Explicit Analytical Derivation of LSL and RSR Dubins' Paths for Intercepting a Uniformly Moving Target - Appendix

September 15, 2025

A Appendix

A.1 Proof of Theorem 1

Since (11) and (15), one can compute C_1, S_2 and C_3 systems such as

$$
\begin{cases}
\psi_1 = \psi_0 + \alpha_1 \\
X_1 = X_p^0 + \delta_1 R \begin{pmatrix} -\sin(\psi_0) + \sin(\psi_1) \\ \cos(\psi_0) - \cos(\psi_1) \end{pmatrix} \\
t_1 = t_0 + \frac{R}{v_p}\delta_1\alpha_1
\end{cases} \tag{1}
$$

$$
\begin{cases}
\psi_2 = \psi_1 \\
X_2 = X_1 + d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} \\
t_2 = t_1 + \frac{d_2}{v_p}
\end{cases} \tag{2}
$$

$$
\begin{cases}
\psi_3 = \psi_2 + \alpha_3 \\
X_3 = X_2 + \delta_3 R \begin{pmatrix} -\sin(\psi_2) + \sin(\psi_3) \\ \cos(\psi_2) - \cos(\psi_3) \end{pmatrix} \\
t_3 = t_2 + \frac{R}{v_p}\delta_3\alpha_3
\end{cases} \tag{3}
$$

Note that:

- t_3 is the interception time noted t_f;

- during straight line, heading does not change so $\psi_2 = \psi_1$;

- pursuer and target final heading are equal so $\psi_3 = \psi_t$;

- pursuer and target final position are equal so $X_3 = X(t_f) = X_f$

Thus, one gets

$$
\begin{aligned}
\psi_t &= \psi_2 + \alpha_3 \\
&= \psi_1 + \alpha_3 \\
&= \psi_0 + \alpha_1 + \alpha_3
\end{aligned} \tag{4}
$$

and

$$
\begin{aligned}
X_f &= X_p^0 + \delta_1 R \begin{pmatrix} -\sin(\psi_0) + \sin(\psi_1) \\ \cos(\psi_0) - \cos(\psi_1) \end{pmatrix} \\
&\quad + d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} + \delta_3 R \begin{pmatrix} -\sin(\psi_1) + \sin(\psi_t) \\ \cos(\psi_1) - \cos(\psi_t) \end{pmatrix}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
X_f =& X_p^0 + d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} \\
&+ R \begin{pmatrix} -\delta_1\sin(\psi_0) + (\delta_1 - \delta_3)\sin(\psi_1) + \delta_3\sin(\psi_t) \\ \delta_1\cos(\psi_0) - (\delta_1 - \delta_3)\cos(\psi_1) - \delta_3\cos(\psi_t) \end{pmatrix}
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
t_f &= t_2 + \frac{R}{v_p}\delta_3\alpha_3 \\
&= t_1 + \frac{d_2}{v_p} + \frac{R}{v_p}\delta_3\alpha_3 \\
&= t_0 + \frac{d_2}{v_p} + \frac{R}{v_p}(\delta_1\alpha_1 + \delta_3\alpha_3).
\end{aligned} \tag{7}
$$

Thus, one gets the system:

$$
\begin{cases}
\psi_t - \psi_0 = \alpha_1 + \alpha_3 \\
X_f = X_p^0 + d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} \\
\quad + R \begin{pmatrix} -\delta_1\sin(\psi_0) + (\delta_1 - \delta_3)\sin(\psi_1) + \delta_3\sin(\psi_t) \\ \delta_1\cos(\psi_0) - (\delta_1 - \delta_3)\cos(\psi_1) - \delta_3\cos(\psi_t) \end{pmatrix} \\
t_f - t_0 = \frac{d_2}{v_p} + \frac{R}{v_p}(\delta_1\alpha_1 + \delta_3\alpha_3)
\end{cases} \tag{8}
$$

According to the target dynamics, X_f can also be expressed as

$$
X_f = X_t^0 + v_t \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \cdot (t_f - t_0) \tag{9}
$$

Combining (??) with (??), one gets:

$$
X_f = X_t^0 + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} (d_2 + R(\delta_1\alpha_1 + \delta_3\alpha_3)). \tag{10}
$$

Moreover, using (??), one get $\alpha_3 = \psi_t - \psi_0 - \alpha_1$, and so

$$
\begin{aligned}
X_f =& X_t^0 + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \\
&\times (d_2 + R\alpha_1(\delta_1 - \delta_3) + R\delta_3(\psi_t - \psi_0))
\end{aligned} \tag{11}
$$

Finally, (??) and (??) give:

$$
\begin{aligned}
&X_t^0 + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} (d_2 + (\delta_1 - \delta_3)R\alpha_1 + R\delta_3(\psi_t - \psi_0)) \\
&= X_p^0 + R \begin{pmatrix} -\delta_1\sin(\psi_0) + (\delta_1 - \delta_3)\sin(\psi_0 + \alpha_1) + \delta_3\sin(\psi_t) \\ \delta_1\cos(\psi_0) - (\delta_1 - \delta_3)\cos(\psi_0 + \alpha_1) - \delta_3\cos(\psi_t) \end{pmatrix} \\
&\quad + d_2 \begin{pmatrix} \cos(\psi_0 + \alpha_1) \\ \sin(\psi_0 + \alpha_1) \end{pmatrix}
\end{aligned} \tag{12}
$$

1

By putting the left part of the equality independent of α_1 and d_2, one gets:

$$0 = - A + d_2 \left(\begin{pmatrix} \cos(\psi_0 + \alpha_1) \\ \sin(\psi_0 + \alpha_1) \end{pmatrix} - \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right)$$
$$+ R(\delta_1 - \delta_3) \left(\begin{pmatrix} \sin(\psi_0 + \alpha_1) \\ -\cos(\psi_0 + \alpha_1) \end{pmatrix} - \alpha_1 \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right) \tag{13}$$

with

$$A = X_t^0 - X_p^0 - R \begin{pmatrix} -\delta_1 \sin(\psi_0) + \delta_3 \sin(\psi_t) \\ \delta_1 \cos(\psi_0) - \delta_3 \cos(\psi_t) \end{pmatrix}$$
$$+ \frac{v_t}{v_p} R \delta_3 (\psi_t - \psi_0) \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \tag{14}$$

A.2 Proof of Theorem 2

Since $\delta_1 = \delta_3 = \delta$, (??) becomes

$$t_f - t_0 = \frac{d_2}{v_p} + \delta \frac{R}{v_p} (\alpha_1 + \alpha_3)$$

and by definition

$$\alpha_1 = \psi_1 - \psi_0 \mod (2\pi)$$

$$\alpha_3 = \psi_f - \psi_1 \mod (2\pi)$$

Then, if $\delta = \delta_1 = \delta_3$, according to Theorem 1, one gets:

$$A = d_2 \left(\begin{pmatrix} \cos(\psi_0 + \alpha_1) \\ \sin(\psi_0 + \alpha_1) \end{pmatrix} - \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right) \tag{15}$$

Let us define

$$A = [A_x, A_y]^\top$$
$$= X_t^0 - X_p^0 - \delta R \begin{pmatrix} -\sin(\psi_0) + \sin(\psi_t) \\ \cos(\psi_0) - \cos(\psi_t) \end{pmatrix}$$
$$- \frac{v_t}{v_p} R \delta (\psi_t - \psi_0) \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix}. \tag{16}$$

Note that A is a 2D vector independent from ψ_1 and d_2. Let's define $l = \frac{1}{d_2}$ and suppose $d_2 > 0$ (as it will be shown below). One gets:

$$\begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} = Al + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \tag{17}$$

By taking the square norm of (??), one gets

$$1 = \left(A_x l + \frac{v_t}{v_p} \cos(\psi_t) \right)^2 + \left(A_y l + \frac{v_t}{v_p} \sin(\psi_t) \right)^2 \tag{18}$$

$$0 = (A_x^2 + A_y^2) l^2 + \left(\frac{v_t}{v_p} \right)^2$$
$$+ 2 \frac{v_t}{v_p} (A_x \cos(\psi_t) + A_y \sin(\psi_t)) l - 1 \tag{19}$$

(??) is a quadratic equation with the following discriminant

$$\Delta = 4 \left(\frac{v_t}{v_p} \right)^2 (A_x \cos(\psi_t) + A_y \sin(\psi_t))^2$$
$$- 4 (A_x^2 + A_y^2) \left(\frac{v_t^2}{v_p^2} - 1 \right) \tag{20}$$

Note that $(A_x^2 + A_y^2) = \|A\|^2$ and $(A_x \cos(\psi_t) + A_y \sin(\psi_t))^2 = \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle^2$, thus (??) can be rewritten as

$$\Delta = 4 \left(\frac{v_t}{v_p} \right)^2 \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle^2 + 4 \|A\|^2 \left(1 - \frac{v_t^2}{v_p^2} \right) \tag{21}$$

$$\Delta = 4 \left(\frac{v_t}{v_p} \right)^2 \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle^2 + \frac{4}{v_p^2} \|A\|^2 (v_p^2 - v_t^2) \tag{22}$$

Note that $\Delta \geq 0$ as $v_p > v_t \geq 0$, so l is equal to l_1 or l_2 where

$$\begin{cases} l_1 = \dfrac{-2 \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + 2\sqrt{\Delta}}{2\|A\|^2} \\ l_2 = -\left(\dfrac{2 \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + 2\sqrt{\Delta}}{2\|A\|^2} \right) \end{cases} \tag{23}$$

Since $d_2 = l^{-1}$ is a distance, the solution must be positive. One so has $l_2 < 0$. Let's check if $l_1 > 0$. l_1 is positive iff

$$-2 \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + 2\sqrt{\Delta} > 0$$

$$\sqrt{\left(\frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle \right)^2 + \left(\frac{\|A\|}{v_p} \right)^2 (v_p^2 - v_t^2)}$$
$$> \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle$$

$$\sqrt{B^2 + C} > B$$

with $C = \left(\frac{\|A\|}{v_p} \right)^2 (v_p^2 - v_t^2) > 0$ since $v_p > v_t$ and $B = \left(\frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle \right)^2$. Since $C > 0$, one has $\sqrt{B^2 + C} > \sqrt{B^2} > B$. So $l_1 > 0$ and so $d_2 = l_1^{-1}$ is positive and can be expressed as

$$d_2 = \frac{\|A\|^2}{-\frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + \sqrt{\Delta}}. \tag{24}$$

Let's go back to (??). (??) can be split in two lines:

$$\begin{cases} \cos\left(\psi_1\right) = \frac{A_x}{d_2} + \frac{v_t}{v_p}\cos\left(\psi_t\right) \\ \sin\left(\psi_1\right) = \frac{A_y}{d_2} + \frac{v_t}{v_p}\sin\left(\psi_t\right) \end{cases} \qquad (25)$$

From (??), one gets

$$\psi_1 = \arctan 2\left(\frac{A_y}{d_2} + \frac{v_t}{v_p}\sin\left(\psi_t\right), \frac{A_x}{d_2} + \frac{v_t}{v_p}\cos\left(\psi_t\right)\right). \quad (26)$$

A.3   Proof of Theorem 3

If $\delta = \delta_1 = -\delta_3$, according to Theorem 1, one gets

$$t_f - t_0 = \frac{d_2}{v_p} + \delta\frac{R}{v_p}\left(\alpha_1 - \alpha_3\right)$$

$$\alpha_1 = \psi_1 - \psi_0 \mod (2\pi)$$

$$\alpha_3 = \psi_f - \psi_1 \mod (2\pi)$$

Then, with $\delta = \delta_1 = -\delta_3$, (??) becomes

$$0 = d_2\left(\begin{pmatrix}\cos(\psi_0+\alpha_1)\\\sin(\psi_0+\alpha_1)\end{pmatrix} - \frac{v_t}{v_p}\begin{pmatrix}\cos\left(\psi_t\right)\\\sin\left(\psi_t\right)\end{pmatrix}\right)$$

$$+ 2R\delta\left(\begin{pmatrix}\sin\left(\psi_0+\alpha_1\right)\\-\cos\left(\psi_0+\alpha_1\right)\end{pmatrix} - \alpha_1\frac{v_t}{v_p}\begin{pmatrix}\cos\left(\psi_t\right)\\\sin\left(\psi_t\right)\end{pmatrix}\right) - A \quad (27)$$

with

$$A = X_t^0 - X_p^0 - R\delta\begin{pmatrix}-\sin\left(\psi_0\right) - \sin\left(\psi_t\right)\\\cos\left(\psi_0\right) + \cos\left(\psi_t\right)\end{pmatrix}$$

$$- \frac{v_t}{v_p}R\delta\left(\psi_t - \psi_0\right)\begin{pmatrix}\cos\left(\psi_t\right)\\\sin\left(\psi_t\right)\end{pmatrix}. \qquad (28)$$