SF1685: Calculus

Limits

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Limits — intuition

Goal: Describe what happens as we make something smaller or larger.

Limits *formalizes* behavior which we in many (but not all) cases can understand intuitively.

In our case, behavior is captured some function, $f(x)$, and we want to understand how f behaves close some point a.

$\in \mathbb{R}$ or $\pm\infty$

Limit — From the book

An informal definition of limit

If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, but not equal to a, we say that the function f approaches the **limit** L as x approaches a, and we write

$$\lim_{x \to a} f(x) = L.$$

Note that the limit L does not care about $f(a)$!

Might not be defined at a.

Classical example

Question

What is the number $0.999\ldots$?

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What is the number $0.\overbrace{999\dots}^{\text{"infinite"}}$?

We need to make sense of "..."

Consider

$$0.9,\ 0.99,\ 0.999,\ \overbrace{0.9999}^{\text{finite!}},\ \dots,\ 1-10^{-n},\dots.$$

It makes sense to **define**

$$0.999\ldots := \lim_{n\to\infty} 1-10^{-n}.$$

Classical example

Question

What is the number $0.999\ldots$?

We need to make sense of "$\ldots$"
Consider

$$0.9,\ 0.99,\ 0.999,\ 0.9999,\ \ldots,\ 1-10^{-n},\ldots.$$

It makes sense to **define**

$$0.999\ldots := \lim_{n\to\infty} 1-\underbrace{10^{-n}}_{\to 0}.$$

By choosing n sufficiently large, we can make 10^{-n} as close to 0 as we like.
Hence, $0.999\ldots = 1$

Another example

Let

$$f(x) \coloneqq \frac{x^2 - 4}{x - 2}.$$

We saw that

$$f(x) = \begin{cases} x + 2 & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2. \end{cases}$$

Another example

Let

$$f(x) := \frac{x^2 - 4}{x - 2}.$$

$x = 1.\underbrace{99\ldots9}_{100}0 \neq 2$

We saw that

$$f(x) = \begin{cases} x + 2 & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2. \end{cases}$$

Then,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \overbrace{\lim_{x \to 2} x + 2}^{\text{only cares about } x \neq 2.} = 4.$$

Reading this out loud: *“The limit of f-of-x, as x goes to two, is equal to four.”*

Notation

Different ways:

$$\lim_{x\to 2} f(x) = \cancel{2}\,4.$$

or

$$\overbrace{f(x)}^{\text{function}} \to \overset{\text{number}}{4} \text{ as } x \to 2$$

"f-of-x approaches four, as x approaches two."

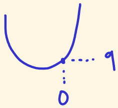
We **do not** write

$$\underbrace{\lim_{x\to 2}}_{\text{a number, doesn't change!}} f(x) \to 4.$$

Types
- Exceptions
- integers +∞
- lists
- maps

Units
- kg
- m
- liters

More examples



$$f(x) := (x+3)^2, \qquad g(x) := \begin{cases} \sin(x) & \text{if } x \neq 0 \\ 7 & \text{if } x = 0 \end{cases}, \qquad h(x) := \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

What is

$$\lim_{x \to 0} f(x) = \underline{\quad 9 \quad} \qquad \lim_{x \to 0} g(x) = \underline{\sin(0) = 0} \qquad \lim_{x \to 0} h(x) = \underline{\text{D.N.E}}$$

One-sided limits

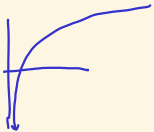
In the definition of limits, we approach *a* from both sides. By restricting to approaching from above, or from below, we can still make sense of the situation:

$$\lim_{x\to 0^+} \frac{x}{|x|} = \underline{\quad 1 \quad} \qquad \lim_{x\to 0^-} \frac{x}{|x|} = \underline{\quad -1 \quad}$$

If points to the left (right) of *a* are outside the domain, we are sloppy:

$$\lim_{x\to 0^+} \sqrt{x} = \underline{\quad 0 \quad} \qquad \lim_{x\to 0^+} \frac{1}{\log(x)} = \underline{\quad 0 \quad}$$

undefined if $x<0$.

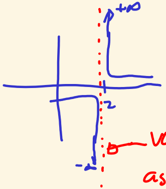


Infinite limits

Functions might sometimes grow without bounds as we approach a:

$$\lim_{x\to 0} x^{-2} = \infty \qquad \lim_{x\to 0} \log(x) = -\infty \qquad \lim_{x\to 2^+} \frac{1}{x-2} = \infty$$

Can be made arbitrarily large.



vertical asymptote

Rules for computing limits

Adding and multiplying functions with limits

Suppose $\underbrace{\lim_{x\to a} f(x) = A, \lim_{x\to a}}_{\text{same!}} g(x) = B$ (the limits exist and are finite). Then

$$\lim_{x\to a} \underset{\infty}{f(x)} + \underset{-\infty}{g(x)} = \underbrace{A+B}_{?} \qquad \lim_{x\to a} \underbrace{f(x)}_{\infty}\underbrace{g(x)}_{0} = \underbrace{AB}_{?} \qquad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B} \text{ if } B \neq 0.$$

There are generalizations to cases where A or B is $\pm\infty$.

If $A=B=0$
can be anything,
depends on f, g.

Composition of functions with limits

Suppose $\lim_{x\to a} f(x) = A$ and $\lim_{x\to A} h(x) = E$, then

$$\lim_{x\to a} \overbrace{h(\underbrace{f(x)}_{\approx A})}^{\approx E} = E.$$

Computing limits using algebra

Show that

$$\lim_{x\to 4} \frac{x^2 - 3x - 4}{\sqrt{x} - 2} = 20.$$

Type "$\frac{0}{0}$"

factor: $x^2 - 3x - 4 = (x-4)(x+1) = (x+1)(\sqrt{x}-2)(\sqrt{x}+2)$

$$= \lim_{x\to 4} \frac{(x+1)(\sqrt{x}+2)\cancel{(\sqrt{x}-2)}}{\cancel{\sqrt{x}-2}} = \lim_{x\to 4} \underbrace{(x+1)}_{5} \cdot \underbrace{(\sqrt{x}+2)}_{4} = 20.$$

$f(x) = \sqrt{x}$

$f(4) = 2$

Computing limits using algebra

Show that

$$(*) \quad \lim_{x \to 1} \frac{\sqrt{2x-1}-1}{\sqrt{x}-1} = 2. \qquad \text{"}\frac{0}{0}\text{"}$$

Trick: multiply w. the conjugate, $\sqrt{x}+1$. $\qquad \sqrt{2x-1}+1$.

$$\left[\left(\sqrt{2x-1}-1\right)\left(\sqrt{2x-1}+1\right) = (2x-1)-1 = 2x-2 = 2(x-1). \right]$$

we have, if $x \neq 1$, that

$$\frac{\sqrt{2x-1}-1}{\sqrt{x}-1} = \frac{\left(\sqrt{2x-1}-1\right)\left(\sqrt{x}+1\right)}{x-1} =$$

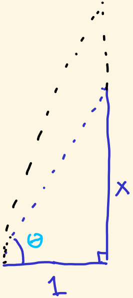
mult. by $\sqrt{2x-1}+1$ top and bottom

$$= \frac{2\cancel{(x-1)}\left(\sqrt{x}+1\right)}{\left(\sqrt{2x-1}+1\right)\cancel{(x-1)}}$$

$\Rightarrow$ $(*)$ is equal to

$$\lim_{x \to 1} \frac{2\cdot\left(\sqrt{x}+1\right)}{\sqrt{2x-1}+1} = \frac{2\cdot 2}{2} = 2 \qquad \blacksquare$$

Limit problems — discussion

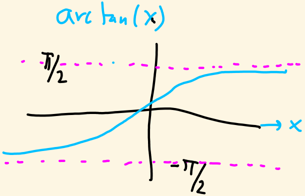


$\theta = \arctan(x)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$(x-3)(x^2 + 3x + 9)$

1. $$\lim_{x\to 3} \frac{x^3 - 27}{x - 3} = \underline{\quad 24 \quad}$$



2. $$\lim_{x\to\infty} \arctan(x) = \underline{\quad \pi/2 \quad}$$

3. $$\lim_{x\to 2} \frac{\sqrt{x} - \sqrt{2}}{\sqrt{2x} - 2} = \underline{\quad 1/\sqrt{2} \quad}$$

4. $$\lim_{x\to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2} = \underline{\qquad\qquad}$$

Mult. by $\sqrt{x^2+3} + 2$ $\rightarrow$ Den. $= x^2 + 3 - 4 = x^2 - 1$

$= (x+1)\underline{(x-1)}$

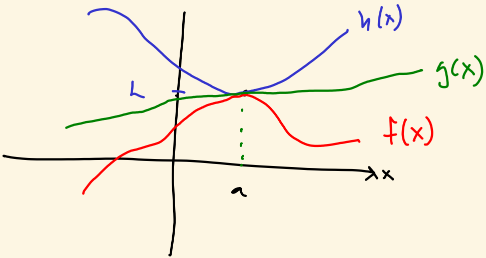
Limit problems — notes

If $x \neq 2$,

$$\frac{\sqrt{x}-\sqrt{2}}{\sqrt{2x}-2} = \frac{\sqrt{x}-\sqrt{2}}{\sqrt{2}\cdot\sqrt{x}-\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{x}-\sqrt{2}}{\sqrt{2}(\sqrt{x}-\sqrt{2})} = \frac{1}{\sqrt{2}}$$

Squeeze theorem, (p.86)

Suppose $f(x) \leq g(x) \leq h(x)$ near a, and that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$. Then $\lim_{x \to a} g(x) = L$.



Standard limits

You need to learn these eventually

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \quad \leftrightarrow \quad \lim_{x\to 0} \frac{\ln(x+1)}{x} = 1 \qquad \lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

Definition of e

$$e := \lim_{n\to\infty} \underbrace{\left(1 + \frac{1}{n}\right)}_{\to 1}^{n \to \infty} \quad \leftrightarrow \quad \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$$

$x = 1/n$

$e = 2.718281 59\ldots$

More limit problems

1. $$\lim_{x\to\infty} \sqrt{x^2+x} - x = \underline{\hspace{4cm}}$$

2. $$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2-1} = \underline{\hspace{4cm}}$$

3. $$\lim_{x\to\infty} \frac{x^5+2x-1}{100x^4} = \underline{\hspace{4cm}}$$

4. $$\lim_{x\to 1} \frac{x-1}{\sqrt[3]{x}-1} = \underline{\hspace{4cm}}$$

In the last problem, use the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, for the values $a = \sqrt[3]{x}$ and $b = 1$.

More limit problems II

1. $$\lim_{x\to 0} \frac{e^{2x}-1}{x} = \underline{\hspace{4cm}}$$

2. $$\lim_{x\to 0} \frac{\sin(x^2)}{x^2+x^3} = \underline{\hspace{4cm}}$$

3. $$\lim_{x\to 0} \frac{\cos(2x)-1}{\sin(x)} = \underline{\hspace{4cm}}$$

4. $$\lim_{x\to \infty} \left(1+\frac{3}{x}\right)^x = \underline{\hspace{4cm}}$$