SF1686 Flervariabelanalys
Exam
October 21, 2020

Time: 8:00-11:00
No books/notes/calculators etc. allowed
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This exam consists of three parts, each worth 12 points. The bonus points from the seminars will be automatically added to the total score of part A, which however cannot exceed 12 points. The problems in part C are mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

Part A is multiple choice and you do not have to motivate your answers. Just check all the correct alternatives.

For full score on any question in part B and part C of the exam, you have to provide a full, motivated, well-presented and easy to follow solution. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument.

Part A.

For questions 1-6 in PART A you should state all correct/true options/statements in your answer sheet. Only answers should be on your answer sheet. No justifications are taken into account.

Each problem has a maximum of 2 points. For each wrong/missing option in each problem one point will be subtracted.

Question 1. Let $f(x,y) = xe^y + y^2$. Find all true statements.

A) The directional derivative at $(x,y) = (1,1)$ in the direction $\frac{1}{\sqrt{5}}(2,-1)$ is calculated

$$(1,1)\cdot\nabla f\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right).$$

B) The directional derivative at $(x,y) = (0,1)$ in the direction $\nu = \frac{1}{\sqrt{2}}(1,1)$ is $\frac{e+2}{\sqrt{2}}$.

C) The directional derivative at $(x,y) = (0,1)$ in the direction $\nu = \frac{1}{\sqrt{2}}(1,1)$ is $\frac{e+3}{\sqrt{2}}$.

Solution question 1:[1] Correct answer is B.

(a) The correct formula is

$$\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)\cdot\nabla f(1,1).$$

Therefore this alternative is wrong.

[1]On part A no motivations are required. But we have supplied some brief motivations for pedagogical purposes.

(b) This alternative is correct. Calculating

$$\nabla f(x,y) = (e^y, xe^y + 2y),$$

and therefore

$$\frac{1}{\sqrt{2}}(1,1)\cdot\nabla f(0,1) = \frac{1}{\sqrt{2}}(1,1)\cdot(e,2) = \frac{e+2}{\sqrt{2}}.$$

(c) This is false in view of the previous point.

Question 2. Let $\mathcal{C}$ be the curve parametrized by $(x(t), y(t)) = \left(\frac{3}{2}t^2, t^3\right)$ for $a \leq t \leq b$. For which values of a and b is the arc-length equal to 1. Find <u>all</u> correct alternatives.

A) $a = \sqrt{3^{1/3} - 1}$ and $b = \sqrt{4^{1/3} - 1}$.

B) $a = \sqrt{3^{2/3} - 1}$ and $b = \sqrt{4^{2/3} - 1}$.

C) $a = \sqrt{2^{2/3} - 1}$ and $b = \sqrt{3^{2/3} - 1}$.

Solution question 2: Correct answers are B and C. The arcleangth of $(x(t), y(t))$ and $a \leq t \leq b$ is calculated

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt =$$

$$3\int_a^b \sqrt{t^2 + t^4}dt.$$

Making the change of variables $s = t^2$ results in the arc-length being

$$\frac{3}{2}\int_{a^2}^{b^2} \frac{\sqrt{s+s^2}}{\sqrt{s}}ds =$$

$$\frac{3}{2}\int_{a^2}^{b^2} \sqrt{1+s}ds = (1+b^2)^{3/2} - (1+a^2)^{3/2}.$$

Inserting the choices it follows that B, and C alternatives are correct.

Question 3. Let

$$I = \int\int\int_V \left(x(x^2+z^2) + 2x^2 - y^2\right)dV$$

where

$$V = \{(x,y,z);\ x^2 + y^2 + z^2 \leq 1\}.$$

Find <u>all</u> correct answers.

A) $I = 4\pi$, B) $I = \frac{15}{4}\pi$, C) $I = \frac{4\pi}{3}\pi$, D) $I = \frac{4\pi}{15}$, E) The integral I is divergent.

Solution question 3: Correct answer is D.
First we notice that V is even and $x(x^2+z^2)$ is an odd function so that part of the integral is zero. Furthermore, by symmetry,

$$\int\int\int_V x^2 dV = \int\int\int_V y^2 dV = \int\int\int_V z^2 dV.$$

It follows that

$$I = \int\int\int_V z^2 dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 r^4\cos^2(\theta)\sin(\theta)drd\phi d\theta = \frac{4\pi}{15}.$$

Question 4. Let $\mathcal{C}$ be the curve defined implicitly by

$$f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{3}y^3 + \frac{1}{2}y^2 = 2.$$

Mark <u>all</u> points (x,y) where $\mathcal{C}$ has a normal pointing in the same direction as $(1,1)$.

A) $(x,y) = (2,0)$ B) $(x,y) = (1,0)$ C) $(x,y) = (0,1)$

D) $(x,y) = (0,2)$ E) $(x,y) = (-2,0)$.

Solution question 4: The correct answers are A, and E. The normal points in the direction $\nabla f(x,y) = (x+y, x+y^2+y)$. Thus a normal of $\mathcal{C}$ points in the direction $(1,1)$ if

$$\nabla f(x,y) = (x+y, x+y^2+y) = (a,a),$$

for some $a \in \mathbb{R}$. But if $x+y=a$ and $x+y^2+y=a$ then $y=0$ and $x=a$.
But $(x,y)=(a,0)$ is a point on the curve only if $f(a,0) = \frac{1}{2}a^2 = 2$, that is if $x = \pm 2$. It follows that the only two points where the normal points in the same direction as $\pm(1,1)$ are $(\pm 2, 0)$. Hence the correct answers are A, and E.

Question 5. Let

$$I = \int_0^{\sqrt{3}} \left(\int_{y/\sqrt{3}}^{\sqrt{4-y^2}} e^{-x^2-y^2}\, dx \right) dy.$$

Find <u>all</u> correct answers.

A) $I = \int\int_D e^{-x^2-y^2}\, dxdy$ where D is a rectangular area in $\mathbb{R}^2$.

B) $I = \int_0^1 \left(\int_0^{\sqrt{3}x} e^{-x^2-y^2}\, dy \right) dx.$

C) $I = \int_0^{\pi/3} \left(\int_0^2 e^{-r^2} r\, dr \right) d\theta.$

D) $I = \frac{\pi}{6}(1 - e^{-4}).$

Solution question 5: True answers are A, C, and D.

A) It is easy to see that if D is small enough then the integral will be small enough and if D contains the domain of integration in I the integral satisfies $I = \int\int_D e^{-x^2-y^2}\, dxdy$. Since the integral changes continuously with the domain there mus be a D inbetween such that

$$I = \int\int_D e^{-x^2-y^2}\, dxdy.$$

It follows that the alternative A is true.

B) This is not true.

C) True (this is just a change to polar coordinates).

D) True by a simple calculation.

Question 6. Consider the equation $y^5 + xy = 4$. Find <u>all</u> correct statements.

A) The points $(x,y) = (3,1)$ and $(x,y) = (-5,-1)$ are solutions to the equation.

B) In a small enough region around $(3,1)$ we can write the solutions to the equation as $(x, f(x))$ for some function $f(x)$ where $f(3) = 1$ and $f(x)$ is differentiable in $x = 3$.

C) If the function f from the previous point exists then $f'(3) = -1/2$.

Solution to question 6: A, and B are true.

A) This is true as is seen by inserting $(x, y) = (3, 1)$ and $(x, y) = (-1, -5)$ in the equation.
B) This is true by the implicit function theorem since $D_y(y^5 + xy) \neq 0$ at $(x, y) = (3, 1)$.
C) This is false since if $f^5(x) + xf(x) = 4$ then we get, by differentiating w.r.t. x,

$$5f^4(x)f'(x) + f(x) + xf'(x) = 0 \Rightarrow f'(3) = -\frac{f(3)}{5f^4(3) + 3} = -\frac{1}{8}.$$

Part B.

Question 7. The hull of a certain submarine can be described as all points in an ellipsoid

$$\frac{x^2}{10^4} + \frac{y^2}{10^2} + \frac{z^2}{10^2} \leq 1.$$

The formula is interpreted in the unit meter so that the submarine is 200m long.
When the submarine descends to the depth $d \geq 0$ the maximal external pressure P_e at a point of the hull, expressed in the unit of atmospheric pressures a.p., as

$$P_e = \frac{d}{10}.$$

To balance the external pressure the captain of the boat can release $R \geq 0$ cubic meters of air at atmospheric pressure. The internal pressure P_i of the submarine can then be calculated by

$$P_i = \frac{R}{V},$$

where V is the volume of the hull in cubic meters.
The hull will be destroyed (either be crushed or burst) if

$$|P_e - P_i| \geq 10.$$

a) Find the volume V of the submarine.

[3 marks]

b) Calculate the least volume R that the captain must release at the depth $d \geq 0$ so that the submarine is not destroyed.

[3 marks]

Solution to question 7: a) By choosing cylindrical coordinates $x = 1$, $y = r\cos(\phi)$ and $z = r\sin(\phi)$ we get that the submarine is determined by the equation

$$r^2 = 100 - \frac{x^2}{100}.$$

We can calculate the volume

$$V = \int_{-100}^{100} \int_0^{2\pi} \int_0^{\sqrt{100-\frac{x^2}{100}}} r dr d\phi dx =$$

$$= \pi \int_{-100}^{100} (100 - \frac{x^2}{100}) dx = \frac{4\pi}{3} \cdot 10^4.$$

Alternatively one use change of variables $(x, y, z) \rightarrow (10^2 x', 10y', 10z')$ which makes the volume element dV to be $10^4 dV'$, and the ellipsoid is transformed to a ball of radius 1, which has volume $4\pi/3$. Hence the volume of the ellipsoid is $\frac{4\pi}{3} \cdot 10^4$.

b) The condition $|P_e - P_i| \leq 10$ can be written

$$-10 \leq P_e - P_i \leq 10 \Rightarrow -10 \leq \frac{d}{10} - \frac{R}{V} \leq 10. \tag{1}$$

From the fact that $-R/V$ is decreasing in R it follows that it is the right inequality that puts the restriction on R.
It follows that the least $R \geq 0$ to assure that the submarine isn't destroyed is

$$R = \begin{cases} 0 & \text{for } 0 \leq d \leq 100 \\ \frac{4\pi}{3} \cdot 10^3(d-100) & \text{for } d > 100. \end{cases}$$

Question 8. Consider the vector field $\mathbf{F}(x,y,z) = (y^2 + \alpha z, 2xy, 3z^2 - x)$ on $\mathbb{R}^3$ where $\alpha \in \mathbb{R}$ is a fixed constant.

(a) Given an open and connected domain D and a continuously differentiable vector field **F**. State conditions that assures that **F** is conservative (i.e. has a potential function).

[1 marks]

(b) Decide for which α the vector field **F** is conservative and calculate a corresponding potential.

[3 marks]

(c) Calculate the work done by the vector field (with any α from the previous part of the question) along the curve

$$\gamma(\theta) = \left(3\cos(\theta), 2\sin(\theta), \frac{\theta}{2\pi}\right) \qquad \text{where } 0 \leq \theta \leq 2\pi.$$

[2 marks]

Solution to question 8:

a) There are two main conditions: i) D needs to be simply connected, ii) curl(**F**) $= (0,0,0)$.
b) Since $\mathbb{R}^3$ is simply connected we know that **F** is conservative if and only if curl(**F**) $= (0,0,0)$. Therefore we calculate

$$\text{curl}(\mathbf{F})(x,y,z) =$$

$$\left(\frac{\partial(3z^2-x)}{\partial y} - \frac{\partial(2xy)}{\partial z}, \frac{\partial(y^2-\alpha z)}{\partial z} - \frac{\partial(3z^2-x)}{\partial x}, \frac{\partial(2xy)}{\partial x} - \frac{\partial(y^2+\alpha z)}{\partial y}\right)(0, \alpha+1, 0).$$

This is zero only if $\alpha = -1$.
In order to find a potential function $\Psi(x,y,z)$ so that

$$\nabla\Psi(x,y,z) = \mathbf{F}(x,y,z) \tag{2}$$

we start by integrating the first component of equation (2)

$$\frac{\partial\Psi}{\partial x}(x,y,z) = y^2 - z \Rightarrow \Psi(x,y,z) = xy^2 - xz + h(y,z).$$

If we insert this in the second component of equation (2) we get that

$$\frac{\partial h}{\partial y}(y,z) = 0.$$

Thus $h(y,z)$ does not depend on y; we may write $h(z)$ for $h(y,z)$. The third component in (2) implies that

$$h(z) = z^3 + C,$$

where C is an arbitrary constant that we can chose to be 0.
So the **answer** to the first part of the question is $\alpha = -1$ and a potential function is

$$\Psi(x,y,z) = z^3 - xz + xy^2.$$

c) To calculate the work done by the vectorfield we simply use the formula

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \Psi(\gamma(2\pi)) - \Psi(\gamma(0)) = -2.$$

Part C.

Question 9. Calculate the circulation of $\mathbf{F} = (-y^2 + e^z, x, z + \sin(x^2))$ around the oriented surface given by

$$x^2 + y^2 + 4(z-1)^2 = 20 \text{ and } z \geq 0,$$

with normal pointing in toward the $z-$axis.

[6 marks]

Solution to question 9: We will use the Stokes Theorem. For that we need to parametrize the boundary of the surface we integrate over, lets call the surface S. The boundary is the set when $z = 0$. That is, when

$$x^2 + y^2 = 16 \text{ and } z = 0,$$

which corresponds to a circle of radius 4 in the plane $z = 0$. We may parametrize the boundary of the surface by

$$\gamma(t) = (x(t), y(t), z(t)) = (4\cos(-t), 4\sin(-t), 0) = (4\cos(t), -4\sin(t), 0), \quad 0 \leq t \leq 2\pi,$$

where we have taken the orientation of the boundary into consideration by having $-t$ as an argument.

Then, by Stokes Theorem and that

$$d\mathbf{r} = (-4\sin(t), -4\cos(t), 0)dt,$$

we have that

$$\int\int_S \mathbf{curl}(\mathbf{F}) \cdot \mathbf{N} dS = \int_\gamma \mathbf{F} \cdot d\mathbf{r} =$$

$$= \int_0^{2\pi} \big(-16\sin^2(t) + 1, 4\cos(t), \sin(16\cos^2(t)), 0\big) \cdot \big(-4\sin(t), -4\cos(t), 0\big) dt =$$

$$\int_0^{2\pi} \big(64\sin^3(t) - 4\sin(t) - 16\cos^2(t)\big)\, dt =$$

$$\int_0^{2\pi} \left(64\sin(t)(1-\cos^2(t)) - 4\sin(t) - 16\left(\frac{1+\cos(2t)}{2}\right)\right) dt =$$

$$\underbrace{\left[64\cos(t) - \frac{64}{3}\cos^3(t) + 4\cos(t) - 4\sin(2t)\right]_{t=0}^{t=2\pi}}_{=0} \underbrace{-8\,[t]_{t=0}^{t=2\pi}}_{=-16\pi} = -16\pi.$$

So the circulation is zero -16π.

Question 10. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x-y}{(x+y)^3} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Furthermore we let $D = [0,2] \times [0,2]$.

(a) Is f continuous on D? Motivate your answer!

[2 marks]

(b) Is f Riemann integrable on D. You must prove your answer.

[4 marks]

Solution to question 10:

a) We begin by showing that the function is not continuous. It is enough to show that $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$. To that end we choose $(x,y) = (t,0)$ and consider

$$\lim_{t\to 0^+} f(t,0) = \lim_{t\to 0^+} \frac{t-0}{(t+0)^3} = \infty. \tag{3}$$

It follows that f is not continuous at the origin.

b) We also claim that f is not Riemann integrable on the square. To prove this we choose an arbitrary $\epsilon > 1$, say $\epsilon = 1$. It is enough to show that there isn't any I and $\delta > 0$ such that

$$|R(f,P) - I| < \epsilon = 1 \tag{4}$$

for every partition P satisfying $\|P\| < \delta$ and choice of tag points (x^*_{ij}, y^*_{ij}). Remember that a partition is a set

$$0 = x_0 < x_1 < x_2 < .... < x_m = 2$$
$$0 = y_0 < y_1 < y_2 < ... < y_n = 2$$

and the tag points satisfy $x_{i-1} \leq x^*_{ij} \leq x_i$ and $y_{j-1} \leq y^*_{ij} \leq y_j$. Furthermore

$$\|P\| = \max_{1\leq i\leq m,\ 1\leq j\leq n} \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}.$$

Also, by definition,

$$R(f,P) = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})(x_i - x_{i-1})(y_j - y_{j-1}).$$

To show that (4) does not hold for any I and $\delta > 0$ we first notice that since $0 = x_0 < x_1$ and $0 = y_0 < y_1$ it follows that for any partition P that $(x_1 - x_0)(y_1 - y_0) > 0$. Next we notice that f is bounded on

$$[0,2] \times [0,2] \setminus [0,x_1) \times [0,y_1),$$

since f is continuouos away from the origin.

Now fix any I and $\delta > 0$ and take any partition P. We will show that (4) cannot hold for every set of tag points (x^*_{ij}, y^*_{ij}). For this partition we have that

$$\Bigg| \sum_{\substack{1\leq i\leq m \\ 1\leq j\leq n \\ (i,j)\neq(1,1)}} f(x^*_{ij}, y^*_{ij})(x_i - x_{i-1})(y_j - y_{j-1}) - I \Bigg| \leq M,$$

for some constant M that is independent of the tag-points, this since the function f is bounded on $[0,2] \times [0,2] \setminus [0,x_1) \times [0,y_1)$. Therefore, using the reverse triangle inequality,

$$|R(f,P) - I| \geq \big| f(x^*_{11}, y^*_{11}) \underbrace{(x_1 - x_0)(y_1 - y_0)}_{>0} \big| - M \tag{5}$$

for every choice of the tag (x^*_{11}, y^*_{11}). But chosing $t > 0$ small enough, $(x^*_{11}, y^*_{11}) = (t,0)$, the limit (3) implies that

$$\big| f(x^*_{11}, y^*_{11}) \underbrace{(x_1 - x_0)(y_1 - y_0)}_{>0} \big| - M > 1 \tag{6}$$

Putting the inequalities (5) and (6) together we have shown that no matter how we chose $I, \delta > 0$ and P we can always find a set of tag points such that

$$|R(f,P) - I| \geq \big| f(x^*_{11}, y^*_{11})(x_1 - x_0)(y_1 - y_0) \big| - M > 1.$$

But this implies that the function f is not Riemann integrable on $[0,2] \times [0,2]$.