SF1686 Flervariabelanalys
Exam
Wednesday, October 21, 2020

Time: 8:00-11:00
No books/notes/calculators etc. allowed
Examiner: John Andersson and Henrik Shahgholian

This exam consists of three parts, each worth 12 points. The bonus points from the seminars will be automatically added to the total score of part A, which however cannot exceed 12 points. The problems in part C are mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

For full score on any question in part B and part C of the exam, you have to provide a full, motivated, well-presented and easy to follow solution. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument.

Part A.

For questions 1-6 in PART A you should state all correct/true options/statements in your answer sheet. Only answers should be on your answer sheet. No justifications are taken into account.

Each problem has a maximum of 2 points. For each wrong/missing option in each problem one point will be subtracted.

Question 1. Let $f(x,y) = xe^y + y^2$. Find <u>all</u> true statements.

A) The directional derivative at $(x,y) = (1,1)$ in the direction $\frac{1}{\sqrt{5}}(2,-1)$ is calculated

$$(1,1)\cdot\nabla f\left(\frac{2}{\sqrt{5}},\frac{-1}{\sqrt{5}}\right).$$

B) The directional derivative at $(x,y) = (0,1)$ in the direction $\nu = \frac{1}{\sqrt{2}}(1,1)$ is $\frac{e+2}{\sqrt{2}}$.

C) The directional derivative at $(x,y) = (0,1)$ in the direction $\nu = \frac{1}{\sqrt{2}}(1,1)$ is $\frac{e+3}{\sqrt{2}}$.

Question 2. Let C be the curve parametrized by $(x(t),y(t)) = \left(\frac{3}{2}t^2, t^3\right)$ for $a \le t \le b$. For which values of a and b is the arc-length equal to 1. Find <u>all</u> correct alternatives.

A) $a = \sqrt{3^{1/3}-1}$ and $b = \sqrt{4^{1/3}-1}$.

B) $a = \sqrt{3^{2/3}-1}$ and $b = \sqrt{4^{2/3}-1}$.

C) $a = \sqrt{2^{2/3}-1}$ and $b = \sqrt{3^{2/3}-1}$.

Question 3. Let
$$I = \int\int\int_V \left(x(x^2+z^2) + 2x^2 - y^2 \right) dV$$
where
$$V = \{(x,y,z);\ x^2+y^2+z^2 \le 1\}.$$
Find all correct answers.

A) $I = 4\pi$, B) $I = \frac{15}{4}\pi$, C) $I = \frac{4\pi}{3}\pi$, D) $I = \frac{4\pi}{15}$, E) The integral I is divergent.

Question 4. Let $\mathcal{C}$ be the curve defined implicitly by
$$f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{3}y^3 + \frac{1}{2}y^2 = 2.$$
Mark all points (x,y) where $\mathcal{C}$ has a normal pointing in the same direction as $(1,1)$.

A) $(x,y) = (2,0)$ B) $(x,y) = (1,0)$ C) $(x,y) = (0,1)$

D) $(x,y) = (0,2)$ E) $(x,y) = (-2,0)$.

Question 5. Let
$$I = \int_0^{\sqrt{3}} \left(\int_{y/\sqrt{3}}^{\sqrt{4-y^2}} e^{-x^2-y^2} dx \right) dy.$$
Find all correct answers.

A) $I = \int\int_D e^{-x^2-y^2} dxdy$ where D is a rectangular area in $\mathbb{R}^2$.

B) $I = \int_0^1 \left(\int_0^{\sqrt{3}x} e^{-x^2-y^2} dy \right) dx$.

C) $I = \int_0^{\pi/3} \left(\int_0^2 e^{-r^2} r dr \right) d\theta$.

D) $I = \frac{\pi}{6}(1-e^{-4})$.

Question 6. Consider the equation $y^5 + xy = 4$. Find all correct statements.

A) The points $(x,y) = (3,1)$ and $(x,y) = (-5,-1)$ are solutions to the equation.

B) In a small enough region around $(3,1)$ we can write the solutions to the equation as $(x, f(x))$ for some function $f(x)$ where $f(3) = 1$ and $f(x)$ is differentiable in $x = 3$.

C) If the function f from the previous point exists then $f'(3) = -1/2$.

Part B.

Question 7. The hull of a certain submarine can be described as all points in an ellipsoid

$$\frac{x^2}{10^4} + \frac{y^2}{10^2} + \frac{z^2}{10^2} \leq 1.$$

The formula is interpreted in the unit meter so that the submarine is 200m long.

When the submarine descends to the depth $d \geq 0$ the maximal external pressure P_e at a point of the hull, expressed in the unit of atmospheric pressures a.p., as

$$P_e = \frac{d}{10}.$$

To balance the external pressure the captain of the boat can release $R \geq 0$ cubic meters of air at atmospheric pressure. The internal pressure P_i of the submarine can then be calculated by

$$P_i = \frac{R}{V},$$

where V is the volume of the hull in cubic meters.

The hull will be destroyed (either be crushed or burst) if

$$|P_e - P_i| \geq 10.$$

a) Find the volume V of the submarine.

[3 marks]

b) Calculate the least volume R that the captain must release at the depth $d \geq 0$ so that the submarine is not destroyed.

[3 marks]

Question 8. Consider the vector field $\mathbf{F}(x, y, z) = (y^2 + \alpha z, 2xy, 3z^2 - x)$ on $\mathbb{R}^3$ where $\alpha \in \mathbb{R}$ is a fixed constant.

(a) Given an open and connected domain D and a continuously differentiable vectorfield **F**. State conditions that assures that **F** is conservative (i.e. has a potential function).

[1 marks]

(b) Decide for which α the vector field **F** is conservative and calculate a corresponding potential.

[3 marks]

(c) Calculate the work done by the vector field (with any α from the previous part of the question) along the curve

$$\gamma(\theta) = \left(3\cos(\theta), 2\sin(\theta), \frac{\theta}{2\pi}\right) \qquad \text{where } 0 \leq \theta \leq 2\pi.$$

[2 marks]

Part C.

Question 9. Calculate the circulation of $\mathbf{F} = (-y^2 + e^z, x, z + \sin(x^2))$ around the oriented surface given by

$$x^2 + y^2 + 4(z-1)^2 = 20 \text{ and } z \geq 0,$$

with normal pointing in toward the $z-$axis.

[6 marks]

Question 10. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x-y}{(x+y)^3} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Furthermore we let $D = [0,2] \times [0,2]$.

(a) Is f continuous on D? Motivate your answer!

[2 marks]

(b) Is f Riemann integrable on D. You must prove your answer.

[4 marks]