KTH Teknikvetenskap

SF1686 Calculus in several variables
Solutions to exam
2021-10-27

1. Let $f(x, y, z) = x^2 + xy + 2y^2 - z^2$.
A. Compute the maximal directional derivative of f at the point $(0, 1, 1)$.
B. Find an equation for the tangent plane to the level surface $f(x, y, z) = 1$ at the point $(0, 1, 1)$.

Lösningsförslag. A. We have $\nabla f = \begin{pmatrix} 2x + y \\ x + 4y \\ -2z \end{pmatrix}$ and so $\nabla f(0, 1, 1) = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$. The maximal directional derivative at the point is given by the norm of the gradient at that point and hence is $\sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{21}$.

B. The tangent plane is orthogonal to the gradient and is given by $\nabla f(0, 1, 1) \cdot \begin{pmatrix} x - 0 \\ y - 1 \\ z - 1 \end{pmatrix} = 0$ i.e. $x + 4(y - 1) - 2(z - 1) = 0$.

Answer: A. $\sqrt{21}$. B. $x + 4(y - 1) - 2(z - 1) = 0$

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2. We study the line integral

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r}$$

where the vector field $\mathbf{F} = (yz, xz, xy + 1)$ and γ is the spiral curve parametrized by $(x, y, z) = (\cos t, \sin t, t)$ where t runs from 0 to $\pi/4$.
A. Compute the line integral directly using the parametrization of the curve.
B. Compute the line integral using a potential for $\mathbf{F}$.

Lösningsförslag. A. Using the given parametrization we get

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/4} (-t \sin^2 t + t \cos^2 t + \cos t \sin t + 1)\, dt = \int_0^{\pi/4} (t \cos 2t + \frac{\sin 2t}{2} + 1)\, dt = \frac{3\pi}{8}$$

B. We see that the vector field $\mathbf{F}$ is conservative with potential $\varphi(x, y, z) = xyz + z$ and so we get

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = [xyz + z]_{(1,0,0)}^{(1/\sqrt{2}, 1/\sqrt{2}, \pi/4)} = \frac{3\pi}{8}$$

Answer: $3\pi/8$

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3. Determine the maximum and minimum values (if they exist) of $f(x, y) = xy^2$ when (x, y) varies in the region given by the inequality $\dfrac{x^2}{3} + \dfrac{y^2}{2} \leq 1$.

Lösningsförslag. The region is compact and the function is continuous and so maximum and minimum values exist. They can be attained at critical points, singular points or boundary points. We have

$$\frac{\partial f}{\partial x} = y^2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2xy$$

and so critical points are given by $y = 0$. In all those points the value of the function is 0. There are no singular points. At the boundary$2x^2 + 3y^2 = 6$ we have $y^2 = 2 - 2x^2/3$ and so at all boundary points $f(x, y) = x(2 - 2x^2/3) = g(x)$, $-\sqrt{3} \leq x \leq \sqrt{3}$. We examine this function g which is continuous on the closed and bounded interval, hence it attains maximum and minimum values. At the boundary points $\pm\sqrt{3}$ the value of g is 0. Critical points are points x where $g'(x) = 0$ i.e. $x = \pm 1$.No singular points. The maximum is $g(1) = 4/3$ and the minimum is $g(-1) = -4/3$. Which therefore are the maximum and minimum respectively of f at the boundary. We have shown that f must attain its maximum and minimum at critical points or at the boundary and we have examined all such points and comparing we see that the maximum is $4/3$ and the minimum is $-4/3$.

(Alternatively, one can solve this problem using Lagrange multipliers or using a parametrization of the ellipse)

Answer: Maximum $4/3$, minimum $-4/3$

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4. Compute the volume of the region K given by the inequalities $x^2 \leq z \leq 4 - y^2$.

Lösningsförslag. The intersection of $z = x^2$ with $z = 4 - y^2$ is obtained when $x^2 = 4 - y^2$ i.e $x^2 + y^2 = 4$. The projektionen of K onto the xy-plane is therefore the disc D with radius 2 around the origin. The volume of K can now be computed using polar coordinates:

$$\iiint_K dV = \iint_D (4 - x^2 - y^2)\, dx\, dy = \int_0^{2\pi} \int_0^2 (4 - r^2) r\, dr\, d\theta = 8\pi$$

Answer: 8π

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5. Compute the flux of the vector field $\mathbf{F} = (x^2, x^2 + y^2, x^2 + y^2 + z^2)$ out of the region K given by the inequalities $\sqrt{x^2 + y^2} \leq z \leq \sqrt{2 - x^2 - y^2}$.

Lösningsförslag. We may use Gauss' theorem (the divergence theorem), since the vector field is C^1 and the region has piecewise C^1 boundary, and we get that the flux is egual to:

$$\iint_{\partial K} \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_K \operatorname{div} \mathbf{F}\, dV = \iiint_K (2x + 2y + 2z)\, dV.$$

In the last integral $2x$ and $2y$ does not give a contribution since their integrals are zero due to symmetry. The intersection between the two boundary surfaces of K is obtained when $x^2 + y^2 = 1$. K can be described using spherical coordinates R, φ, θ by $0 \leq R \leq \sqrt{2}$, $0 \leq \varphi \leq \pi/4$ och $0 \leq \theta \leq 2\pi$. The flux is

$$\iiint_K 2z\, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} 2R\cos\varphi R^2 \sin\varphi\, dR\, d\varphi\, d\theta = 2\pi \int_0^{\pi/4} \sin 2\varphi\, d\varphi \int_0^{\sqrt{2}} R^3\, dR = \pi$$

Svar: π

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6. Give a proof of the theorem that states that a real-valued function of two variables with continuous partial derivatives in a neighborhood of a point must be differentiable at that point.

Lösningsförslag. See the textbook.