SF1685: Calculus

Inverse trigonometric functions and their derivatives

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A few more limits

$$\lim_{x\to\infty} \frac{x^a}{e^x} = 0 \qquad \text{for any } a$$

$$\lim_{x\to\infty} \frac{\log(x)}{x^a} = 0 \qquad \text{for any } a > 0$$

$$\lim_{x\to 0^+} x^a \log(x) = 0 \qquad \text{for any } a > 0$$

We remember the inverse trigonometric functions,

$\boxed{\sin^{-1}(x)}$

$\cos^{-1}(x)$

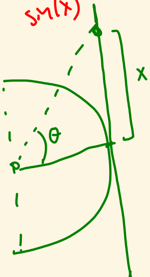
$\tan^{-1}(x)$

$$\left.\begin{aligned} \arcsin(x) &: [-1,1] \to [-\pi/2, \pi/2] \\ \arccos(x) &: [-1,1] \to [0, \pi] \end{aligned}\right] \text{By definition!}$$

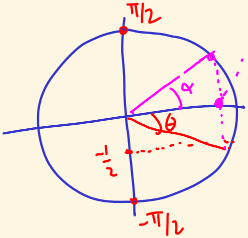
$$\arctan(x) : (-\infty, \infty) \to (-\pi/2, \pi/2)$$

$\sin^2(x) = (\sin(x))^2$

$\sin^{-1}(x) \stackrel{?}{=} \frac{1}{\sin(x)}$ Not the case!



$\arctan(x) = \theta$



$\arccos(\sqrt{3}/2) = \alpha$

$\arcsin(-1/2) = \theta$

Derivative of arcsin

Let $-1 < x < 1$. Then

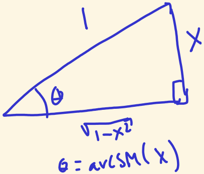
$$\sin(\arcsin(x)) = x,$$

so

$$\cos(\arcsin(x))\underline{D[\arcsin(x)]} = 1$$

Now, $\cos(\arcsin(x)) \geq 0$ (why?), so $\cos(\arcsin(x)) = \sqrt{1-x^2}$. Hence,

$$D[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}.$$



Derivative of arccos

Let $-1 < x < 1$. Then

$$\cos(\arccos(x)) = x,$$

so

$$-\sin(\arccos(x))D[\arccos(x)] = 1$$

Now, $\sin(\arccos(x)) \geq 0$ (why?), so $\sin(\arccos(x)) = \sqrt{1 - x^2}$. Hence,

$$D[\arccos(x)] = -\frac{1}{\sqrt{1 - x^2}}.$$

Derivative of arctan

Recall that

$$D[\tan(x)] = \frac{1}{\cos^2(x)} = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = 1 + \tan^2(x).$$

Thus, taking derivatives on both sides of $\tan(\arctan(x)) = x$, we have that

$$\underbrace{(1 + \tan^2(\arctan(x)))}_{1+x^2} \cdot D[\arctan(x)] = 1,$$

so

$$D[\arctan(x)] = \frac{1}{1 + x^2}.$$

Extremely important!

$$\int \frac{2}{4x^2+3}\,dx = \text{Some arctan thing} \ldots + C$$

Hyperbolic functions

$D[e^{kx}] = k \cdot e^{kx}$

We define

$$\sinh(x) := \frac{e^x - e^{-x}}{2} \qquad \cosh(x) := \frac{e^x + e^{-x}}{2}.$$

These are called **hyperbolic sine** and **hyperbolic cosine**.

Question

Compute $\sinh^2(x) - \cosh^2(x)$. $= -1$

$\left[\cosh^2(x) - \sinh^2(x) = 1\right]$

compare w. $\cos^2 x + \sin^2 x = 1$

Question

Compute $D[\sinh(x)]$ and $D[\cosh(x)]$.

- $D[\sinh x] = \cosh(x)$
- $D[\cosh x] = \sinh(x)$

- $D[\sin x] = \cos x$
- $D[\cos x] = -\sin x$

Notes

$$\int \frac{1}{x^2+2^2}\,dx = \left\{ \begin{bmatrix} x = 2\sinh t \\ dx = 2\cdot\cosh t\,dt \end{bmatrix} \right.$$

Might not be the best approach ...

$$\cosh^2(x) = \sinh^2(x) + 1$$

$$(2\cosh(x))^2 = (2\sinh x)^2 + 2^2$$

$$\int \frac{1}{(2\sinh t)^2+2^2}\cdot 2\cosh t\,dt = \int \frac{\cancel{2\cosh t}}{(2\cosh t)^{\cancel{2}}}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\cosh t}\,dt$$

Ok let's do something with complex numbers

Recall,

$$\sinh(x) := \frac{e^x - e^{-x}}{2} \qquad \cosh(x) := \frac{e^x + e^{-x}}{2}.$$

Moreover, you might also remember that $e^{ix} = \cos(x) + i\sin(x)$.

Question

Compute $\sinh(ix)$ and $\cosh(ix)$.

$re^{i\theta} = r\cdot\cos(\theta) + r\cdot i\cdot\sin(\theta)$

$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = \frac{[\cancel{\cos(x)} + i\sin(x)] - [\cancel{\cos(x)} - i\sin(x)]}{2}$$

$$= i\cdot\sin(x).$$

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \frac{(\cos(x) + \cancel{i\sin x}) + (\cos x - \cancel{i\sin x})}{2} = \cos(x).$$

Differential equations

Why differential equations?

- Physics — physicists love differential equations.
- Engineering in general (finite element method).
- Finance — Black–Scholes model, pricing stock options.
- Simulation of water, wind and nature (good-looking computer science graphics!)
- See also: Wikipedia, *List of named differential equations*.
- My most recent experience: Counting problems in combinatorics.

Sparkonto

Avanza

Youtube

2 minute papers

Notable history, about the 3-body problem

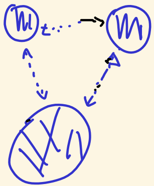
In 1887, in honour of his 60th birthday, Oscar II, King of Sweden, advised by *Gösta Mittag-Leffler*, established a prize for anyone who could find the solution to the 3-body problem. Prize was awarded to *Henri Poincaré*, 1889.

Special relativity theory,

Poincaré conj.

Perelman solved this.

positions, velocities

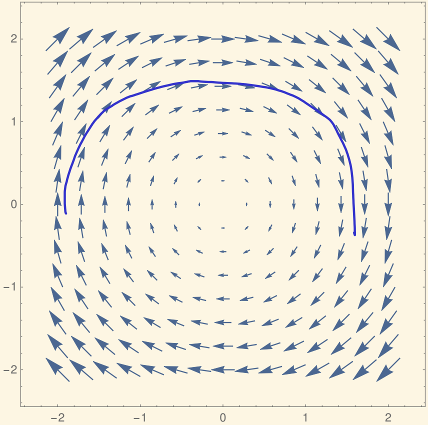


Butterfly effect,
Small changes
turn into
large differences
over time.

Chaos theory
Butterfly effect
Dynamical systems
[Lorenz attractor]
Diff. eq.

A differential equation by picture

Let us consider $y' = -\frac{x}{y}$, and suppose $y_1(x)$ is a solution. This means that the **tangent** to $y_1(x)$ at (x, y) has slope $-x/y$.



Circle?

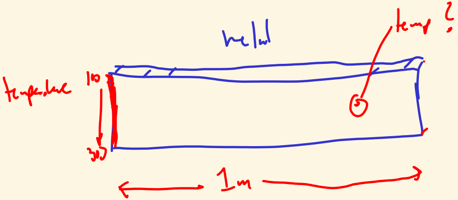
Differential equations I

Easy one: $y' = y$. Usually, we want something like $y(0) = A$.

$y(x) = e^x$ General solution : $y(x) = C \cdot e^x$

C some constant. (depends on bdd cond)

$y(x) \equiv 0$



Heat equation, diff. eq.

Differential equations II

What solutions do we have to the equation

$$y' = ry$$

where r is a constant?

$$D[\underbrace{e^{r\cdot x}}_{y}] = r\cdot \underbrace{e^{r\cdot x}}_{y}$$

General solution: $\underline{C\cdot e^{r\cdot x}}$.

Linear differential equations

A (homogeneous) **linear differential equation** can be expressed in the form

$$y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_2 y'' + c_1 y' + c_0 y = 0.$$

Example: $y'' + y = 0$, $A \cdot \cos(x)$, $B \cdot \sin(x)$ are solutions

$\underline{A \cdot \cos(x) + B \sin(x)}$

$\cos x$ and $\sin x$ <u>span</u> the set of solutions!

Example II $y'' - y = 0$, $A \cdot \cosh(x)$, $B \cdot \sinh(x)$

$\underline{A \cdot \cosh(x) + B \sinh(x)}$

Linear differential equations and linear algebra

Let us consider two solutions, y_1 and y_2 to

$$y'' + 5y' + 6y = 0.$$

Question

What can we say about $\lambda y_1 + \mu y_2$, where $\lambda, \mu \in \mathbb{R}$?

$$D^2[\lambda y_1 + \mu y_2] = \lambda y_1'' + \mu y_2''$$

$$D[\lambda y_1 + \mu y_2] = \lambda y_1' + \mu \cdot y_2'$$

$$(\lambda y_1 + \mu y_2)'' + 5(\lambda y_1 + \mu y_2)' + 6(\lambda y_1 + \mu y_2) =$$

$$\lambda\underbrace{(y_1'' + 5y_1' + 6y_1)}_{0} + \mu\underbrace{(y_2'' + 5y_2' + 6y_2)}_{0} = 0$$

Conclusion: [Subspace!]

If y_1, y_2 are solutions, then $\lambda y_1 + \mu y_2$ is also a solution!

Extra exercises

Question

Recall that $\sec(x) := \frac{1}{\cos(x)}$, and that $\csc(x) := \frac{1}{\sin(x)}$. Find the derivative of $\operatorname{arcsec}(x)$ and $\operatorname{arccsc}(x)$.

Question

Verify that the curves defined by $y^2 + x^2 = C$ are solutions to $y' = -\frac{x}{y}$.
Hint: Use implicit differentiation, and solve for y'.