Exam for Discrete Mathematics SF1610 for TCOMK, May 31st, 8:00-13:00,

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Course responsible: Stephanie Ziegenhagen

No books/calculators or other forms of help are allowed.

Grading: : (OBS: the total amount of points in this exam is 37p.)

13 points or more gives the Fx grade
15 points or more are required for the grade E
18 points or more are required for the grade D
22 points or more are required for the grade C
28 points or more are required for the grade B
32 points or more are required for the grade A

Note: For getting the full amount of points a fully motivated solution needs to be given for each exercise.

Part I

Each of the exercises 1 to 5 correspond to the Partial Exam with the same number. For x = 1, 2, 3, 4, 5, if you passed Partial Exam x then you immediately receive 3 points for question x below, and you cannot gain any further points on that question.

1. (3p) Give all solutions x in $\mathbb{Z}/36\mathbb{Z}$ of the equation

$$15 \cdot x + 33 = 0.$$

Solution: We solve the diophantine equation

$$15x + 36y = -33.$$

To determine $\gcd(15, 36)$ we use the Euclidean algorithm:

$$\begin{aligned} 36 &= 2 \cdot 15 + 6 \\ 15 &= 2 \cdot 6 + 3 \\ 6 &= 2 \cdot 3, \end{aligned}$$

hence $\gcd(15, 36) = 3$. Since 3 divides -33, the equation has solutions. We backtrack the Euclidean algorithm to write $\gcd(15, 36)$ as a linear combination of 15 and 36 and get

$$3 = 15 - 2 \cdot 6 = 15 - 2 \cdot (36 - 2 \cdot 15) = 5 \cdot 15 + (-2) \cdot 36.$$

Since
$$\mathrm{lcm}(15,36) = \frac{15 \cdot 36}{\gcd(15,36)} = 15 \cdot 12 = 5 \cdot 36,$$
the set of solutions to the diophantine equation above is given by
$$\{(\frac{-33}{3} \cdot 5 + 12 \cdot k, \frac{-33}{3} \cdot (-2) - 5 \cdot k)\}.$$
Hence any x of the form $x = -55 + 12 \cdot k$ satisfies $15 \cdot x + 33 = 0$ modulo 36. In $\mathbb{Z}/36\mathbb{Z}$ we hence have the different solutions $x = 5, 17$ and 29.

2. (3p) A group consisting of 7 IT-experts and 9 mathematicians wants to form a work group consisting of 4 people. There should be at least one IT-expert and at least one mathematician in the work group. How many different work groups can they form?
(**Your answer should be a sum and/or product of integers**).

Solution: Without any restrictions we can form $\binom{7+9}{4}$ different work groups. From those we have to substract the work groups that consist entirely of IT-experts (there are $\binom{7}{4}$ such groups) and the work groups that consists entirely of mathematicians (there are $\binom{9}{4}$ such groups). Hence the answer is that there are
$$\begin{aligned}
\binom{16}{4} - \binom{7}{4} - \binom{9}{4} &= \frac{16 \cdot 15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} - \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} - \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} \\
&= 4 \cdot 5 \cdot 7 \cdot 13 - 7 \cdot 5 - 3 \cdot 7 \cdot 6 \\
&= 7 \cdot (260 - 5 - 18) \\
&= 7 \cdot 237 \\
&= 1659.
\end{aligned}$$
admissible groups.

3. (3p) Consider the group $G = (\mathbb{Z}/24\mathbb{Z}, +)$.

(a) (1p) Determine the subgroup generated by 9.

(b) (1p) Is there an element of order 4 in G? If yes, write down such an element. Otherwise, explain why there is no element of order 4.

(c) (1p) Is there an element of order 5 in G? If yes, write down such an element. Otherwise, explain why there is no element of order 5.

Solution:

(a) The subgroup generated by 9 is
$$< 9 >= \{0, 9, 18, 3, 12, 21, 6, 15\}.$$

(b) Yes, 6 has order 4:
$$< 6 >= \{0, 6, 12, 18\}.$$

(c) Since the order of any element has to divide $|G|$, and 5 does not divide $|G| = 24$, there can be no such element.

4. (3p) Determine the number of different Boolean functions $g(x, y, z)$ such that for all x, y, z the equation

$$\bar{x}\bar{y}(yz + \bar{x})g(x, y, z) = 0$$

is fulfilled.

Solution: We first rewrite $\bar{x}\bar{y}(yz + \bar{x})$ to a simpler form:

$$\begin{aligned} \bar{x}\bar{y}(yz + \bar{x}) &= \bar{x}\bar{y}yz + \bar{x}\bar{y}\bar{x} \\ &= \bar{x} \cdot 0 \cdot z + \bar{x}\bar{x}\bar{y} \\ &= 0 + \bar{x}\bar{y} \\ &= \bar{x}\bar{y}. \end{aligned}$$

So we want to know how many Boolean functions there are such that

$$\bar{x}\bar{y}g(x, y, z) = 0$$

for all x, y, z. To do this, we write down the truth table for $\bar{x}\bar{y}$ and check what values g can take if $\bar{x}\bar{y}g(x, y, z)$ has to be zero:

x	y	z	$\bar{x}\bar{y}$	$g(x, y, z)$ has to be...
0	0	0	1	0
0	0	1	1	0
0	1	0	0	0 or 1
0	1	1	0	0 or 1
1	0	0	0	0 or 1
1	0	1	0	0 or 1
1	1	0	0	0 or 1
1	1	1	0	0 or 1

Hence we are forced to set $g(0, 0, 0) = 0$ and $g(0, 0, 1) = 0$, but can choose the value of g freely otherwise. Since there are 6 combinations of arguments (x, y, z) for which we can choose $g(x, y, z)$ freely and since for each such argument we have 2 choices, this gives us 2^6 choices.

5. (3p) A planar graph G has 2 connected components and 25 vertices. There is a planar drawing of the graph in which, if one adds an edge joining the two components such that the resulting graph is planar, there are 13 regions (the outer region included). How many edges does G have?

Solution: Let G' be the graph obtained by drawing an edge connecting the connected components of G. Then G' is a planar connected graph. Hence if e', v' and f' denote the number of edges, vertices and faces of G', we have that

$$v' - e' + f' = 2.$$

But we know that $v' = 25$ and $f' = 13$. Since the number of edges in G is $e' - 1$, we can calculate the number of edges of G as

$$e' - 1 = v' + f' - 3 = 25 + 13 - 3 = 35.$$

Part II

6. (4p) Consider the symmetric group $(S_9, \circ)$ which consists of permutations of the set $\{1, ..., 9\}$. Let $\sigma \in S_9$ be the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 5 & 8 & 7 & 4 & 9 & 2 & 1 \end{bmatrix}.$$

   (a) (3p) Is there a permutation $\tau \in S_9$ such that

$$\sigma^2 \circ \tau^2 = \tau^2 \circ \sigma?$$

   If there is such a τ, write it down. Otherwise explain why no such τ exists.

   (b) (1p) Is there a permutation $\omega \in S_9$ such that

$$\sigma^2 \circ \omega^{-1} = \sigma^3?$$

   Again, if there is such a ω, write it down. Otherwise explain why no such ω exists.

Solution:

   (a) We write σ in cycle notation as

$$\sigma = (13579) \circ (2648).$$

   Since $\mathrm{sgn}((13579)) = 1$ and $\mathrm{sgn}((2648)) = -1$, the permutation σ is odd. But regardless of how we define τ, for any $\tau \in S_9$

$$\mathrm{sgn}(\sigma^2 \circ \tau^2) = \mathrm{sgn}(\sigma)^2 \mathrm{sgn}(\tau)^2 = 1 \quad \text{and} \quad \mathrm{sgn}(\tau^2 \circ \sigma) = \mathrm{sgn}(\tau)^2 \mathrm{sgn}(\sigma) = -1.$$

   Hence there is no τ satisfying the above equations.

   (b) Using that S_9 is a group, we see that

$$\begin{aligned} & \sigma^2 \circ \omega^{-1} = \sigma^3 \\ \Leftrightarrow \quad & \sigma^2 = \sigma^3 \circ \omega \\ \Leftrightarrow \quad & \sigma^{-3} \circ \sigma^2 = \omega \\ \Leftrightarrow \quad & \sigma^{-1} = \omega. \end{aligned}$$

   Hence ω is the inverse of σ, which is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 1 & 6 & 3 & 2 & 5 & 4 & 7 \end{bmatrix}.$$

7. For each of the graphs (A), (B) and (C) determine whether the graph...

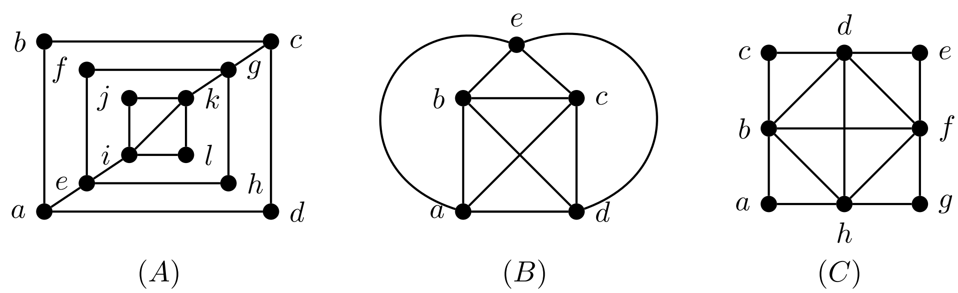
   (a) (1p) ... has a Eulerian trail,

   (b) (1p) ... has a Eulerian circuit,

(c) (1p) ... has a Hamiltonian cycle,

(d) (1p) ... is planar.

In each of these cases, if the answer is yes, write down or draw respectively a Eulerian trail, Eulerian circuit, Hamiltonian cycle or a planar drawing of the graph. If the answer is no, explain why not. (You can refer to results from the course.)



Solution:

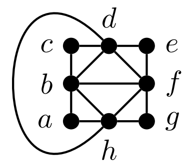
(a)
- (A): Yes, (A) has a Eulerian trail, for example by visiting the vertices (a,b,c,d,a,e,f,g,h,e,i,j,k,i,l,k,g,c)
- (B) Yes, (B) has a Eulerian trail, for example (e,d,c,e,a,d,b,a,c,b,e)
- (C) No, (C) has no Eulerian trail because the vertices b, d, f and h all have odd degree.

(b)
- (A) No, (A) does not have a Eulerian trail, since a and c both are vertices of odd degree.
- (B) Yes, (B) has a Eulerian circuit. The Eulerian trail in (a) already is a Eulerian circuit.
- (C) No: Since (C) doesn't have a Eulerian trail by (a), it can't have a Eulerian cycle.

(c)
- (A) No, (A) does not have a Hamiltonian cycle: If it had a Hamiltonian cycle, this cycle would have to use the edge between a and b and the edge between b and c. Similarly, the cycle would have to travel along the edge between a and c and the edge between c and d. Furthermore, it is not possible to visit b and then leave the outer cycle of the graph and visit d later, or the other way around. Hence any trail has to contain a cycle consisting of a,b,c and d, for example (a,b,c,d,a) or (b,a,d,c,a) etc. But this means we arrive back at our starting point before being able to visit any other vertex.
- (B) Yes, (B) has the Hamiltonian cycle (e,c,d,a,b).
- (C) Yes, (C) has the Hamiltonian cycle (a,b,c,d,e,f,g,h,a).

(d) (A) Yes, (A) is planar since the drawing in the exam already is a planar drawing. (B) No, (B) is not planar because it is the complete graph on 5 vertices, K_5.

(C) Yes, (C) is planar:



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8. (4p) Find a 1-error-correcting code C of length 10, with 64 codewords, and such that C contains the word `1111111111` and corrects the word `1111000000` to `1111000001` (You don't need to write down all codewords if you write down a precise definition of C). If you construct a code which satisfies only some of the required properties you will get a part of the points for this question.

   Solution: We will construct a check matrix H such that we can set C to be the associated code, i.e. we set
   $$C = \{x \in (\mathbb{Z}/2\mathbb{Z})^{\times 10} | Hx = 0\}.$$

   To construct H, note the following:

   - Since the code should be of length 10, H has to have 10 columns.
   - Since C should contain $64 = 2^6$ codewords, we want that the rank $r(H)$ of the matrix H satisfies
     $$10 = 6 + r(H).$$
     Hence H has to have rank 4.

   Hence we can try to find a matrix H with 4 rows and 10 columns. To start, we make sure that H has rank 4 by defining the first four columns as follows:
   $$H = \begin{pmatrix} 1 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\ 0 & 1 & 0 & 0 & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 1 & 0 & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 1 & ? & ? & ? & ? & ? & ? \end{pmatrix}.$$

   Now let's consider the other properties our code should have:

   - It should be 1-error-correcting, hence H should have no zero column and no column should appear twice.

- The codeword $\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$ should be in C, meaning that

$$H \cdot \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix} = 0.$$

This is the case if and only if there is an even number of 1's in every row of H.

- The word $\begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}$ should be corrected to $\begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\\0\\1\end{pmatrix}$. This means that $\begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}$ has an error in the 10th digit. Hence if we calculate

$$H \cdot \begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

the result has to be the 10th column of H. Since we already defined the first four columns of H, we can actually calculate that.

$$H \cdot \begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

Hence $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ has to be the 10th column of H.

Playing around with possible values for the 5th, 6th, 7th, 8th and 9th column of H we find that if we set

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

the associated code C satisfies all the requirements above.

(Part III is on the last page)

Part III

If in this part you wish to use or refer to a theorem from the course, you must give a statement of the theorem. (Your phrasing does not have to match the phrasing in the course word-for-word.) Particular weight will be placed on your motivations in your answer to these questions.

9. A sports club consists of 30 members: The basketball team has 9 members, the volleyball team has 12 members and the soccer team has 9 members. There are three administrative groups organizing the club: The "accounting group", the "scheduling group" and the "homepage group". Every team has to send 2 representatives who volunteer to work in one of these groups (but representatives from the same team do not necessarily have to work in the same group).

   (a) (3p) How many possible combinations of forming these administrative groups are there, if there has to be at least one person in each group?
   (**Your answer should be given as a sum and/or product of integers**)

   (b) (2p) If, in addition to the requirements in part (a), the accounting group should contain members of at least two different teams, how many possibilities are there to form the administrative groups?
   (**Your answer should be given as a sum and/or product of integers**)

   Solution:

   (a) For a team with n members, there are $\binom{n}{2}$ ways of choosing to 2 people. Hence there are $\binom{9}{2} \cdot \binom{12}{2} \cdot \binom{9}{2}$ different ways of choosing the representatives in total.
   Since the representatives are different from each other, and the groups are named, but the representatives are not "ordered within the groups", there are $S(6,3) \cdot 3!$ ways of distributing the representatives among the groups.
   Hence there are

$$\begin{aligned} & \left(\binom{9}{2} \cdot \binom{12}{2} \cdot \binom{9}{2}\right) \cdot S(6,3) \cdot 3! \\ = \ & \frac{9 \cdot 8}{2} \cdot \frac{12 \cdot 11}{2} \cdot \frac{9 \cdot 8}{2} \cdot 90 \cdot 6 \\ = \ & 36 \cdot 66 \cdot 35 \cdot 90 \cdot 6 \end{aligned}$$

   ways of forming the groups.

   (b) We have to subtract the number of options where the accounting team only consists of members from one team from our previous result. We can still follow the same procedure as before for choosing representatives, the restriction only arises when forming the groups.
   Let's count the non-admissible options: If the accounting team consists of two people, and they are both from the same team, there are 3 options for which team they are from. The other 4 representatives are distributed among the other 2 groups, giving $S(4,2) \cdot 2!$ options. On the other hand, if the accounting team consists of only one person, there are 6 options for who this person could be, since we can choose that person from the group of representatives we already chose. Then there are $S(5,2) \cdot 2!$ ways of distributing the

remaining 5 representatives among the other two groups.
Hence there are

$$\left(\binom{9}{2}+\binom{12}{2}+\binom{9}{2}\right)\cdot(S(6,3)\cdot 3!-3\cdot S(4,2)\cdot 2!-6\cdot S(5,2)\cdot 2!)$$

ways of doing this.

10. (5p) Prove that all subgroups of $(\mathbb{Z},+)$ are cyclic. (Be very careful with explaining your argument and which definitions and results you use!) Hint: If H is a subgroup of $(\mathbb{Z},+)$, what might be a candidate for a generator?

Solution: Let H be any subgroup of $(\mathbb{Z},+)$. We want to prove that H is cyclic, i.e. we want to show that there is an $a\in\mathbb{Z}$ such that

$$\langle a\rangle = H.$$

Recall that for a group $(G,*)$ and $g\in G$, the subgroup $\langle g\rangle$ generated by g consists of all powers of g, i.e.

$$\langle g\rangle = \{g^k | k\in\mathbb{Z}\}.$$

Since in our case the group operation is integer addition, for $a\in\mathbb{Z}$ we have that

$$\langle a\rangle = \{k\cdot a | k\in\mathbb{Z}\}.$$

For a given subgroup H of $\mathbb{Z}$ we distinguish two cases and show that H is cyclic in either case:

- If $H=\{0\}$, then H is clearly cyclic because $\{0\}=\langle 0\rangle$.
- If $H\neq\{0\}$, then H contains an element $b\neq 0$. Since H is a subgroup, both b and its inverse $-b$ are in H. Hence H contains a positive number. Let a be the smallest positive number contained in H.
  We prove by contradiction that $H=\langle a\rangle$: Since a is in H, we know that $\langle a\rangle\subset H$. Hence if $H\neq\langle a\rangle$, this means that there is an element $c\in H$ such that $c\notin\langle a\rangle$. This means that c is not of the form $k\cdot a$ for any $k\in\mathbb{Z}$, or equivalently that a does not divide c. If a does not divide c, then $\gcd(a,c)<a$. But we know that we can write the greatest common divisor of two numbers as a linear combination of these numbers, meaning in our case that there are $x,y\in\mathbb{Z}$ such that

  $$\gcd(a,c)=a\cdot x+c\cdot y.$$

  Since a and c are in H, so are $a\cdot x$, $c\cdot y$ and their sum $a\cdot x+c\cdot y$, hence $\gcd(a,c)$ is an element in H. But $1\leq\gcd(a,c)<a$, which contradicts our choice of a as the smallest positive element in H.

Since H is cyclic in each of these cases, H is always cyclic.