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Optimal pricing and composition of multiple bundles: A two-step approach

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ABSTRACT

This paper analyzes the problem facing a firm that must determine the optimal composition and pricing of multiple bundles it supplies to a single market segment. It is assumed the firm's competitors does not react in the short run to its decisions and that their bundles' prices and characteristics are known. It is further assumed that potential consumers of bundles are rational and maximize a random utility function. The problem is modeled as a mixed integer non-linear program. Though such programs are normally hard to solve using traditional methods, the difficulties are circumvented here by taking advantage of the problem's mathematical structure to develop a novel two-phase solution approach. Two aspects of the solution are particularly worthy of note: (1) we demonstrate that the optimal price of a bundle marketed by a firm depends on the composition of all the firm's bundles, and not on their prices, and on the composition and price of all the competitors' bundles in the market, and (2) that if the consumer choice behavior is described by a logit model, then the firm's bundles should be very similar to each other.

1. Introduction

In its simplest form, bundling consists in grouping goods and services into a package and selling it at a global price that is generally more attractive to the consumer than the prices the goods and services would sell at if they were sold separately (Guiltinan & Gordon, 1988). Dukart (2000) and Swartz (2000) have suggested that bundled services are of greater interest to business customers than private individuals because the former require more services and prefer paying them on a single bill.

The practice of bundling product and services is growing in importance in many industries and some service sector firms now base their business strategies on this tool (Simon and Wuebker in Fuerderer, Huchzermeier, & Schrage, 1999). Its significance compared to other strategies has been studied empirically by Schoenherr and Mabert (2010), while the impacts of different bundling strategies (single, pure and mixed) for information products have been compared by Li, Feng, Chen, and Kou (2013).

An example of the application of bundling is a cable television provider that offers a package of three movie channels (HBO, Cinemax and Cinecanal), two sports channels (ESPN and FoxSport) and three cultural channels (NatGeo, Discovery and History). Alternatively, it could offer four movie channels (HBO, Cinemax, Sony and Warner), no sports channels and two cultural channels (NatGeo and History). If a provider distributed 15 movie channels, 10 sports channels and 5 cultural channels, the number of different bundles it could design would be $2^{30} > 10^{10}$. Clearly, coming up with a design for a good group of, say, five bundles and deciding how to price them for a given market segment is not an easy matter.

Some other cases of product composition are holiday packages (return flight, hotel stay and car rental), restaurant menus (entrée, main dish and dessert) and telecommunications packages (local calling, long distance calling, Internet access and cellular phone).

Generally speaking, firms that design more than one bundle do so because they intend to supply them to different market segments. In this article, however, we present the case of a business that seeks to market multiple product bundles even though it will supply them to a single market segment. This is a significant problem that arises often when firms face restrictions on production levels, supply of inputs or storage space that prevent them from producing the quantity required for the optimal bundle. In such a situation, if all of the company's optimal bundles have very similar composition so that only a few of the bundle components differ from one bundle to the next, its total profit will be very similar to what it would obtain from its segment with a single (optimal) bundle.

The present paper attempts to determine the optimal composition and prices of a set of b bundles a firm intends to supply to its markets. As we will see in the next section, the proposed analysis is a natural extension of the methodology suggested by

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Bitran and Ferrer (2007) and contributes to the state of the art in that it develops a model and a solution approach for the multiple bundles case.

The remainder of this article is organized as follows. Section 2 reviews the state of the art in pricing and composition of bundles; Section 3 defines the problem to be addressed; Section 4 introduces a customer choice model based on utility maximization and builds a model to solve the optimization problem specified in Section 3; Section 5 solves the problem by breaking it down into two consecutively solved subproblems consisting of setting optimal prices of the multiple bundles and then determining their optimal composition, and also provides someone managerial insight; and finally, Section 6 presents our conclusions.

2. Literature survey

The pricing and composition of bundles have been widely investigated in the contexts of consumer behavior. Studies such as Yadav and Monroe (1993) and Yadav (1994) focus on the way consumers evaluate different product packages. Suri and Monroe (2000) examine the effects of contextual factors on consumer intentions to buy product packages. Their research shows that giving price discounts on individual products may significantly reduce the attractiveness to consumers of bundles containing the same goods. Herrmann, Huber, and Higie Coulter (1997) found that discounts and complementarity of package components are factors that usually increase consumers' purchase intentions.

Other works in the literature attempt to develop and empirically test a general choice model for bundles that takes into account the interdependencies among the items composing them Chung and Rao (2003). Various studies conclude that incorrect modeling of reservation prices may result in large losses for the seller and that offering mid-season packages may be more effective than individual product price reductions (Bulut, Gürler, & Sen, 2009; Gülder, Öztop, & Sen, 2009).

In the case of markets where customers have varying levels of knowledge of the bundle components, Basu and Vitharana (2009) demonstrate that those with more knowledge exhibit greater variability of reservation price. The authors use an analytic model to determine the conditions for obtaining the maximum benefits on each of three sale strategies, the first based on individual components (no bundling), the second using bundles only (pure bundling) and the third a mixed approach. Chakravarty, Mild, and Taudes (2013) compare the bundling set and bundling gain when the production and retail functions are integrated in a single firm with the bundling set and bundling gain of three supply chain scenarios with different levels of coordination. In the first-best scenario, bundle margins are determined so as to optimize the profit of the whole supply chain.

Hui, Yoo, Choudhary, and Tam (2012) extend the previous literature on bundling, which usually assumes consumer heterogeneity along a single consumer attribute, showing that an individual consumer's demand function can be expressed as the interaction of the demand function's intercept (indicating the consumer's initial willingness to pay, that is, to pay for the first unit of a product) and slope (representing the consumer's appetite, that is, the quantity consumed when the product is free). Using a combination of analytical and numerical methods, they demonstrate that appetite heterogeneity favors mixed bundling while initial willingness-to-pay heterogeneity may reduce its profitability relative to pure bundling. Banciu and Odegaard (2016) analyze the problem that arises when the valuation of a bundle's components are dependant on each other. By modeling the combined density of the reservation price, they are able to show under which circumstances it is more profitable to supply just the bundle or the entire line of products (individual products and bundles).

Hitt and Chen (2005) describe the concept of customized bundling in which customers can choose a bundle composed of M out of a total of N products for a fixed price. They show that this approach outperforms pure bundling and no bundling strategies when there are positive marginal costs and consumer valuations are heterogeneous. More recently, Wu, Hitt, Chen, and Anandalingam (2008) and Yang and Ng (2010) find that customized bundling increases benefits to the firm compared to pure and no bundling approaches when consumers do not value all of the component goods positively. On the other hand, Armstrong (2013) shows that when consumer valuations of products in a bundle are non-additive and when the products are supplied by separate sellers, a seller has a unilateral incentive to offer a discount for the purchase of the bundle if its customers buy products from the competition.

Yet other studies suggest an optimization approach that simultaneously decides bundle design and pricing. Hanson and Martin (1990) propose a mathematical programming formula for determining the bundle configuration and price that maximize benefits without explicitly considering the entire range of feasible solutions. In a later article, Venkatesh and Mahajan (1993) consider two dimensions of the consumer decision-making process (time and money) in determining the optimal price for a given bundle under different bundling strategies. A pair of publications by Green, Krieger, and Agarwal (1991), Green, Krieger, and Agarwal (1992) present an algorithm that solves a stochastic mixed integer programming model to determine a bundle's price and configuration. In more recent works, Proano, Jacobson, and Zhang (2012) build a mixed integer non-linear programming model to identify the number of vaccines in bundles of antigens and the range of feasible prices that maximize the sum of producer benefits and consumer surplus, and Ferrer, Mora, and Olivares (2010) address the problem facing a seller of bundles composed of a service and a related product offered for a monthly flat rate plus a subscription fee to a customer base that changes over time. In the latter article, pricing policy is determined via a dynamic programming approach that identifies the long-run optimal number of customers for each bundle. Meyer and Shankar (2016) and Odegaard and Wilson (2016), on the other hand, tackle the problem of determining the optimal price for a bundle composed of assets and services (hybrid bundle). They show how the optimal price changes with variations in the quality of the services or the products in the bundle.

Finally, the problem of how to determine the composition and price of a single bundle that maximize the total expected benefits to the firm in a competitive market is analyzed by Bitran and Ferrer (2007) and Perez, López-Ospina, Cataldo, and Ferrer (2016). Bitran and Ferrer (2007) study formulates a mixed integer nonlinear programming model and shows that an optimal policy for bundle composition and price determination can be identified. The optimal pricing problem is first solved with a closed-expression that depends on the composition. The bundle composition problem is then handled with an algorithm that builds the bundle component by component considering the contribution made by each possible component alternative to the objective function. Noteworthy in this approach is that the algorithm solves to optimality in the same number of iterations as there are components in the bundle. Perez et al. (2016) extend the results of Bitran and Ferrer (2007), proposing a constrained multinomial logit model that incorporates a budget constraint into the consumer decision-making process.

3. Definition of the problem

Consider a hypothetical firm faced with the problem of determining the composition and prices of a set of bundles to be supplied to a single segment of homogeneous customers. The company aims to define a set of bundles and find the optimal price for each one so as to maximize its benefits. The individual bundles each consist of a set of component products or services, and for each such set there is a group of known choice alternatives. Certain components must always be selected for inclusion in a given bundle while others may or may not be. The non-selection of a particular component in a given case is considered to be a valid option and can therefore be treated in a similar manner to the other alternatives.

The two main considerations that determine the composition of possible bundles are technological and competitive feasibility. Technological feasibility refers to certain technical specifications and requirements that bundles must satisfy such as containing certain components or containing them in certain quantities. Competitive feasibility denotes the requirement that the bundles offered by a firm be competitive with those supplied by its rivals. Though the firm's decision makers obviously do not control the composition or prices of competing bundles, they must take them into account.

A simpler version of this problem was posed and solved by Bitran and Ferrer (2007), who devised a method of finding the optimal composition and price of a single bundle. They showed that the optimal composition could be determined by building a space of all feasible bundles and then finding the optimal one within it. The model developed here in the next section extends their results.

4. Specification of the model

To specify our model we begin by assuming that the number of bundles to be supplied by our hypothetical firm to a single market segment is exogenously defined as b, where $b \geq 1$. As in Bitran and Ferrer (2007), a single consumer segment is considered. This means that no coupling constraint will be specified, thus dispensing with the need to determine the optimal bundle composition simultaneously over all market segments.

We also assume that a consumer's willingness to pay, known as the reservation price, varies from bundle to bundle given that the attractiveness of each bundle to the consumer is different. Following Bitran and Ferrer (2007) we call such valuations the *bundle attraction factor*. Note that it does not include price, which is treated as a separate attribute. This factor is measured by the weighted sum of the individual attraction factors of each bundle component. As with Green et al. (1992), we assume further that there are no interaction factors between the components.

The necessary notation for the model's sets, indexes, parameters and variables is set out below.

Sets:

- $\mathcal{N}$: the set of bundles offered on the market by the competition. By abuse of notation, we say that $Card(\mathcal{N}) = \mathcal{N}$.
- $\mathcal{M}$: set of components in a bundle. By abuse of notation, we say that $Card(\mathcal{M}) = \mathcal{M}$.
- $\mathcal{S}_j$: set of alternatives for component j of a bundle.

Indexes:

- k, l, t: indexes for the bundles offered by the firm.
- n : index for the bundles offered by the competition.
- j : index for the set of components of a bundle.
- u : index for the possible choice alternatives of a component.
- i : index for all bundles offered on the market.

Parameters:

- b : number of bundles to be constructed.
- $\hat{p}_n$: price at which bundle n is offered by the competition.
- c_{uj} : cost of alternative u of the set of components j.
- I_{uj} : attractiveness of alternative u of the set of components j.
- a_j : attractiveness weight of component j in the bundle composition.
- γ : utility of the bundles offered by the competition and of non-purchase.
- g : number of bundles offered by the competition.
- β : sensitivity of utility to bundle price.

Variables:

- x_{ujk} : binary variable indicating whether or not alternative u of the set of components j is chosen for the composition of bundle k.
- X_k : binary matrix representing the composition of bundle k.
- p_k : price of bundle k.
- q_k : probability that a consumer chooses bundle k.
- c_{X_k} : binary matrix representing the cost to the firm of bundle k.
- I_{X_k} : binary matrix representing the attractiveness of bundle k.

Decisions that are under the firm's control are the composition of the b bundles it offers to its customer segment and the prices it sets for them so as to maximize profits (recall that b is an integer greater than or equal to 1). Thus, these decisions are described by the attractiveness of the bundle $I_{X_k} = \sum_{j\in\mathcal{M}} a_j \sum_{u\in\mathcal{S}_j} I_{uj}x_{ujk}$, parameter $a_j > 0$ for all $j \in \mathcal{M}$, the bundle's cost to the firm $c_{X_k} = \sum_{j\in\mathcal{M}} \sum_{u\in\mathcal{S}_j} c_{uj}x_{ujk}$, parameter $c_{uj} \geq 0$ for all $u \in \mathcal{S}_j$, $j \in \mathcal{M}$ and the bundle's price p_k for $k = 1, \ldots, b$. Bundles not under the control of the firm are considered as given information and are characterized by attractiveness I_{X_n} where $n \in \mathcal{N}$ and price $\hat{p}_n$ where $n \in \mathcal{N}$. Since we assume that competitors do not react in the short run to the firm's decisions, the model is static rather than dynamic.

Consumers may choose any one of the bundles offered on the market, or decline to choose any. They are considered to be rational and random utility maximizers, the latter an increasingly common assumption in consumer choice models. This implies, first of all, that consumers will prefer the option giving them the greatest perceived utility. According to McFadden (1974) they choose over a set of attributes, making an overall evaluation of every possible alternative on the basis of a random utility function and then picking the one that confers the highest value.

The random utility maximizer assumption also implies that utility is divided into a deterministic component and a stochastic one. Early work on this component-based approach in bundle choice models was developed by Hanson and Martin (1990) and then Venkatesh and Mahajan (1993). In our formulation, the perceived utility conferred by bundle i on a given consumer in the single consumer segment is specified as the sum of a deterministic component denoted $V(\cdot)$ that depends on the price and composition of the bundle, and a stochastic component expressed by an independent disturbance term ε_i that includes all factors preventing the consumer from determining a good's exact utility. The random utility model is $U_i = V(p_i, X_i) + \varepsilon_i$ for all $i = 1, \ldots, \mathcal{N} + b + 1$, where the deterministic utility is $V(p_i, X_i) = I_{X_i} + \beta p_i$ and parameter $\beta < 0$ is a scalar that expresses the sensitivity of utility to price.

Ben-Akiva and Lerman (1985) and McFadden (1974) show that the probability of choosing a bundle from among a set of alternatives is given by the closed-form expression $q(p_i, X_i) = \frac{e^{V(p_i,X_i)}}{\xi + e^{V(p_i,X_i)}}$, where ξ is the utility of all the bundles offered to the consumer segment plus the non-purchase utility. Since the firm supplies b bundles to the market, the probability that the firm's bundle i is chosen is given by

$$q_i(p_1, \ldots, p_b, X_1, \ldots, X_b) = \frac{e^{V(p_i,X_i)}}{\gamma + \sum_{k=1}^{b} e^{V(p_k,X_k)}} \tag{1}$$

The probability of choosing a bundle is thus represented by a multinomial logit (MNL) model (Ben-Akiva & Lerman, 1985; Zhang, 2015). If we let V_0 be the deterministic non-purchase utility, then $\gamma = e^{V_0} + \sum_{n\in\mathcal{N}} e^{V(\hat{p}_n,X_n)}$. Among the properties of (1), as explained in Bitran and Ferrer (2007), is that $lim_{p_i \to \infty} q_i(p_1, \ldots, p_b, X_1, \ldots, X_b) = 0$, $0 \le q_i(p_1, \ldots, p_b, X_1, \ldots, X_b) \le 1$ and $0 \le \sum_{i=1}^{b} q_i(p_1, \ldots, p_b, X_1, \ldots, X_b) \le 1$.

Given that we are using a logit model to simultaneously determine the composition of multiple bundles which will all be different from, and perfect substitutes for, each other, the solution must also satisfy the independence of irrelevant alternatives (IIA) property. This condition is handled by Eq. (8), which makes pairwise comparisons of all the chosen bundles to check that there are no more than $\mathcal{M} - 1$ identical components, thus ensuring each of the b bundles differs from every other one in at least one component.

With the foregoing as our basis, we now set out our proposed mixed integer non-linear programming model for solving the *pricing and composition of multiple bundles problem* (MBP) model in order to determining the composition and pricing of the b bundles that will be supplied to a single market segment.

$$\text{(MBP)} \quad \max_{p_1,\ldots,p_b,X_1,\ldots,X_b} \Pi(p_1, \ldots, p_b, X_1, \ldots, X_b) = \sum_{k=1}^{b} q_k(p_1, \ldots, p_b, X_1, \ldots, X_b)(p_k - c_{X_k}) \tag{2}$$

$$\text{s.t: } q_k(p_1, \ldots, p_b, X_1, \ldots, X_b) = \frac{e^{I_{X_k} + \beta p_k}}{\gamma + \sum_{l=1}^{b} e^{I_{X_l} + \beta p_l}} \quad \forall k = 1, \ldots, b. \tag{3}$$

$$\gamma = e^{V_0} + \sum_{n \in \mathcal{N}} e^{I_{X_n} + \beta \hat{p}_n} \tag{4}$$

$$c_{X_k} = \sum_{j \in \mathcal{M}} \sum_{u \in \mathcal{S}_j} c_{uj} x_{ujk} \quad \forall k = 1, \ldots, b. \tag{5}$$

$$I_{X_k} = \sum_{j \in \mathcal{M}} a_j \sum_{u \in \mathcal{S}_j} I_{uj} x_{ujk} \quad \forall k = 1, \ldots, b. \tag{6}$$

$$\sum_{u \in \mathcal{S}_j} x_{ujk} = 1 \quad \forall j \in \mathcal{M}; \ \forall k = 1, \ldots, b. \tag{7}$$

$$\sum_{j \in \mathcal{M}} \sum_{u \in \mathcal{S}_j} x_{ujk} x_{ujl} \le \mathcal{M} - 1 \quad \forall k = 1, \ldots, b; \ \forall l = k+1, \ldots, b. \tag{8}$$

$$x_{ujk} \in \{0, 1\} \quad \forall j \in \mathcal{M}; \ \forall u \in \mathcal{S}_j; \ \forall k = 1, \ldots, b. \tag{9}$$

$$p_k \ge 0 \quad \forall k = 1, \ldots, b. \tag{10}$$

The various constraints in the model may be described briefly as follows. Constraint (3) determines the probability an individual chooses one of the firm's bundles given its composition and price. Constraint (4) determines the consumer utility of the bundles offered by the competition and of non-purchase. Constraints (5) and (6) determine the cost and attractiveness of each bundle designed. Constraint (7) imposes that in the design of each bundle a single alternative is chosen for each component. Constraint (8), as already noted above, ensures that no two bundles have identical compositions. Finally, constraints (9) and (10) define the nature of the variables.

As can be appreciated, the model as a whole is a non-linear programming formulation with continuous and binary variables. Dobson and Kalish (1993) have shown that for both product selection and pricing problems, such models are usually NP-hard given that the product selection problem is a set covering problem. Nevertheless, Bitran and Ferrer (2007) were able to solve the problem for the case where $b = 1$. In the next section, we develop an approach that solves the MBP to optimality in a reasonable amount of time.

5. Solving the multiple bundles problem

The structure of the MBP is designed so that the solution process can be divided into two phases, each one solving a single subproblem. In the first phase, the optimal price (p_k) is determined for each of the b bundles on the assumption that their respective optimal compositions (X_k) are already known. Then, in the second phase, the optimal prices obtained in the first phase are used to generate the optimal compositions that maximize the benefits to the firm.

5.1. Phase 1: multiple bundle optimal price subproblem

Given our assumption for this phase that the optimal composition of each bundle is already known, the decisions and constraints relating to composition are temporarily set aside. Substituting (3) above into the objective function (2), the unconstrained price optimization subproblem becomes

$$\Pi = \max_{p_1,\ldots,p_b \ge 0} \sum_{k=1}^{b} \frac{e^{I_{X_k} + \beta p_k}}{\gamma + \sum_{l=1}^{b} e^{I_{X_l} + \beta p_l}} (p_k - c_{X_k}). \tag{11}$$

This leads to the following proposition:

Proposition 1. *The optimal price for bundle k is given by the closed form expression*

$$p_k^* = c_{X_k} - \frac{1}{\beta}\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l} + \beta c_{X_l} - 1}\right)\right) \quad \forall k = 1, \ldots, b, \tag{12}$$

where $W(\cdot)$ is the Lambert W-function. Thus, the optimal price of bundle k depends on the composition of all b bundles. This price will be greater than or equal to (c_{X_k}), the cost to the firm of bundle k, whenever $\beta < 0$ and $W(\cdot) \ge 0$, the latter being the case if the W-function's argument is positive. In (12) all of these conditions are satisfied. The foregoing also implies that prices are always non-negative.

Proof. See A.1. □

Substituting the optimal price into the benefit function, we obtain the following corollaries:

Corollary 1. *The optimal expected benefit Π^* is*

$$\Pi^* = \frac{-1}{\beta} W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l} + \beta c_{X_l} - 1}\right). \tag{13}$$

Proof. See A.2. □

Corollary 2. *The probability of choosing bundle k when optimal price is p_k^* is*

$$q_k^* = \frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l} + \beta c_{X_l} - 1}\right)}{\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l} + \beta c_{X_l} - 1}\right)\right)} \left(\frac{e^{I_{X_k} + \beta c_{X_k} - 1}}{\sum_{l=1}^{b} e^{I_{X_l} + \beta c_{X_l} - 1}}\right), \tag{14}$$

where the first factor is the probability a consumer chooses one of the b bundles offered by the firm and the second factor is the probability the bundle chosen is k.

Proof. See A.3. □

When $b = 1$, the expressions for price, probability of purchase and optimal benefit reduce to those given in Bitran and Ferrer (2007).

5.2. Phase 2: multiple bundle optimal composition subproblem

Having just obtained a closed-form expression for the optimal prices of the multiple bundles we now address the second subproblem, which is the determination of the bundles' optimal composition and thus the maximization of total benefits.

Consider the definition of *Pareto bundles* posited by Bitran and Ferrer (2007). For any two bundles k and l, k is said to dominate l if $I_{X_k} \geq I_{X_l}$ and $c_{X_k} \leq c_{X_l}$. This implies that instead of having to check the entire feasible bundle space Ω, we can confine our search for the solution to the Pareto-efficient bundle frontier Ω^*, which is constructed with the non-dominated bundles.

Thus, to determine the composition of one of the b bundles we could explicitly construct Ω^* and then search for the best bundle. However, Garey and Johnson (1979) have shown that finding any point on a Pareto-efficient frontier is itself an NP-complete problem, and similar conclusions were drawn by Warburton (1987) and Beasley and Christofides (1989). It would appear, then, that the explicit construction of such a frontier is not in fact the best approach. In light of this, Bitran and Ferrer (2007) propose, for the single bundle case, an $O(\mathcal{HM})$-order pseudo-polynomial time algorithm, where $\mathcal{H}$ is the cardinality of the largest $\mathcal{S}_j$ set. This algorithm, which we will call BF, does not require explicit construction of the Pareto-efficient frontier.

But whereas the Pareto-efficient frontier for the single bundle problem is built of points corresponding to individual bundles, for the multiple bundle problem the points are groups of feasible bundles. It is possible, however, to identify the contribution of each of these bundles to the total objective function. The following proposition states a key result regarding this contribution:

Proposition 2. *The partial derivatives of the objective function Π^* with respect to I_{X_k} and c_{X_k} are proportional to q_k^* for any arbitrary points I_{X_k} and c_{X_k}. More precisely,*

$$\nabla \Pi^* = \left(\frac{\partial \Pi^*}{\partial I_{X_k}}, \frac{\partial \Pi^*}{\partial c_{X_k}} \right) = \left(\frac{-1}{\beta}, -1 \right) q_k^*, \tag{15}$$

where q_k^* is the probability that bundle k is chosen when its price is p_k^*.

Proof. See A.4. □

The results in Proposition 2 are of fundamental importance, for they show that an increment in the objective function with respect to I_{X_k} is proportional to $-1/\beta$ while an increment with respect to c_{X_k} is proportional to -1. This implies that the point on the Pareto-efficient frontier for the multiple bundles problem that maximizes utility is the one whose bundles confer the largest utility increment. The determination of the optimal composition of multiple bundles therefore reduces to identifying the composition of those that maximize the contribution to the objective function. This conclusion is expressed as follows:

$$\max_{X_1, X_2, \ldots, X_b \in \Omega} \sum_{k=1}^{b} \frac{-I_{X_k}}{\beta} - c_{X_k}. \tag{16}$$

One of the bases of our solution approach is the separation of (16) into b stages, in each of which the optimal composition of only one bundle is determined. This means that in each stage k we must also consider the optimal composition of the bundles constructed in the previous stages ($l = 1, \ldots, k$). For this purpose we will need the following definition:

Definition 1 (Inner adjacent frontier)**.** Let Ω_1 be the set of all feasible bundles and Ω_1^* its Pareto-efficient frontier. Select a bundle $\bar{b}$ belonging to Ω_1^* and eliminate it from the set Ω_1. The result is a new set Ω_2 of all the feasible bundles, whose Pareto-efficient frontier is denoted Ω_2^*. This new construction is called the inner adjacent frontier of Ω_1^* under $\bar{b}$.

The significance of Definition 1 is the linkage it specifies between successive pairs of optimal bundle composition problems. Thus, if we want to obtain the optimal composition of any two bundles in a set of feasible bundles, we look for the first composition (X_1^*) on the Pareto-efficient frontier of the original problem and the second composition on that frontier's inner adjacent frontier under X_1^*.

With this definition we now describe the separation into stages, in each of which a single optimal bundle is determined. In stage 1 we obtain the optimal composition of a single bundle in the space Ω_1 containing all feasible bundles and therefore also on the Pareto-efficient frontier Ω_1^*. Let us call this problem $P(1)$. The solution of $P(1)$ is denoted X_1^* and indicates the composition of the first of the b optimal bundles. The information on this optimal composition X_1^* is passed on from stage 1 to stage 2 so that the second bundle is not given the same composition. This is ensured simply by eliminating X_1^* from the feasible bundle space Ω_1, thus obtaining a new Pareto-efficient frontier Ω_2^* which by construction is the inner adjacent frontier of Ω_1^* under X_1^*.

Now let us call $P(2)$ the problem of determining the optimal composition of a single bundle located in the feasible bundle space Ω_2, and therefore also on Ω_2^*. The solution to $P(2)$ will give the optimal composition X_2^*, the composition of the second bundle. This second bundle is then eliminated from the feasible bundle space Ω_2, resulting in the construction of a new Pareto-efficient frontier that will be the inner adjacent frontier of Ω_2^* under X_2^*. The optimal composition and the new Pareto-efficient frontier are then passed on to the next stage and the third bundle is determined in the same fashion as the previous two. Iterating this process b times will identify the optimal composition of all b bundles.

The solution approach just outlined can be considered an application of dynamic programming given that it divides the original problem into b sequential subproblems and uses the solution of each of them to obtain the solution of the next one. We can therefore reformulate (16) as the following dynamic programming problem:

$$\underbrace{\max_{X_b \in \Omega_b^*} \frac{-I_{X_b}}{\beta} - c_{X_b}}_{\text{stage } b} + \Bigg\{ \underbrace{\max_{X_{b-1} \in \Omega_{b-1}^*} \frac{-I_{X_{b-1}}}{\beta} - c_{X_{b-1}}}_{\text{stage } b-1} + \cdots + \Bigg\{ \underbrace{\max_{X_1 \in \Omega_1^*} \frac{-I_{X_1}}{\beta} - c_{X_1}}_{\text{stage } 1} \Bigg\} \cdots \Bigg\}, \tag{17}$$

where Ω_{k+1}^* is the inner adjacent frontier of Ω_k^* under X_k^* for $k = 1, \ldots, b-1$. This in turn can be rewritten as a formulation of the Bellman equation for each stage k:

$$F_k^*(\Omega_k^*) = \max_{X_k \in \Omega_k^*} \left\{ \frac{-I_{X_k}}{\beta} - c_{X_k} + F_{k-1}^*(\Omega_{k-1}^*) \right\}, \tag{18}$$

where $F_0^*(\cdot) = 0$.

Bellman's optimality principle (Bellman, 1957) ensures that the optimal compositions $X_1^*, X_2^*, \ldots, X_b^*$, each one obtained as the solution of its respective stage-level problem, constitute the optimal solution of the complete subproblem. To solve (18), the Pareto-efficient frontiers must be constructed for each of the stages $k = 1, \ldots, b$. But as has already been observed, their construction is highly complex. However, they can in fact be identified by making use of the inner adjacent dependency existing between the frontiers of two successive stages (k and $k+1$).

To understand this dependency notion, assume, recalling our example in Section 1, that a cable TV company wants to build two bundles, each containing a movie channel (component A), a sports channel (component B) and a cultural channel (component C). The

Table 1
Data for cable TV example.

Component	Alternative	Attractiveness	Cost	Index BF
A: movie channels	1 (HBO)	2	110	175.71
	2 (Cinemax)	5	240	474.29
	3 (Cinacanal)	7	900	100.00
B: sport channels	1 (ESPN)	5	400	314.29
	2 (FoxSport)	7	800	200.00
C: cultural channels	1 (NatGeo)	4	230	341.43
	2 (Discovery)	5	500	214.29
	3 (History)	6	560	297.14

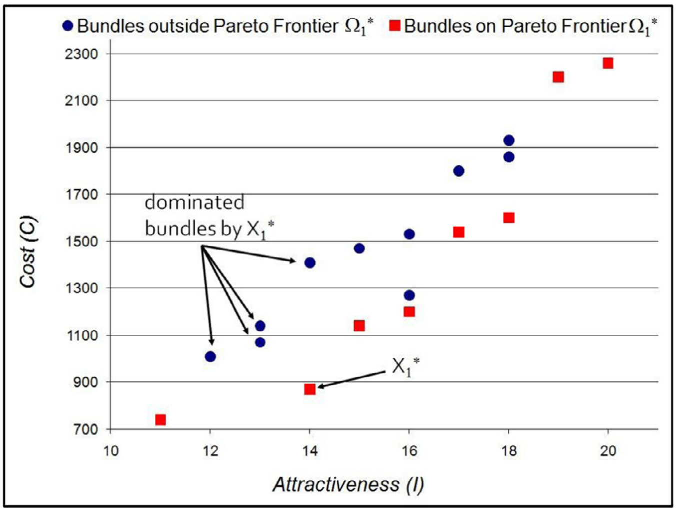

Fig. 1. Efficient Pareto frontier Ω_1^*.

choice alternatives for each component and their attractiveness and cost levels are as given in Table 1, with sensitivity of utility to price β equal to -0.007. The Pareto-efficient frontier for stage 1 and the bundles not on it (together constituting the set of all feasible bundles) are shown in Fig. 1. The optimal composition of the first bundle, located on Ω_1^*, is X_1^*. The Pareto-efficient frontier for stage 2 includes two bundles that were dominated by X_1^* in stage 1 but are now on the frontier because X_1^* no longer belongs to the feasible bundle space and no other bundle still in the space dominates them. The other two of the four bundles that were dominated by X_1^* in stage 1 continue to be dominated in stage 2 even though X_1^* has been eliminated, as is shown in Fig. 2.

With this example in mind, we present the next proposition.

Proposition 3. *Let Ω be the feasible bundle space. Also, let Ω_1^* be a Pareto-efficient frontier containing d bundles whose optimal bundle has the composition X_1^*. Finally, let Ω_2^* be the inner adjacent frontier of Ω_1^* under bundle X_1^*. Then Ω_2^* will contain the $d-1$ bundles other than X_1^* that were on Ω_1^* as well as some of the bundles in Ω that were not on Ω_1^* because they were dominated exclusively by X_1^*.*

Proof. See A.5. □

Proposition 3 leads us to conclude that constructing inner adjacent frontiers is also a complex task. Even though $d-1$ bundles on the Pareto-efficient frontier are known, it is not clear which bundles or how many of them will appear on it once the bundle that dominated them has been eliminated from the feasible bundle space.

In view of the above, we set out in what follows an alternative procedure for solving (17) without having to build either the Pareto-efficient or the inner adjacent frontier at each stage. We start with two useful definitions:

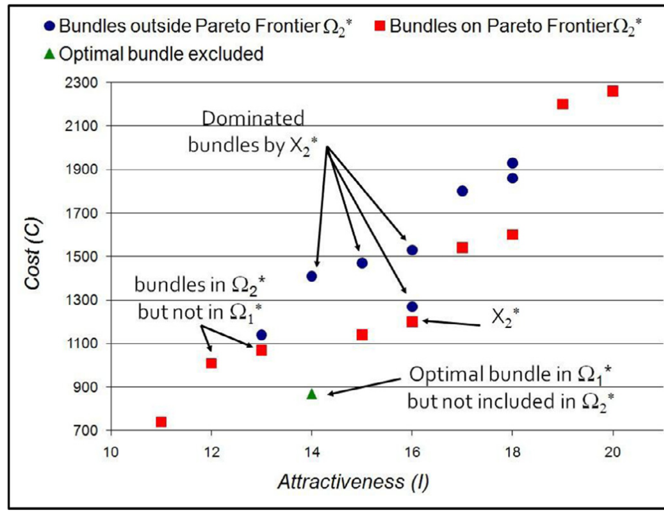

Fig. 2. Efficient Pareto frontier Ω_2^*.

Definition 2 (Ranked list of a component)**.** If we calculate the contribution of each component alternative to the gradient of the objective function using the expression $f(I,c) = \frac{-I}{\beta} - c$, a descending ordered list of the alternatives can be constructed as a function of the value of $f(I, c)$. This list is called the ranked list of a component.

Definition 3 (Adjacent bundle)**.** Let k and l be two bundles. Bundle l is said to be adjacent to bundle k if they are the same for all components except one and their respective chosen alternatives for that exception are in immediately adjacent places on the ranked list. In the special case where bundle k's chosen alternative is the last one on the list, bundle l's chosen alternative must be the first one on the list (boundary condition to properly define the algorithm and its correctness).

It follows from Definition 3 that a bundle will have as many adjacent bundles as it has components. This leads to the definition of what we call a candidate bundle set:

Definition 4 (Candidate bundle set)**.** Let $X_1^*, X_2^*, \ldots, X_k^*$ be the optimal composition of the bundles obtained in stages 1 to k, respectively. Also, let $L(X_1^*)$ be the set of bundles adjacent to X_1^*, $L(X_2^*)$ the set of bundles adjacent to X_2^*, and so on up to $L(X_k^*)$, the set of bundles adjacent to X_k^*. Now construct a new set $T(X_{k+1})$ as the union of the k sets of adjacent bundles. All of its bundles should have an index value $f(I,c) = \frac{-I}{\beta} - c$ that is less than or equal to the optimal composition X_k^* obtained in the previous stage. Thus, $T(X_{k+1})$ is the set of candidate bundles for solving stage k+1.

With Definitions 2–4 we can now develop the following proposition:

Proposition 4. *Let Ω_k be the feasible bundle space in stage k. Also, let Ω_k^* be the Pareto-efficient frontier of Ω_k and X_k^* the optimal composition of bundle k. Then the optimal composition of bundle k+1 bundle with $k = 1, \ldots, b-1$, denoted X_{k+1}^*, is the bundle in $T(X_{k+1})$ that has the highest index as given by $f(I,c) = \frac{-I}{\beta} - c$.*

Proof. See A.6. □

The importance of Proposition 4 lies in the fact that when we want to solve stage k+1 of (17), we need only identify the optimal composition of bundle k+1 in the set $T(X_{k+1})$; there is no need to determine the Pareto-efficient frontier Ω_{k+1}^*. Note that since the number of adjacent bundles for X_k^* is $\mathcal{M}$, the number of bundles in $T(X_{k+1})$ is bounded above by $k \cdot \mathcal{M}$.

Table 2
All feasible bundles for cable TV example.

No.	A	B	C	Attractiveness	Cost	No.	A	B	C	Attractiveness	Cost
1	1	1	1	11	740	10	2	2	1	16	1270
2	1	1	2	12	1010	11	2	2	2	17	1540
3	1	1	3	13	1070	12	2	2	3	18	1600
4	1	2	1	13	1140	13	3	1	1	16	1530
5	1	2	2	14	1410	14	3	1	2	17	1800
6	1	2	3	15	1470	15	3	1	3	18	1860
7	2	1	1	14	870	16	3	2	1	18	1930
8	2	1	2	15	1140	17	3	2	2	19	2200
9	2	1	3	16	1200	18	3	2	3	20	2260

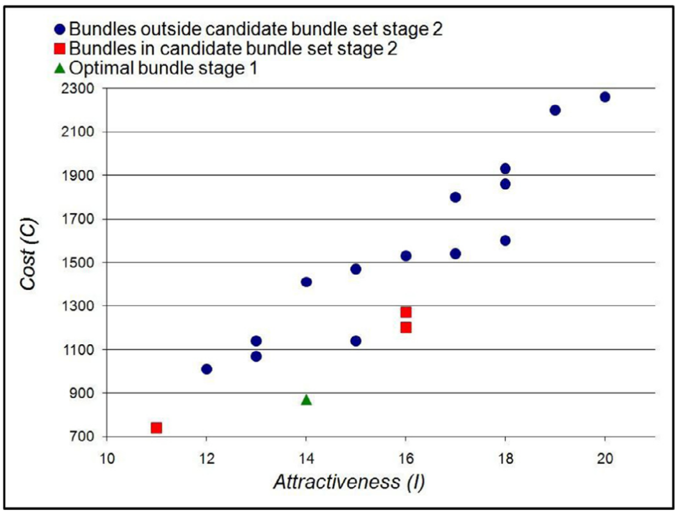

Fig. 3. Results for stage 1 and sets for stage 2.

Thus, Proposition 4 ensures that once we know the composition of the first bundle, we can obtain the optimal composition of the remaining $b-1$ bundles iteratively using the recurrence function (18). We therefore introduce the following proposition:

Proposition 5. *The optimal composition of the stage 1 bundle is found by applying the BF algorithm to the optimal composition problem for a single bundle in a single market segment.*

Proof. See A.7. □

We are now able to set out our proposed solution approach, for which we will again use our cable TV case as an example (see Table 1) and add the following data: first, the utility of the bundles offered by the competition is $\gamma = 12{,}000$, and second, the number of cable TV bundles is $b = 3$.

To find the optimal composition of these 3 bundles, the first step is to build the ranked list for all of the components. The first bundle's composition is obtained by solving the stage 1 problem of (17) to get X_1^*. This solution is generated using the *BF* algorithm. In our example, X_1^* is the composition of bundle 7 (see Table 2), which consists of the alternatives 2 (Cinecanal), 1 (ESPN) and 1 (NatGeo) for components *A*, *B* and *C*, respectively. To determine the optimal composition of the second bundle X_2^*, we obtain the list of bundles adjacent to bundle 7 and build the candidate bundle set $T(X_2)$. According to Table 2, these bundles could be 1, 8, 9, 10 and 13, but by using Definition 3 we find that they in fact are 1, 9 and 10, and since we are in the first stage, these three are the only ones containing $T(X_2)$ (see Fig. 3). By Proposition 4, the optimal composition of the stage 2 bundle is the bundle in $T(X_2)$ with the highest index value for $f(I,c) = \frac{-I}{\beta} - c$. The optimal composition X_2^* in our example is alternatives 2 (Cinecanal), 1 (ESPN) and 3 (History) for components *A*, *B* and *C*, respectively, which is bundle 9. Finally, to determine the optimal composition of the third bundle X_3^* we build the candidate bundle set for stage 3, obtaining bundle 10 which consists of alternatives 2 (Cinecanal), 2 (FoxSport) and 1 (NatGeo) for components *A*, *B* and *C*, respectively.

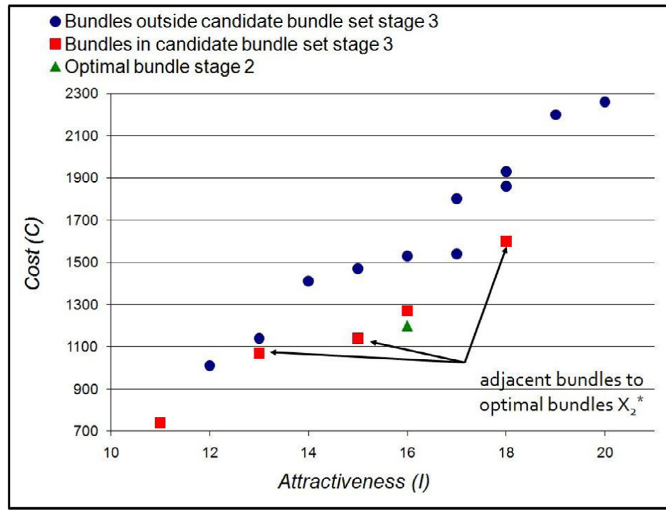

Fig. 4. Results for stage 2 and sets for stage 3.

The change in the candidate bundle set between stage 2 and stage 3 is shown in Fig. 4, which also indicates the optimal bundle for the latter stage. Note that the set $T(X_3)$ contains the $T(X_2)$ and $L(X_2^*)$ bundles.

The above-described procedure is captured in the following solution algorithm for the multiple bundle composition problem.

Multiple bundle composition algorithm

Begin
  Do $Rank[j,u] := 0$ for all $j \in \mathcal{M}$ and $u \in \mathcal{S}_j$; (Ranked list building)
  $k := 1$;
  While $k \le b$ do
    If $k = 1$ then
      Solve stage 1 with algorithm $BF \mapsto X_1^*$
    End If
    If $k > 1$ then
      Determine $L(X_{k-1}^*)$
      Build $T(X_k)$
      Solve stage k:

$$X_k^* := \arg \max_{X_k \in T(X_k)} f(I_{X_k}, c_{X_k}) = \frac{-I_{X_k}}{\beta} - c_{X_k}$$

    End If
    $k = k + 1$
  While End
End

The iterated application of this algorithm ensures that the b bundles will be obtained in exactly b stages. This procedure has the notable advantage of obviating the need to explicitly construct Pareto-efficient frontiers in each stage. Thus, on the basis of Eqs. (12), (13) and (14) associated respectively with Proposition 1, Corollary 1 and Corollary 2, all developed in Phase 1, the algorithm built in Phase 2 ensures the optimal composition and price will be obtained for the three bundles supplied by a firm to a single market segment. The results generated by the algorithm for the cable TV example are displayed in Table 3. As can be seen, the expected utility was 22.27 and the market share attained was 13.09%.

The proposed algorithm will function correctly as long as all of the component sets $\mathcal{S}_j$ are well-defined in the sense that all of the components have non-negative cost and attractiveness factor

Table 3
Cable TV example results ($b = 3$).

No.	A	B	C	Attractiveness	Cost	Price	q_i [%]	Π_i
7	2	1	1	14	870	1035.1	6.18	10.20
9	2	1	3	16	1200	1365.1	4.53	7.48
10	2	2	1	16	1270	1435.1	2.78	4.59

values. The computational complexity of the algorithm is of order $O(\mathcal{HM} + (b-1)\mathcal{M})$, where b is the number of bundles to be composed, $\mathcal{H}$ is the cardinality of the largest set $\mathcal{S}_j$ and $\mathcal{M}$ is the number of components in a bundle.

5.3. Evaluation of algorithm performance: comparison with an optimization software

To evaluate the proposed algorithm we compared its performance to that of an optimization software. For this purpose we constructed six test cases, each with three components, based on the cable TV data in Table 1. The only change was the number of bundles to be designed. Two more complex cases having many more possible solutions were also tested. For these two, the values for cost and attractiveness were chosen randomly, the former ranging between 1 and 1000 and the latter between 10 and 100. The first one had four components with 5, 4, 3, and 5 choices, respectively while the second one had five components with 4, 4, 5, 6, and 4 choices, respectively. For all eight test cases, $\gamma = 12.000$ and $\beta = -0{,}007$.

The eight cases were then solved by both the algorithm and the GAMS IDE version 23.8.1 optimization software with the Bonmin solver, which is designed for integer non-linear programming problems. The results are given in Table 4. The software solutions are the best ones found within 90 minutes of running time.

As can be seen, the more possible solutions there were, the greater was the algorithm's advantage in processing time over the optimization software. In the most complex case, set out in the second-to-bottom row of the table, the algorithm took only 42 seconds to find the solution whereas the software needed 5400 seconds. Also, the utility of the algorithm solution was always greater than that of the software. Specifically, in the most complex case 3% higher (46,055.43 versus 44,679.20).

5.4. Relationship between optimal solution for b y $b+1$ bundles

Based on the analysis of Phase 2, we are able to establish the following proposition:

Proposition 6. *In the absence of administration costs, the optimal composition for $b+1$ bundles contains the optimal solution for b bundles.*

Proof. The demonstration is straightforward, following directly from Eqs. (17) and (18). Clearly, when solving the problem of $b+1$ bundles the same b first stages are consecutively solved as for the problem of b bundles (Eq. (17)). Only the solution of stage $b+1$ will be different, and it is found by solving over the set Ω^*_{b+1} as shown in (18). Consequently, the bundles that comprise the solution of the b bundles case must necessarily include all those in the solution of any $\tilde{b}$ bundles case where $1 \le \tilde{b} < b$. However, enlarging the problem from b bundles to $b+1$ may change the b bundles' optimal prices given that the optimal price of each bundle is dependent on the composition of all the others (Proposition 1). □

5.5. Optimal number of bundles

From our analysis so far we are now in a position to answer the question whether there exists an optimal number of bundles to be marketed. Assuming there are no administration costs, we propose the following:

Proposition 7. *The expected utility is increasing with respect to the number of bundles to be marketed when there are no administration costs depending on the number of bundles to be designed.*

Proof. See A.8. □

The variation in utility as the number of bundles to be supplied increases from 1 to all 18 that can possibly be composed in our cable TV example (see Table 1) is shown as a continuous line in Fig. 5. The dashed line in the figure is the trend in the marginal utility of adding a new bundle. The two curves are consistent with Proposition 6 according to which utility (Π) grows as the number of bundles (b) to be composed increases but at a decreasing marginal rate.

If we include bundle administration costs, the optimal number of bundles to be put on the market will be determined by the interaction of the marginal cost function with the marginal utility function. For example, if the cost function is linear – say, 5 per bundle – the optimal number will be 2 given that the marginal utility of marketing the second bundle is 5.21 whereas that of the third is only 4.06.

6. Conclusions and final comments

This paper addresses the problem of simultaneously determining the optimal composition and pricing of a set b of bundles marketed by a firm whose objective is to ensure its composition and pricing decisions maximize total benefits. The decision to offer more than one bundle to a single market segment is motivated by the existence of production, inputs supply or storage limits. The firm's benefits are directly related to consumer preferences over both its own products and those of its competitors. These preferences are assumed to maximize consumer utility as defined by a random utility model. It is also assumed that demand can be described in terms of the price and attributes of all of the firm's bundles. The random utility model is a multinomial logit formulation.

To solve the problem an integer non-linear optimization model is proposed. Though such models are normally difficult to solve, in

Table 4
Results of proposed algorithm and optimization software for 8 test cases.

Components	Bundles to compose	Possible solutions	Algorithm utility	Algorithm time (second)	Software utility	Software time (second)
3	2	153	18.21	0.00	18.21	0.51
3	3	816	22.27	0.00	22.27	1.53
3	4	3060	25.82	0.00	25.82	7.03
3	5	8568	28.56	0.00	25.56	22.21
3	6	18,564	30.04	0.03	30.04	2,042.20
3	7	31,824	31.03	0.04	31.03	1,913.02
4	3	4,455,100	46,055.43	42.90	44,679.20	5400
5	2	1,842,240	58,693.40	12.90	58,656.41	5400

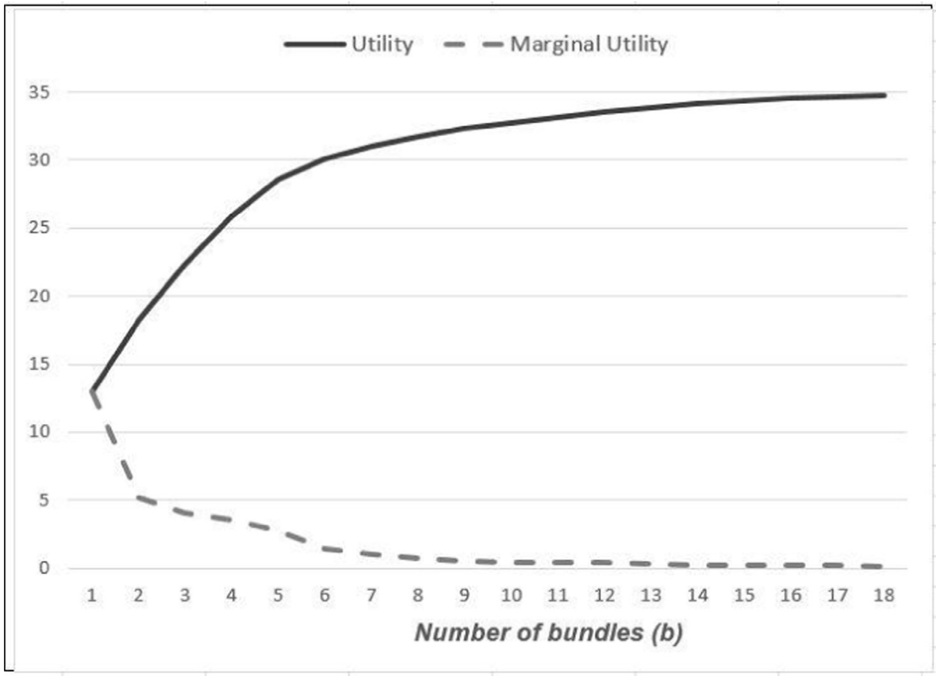

Fig. 5. Variation in utility as the number of bundles to be marketed varies.

the case studied the problem's mathematical structure is such that it can be addressed in two phases. In the first phase, it is assumed the optimal composition of the b bundles offered by the firm is known, and a closed expression is obtained that determines the optimal price for each bundle. In the second phase, the optimal composition of all b bundles is derived by substituting the closed price expression into the original problem formulation, thus obtaining a new optimization model that is pre-optimized for price.

This new model is rewritten as a dynamic programming problem that in each stage (or subproblem) determines the optimal composition of one of the b bundles. The optimal solution of the global problem is generated by an algorithm constructed around the idea of a Pareto-efficient frontier, first described by Bitran and Ferrer, in combination with the novel concepts of inner adjacent frontier, ranked list of a component, adjacent bundle and candidate bundle set. The algorithm is a pseudo-polynomial of order $O(\mathcal{HM} + (b-1)\mathcal{M})$. It requires an initial bundle, designed following the approach also developed by Bitran and Ferrer for the single bundle composition problem in a single market segment. It is demonstrated that this bundle will always be one of the b bundles chosen.

Two aspects of the solution are particularly worthy of note. First, the optimal price of each bundle designed by a firm depends on the composition of all of its other bundles, but not on their prices; and second, the bundles designed are very similar in their composition. The fact that any one bundle's price does not depend directly on the prices of the others – though it does indirectly through its composition – ensures that there exists a closed expression to calculate it. The similarity of the bundles' composition is due to the fact that an MNL consumer choice model was used.

Since the bundles are supplied to a single market segment, the fact that they are similar reduces the problems arising from inaccurate estimates of consumer preferences. However, there are certain contexts in which marketing multiple bundles in a single segment is problematic. This is the case, for example, when there are quantity-of-production limits on certain bundle components, or when the reserve prices for different bundles over the market segment are randomly distributed rather than deterministic.

It was demonstrated that the marginal utility of composing and marketing an additional bundle is always non-negative. The value of this bundle therefore depends on the marginal utility relative to the marginal administration cost function for bundle marketing. Thus, the optimal number of bundles to be designed and marketed is given by the point at which the two marginal functions are equal.

It might seem intuitive to assume that if a firm wants to offer more than one bundle in a single market, then the firm should offer bundles that are different from each other, hoping to increase its profits based on the variety of options offered to consumers. But, one of the economic implications of our work is that it demonstrates that, under certain conditions, a company must offer bundles consisting of similar products in order to maximize profits. We also show that the decisions regarding the optimal price for each bundle can be separated from those regarding bundles composition, and that contrary to what one might think, if two or more bundles are offered in the market, the optimal price for each of these bundles does not depend on the price of the others.

A task for future research is to solve the problem of the optimal composition of multiple bundles to be supplied to multiple market segments. Preliminary analysis suggests, however, that finding a closed expression will not be possible and that the adjacency properties employed in the approach proposed here will not hold for the multiple segment case. Other avenues for further research include dropping the assumption that there is interaction among the different components of the bundle, using a different consumer choice model, or including competitors' possible reactions to the firm's choices of bundle compositions and prices.

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Appendix A. Proofs

A1. Proof of Proposition 1

Recalling (11) in the main text, we differentiate Π with respect to the price of bundle k and set the derivative to 0 to obtain the first-order conditions. Thus, for all $k = 1, \ldots, b$,

$$\frac{\partial \Pi}{\partial p_k} = \frac{e^{I_{X_k}+\beta p_k}\left[\left(1+\beta p_k - \beta c_{X_k}\right)\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right) - \beta \sum_{l=1}^{b}(p_l - c_{X_l})e^{I_{X_l}+\beta p_l}\right]}{\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right)^2} = 0 \tag{A.1}$$

Since $(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}) > 0$ and $e^{I_{X_k}+\beta p_k} > 0$, then

$$\left(1+\beta p_k - \beta c_{X_k}\right)\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right) - \beta \sum_{l=1}^{b}(p_l - c_{X_l})e^{I_{X_l}+\beta p_l} = 0 \tag{A.2}$$

If we rewrite (A.2) in terms of bundle w, subtract it from (A.2) and then divide by β, we are left with

$$\left(p_k - c_{X_k} - (p_w - c_{X_w})\right)\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right) = 0 \tag{A.3}$$

Since $\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l} > 0$, then $p_k - c_{X_k} = p_w - c_{X_w}$. Now define $r = p_k - c_{X_k}$ for all $k = 1, \ldots, b$. Eq. (A.2) then becomes $(1+\beta r)(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta r+\beta c_{X_l}}) - \beta \sum_{l=1}^{b} r e^{I_{X_l}+\beta r+\beta c_{X_l}} = 0$. Grouping terms, this reduces to $(1+\beta r)\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta r+\beta c_{X_l}} = 0$, and if we let $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}}$, which is constant for a given set of bundles, we obtain $(1+\beta r)\gamma + e^{\beta r}Q = 0$. Collecting terms, we have $(-1-\beta r)e^{(-1-\beta r)} = \frac{Qe^{-1}}{\gamma}$. If we then apply the Lambert W-function and define $W(z) = (-1-\beta r)$ and $z = \frac{Qe^{-1}}{\gamma}$, we get $(-1-\beta r) = W(\frac{1}{\gamma}Qe^{-1})$, where $r = \frac{-1}{\beta}(1 + W(\frac{1}{\gamma}Qe^{-1}))$. Finally, since $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}}$ we arrive at the following closed-form expression for the optimal price:

$$p_k^* = c_{X_k} - \frac{1}{\beta}\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right) \quad \forall k = 1, \ldots, b.$$

A2. Proof of Corollary 1

If (11) is expressed as a function of p_k^*, we have

$$\Pi^* = \sum_{k=1}^{b} \frac{e^{I_{X_k}+\beta p_k^*}}{\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l^*}}(p_k^* - c_{X_k}) \tag{A.4}$$

Substituting the result of (12) into (A.4), $\Pi^* = \sum_{k=1}^{b} \frac{e^{I_{X_k}+\beta G_k}}{\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta G_l}}(G_k - c_{X_k})$, where $G_k = c_{X_k} - \frac{1}{\beta}(1 + W(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}))$. Again letting $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$ (all of which are constants), we have

$$\Pi^* = \frac{-1}{\beta}(W(z)+1)\sum_{k=1}^{b} \frac{e^{I_{X_k}+\beta c_{X_k}-1-W(z)}}{\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1-W(z)}\right)} \tag{A.5}$$

Since $(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1-W(z)})$ is independent of k,

$$\Pi^* = \frac{-1}{\beta}(W(z)+1)\frac{\sum_{k=1}^{b} e^{I_{X_k}+\beta c_{X_k}-1-W(z)}}{\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1-W(z)}\right)}$$

Recalling that $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$, we have $\Pi^* = \frac{-1}{\beta}(W(z)+1)(\frac{Qe^{-W(z)}}{\gamma+Qe^{-W(z)}})$, and since $z = \frac{Q}{\gamma}$, then $\Pi^* = \frac{-1}{\beta}(W(z)+1)(\frac{ze^{-W(z)}}{1+ze^{-W(z)}})$.

Finally, by definition of the Lambert W-function we know that $(\frac{ze^{-W(z)}}{1+ze^{-W(z)}}) = \frac{W(z)}{W(z)+1}$, which leaves

$$\Pi^* = \frac{-1}{\beta}(W(z)+1)\frac{W(z)}{W(z)+1} \tag{A.6}$$

thus proving Corollary 1

$$\Pi^* = \frac{-1}{\beta}W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)$$

A3. Proof of Corollary 2

By virtue of (A.5) and (A.6) we have $\frac{W(z)}{W(z)+1} = q$, where q is the probability that one of the b bundles offered by the firm will be chosen. We therefore have

$$\begin{aligned} q &= \frac{W(z)}{W(z)+1}\left(\frac{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right) \\ &= \frac{W(z)}{W(z)+1}\left(\frac{e^{I_{X_1}+\beta c_{X_1}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}} + \ldots + \frac{e^{I_{X_b}+\beta c_{X_b}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right) \end{aligned}$$

where $\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}$ is defined as the proportion of q contributed by bundle k. Finally, again letting $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$, we have the proof of Corollary 2.

$$q_k^* = \frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)}{\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)}\left(\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right)$$

A4. Proof of Proposition 2

The derivative of Π^* with respect to the composition of bundle k is given by

$$\begin{aligned} \frac{\partial \Pi^*}{\partial I_{X_k}} &= \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial I_{X_k}}\right) = \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial z}\frac{\partial z}{\partial I_{X_k}}\right) \\ &= \frac{-1}{\beta}\frac{W(z)}{z(1+W(z))}\frac{1}{\gamma}e^{I_{X_k}+\beta c_{X_k}-1} \end{aligned}$$

Recalling that $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$, we have

$$\frac{\partial \Pi^*}{\partial I_{X_k}} = \frac{-1}{\beta}\frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)}{\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)}\left(\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right)$$

The derivative of Π^* with respect to the cost c_{X_k} of bundle k is

$$\begin{aligned} \frac{\partial \Pi^*}{\partial c_{X_k}} &= \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial c_{X_k}}\right) = \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial z}\frac{\partial z}{\partial c_{X_k}}\right) \\ &= \frac{-1}{\beta}\frac{W(z)}{z(1+W(z))}\frac{\beta}{\gamma}e^{I_{X_k}+\beta c_{X_k}-1} \end{aligned}$$

since $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$, we have

$$\frac{\partial \Pi^*}{\partial c_{X_k}} = -\frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)}{\left(1+W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)}\left(\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right)$$

Finally, using the expression for q_k^* from Corollary 2, we obtain

$$\nabla \Pi^* = \left(\frac{\partial \Pi^*}{\partial I_{X_k}}, \frac{\partial \Pi^*}{\partial c_{X_k}}\right) = \left(\frac{-1}{\beta}, -1\right) q_k^*$$

A5. Proof of Proposition 3

We may state without loss of generality that the feasible bundle space Ω contains the following f+1 bundles: $X_{b_1}, X_{b_2}, \ldots, X_{b_d}, \ldots, X_{b_f}$ and X_1^*. We also assume that the Pareto-efficient frontier Ω_1^* is composed of the d bundles $X_{b_1}, X_{b_2}, \ldots, X_{b_{d-1}}$ and X_1^*. Then for all $j = b_d, \ldots, b_f$, it must be the case that $I_{X_j} \le I_{X_i}$ and $c_{X_j} \ge c_{X_i}$ for at least one i in $\{b_1, \ldots, b_{d-1}, X_1^*\}$.

The construction Ω_2^*, which is the inner adjacent frontier of Ω_1^* under X_1^*, will contain the d −1 bundles b_1, ...,b_{d-1} since the latter are not dominated by any bundle and will not be in any way affected by the extraction of bundle X_1^*. In addition, a bundle j in $\{b_d, \ldots, b_f\}$ will be incorporated into Ω_2^* provided $I_{X_j} \le I_{X_i}$ and $c_{X_j} \ge c_{X_i}$ are not satisfied for all i in $\{b_1, \ldots, b_{d-1}\}$. This ensures that any bundle that was originally dominated only by X_1^* will be on the new Pareto-efficient frontier when the latter is eliminated from the feasible bundle set. Therefore, Ω_2^* will contain the d−1 bundles inherited from the previous frontier Ω_1^* and the bundles in $\{b_1, \ldots, b_{d-1}\}$ to be incorporated.

Proposition 3 is therefore proved.

A6. Proof of Proposition 4

Proposition 2 states that the contribution of any given bundle to the objective function is independent of the contribution of every other bundle. Therefore, the optimal composition of the stage k+1 bundle is independent of the term $F_k^*(\Omega_k^*)$ given by (18), which is the contribution made by all of the already formed bundles to optimal utility.

It follows from the above that determining the composition of the optimal bundle in subproblem k+1 requires only the state information of the candidate bundle set $T(X_{k+1})$ for that stage. As was explained in Definition 4, building the set $T(X_{k+1})$ only requires information on the optimal bundles of the previous stages, that is, X_1^*, X_2^*,..., X_k^*, which is always present by virtue of the method used.

Therefore, to determine the optimal composition of the stage k+1 bundle we must solve

$$F_{k+1}^*(\Omega_{k+1}^*) = \max_{X_{k+1}\in T(X_{k+1})} \left\{ \frac{-I_{X_{k+1}}}{\beta} - c_{X_{k+1}} \right\} \tag{A.7}$$

thus proving Proposition 4.

A7. Proof of Proposition 5

We define C as the total number of bundles that can be formed. Thus, $C = Card(\Omega)$. Rewriting (16) so as to choose the best b bundles, we obtain the following knapsack problem:

$$\max_{Y_1, Y_2, \ldots, Y_C} \sum_{k=1}^{C} \left(\frac{-I_k}{\beta} - c_k\right) Y_k$$

$$s.t: \quad \sum_{k=1}^{C} Y_k \le b$$

$$Y_1, Y_2, \ldots, Y_C \in \{0, 1\}$$

where Y_k is a binary variable that indicates whether the kth bundle should be chosen and $(-I_k/\beta) - c_k = d_k$ is the benefit obtained by choosing bundle k. The number of bundles to be chosen is limited by the budget constraint, the cost a_k in resources of choosing any given bundle being set at 1 $\forall k$. We then construct the quotient $\nu_k = d_k/a_k$ and order the C bundles so that $\nu_1 \ge \nu_2 \ge \nu_3 \ge \ldots \ge \nu_C$. Given this ordering and the fact that all $a_k = 1$, the bundle designed by the BF algorithm (Bitran & Ferrer, 2007) will be the one that is associated with ν_1. The second turnpike theorem, described in detail by Garfinkel and Nemhauser (1972), state that if $\nu_1 > \nu_2$ and there exists an $h = (a_1 - 1) \cdot \max_{k\ge 2}\{a_k\}$ such that $b > h$ (where b is the knapsack-type constrained resource), then the article associated with ν_1 is the optimal choice among all the choosable articles. In our case, $h = 0$ given that $a_1 = 1$, and $b \ge 1$ is the number of bundles to be formed. The conditions set by the theorem are thus satisfied and the first bundle to be chosen must be the one constructed by the BF algorithm for the single-bundle optimal composition problem.

A8. Proof of Proposition 6

By (13) we have that: $\Pi^* = \frac{-1}{\beta} W(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1})$. If an additional bundle is composed, the benefit can be written as

$$\Pi^* = \frac{-1}{\beta} W\left(\frac{1}{\gamma}\sum_{l=1}^{b+1} e^{I_{X_l}+\beta c_{X_l}-1}\right)$$
$$= \frac{-1}{\beta} W\left(\frac{1}{\gamma}\left(e^{I_{X_{b+1}}+\beta c_{X_{b+1}}-1} + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right),$$

where $I_{X_{b+1}} + \beta c_{X_{b+1}} - 1 > -1$ given that $I_{X_{b+1}} + \beta c_{X_{b+1}} > 0$ for every case. Then $e^{I_{X_{b+1}}+\beta c_{X_{b+1}}-1} > e^{-1} > 0$, implying in turn that $e^{I_{X_{b+1}}+\beta c_{X_{b+1}}-1} + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1} > \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$. Since the Lambert W-function is strictly increasing, the utility of b bundles is always less than the utility of $b+1$ bundles provided there are no administration costs.

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