SF1685: Calculus

Series,

but mainly sequences.

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Sequences — intro

A **sequence** is a list of numbers,

$$a_1, a_2, a_3, \ldots,$$

and we write $\{a_k\}_{k=1}^{\infty}$ as shorthand.

Properties of sequences

A sequence $\{a_k\}_{k=1}^{\infty}$ can be

- **bounded above by** M if $a_k \leq M$ for all k,
- **bounded below by** L if $L \leq a_k$ for all k,
- **bounded** if there is some B such that $|a_k| \leq B$ for all k,
- **increasing** if $a_1 \leq a_2 \leq a_3 \leq \cdots$,
- **decreasing** if $a_1 \geq a_2 \geq a_3 \geq \cdots$,
- **monotonic** if it is either increasing or decreasing,
- **alternating** if every other element is positive, and remaining elements are negative.

$2, -3, 5, -7, 4, -16, 7, -1,$

alternating.

Limits of sequences

A sequence can have a limit:

$$\lim_{n\to\infty} a_n = L.$$

The limit exists if for every $\varepsilon > 0$, there is an N, such that

$$n > N \implies |a_n - L| < \varepsilon.$$

That is, when n gets large, a_n gets closer and closer to L.

A sequence **diverges** if it does not have a limit.

Can have ∞, or $-\infty$ as limits, Consider the following sequences. Determine which **have a limit**, **are alternating**, **are bounded (in what sense)**, **are decreasing/increasing**.

- $f(x) = \frac{1}{2^x}$
  $$\{2^{-n}\}_{n=0}^{\infty} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$
  has limit 0.
  bounded from below by 0, above by 1.
  decreasing
- $$\{(-1)^n\}_{n=0}^{\infty} = 1, -1, 1, -1, 1, -1, \ldots$$
  Alternating, no limit, bounded from below by -1, from above by 1.
- $$\{\sin(n)\}_{n=1}^{\infty} = \sin(1), \sin(2), \sin(3), \sin(4), \ldots$$
  bounded from below/above by ±1, no limit.
- $$a_0, a_1, a_2, \ldots = 1000, 502, 253, \ldots, \text{ where } a_0 = 1000, a_{n+1} = \frac{a_n + 4}{2}.$$
  Decreasing, has a limit; 4 and $4 \le a_n \le 1000$ for all n.
  average of a_n and 4
- $$\{(-1)^{1-n}/n\}_{n=1}^{\infty} = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$$

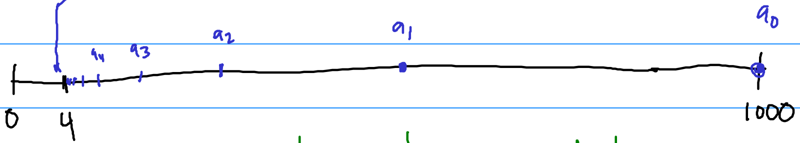
$$a_{n+1} = \frac{a_n}{2} + 2$$

$$\left[a_n = 4 + 249 \cdot 2^{3-n} \right]$$

Closed-form formula

Not easy to find $f(x)$, s.t. $f(n) = a_n$ in this case.

The limit.



Can be proved using induction,

Matrices

$$\lim_{n\to\infty} 4 + 249 \cdot 2^{3-n} = 4, \quad \underbrace{\begin{bmatrix} a_{n+1} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} a_n \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} a_0 \\ 1 \end{bmatrix}$$

Enough to compute matrix powers.

Diagonalization! Eigenvalues etc.

If $\lim_{x\to\infty} f(x) = L$, then the sequence $\{f(n)\}_{n=1}^{\infty}$ also has the limit L.

Give an example of a function f, such that $\{f(n)\}_{n=1}^{\infty}$ has limit 0, but $\lim_{x\to\infty} f(x)$ diverges.

Take

$$f(x) = \sin(2\pi x).$$

$$\boxed{\text{But } \lim_{x\to\infty} \sin(2\pi x) \text{ does not exist!}}$$

If $a_n = \sin(2\pi n)$, for $n = 1, 2, 3, \ldots$

then $\lim_{n\to\infty} a_n = 0$,

$f(1) = \sin(2\pi) = 0$

$f(2) = \sin(4\pi) = 0$

$\vdots$

$f(n) = 0$ for all integers, n, and $0, 0, 0, 0 \ldots$ has 0 as limit.

Properties of sequences and limits

Suppose $\lim_{n\to\infty} a_n = 5$

The usual rules for limits also hold for sequences;

then $\lim_{n\to\infty} f(a_n) = f(5)$ if f is continuous close to 5.

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n$$

provided the limits in the right-hand-side exist. Similar for multiplication, quotients and composition with a continuous function.

We also have a version of the **squeeze theorem**.

Suppose $a_n \le b_n \le c_n$ for all integers $n = 1, 2, 3, \ldots$

and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$\lim_{n\to\infty} b_n = L$

Properties of sequences and limits

Suppose $\{a_n\}_{n=1}^{\infty}$ is bounded from above by B, and there is some M such that

$$a_M \leq a_{M+1} \leq a_{M+2} \leq \cdots$$

B' B

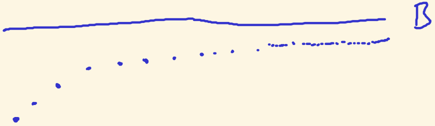
smallest upper bound, this will be the limit.

(i.e., the sequence is **ultimately increasing**), then the limit

$$\lim_{n\to\infty} a_n$$

exists (and is less than or equal to B).

sketch of proof:



A similar statement is true for sequences bounded from below and ultimately decreasing.

$a_2 = \sqrt{2\cdot 1+3} = \sqrt{5} > 2$. So $a_1 < a_2$.

Let $a_1 = 1$ and let $a_n = \sqrt{2a_{n-1}+3}$. Show that the limit $\lim_{n\to\infty} a_n$ exists. **Can we compute the limit also?**

Need to show 1) upper bound, 2) increasing.

Let's see if 4 is an upper bound. Want to show $a_n \le 4$ for all n.

Suppose $a_n \le 4$. Then $a_{n+1} = \sqrt{2\cdot a_n + 3} \le \sqrt{2\cdot 4+3} = \sqrt{11} \le 4$.

So, $a_n \le 4 \Rightarrow a_{n+1} \le 4$. By induction, all $a_n \le 4$. So 4 is an upper bound.

2) Want to show $a_{n+1} \geq a_n$ for all n.

Suppose $[a_n \geq a_{n-1}]$. Then $a_{n+1} = \sqrt{2\cdot a_n + 3} \geq \sqrt{2a_{n-1}+3} = a_n$

[Since $a_n \geq a_{n-1}$] ind. hyp

we have proved this,

So, $[a_{n+1} \geq a_n]$

So by induction, we're done.

Can we find a better upper bound? Maybe 3 works?

Suppose $a_n \le 3$. Then

$$a_{n+1} = \sqrt{2 \cdot a_n + 3} \le \sqrt{2 \cdot 3 + 3} = \sqrt{9} = 3.$$

So $a_{n+1} \le 3$ also. So 3 also works!

We know $1 \le L \le 3$

Suppose $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} \underbrace{\sqrt{2 \cdot a_n + 3}}_{a_{n+1}} = L$

The function $f(x) = \sqrt{2x+3}$ is continuous, for $x > 0$

So, $\sqrt{2 \cdot L + 3} = L$, an equation for L.

$\Rightarrow 2 \cdot L + 3 = L^2 \Rightarrow L^2 - 2L - 3 = 0.$ $\begin{cases} L = -1 \\ \underline{L = 3} \end{cases}$

So the limit is in fact $\underline{3}$.

Special limits we should know

- If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$.
- For **any** real number x, we have that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

That is, factorial outgrows every exponential function.

Proof: Suppose $0 < x < M$, and $n = M + m$

$$\frac{x^n}{n!} = \underbrace{\frac{x \cdot x \cdots x}{1\cdot 2\cdot 3 \cdot M}}\cdot\frac{x\cdot x\cdot x\cdots x}{(M+1)(M+2)\cdots(M+m)} \le \frac{x^M}{M!}\cdot\frac{x^m}{M^m} = \underbrace{\frac{x^M}{M!}}_{\text{Fixed}}\cdot\underbrace{\left(\underbrace{\frac{x}{M}}_{<1}\right)^m}_{\to 0}$$

as $\begin{cases} m \to \infty \\ n \to \infty \end{cases}$

as $n \to \infty$, $m \to \infty$, and $\left(\frac{x}{M}\right)^m \to 0$.

∎

Series

A **series** is a limit, involving a sum

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \qquad \text{shorthand: } \sum_{k=1}^{\infty} a_k.$$

The first example is the **geometric series**, where $a_k = cr^{k-1}$. We have that the **partial sum**

$$\underbrace{S_n = \sum_{k=1}^{n} cr^{k-1} \text{ is equal to } c\frac{1-r^n}{1-r}.}_{\text{High-school}}$$

Also true if $r, c \in \mathbb{C}$.

If $|r| < 1$, what is the limit?

The limit $\lim_{n\to\infty} S_n = \lim_{n\to\infty} c\frac{1-r^n}{1-r} = \frac{c}{1-r}$.

$\left[\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots \right]$ whenever $|r| < 1$.

$(I-A)^{-1} = I + A + A^2 + A^3 + \cdots$

provided that all eigenvalues < 1...

Problem

$(I-A)^{-1} = I + A + A^2 + \dots$ is easy to verify if A is a diagonal matrix.

Let $a_1 = 1$ and let $a_n = \sqrt{4a_{n-1}+5}$. Show that the limit $\lim_{n\to\infty} a_n$ exists, and compute it.

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$$

Then $I - A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2/3 \end{pmatrix}$

$$(I-A)^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 3/2 \end{pmatrix}. \quad \text{LHS}$$

True for diagonal matrices,

for a general matrix

with $A = TDT^{-1}$

$A^n = T D^n T^{-1}$

$1 + A + A^2 + \dots$

$= T(\underbrace{I + D + D^2 + \dots}_{\text{can compute}}) T^{-1}$

$$\text{RHS} = I + A + A^2 + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}^2 + \dots = \begin{pmatrix} \frac{1}{1-1/2} & 0 \\ 0 & \frac{1}{1-1/3} \end{pmatrix}$$

Problem

Let $a_1 = 1$ and let $a_{n+1} = \frac{1}{2}a_n + \sqrt{a_n}$. Show that the limit $\lim_{n\to\infty} a_n$ exists, and compute it.

A bounded series — a challenge (maybe for the exercise session)

Question

Let

$$S_n := \sum_{k=1}^{n} \sin(k).$$

Show that $\{S_n\}_{n=1}^{\infty}$ is bounded.

Hint:

1) Geometric series

2) Complex numbers!

$$\frac{e^{i\alpha k} - e^{-i\alpha k}}{2i} = \sin(\alpha k)$$

The following generalization is perhaps easier:

Sum these for different k = Sum these for different k,

Question

Let $\alpha > 0$ be fixed, and let

$$A_n := \sum_{k=1}^{n} \sin(\alpha k).$$

what is $\sum_{k=1}^{n} (e^{i\alpha})^k$?

Geometric series!

Show that $\{|A_n|\}_{n=1}^{\infty}$ is bounded from above by $\frac{2}{\sqrt{\sin^2(\alpha)+(1-\cos(\alpha))^2}}$.