ATS 601: Atmospheric Dynamics I
HW2: Momentum Equations

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Estimate of Time to Completion: 5 hours

Maximum Allotted Time to Completion: 10 hours

Actual Time to Completion: 7 hours

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ATS 601: HW #2 Due September 3^{rd}

1.

(a) Starting from mass continuity $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{u}) = 0$

$$\frac{\partial \psi}{\partial t} + \nabla\cdot(\psi\vec{u}) = 0$$

$$\frac{\partial \psi}{\partial t} + \psi\nabla\cdot\vec{u} + \vec{u}\cdot\nabla\psi = 0$$

• given that $\vec{u}(x,t) = U$ ∴ $\cancel{\nabla} u = 0$

$$\frac{\partial \psi}{\partial t} + U\cdot\nabla\psi = 0$$

$u = u(x,t)$ only x-dimension

$$\boxed{\frac{\partial \psi}{\partial t} + U\frac{\partial \psi}{\partial x} = 0}$$ ✓

(b) Show that (a) $= \frac{D\psi}{Dt} = 0$

From the material derivative: $\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + \vec{u}\cdot\nabla\psi$

∴ $\frac{\partial \psi}{\partial t} = \frac{D\psi}{Dt} - u\cdot\nabla\psi$

Plug into (a)

$$\frac{\partial \psi}{\partial t} + U\frac{\partial \psi}{\partial x} = 0$$

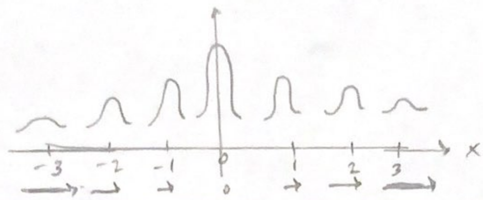
$$\left(\frac{D\psi}{Dt} - \cancel{u\cdot\nabla\psi}\right) + \cancel{U\frac{\partial \psi}{\partial x}} = 0$$

$$\boxed{\frac{D\psi}{Dt} = 0}$$ ✓

(c) The solution to the advection equation for $u(x,t) = U$ is $\psi(x,t) = \psi_0(x-Ut)$ w/ I.C. $\psi(x,t=0) = \psi_0(x)$.

⇒ The blob will be advected to the right @ some constant U, given by $\psi(x,t) = \psi_0(x-Ut)$. Over some time we see $\psi(x,1) = \psi_0(x-U)$ and $\psi(x,2) = \psi_0(x-2U)$ and so on.

⇒ The eulerian perspective we are watching from a stationary point and see the blob move right wrt us. The lagrangian perspective is when we are moving along w/ the blob to the right but we are stationary from this point of view.



abs of x

(d) consider $u(x,t) = \Lambda|x|$ $\Lambda \to$ constant > 0 $u(x,t) = \Lambda|x|$

⇒ The tracer blob will be moving to the right however it will slow down as x approaches zero, stop @ $x=0$ and then speed up past $x=0$. Due to mass conservation, the blob will squish & squeeze

$$\Rightarrow \frac{\partial\psi}{\partial t} + \nabla\cdot(\psi\vec{u}) = 0$$

$$\frac{\partial\psi}{\partial t} + \psi\frac{\partial}{\partial x}(u) + u\frac{\partial}{\partial x}(\psi) = 0$$

$$\frac{\partial\psi}{\partial t} + \psi\frac{\partial}{\partial x}(\Lambda|x|) + (\Lambda|x|)\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\psi}{\partial t} + \psi\Lambda\frac{x}{|x|} + \frac{\partial\psi}{\partial x} = 0$$

$$\boxed{\begin{aligned} &\frac{\partial\psi}{\partial t} + \psi\Lambda + \Lambda|x|\frac{\partial\psi}{\partial x} = 0 \quad x > 0 \\ &\frac{\partial\psi}{\partial t} - \psi\Lambda + \Lambda|x|\frac{\partial\psi}{\partial x} = 0 \quad x < 0 \end{aligned}}$$

⇒ This new continuity equation has an extra ψ term scaled by Λ and $\frac{\partial\psi}{\partial x}$ is scaled by a non-constant term $\Lambda|x|$ which will always be positive since $\Lambda > 0$ and $|x| \geq 0$. I think its also useful to equate some terms instead of having it equal to zero: $\frac{\partial\psi}{\partial t} = -\Lambda|x|\frac{\partial\psi}{\partial x} - \psi\Lambda$ which says the change in density w/ time equals the negative rate of change of density w/ spatial x, scaled by a $\Lambda|x|$, minus some other term $-\psi\Lambda$.

$\frac{\partial\psi_j^i}{\partial t} \approx \frac{\psi_j^{i+1} - \psi_j^i}{\Delta t}$

$\frac{\partial\psi_j^i}{\partial x} \approx \frac{\psi_j^i - \psi_{j-1}^i}{\Delta x}$

(e) Plugging in these terms on the left to our continuity eqn from (d):

$$\Rightarrow \frac{\psi_j^{i+1} - \psi_j^i}{\Delta t} + \psi_j^i\Lambda + \Lambda|x|\left(\frac{\psi_j^i - \psi_{j-1}^i}{\Delta x}\right) = 0$$

$$\psi_j^{i+1} - \psi_j^i + \Delta t\,\psi_j^i\Lambda + \frac{\Delta t\Lambda|x|}{\Delta x}(\psi_j^i - \psi_{j-1}^i) = 0$$

$$\boxed{\begin{aligned} &\psi_j^{i+1} = \psi_j^i - \Delta t\Lambda\psi_j^i - \frac{\Delta t\Lambda|x|}{\Delta x}(\psi_j^i - \psi_{j-1}^i) \quad x > 0 \\ &\psi_j^{i+1} = \psi_j^i + \Delta t\Lambda\psi_j^i - \frac{\Delta t\Lambda|x|}{\Delta x}(\psi_j^i - \psi_{j-1}^i) \quad x < 0 \end{aligned}}$$

ATS 601: Atmospheric Dynamics I

Tyler Barbero | Due 9/3

In [1]:
```
%matplotlib inline
```

In [2]:
```
import numpy as np
import math
import xarray as xr
import matplotlib.pyplot as plt
import warnings
```

```
/anaconda3/lib/python3.7/site-packages/dask/config.py:168: YAMLLoadWarni
ng: calling yaml.load() without Loader=... is deprecated, as the default
Loader is unsafe. Please read https://msg.pyyaml.org/load for full detai
ls.
  data = yaml.load(f.read()) or {}
/anaconda3/lib/python3.7/site-packages/distributed/config.py:20: YAMLLoa
dWarning: calling yaml.load() without Loader=... is deprecated, as the d
efault Loader is unsafe. Please read https://msg.pyyaml.org/load for ful
l details.
  defaults = yaml.load(f)
```

In [3]:
```
large = 28; med = 24; small = 15
params = {'axes.titlesize': small,
          'legend.fontsize': med,
          'figure.figsize': (8, 4),
          'axes.labelsize': small,
          'axes.titlesize': small,
          'xtick.labelsize': small,
          'ytick.labelsize': small,
          'figure.titlesize': med,
          'axes.titlepad': 10,
         'axes.facecolor': 'black'} # For electric background plots
plt.rcParams.update(params)
```

Problem 1: Passive tracer advection

1e) Step 2: Convert your equation from (d) into numerical form, and solve for the density ψ_j^{i+1} at a future time $i+1$ at location j.

i,j notational form of continuity equation

$\psi_j^{i+1} = \psi_j^i - \Delta t \Lambda \psi_j^i - \frac{\Delta t \Lambda |x|}{\Delta x}(\psi_j^i - \psi_{j-1}^i)$ for x > 0 and

$\psi_j^{i+1} = \psi_j^i + \Delta t \Lambda \psi_j^i - \frac{\Delta t \Lambda |x|}{\Delta x}(\psi_j^i - \psi_{j-1}^i)$ for x < 0.

Numerical form of continuity equation

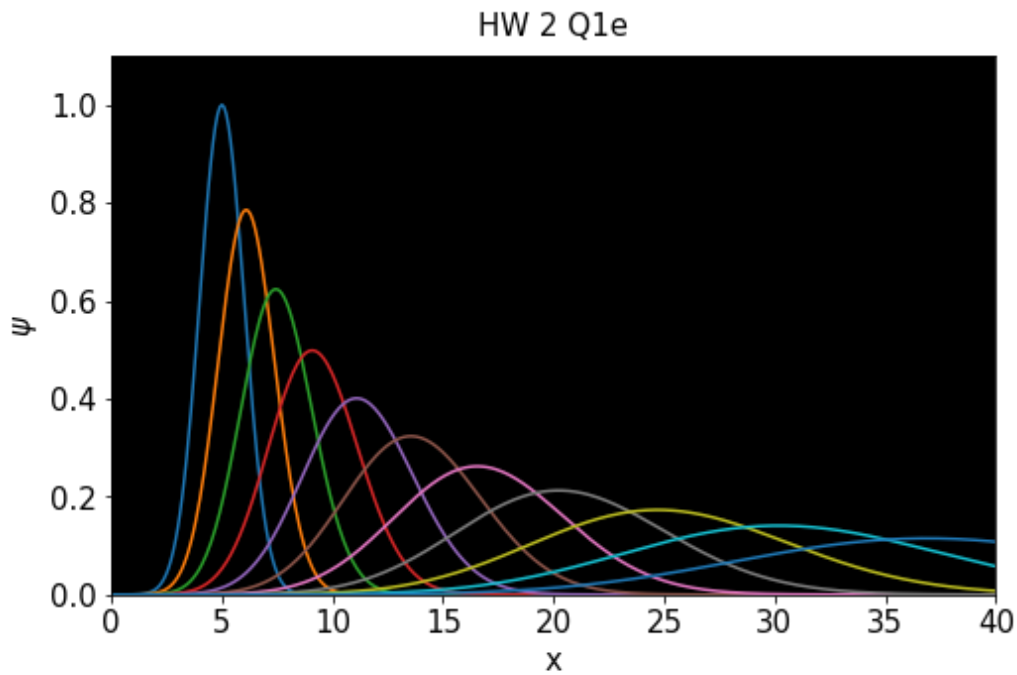
$psi[xi, ti + 1] = psi[xi, ti] + (dt * A * psi[xi, ti]) - (A * np.abs(x[xi]) * (dt/dx) * (psi[xi, ti] - psi[xi - 1$
for x > 0 and

$psi[xi, ti + 1] = psi[xi, ti] - (dt * A * psi[xi, ti]) - (A * np.abs(x[xi]) * (dt/dx) * (psi[xi, ti] - psi[xi - 1$
for x < 0.

In [4]:
```
dt = 0.01
dx = 0.10
mu = 5
A = 0.1 #lambda
sigma = 1


# Define time and space and psi arrays
t = np.arange(0,22,dt)
x = np.arange(-100,100,dx)
```

In [5]:
```
# assign array for psi
psi = np.zeros([len(x),len(t)])

# intial condition
psi_0 = np.exp(-1*((x-mu)**2)/(2*sigma**2))
psi[:,0] = psi_0

# code up numerical equation
# tmp = psi[xi,ti]+(dt*A*psi[xi,ti])-(A*np.abs(x[xi])*dt/dx*(psi[xi,ti]-p
si[xi-1,ti]))

# emnumerate gets the index and value
for ti,tv in enumerate(t[1:]):
    for xi,xv in enumerate(x[1:]):

        # calculating psi at each x and at each t index

        # case where value of x < 0
        if xv < 0:
            psi[xi,ti+1] = psi[xi,ti]+(dt*A*psi[xi,ti])-(A*np.abs(x[xi])*
(dt/dx)*(psi[xi,ti]-psi[xi-1,ti]))

        # case where value of x > 0
        elif xv > 0:
            psi[xi,ti+1] = psi[xi,ti]-(dt*A*psi[xi,ti])-(A*np.abs(x[xi])*
(dt/dx)*(psi[xi,ti]-psi[xi-1,ti]))
```

In [6]:
```python
plt.figure(figsize=(8,5))
for i in np.arange(0,len(t),200):
    plt.plot(x,psi[:,i]);
#     plt.legend([i])

plt.xlabel('x');
plt.ylabel('$\psi$');
plt.xlim([0,40])
plt.ylim([0,1.1])
plt.title('HW 2 Q1e');
```



Problem 2: Reading-in global atmospheric data

In [7]:
```python
ds = xr.open_dataset('/Users/tyler/Desktop/601/homework_2_ERA-interim_300hPa_2014-Jan1-8.nc')
```

In [8]:
```python
u = ds.u
v = ds.v
```

In [9]:
```python
latitude = ds.latitude
longitude = ds.longitude
time = ds.time
```

Problem 3: Approximations to the momentum equations (gravity, viscosity)

(a) Using the data from Problem 2, estimate typical values* of the local accelerations of the zonal and meridional wind in the Northern midlatitudes (30-60 N), namely: $\partial u/\partial t$ and $\partial v/\partial t$

In [10]:
```
u_midlat = u.sel(latitude=slice(60,30))
v_midlat = v.sel(latitude=slice(60,30))
```

In [11]:
```
dudt = np.zeros(np.shape(u_midlat))
dvdt = np.zeros(np.shape(v_midlat))

for i in np.arange(1,np.shape(dudt)[0]-1):
    dudt[i-1,:,:] = (u_midlat[i+1,:,:] - u_midlat[i-1,:,:])/(2*6*60*60)
    dvdt[i-1,:,:] = (v_midlat[i+1,:,:] - v_midlat[i-1,:,:])/(2*6*60*60)
```

In [12]:
```
# calculate typical values using RMS
def RMS(matrix):
    tmp = matrix**2
    tmp = np.mean(tmp)
    tmp = np.sqrt(tmp)
    return tmp
```

In [13]:
```
print('The RMS or typical values of the zonal wind is: '+str(RMS(dudt))+' m/s')
print('The RMS or typical values of the meridional wind is: '+str(RMS(dvdt))+' m/s')
```

```
The RMS or typical values of the zonal wind is: 0.00024879785447811195
m/s
The RMS or typical values of the meridional wind is: 0.00034777278712432
18 m/s
```

(b) Estimate the magnitude of the wind tendencies due to viscosity in the momentum equations for the atmosphere. You can do this a variety of ways, some easier than others. I would suggest following the scaling method in Vallis (link to textbook on Canvas) that uses the Reynolds number (section 1.11.1, page 44), as this will provide a ballpark solution in record time. Alternatively, you can try and calculate the laplacian of the wind on the sphere and compute the actual term (much more difficult).

- I found a typical value of viscosity for air to be $\nu = 1.81e-5$ at 15 degrees C. From this page (https://www.ncdc.noaa.gov/sotc/global/202004 (https://www.ncdc.noaa.gov/sotc/global/202004)), NOAA says the average global temperature from the 20th century was 13.7C (I couldnt find global average for the 2000-2021.) It will probably be higher in 2000-2021 so I think this is close enough.
- I chose a typical characteristic length scale as $L = 10^7 m$ because from lecture I think from lecture it said synoptic scales were between 10^6 and $10^8 m$. And definitely we're looking at midlatitudes between 30 and 60N which is definitely a synoptic scale.
- For a characteristic velocity scale I took the average of the RMS of the zonal and meridional component of wind speeds. For a typical velocity, I found $U = 35.7 m/s$.
- Finally, using the Force due to viscosity scaling in Vallis: ($\nu \frac{U}{L^2}$)

In [14]:

```
# Typical value of viscoity (https://www.engineersedge.com/physics/viscos
ity_of_air_dynamic_and_kinematic_14483.htm)
v = 1.81e-5 # units: kg/(m*s)

# Typical characteristic length scale (L) (between 10^6 and 10^8)
L = 10**7 # units: meters

# Typical characteristic velocity scale (average both du_dt and dv/dt)
U = np.sqrt(RMS(u_midlat)**2 + RMS(v_midlat)**2)

# Contribution due to viscosity is v*U/L^2
F_v = v*U/(L**2)
print('Typical force due to viscosity is: '+str(np.array((F_v))))
```

```
Typical force due to viscosity is: 6.460452476501464e-18
```

- The typical time tendencies due to force of viscosity is $6.5e - 18kg/m^2s^2$. (F/m = acceleration)
- Compared to time tendecies of wind (du/dt and dv/dt), from a) those values were magnitude $10^{-4}m/s^2$.
- Therefore, I since the difference in magnitudes is so large, I think it is okay to neglect viscosity in this course.