SF1900/SF1912 Probability Theory and Statistics
6.0 credits
Lecture 14
Theory of Inference
Statistical Hypothesis Testing
Use of Normal Approximation

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Normal Approximation

$\theta^* \in N(\theta, D(\theta^*))$

$\frac{\theta^* - \theta}{D(\theta^*)} \in N(0, 1)$

The alternative hypothesis determines the decision rule (the critical region).

If we cannot use the standard deviation $D(\theta^*)$ we estimate it with $d(\theta^*)$.

Then we use the test statistic

$\frac{\theta^* - \theta}{d(\theta^*)} \in N(0, 1)$

Example of Normal Approximation of A Test Statistic

The number of ships that, during a time interval of length t (unit: minute), pass Elsinore on their way to the south through the sound between Sweden and Denmark is considered to be $Po\left(\lambda t\right)$.

The number of ships passing in non-overlapping intervals are assumed to be independent.

A person wishes to estimate λ and test an idea about it. The person counts the number of ships in three different time intervals.

Period of Observation	30	30	40
Number of ships	10	12	18

Test the hypothesis $H_0 : \lambda = 0.5$ against $H_1 : \lambda < 0.5$ at significance level of 1%.

Model for the Random Experiment

The model for the random experiment of counting the number of passing ships.

Period of Observation	$t_1 = 30$	$t_2 = 30$	$t_3 = 40$
Number of ships	$x_1 = 10$	$x_2 = 12$	$x_3 = 18$

Each of the x_1, x_2 and x_3 are observations on X_1, X_2 and X_3

$X_1 \sim Po\,(30\lambda)$ $\qquad X_2 \sim Po\,(30\lambda)$ $\qquad X_3 \sim Po\,(40\lambda)$

$\lambda^*_{\text{obs}} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} t_i}$ (SF1900 p. 209) (SF1912 s. 270)

$\frac{\lambda^* - \lambda}{d(\lambda^*)} \sim N\,(0, 1)$ $\qquad H_0 : \lambda = 0.5 \qquad H_1 : \lambda < 0.5 \qquad \alpha = 0.01$

Testing Hypotheses on the Binomial Distribution

We roll a die $n = 18$ times. Our A is "even result".

The die shows an even result with the unknown probability p.

Test the hypothesis $H_0 : p = \frac{1}{2}$ against $H_1 : p \neq \frac{1}{2}$.

X=number of even outcomes (The test statistic)

Result: $x = 3$

$X \sim Bin\,(n, p)$

Testing Hypotheses on the Binomial Distribution

When H_0 is true $X \sim Bin\left(18, \frac{1}{2}\right)$ $\qquad n = 18$

Since $H_1 : p \neq \frac{1}{2}$ we will reject H_0 for low values of the test statistic <u>and</u> for high values.

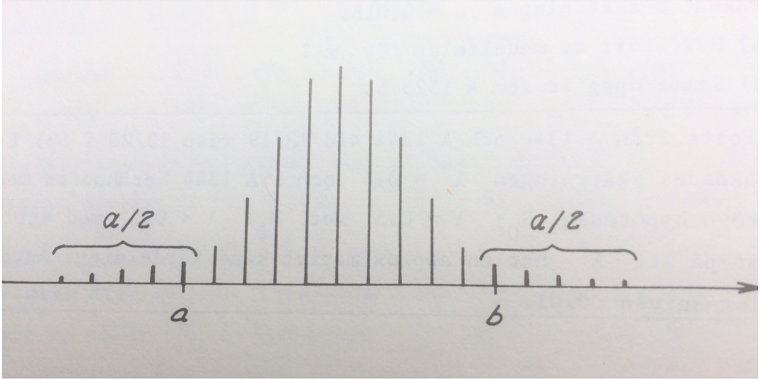
We will use the P method in this example. $\qquad npq = 4.5 < 10$ No normal approx. $p = \frac{1}{2}$ No Poisson approx.

To find the significance level α we calculate

$$\begin{aligned}
\alpha &= P\left(\text{rejecting } H_0 \text{ when } H_0 \text{ is true}\right) \\
&= P\left(X \leq a \quad \text{or} \quad X \geq b\right) \qquad \text{disjoint} \\
&= \sum_{k=0}^{a} \binom{18}{k} \left(\frac{1}{2}\right)^{18} + \sum_{k=b}^{18} \binom{18}{k} \left(\frac{1}{2}\right)^{18}
\end{aligned}$$

Choose a and b to be symmetric: $\qquad b = 18 - a$

The Null Distribution: The Distribution of the Test Statistic When the Null Hypothesis H_0 is True – $Bin(18, 0.5)$



$a = 4$ and $b = 18 - 4 = 14$

Reject H_0 if $X \leq 4$ or $X \geq 14$.

Table of the Observed Significance Level (the P Value) for $Bin(18, 0.5)$

By calculating

$$\alpha = 2\sum_{k=0}^{a} \binom{18}{k} \left(\frac{1}{2}\right)^{18}$$

for $a = 0, 1, \ldots, 5$

	$b = 17$	$b = 16$	$b = 15$	$b = 14$	$b = 13$
a	1	2	3	4	5
α	0.00014	0.00132	0.00754	0.03088	0.09626
	$***$	$**$	$**$	$*$	

* (5% significance level) ** (1% significance level) *** (0.1% significance level)

The result was $x = 3$, so we reject H_0 at 1% significance level, since the P value is $0.00754 < 0.01$.

Testing Hypotheses on the Binomial Distribution – Calculating the P Value By Normal Approximation

If $n = 150$ we can approximately calculate the probabilities on the $Bin\left(150, \frac{1}{2}\right)$-distribution.

$npq = \frac{150}{4} = 37.5 > 10$ Normal approximation works

$H_0 : p = \frac{1}{2}$ versus $H_1 : p \neq \frac{1}{2}$.

$n = 150$ and $x = 60$.

Calculate the P value approximatively.

Non Symmetric Null Distribution

When $p = \frac{1}{2}$ the binomial distribution is symmetric. This made some calculations easier.

A person is asked to utter a number $0, 1, \ldots, 9$ randomly.

$A =$ the person utters "3" or "7".

Eighteen persons are asked ($n = 18$) to do this. X=the number of times A occurs.

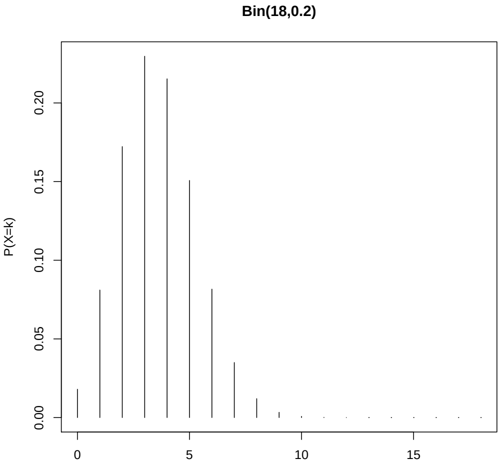
$p = P(A) = \frac{2}{10}$ if all numbers are equally likely.

The experimenter doesn't believe that all numbers have the same probability of appearing ($p \neq 0.2$)

Non Symmetric Null Distribution

$H_0 : p = 0.2$ versus $H_1 : p \neq 0.2$.

H_0 true $X \sim Bin(18, 0.2)$ (the null distribution)



Non Symmetric Null Distribution

Reject H_0 if $x \leq a$ or $x \geq b$.

$$\alpha = \sum_{k=0}^{a} \binom{18}{k} (0.2)^k (0.8)^{18-k} + \sum_{k=b}^{18} \binom{18}{k} (0.2)^k (0.8)^{18-k}$$

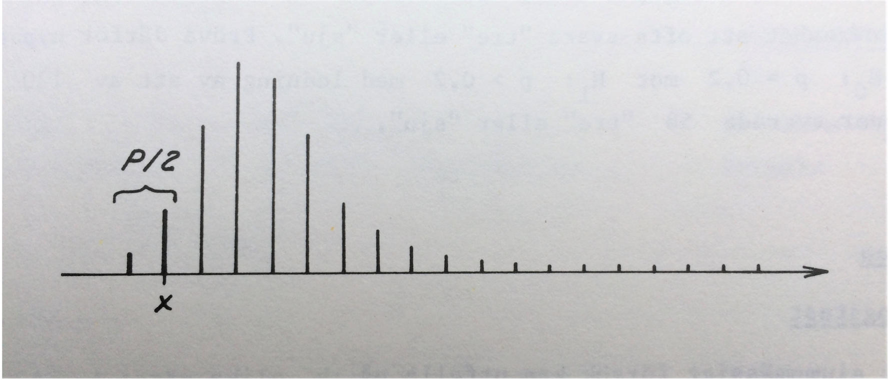
In this case it is unnatural to choose a and b symmetrically.

The sum are not necessarily equal.

Non Symmetric Null Distribution – Modified P Value Method

If the result of the random experiment x lies in the left tail of the null distrubution we calculate

$$\frac{P}{2} = \sum_{k=0}^{x} \binom{18}{k} (0.2)^k (0.8)^{18-k}$$



Non Symmetric Null Distribution – Modified P Value Method

If x belongs in the right tail

$$\frac{P}{2} = \sum_{k=x}^{18} \binom{18}{k} (0.2)^k (0.8)^{18-k}$$