



Machine Learning

# Support Vector Machines

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Optimization  
objective

# Alternative view of logistic regression

$$\rightarrow h_{\theta}(x) = \frac{1}{1 + e^{-\theta^T x}}$$



If  $y = 1$ , we want  $h_{\theta}(x) \approx 1$ ,  $\theta^T x \gg 0$   
If  $y = 0$ , we want  $h_{\theta}(x) \approx 0$ ,  $\theta^T x \ll 0$

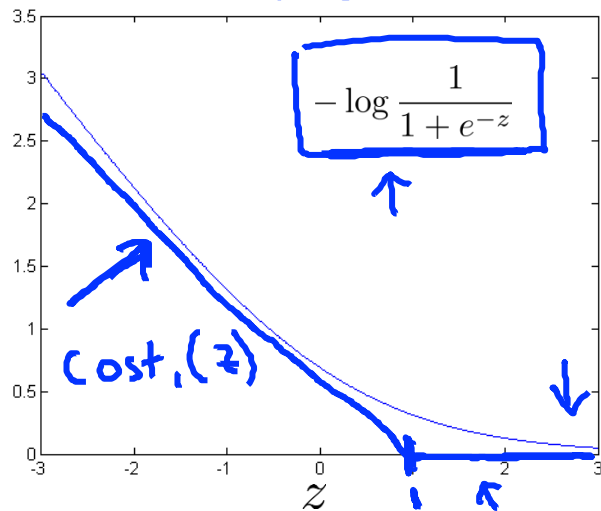
# Alternative view of logistic regression

Cost of example:  $-(y \log h_{\theta}(x) + (1 - y) \log(1 - h_{\theta}(x)))$   $\leftarrow$

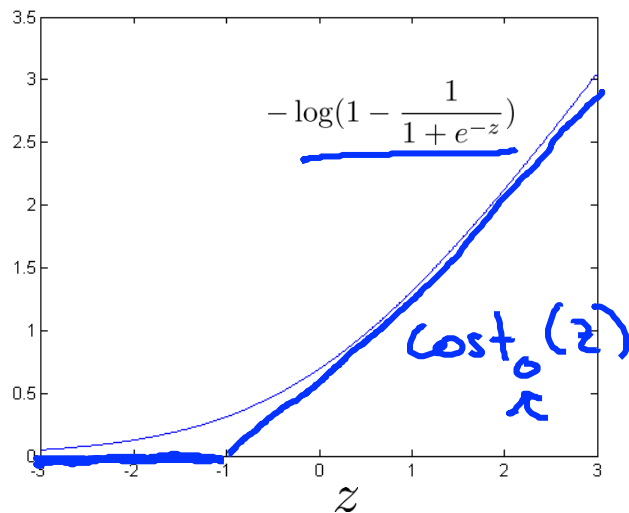
$$= -y \log \frac{1}{1 + e^{-\theta^T x}} - (1 - y) \log(1 - \frac{1}{1 + e^{-\theta^T x}})$$

If  $y = 1$  (want  $\theta^T x \gg 0$ ):

$$z = \theta^T x$$



If  $y = 0$  (want  $\theta^T x \ll 0$ ):



# Support vector machine

Logistic regression:

$$\min_{\theta} \frac{1}{m} \left[ \sum_{i=1}^m y^{(i)} \left( -\log h_{\theta}(x^{(i)}) \right) + (1 - y^{(i)}) \left( -\log(1 - h_{\theta}(x^{(i)})) \right) \right] + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$$

Support vector machine:

$$\min_{\theta} C \sum_{i=1}^m \left[ y^{(i)} cost_1(\theta^T x^{(i)}) + (1 - y^{(i)}) cost_0(\theta^T x^{(i)}) \right] + \frac{1}{2} \sum_{j=1}^n \theta_j^2$$

## SVM hypothesis

$$\min_{\theta} C \sum_{i=1}^m \left[ y^{(i)} cost_1(\theta^T x^{(i)}) + (1 - y^{(i)}) cost_0(\theta^T x^{(i)}) \right] + \frac{1}{2} \sum_{j=1}^n \theta_j^2$$

Hypothesis:



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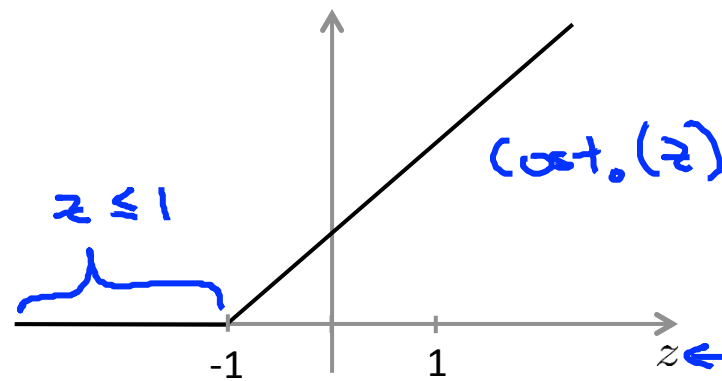
# Support Vector Machines

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## Large Margin Intuition

# Support Vector Machine

$$\rightarrow \min_{\theta} C \sum_{i=1}^m \left[ y^{(i)} \underline{\text{cost}_1(\theta^T x^{(i)})} + (1 - y^{(i)}) \underline{\text{cost}_0(\theta^T x^{(i)})} \right] + \frac{1}{2} \sum_{j=1}^n \theta_j^2$$



$\rightarrow$  If  $y = 1$ , we want  $\theta^T x \geq 1$  (not just  $\geq 0$ )

$$\theta^T x \geq 1$$

$\rightarrow$  If  $y = 0$ , we want  $\theta^T x \leq -1$  (not just  $< 0$ )

$$\theta^T x \leq -1$$

$$C = 100,000$$

## SVM Decision Boundary

$$\min_{\theta} C \sum_{i=1}^m \left[ y^{(i)} \text{cost}_1(\theta^T x^{(i)}) + (1 - y^{(i)}) \text{cost}_0(\theta^T x^{(i)}) \right] + \frac{1}{2} \sum_{j=1}^n \theta_j^2$$

$= 0$

Whenever  $y^{(i)} = 1$ :

$$\theta^T x^{(i)} \geq 1$$

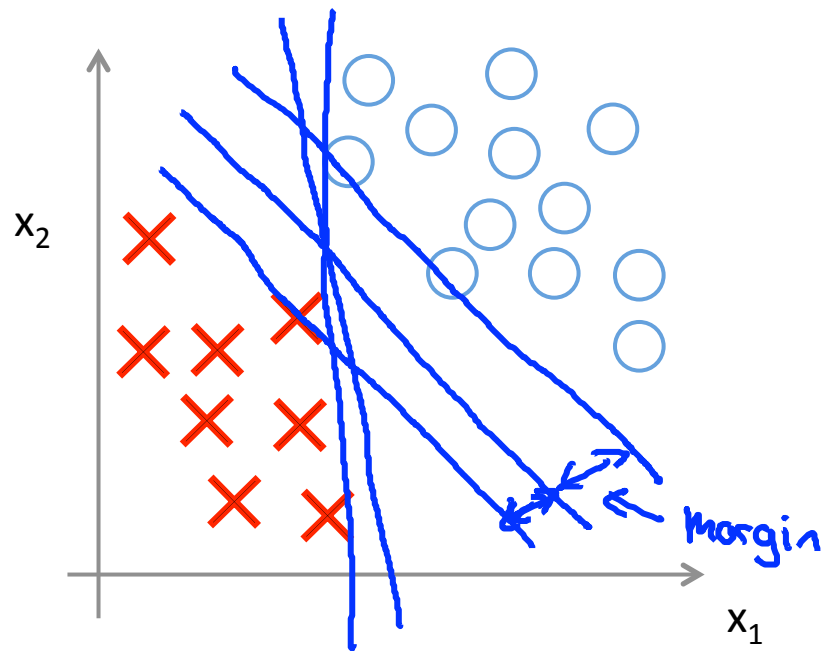
Whenever  $y^{(i)} = 0$ :

$$\theta^T x^{(i)} \leq -1$$

$$\begin{aligned} \min_{\theta} \quad & C \sum_{i=1}^m \theta_j + \frac{1}{2} \sum_{j=1}^n \theta_j^2 \\ \text{s.t.} \quad & \theta^T x^{(i)} \geq 1 \quad \text{if } y^{(i)} = 1 \\ & \theta^T x^{(i)} \leq -1 \quad \text{if } y^{(i)} = 0 \end{aligned}$$



## SVM Decision Boundary: Linearly separable case



Large margin classifier

# Large margin classifier in presence of outliers





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# Support Vector Machines

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The mathematics  
behind large margin  
classification (optional)

# Vector Inner Product



$$\rightarrow u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \rightarrow v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$u^T v = ? \quad [u_1 \ u_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\|u\| = \text{length of vector } u \\ = \sqrt{u_1^2 + u_2^2} \in \mathbb{R}$$

$p =$  length of projection of  $v$  onto  $u$ .

$$\begin{aligned} u^T v &= \underline{p} \cdot \underline{\|u\|} \leftarrow = v^T u \\ \text{Signed} \quad &= u_1 v_1 + u_2 v_2 \leftarrow p \in \mathbb{R} \end{aligned}$$

$$u^T v = p \cdot \|u\|$$

$$p < 0$$

$$\omega = (\sqrt{\omega'})^2$$

## SVM Decision Boundary

$$\min_{\theta} \frac{1}{2} \sum_{j=1}^n \theta_j^2 = \frac{1}{2} (\theta_1^2 + \theta_2^2) = \frac{1}{2} \left( \sqrt{\theta_1^2 + \theta_2^2} \right)^2 = \frac{1}{2} \|\theta\|^2$$

$$\text{s.t. } \theta^T x^{(i)} \geq 1 \quad \text{if } y^{(i)} = 1$$

$$\rightarrow \theta^T x^{(i)} \leq -1 \quad \text{if } y^{(i)} = 0$$

Simplification:  $\theta_0 = 0$   $n=2$

$$= \|\theta\|$$

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad \theta_0 = 0$$

$$\theta^T x^{(i)} = ?$$

↑ ↑  
 $u^T v$



$$\theta^T x^{(i)} = p^{(i)} \cdot \|\theta\| \leftarrow$$

$$= \theta_1 x_1^{(i)} + \theta_2 x_2^{(i)} \leftarrow$$

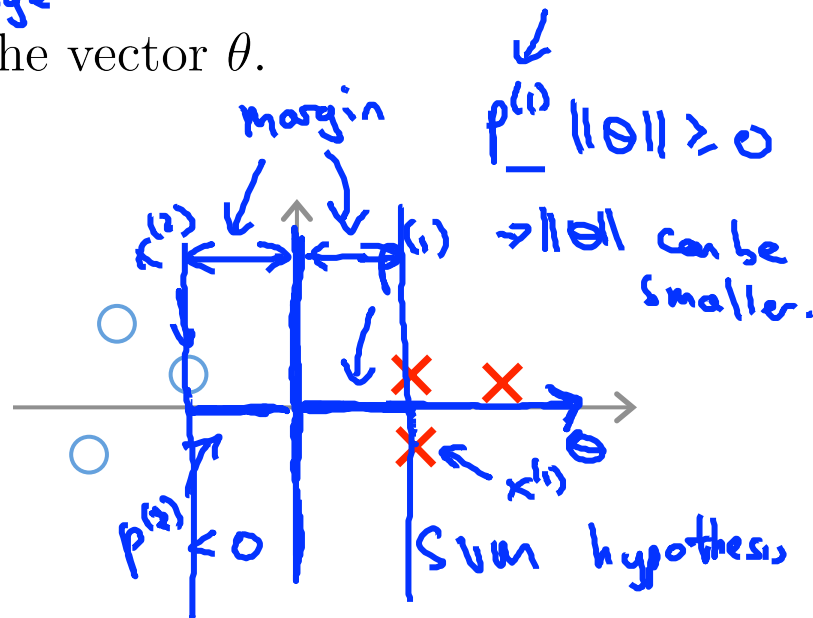
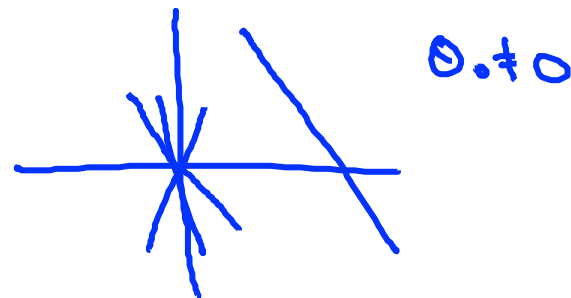
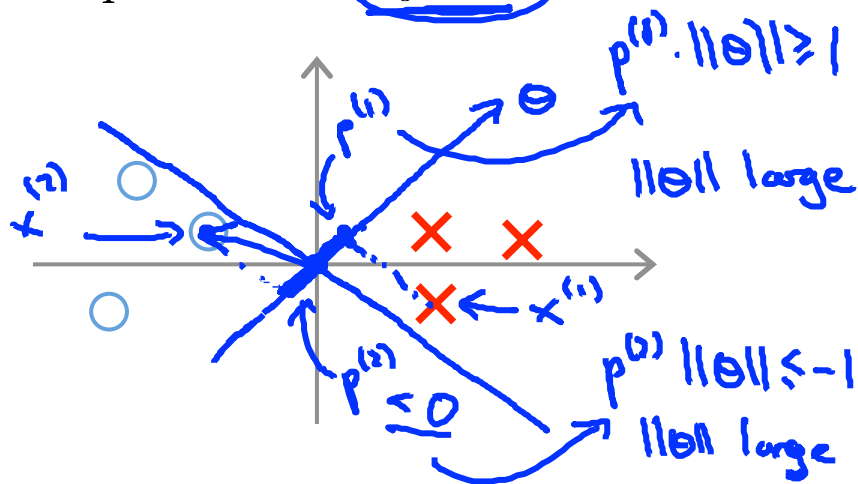
# SVM Decision Boundary

$$\Rightarrow \min_{\theta} \frac{1}{2} \sum_{j=1}^n \theta_j^2 = \frac{1}{2} \|\theta\|^2 \leftarrow$$

$$\text{s.t. } \left. \begin{array}{l} p^{(i)} \cdot \|\theta\| \geq 1 \quad \text{if } y^{(i)} = 1 \\ p^{(i)} \cdot \|\theta\| \leq -1 \quad \text{if } y^{(i)} = -1 \end{array} \right\} C \text{ very large}$$

where  $p^{(i)}$  is the projection of  $x^{(i)}$  onto the vector  $\theta$ .

Simplification:  $\theta_0 = 0$





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## Kernels I

## Non-linear Decision Boundary



Predict  $y = 1$  if

$$\rightarrow \theta_0 + \theta_1 \underline{x_1} + \theta_2 \underline{x_2} + \theta_3 \underline{x_1 x_2} \\ + \theta_4 \underline{x_1^2} + \theta_5 \underline{x_2^2} + \dots \geq 0$$

$$h_0(x) = \begin{cases} 1 & \text{if } \theta_0 + \theta_1 x_1 + \dots \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow \theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 + \dots$$

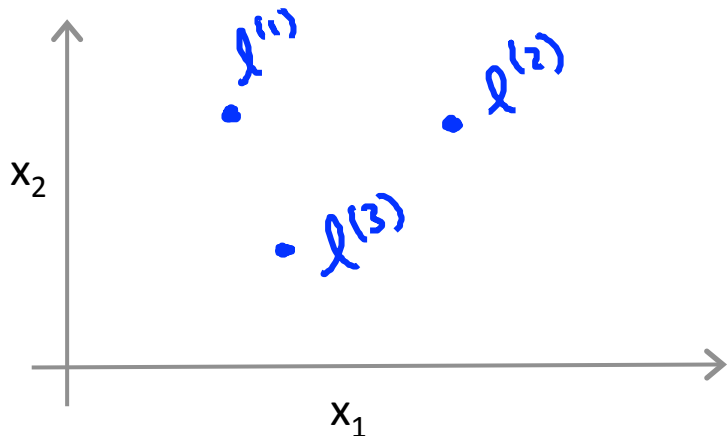
$$f_1 = x_1, \quad f_2 = x_2, \quad f_3 = x_1 x_2, \quad f_4 = x_1^2, \quad f_5 = x_2^2, \dots$$

Is there a different / better choice of the features  $f_1, f_2, f_3, \dots$ ?



# Kernel

Given  $x$ , compute new feature depending on proximity to landmarks  $l^{(1)}, l^{(2)}, l^{(3)}$



Given  $x$ :

$$f_1 = \text{similarity}(x, l^{(1)}) = \exp\left(-\frac{\|x - l^{(1)}\|^2}{2\sigma^2}\right)$$

$$f_2 = \text{similarity}(x, l^{(2)}) = \exp\left(-\frac{\|x - l^{(2)}\|^2}{2\sigma^2}\right)$$

$$f_3 = \text{similarity}(x, l^{(3)}) = \exp(\dots)$$

$\underbrace{\hspace{10em}}_{\text{Kernel (Gaussian kernels)}} \quad k(x, l^{(i)})$

## Kernels and Similarity

$$f_1 = \text{similarity}(x, \underline{l^{(1)}}) = \exp\left(-\frac{\|x - l^{(1)}\|^2}{2\sigma^2}\right)$$

If  $x \approx l^{(1)}$  :

$$f_1 \approx \exp\left(-\frac{0^2}{2\sigma^2}\right) \approx 1$$

If  $x$  if far from  $l^{(1)}$  :

$$f_1 = \exp\left(-\frac{(\text{large number})^2}{2\sigma^2}\right) \approx 0.$$

$$\begin{array}{ccc} l^{(1)} & \rightarrow & f_1 \\ l^{(2)} & \rightarrow & f_2 \\ l^{(3)} & \rightarrow & f_3 \end{array}$$

↑                      ↑  
                              X

Example:

$$\rightarrow l^{(1)} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$f_1 = \exp \left( -\frac{\|x - l^{(1)}\|^2}{2\sigma^2} \right)$$

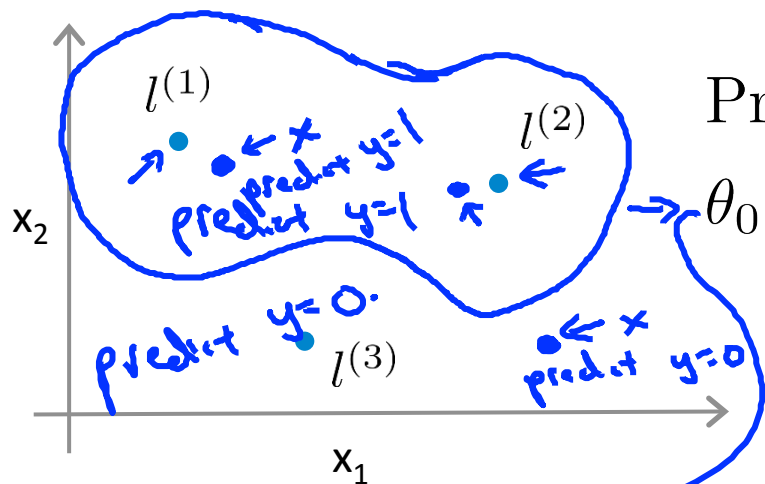
$$\rightarrow \sigma^2 = 1$$

$$x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\sigma^2 = 0.5$$

$$\sigma^2 = 3$$





Predict "1" when

$$\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 \geq 0$$



$$\underline{\theta_0 = -0.5, \theta_1 = 1, \theta_2 = 1, \theta_3 = 0}$$

$$f_1 \approx 1, f_2 \approx 0, f_3 \approx 0.$$

$$\begin{aligned} \rightarrow \theta_0 + \theta_1 \times 1 + \theta_2 \times 0 + \theta_3 \times 0 \\ = -0.5 + 1 = 0.5 \geq 0 \end{aligned}$$

$$f_1, f_2, f_3 \approx 0$$

$$\rightarrow \underline{\theta_0} + \theta_1 \underline{f_1} + \dots \approx -0.5 < 0$$



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## Kernels II

## Choosing the landmarks



Given  $x$ :

$$\begin{aligned} \rightarrow f_i &= \text{similarity}(x, l^{(i)}) \\ &= \exp\left(-\frac{\|x - l^{(i)}\|^2}{2\sigma^2}\right) \leftarrow \end{aligned}$$

Predict  $y = 1$  if  $\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 \geq 0$   $\leftarrow$

Where to get  $l^{(1)}, l^{(2)}, l^{(3)}, \dots$ ?



## SVM with Kernels

- Given  $(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})$ ,
- choose  $l^{(1)} = x^{(1)}, l^{(2)} = x^{(2)}, \dots, l^{(m)} = x^{(m)}$ .

Given example  $\underline{x}$ :

$$\begin{aligned} \rightarrow f_1 &= \text{similarity}(x, l^{(1)}) \\ \rightarrow f_2 &= \text{similarity}(x, l^{(2)}) \\ &\vdots \end{aligned}$$

$$f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \quad f_0 = 1$$

For training example  $(x^{(i)}, y^{(i)})$ :

$$\underline{x^{(i)}} \rightarrow \begin{bmatrix} f_1^{(i)} \\ f_2^{(i)} \\ \vdots \\ f_m^{(i)} \end{bmatrix} \quad \begin{aligned} f_1^{(i)} &= \text{sim}(x^{(i)}, l^{(1)}) \\ f_2^{(i)} &= \text{sim}(x^{(i)}, l^{(2)}) \\ &\vdots \\ f_i^{(i)} &= \text{sim}(x^{(i)}, l^{(i)}) = \exp\left(-\frac{0}{2\sigma^2}\right) = 1 \\ &\vdots \\ f_m^{(i)} &= \text{sim}(x^{(i)}, l^{(m)}) \end{aligned}$$

$$\underline{x^{(i)}} \in \mathbb{R}^{n+1} \quad (\text{or } \mathbb{R}^n)$$

$$f^{(i)} = \begin{bmatrix} f_0^{(i)} \\ f_1^{(i)} \\ f_2^{(i)} \\ \vdots \\ f_m^{(i)} \end{bmatrix} \quad f_0^{(i)} = 1$$

# SVM with Kernels

Hypothesis: Given  $x$ , compute features  $f \in \mathbb{R}^{m+1}$

→ Predict "y=1" if  $\theta^T f \geq 0$

$$\theta_0 f_0 + \theta_1 f_1 + \dots + \theta_m f_m$$

$$\theta \in \mathbb{R}^{n+1}$$

Training:

$$\rightarrow \min_{\theta} C \sum_{i=1}^m y^{(i)} \text{cost}_1(\theta^T f^{(i)}) + (1 - y^{(i)}) \text{cost}_0(\theta^T f^{(i)}) + \frac{1}{2} \sum_{j=1}^m \theta_j^2$$

*Handwritten notes:*  $\theta^T f^{(i)}$ ,  $\theta_0$ ,  $n=m$

$$\rightarrow \sum_{j=1}^m \theta_j^2 = \theta^T \theta \leftarrow \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix}$$

*Handwritten notes:*  $\theta^T M \theta$ ,  $\|\theta\|^2$ , (ignore  $\theta_0$ ),  $m = 10,000$



## SVM parameters:

$C \left( = \frac{1}{\lambda} \right)$ .  $\rightarrow$  Large  $C$ : Lower bias, high variance.  
 $\rightarrow$  Small  $C$ : Higher bias, low variance.

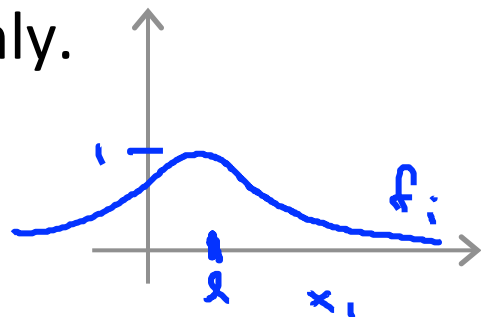
(small  $\lambda$ )

(large  $\lambda$ )

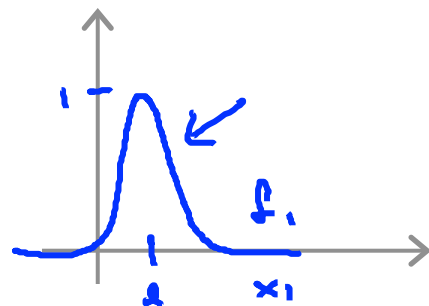
$\sigma^2$  Large  $\sigma^2$ : Features  $f_i$  vary more smoothly.

$\rightarrow$  Higher bias, lower variance.

$$\exp\left(-\frac{\|x - \mu^{(i)}\|^2}{2\sigma^2}\right)$$



Small  $\sigma^2$ : Features  $f_i$  vary less smoothly.  
Lower bias, higher variance.





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## Using an SVM

Use SVM software package (e.g. liblinear, libsvm, ...) to solve for parameters  $\theta$ .



Need to specify:

→ Choice of parameter C.

Choice of kernel (similarity function):

E.g. No kernel ("linear kernel")

Predict " $y = 1$ " if  $\theta^T x \geq 0$

$$\theta_0 + \theta_1 x_1 + \dots + \theta_n x_n \geq 0 \quad \rightarrow \quad \underline{n \text{ large}}, \quad \underline{m \text{ small}} \quad \underline{x \in \mathbb{R}^{n+1}}$$

→ Gaussian kernel:

$$f_i = \exp\left(-\frac{\|x - l^{(i)}\|^2}{2\sigma^2}\right), \text{ where } l^{(i)} = x^{(i)}.$$

Need to choose  $\sigma^2$ .



$x \in \mathbb{R}^n$ ,  $n$  small  
and/or  $n$  large



Kernel (similarity) functions:

function  $f = \text{kernel}(\underline{x1}, \underline{x2})$

$$f = \exp\left(-\frac{\|\underline{x1} - \underline{x2}\|^2}{2\sigma^2}\right)$$

return

$x \rightarrow \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{matrix}$

→ Note: Do perform feature scaling before using the Gaussian kernel.

$$\rightarrow \boxed{\|x - l\|^2}$$

$$v = x - l$$

$$\|v\|^2 = v_1^2 + v_2^2 + \dots + v_n^2$$

$$= (x_1 - l_1)^2 + (x_2 - l_2)^2 + \dots + (x_n - l_n)^2$$

1000 feet<sup>2</sup>      1-5 bedrooms

## Other choices of kernel

Note: Not all similarity functions  $\text{similarity}(x, l)$  make valid kernels.

→ (Need to satisfy technical condition called “Mercer’s Theorem” to make sure SVM packages’ optimizations run correctly, and do not diverge).

Many off-the-shelf kernels available:

- Polynomial kernel:

$$k(x, l) = (x^T l)^3, \quad (x^T l)^2 + 1, \quad (x^T l + 5)^4$$

Handwritten annotations:   
 - An arrow points from  $(x^T l)^3$  to the text  $(x^T l)^2 + 1$ .   
 - An arrow points from  $(x^T l)^2 + 1$  to  $(x^T l + 5)^4$ .   
 - An arrow points from  $(x^T l + 5)^4$  to the word "degree" with an upward arrow.   
 - A circled "3" is under  $(x^T l + 1)$  and a circled "4" is under  $(x^T l + 5)$ .

- More esoteric: String kernel, chi-square kernel, histogram intersection kernel, ...
- Handwritten annotation:  $\text{sim}(x, l)$  with an arrow pointing to it from above.

## Multi-class classification



$$y \in \{1, 2, 3, \dots, K\}$$

↑

Many SVM packages already have built-in multi-class classification functionality.

→ Otherwise, use one-vs.-all method. (Train  $K$  SVMs, one to distinguish  $y = i$  from the rest, for  $i = 1, 2, \dots, K$ ), get  $\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(K)}$   
Pick class  $i$  with largest  $(\theta^{(i)})^T x$

↑  
 $y=1$     ↑  
 $y=2$     ...    ↑  
                          $\theta = K$

## Logistic regression vs. SVMs

$n$  = number of features ( $x \in \mathbb{R}^{n+1}$ ),  $m$  = number of training examples

→ If  $n$  is large (relative to  $m$ ): (e.g.  $n \geq m$ ,  $n = \underline{10,000}$ ,  $m = \underline{10} \dots \underline{1000}$ )

→ Use logistic regression, or SVM without a kernel ("linear kernel")

→ If  $n$  is small,  $m$  is intermediate:

( $n = \underline{1-1000}$ ,  $m = \underline{10-10,000}$ ) ←

→ Use SVM with Gaussian kernel

If  $n$  is small,  $m$  is large: ( $n = \underline{1-1000}$ ,  $m = \underline{50,000+}$ )

→ Create/add more features, then use logistic regression or SVM without a kernel



→ Neural network likely to work well for most of these settings, but may be slower to train.