Solution of the Final Exam – Part C – SF1610 Discrete Mathematics - TCOMK

Examiner: Armin Halilovic
Course responsible: Ivan Martino
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You may join for the second part of the exam only if you have attended the first one.
Part C – Time: 11:30 – 13:30 (Extra-time students: 13:00 – 16:00)
Right after, you have 20 minutes to take pictures and upload your solution on Canvas.

Short summary of the rules of the exam:

1. Use your computer only to read the questions of the exam.
2. If you may, use your phone for the Zoom-meeting call and place it so that your desk is visible; if you are not using your phone for the Zoom-meeting call, then you cannot use your phone for the whole exam.
3. No calculator, books, notes, lecture notes are allowed.
4. You may use your phone during the 20 minute break to take picture of your solution. You still need to be visible while doing so, hence you need to be in the Zoom-call from another device.

The exam consists of 10 questions in three parts: A, B and C. **Problems are not ordered by difficulty.** This exam provides a total of 39 points, plus any extra bonus points from the partial exams. The bonus point collected during the semester will be extra to the final exam score.

IMP. The full points will be given only to complete and fully explained solutions.

Evaluation table:
13 points total or more – give at least the grade of Fx
15 points total or more – give at least the grade of E
18 points total or more – give at least the grade of D
22 points total or more – give at least the grade of C
27 points total or more – give at least the grade of B
32 points total or more – give at least the grade of A

You should use paper and pen to solve the following exercises. You scan/take pictures of your solutions (jpg, jpeg, png, pdf format). Then, you have to upload your solutions gathered in a folder and compressed (as a zip or rar file) to Canvas / tasks / Final Exam – May 27, 2020 – Part 2: C.

Extra-time students should use a different folder Canvas / tasks / Final Exam – May 27, 2020 – Part 2: C – Extra time students.

IMPORTANT: The folder name should contain your last name and name; in other words use the NAME_FIRSTNAME as name of the folder with your solutions.

Write names and social security numbers on each sheet. On the first sheet write "I have done this Final exam by myself" and sign it. So you declare that you have made final exam by yourself.

The parameters p and q in the information below are the last two digits of your social security number. For example: If your social security number is 751332 2248 then p = 4 and q = 8.

PART C

9. (2p) Let G be a graph with n = 7+ (p mod 3) verticies.

a) Determine the minimum number of edges that G can have to be a non-planar (connected or not-connected) graph with n verticies.

b) Determine the minimum number of edges that G can have to be non-planar connected graph with n verticies.

Note. A not-connected graph could be a planar graph.

IMP. Only a full explained complete solution will get points.

Solution for n = 9

A graph is non-planar if and only if it contains $K_{3,3}$ or K_5 as minor (see Kutatowski's Theoerm in the Book). Note that $K_{3,3}$ has 6 verticies and 9 edges while K_5 has 5 verticies and 10 edges.

a) In our case we have n = 9 verticies. We can draw $K_{3,3}$ and add 3 isolated verticies. Then we get a (non-connected) non-planar graph with 9 edges.

Answer a): 9 edges

b) Solution for n = 9. We can draw $K_{3,3}$ (which has 6 nodes and 9 edges) and add 3 further edges to connect the 3 verticies, the ones not in $K_{3,3}$. Then, we get a connected non-planar graph with 9 + 3 = 12 edges.

Answer b): 12 edges

Answer b) for generic n:

If n = 8, minimum = 9 + 2 = 11 edges.

If n = 7, the minimum = 9 + 1 = 10 edges.

Grading: 1p for each item

10. (2p) Let K= 20+p. Compute the following sums

a) $$\sum_{0\le r\le n\le K} \binom{K}{n} \cdot \binom{n}{r}$$

b) $$\sum_{0\le r\le n\le m\le K} \binom{m}{n} \cdot \binom{n}{r}$$

IMP. Only a full explained complete solution will get points.

Solution:

a) $$\sum_{0\le r\le n\le K} \binom{K}{n} \cdot \binom{n}{r} = \sum_{n=0}^{K}\sum_{r=0}^{n} \binom{K}{n}\binom{n}{r} = \sum_{n=0}^{K} \binom{K}{n} \sum_{r=0}^{n} \binom{n}{r} = \sum_{n=0}^{K} \binom{K}{n} 2^n = (1+2)^K = 3^K$$

b) $$\sum_{0\le r\le n\le m\le K} \binom{m}{n} \cdot \binom{n}{r} = \sum_{m=0}^{K} \sum_{0\le r\le n\le m} \binom{m}{n} \cdot \binom{n}{r} = \text{(enligt a delen)} = \sum_{m=0}^{K} 3^m = \frac{3^{K+1}-1}{2}$$

"enligt a delen" means "using part a)"

Grading: 1p for each item

11. (3p) Let K=3+(q mod 2). Find all integers solutions x to the following systerm of modular equations:

$$\begin{cases} x \equiv 2 (\bmod K) \\ x \equiv 3 (\bmod 7) \\ x \equiv 5 (\bmod 11). \end{cases}$$

Note. In other words, x must be a solution of all the above modular equations.

IMP. Only a full explained complete solution will get points.

Solution for K=3

We must determine x such that

$$\begin{cases} x \equiv 2 (\bmod 3) \\ x \equiv 3 (\bmod 7) \\ x \equiv 5 (\bmod 11). \end{cases} \quad \text{(system 1)}$$

Thus, we solve the associated equation system with diophantine equations.

$$\begin{cases} x = 2 + 3r \\ x \equiv 3 + 7s \\ x \equiv 5 + 11t \end{cases} \quad \text{(system 2)}$$

In system 2, r,s, and t are integer parameters that defining x.

Subtracting to the first equation the second equation

$$\begin{cases} x = 2 + 3r \\ x \equiv 3 + 7s \end{cases} \quad (*)$$

one gets

$$0 = -1 + 3r - 7s$$

that is

$$3r - 7s = 1. \quad (**)$$

We use Euclid's algorithm to solve (**) and we get
r = 5 + 7k,
s = 2 + 3k.
Now, we also use (*) and we obtain
x = 2 + 3r = 2 + 3 (5 + 7k) = 17 + 21k, where k in an integer.
Note that we can get equivalent solution if we use the second equation in (*) and s = 2 + 3k, for instance.

Now, we substitute x=17 + 21k instaed of the first two equations in the system 1

$$\begin{cases} x = 17 + 21k \\ x \equiv 5 + 11t \end{cases} \quad (****).$$

With the same method we get the following relation among the integers k and t.

$$11t - 21k = 12$$

and solving this equation we get

k=1+11j,
t=3+21j, for a generic integer j.
We can now use all the information on k and t, together with the system (****):
x=17+21k=17+21(1+11j) =38+231j

Solution for K=3: x=38+231j
Solution for K=4: x=38+308j

Grading:
1p for correct (**)
1p for correct (****)
3p for everything correct.

12. (5p) Two groups $(G_1,*)$ and $(G_2,\circ)$ are isomorphic if there is a bijection $f: G_1 \to G_2$ such that f(x * y) = f(x) ∘ f(y) for every x and y in G_1.
Let us consider the groups $(G_1,*)$ and $(G_2,\circ)$, defined by the following operation tables:

G_1

*	x	y	z	w
x	x	y	z	w
y	y	x	w	z
z	z	w	x	y
w	w	z	y	x

G_2

∘	a	b	c	d
a	a	b	c	d
b	b	d	a	c
c	c	a	d	b
d	d	c	b	a

a) (3p) Show that $(G_1,*)$ and $(G_2,\circ)$ **are not** isomorphic.

b) (2p) Let M={***-i, i, -1,1***} where ***i*** is the imaginary complex number. This is a complex number such that $i^2 = -1$. Show that $(M,\cdot)$ is a group that **is** isomorphic to $(G_2,\circ)$; here the operation · is the complex multiplication.

IMP. Only a full explained complete solution will get points.

Solution:
Assume by contradiction that such isomorphism between G_1 and G_2 exists. In other words, there exists an injective and bijective map $f: G \to H$ such that f(p*q)=f(p)∘f(q) for every p and q in G_1.

Using the tables that define the operations for the groups $(G_1,*)$ and $(G_2,\circ)$ we know that x is the neutral element for $(G_1,*)$ and a is the neutral element for $(G_2,\circ)$.
Hence, the elements y, z, and w have order 2 in $(G_1,*)$. Indeed their square is the neutral element x.
Consider a generic non-trivial element p in $(G_1,*)$. Then, f(x)=f(p*p)=f(p)∘f(p).
The element f(p) belongs to $(G_2,\circ)$, but in such group there is only one element of other two; precisely d. Thus, every p maps to d, and the map f is not injective.
Thus, $(G_1,*)$ and $(G_2,\circ)$ are not isomphic.

b) We want to define an isomophism g from $(G_2,\circ)$ to $(M,\cdot)$. This means that we have to provide a map injective and bijective map g from $(G_2,\circ)$ to $(M,\cdot)$ such that g(p*q)=g(p)∘g(q) for every p and q in G_2.
Consider
g(a)=1
g(b)=i
g(c)=-i

$g(d)=-1$.

It is easy to check that this is a bijection. Indeed, it is injective, and every injective function from a finite set to a finite set (of the same cardinality) is also surjective. From the tables we may also verify that $g(p*q)=g(p)\circ g(q)$ holds for every p and q in G_2.

Grading:
3p for item a
2p for item b