Explicit Analytical Derivation of LSL and RSR Dubins' Paths for Intercepting a Uniformly Moving Target - Appendix

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A Appendix

A.1 Proof of Theorem 1

Since (11) and (15), one can compute C_1, S_2 and C_3 systems such as

$$\begin{cases} \psi_1 = \psi_0 + \alpha_1 \\ X_1 = X_p^0 + \delta_1 R \begin{pmatrix} -\sin(\psi_0) + \sin(\psi_1) \\ \cos(\psi_0) - \cos(\psi_1) \end{pmatrix} \\ t_1 = t_0 + \frac{R}{v_p}\delta_1\alpha_1 \end{cases} \tag{1}$$

$$\begin{cases} \psi_2 = \psi_1 \\ X_2 = X_1 + d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} \\ t_2 = t_1 + \frac{d_2}{v_p} \end{cases} \tag{2}$$

$$\begin{cases} \psi_3 = \psi_2 + \alpha_3 \\ X_3 = X_2 + \delta_3 R \begin{pmatrix} -\sin(\psi_2) + \sin(\psi_3) \\ \cos(\psi_2) - \cos(\psi_3) \end{pmatrix} \\ t_3 = t_2 + \frac{R}{v_p}\delta_3\alpha_3 \end{cases} \tag{3}$$

Note that:

- t_3 is the interception time noted t_f;
- during straight line, heading does not change so $\psi_2 = \psi_1$;
- pursuer and target final heading are equal so $\psi_3 = \psi_t$;
- pursuer and target final position are equal so $X_3 = X(t_f) = X_f$

Thus, one gets

$$\begin{aligned} \psi_t &= \psi_2 + \alpha_3 \\ &= \psi_1 + \alpha_3 \\ &= \psi_0 + \alpha_1 + \alpha_3 \end{aligned} \tag{4}$$

and

$$\begin{aligned} X_f = X_p^0 &+ \delta_1 R \begin{pmatrix} -\sin(\psi_0) + \sin(\psi_1) \\ \cos(\psi_0) - \cos(\psi_1) \end{pmatrix} \\ &+ d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} + \delta_3 R \begin{pmatrix} -\sin(\psi_1) + \sin(\psi_t) \\ \cos(\psi_1) - \cos(\psi_t) \end{pmatrix} \end{aligned} \tag{5}$$

$$\begin{aligned} X_f =& X_p^0 + d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} \\ &+ R \begin{pmatrix} -\delta_1 \sin(\psi_0) + (\delta_1 - \delta_3)\sin(\psi_1) + \delta_3 \sin(\psi_t) \\ \delta_1 \cos(\psi_0) - (\delta_1 - \delta_3)\cos(\psi_1) - \delta_3\cos(\psi_t) \end{pmatrix} \end{aligned} \tag{6}$$

and

$$\begin{aligned} t_f &= t_2 + \frac{R}{v_p}\delta_3\alpha_3 \\ &= t_1 + \frac{d_2}{v_p} + \frac{R}{v_p}\delta_3\alpha_3 \\ &= t_0 + \frac{d_2}{v_p} + \frac{R}{v_p}(\delta_1\alpha_1 + \delta_3\alpha_3). \end{aligned} \tag{7}$$

Thus, one gets the system:

$$\begin{cases} \psi_t - \psi_0 = & \alpha_1 + \alpha_3 \\ X_f = & X_p^0 + d_2 \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} \\ & + R \begin{pmatrix} -\delta_1 \sin(\psi_0) + (\delta_1 - \delta_3)\sin(\psi_1) + \delta_3 \sin(\psi_t) \\ \delta_1 \cos(\psi_0) - (\delta_1 - \delta_3)\cos(\psi_1) - \delta_3\cos(\psi_t) \end{pmatrix} \\ t_f - t_0 = & \frac{d_2}{v_p} + \frac{R}{v_p}(\delta_1\alpha_1 + \delta_3\alpha_3) \end{cases} \tag{8}$$

According to the target dynamics, X_f can also be expressed as

$$X_f = X_t^0 + v_t \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} . (t_f - t_0) \tag{9}$$

Combining (9) with (7), one gets:

$$X_f = X_t^0 + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} (d_2 + R(\delta_1\alpha_1 + \delta_3\alpha_3)). \tag{10}$$

Moreover, using (4), one get $\alpha_3 = \psi_t - \psi_0 - \alpha_1$, and so

$$\begin{aligned} X_f =& X_t^0 + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \\ &\times (d_2 + R\alpha_1(\delta_1 - \delta_3) + R\delta_3(\psi_t - \psi_0)) \end{aligned} \tag{11}$$

Finally, (11) and (8) give:

$$\begin{aligned} & X_t^0 + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} (d_2 + (\delta_1 - \delta_3) R\alpha_1 + R\delta_3(\psi_t - \psi_0)) \\ &= X_p^0 + R \begin{pmatrix} -\delta_1 \sin(\psi_0) + (\delta_1 - \delta_3)\sin(\psi_0 + \alpha_1) + \delta_3 \sin(\psi_t) \\ \delta_1 \cos(\psi_0) - (\delta_1 - \delta_3)\cos(\psi_0 + \alpha_1) - \delta_3\cos(\psi_t) \end{pmatrix} \\ &+ d_2 \begin{pmatrix} \cos(\psi_0 + \alpha_1) \\ \sin(\psi_0 + \alpha_1) \end{pmatrix} \end{aligned} \tag{12}$$

By putting the left part of the equality independent of α_1 and d_2, one gets:

$$\begin{aligned}0 =& - A + d_2 \left(\begin{pmatrix} \cos(\psi_0 + \alpha_1) \\ \sin(\psi_0 + \alpha_1) \end{pmatrix} - \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right) \\ &+ R(\delta_1 - \delta_3) \left(\begin{pmatrix} \sin(\psi_0 + \alpha_1) \\ -\cos(\psi_0 + \alpha_1) \end{pmatrix} - \alpha_1 \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right)\end{aligned} \tag{13}$$

with

$$\begin{aligned}A =& X_t^0 - X_p^0 - R \begin{pmatrix} -\delta_1 \sin(\psi_0) + \delta_3 \sin(\psi_t) \\ \delta_1 \cos(\psi_0) - \delta_3 \cos(\psi_t) \end{pmatrix} \\ &+ \frac{v_t}{v_p} R \delta_3 (\psi_t - \psi_0) \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix}\end{aligned} \tag{14}$$

A.2 Proof of Theorem 2

Since $\delta_1 = \delta_3 = \delta$, (7) becomes

$$t_f - t_0 = \frac{d_2}{v_p} + \delta \frac{R}{v_p} (\alpha_1 + \alpha_3)$$

and by definition

$$\alpha_1 = \psi_1 - \psi_0 \mod (2\pi)$$

$$\alpha_3 = \psi_f - \psi_1 \mod (2\pi)$$

Then, if $\delta = \delta_1 = \delta_3$, according to Theorem 1, one gets:

$$A = d_2 \left(\begin{pmatrix} \cos(\psi_0 + \alpha_1) \\ \sin(\psi_0 + \alpha_1) \end{pmatrix} - \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right) \tag{15}$$

Let us define

$$\begin{aligned}A &= [A_x, A_y]^\top \\ &= X_t^0 - X_p^0 - \delta R \begin{pmatrix} -\sin(\psi_0) + \sin(\psi_t) \\ \cos(\psi_0) - \cos(\psi_t) \end{pmatrix} \\ &\quad - \frac{v_t}{v_p} R \delta (\psi_t - \psi_0) \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix}.\end{aligned} \tag{16}$$

Note that A is a 2D vector independent from ψ_1 and d_2. Let's define $l = \frac{1}{d_2}$ and suppose $d_2 > 0$ (as it will be shown below). One gets:

$$\begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1) \end{pmatrix} = Al + \frac{v_t}{v_p} \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \tag{17}$$

By taking the square norm of (17), one gets

$$1 = \left(A_x l + \frac{v_t}{v_p} \cos(\psi_t) \right)^2 + \left(A_y l + \frac{v_t}{v_p} \sin(\psi_t) \right)^2 \tag{18}$$

$$\begin{aligned}0 =& \left(A_x^2 + A_y^2\right) l^2 + \left(\frac{v_t}{v_p} \right)^2 \\ &+ 2 \frac{v_t}{v_p} (A_x \cos(\psi_t) + A_y \sin(\psi_t)) l - 1\end{aligned} \tag{19}$$

(19) is a quadratic equation with the following discriminant

$$\begin{aligned}\Delta =& 4 \left(\frac{v_t}{v_p} \right)^2 (A_x \cos(\psi_t) + A_y \sin(\psi_t))^2 \\ &- 4 \left(A_x^2 + A_y^2\right) \left(\frac{v_t^2}{v_p^2} - 1 \right)\end{aligned} \tag{20}$$

Note that $(A_x^2 + A_y^2) = \|A\|^2$ and $(A_x \cos(\psi_t) + A_y \sin(\psi_t))^2 = \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle^2$, thus (20) can be rewritten as

$$\Delta = 4 \left(\frac{v_t}{v_p} \right)^2 \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle^2 + 4 \|A\|^2 \left(1 - \frac{v_t^2}{v_p^2} \right) \tag{21}$$

$$\Delta = 4 \left(\frac{v_t}{v_p} \right)^2 \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle^2 + \frac{4}{v_p^2} \|A\|^2 \left(v_p^2 - v_t^2 \right) \tag{22}$$

Note that $\Delta \geq 0$ as $v_p > v_t \geq 0$, so l is equal to l_1 or l_2 where

$$\begin{cases} l_1 = \dfrac{-2 \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + 2\sqrt{\Delta}}{2\|A\|^2} \\ l_2 = -\left(\dfrac{2 \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + 2\sqrt{\Delta}}{2\|A\|^2} \right) \end{cases} \tag{23}$$

Since $d_2 = l^{-1}$ is a distance, the solution must be positive. One so has $l_2 < 0$. Let's check if $l_1 > 0$. l_1 is positive iff

$$-2 \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + 2\sqrt{\Delta} > 0$$

$$\begin{aligned}&\sqrt{\left(\frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle \right)^2 + \left(\frac{\|A\|}{v_p} \right)^2 \left(v_p^2 - v_t^2 \right)} \\ &\qquad > \frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle\end{aligned}$$

$$\sqrt{B^2 + C} > B$$

with $C = \left(\frac{\|A\|}{v_p} \right)^2 \left(v_p^2 - v_t^2 \right) > 0$ since $v_p > v_t$ and $B = \left(\frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle \right)^2$. Since $C > 0$, one has $\sqrt{B^2 + C} > \sqrt{B^2} > B$. So $l_1 > 0$ and so $d_2 = l_1^{-1}$ is positive and can be expressed as

$$d_2 = \frac{\|A\|^2}{-\frac{v_t}{v_p} \left\langle A, \begin{pmatrix} \cos(\psi_t) \\ \sin(\psi_t) \end{pmatrix} \right\rangle + \sqrt{\Delta}}. \tag{24}$$

Let's go back to (17). (17) can be split in two lines:

$$\left\{ \begin{array}{l} \cos(\psi_1) = \frac{A_x}{d_2} + \frac{v_t}{v_p}\cos(\psi_t) \\ \sin(\psi_1) = \frac{A_y}{d_2} + \frac{v_t}{v_p}\sin(\psi_t) \end{array} \right. \tag{25}$$

From (25), one gets

$$\psi_1 = \arctan 2\left(\frac{A_y}{d_2} + \frac{v_t}{v_p}\sin(\psi_t), \frac{A_x}{d_2} + \frac{v_t}{v_p}\cos(\psi_t)\right). \tag{26}$$

A.3 Proof of Theorem 3

If $\delta = \delta_1 = -\delta_3$, according to Theorem 1, one gets

$$t_f - t_0 = \frac{d_2}{v_p} + \delta\frac{R}{v_p}(\alpha_1 - \alpha_3)$$

$$\alpha_1 = \psi_1 - \psi_0 \mod (2\pi)$$

$$\alpha_3 = \psi_f - \psi_1 \mod (2\pi)$$

Then, with $\delta = \delta_1 = -\delta_3$, (13) becomes

$$\begin{aligned} 0 = d_2 &\left(\begin{pmatrix}\cos(\psi_0+\alpha_1) \\ \sin(\psi_0+\alpha_1)\end{pmatrix} - \frac{v_t}{v_p}\begin{pmatrix}\cos(\psi_t) \\ \sin(\psi_t)\end{pmatrix}\right) \\ &+ 2R\delta\left(\begin{pmatrix}\sin(\psi_0+\alpha_1) \\ -\cos(\psi_0+\alpha_1)\end{pmatrix} - \alpha_1\frac{v_t}{v_p}\begin{pmatrix}\cos(\psi_t) \\ \sin(\psi_t)\end{pmatrix}\right) - A \end{aligned} \tag{27}$$

with

$$\begin{aligned} A = X_t^0 - X_p^0 - R\delta &\begin{pmatrix}-\sin(\psi_0) - \sin(\psi_t) \\ \cos(\psi_0) + \cos(\psi_t)\end{pmatrix} \\ &- \frac{v_t}{v_p}R\delta(\psi_t - \psi_0)\begin{pmatrix}\cos(\psi_t) \\ \sin(\psi_t)\end{pmatrix}. \end{aligned} \tag{28}$$