SF1685: Calculus

Continuity

Lecturer: Per Alexandersson, perale@kth.se

Continuity

Let $f(x)$ be a function, and (a, b) be an open interval in its domain. For $c \in (a, b)$ we say that f is **continuous at c** if

$$\lim_{x \to c} f(x) = f(c).$$

If this is not the case, f is **discontinuous**.

Roughly, the function value at c agrees with the limit. A function is **continuous** if it is continuous at every point in its domain.

Endpoints

$[a, b)$

The function f is **right-continuous** at c if

$$\lim_{x \to c^+} f(x) = f(c),$$

that is, approaching from the right gives the function value at c.
We have a similar notion for **left-continuous**

This notion is used at the endpoints of the domain.

Questions

$\lim_{x \to 0} \frac{1}{x} = $ D.N.E

Is $\lim_{x \to 0} \frac{\sin x}{x} = 1$?

Yes!

Let

$$f(x) := \frac{1}{x}, \qquad g(x) := \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \qquad h(x) := \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Domain: $x \neq 0$. Domain: $\mathbb{R}$ Domain: $\mathbb{R}$

Question

Which of these are continuous on their domains?

$f(x)$ is continuous for all $x \neq 0$, i.e, all x in the domain.

$g(x)$ is not cont. at $x = 0$,

$h(x)$ is continuous! 1 has been carefully chosen!

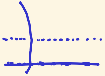
Continuity continued

Most functions you have seen are continuous; polynomials, trig functions, exponential functions, $x \mapsto |x|$.

Moreover, sums, differences, products, and composition preserves continuity.

Some strange functions

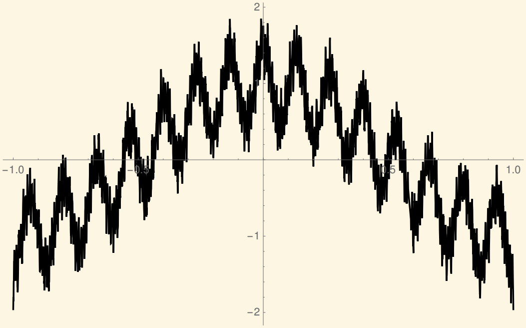
Is $\lim_{x \to 0} f(x) = 1$? No.



$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{2^n}\cos(13^n \pi x)}_{\text{continuous.}}$$

The first function is discontinuous everywhere.

The second function (a Weierstrass function) looks roughly like



and is in fact continuous, but its derivative discontinuous everywhere.

Max/min theorem

Suppose f is continuous (left/right at endpoints) on a **closed interval** $[a, b]$. Then there are some points $x_{max}, x_{min} \in [a, b]$ such that

$$f(x_{min}) \leq f(x) \leq f(x_{max})$$

for all $x \in [a, b]$.

In other words, ***f* attains its maximum and minimum.**

Examples to discuss

Let $f(x) : [0,2] \to \mathbb{R}$, $g(x) : (0,1] \to \mathbb{R}$ and $h(x) : [0,1] \to \mathbb{R}$ be functions defined as

$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \neq 0.$

$\underline{g(1)=1}$

$$f(x) := \begin{cases} x & \text{if } 0 \le x < 1 \\ x/2 & \text{if } 1 \le x < 2 \\ 0 & \text{if } x = 2 \end{cases} \qquad g(x) := x^2 \qquad h(x) := \begin{cases} \frac{e^x - 1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

There's no x, s.t $g(x)=0$.
$g(x)>0$.

Not cont.!

- infimum
- supremum

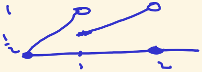
Question

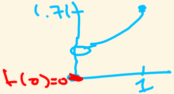
Which of these are continuous? Which attain their minima/maxima? What are the minima and maxima?

$\inf = 0$, $f(0)=0$

$f(x)<1$ for all x

But no pt s.t $f(x)=1$.





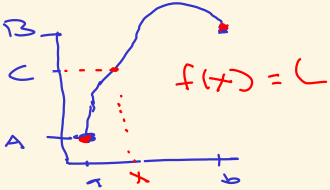
$f(0)=0$, minimum

$f(1)=e-1$, max.

Intermediate value theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose $f(a) = A$ and $f(b) = B$ with $A < B$. If $A \leq C \leq B$, then there is some $x \in [a, b]$ such that $f(x) = C$.

In other words, if f attains the values A and B, it also attains all values between A and B.



Super obvious!

Intuitively clear!

Problem I

Question

Show that $\cos(x) = x$ has a solution with $x \in [0, 1]$. *Can we approximate it?*

Look at $f(x) := \cos(x) - x$. continuous! $[0, \tfrac{1}{2}]$ $[\tfrac{1}{2}, 1]$

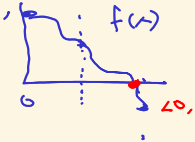
If $f(x) = 0$, then x solves $\cos(x) = x$. Interval halving method,

Try in Python!

$f(0) = \cos(0) - 0 = 1$

$f(1) = \underline{\cos(1) - 1 < 0}$

for sure $\cos(1) < 1$



Must be some pt $x \in [0,1]$ s.t $f(x) = 0$ by IVT.

Problem II

Question

$t=1$ $t=0$

A professor has her house full of empty blackboards. At 06.00, she starts to continuously scribble calculations until midnight, when all boards are covered. The next morning, (06.00) she realizes that she forgot a minus sign somewhere, and starts erasing the boards until they have been cleaned.
Show that there was some time during both days, where the boards were covered with the same amount of chalk.

$f(t)$ percentage of chalk on the board day 1
0% → 100%

$g(t)$ - - - - - - day 2

$f(0) = 0$, $f(1) = 100$

$g(0) = 100$, $g(1) = 0$

Want to find $t \in [0,1]$ s.t. $f(t) = g(t)$

Look at $f(t) - g(t) =: h(t)$

$h(0) = -100$ $h(1) = 100$.

Hint: Let $f(t)$ denote the amount of chalk on the board the first day at time t, and $g(t)$ be the amount of chalk remaining on the board on the second day.

Problem III — A cake problem

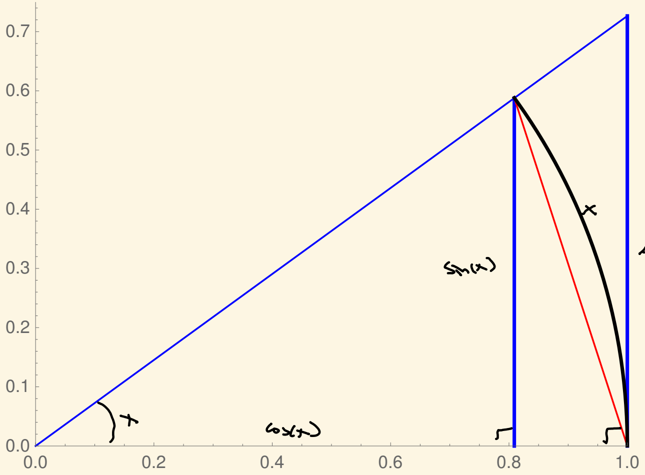
Question

Two kids have made a cake covered with frosting. The cake is a bit deformed, and the frosting only covers some parts of the cake. The two kids want to cut the cake into two pieces, such that both pieces have the same cake area, and the same area of frosting.

Show that this can be done. See also **Ham sandwich theorem**

Limit proof

Proof that $\sin(x)/x \to 1$ as $x \to 0$. We let $x > 0$ denote an angle in radians.



Area small: $\frac{\sin(x)\cdot\cos(x)}{2}$

Area circle sector: $\frac{x}{2}$

Area big: $\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{2}$

Whenever $0 < x < \pi/2$

$$\sin(x)\cdot\cos(x) \le x \le \frac{\sin(x)}{\cos x}.$$

Divide by $\sin(x) > 0$

$a, b > 0$

$\frac{1}{a} > \frac{1}{b}$

$\Updownarrow$

$a < b$

$$\cos(x) \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)}$$

algebra

$$\underset{\to 1}{\cos(x)} \le \underset{\to 1}{\frac{\sin(x)}{x}} \le \underset{\to 1}{\frac{1}{\cos x}}$$

take limit $x \to 0$.

Squeeze thm!

Leftover problems from lecture 1

Partial fraction decomp.

Question

Find numbers *a* and *b* so that $\frac{x+1}{x^2+5x+6} = \frac{a}{x+2} + \frac{b}{x+3}$.

$x^2+5x+6 = (x+2)(x+3)$

$[x \neq -2, -3]$

$$\frac{x+1}{(x+2)(x+3)} = \frac{a}{x+2} + \frac{b}{x+3}.$$

multiply by lcd.

$\Longleftrightarrow$

$$x+1 = a(x+3) + b(x+2)$$
$$= (a+b)x + 3a + 2b$$

should hold for all x (identity)

Coeff must agree

$$\begin{cases} 1 = a+b \\ 1 = 3a+2b \end{cases} \Rightarrow$$

solve

$$\begin{cases} a = -1 \\ b = 2 \end{cases}$$

$1 = 3(1-b) + 2b$

$1 = 3 - b$

Answer: $\frac{-1}{x+2} + \frac{2}{x+3}$

Leftover problems from lecture 1

Question

Let $f(x)$ be defined as $\sqrt{1-x}$ if $x \leq 1$ and $-\sqrt{x-1}$ otherwise. What is the domain and range of f? Is f **invertible**?

Notes?