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These two objectives may seems quite far one from the other but actually they can be linked by... **Quadtrees**.

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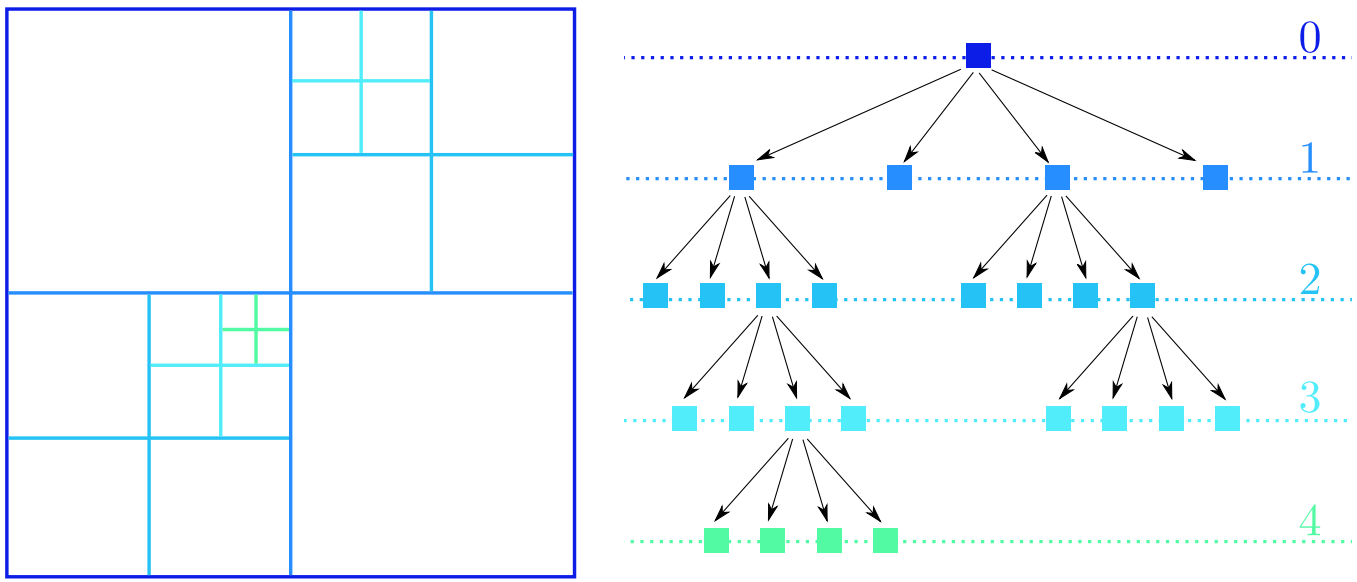

Image from: T. Bellotti, M. Theillard - A coupled level-set and reference map method for interface representation with applications to two-phase flows simulation - 2018 in press: J. Comput. Phys.

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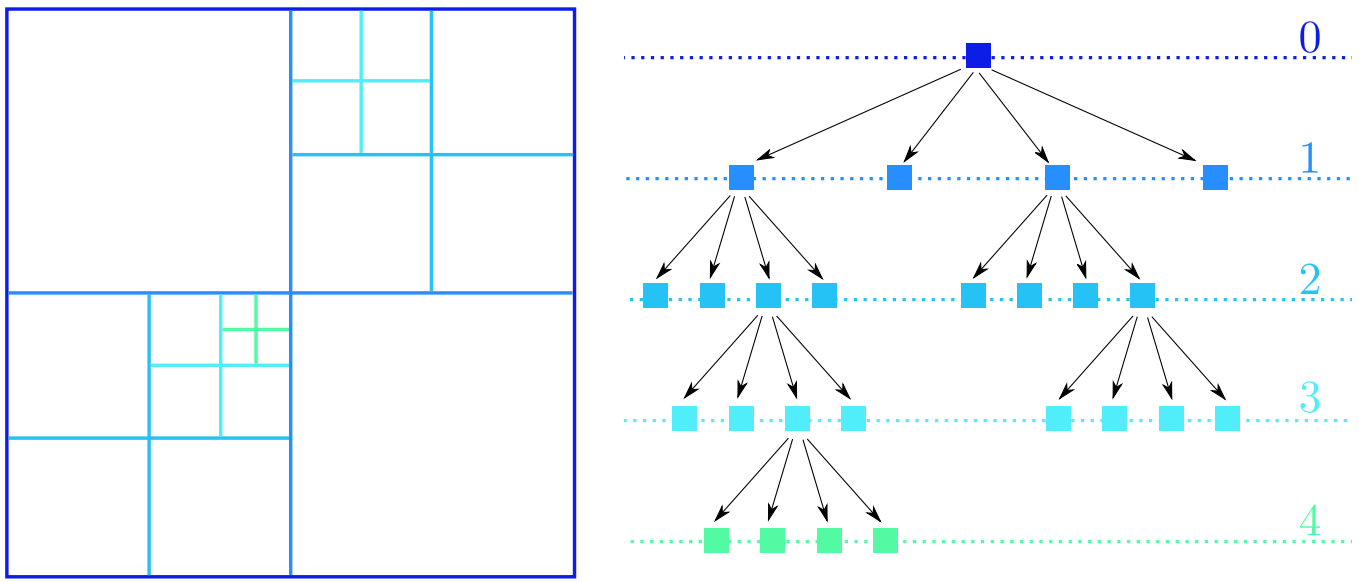

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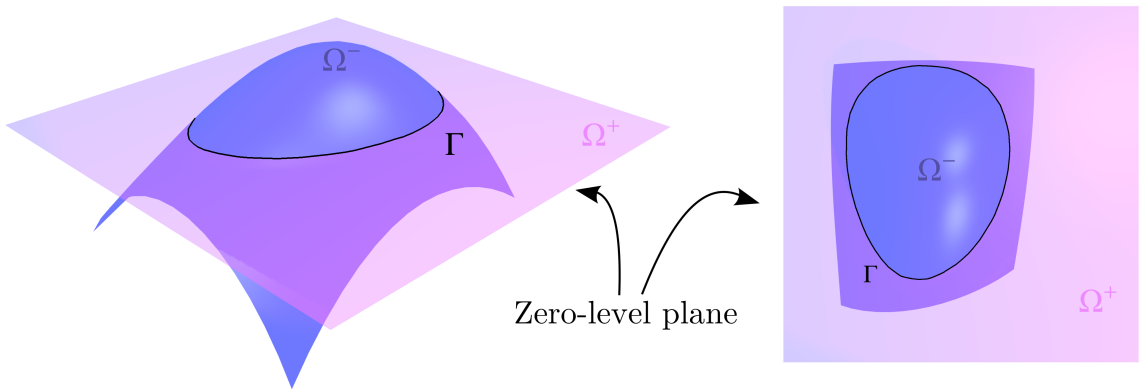
It has a clear geometrical counterpart.

Some terminology

- Leaf.
- Level
- Maximum level
- Minimum level.

Level-set theory

In this work, we only see it as a **strategy to build** beautiful and meaningful **adaptive meshes** (there is more than this).



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$$\Gamma = \{\mathbf{x} \in \Omega \ : \ \phi(\mathbf{x}) = 0\},$$
$$\Omega^- = \{\mathbf{x} \in \Omega \ : \ \phi(\mathbf{x}) < 0\},$$
$$\Omega^+ = \{\mathbf{x} \in \Omega \ : \ \phi(\mathbf{x}) > 0\}.$$

Level-set theory

It is probably the easiest way of **representing an interface** in a computationally efficient manner.

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Example

In 2D, a circle centered in (x_0, y_0) with radius R can be represented by:

$$\phi(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2} - R.$$

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Notice that, given Γ, the level-set is not unique, but this is not our problem now. It is unique if we assume (and we will do so) that $\phi(x, y)$ is nothing but the **signed distance** of (x, y) from Γ.

$$\text{split } \mathcal{C} \text{ if } : \quad |\phi(\mathbf{x}_\mathcal{C})| \leq \text{Lip}(\phi) \cdot \text{diag}(\mathcal{C}) \qquad \text{and} \qquad \text{level}(\mathcal{C}) \leq \max_{\text{level}},$$

This naively means that we split a cell wheather its diagonal length exceeds the distance from the interface Γ represented by the level-set ϕ.

Gaussian quadrature

Considering a stardard quadrilateral domain $\Omega = [-1, 1]^2$ and given a possibly smooth function $f : \Omega \to \mathbb{R}$, we want to compute:

$$\int_\Omega f(x, y)\, dxdy.$$

This can be done by quadrature formulae, based on simple evaluation of the function f at certain points of the domain. For our purpose, we use two choices:

1. **"Naive" formula**, with only one function evaluation.

$$\int_\Omega f(x, y)\, dxdy \simeq |\Omega| f(0, 0) = 4f(0, 0)$$

2. **3rd order Gaussian formula**, with nine function evaluations (plus extra computations to adapt to a generic domain).

$$\begin{aligned}\int_\Omega f(x, y)\, dxdy \simeq \frac{5}{81}&\left[5f\left(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) + 8f\left(0, -\sqrt{\frac{3}{5}}\right) + 5f\left(\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right)\right] \\ &+ \frac{8}{81}\left[5f\left(-\sqrt{\frac{3}{5}}, 0\right) + 8f\left(0, 0\right) + 5f\left(0, \sqrt{\frac{3}{5}}\right)\right] \\ &+ \frac{5}{81}\left[5f\left(-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right) + 8f\left(0, \sqrt{\frac{3}{5}}\right) + 5f\left(\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right)\right].\end{aligned}$$

Rgb colors and their "distance"

It is probably the simplest way of representing colors, as a combination of three components, **red, green and blue**. Thus, a color C is nothing else than a triplet:

$$C = [C_{\text{red}}, C_{\text{green}}, C_{\text{blue}}],$$

where $C_{\text{red}}, C_{\text{green}}, C_{\text{blue}} \in \{0, \ldots, 255\}$. In this way, we can represent 16'777'216 different tones.
For our purpose, it is useful to define the notion of **distance** between two colors, which can be defined in many ways. In our case, we use the simple "corrected" formula:

$$d(C^1, C^2) = \sqrt{2\left(C^1_{\text{red}} - C^2_{\text{red}}\right)^2 + 4\left(C^1_{\text{green}} - C^2_{\text{green}}\right)^2 + 3\left(C^1_{\text{blue}} - C^2_{\text{blue}}\right)^2}.$$

The mean color between $\left\{C^1, \ldots, C^N\right\}$ is defined by:

$$\overline{C}\left(\left\{C^1, \ldots, C^N\right\}\right) = \left[\frac{1}{N}\sum_{n=1}^{N} C^n_{\text{red}}, \frac{1}{N}\sum_{n=1}^{N} C^n_{\text{green}}, \frac{1}{N}\sum_{n=1}^{N} C^n_{\text{blue}}\right],$$

and a sort of standard deviation as:

$$\sigma\left(\left\{C^1, \ldots, C^N\right\}\right) = \frac{1}{N}\sqrt{\sum_{n=1}^{N} d(C_n, \overline{C})^2} \in [0, 765].$$

How to compress an image?

Let $\mathcal{P}$ be a preleave whose children are $\mathcal{L}(\mathcal{P})$. We define a tolerance $0 < \epsilon \ll 1$. Then:

$$\text{merge} \quad \mathcal{P} \quad \text{if} \ : \ \sigma\left(\mathcal{L}(\mathcal{P})\right) \leq 765 \cdot \epsilon.$$

In the case of merging, we consider:

$$C_{\mathcal{P}} = \overline{C}\left(\mathcal{L}(\mathcal{P})\right).$$

The Cell class

	Cell<T>
#	const Point<T> base_point
#	const T dx
#	const T dx
#	const unsigned char level
#	std::shared_ptr<Cell> l_l
#	std::shared_ptr<Cell> l_r
#	std::shared_ptr<Cell> u_l
#	std::shared_ptr<Cell> u_r
+	Cell(Point<T>, T, T, unsigned char)
+	virtual ~Cell()
+	Point<T> getBasePoint() const
+	unsigned char getLevel() const
+	bool isLeaf() const
+	Point<T> getCenter() const
+	T getDx() const
+	T getDy() const
+	T getDiagonal() const
+	T cellSurface() const
+	virtual void splitCell()
+	virtual void mergeCell()
+	void refineCell(const RefinementCriterion<T> & , const unsigned char)
+	void simplifyCell(const RefinementCriterion<T> &, const unsigned char)
+	void updateCell(const RefinementCriterion<T> &, const unsigned char, const unsigned char)
+	std::vector<Point<T>> getVertices() const
+	void getLeaves(std::vector<std::shared_ptr<Cell<T>>> &)
+	void getPreLeaves(std::vector<std::shared_ptr<Cell<T>>> &) const
+	std::string tikzDot() const
+	std::string tikzSquare(const RGBColor color, const bool) const
+	T zeroOrderIntegration(const std::function<T(Point<T>)> &) const
+	T thirdOrderGaussianIntegration(const std::function<T(Point<T>)> &) const

The Pixel class inheriting from Cell

The “generalized” pixel is basically a Cell...

	Pixel<T> : public Cell<T>
#	RGBColor field
+	Pixel(Point<T>, T, T, unsigned char)
+	Pixel(Point<T>, T, T, unsigned char, const RGBColor &)
+	virtual ~Pixel()
+	void setField(const RGBColor &)
+	RGBColor getField() const
+	virtual void splitCell() override
+	RGBColor meanField()
+	double stdDevField()
+	virtual void mergeCell() override

plus a **color**. We have to be careful to **cast pointers** when we need to extract information from the Pixel.

The QuadTree class

This class is mostly a **wrapper** of the Cell class.

	QuadTree <T>
#	const T x_size
#	const T y_size
#	unsigned char min_level
#	unsigned char max_level
#	std::shared_ptr<Cell<T>> parent_cell
#	std::vector<std::shared_ptr<Cell<T>>> getLeaves() const
+	QuadTree(Point<T>, T, T, unsigned char, unsigned char)
+	virtual ~QuadTree()
+	T simpleIntegration(std::function<T(Point<T>)> &) const
+	unsigned getMinLevel() const
+	unsigned getMaxLevel() const
+	size_t numberOfLeaves() const
+	void buildUniform()
+	void buildUniform(unsigned)
+	void clear()
+	void updateWithLevelSet(const LipschitzFunction<T> &)
+	void updateQuadTree(const RefinementCriterion<T> &)
+	void updateQuadTree(const RefinementCriterion<T> &, const unsigned char, const unsigned char)
+	std::vector<Point<T>> getCenters()
+	void exportCentersTikz(const std::string &) const
+	void exportMeshTikz(const std::string &, bool) const
+	T simpleIntegration(const std::function<T(Point<T>)> &) const
+	T thirdOrderGaussianIntegration(const std::function<T(Point<T>)> &) const
+	T simpleIntegration(const std::function<T(std::shared_ptr<Cell<T>>)> &) const

The Image class

This class is mostly a **wrapper** of the Pixel class. We distinguished it from the QuadTree class (it does not inherit from it) because there are many features that they do not share.

	Image<T>
#	T x_size
#	T y_size
#	unsigned char min_level
#	unsigned char max_level
#	std::shared_ptr<Pixel<T>> parent_cell
+	Image()
+	Image(T, T, unsigned char, unsigned char)
+	~Image()
+	unsigned int getMinLevel() const
+	unsigned int getMaxLevel() const
+	size_t numberOfPixels() const
+	void clear()
+	void simplifyImage(double)
+	void buildUniform(unsigned char)
+	void createFromFile(std::string)
+	void saveImage(const std::string & filename) const

The way we update the mesh: the RefinementCriterion class

This is a very simple **abstract class** with an operator telling us if we have to split a Cell or not.

	RefinementCriterion <T>
+	RefinementCriterion()
+	virtual ~RefinementCriterion()
+	virtual bool operator()(std::shared_ptr<Cell<T>>) const = 0

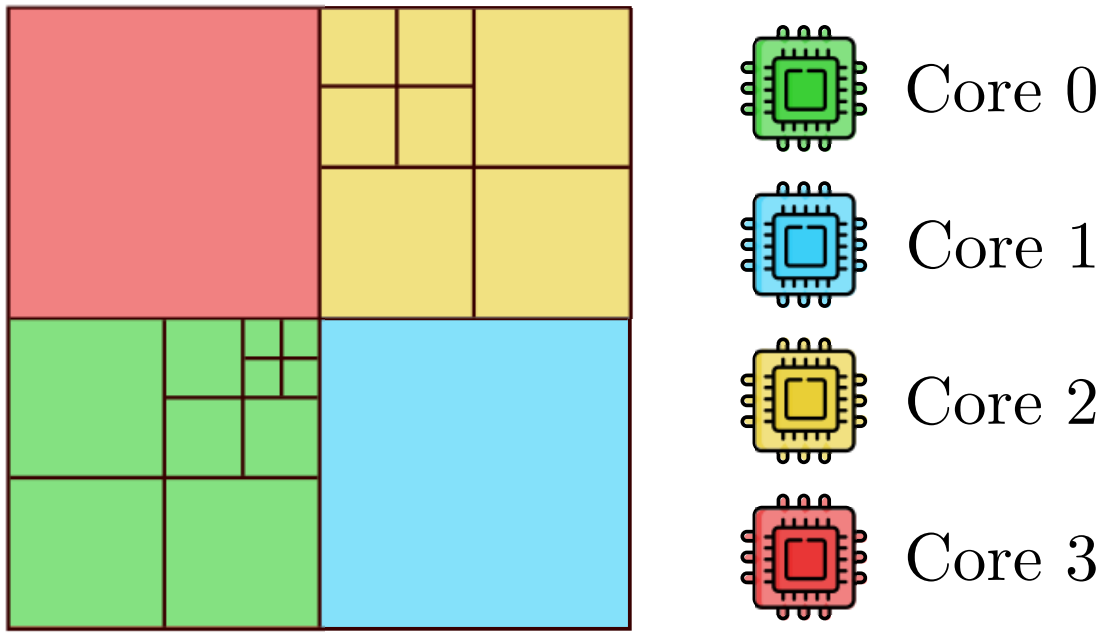
Many important criteria inherit from it:

	RefineAlwaysCriterion<T> : public RefinementCriterion<T>
+	RefineAlwaysCriterion()
+	virtual ~RefineAlwaysCriterion()
+	virtual bool operator()(std::shared_ptr<Cell<T>>) const override

	LevelSetCriterion<T> : public RefinementCriterion<T>
-	const LipschitzFunction<T> & level_set
+	LevelSetCriterion(const LipschitzFunction<T> &)
+	virtual ~LevelSetCriterion()
+	virtual bool operator()(std::shared_ptr<Cell<T>>) const override

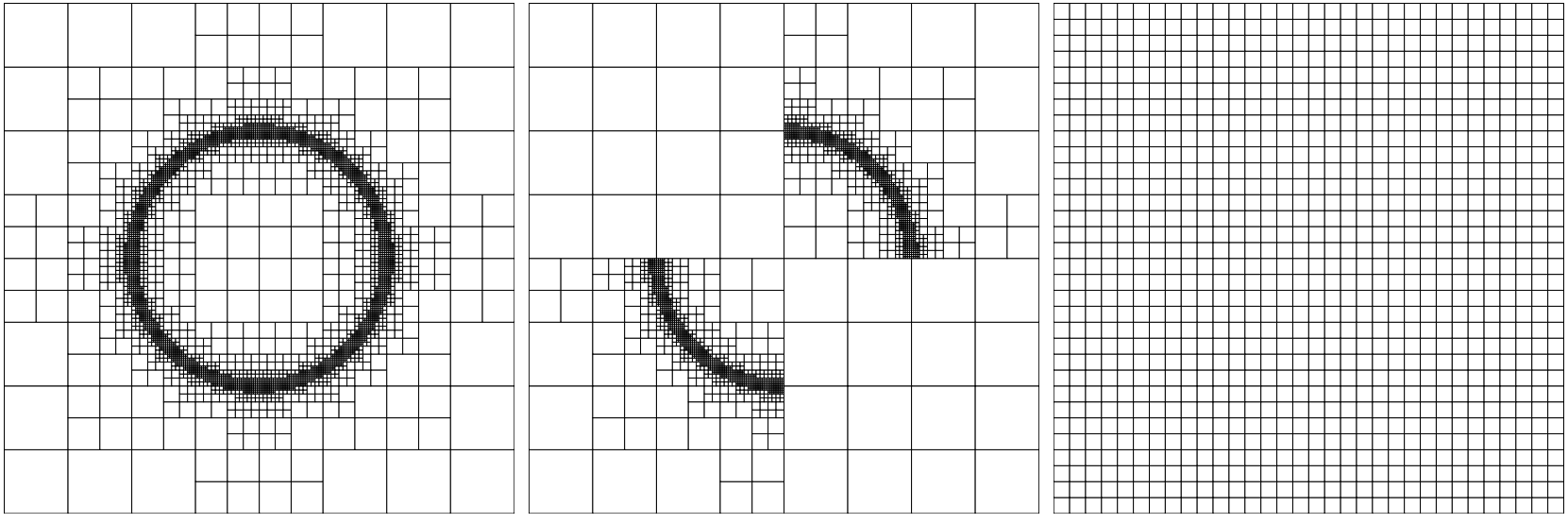
	CriterionVariance<T> : public RefinementCriterion<T>
-	double thr
+	CriterionVariance()
+	virtual ~CriterionVariance()
+	virtual bool operator()(std::shared_ptr<Cell<T>>) const override

The key idea is to have a number of core which is a power of 4 and have a minimum level large enough, so that we can avoid communications between cores and each of them has a local tree (no shared memory).



And each subtree (core) integrates independently. At the very end, the sub-integrals are summed with a **reduce** procedure.

$$\Omega = [-2, 2]^2 \quad \phi(x, y) = \sqrt{x^2 + y^2} - 1 \quad f(x, y) = \mathbb{I}_{\{\phi(x,y) \leq 0\}} \quad \int_\Omega f(x, y)\, dxdy = \pi.$$

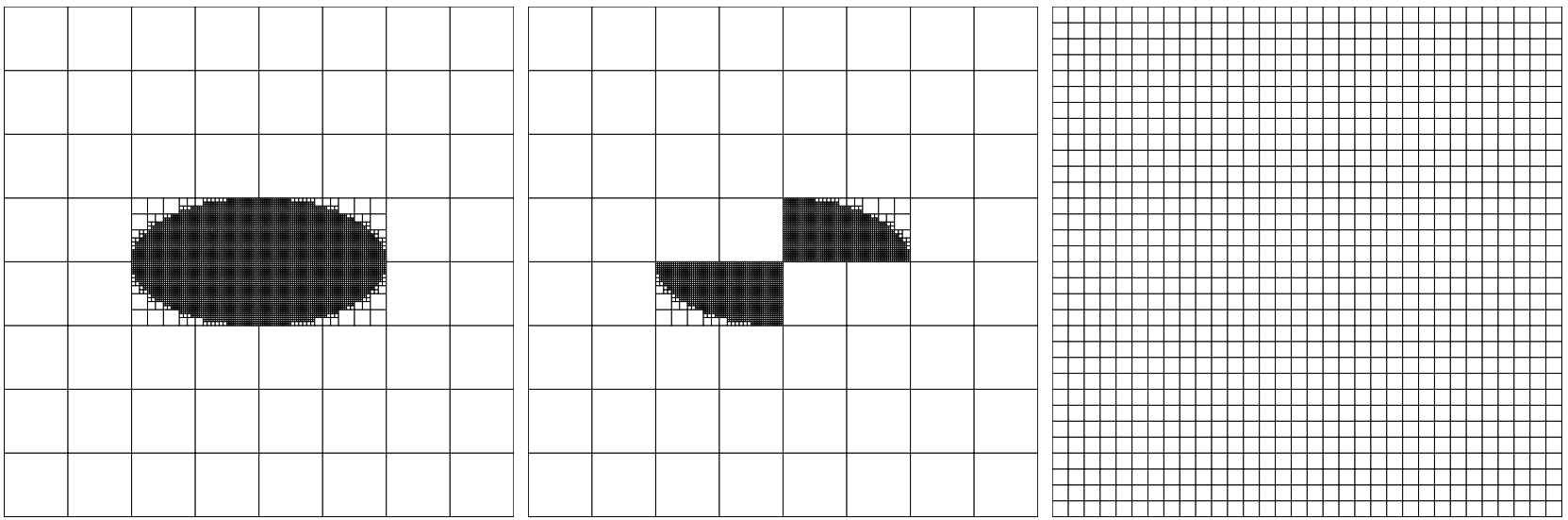


Naive	Mesh 1		Mesh 2		Mesh 3	
# of cores	Time [s]	Speedup	Time[s]	Speedup	Time [s]	Speedup
1	10.2	-	76.3	-	83.9	-
4	10.2	**54.4**	76.3	**45.2**	83.9	**54.5**
16	10.2	**87.0**	76.3	**43.2**	83.9	**0.34**

3rd Gaussian	Mesh 1		Mesh 2		Mesh 3	
# of cores	Time [s]	Speedup	Time[s]	Speedup	Time [s]	Speedup
1	10.2	-	76.3	-	83.9	-
4	10.2	**54.4**	76.3	**45.2**	83.9	**54.5**
16	10.2	**87.0**	76.3	**43.2**	83.9	**0.34**

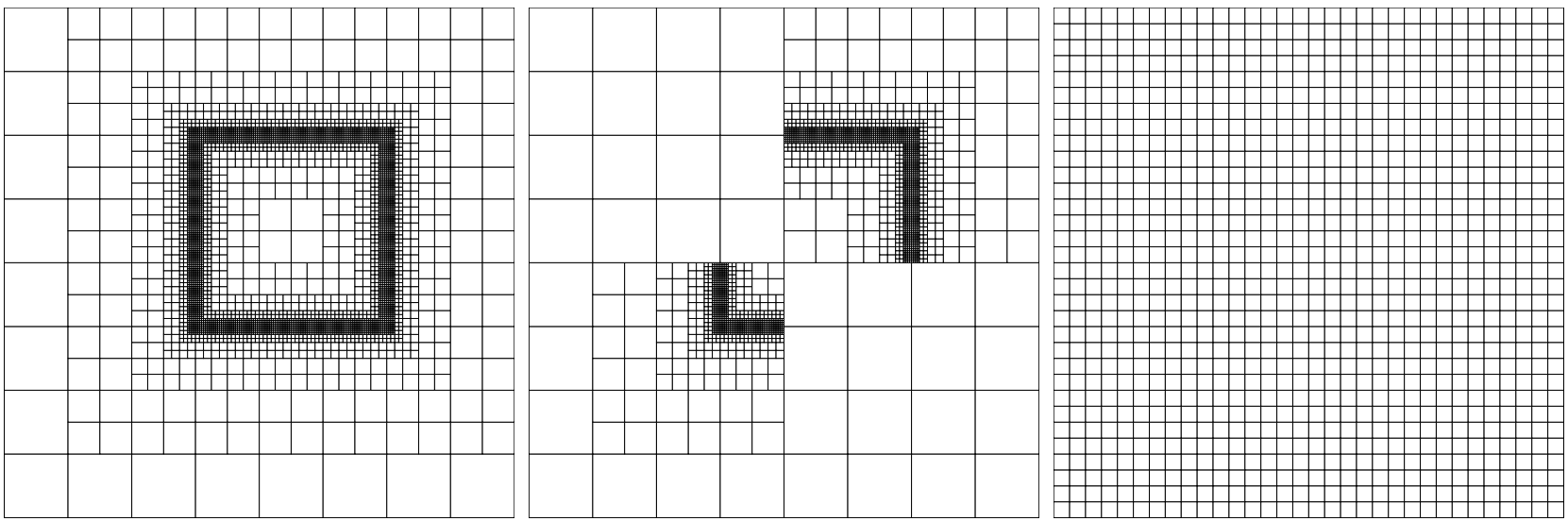
$$\Omega = [-2, 2] \quad f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)} \quad \sigma_x = 0.1 \quad \sigma_y = 0.05 \quad \int_\Omega f(x, y)\, dxdy \simeq 1.$$

We refine in the ellipse within 10 standard deviations.



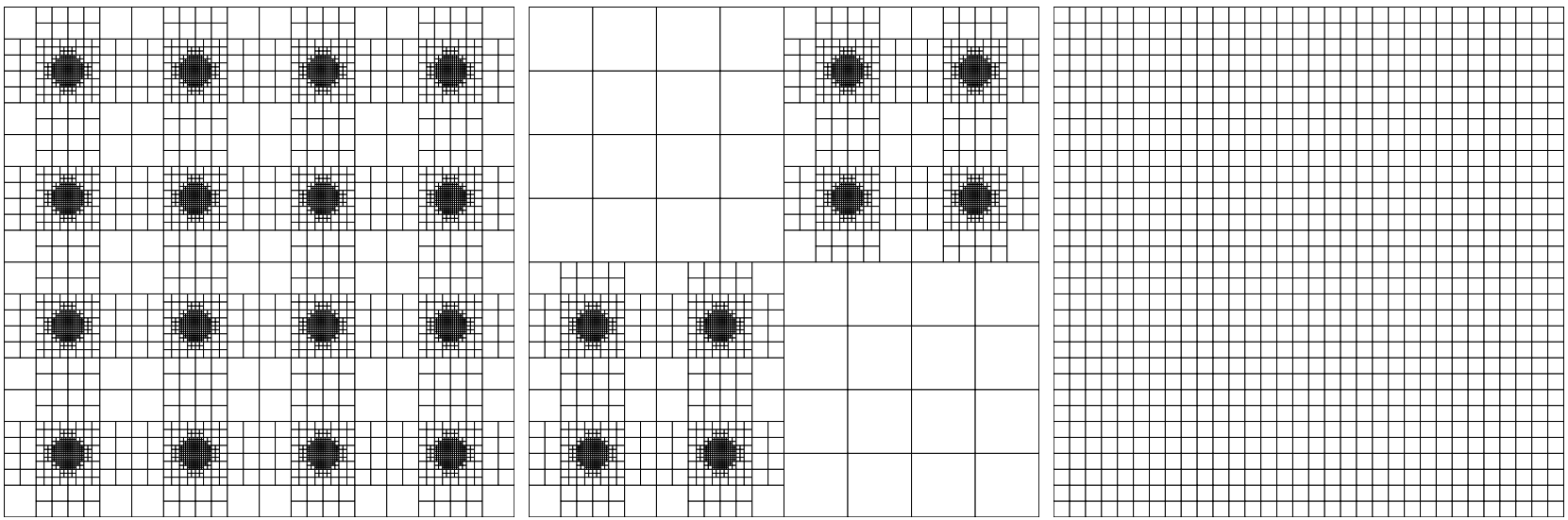
$$\Omega = [-2, 2]^2 \quad \phi(x, y) = \max\{|x - 0.25| - 0.75, |y - 0.25| - 0.75\}$$

$$f(x, y) = (x^2 + y^2)\left[\cos(\pi x) + \sin(\pi y)\right] \mathbb{I}_{\{\phi(x,y) \leq 0\}} \quad \int_\Omega f(x, y)\, dxdy = \frac{3(-16 - 12\pi + 7\pi^2)}{8\pi^3} \simeq 0.186109$$



$$\Omega = [0,8]^2 \quad \phi(x,y) = \min_{i,j=0,\dots,3} \left(\sqrt{(x-(2i+1))^2 + (y-(2j+1))^2} - 0.15 \right) \quad f(x,y) = \mathbb{I}_{\{\phi(x,y) \leq 0\}}$$

$$\int_\Omega \psi(x,y)\,dxdy = \frac{9\pi}{25}$$



Three different ϵ: $\epsilon = 0.012, 0.024, 0.048$.

Original figures size: 262144 px.



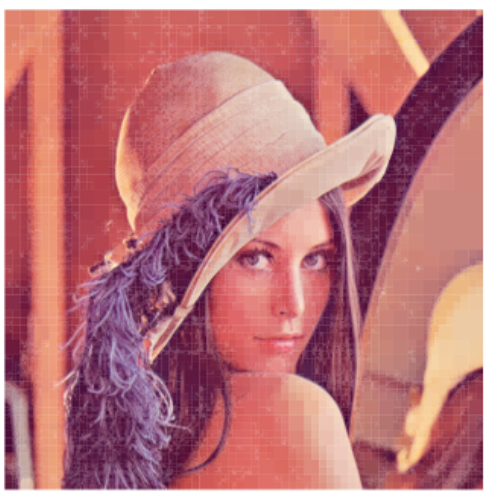
131509 px - comp. ratio = 1.99 84808 px - comp. ratio = 3.09 38152 px - comp. ratio = 6.87

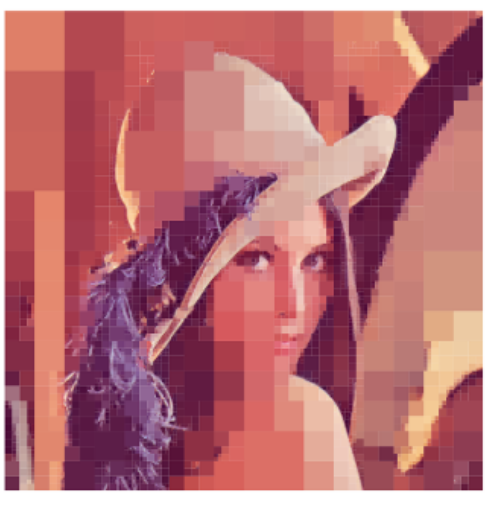


115438 px - comp. ratio = 2.27 69259 px - comp. ratio = 3.78 27394 px - comp. ratio = 9.57



32188 px - comp. ratio = 8.14 16405 px - comp. ratio = 15.98 6571 px - comp. ratio = 39.89





75172 px - comp. ratio = 3.49 29890 px - comp. ratio = 8.77 8902 px - comp. ratio = 29.45



171517 px - comp. ratio = 1.53 85258 px - comp. ratio = 3.07 24466 px - comp. ratio = 10.71