SF1685: Calculus

Series III

Lecturer: Per Alexandersson, perale@kth.se

Convergence tests

- First verify that the terms converge to 0.
- Compare with the corresponding integral (function must be decreasing).
- Comparison test
- Ratio test
- Root test
- Remember the *p*-series, as standard series to compare with.

Make your own "cheat-sheet"

Alternating series

Remember, a sequence $a_0, a_1, \ldots,$ is **alternating**, if the signs are alternating, i.e., $a_n a_{n+1} < 0$ for all n.
Suppose $b_n \geq 0$ for all n. Then the *alternating series*

$$\sum_{n=0}^{\infty} (-1)^n b_n$$

converges if $b_n \to 0$ as $n \to \infty$.

Moreover, if $b_n \not\to 0$, it does not converge.

Alternating series

For example, $\sum_{n=1}^{\infty}(-1)^{n}/n$ converges. What is the limit?

$$-1 + \tfrac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \cdots$$

$$\log(1-x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

Put $x=-1$, we get $\underline{\underline{\log(2)}}$.

Taylor series

Remember that we looked at Taylor polynomials about some point a. We can define the **Taylor series** of $f(x)$ at a, as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

[No error term!]

provided that all derivatives of f exist at a, and the series converges at x.

(error $\to 0$)

$f''(0)$ does not exist

If $a = 0$ we say that it is the Maclaurin series of f.

$$\underbrace{f''(x) = \begin{cases} 0 & \text{if } x>0 \\ 2 & \text{if } x>0 \end{cases}}_{\text{Not cont.}}$$

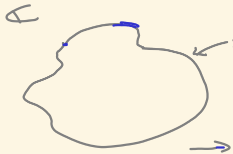
But $f^{(3)}(0)$ does not exist!

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

continuous, $f'(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases}$

Analytic functions

extremely rigid functions.

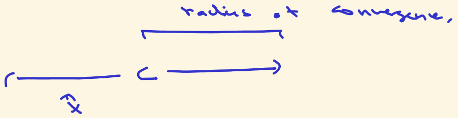


Heat eq. change, specify values at the bdd of some region, → unique analytic function in the region.

A function f is called **analytic** at c, if all the derivatives of f exist at c, and its Taylor series converges on some interval I containing c.

For analytic functions, we only know $f(c), f'(c), f''(c), \dots\dots,$ but we can compute $f(x)$ for all $x \in I$. Magic!

The maximal distance from c such that the series still converges is called the **radius of convergence**.



Radius of convergence , near $a=0$, Maclaurin case.

What is the radius of convergence for

- Ratio test:

$$\frac{a_{n+1}}{a_n} \qquad \frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!} = \frac{x}{n+1}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

First, we have shown $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any $x \in \mathbb{R}$.

- This is eventually < 1, by ratio test, converges. Radius is ∞.

Ratio test. $\frac{x^{n+1}}{x^n} = x$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Geometric series, converges if $-1 < x < 1$.

so convergence if $-1 < x < 1$,
$\Rightarrow$ Radius of convergence is 1.

-

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$$

Hint: Use the ratio test.

Ratio test: $\frac{(2x)^{n+1}}{(2x)^n} = 2x$,

converges if $-1 < 2x < 1$, so $-1/2 < x < 1/2$

Radius of convergence $= 1/2$.

Uniqueness of Taylor series

Suppose that there is some sequence $\{a_n\}$ such that in some interval I containing c, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n(x-c)^n}{n!} \text{ for all } x \in I.$$

"Infinite-dimensional vector space"

Then $a_n = f^{(n)}(c)$ for all n.

We cannot have two different taylor series for the same function!

Similar phenomenon:
suppose $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in \mathbb{R}^3$ is a basis, and $\bar{v} \in \mathbb{R}^3$, then there are unique coeffs, c_1, c_2, c_3 s.t.

$$\bar{v} = c_1\bar{u}_1 + c_2\bar{u}_2 + c_3\bar{u}_3$$

The proof is more or less same as comparing coefficients.

Using derivatives and integrals to find new Taylor series

The point: efficient way of computing Taylor series!

Warning: these operations may change the radius of convergence!

Example: Find Taylor series of $\log(1-x) = -\int \frac{1}{1-x}\,dx$.

We know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ if $-1 < x < 1$,

Let's integrate both sides!

$$\int \frac{1}{1-x}\,dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$\Longleftrightarrow$

So $-\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots -$

$\Longleftrightarrow$ (set all $x = -x$)

$-\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\Longleftrightarrow$

Larger interval: $[-1 < x \le 1]$

$\ln(1+x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots$

$\ln(2) = \ln(1+1)$

$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$

$\cdots \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$,

which we had before.

Arctan Taylor series

Example: Find Taylor series of $\arctan(x) = \int \frac{1}{1+x^2}\,dx$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \cdots$$

so $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \cdots$$

and $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$$

Finally $$\int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + x^8 \cdots\, dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots$$

Which is the Taylor series for arctan!

Application of series: some probability

Maria asks the computer for a uniform random real number between 0 and 1 and call this x_1. She then asks for a second number in the same manner, x_2. If $x_1 > x_2$, she stops, otherwise, she asks for a third number, x_3, and stops unless this number is the largest seen so far.

What is the expected number of steps before the process ends? The last query counts as a step as well.

$$x_1 < x_2 < x_3 < \dots < x_n > x_{n+1}$$

stop here.

How many steps, on average?

Solution

First: What is the probability that it takes at least $n+1$ steps?

This means, we have $x_1 < x_2 < x_3 < x_4 < \dots < x_n$

Among n different #'s, there's only one way out of $n!$ permutations where they are ordered increasingly.

Prob{at least $n+1$ steps} is $\frac{1}{n!}$.

$\Rightarrow$ Prob{exactly n steps is} is $\frac{1}{(n-1)!} - \frac{1}{n!}$

Expected # of steps: $\sum_{n=1}^{\infty} n \cdot \Big(\underbrace{\frac{n}{n(n-1)!} - \frac{1}{n!}}_{\frac{(n-1)}{n!}} \Big)$

$$= \sum_{n=1}^{\infty} \frac{n\cdot(n-1)}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e^1 \approx 2.71\dots$$

$n=1$ gives 0

Solving differential equations using Taylor series

Let $y(x)$ be a ~~differentiable~~ analytic function which satisfies $y(0) = 0$ and $dy/dx = 2xy + 2x$.
Find a_0, a_1, a_2, a_3, a_4 in the Taylor expansion

$$(*) \qquad y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

$a_0 = 0$ since $y(0) = 0$

Then, verify your answer by first checking that $y(x) = e^{x^2} - 1$ is the solution.

(*)

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Diff. eq states

$$\underbrace{a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots}_{= y'(x)} = 2x + 2x(\overbrace{a_1 x + a_2 x^2 + a_3 x^3 + \cdots}^{y(x)})$$

uniqueness of taylor series: compare coeffts!

$2a_2 = 2 \Rightarrow a_2 = 1$

$3a_3 = 2a_1 \Rightarrow a_3 = 0$

$4a_4 = 2a_2 \quad a_4 = 2/4 \cdot 1$

$a_1 = 0$, const

$a_2 = 1$

$a_3 = 0$

$a_4 = 1/2$

$a_5 = 0$

$a_6 = \cdots$

We can continue and get as many terms as we want!

Solving differential equation $dy/dx = 2xy + 2x$, $y(0) = 0$.

$e^{x^2} - 1$,

$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$

$e^{x^2} - 1 = \underline{1} x^2 + \frac{1}{2} x^4 + \frac{1}{\underline{6}} x^3 + \cdots$

a_2 a_4

Recap

- Limits
- Derivatives, min/max,
- Drawing graphs, asymptotes
- Differential equations , ansatz for y_p
- Taylor series
- Integration, surface, volume, arclength of parametric curves.
- Sequences, series