1 Empirical Orthogonal Functions

Empirical Orthogonal Function (EOF) analysis is used to decompose a two-dimensional dataset, typically with dimensions of space and time, into orthogonal basis functions or modes. Unlike a Fourier analysis and other decompositions in which the basis functions are specified, the EOF basis functions are determined directly from the data, or empirically. EOFs not only provide an orthogonal basis set, but the modes also are efficient in that the first mode explains the dominant covarying pattern, the second mode the next dominant pattern of the residual (mode 1 removed) that is orthogonal to the first mode, and so on. EOFs often can compress a large dataset into a small subset of modes that account for much of the overall variance.

Given a variable $h(x,t)$ that is a specified at N_x space points and N_t time points, the EOF analysis represents the data as a sum of the product of spatial and temporal functions,

$$h(x,t) = \sum_{k=1}^{N} a_k(t) e_k(x) \tag{1}$$

where $e_k(x)$ is the spatial basis function for mode k, and $a_k(t)$ is the temporal expansion function for mode k, and $N = min(N_x, N_t)$. An orthogonality condition is imposed such that the spatial modes are orthonormal over x,

$$\sum_x e_j(x) e_k(x) = \delta_{jk}, \tag{2}$$

and that the temporal expansion functions are uncorrelated over t,

$$< a_j a_k >_t = \delta_{jk} < a_k^2 >_t, \tag{3}$$

where $< .. >_t$ represents a time average. The e_k's are the eigenfunction solutions of

$$\mathbf{Ce} = \lambda \mathbf{e}, \tag{4}$$

where $\mathbf{C}$ is the *covariance matrix* of $h(x,t)$ with elements

$$C_{mn} = < h'(x_m, t) h'(x_n, t) >_t, \tag{5}$$

where the prime indicates departures from a mean, e.g., the temporal mean at each grid point or the spatial mean at each time. The e_k's are the column vectors of $\mathbf{e}$, and λ is a diagonal matrix of eigenvalues, which represent the variance accounted for by each mode

$$\lambda_k = < a_k^2 >_t \, . \tag{6}$$

The eigenfunctions are ordered such that $\lambda_1 > \lambda_2 > .. > \lambda_N$. The total variance of the dataset is represented by the sum of the eigenvalues

$$\sum_x < h(x,t)^2 >_t = \sum_{k=1}^{N} \lambda_k. \tag{7}$$

For each spatial mode we compute the corresponding temporal expansion function as,

$$a_k(t) = \sum_x h(x,t) e_k(x). \tag{8}$$

The above applies when $N_t > N_x$, in which case the spatial functions are orthonormal, and the temporal expansions have the same physical unit as $h(x,t)$. If $N_x > N_t$, the dimensions can be switched such that $h(x,t)$ is represented by a set of temporal basis functions with associated spatial expansion functions,

$$h(x,t) = \sum_{k=1}^{N} a_k(x) e_k(t). \tag{9}$$

The orthonormal temporal modes,

$$\sum_t e_j(t) e_k(t) = \delta_{jk}, \tag{10}$$

are the eigenfunctions of $\mathbf{C}$, computed using spatial averages,

$$C_{ij} =< h'(x,t_i) h'(x,t_j) >_x, \tag{11}$$

and the spatial expansion functions are obtained by projecting the temporal modes on to the data,

$$a_k(x) = \sum_t h(x,t) e_k(t). \tag{12}$$

Whether $N_t > N_x$ or $N_x > N_t$, the spatial and temporal functions described by the $N = min(N_t, N_x)$ modes are equivalent. It is more efficient computationally to form C so that it has the smaller dimension.

There are a number of variations to the standard EOF analysis described above. For example, the eigenfunctions can be obtained from the correlation matrix (normalizes all data to equal variance), the cross-spectral matrix (complex eigenfunctions with amplitude and phase information), the lagged-covariance matrix (not restricted to standing patterns), etc. Since the EOFs form a basis set, they can be rotated to emphasize signals in sub-domains of the data.

The EOFs also can be obtained directly from the dataset (i.e., you do not need to compute C) using a *Singular Value Decomposition* (SVD). Given an invertible, real m x n matrix $\mathbf{A}$ with $m > n$, then the singular value decomposition of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathbf{T}}, \tag{13}$$

where $\mathbf{U}$ is a m x m matrix, $\mathbf{D}$ is m x n, and $\mathbf{V}$ is n x n. $\mathbf{U}$ and $\mathbf{V}$ have orthogonal columns so that

$$\mathbf{U}^{\mathbf{T}}\mathbf{U} = \mathbf{I} \tag{14}$$

and

$$\mathbf{V}^{\mathbf{T}}\mathbf{V} = \mathbf{I}. \tag{15}$$

So if $\mathbf{A}$ is our original data array $h(x,t)$, then the columns of $\mathbf{V}$ are equivalent to the eigenfunctions obtained from the covariance matrix, $\mathbf{C}$, and the columns of $\mathbf{UD}$ are the expansion functions for each mode. $\mathbf{D}$ is a diagonal matrix whose elements are the *singular values* of $\mathbf{A}$, which are related to the eigenvalues of the covariance matrix of $\mathbf{A}$ by $d_k = \lambda_k^{1/2}$.