SF1685: Calculus

Linear differential equations

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Linear differential equations

A (homogeneous) **linear differential equation** can be expressed in the form

$$y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_2 y'' + c_1 y' + c_0 y = 0.$$

General second-degree differential equation

Let $p, q \in \mathbb{R}$, and lets consider

$$y'' + py' + qy = 0.$$

Now, suppose α and β are two *different* roots of $t^2 + pt + q = 0$ (characteristic equation).

Claim: $e^{\alpha x}$ and $e^{\beta x}$ are solutions, and the general solution is

$$Ae^{\alpha x} + Be^{\beta x}, \qquad A, B \in \mathbb{R}$$

We check:

$$\begin{aligned} D^2[e^{\alpha x}] + pD[e^{\alpha x}] + qe^{\alpha x} &= \alpha^2 e^{\alpha x} + p\alpha e^{\alpha x} + qe^{\alpha x} \\ &= e^{\alpha x}\left(\underbrace{\alpha^2 + p\alpha + q}_{=0 \text{ since } \alpha \text{ is a root}}\right) = 0 \end{aligned}$$

And what do we know about $\alpha^2 + p\alpha + q$?

Two questions: First, what if the roots are complex?

Suppose $\alpha = a \pm ib$ are the two roots. Then

α_n

$$e^{(a+ib)x} \text{ and } e^{(a-ib)x}$$

"vector space observation"

are solutions. But, by the argument last class, so are

$$e^{(a+ib)x} + e^{(a-ib)x} \text{ and } e^{(a+ib)x} - e^{(a-ib)x}$$

Now,

$[e^{i\theta} = \cos\theta + i\sin\theta]$

$$\begin{aligned} e^{(a+ib)x} + e^{(a-ib)x} &= e^{ax}\left(e^{ibx} + e^{-ibx}\right) \\ &= e^{ax}\left(\cos(bx) + \cancel{i\sin(bx)} \quad + \cos(bx) - \cancel{i\sin(bx)}\right) \\ &= \underbrace{2e^{ax}\cos(bx)}_{\text{Real}}. \end{aligned}$$

Same manner: $2e^{ax} \cdot \sin(bx)$ is also a solution.

In conclusion

We want to solve $y'' + py' + qy = 0$. If the characteristic equation $t^2 + pt + q = 0$ has the complex roots $a \pm ib$, then the general (real) solution is of the forms

$$Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)$$

where $A, B \in \mathbb{R}$.

See Adam's p. 961 for exercises.

Second question: What if we have a double root?

We want to solve $y'' + py' + qy = 0$. If the characteristic equation $t^2 + pt + q = 0$ has the *double root* ρ, then the general solution is of the forms

$$(A + Bx)e^{\rho x}$$

where $A, B \in \mathbb{R}$.

Similar theory for higher degree equations

E.g. $y''' + 3y'' + 3y' + y = 0$

consider: $t^3 + 3t^2 + 3t + 1 = 0$ $\quad (t+1)^3 = 0$, triple root.

$\underline{t = -1}$

Solutions are $(A + Bx + Cx^2)e^{-x}$

Some typical questions I

Question

Find the solution to $y'' + 5y' + 6y = 0$ which satisfies $y(0) = 1$, $y'(0) = 2$.

First consider $t^2 + 5t + 6 = 0$. This eq. has the roots -2 and -3.

Thus

$y = A \cdot e^{-2x} + Be^{-3x}$ is the general solution.

$y' = -2Ae^{-2x} - 3Be^{-3x}$

we want $y(0) = 1$, so $A + B = 1$, and

$y'(0) = 2$, so $-2A - 3B = 2$.

<u>Solve:</u> $\begin{cases} A + B = 1 \\ 0 \quad -B = 4 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = -4 \end{cases}$.

Finally, $y = 5 \cdot e^{-2x} - 4e^{-3x}$ is the solution we seek.

Some typical questions II

Question

Find the solution to $y'' - 4y' + 4y = 0$ which satisfies $y(0) = 1$, $y'(0) = 4$.

Consider $t^2 - 4t + 4 = 0 \Leftrightarrow (t-2)^2 = 0$, so $t = 2$ is a double zero.

The general solution is $y = (A + Bx) \cdot e^{2x}$.

$y' = 2Ae^{2x} + B e^{2x} + 2Bx \cdot e^{2x}$

Bdd cond: $y(0) = 1 \Rightarrow A = 1$

$y'(0) = 4 \Rightarrow 2A + B = 4$, so $B = 2$.

Hence, $y = (1+2x)e^{2x}$ is the solution we seek.

Some typical questions II

Question

Find the solution to $y'' + 2y' + 10y = 0$ which satisfies $y(0) = 2$, $y'(0) = 7$.

Consider $t^2 + 2t + 10 = 0$, $t = \frac{-2}{2} \pm \sqrt{1-10}$.

So, $-1 \pm 3i$ are the solutions.

The general solution is $y = e^{-x} \cdot (A\cos 3x + B\sin 3x)$

$y' = -e^{-x} \cdot (A\cos 3x + B\sin 3x) + e^{-x}(-3A \cdot \sin(3x) + 3B \cdot \cos(3x))$

Now, $y(0) = 2 \Rightarrow e^0 (A \cdot \cos(0) + B\sin(0)) = 2 \Rightarrow A = 2.$

$y'(0) = 7, \Rightarrow -A + 3B = 7 \Rightarrow B = 3.$

Thus, $e^{-x}(2 \cdot \cos 3x + 3\sin 3x)$ is the solution we seek.

Inhomogeneous equations

What if we want to solve

$$y'' + 5y' + 6y = 6x + 11, \qquad y'' + 5y' + 6y = 6e^{-x}, \qquad y'' - 3y' + 2y = 5\sin(2x)$$

$y = Ax + B$ $\qquad$ $y = A \cdot e^{-x}$ $\qquad$ $y = A \cdot \sin 2x + B\cos 2x$.

The idea is to first find the **general solution**, y_h, to the **homogeneous equation**, $y'' + 5y' + 6y = 0$, first, and then find a **particular solution**, y_p.
Then, the solution we seek is $y := y_h + y_p$.

So how do we find y_p?
Make an anzats, which is similar-looking to the right-hand side.

"educated guess", the derivatives should

Question

Find the solution to $y'' + 5y' + 6y = 6x + 11$ which tangents the line $7x + 2$ at the origin.

The solution to the hom. eq is $y_h = Ae^{-2x} + Be^{-3x}$.

Now, we suppose $\begin{cases} y_p = ax+b, \\ y_p' = a \\ y_p'' = 0 \end{cases}$, so $\underline{5a} + 6(ax+b) = 6x + 11$.

We need that $\underbrace{6ax = 6x}_{\Rightarrow a=1}$ and $\underbrace{5a + 6b = 11}_{a=1 \text{ so } b=1}$.

Hence, $\underline{y_p = x+1}$, and $y = x + 1 + Ae^{-2x} + Be^{-3x}$ is the gen. sol.

Tangent to $7x+2$ at $x=0$, same as $y(0) = 2$, $y'(0) = 7$, $\begin{cases} 1 + A + B = 2 \\ 1 - 2A - 3B = 7 \end{cases}$

Solve now for A and B ...

Question

Find the solution to $y'' + 5y' + 6y = 6e^{-x}$ $\boxed{e^{-2x}}$ which satisfies $\cancel{\lim_{x\to\infty} y(x) = 0}$ $y'(0) = 0$. and $y(0) = 2$.

As before, $y_h = Ae^{-2x} + Be^{-3x}$. Now, ansatz, $y_p = C \cdot e^{-x}$

$y_p' = -C \cdot e^{-x}$, so $C \cdot e^{-x} - 5 \cdot Ce^{-x} + 6 \cdot Ce^{-x} = 6 \cdot e^{-x}$

$y_p'' = C \cdot e^{-x}$

$\Rightarrow \quad 2 \cdot C = 6$, so $\underline{C = 3}$.

[$y_p = C \cdot e^{-2x}$, not good!
Instead, $\underline{C \cdot x \cdot e^{-2x}}$]

Hence, $y_p = 3 \cdot e^{-x}$, and $y = 3 \cdot e^{-x} + Ae^{-2x} + Be^{-3x}$.

Solve now for A and B:

$$\begin{cases} y(0) = 2: & 3 + A + B = 2 \\ y'(0) = 0: & -3 - 2A - 3B = 0 \end{cases} \quad \ldots \quad \text{solve!}$$

Question

Find the solution to $y'' - 3y' + 2y = 5\sin(2x)$ which satisfies $y(0) = y'(0) = 0$.

Solve $t^2 - 3t + 2 = 0$. we get $t = 2, t = 1$ So $y_h = A \cdot e^{2x} + Be^{x}$.

Ansatz: $y_p = C\sin(2x) + D\cos(2x)$.

$y_p' = 2 \cdot C \cdot \cos(2x) - 2D \cdot \sin(2x)$

$y_p'' = -4 \cdot C \cdot \sin(2x) - 4 \cdot D \cdot \cos(2x)$.

Plug in into eq:

$$-4 \cdot C \cdot \sin(2x) - 4D \cdot \cos 2x - 3(2 \cdot C \cdot \cos 2x - 2D \cdot \sin 2x) + 2(C \cdot \sin 2x + D\cos 2x) = 5 \cdot \sin 2x.$$

Compare coeffts:

$$\begin{cases} \sin 2x: & -4C + 6D + 2C = 5 \\ \cos 2x: & -4D - 6C + 2D = 0 \end{cases} \iff \begin{cases} -2C + 6D = 5 \\ -6C - 2D = 0 \end{cases} \quad \begin{aligned} C &= 1/4 \\ D &= -3/4 \end{aligned}$$

Gen. sol: $y = Ae^{2x} + Be^{x} + \frac{1}{4}\sin(2x) - 3/4\cos(2x)$, now find A and B.

Side track — More linear algebra?

Let $u(x)$ be an unknown function, such that $u'' - 3u' + 2u = 0$. Introduce $v(x) := u'(x)$. Then,

$$v'(x) = u''(x) = 3u' - 2u = 3v - 2u.$$

Diagonal matrices, easy!

In matrix format,

$$\begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}}^{A} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}.$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

What is the characteristic polynomial of the matrix?

$$\left.\begin{aligned} u' &= 2u \\ v' &= v \end{aligned}\right\} \quad \begin{aligned} u &= C \cdot e^{2x} \\ v &= D \cdot e^{x} \end{aligned}$$

We have

$$\begin{vmatrix} 0 - r & 1 \\ -2 & 3 - r \end{vmatrix} = -r(3 - r) + 2 = r^2 - 3r + 2.$$

$r_1 = 2$
$r_2 = 1$

Diagonalize:

$A = T^{-1} D \cdot T$

$$T \begin{pmatrix} u' \\ v' \end{pmatrix} = D \cdot T \begin{pmatrix} u \\ v \end{pmatrix}$$