SF1685: Calculus

Parametric curves

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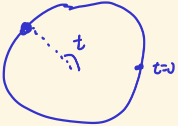
Parametric curves

A function graph describes a curve in the plane, $(x, f(x))$. But we can generalize this to describe a **parametric curve**, by allowing the x-coordinate to also depend on some parameter:

$$(f(t), g(t)). \qquad t \in [a,b] \quad , -\infty < t < \infty.$$

We have the **parameter** t and the **parametric equations** f and g.
It is convenient to think of t as a **time parameter**, which gives a direction.

The first example of a parametric curve

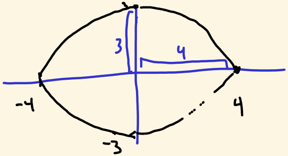


The unit circle,

$$(\cos(t), \sin(t)), \qquad t \in [0, 2\pi]$$

or an ellipse

$$(4\cos(t), 3\sin(t)), \qquad t \in [0, 2\pi]$$

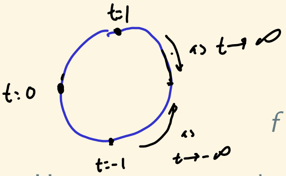


Ellipse w/ x-radius a, y-radius b is given by

$(a\cos(t), b\cdot\sin(t))$.

Alt. $\longrightarrow$ $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Another example:



$$f(t) = \frac{t^2-1}{t^2+1}, \qquad g(t) = \frac{2t}{t^2+1}, \qquad -\infty < t < \infty$$

How can we see what curve this is?

$$f(t)^2 + g(t)^2 = \frac{(t^2-1)^2 + (2t)^2}{(t^2+1)^2} = \frac{t^4 \cancel{-2t^2} + 1 + \overset{2}{\cancel{4}}t^2}{(t^2+1)^2} = \frac{(t^2+1)^2}{(t^2+1)^2} = 1$$

Distance from $(f(t), g(t))$ to $(0,0)$ is always 1.

Example

$x = t^2$ $\qquad$ $\sqrt{x} = t$

$y = 2t + t^2 - 1$ $\qquad$, $\boxed{y = 2\sqrt{x} + x - 1}$

$$f(t) = t^2, \qquad g(t) = 2t + t^2 - 1, \qquad t \geq 0$$

Here, f^{-1} exists, so this is the same curve as $(x, 2\sqrt{x} + x - 1)$ by setting $t^2 = x$. So,

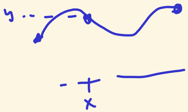
whenever f has an inverse, we can also express the curve as $(x, g(f^{-1}(x)))$.

Param. of unit circle, coord. dep. of time

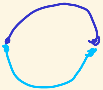
$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) = \pm\sqrt{1-\cos^2(t)} \end{cases}$$

Graph relation

$[y = \pm\sqrt{1-x^2}]$



From parametric to graph



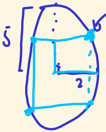
Example inspired by previous final

Parametrize the curve $x^2/4+y^2/25=1$. Sketch the curve. How big area can a rectangle have if its 4 corners are on the curve, and its sides are parallel to the coordinate axes.

Ellipse, parametrize as $\left(\frac{x}{2}\right)^2+\left(\frac{y}{5}\right)^2=1$

$$\begin{cases} x = 2\cdot\cos(t) \\ y = 5\cdot\sin(t) \end{cases}$$

$0\le t\le 2\pi$



area, as a function of this corner location.

$$\text{Area} = 4\cdot x\cdot y = 4\cdot 2\cdot 5\cdot\cos t\cdot\sin t$$
$$= 20\cdot\underline{2\cdot\cos t\cdot\sin t}$$
$$= 20\cdot\sin 2t$$

$0\le t\le\pi/2$

Want to maximize $20\cdot\underbrace{\sin(2t)}_{\text{max value } 1}$, so $2t=\pi/2$ $[t=\pi/4]$

Maximal area is 20, with a corner at $\left(\frac{2\cdot\sqrt{2}}{2}, 5\cdot\frac{\sqrt{2}}{2}\right)$

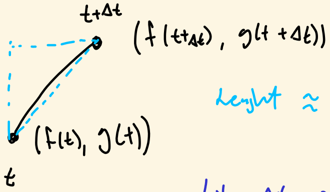
Arc length of a parametrized curve

Let $r(t) = (f(t), g(t))$ be a curve defined on some interval $a \leq t \leq b$. Then the arc length of $r(t)$ is

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt.$$

Note that we recover the previous formula we saw in the when $r(t) = (t, g(t))$.

Special case $(x, f(x))$, graph. Arc length is $\int_a^b \sqrt{1+(f'(x))^2}\,dx$



length $\approx \sqrt{\underbrace{\frac{(f(t+\Delta t)-f(t))^2}{\Delta t^2}}_{\to (f'(t))^2} + \underbrace{\frac{(g(t+\Delta t)-g(t))^2}{\Delta t^2}}_{(g'(t))^2}} \cdot \Delta t$

let $\Delta t \to 0$

Example from previous final

Show that the curve $r(t) = (\cos(t)/t^2, \sin(t)/t^2)$, with $\pi/2 \leq t < \infty$, has finite length.

It suffices to show that the length is finite. First, $D\left[\frac{\cos(t)}{t^2}\right] = -\frac{\sin(t)}{t^2} - 2\frac{\cos(t)}{t^3} = \frac{-t\sin(t) - 2\cos t}{t^3}$

and $D\left[\frac{\sin(t)}{t^2}\right] = \frac{\cos(t)}{t^2} - 2\frac{\sin(t)}{t^3} = \frac{t\cdot\cos(t) - 2\sin t}{t^3}$.

Now, arc length formula gives $\int_{\pi/2}^{\infty} \sqrt{\left(\frac{-t\sin t - 2\cos t}{t^3}\right)^2 + \left(\frac{t\cos t - 2\sin t}{t^3}\right)^2}\,dt$.

This equals,

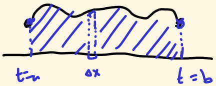
$$\int_{\pi/2}^{\infty} \frac{1}{t^3}\sqrt{t^2\sin^2 t + 4t\sin t\cos t + 4\cos^2 t + t^2\cos^2 t - 4t\sin t\cos t + 4\sin^2 t}\,dt = \int_{\pi/2}^{\infty} \frac{1}{t^3}\sqrt{t^2+4}\,dt$$

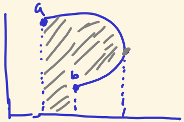
want t^2 instead,

Finite, P-integral.

Now, this is equal to $\int_{\pi/2}^{2} \frac{1}{t^3}\sqrt{t^2+4}\,dt + \int_{2}^{\infty} \frac{1}{t^3}\sqrt{t^2+4}\,dt$. First integral is finite, and second $\leq \int_{2}^{\infty} \frac{1}{t^3}\sqrt{t^2+t^2}\,dt = \int_{2}^{\infty} \sqrt{2}\cdot\frac{1}{t^2}\,dt$

Area under parameterized curve





Let us consider some curve $x = f(t)$, $y = g(t)$, with $a \leq t \leq b$. We know that area under the graph is

$$A = \int_{x(a)}^{x(b)} y dx.$$

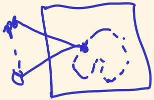
Let's do the change of variables, $y = g(t)$. Moreover $dx/dt = f'(t)$, so

$$A = \int_a^b g(t) f'(t) dt.$$

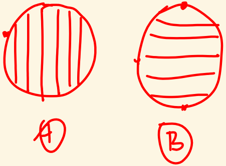
Here, we have assumed that $f'(t) > 0$ (increasing), and that $g(t) \geq 0$.
If we instead traverse the curve in the opposite direction, we get a negative contribution.

Area of a region bounded by a curve.

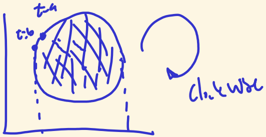
Suppose $(f(t), g(t))$, with $t \in [a, b]$ determines a **simple closed curve**, traversed clockwise. Then the area A it encloses is given by both the following expressions:

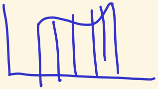


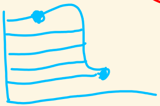
$$A = \int_a^b \underbrace{g(t)f'(t)dt}_{A} = -\int_a^b \underbrace{g'(t)f(t)dt}_{B}.$$



(If we traverse counterclockwise, the sign changes).







Greens formula for area

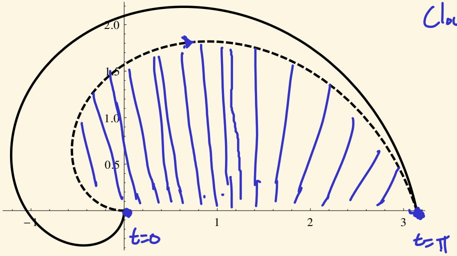
Sometimes, it is convenient to take the average of the two previous expressions, as ugly things may cancel. Thus,

$$A = \frac{1}{2}\int_a^b \big(g(t)f'(t) - g'(t)f(t)\big)\,dt,$$

when traversing the curve clockwise.

Example

Consider the curve (dashed in the figure) $r(t) = (-t\cos(t), t\sin(t))$, $0 \leq t \leq \pi$. Determine the area it bounds together with the x-axis.

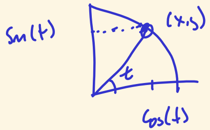


Clockwise

The other curve is given by $(-t\sin(3t/2), -t\cos(3t/2))$, in case someone is interested.

Solution

We need to compute $\frac{1}{2}\int_a^b (g(t)f'(t) - g'(t)f(t))\, dt$ for $(f(t), g(t)) = (-t\cos(t), t\sin(t))$, $a = 0$, $b = \pi$.



$f'(t) = t\cdot\sin(t) - \cos(t)$

$g'(t) = t\cdot\cos(t) + \sin(t)$

we get $t\cdot\sin(t)[t\cdot\sin t - \cos t] + t\cdot\cos(t)[t\cdot\cos t + \sin t]$

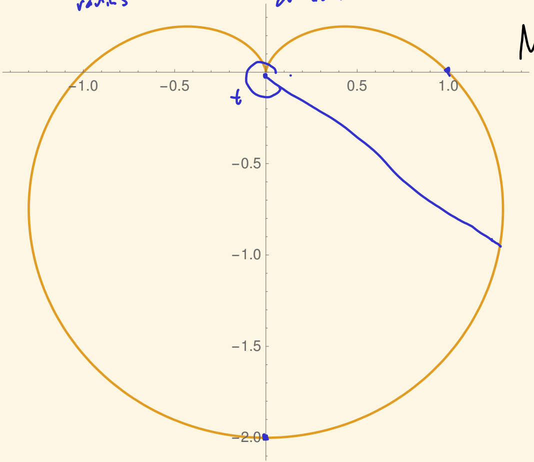
$= t\sin^2 t + t\cos^2 t = t.$

So area is $\frac{1}{2}\int_0^\pi t\, dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_0^\pi = \frac{\pi^2}{4}$ $\blacksquare$.

Cardiodid

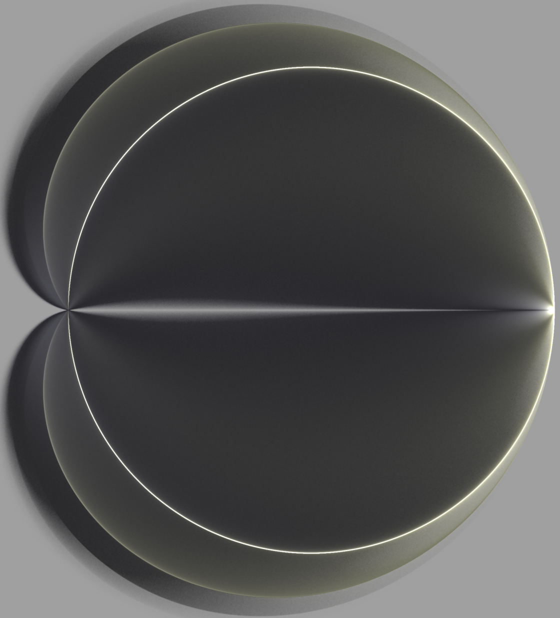
The **cardiodid** has the parametrization below. Find its area.

$$r(t) = \underbrace{(1 - \sin(t))}_{\text{radius}} \cdot \underbrace{(\cos(t), \sin(t))}_{\text{direction.}}, \qquad 0 \le t \le 2\pi$$



Note:

Counter Clockwise!



Cardiodid calculations

$$(f(t),g(t)) = (1-\sin(t))\cdot(\cos(t),\sin(t)) \qquad \frac{1}{2}\int_0^{2\pi}\left(g(t)f'(t)-g'(t)f(t)\right)dt$$

$f'(t) = (1-\sin t)\cdot(-\sin t) + (-\cos t)\cdot\cos(t)$

$g'(t) = (1-\sin t)\cos t + (-\cos t)\cdot\sin(t)$

$g(t)\cdot f'(t) = (1-\sin t)\cdot\sin(t)\cdot[\sin^2 t - \sin t - \cos^2 t)]$

$f(t)\cdot g'(t) = (1-\sin t)\cdot\cos t\,[\cos t - \sin t\cdot\cos t - \cos t\cdot\sin t]$

$\cos^2 t \overset{?}{=} \frac{1 \pm \sin 2t}{2}$

Difference : $(1-\sin t)[\sin^3 t - \sin^2 t - \sin t\cdot\cos^2 t - \cos^2 t + \sin t\cdot\cos^2 t + \cos^2 t\cdot\sin t]$

$= (1-\sin t)\,(\sin t(\underbrace{\sin^2 t + \cos^2 t}_{1}) - 1)$

$= (1-\sin t)(\sin t - 1) = -(1-\sin t)^2$

Counter CW

Area $\frac{1}{2}\int_0^{2\pi}(1-\sin t)^2\,dt$

$= \frac{1}{2}\int_0^{2\pi}\cos^2 t - 2\sin t\,dt = \underbrace{\frac{1}{2}\int_0^{2\pi}\cos^2 t\,dt}_{\text{Easy,}}$

Answer: $3\pi/2$?

Preview of next week if time

Series, infinite sums,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$$

geometric series.

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \longrightarrow \infty$$

this diverges.

Compare w. $\underbrace{\int_1^{\infty} \frac{1}{x}\,dx}_{}$

$$\lim_{k \to \infty} \left(\sum_{n=1}^{k} \frac{1}{n} - \underbrace{\int_1^{k} \frac{1}{x}\,dx}_{\ln(k)} \right) \stackrel{?}{=}$$