math 1704 HW #6

1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) show that the characteristic Eqn of A is $\lambda^2 - 2\lambda + 1 = 0$.

we have: $Av = \lambda v$

$\Leftrightarrow Av - \lambda v = 0$

$\Leftrightarrow (A - \lambda I)v = 0$

$\Leftrightarrow \det(A - \lambda I) = 0$

$\rightarrow \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\Leftrightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 = 0$

$\boxed{1 - 2\lambda + \lambda^2 = 0}$ ✓

roots:

$(\lambda - 1)(\lambda - 1) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 1$ ✓

(b) perturb one coefficient of the characteristic polynomial slightly & consider the eqn $\lambda^2 - 2\lambda + (1-\varepsilon) = 0$, where $0 < \varepsilon << 1$. Solve the eqn for the roots of $\hat{\lambda}_1, \hat{\lambda}_2$

quadratic Eqn? $\lambda = \frac{+2 \pm \sqrt{4 - (4)(1)(1-\varepsilon)}}{2}$ $\quad 4 - (4 - 4\varepsilon) = 4 - 4 + 4\varepsilon$

$\lambda = \frac{+2 \pm 2\sqrt{\varepsilon}}{2} = -1 \pm \sqrt{\varepsilon}$

$\boxed{\hat{\lambda}_1 = 1 + \sqrt{\varepsilon}\,;\ \hat{\lambda}_2 = 1 - \sqrt{\varepsilon}}$

(c) show that when $\varepsilon = 10^{-12}$, $|\hat{\lambda}_1 - \lambda_1|$ and $|\hat{\lambda}_2 - \lambda_2|$ are one million times bigger than ε.

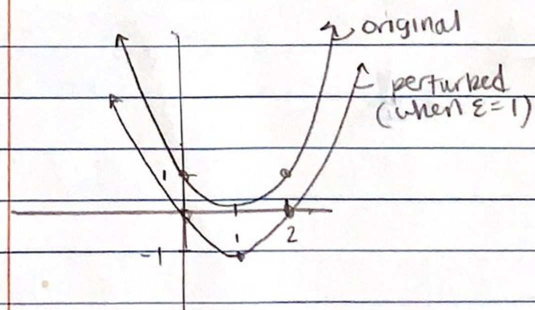
$|\hat{\lambda}_1 - \lambda_1| = |1 + \sqrt{10^{-12}} - 1| = |1e-6| = 1e-6$

$|\hat{\lambda}_2 - \lambda_2| = |1 - \sqrt{10^{-12}} - 1| = |-1e-6| = 1e-6$

$\Rightarrow$ $\varepsilon \times$ one million $= 1e-6$

thus $|\hat{\lambda}_1 - \lambda_1|$ & $|\hat{\lambda}_2 - \lambda_2|$ are one million times bigger than ε. $\boxed{QED}$ ✓

(d) sketch the graphs of original & perturbed polynomials using ε bigger than 10^{-12} to understand why roots are so sensitive to ε pertubation



original has one root at $x=1$ as ε goes from 0 to 1, the original polynomial shifts downward to position of perturbed when $\varepsilon = 1$. There are two roots that continuously change as $\varepsilon \rightarrow 0$ to 1. When $\varepsilon = 1$ $1 - \varepsilon = 1 - 1 = 0$ so $\lambda^2 - 2\lambda = 0$ and the system has roots @ $\lambda_1 = 2$ and $\lambda_2 = 0$.

2. An $(n\times n)$ matrix A is called semi-simple if it has n linearly independent eigenvectors, otherwise it is called defective. For each, prove a 2×2 matrix that satisfies the requested properties

(a) A has 2 distinct eigenvalues & is semisimple:

→ construct A. we know $A = V^T \Sigma V$

Let $\lambda_1 = 2,\ \lambda_2 = 5$ $\quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ← linearly indep eigenvectors

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}}$$

check!
$$\left|\begin{pmatrix} 2-\lambda & 0 \\ 0 & 5-\lambda \end{pmatrix}\right| = 0$$
$(2-\lambda)(5-\lambda)$
$10 - 7\lambda + \lambda^2 = 0$
$(\lambda-5)(\lambda-2) = 0$

$\lambda_1 = 5,\ \lambda_2 = 2$ ✓

$(A - 5I)v = 0$
$\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} v = 0$
$\left(\begin{array}{cc|c} -3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$
$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$(A - 2I)v = 0$
$\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} v = 0$
$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}\right)$
$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) A matrix has two eigenvalues that are the same, & is defective

Let $\lambda_1 = \lambda_2 = 1$ Let $V = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ ← linearly dependent eigenvectors

$A = V^T \Sigma V$
$= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$
$= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$$\boxed{A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}$$

check
$\det(A - \lambda I) = 0$
$\left|\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}\right|$
$(1-\lambda)(4-\lambda) - 4 = 0$
$4 - 5\lambda + \lambda^2 - 4$
$\lambda^2 - 5\lambda = 0$

$\lambda = 0,\ \lambda = 5$ ✓

$\lambda = 0$: $\begin{pmatrix} 1-0 & 2 \\ 2 & 4-0 \end{pmatrix} v = 0$
$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$
defective linearly dependent

$\lambda = 5$
$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} = \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$
lin. dependent

(c) A matrix $\lambda_1 = \lambda_2$ and is semisimple.

try constructing A $\quad A = V^T \Sigma V$ $\quad$ let $V^T = V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\lambda_1 = \lambda_2 = 1$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

two linearly independent eigenvectors namely the standard basis vectors for $\mathbb{R}^2$.

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

check: $\det(A - \lambda I) = 0$
$$\left|\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}\right| = 0$$
$(1-\lambda)(1-\lambda) = 0$
$1 - 2\lambda + \lambda^2 = 0$
$(\lambda-1)(\lambda-1) = 0$
$\lambda_1 = \lambda_2 = 1$ ✓

$(A - 1I)v = 0$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} v = 0$ $\quad$ so v_1 can be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, v_2 can be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✓
$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$
and are linearly independent eigenvectors

Formulas

① $q_0 = q$

② $q_{j+1} = \frac{1}{s_{j+1}} A^j q$

③ $s_{j+1} = \|A^j q\|_\infty$ ← vector ∞-norm

$\det(A - \lambda I) = 0$

$\left|\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\right| = 0$

$(-\lambda)(-\lambda) - 1 = 0$

$\lambda^2 - 1 = 0$

$\sqrt{\lambda^2} = \sqrt{1}$

$\boxed{\lambda = \pm 1}$

3 Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Carry out power method starting w/ vector $q_0 = [a, b]^T$ $a, b \geq 0$, $a \neq b$. Explain why sequence fails to converge.

Let $a = 2$ $b = 4$ $a \neq b$, $a, b \geq 0$ ✓ $q = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$Aq_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $s_1 = \max\{4, 2\} = 2$

$q_1 = \frac{1}{s_1} Aq_0 = \frac{1}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$Aq_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $s_2 = \max\{1, 2\} = 2$

$q_2 = \frac{1}{s_2} Aq_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $s_3 = \max\{0.5, 1\} = 1$

$Aq_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ $s_4 = \max\{0.5, 1\} = 1$

$q_3 = \frac{1}{s_3} Aq_2 = \frac{1}{1}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ $s_5 = \max\{1, 0.5\} = 1$

$Aq_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $s_6 = \max\{0.5, 1\} = 1$

$q_4 = \frac{1}{s_4} Aq_3 = \frac{1}{1}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $s_7 = \max\{0.5, 1\} = 1$

$Aq_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ ...... and so on.

The sequence fails to converge, rather it oscillates between $q = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ and $q = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ for my set of values. I calculated the eigenvalues to be $-1, 1$. By the convergence rate $\frac{\lambda_2}{\lambda_1}$, our sequence will not converge because $\longrightarrow$ $\frac{\lambda_2}{\lambda_1} = -\frac{1}{1} = -1$, and as $j \to \infty$ will diverge to $-\infty$.

4. Show that if A is positive definite & symmetric then all its eigenvalues are real and positive.

① Let λ be an eigenvalue w/ eigenvector v. Consider $v^T A v$ and deduce result from this

so $Av = \lambda v$

$v^T A v = \lambda v^T v$

$v^T A v = \lambda \|v\|_2$ by definition of positive definite

A is positive definite; $\vec{v} \neq 0$; thus LHS is strictly positive

$\|v\|_2$ is the 2-norm which returns a real positive scalar

thus λ has to be positive and thus real. ✓ $\boxed{\text{QED}}$

② consider the cholesky decomposition of the positive definite matrix A and the SVD of its cholesky factor R.

recall cholesky $A = R^T R$

sym. lower upper

$R = U \Sigma V^T$

square assumption?

$A = R^T R = (U \Sigma V^T)^T U \Sigma V^T$

$= (V^T)^T \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T$ since U orthog

$= V \Sigma^T \Sigma V^T$

Σ diagonal thus $\Sigma^2 = \Sigma^T \Sigma$

$A = V \Sigma^2 V^T$

V^T orthog ∴ can switch

$V^T A = \underbrace{V^T V}_{I} \Sigma^2 V^T$

$A V^T = \Sigma^2 V^T$

$\sigma_i^2 = \lambda_i$

assume A positive def, V^T is orthog.

σ_i^2 always positive and real because A has real entries and is pos. def. Therefore by $\sigma_i^2 = \lambda_i$, the eigenvalues are positive & real. $\boxed{\text{QED}}$ ✓

5.

(a) what are the eigenvalues of B, C in terms of eigenvalues of A.

doing eig(A) in MATLAB the eigenvalues of A are $-4, -1, \frac{1}{2}, 3, 2$.

B is given as $B = A - 1.5I$

$$Av = \lambda v \Rightarrow A = v^T \lambda v$$

$$B = v^T \lambda v - 1.5I$$

$$Bv = \lambda v - 1.5Iv$$

$$Bv = (\lambda - 1.5)v$$

where the eigenvalues of B are given by λ (Eigenvalues of A) $- 1.5$.
↑ diag 5x5 matrix.

$C = inv(B) = B^{-1}$

$B^{-1} = (v^T \lambda v - 1.5I)^{-1}$

$= (v^T)^{-1} \lambda^{-1} v^{-1} - 1.5I^{-1}$

$C = v \lambda^{-1} v^T - 1.5I$

$Cv^T = \lambda^{-1} v^T - 1.5Iv^T$

$Cv^T = (\lambda^{-1} - 1.5I)v^T$

inverse of eigenvalues of B − 1.5.

thus $\boxed{\text{Eigenvalues of B and C are } -5.5, -2.5, -1, 1.5, 0.5}$

(b) use reasoning (and show it); q is a very good approximation for an eigenvector of A. What is the corresponding eigenvalue?

Eigenvalue of C follow λ^{-1} (eigenvalues of A) -1.5

$\frac{1}{-4} - 1.5 = -0.25 - 1.5 = \boxed{-1.75 = \lambda_1 \text{ of } C}$

eigenvector corr. to λ_1

we get Cv_1 due to convergence of the sequence.

with $\boxed{\lambda_2 = -2.5 \quad \lambda_3 = 0.5 \quad \lambda_4 = -1 \quad \lambda_5 = 1.2}$

c) If we replaced line 4 of script w/ 0.25 instead of 1.5

λ of A = $-4, 3, 2, 0.5, -1$

λ of B = A − 0.25

λ of C = $A^{-1} - 0.25 = -0.5, -1.25, 1.75, 0.0833, 0.25$

The answer is +1.25 higher than in b).

d) By plugging norm((A-x-eye(5))*q) → 12.9642

with $x = -1.75$. I think that means the approx is not very good.