SF1685: Calculus

The chain rule

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Side note on D

Recall that we have that D represents taking the derivative of a function.

D : functions $\to$ functions. This called an **operator**, and it acts on the vector space of (differentiable) functions.

You might want compare this with *higher-order functions*, e.g., `map`, `fold`.

$f(x) = x+2$

$$\texttt{map} : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R}^n \to \mathbb{R}^n).$$

$(x, y, z, \ldots) \longrightarrow (x+2, y+2, z+2, \ldots)$

The product rule

We have that $D[f \cdot g] = D[f]g + D[g]f$. Not! $D[f] \cdot D[g]$

By definition:

$$D[f \cdot g] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Subtract and add $f(x+h)g(x)$ in the numerator:

$$\frac{f(x+h)g(x+h) - f(x+h)g(x) \quad + \quad f(x+h)g(x) - f(x)g(x)}{h}$$

Rewrite

$\to f(x)$ $\quad \to g'(x)$ $\quad \to f'(x)$

$$f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}$$

Now recognize the derivatives.

Application of the product rule (proof by induction)

Last time.

$D[x] = 1$.

For *integers* $n \geq 1$, we have that $D[x^n] = nx^{n-1}$.

induction hypothesis

We proved statement is true for $n = 1$. Assume true for some fixed $n \geq 1$. Then,

$$D[x^{n+1}] = D[x \cdot x^n] = \{\text{by prod. rule}\} = D[x] \cdot x^n + x \cdot D[x^n]$$

Now

$$D[x] \cdot x^n + x \cdot D[x^n] = 1 \cdot x^n + x \cdot (n)x^{n-1} = (n+1)x^n.$$

Hence, *formula true for n implies that the formula is true for n + 1*.

$$\left\{\begin{array}{l} D[x^n] = n \cdot x^{n-1} \\ D[x] = 1 \end{array}\right. \Longrightarrow D[x^{n+1}] = (n+1)x^n.$$

We now know how to compute derivatives of polynomials.

Last missing rule is that for function composition, $D[f(g(x))]$. The chain rule states that

$$D[f(g(x))] = f'(g(x)) \cdot g'(x)$$

The chain rule (incomplete proof)

We assume f, g are differentiable functions. → f and g are continuous!

$$D[f(g(x))] = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

constant

Multiply numerator and denominator:

if $=0$, it's bad!

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{\boxed{g(x+h) - g(x)}} \cdot \frac{\boxed{g(x+h) - g(x)}}{h}$$

e.g. $g(x) = c$

Now let $k := g(x+h) - g(x)$, and $y = g(x)$. As $h \to 0$, we know that $k \to 0$ (why?).
Then,

g is continuous!

$$\frac{f(y+k) - f(y)}{k} \cdot \frac{g(x+h) - g(x)}{h}$$

$$f'(g(x)) \cdot g'(x)$$

Discussion

Question

Why is the proof above sometimes **invalid**? Discuss!

Question

Show that $Dx - xD = 1$. That is, verify that for all differentiable $f(x)$,

$D[x \cdot f] - x \cdot D[f] = f$

$$D[x \cdot f(x)] - x \cdot D[f(x)] = f(x).$$

Compare with $A = B$ for matrices iff $A\mathbf{v} = B\mathbf{v}$ for all vectors of appropriate dimension.

$D[x \cdot f] = 1 \cdot f + \underline{x \cdot f'}$

$x \cdot D[f] = \underline{x \cdot f'}$

$\Rightarrow \quad D[x \cdot f] - x \cdot D[f] = f$

$\underbrace{Dx - xD}_{\text{operator}} = \underbrace{1}_{\text{operator}}$

The chain rule, consequences

$$f/g = f \cdot \frac{1}{g}$$

We can easily prove the *quotient rule*,

$$D[f/g] = \frac{f' \cdot g - g' \cdot f}{g^2}$$

by computing $D[1/g]$ first using the chain rule, and then using the product rule.

Chain rule questions — Discussions

Question

Compute the derivatives of the following functions:

$$\sin(x^3), \qquad e^{f(x^2+3x)}, \qquad f(g(x)^5).$$

$D[\sin(x^3)] = \cos(x^3)\cdot 3x^2.$

$$D[e^{f(x^2+3x)}] = e^{f(x^2+3x)}\cdot D[f(x^2+3x)]$$
$$= e^{f(x^2+3x)}\cdot (f'(x^2+3x)\cdot(2x+3))$$

$$D[f(g(x)^5)] = f'(g(x)^5)\cdot D[g(x)^5]$$
$$= f'(g(x)^5)\cdot 5\cdot g(x)^4\cdot g'(x)$$

outer function is x^5

$D[x^5] = 5x^4.$

Problem

Question

How do we compute $D[x^x]$? (Defined whenever $x > 0$).

Note that $x^x = \exp(\log(x^x)) = \exp(x \log(x))$. Now we can apply the chain and product rule! We only need to know how to compute $D[e^x] = e^x$ and $D[\log(x)] = 1/x$ (we shall prove this later). We get

$$\begin{aligned} D[x^x] &= D[\exp(x \log(x))] \\ &= \exp(x \log(x)) \cdot D[x \log(x)] \\ &= x^x \cdot (D[x] \cdot \log(x) + x \cdot D[\log(x)]) \\ &= x^x \cdot (\log(x) + 1) \,. \end{aligned}$$

Trick!

$$f(x)^{g(x)} = e^{g(x) \cdot \log(f(x))}$$

For $\alpha \in \mathbb{R}$, we have that $D[x^\alpha] = \alpha x^{\overset{\alpha-1}{\cancel{n-\alpha}}}$.

Rule:

$\underline{\log(x^\alpha) = \alpha \cdot \log(x)}$

We have

$$
\begin{aligned}
D[x^\alpha] &= D[\exp(\alpha \log(x))] \\
&= \exp(\alpha \log(x)) \cdot D[\alpha \log(x)] \\
&= x^\alpha \cdot \left(\alpha \cdot x^{-1}\right) \\
&= \alpha x^{\alpha-1}.
\end{aligned}
$$

Higher order derivatives

Taking derivative a second time, gives $f''(x)$. In general, we shall use the notation below for the second derivative,

$$f''(x), \quad D^2[f], \quad \frac{d^2 f}{dx^2}$$

$s(t)$ total distance traveled

$s'(t)$ velocity at time t

$s''(t)$ acceleration.

and

$$f^{(n)}(x), \quad D^n[f], \quad \frac{d^n f}{dx^n}$$

for the nth derivative.

Solution to actually computing $\sin(0.2)$

$\sin(x) \approx x - \frac{x^3}{6} + \dots$

Application of the third derivative in presidential elections

From wikipedia:
When campaigning for a second term in office[1], U.S. President Richard Nixon announced that the rate of increase of inflation was decreasing, which has been noted as *the first time a sitting president used the third derivative to advance his case for reelection.*

$$\text{inflation} := \text{rate at which purchasing power of money decreases.}$$

Thus, Nixon was talking about $P^{(3)}(1972)$, where $P(t)$ is the purchasing power of money at time t.

[1] Nixon was reelected on November 7, 1972, in one of the largest landslide election victories in American history.

Complicated formula?

Question

$D^{n+1}[x^2e^x] = x^2e^x + 2(n+1)xe^x + (n+1)ne^x$

$\Uparrow$

Prove that for $n \geq 1$,

$$D^n[x^2e^x] = x^2e^x + 2n \cdot x \cdot e^x + \underbrace{(n^2 - n)}_{n(n-1)} \cdot e^x. \qquad (*)$$

Method: induction! Base case: $n=1$

$D[x^2 \cdot e^x] = 2x \cdot e^x + x^2 \cdot e^x$! Rhs$(n=1)$: $x^2 e^x + 2x \cdot e^x$ ok, same!

Ind. hyp: Assume $D^n[x^2e^x] = x^2e^x + 2nxe^x + (n^2-n)e^x$ $(*)$.

$\Rightarrow D^{n+1}[x^2 \cdot e^x] = D[x^2e^x] + 2n\, D[x \cdot e^x] + (n^2-n)\, D[e^x]$

$= 2xe^x + x^2e^x + 2n(e^x + x \cdot e^x) + (n^2-n)e^x$

$= x^2 \cdot e^x + 2(n+1)\, x \cdot e^x + \underbrace{(n^2+n)}_{(n+1)n} e^x$ $(*)$ for $n+1$.

Proof of formula

Mathematica?

If there is time and interest, we can do some calculations in Mathematica.

Complicated formula II — Homework

Question

Prove that for $n \geq 1$,

$$D^n[x^3 e^x] = x^3 e^x + 3nx^2 e^x + 3n(n-1) \cdot x \cdot e^x + n(n-1)(n-2) \cdot e^x.$$

Hint: Use proof by induction.

Question

Show that

$$D^2 x^2 - 2xD^2 x + x^2 D^2 = 2.$$

That is, verify that for any function f with a second derivative,

$$D^2[x^2 \cdot f(x)] - 2xD^2[x \cdot f(x)] + x^2 D^2[f(x)] = 2 \cdot f(x).$$

Hint: Use $Dx - xD = 1$ a few times.