SF1685: Calculus

The derivative

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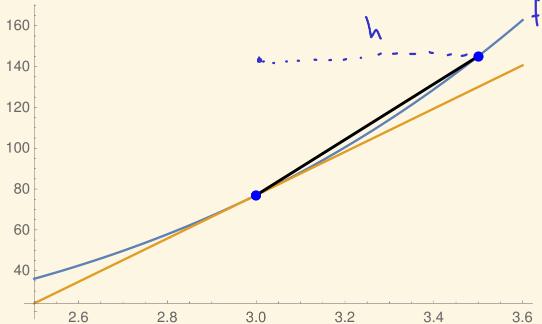
Derivative, intuition



f(t), total distance traveled at time t.

Let $f(t)$ be the total distance traveled at time t. Then $\frac{f(t+h)-f(t)}{h}$ is the **average speed** between time t and time $t+h$.

*We want a notion of speed **at** time t.*

Derivative, definition

We say that the derivative of $f(t)$ at time t_0 is the *limit*

$$\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}$$

We can view this as a function of t_0, so we can define a **new function**, **the derivative** as

$$f'(t) := \frac{f(t+h) - f(t)}{h}, \quad \text{as } t \to 0.$$

This new function is only defined where the limit exists! A function is **differentiable** if the derivative exists everywhere in its domain.

Derivative, notation

The derivative of f with respect to parameter t, can be expressed in several ways.

$$f'(t), \qquad D[f], \qquad f'_t(t), \qquad \frac{df}{dt}, \qquad \dot{f}(t).$$

Liebniz notation

The notation with a dot is most common in physics, and is derivative with respect to *time*.

Notation will be more important later, when dealing with functions depending on several variables.

Using

$$D[\cdots]$$

as notation for derivative is *extremely convenient*!

$$f'(t_0) \neq D[\underbrace{f(t_0)}_{\text{constant}}]$$

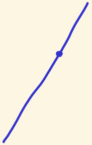
Derivative of a linear function

x	$f(x)$
0.01	2
0.02	2.3
0.03	2.6
$\vdots$	$\vdots$

Can we interpret $f'(x)$?

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Let $f(x) := ax + b$, a, b constants. Then



$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[a(\cancel{x} + h) + \cancel{b}] - [a\cancel{x} + \cancel{b}]}{h} \\ &= \lim_{h\to 0} \frac{a\cancel{h}}{\cancel{h}} \\ &= a. \end{aligned}$$

Derivative of a sum

Let $f(x) \coloneqq r(x) + s(x)$, where r, s are *differentiable*. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[r(x+h) + s(x+h)] - [r(x) + s(x)]}{h} \\ &= \lim_{h\to 0} \frac{r(x+h) - r(x)}{h} + \frac{s(x+h) - s(x)}{h} \\ &= r'(x) + s'(x). \end{aligned}$$

Here, we use properties of limits. Conclusion: $D[f(x) + g(x)] = D[f(x)] + D[g(x)]$.

Derivative of x^2

$$
\begin{aligned}
D[x^2] &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{2\cancel{h}x}{\cancel{h}} + \underbrace{h}_{\to 0} \\
&= 2x.
\end{aligned}
$$

We shall later show that $D[x^n] = nx^{n-1}$, for all $n \in \mathbb{R}$.

Derivative of sin(x)

Need the addition rule, $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$.

$$
\begin{aligned}
D[\sin(x)] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)\cdot 1}{h} \\
&= \lim_{h\to 0} \sin(x)\underbrace{\frac{(\cos(h) - 1)}{h}}_{\to 0} + \cos(x)\underbrace{\frac{\sin(h)}{h}}_{\to 1} \qquad = \cos(x).
\end{aligned}
$$

This is equal to cos(x), by using standard limits we have proved before.

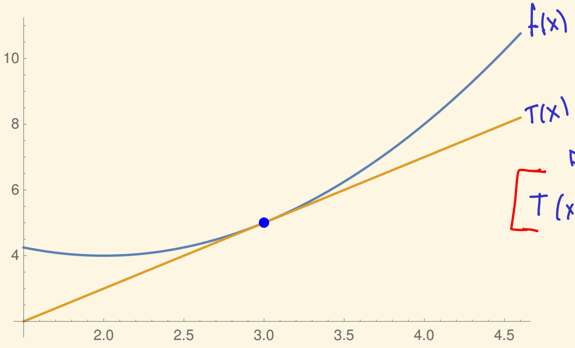
Derivative of $\cos(x)$ — discussion

Need the addition rule, $\cos(x + h) = \cos(x)\cos(h) \cancel{+} - \sin(x)\sin(h)$.

very similar to above!

Tangents

Let $f(x)$ be a function. The **tangent** to $f(x)$ at x_0, is the *line* $\underbrace{ax + b}_{T(x)}$, which passes through $(x_0, f(x_0))$ and has **slope** $f'(x_0)$.



$f(x_0) = T(x_0)$ (*)

$f'(x_0) = T'(x_0) = a$

Formula for tangent line at $x = x_0$.

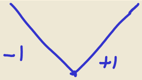
$$\boxed{T(x) = f'(x_0)(x - x_0) + f(x_0)}$$

Discussion questions

Question

Let

$$\text{wait}\begin{cases} \sin(\pi/4) = a\cdot\frac{\pi}{4} + b & \text{cont.} \\ \cos(\pi/4) = a \end{cases}$$



$f(x) = |x|$

$$f(x) := \begin{cases} \sin(x) & \text{if } x < \pi/4 \\ ax + b & \text{otherwise.} \end{cases}$$

Determine a and b so that $f(x)$ is differentiable for all $x \in \mathbb{R}$.

the derivative exists.

Question

Compute the tangent to $f(x) := (x-2)^2 + 4$, at $x = 3$.

$\{\log, \arctan, \dots\}$

$\{\exp, \sin, \cos\}$

Slope: $f'(3) = 2$

$f(x) = x^2 - 4x + 8$, $f(3) = 9 - 12 + 8 = 5$

$f'(x) = 2x - 4$

Tangent line : $2\cdot(x-3) + 5 = 2x - 1$

Notes

Need to check $x=\pi/4$.

$$\lim_{h\to 0^+} \frac{f(\pi/4+h)-f(\pi/4)}{h} = \lim_{h\to 0^+} \frac{a(\cancel{\pi/4}+h)+\cancel{b} - a(\cancel{\pi/4})-\cancel{b}}{h} = a$$

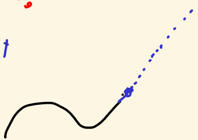
$$=$$

Derivative on the RHS of $\pi/4$ is just a.

Must be the same!

$$\lim_{h\to 0^-} \frac{f(\pi/4+h)-f(\pi/4)}{h} = \lim_{h\to 0^-} \frac{\sin(\pi/4+h) - \frac{a\cdot\pi}{4} - b}{h} = a$$

numerator must go to 0 as $h\to 0$,



$$\Rightarrow \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \underbrace{= \frac{a\cdot\pi}{4}+b}$$

Must hold, $\Leftrightarrow$ f is continuous!

$$\begin{cases} a = \frac{\sqrt{2}}{2} \\ \frac{a\cdot\pi}{4} + b = \frac{\sqrt{2}}{2} \end{cases}$$

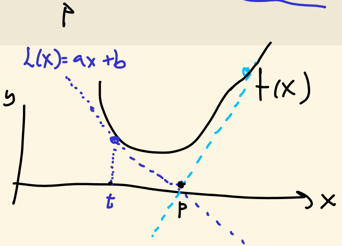
Solve for b.

$\cos(\pi/4) = \frac{\sqrt{2}}{2}$, slope on the left at $\pi/4$, $\Rightarrow a = \frac{\sqrt{2}}{2}$.

Tangents, question II

Question

Is there a line (or lines) through the point $(1, 2)$, which is a tangent to $f(x) := 4x^2 + 13x + 1$?



Alt.
Lines through (1,2)
$l(x) = ax - a + 2$

t: x-coordinate for intersection of f(x) and the line.

$f'(x) = 8x + 13.$

$$
\begin{aligned}
l(1) = 2 \\
l(t) = f(t) \\
l'(t) = f'(t)
\end{aligned}
\iff
\begin{cases}
a + b = 2 & \text{(line should pass the pt)} \\
at + b = 4t^2 + 13t + 1, & (*) \text{ pass through tangent pt.} \\
a = 8t + 13,
\end{cases}
$$

$a = 8t + 13$
$b = 2 - (8t + 13)$

$(*)$ $(8t + 13)t + 2 - (8t + 13) = 4t^2 + 13t + 1$, just one unknown!

Solve: $t_1 = -1$, $t_2 = 3$.

→ this gives a and b

Yes, tangent lines at $x = -1$, and $x = 3$.

Notes

t_1 gives $a_1 = -8+13 = 5$, $b_1 = 2-(-8+13) = -3$ $\quad \underline{L_1(x) = 5x-3}$

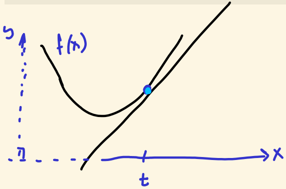
t_2 gives $a_2 = 24+13 = 37$, $b_2 = 2-37 = -35$ $\quad \underline{L_2(x) = 37x-35}$

Answer: yes, we have two such lines, $L_1(x)$ and $L_2(x)$.

Tangents, question III

Question

Determine for which values of a, the function $f(x) := 2x^2 + ax + 3$ is tangent to the line $2x + 1$.



Let t be the point of tangency. $f'(x) = 4x + a$,

$$\begin{cases} 4t + a = 2 & \text{[tangent line slope = derivative at } t\text{]} \\ 2t^2 + at + 3 = 2t + 1 & \text{[need to intersect at } t\text{]} \end{cases}$$

t is x-coord for tangency

$a = 2 - 4t$

$\Rightarrow \quad 2t^2 + (2-4t)\cdot t + 3 = 2t + 1$

$t_1 = -1 \quad t_2 = 1$

Check! $(t = -1)$ $\quad 2 + 6(-1) + 3 \stackrel{?}{=} -2 + 1$

$-4 + 3 \stackrel{?}{=} -1$ yes!

Case $t = -1$ gives $a = 6$,

case $t = 1$ gives $a = -2$

Answer: $a \in \{-2, 6\}$.

$a = 6$, $a = -2$

Tangents, question IV

Question

Let $f(x) = x^2 + 4$, $g(x) = -x^2 - 2x - 1$. Find the common tangent lines of the two functions.

s, t x-coordinates for intersections.

Notes

Derivative

Question

Let

$$f(x) := \begin{cases} x^2 + 3x + 1 & \text{if } x \leq 2 \\ ax + b & \text{otherwise.} \end{cases}$$

Determine a and b so that $f(x)$ is differentiable for all $x \in \mathbb{R}$.

Question

Is there a line (or lines) through the point $(0, 8)$, which is a tangent to $f(x) := x^3 + x + 10$?

Tangents, try yourself

Question

Let $f(x) = x^2 + x + 2$, $g(x) = -x^2 + 3x - 3$. Find the common tangent lines of the two functions.

Question

Let $f(x) = x^4 - 18x^2 + 6x$. Find the tangent line to $f(x)$ which tangents f in two different points.