SF1685: Calculus

Applications of derivatives

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Example

In a server hall, there is one new server added every week. Moreover, in each server, the total storage capacity is increased by 2TB each day. = 2/4 TB/wk.
Today, there are 20 servers, each with 800 *TB* capacity. At what rate is the storage capacity increasing per week?

derivative

Total capacity

$C(t) = S(t) \cdot E(t)$

\# servers   cap./server

Want to find

$C'(0) = S'(0) \cdot E(0) + E'(0) \cdot S(0)$

$= 1 \cdot 800 + \frac{2}{7} \cdot 20$

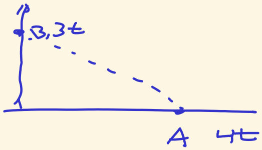
$= 800 + \frac{40}{7}$.

Example: Warm-up — discuss in breakout rooms

The point A is moving along the x-axis with speed $4cm/s$. Similarly, point B moves along the y-axis with speed $3cm/s$. At time 0, both points are located at the origin.

a) What is the distance between the points after t seconds?

b) At what rate is the distance increasing?



Distance: $\sqrt{(4t)^2 + (3t)^2} = 5t$. Answer to a).

b) this is the derivative, $5 cm/s$.

Example: Challenge — discuss in breakout rooms

The point A is moving along the x-axis with speed e^{t^2} (cm/s). Similarly, point B moves along the y-axis with speed $3e^{t^2}$. At time $t = 7$, A is located at $(3, 0)$, while B is located at $(0, 4)$.

At time $t = 7$, at what speed is the distance between the points changing?

Note, speed is not constant: Position of A: $x(t)$
B: $y(t)$

$x(7)=3$
$y(7)=4$

Distance: $\sqrt{x(t)^2+y(t)^2} = D(t)$ we want $D'(7)$.

$$D'(t) = \frac{2x(t)\cdot x'(t) + 2y(t)\,y'(t)}{2\sqrt{x(t)^2+y(t)^2}} \Rightarrow D'(7) = \frac{\cancel{2}\cdot 3\cdot e^{7^2} + \cancel{2}\cdot 4\cdot 3\cdot e^{7^2}}{\cancel{2}\sqrt{5^2}}$$

$$= \frac{e^{49}\cdot 15}{5} = \underline{\underline{3\cdot e^{49}}} \ \text{(cm/s)}$$

L'Hôpital's rule

(Bernoulli ?)

Suppose $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $0/0$ or ∞/∞. Moreover, suppose that both f and g are differentiable near a, and that $g(x) \neq 0$ near a. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

if the limit on the right exists.

Proof sketch

This is only for the case when f, g are differentiable at $x = a$, with a continuous derivative.

Suppose $f(a) = g(a) = 0$. Then

"$\frac{0}{0}$"

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - \overset{=0}{f(a)}}{g(x) - \underset{=0}{g(a)}}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

Recognize as derivatives.

$$= \frac{f'(a)}{g'(a)}.$$

The more general statement requires some more careful analysis.

Examples — see if these can be computed by using l'Hôpital's rule

$$a)\ \lim_{x\to 0} \frac{e^{\sin(2x)} - 1}{x} \qquad b)\ \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad c)\ \lim_{x\to 0} \frac{1}{\sqrt{x}} - \frac{1}{2x + \sqrt{x}} \qquad d)\ \lim_{x\to\infty} \frac{x + \sin(x)}{x}$$

No issue,

we get

$$\frac{\cos(2x)\cdot 2\cdot e^{\sin(x)}}{1}$$

b)

$$\frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$$

Now use l'Hopital's.

we get

$$\frac{2\cancel{e^{2x}}}{2\cancel{e^{2x}}} = 1$$

The limit is 1.

c)

$$\frac{\cancel{2x+\sqrt{x}} - \cancel{\sqrt{x}}}{\sqrt{x}(2x+\sqrt{x})}$$

$$\frac{2\sqrt{x}}{(2x+\sqrt{x})}$$

d)

Derivatives:

$$\frac{1+\cos(x)}{1}$$

limit doesn't exist as $x\to\infty$.

But

$$\lim_{x\to\infty} \frac{1 + \frac{\sin}{x}}{1} = 1$$

with $\frac{\sin}{x} \to 0$

Let's compute

$$\lim_{x \to 0} \frac{\sin x}{x},$$

By l'Hôpital's, we need to compute

Derivative of $\sin(x)$ at 0.

$\|$

$D[\sin x]$ at 0 is $\displaystyle \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h}.$

Now we're back where we started!

limit $\rightarrow$ derivative issue!

This is why we computed this limit with another method.

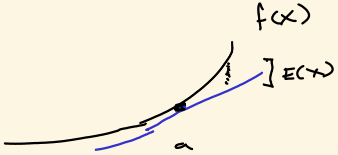
Approximations

Recall that we have $f(x) \approx f(a) + f'(a)(x - a)$, near $x = a$. That is, the tangent line gives an approximation of the function.

Let us consider the **error function**,

$$E(x) \coloneqq f(x) - f(a) - f'(a)(x - a).$$

Clearly, $E(a) = 0$.



Sidetrack — Generalized mean value theorem

If f and g are continuous and differentiable on $[a, b]$ then there is a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Recall:

Standard MVT : $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a,b)$.

Approximations

$E'(x) = f'(x) - f'(a).$

So, consider $E(x) := f(x) - f(a) - f'(a)(x-a)$ on the interval $[a,x]$.

By GMVT (with $f(x) = E(x)$, $g(x) = (x-a)^2$),

$$\frac{E(x)}{(x-a)^2} = \frac{E(x) - \overbrace{E(a)}^{=0}}{(x-a)^2 - (a-a)^2} = \frac{E'(c)}{2(c-a)}$$

for some $c \in (a,x)$. However,

$$\frac{E'(c)}{2(c-a)} = \frac{1}{2}\underbrace{\frac{f'(c)-f'(a)}{c-a}}_{\text{MVT again}} = \frac{1}{2}f''(d)$$

for some $d \in (a,c)$. Thus,

$$\frac{E(x)}{(x-a)^2} = \frac{1}{2}f''(d), \qquad \text{for some } d \in (a,x).$$

We can conclude that

$$|E(x)| \le \frac{1}{2}\max_{d\in[a,x]}|f''(d)| \cdot (x-a)^2.$$

Computation I

Let us compute $e^{0.3}$, by approximation near 0, and determine the error.

$$f(x) \approx f(a) + f'(a)(x - a)$$

Hence,

$$e^{0.3} \approx e^0 + e^0(0.3 - 0) = 1 + 0.3.$$

Computation II

The error is

$$|E(x)| \leq \frac{1}{2} \max_{d \in [a,x]} |f''(d)| \cdot (x-a)^2.$$

so in our case,

$$|E(x)| \leq \frac{1}{2} \max_{d \in [0,0.3]} e^d \cdot (0.3)^2 = \frac{0.09}{2} e^{0.3}$$

Here, we can for sure say that $e^{0.3} < \underset{e}{2.71}$, so $|E(x)| < 2.71 \cdot 0.5 \cdot 0.09 < 0.13$.

Hence, $e^{0.3} = 1.3 \pm 0.13$.

In particular, $e^{0.3} < 1.43$, so we can use that instead in the approximation above

$$E(x) < 1.43 \cdot 0.5 \cdot 0.09 < 0.07.$$

We get a better approximation $e^{0.3} = 1.3 \pm 0.07$.

The calculator gives the value $e^{0.3} = 1.34986\ldots$.

Another question

Approximate log(1.4) and determine the error.

$D[\log x] = \frac{1}{x}$ we use $a=1$, $x=1.4$

$$\log(1.4) = \underbrace{\log(1)}_{=0} + \frac{1}{1.4}(1.4-1) = \frac{0.4}{1.4} = \frac{4}{14} = \frac{2}{7} \approx 0.28.$$

$$E \le \frac{1}{2} \underbrace{\max_{d\in[1,1.4]} \left|\frac{-1}{d^2}\right|}_{=1} \cdot (0.4)^2 = 0.08 \cdot 1$$

Actual value

$\log 1.4 = 0.336\ldots$

Thus $\log(1.4) = 0.28 \pm 0.08$

ok!

$$0.2 \le \log(1.4) \le 0.36.$$

Generalization — main topic for the next lecture

In general,

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where the error is

$$\frac{1}{(n+1)!} \max_{d \in [a,x]} |f^{(n+1)}(d)| \cdot (x-a)^{n+1}.$$

Taylor polynomial

Question

(From an old final)
Compute

$$\lim_{t \to 0} \frac{e^{\sin(t)} - 1 - t}{1 + \log(t^2)}.$$