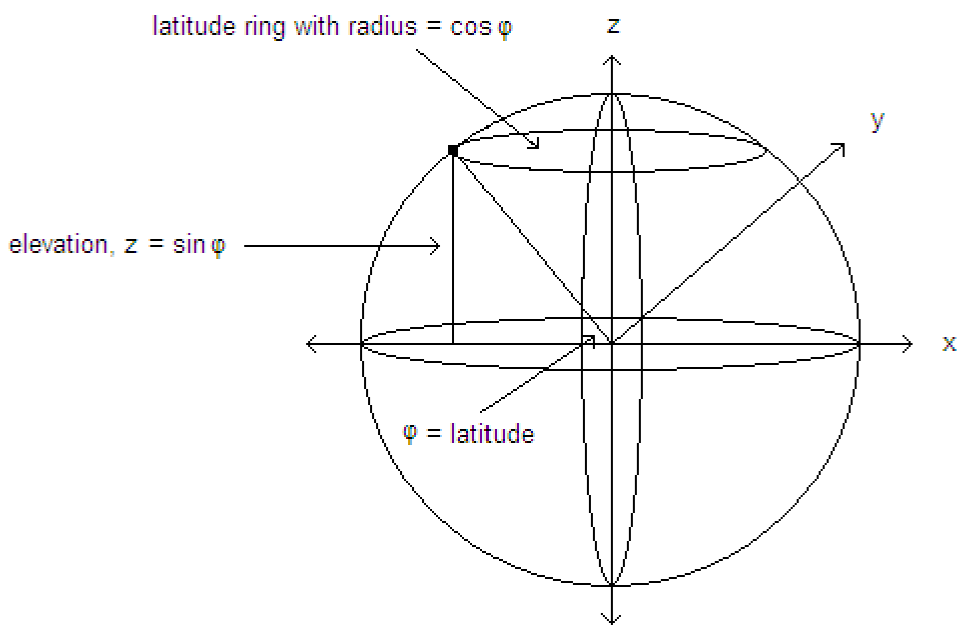
giving equatorial *x* and *y* coordinates of $(\cos\beta_1, \sin\beta_1)$ and $(\cos\beta_2, \sin\beta_2)$, where β_1 is the longitude of point 1 and β_2 is the longitude of point 2. But geographic points are not typically on the equator, so they must be adjusted toward the center of the sphere according their respective latitudes. Each point lies on a latitude ring parallel to the equator with radius determined by the latitude as seen in the following diagram:



Another top-down view illustrates the *x*, *y* coordinates of points 1 and 2, where the longitude lines intersect their respective latitude ring:



Multiplying $(\cos\beta_1, \sin\beta_1)$ by $\cos\varphi_1$ [to produce $(\cos\beta_1\cos\varphi_1, \sin\beta_1\cos\varphi_1)$] moves along longitudinal line 1 from the equator to the correct *x*, *y* coordinates for longitude$_1$ (on latitude ring 1). Similarly, multiplying $(\cos\beta_2, \sin\beta_2)$ by $\cos\varphi_2$ correctly positions point 2 at *x*, *y* coordinates $(\cos\beta_2\cos\varphi_2, \sin\beta_2\cos\varphi_2)$] along longitudinal line 2 (on latitude ring 2). *z* coordinates are simply $\sin\varphi$, as seen here:



And this gives complete *x*, *y*, *z* coordinates for each point, namely:

$$(x_1, y_1, z_1) = (\cos\beta_1 \cos\varphi_1, \sin\beta_1 \cos\varphi_1, \sin\varphi_1)$$

and

$$(x_2, y_2, z_2) = (\cos\beta_2 \cos\varphi_2, \sin\beta_2 \cos\varphi_2, \sin\varphi_2),$$

where β_1 and φ_1 are the longitude and latitude of point 1 and β_2 and φ_2 are the longitude and latitude of point 2, which can be used in

$$\mathrm{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(\cos\beta_2 \cos\varphi_2 - \cos\beta_1 \cos\varphi_1)^2 + (\sin\beta_2 \cos\varphi_2 - \sin\beta_1 \cos\varphi_1)^2 + (\sin\varphi_2 - \sin\varphi_1)^2}$$

to calculate $\theta = 2\arcsin\left(\frac{\mathrm{d}/2}{r}\right)$ and, finally, multiplying θ by 3,959 gives the distance between the points.

Imagine a table or file containing the geographic locations of a large set of significant places (weather stations, cell phone towers, tagged animals in biological experiments, known mineral deposits, etc.). We might want to report the number of points within a specified radius of a given point, or exhaustively calculate the distance between all points and all other points. As a follow up challenge, you might consider creating such a table (or, like me, copy one from the internet – I have used a zip code table with longitude and latitude from zips.sourceforge.net and it has over 43,000 entries – one for each US zip code), then iterating all rows and calculating arc distance to a known point. If successful, you will have assembled relatively simple components (geometry, trigonometry, and possibly SQL for data management) into a powerful information tool for distance measurement, spacial density analysis, boundary sampling, and whatever else your geographic imagination devises.

Have fun!