SF1685: Calculus

The chain rule

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Side note on D

Recall that we have that *D* represents taking the derivative of a function.

D: functions \rightarrow functions. This called an **operator**, and it acts on the vector space of (differentiable) functions.

You might want compare this with higher-order functions, e.g., map, fold.

$$\text{map}: (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R}^n \to \mathbb{R}^n).$$

$$(+,>,2,-) \longrightarrow (\times +^2,5 + 2,2 + 2,...)$$

The product rule

We have that
$$D[f \cdot g] = D[f]g + D[g]f$$
. $N \neq N$

By definition:

Rewrite

$$D[f \cdot g] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Subtract and add f(x + h)g(x) in the numerator:

$$\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} + \frac{f(x+h)g(x)-f(x)g(x)}{h}$$

$$f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}$$

Now recognize the derivatives.

Application of the product rule (proof by induction)

For *integers* $n \ge 1$, we have that $D[x^n] = nx^{n-1}$.

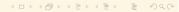
We proved statement is true for n = 1. Assume true for some fixed $n \ge 1$. Then,

$$D[x^{n+1}] = D[x \cdot x^n] = \{ \text{by prod. rule} \} = D[x] \cdot x^n + x \cdot D[x^n]$$

Now

$$D[x] \cdot x^n + x \cdot D[x^n] = 1 \cdot x^n + x \cdot (n)x^{n-1} = (n+1)x^n.$$

Hence, formula true for n implies that the formula is true for n + 1.



We now know how to compute derivatives of polynomials.

Last missing rule is that for function composition, D[f(g(x))]. The chain rule states that

$$D[f(g(x))] = f'(g(x)) \cdot g'(x)$$

The chain rule (incomplete proof)

We assume f, g are differentiable functions.

$$D[f(g(x))] = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

Multiply numerator and denominator:

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \underbrace{g(x+h) - g(x)}_{h}$$

Now let k := g(x + h) - g(x), and y = g(x). As $h \to 0$, we know that $k \to 0$ (why?). Then,

$$\frac{f(y+k)-f(y)}{k} \cdot \frac{g(x+h)-g(x)}{\lambda}$$

$$f'(5(x)) - 5'(x)$$

if=0, it, ball

Discussion

Question

Why is the proof above sometimes invalid? Discuss!

Question

Show that Dx - xD = 1. That is, verify that for all differentiable f(x),

$$D[x \cdot f(x)] - x \cdot D[f(x)] = f(x).$$

Compare with A = B for matrices iff $A\mathbf{v} = B\mathbf{v}$ for all vectors of appropriate dimension.

$$D(x-x) = \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} = \frac{1}{x} + \frac{1}{x} = \frac{1}{x} + \frac{1}{x} = \frac{1}{x} + \frac{1}{x} = \frac{1}{$$

The chain rule, consequences

We can easily prove the quotient rule,

$$D[f/g] = \frac{f' \cdot g - g' \cdot f}{g^2}$$

by computing D[1/g] first using the chain rule, and then using the product rule.

Chain rule questions — Discussions

Question

Compute the derivatives of the following functions:

$$\sin(x^3)$$
, $e^{f(x^2+3x)}$, $f(g(x)^5)$.

Problem

Question

How do we compute $D[x^x]$? (Defined whenever x > 0).

Note that $x^x = \exp(\log(x^x)) = \exp(x \log(x))$. Now we can apply the chain and product rule! We only need to know how to compute $D[e^x] = e^x$ and $D[\log(x)] = 1/x$ (we shall prove this later). We get

$$D[x^{x}] = D[\exp(x \log(x))]$$

$$= \exp(x \log(x)) \cdot D[x \log(x)]$$

$$= x^{x} \cdot (D[x] \cdot \log(x) + x \cdot D[\log(x)])$$

$$= x^{x} \cdot (\log(x) + 1).$$

For $\alpha \in \mathbb{R}$, we have that $D[x^{\alpha}] = \alpha x^{n-\alpha}$.

We have

$$D[x^{\alpha}] = D[\exp(\alpha \log(x))]$$

$$= \exp(\alpha \log(x)) \cdot D[\alpha \log(x)]$$

$$= x^{\alpha} \cdot (\alpha \cdot x^{-1})$$

$$= \alpha x^{\alpha - 1}.$$

Higher order derivatives

Taking derivative a second time, gives f''(x). In general, we shall use the notation below for the second derivative,

for the second derivative,
$$f''(x), \quad D^2[f], \quad \frac{d^2f}{dx^2} \qquad \text{s(4)} \quad \text{distance} \\ \text{s'(4)} \quad \text{ve bodis ad} \\ \text{f(n)}(x), \quad D^n[f], \quad \frac{d^nf}{dx^n} \quad \text{s''(4)} \quad \text{acceleration}$$

for the *n*th derivative.

and

Solution to actually computes SM(0.2)

4 D > 4 B > 4 E > 4 E > 9 Q C

Application of the third derivative in presidential elections

From wikipedia:

When campaigning for a second term in office¹, U.S. President Richard Nixon announced that the rate of increase of inflation was decreasing, which has been noted as the first time a sitting president used the third derivative to advance his case for reelection.

inflation := rate at which purchasing power of money decreases.

Thus, Nixon was talking about $P^{(3)}(1972)$, where P(t) is the purchasing power of money at time t.

¹ Nixon was reelected on November 7, 1972, in one of the largest landslide election victories in American history.

Complicated formula?

Question

Prove that for $n \ge 1$,

$$D^{n}[x^{2}e^{x}] = x^{2}e^{x} + 2n \cdot x \cdot e^{x} + \underbrace{(n^{2} - n)}_{h \cdot (n-1)} \cdot e^{x}.$$

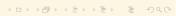
Method: inductions Base case: 1=1

Ind. hyp: Assure D'[x2ex]= x2exx2nxex + (h2-n)ex 0.

$$= x^{2} \cdot e^{x} + 2(n+1) \cdot x \cdot e^{x} + (n^{2}+1) \cdot e^{x}$$



CKO



Proof of formula

Mathematica?

If there is time and interest, we can do some calculations in Mathematica.

Complicated formula II — Homework

Question

Prove that for $n \ge 1$,

$$D^{n}[x^{3}e^{x}] = x^{3}e^{x} + 3nx^{2}e^{x} + 3n(n-1) \cdot x \cdot e^{x} + n(n-1)(n-2) \cdot e^{x}.$$

Hint: Use proof by induction.

Question

Show that

$$D_x^2 - 2xD_x^2 + x^2D_y^2 = 2$$

That is, verify that for any function *f* with a second derivative,

$$D[x^2 \cdot f(x)] - 2xD[x \cdot f(x)] + x^2D[f(x)] = 2 \cdot f(x).$$

Hint: Use Dx - xD = 1 a few times.