SF1685: Calculus

Integration

Register for finals!

Typo/mistake last session

We used $a=1$ to approximate

$$\log(\underbrace{1.4}_{x}) \approx \log(1) + \frac{1}{\underbrace{1}_{\text{Not } 1.4.}}(x-a)$$

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Summation

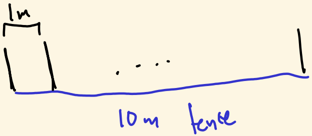
Recall, if $m \le n$, we introduce the notation

$$\sum_{i=m}^{n} f(i) := \overbrace{f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)}^{\#\text{terms is } n-m+1.}.$$

Implementation: For-loop

Watch out for off-by-one errors!

fencepost mistake.



requires 11 poles if 1m between each.

Properties of summation

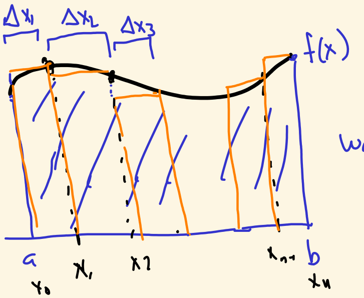
(Some might fail if we extend to infinite # of terms).

- $$\sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$$
- $$\sum_{i=m}^{n} A \cdot f(i) = A \sum_{i=m}^{n} f(i)$$
- $$\sum_{i=m}^{n} f(i) = \sum_{i=0}^{n-m} f(m+i)$$
- $$\sum_{i=1}^{n} 1 = n$$

Area under a curve

Let $a = x_0 < x_2 < x_3 \cdots < x_n = b$, and $\Delta x_i := x_i - x_{i-1}$. Note that $\Delta_1 + \Delta_2 + \cdots + \Delta_n = b - a$. Now assume f is piecewise continuous and non-negative on $[a, b]$. Then

$$\text{Area under } f \approx \sum_{i=1}^{n} \overbrace{f(x_i)}^{\text{height of rectangle } i} \underbrace{\Delta x_i}_{\text{base of rectangle } i}$$



want the area

Upper and lower Riemann sums

We **partition** $[a, b]$ with points, $a = x_0 < x_2 < x_3 \cdots < x_n = b$. Let P be any such partition.

Upper Riemann sum[1]

$$U(f, P) := \sum_{i=1}^{n} \overbrace{f(x_i^\star)}^{\text{largest value of } f(x) \text{ on the interval w. base } \Delta x_i} \underbrace{\Delta x_i}_{\text{base length}}$$

where $f(x_i^\star)$ is chosen so that its the **maximum** over $[x_{i-1}, x_i]$.

Lower Riemann sum

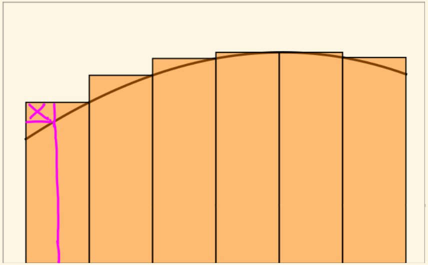
$$L(f, P) := \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where $f(x_i^*)$ is chosen so that its the **minimum** over $[x_{i-1}, x_i]$.

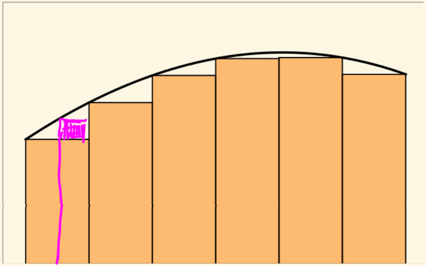
[1] See also the notion of Darboux-sum.

Upper and lower Riemann sums — Illustration

Take maximum on each interval for height



Take the minimum.



Intuitively: By refining the partition P, the difference between upper and lower estimate should decrease.

Integral — definition

Hidden limit: $n \to \infty$ # points in the partition.

0.4926

0.1 0.2 0.445 $\underline{0.4999}$

$\frac{1}{2}$

0.8 0.7 0.62 0.512 $\underline{0.500012}$

Evidently,

$$L(f, P) \leq \text{Area under } f \leq U(f, P)$$

for any choice of partition P. If there is a unique number, I, which lies between $L(f, P)$ and $U(f, P)$, then we say that this is the **integral of $f(x)$ between a and b**. This is denoted by

$$I = \int_a^b f(x)dx$$

"$\int$" Letter S, $\underline{\text{Sum}}$

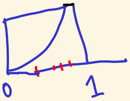
By def. unique number between lower and upper Riemann sums.

How to compute?

$f(x) = x^2$

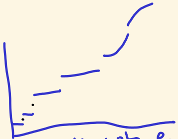
$a = 0$

$b = 1$



Lots of Partitions to check!

Examples



monotone function.

Functions for which $\int_a^b f(x)dx$ exists:

- Functions continuous on $[a, b]$ (not allowed to diverge to infinity).
- Functions which are *monotone* on $[a, b]$.

A function which **cannot** be integrated (by the means defined above):

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

(not monotone! not continuous)

Let's try to find

$$\int_0^1 f(x)\,dx$$

$$U(f,P) = \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{=1} \underbrace{\Delta x_i} = 1 \ (=b-a).$$

each interval contains a rational number

$$L(f,P) = \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{=0} \Delta x_i = 0.$$

$$0 \le \int_0^1 f(x)\,dx \le 1$$

I is not unique!

Comments

Liebniz (came up w. Calculus) $\frac{\partial F}{\partial x}$

C. F. Gauss

Pascal had an idea

Probability!

Kolmogorov axiomatized this

Random variable.

Kolmogorov → V. Arnold → B. Shapiro → P. Alexandersson

complex variables analysis..

S.U

KTH

Integration has many uses.

The above definition is totally useless in practice! We shall fix this later, by **the fundamental theorem of calculus**.

Properties of integration

Can prove from the def!

$$\int_a^b 1\, dx = b-a.$$

$$\int_a^a f(x)dx = 0$$

For convenience!
(similar to how determinants have sign)

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

(Makes sense with area interpretation)

-
- because true for sums

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

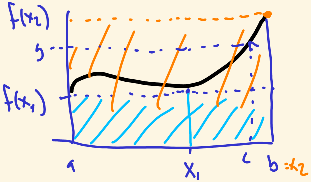
$$\int_a^b Af(x)dx = A\int_a^b f(x)dx$$

Mean value theorem for integrals

If f is continuous, then there is a $c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof



$$\int_a^b f(x)\,dx \geq \underbrace{f(x_1)\,(b-a)}$$

Area of

$$\int_a^b f(x)\,dx \leq \underbrace{f(x_2)\,(b-a)}$$

area of covers area under the graph.

Since f is continuous, there is a c, s.t $f(c) = y$ for any $y \in [f(x_1), f(x_2)]$

Fundamental theorem of calculus

Relates area under a graph with derivatives!

Suppose f is continuous on an interval $[a, b]$. Let

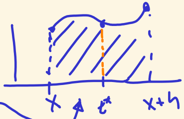
$$F(x) := \int_a^x f(t)dt. \qquad x \in [a,b].$$

Then $F(x)$ is *differentiable* and $F'(x) = f(x)$.

Moreover, let $G(x)$ be any function such that $G'(x) = g(x)$ (we say that G is an **anti-derivative** of g). Then

$$\int_a^b g(x)dx = G(b) - G(a).$$

Fundamental theorem of calculus — proof



We have $F(x) := \int_a^x f(t)dt$, so for $x \in (a,b)$,

$$F(x+h) - F(x) = \int_x^{x+h} f(t)dt = f(t^*)(\underbrace{(x+h) - x}_{h})$$

for some $t^* \in [x, x+h]$ *(why?)*. Mean value theorem!

Therefore,

$$F(x+h) - F(x) = f(t^*) \cdot h \quad \Longrightarrow \quad \frac{F(x+h) - F(x)}{h} = f(t^*).$$

Now we let $h \to 0$.

$F'(x)$ $\qquad$ $f(x)$

Then $t^* \to x$

So $f(t^*) \to f(x)$

since f is continuous.

Conclusion

you are given $f(x) = x^2$.

Find $F(x)$ s.t $F'(x) = x^2$

We say that $F(x)$ is an anti-derivative of x^2.

$F(x) = \frac{x^3}{3} + C$ for any C.

Instead of computing with Riemann sums, it is "enough" to be able to find anti-derivatives.

The remaining week we will do anti-derivatives,

- substitution $\longleftrightarrow$ chain rule
- integration by parts $\longleftrightarrow$ product rule.

Banach - Tarski paradox $\longleftrightarrow$ issue with volume.

Related topics

Example, there is no explicit function whose derivative is e^{x^2}.

Notation for anti-derivative: $\int \frac{1}{1+x^2}\,dx = \arctan(x)+C$

Risch algorithm — 1968. Algebraic method for computing anti-derivatives if they exist. In Mathematica: *“Integrate uses about 500 pages of Wolfram Language code and 600 pages of C code.”*

Recent developments — *DEEP LEARNING FOR SYMBOLIC MATHEMATICS* Lample & Charton, 2020. See also the Quanta Magazine article.

Advantage: Easy to create large data sets!

Create bunch of random functions, and compute their derivatives (Easy!)

NN need to invert this process.

Tricky question (from 2001)

Define $\operatorname{sinc}(x) := \frac{\sin x}{x}$. Then one can show (or numerically be convinced) that

$$\int_0^\infty \operatorname{sinc}\left(\frac{x}{1}\right) dx = \frac{\pi}{2}$$
$$\int_0^\infty \operatorname{sinc}\left(\frac{x}{1}\right)\operatorname{sinc}\left(\frac{x}{3}\right) dx = \frac{\pi}{2}$$
$$\int_0^\infty \operatorname{sinc}\left(\frac{x}{1}\right)\operatorname{sinc}\left(\frac{x}{3}\right)\operatorname{sinc}\left(\frac{x}{5}\right) dx = \frac{\pi}{2}$$
$$\int_0^\infty \operatorname{sinc}\left(\frac{x}{1}\right)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{7}\right) dx = \frac{\pi}{2}$$
$$\int_0^\infty \operatorname{sinc}\left(\frac{x}{1}\right)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{9}\right) dx = \frac{\pi}{2}$$

Guess the value of[2]

$$\int_0^\infty \operatorname{sinc}\left(\frac{x}{1}\right)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{15}\right) dx. \approx \frac{\pi}{2} - 2\cdot 10^{-11}$$

Not rounding error!

[2] See Borwein integral