TYLER BARBERO

Math 170A midterm 2

1. consider the matrix $A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ Denote $A = U\Sigma V^T$ the SVD of A. compute Σ^+.

$$A^+ = (U\Sigma V^T)^+ = (V^T)^+ \Sigma^+ U^+ = V\Sigma^+ U^T$$

find eigenvalues to get σ_1, σ_2

$$AA^T = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

do $\det(AA^T - \lambda I)$

$$\det\left(\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{bmatrix}\right) = 0$$

$$(5-\lambda)(5-\lambda) - (-4)(-4) = 0$$

$$25 - 10\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 10\lambda + 9 = 0$$

$$(\lambda - 1)(\lambda - 9) = 0$$

$$\lambda = 1, \ \lambda = 9$$

$\sigma_i^2 = \lambda_i \Rightarrow \sigma_i = \sqrt{\lambda}$ thus $\sigma_1 = \sqrt{9} = 3$, $\sigma_2 = \sqrt{1} = 1$

$\sigma_1 = 3 \quad \sigma_2 = 1$

$\sigma_1 > \sigma_2 \ldots$

$$\Sigma = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$$

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & \\ & 1/\sigma_2 \end{bmatrix}$$

$A = n\times n \Rightarrow \Sigma = n\times n \therefore \Sigma^+ = \Sigma^{-1}$. $\quad \Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & \\ & 1/\sigma_2 \end{bmatrix}$

$$\Sigma^+ = \Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & 1/\sigma_2 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/1 \end{bmatrix}$$

$$\boxed{\Sigma^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}}$$

2. consider matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. we want to solve $Ax = b$ but can only solve $A\hat{x} = \hat{b}$. Here $\hat{b} = b + \delta b$ and $\hat{x} = x + \delta x$.

Q which bound on the relative error in b, $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ makes sure that the relative error in the solution x, $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$, is less than or equal to 0.01?

- recall $\|\cdot\|_\infty$ (matrix) $\quad \|A\|_\infty = \max \sum_{j=1} |a_{ij}|$ (sum over rows)
- recall $\|\cdot\|_\infty$ (vector norm) $\quad \|x\|_\infty = \max_{i=1,\dots,n} |x_i| \quad x \in \mathbb{R}^n$

- From Lecture we know : $\frac{\|x - \hat{x}\|}{\|x\|} \leq K(A) \frac{\|b - \hat{b}\|}{\|b\|}$

compute $K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$

$$\|A\|_\infty = \overset{\max}{(\Sigma 1, \Sigma 2)} = 2$$

$$\|A^{-1}\| = \frac{1}{2-0}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = \max(\Sigma 1, \Sigma 0.5) = 1$$

$$\therefore K(A) = (2)(1) = 2$$

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq K(A) \cdot \frac{\|\delta b\|_\infty}{\|b\|_\infty} \leq 0.01$$

thus for the rel error on x to be less than or equal to 0.01, do

$$\boxed{\frac{\|\delta b\|_\infty}{\|b\|_\infty} \leq \frac{(0.01)}{K(A)} = \frac{0.01}{2} = 0.005.}$$

3. consider the matrix $A = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$.

compute the full QR decomposition of A.

Find reflector Q that maps column of A to $\|a_1\| \cdot e_1$

define $y = \|a_1\| e_1$

$$\|a_1\| = (3^2 + 0^2 + 4^2)^{\frac{1}{2}} = \sqrt{25} = 5 = \|a_1\| = r_{11}$$

$$y = 5 \cdot e_1 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

$$u := \frac{x-y}{\|x-y\|_2} = \frac{a_1 - y}{\|a_1 - y\|_2} = \frac{\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}}{\left\|\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}\right\|_2} = \frac{\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}}{\sqrt{4+16} = \sqrt{20}} = \frac{1}{\sqrt{20}} \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} = u$$

$$Q = I_3 - 2uu^T$$

$$= I_3 - 2 \frac{1}{\sqrt{20}} \frac{1}{\sqrt{20}} \left[\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} (-2\ 0\ 4) \right]$$

$$= I_3 - \frac{2}{20} \begin{bmatrix} 4 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & 16 \end{bmatrix} =$$

$A = QR$ ← upper

3x1, 3x3, 3x1

$$= I_3 - \frac{1}{10} \begin{bmatrix} 4 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{5} & 0 & -\frac{4}{5} \\ 0 & 0 & 0 \\ -\frac{4}{5} & 0 & \frac{8}{5} \end{bmatrix}$$

$$\boxed{Q = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & -\frac{3}{5} \end{bmatrix}; \quad R = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \quad \text{and } A = QR}$$

check $A = QR$ $\quad QR = \begin{matrix} 5(\frac{3}{5}) = \\ 5(0) = \\ 5(\frac{4}{5}) = \end{matrix} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = A$ ✓