SF1685: Calculus

Integration techniques

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Primitive functions and anti-derivatives

Recall,

$$\int f(x)dx$$

denotes the **anti-derivatives** of $f(x)$, or **the primitive functions** of $f(x)$.

A few examples of primitive functions

$$\int x^2 dx = x^3/3 + C \qquad \int e^x dx = e^x + C \qquad \int \cos(2x) dx = \sin(2x)/2 + C$$

Substitution

Suppose we have to compute

$$\int f(g(x))g'(x)dx.$$

By the chain-rule, we know that

$$D[F(g(x))] = F'(g(x))g'(x).$$

Hence, if we can find $F(x)$, given by

$$F(u) := \int f(u)du$$

then

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

$F'(x) = f(x)$

Examples, substitution

Compute

$$\int \cos(x^3)x^2\,dx \qquad \int \cos(x)e^{\sin(x)}\,dx$$

$$\frac{1}{3}\int \underbrace{\cos(x^3)\,3\cdot x^2}\,dx$$

$$\frac{1}{3}\sin(x^3) + C$$

Substitution.

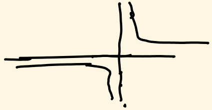
$$\int \underline{\cos x}\, e^{\sin x}\, \underline{dx} = \left[\begin{array}{l} u = \sin x \\ du = \underline{\cos x\, dx} \end{array}\right]$$

$$= \int e^u\, du = e^u + C$$

$$= e^{\sin x} + C$$

valid when $x > 0$ $\quad D[\ln(x)] = \frac{1}{x}$

$$\int \frac{1}{x}\,dx$$



$D[\ln|x|] = \frac{1}{x}$

$x \neq 0$

$\frac{1}{x}$ odd $\quad f(-x) = -f(x)$

Examples, substitution

Compute

$$\int (5x+7)^8\,dx \qquad \int \frac{x}{x^2+1}\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \underbrace{(5x+7)}_{u}{}^8\,dx = \left[\begin{array}{l} u = 5x+7 \\ du = 5\,dx \\ dx = \frac{du}{5} \end{array}\right]$$

$$\int \frac{u^8}{5}\cdot du = \frac{u^9}{9}\cdot\frac{1}{5} + C$$

$$= \frac{(5x+7)^9}{45} + C$$

$$\frac{1}{2}\int \frac{1}{x^2+1}\cdot 2x\cdot dx$$

$u\,x$

$u = x^2+1$

$du = 2x\cdot dx$

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C$$

$$\int (5x+7)^8\,dx = \frac{1}{5}\int \underbrace{(5x+7)}_{g}{}^8\cdot \underbrace{5}_{g'}\,dx$$

$f(u) = u^8$

$g(x) = 5x+7$

$g'(x) = 5$

$$\int f(g(x))\,g'(x)\,dx$$

$$= F(g(x)) + C$$

Examples, substitution

Compute

$$\int \frac{x^2}{x^3+1}\,dx \qquad \int \frac{x}{x^4+1}\,dx$$

$$\int \frac{x^2}{x^3+1}\,dx \qquad \left[\begin{aligned} u &= x^3+1 \\ du &= 3x^2\,dx \\ \frac{du}{3} &= x^2\,dx \end{aligned}\right]$$

$$\int \frac{1}{u}\,\frac{du}{3} = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|x^3+1| + C$$

$$\left[\begin{aligned} u &= x^2 \\ \frac{du}{2} &= x\,dx \end{aligned}\right]$$

$$\frac{1}{2}\int \frac{1}{u^2+1}\,du =$$

$$\frac{1}{2}\arctan(u) + C$$

$$\frac{1}{2}\arctan(x^2) + C$$

Examples

Compute

$$\int xe^{-x^2}\,dx \qquad 2\int (\cancel{4}u+\cancel{2})(\underbrace{6+u+u^2}_{t})^8\,du$$

$2u+1$

$u\ x$

$$\left[\begin{array}{l} u=-x^2 \\ -\frac{du}{2} = x\,dx \end{array}\right]$$

$$-\frac{1}{2}\int e^u\,du$$

$$= -\frac{1}{2}e^{-x^2}+C$$

$$\left[\begin{array}{l} t = 6+u+u^2 \\ dt = 1+2u \end{array}\right]$$

$$2\int t^8\,dt$$

$$= 2\frac{t^9}{9}+C$$

$$= \frac{2}{9}(6+u+u^2)^9$$

Examples, substitution

Compute

$$\int \sin(x)^5 dx \qquad \int x\sqrt{x+1}dx$$

Hint : $\sin^2 x = 1 - \cos^2 x$

$\sin^5(x) = \sin(x) \cdot (\sin^2(x))^2$

$= \sin(x) \cdot (1 - \cos^2 x)^2$

Here use $\left[\begin{array}{l} u = \cos x \\ du = -\sin(x) dx \end{array}\right]$

$\left[\begin{array}{l} x+1 = u^2 \\ dx = 2u \cdot du \\ x = u^2 - 1 \end{array}\right]$

Wishful thinking approach.

Examples, substitution

Compute

$$\int \tan(x)dx \qquad \int \sqrt{1-x^2}dx$$

Examples, substitution

Compute

$$\int \frac{1}{x\sqrt{x-4}}dx \qquad \int \frac{\sin(2x)}{\sqrt{1+\cos^2(x)}}dx \qquad \int \frac{1}{4+9x^2}dx$$

Intermission

Integration by parts

Suppose $f(x) = F'(x)$, $g(x) = G'(x)$. By the product rule,

$$D[F(x)G(x)] = f(x)G(x) + g(x)F(x).$$

Taking anti-derivatives on both sides, we have

$$F(x)G(x) = \int f(x)G(x)dx + \int g(x)F(x)dx.$$

Rewriting a bit gives

$$\int f(x)G(x)dx = F(x)G(x) - \int g(x)F(x)dx$$

ILATE — Guideline for what to derive

I L A T E

Inverse trig., logs, algebraic, trig., exponentials.

x^2

$x^{3/2}$

Examples, integration by parts

Compute

$$\int xe^x\,dx \qquad \int x^2e^x\,dx$$

derive integrate

$$\int x\cdot e^x\,dx = x\cdot e^x - \int 1\cdot e^x\,dx$$

$$= x\cdot e^x - e^x + C$$

$$\int x^2\cdot e^x\,dx = x^2e^x - \int 2x\cdot e^x\,dx$$

$$= x^2\cdot e^x - 2\underbrace{\int x\cdot e^x\,dx}_{\text{use prev. calc.}}$$

$$= x^2\cdot e^x - 2\left(x\cdot e^x - e^x\right) + C$$

Examples, integration by parts

Compute

$$\int \log(x)dx \qquad \int x\log(x)dx$$

$$\int 1\cdot\log(x)\,dx = x\cdot\log(x) - \int x\cdot\frac{1}{x}\,dx$$

$$= x\cdot\log(x) - x + C \qquad \blacksquare$$

$$\int x\cdot\log(x)\,dx = \frac{x^2}{2}\cdot\log x - \int \overbrace{\frac{x^2}{2}\cdot\frac{1}{x}}^{\frac{x}{2}}\,dx$$

$$= \frac{x^2}{2}\cdot\log(x) - \frac{x^2}{4} + C \qquad \blacksquare$$

Examples, integration by parts

Compute

we did before

$$\cancel{\int x^2 e^x dx} \qquad \int x^3 e^{x^2} dx$$

Trick: can't integrate e^{x^2} but $x \cdot e^{x^2}$ is ok.

$$\int x^2 \cdot (x \cdot e^{x^2}) \cdot dx = x^2 \cdot \tfrac{1}{2} e^{x^2} - \int 2x \cdot \tfrac{1}{2} \cdot e^{x^2} dx$$

$$= \frac{x^2}{2} \cdot e^{x^2} - \underbrace{\int x \cdot e^{x^2} dx}_{(*)}$$

$$= \frac{x^2}{2} \cdot e^{x^2} - \frac{1}{2} e^{x^2} + C$$

$$(*) \int x \cdot e^{x^2} dx = \frac{1}{2} \int 2x \cdot e^{x^2} dx = \frac{1}{2} e^{x^2} + C \quad \left(\begin{array}{c}\text{subst.}\\ u = x^2\end{array}\right).$$

Alt: (Does not work!)

$$\int x^3 e^{x^2} dx = \frac{x^4}{4} e^{x^2} - \underline{\int \frac{x^4}{4} \cdot 2x \cdot e^{x^2}} dx$$

x^3: done

$\rightarrow$ now need to compute $\int x^5 e^{x^2} dx$ $\rightarrow \int x^7 e^{x^2} \ldots$

This is worse! The x-power has increased!

Examples, integration by parts

Compute

$$\int \arctan(x)dx \qquad \int x\sin(x)dx$$

$(*)$ $\int 1\cdot\arctan(x)\,dx = x\cdot\arctan x - \underline{\int x\cdot\frac{1}{x^2+1}\,dx}$

Now $\int \frac{x}{x^2+1}\,dx = \left[\begin{array}{l} u = x^2+1 \\ \frac{du}{2} = x\,dx \end{array}\right] = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log|x^2+1| + C$

So $(*)$ is equal to

$x\cdot\arctan(x) - \frac{1}{2}\log|x^2+1| + C$ $\blacksquare$

$\int x\cdot\sin(x)\,dx = -x\cdot\cos(x) - \int 1\cdot(-\cos x)\,dx$

$= -x\cdot\cos(x) + \sin(x) + C$ $\blacksquare$

Examples, integration by parts

$e^{x^2} \neq e^{2x}$, $(e^x)^2 = e^{2x}$

Compute

$$A = \int \sin(x)e^{2x}\,dx \qquad \int \log(x)x^3\,dx$$

$$A = \int \sin(x)\cdot e^{2x}\,dx = \sin(x)\frac{e^{2x}}{2} - \frac{1}{2}\underbrace{\int \cos(x)\cdot e^{2x}\,dx}$$

$$B = \int \cos(x)\cdot e^{2x}\,dx = \cos x\cdot\frac{e^{2x}}{2} + \frac{1}{2}\underbrace{\int \sin(x)\, e^{2x}\,dx}_{A}$$

Solve for A.

$$A = \frac{\sin(x)\cdot e^{2x}}{2} - \frac{1}{2}\left(\cos x\cdot\frac{e^{2x}}{2} + \frac{1}{2}A\right)$$

$$A = \frac{\sin(x)e^{2x}}{2} - \frac{1}{4}\cdot\cos(x)e^{2x} - \frac{1}{4}A$$

$$\frac{5A}{4} = \left(\frac{\sin(x)}{2} - \frac{1}{4}\cos(x)\right)e^{2x}$$

So

$$A = \frac{1}{5}e^{2x}\left(2\sin(x) - \cos(x)\right) + C$$

Needed to solve an equation!

Examples, integration by parts

Compute $\int \sin^2(x)dx$ by integration by parts.

$$A = \int \sin(x)\cdot\sin(x)\,dx = -\sin(x)\cdot\cos(x) + \int \cos(x)\cdot\cos x\,dx$$

$$B := \int \cos^2 x\,dx = \int 1-\sin^2 x\,dx = \underbrace{\int 1\,dx}_{x} - \underbrace{\int \sin^2 x\,dx}_{A} = x - A$$

Hence,

$$A = -\sin(x)\cdot\cos(x) + (x - A)$$

$$A = \frac{-\sin(x)\cos x + x}{2} + C$$

Answer: $\frac{x}{2} - \frac{\sin(x)\cdot\cos(x)}{2} + C$ $\blacksquare$

Example

Compute $\int \sin^2(x)dx$ by using algebra/trig rules.
(*)

Most important trig rule:

$$\left[\cos 2x = \cos^2 x - \sin^2 x\right]$$
$$= (1-\sin^2 x) - \sin^2 x$$
$$= 1 - 2\sin^2 x.$$

solve for $\sin^2 x$:

$$\left[\frac{1-\cos 2x}{2} = \sin^2 x\right]$$

(*) equals

$$\int \frac{1}{2} - \frac{\cos 2x}{2}\, dx =$$

$$\underbrace{\frac{1}{2}\int 1\, dx} - \frac{1}{4}\int \underline{2\cdot\cos 2x}\, dx$$

$$\frac{x}{2} - \frac{1}{4}\cdot \sin 2x + C. \blacksquare$$

$$\sin(2x) = 2\cdot\sin(x)\cdot\cos(x)$$

$$= \frac{x}{2} - \frac{\sin(x)\cdot\cos(x)}{2} + C \blacksquare$$