SF1685: Calculus

Improper integrals

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Integrals

So recall that by the fundamental theorem of calculus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

Question

Compute

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \Big[\, 2\sqrt{x}\,\Big]_0^1$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1}$$

$a \neq -1$

$$\int x^{-1/2}\,dx = 2\cdot x^{1/2} + C$$

$$= 2\sqrt{1} - 2\sqrt{0} = 2.$$

Note: $\frac{1}{\sqrt{x}}$ is not defined at $x=0$!

Technically, we should do $\lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$, $= \lim_{t\to 0^+} 2\sqrt{1} - 2\sqrt{t} = 2.$

But we simply write $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. Hidden limit!

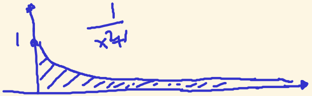
Two types of improper integrals

where c is in the interval where we integrate over.

We have that $\lim_{x\to c} f(x) = \infty$, or the domain where we integrate is unbounded.

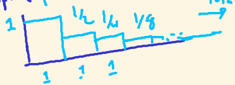
Compute

$$\int_0^\infty \frac{1}{x^2+1}\,dx$$



What does this mean? $\lim_{t\to\infty} \int_0^t \frac{1}{x^2+1}\,dx =$

Compare



total sum $\to$ 2

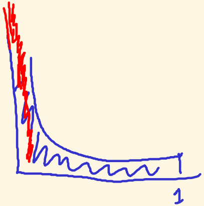
$$\lim_{t\to\infty} \left[\arctan x\right]_0^t = \lim_{t\to\infty} \underbrace{\arctan(t)}_{\to \frac{\pi}{2}} - \underbrace{\arctan(0)}_{=0} = \frac{\pi}{2}.$$

p-integrals

Case $p \neq 0$

$$\underbrace{\lim_{t\to 0}\int_t^1 x^{-p}\,dx}_{*} = \left[-\frac{x^{1-p}}{1-p}\right]_t^1 = \frac{-1}{1-p} + \boxed{\lim_{t\to 0}\frac{t^{1-p}}{1-p}}$$

exist if $p<1$.



We have

$$\int_0^1 \frac{1}{x^p}\,dx$$

* the limit exists and is finite

converges if $p < 1$ and diverges otherwise.

limit is ∞ or d.n.e

On the other hand,

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if $p > 1$.

Case $p=1$

$$\int_t^1 \frac{1}{x}\,dx = [\ln(x)]_t^1$$

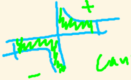
$$= 0 - \ln(t)$$

$$\to \infty \text{ if } t\to 0.$$

These are used for comparisons!

Technical issue?

What about



Cancel!

Breakout room!

a) $$\int_{-2}^{2} \frac{1}{x^2}\,dx \qquad \text{and} \qquad \int_{-2}^{2} \frac{1}{\sqrt[3]{x}}\,dx$$

$=0$

$$\int_{-2}^{0} \frac{1}{\sqrt[3]{x}}\,dx = -\frac{3\sqrt[3]{4}}{2}$$

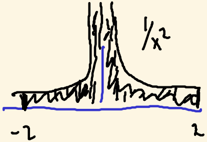
$$\int_{0}^{2} \frac{1}{\sqrt[3]{x}}\,dx = \frac{3\sqrt[3]{4}}{2}$$

a) First try: $$\int_{-2}^{2} \frac{1}{x^2}\,dx = \left[-x^{-1}\right]_{-2}^{2}$$

$$= -\frac{1}{2} - \left(\frac{1}{2}\right) = -1.$$

Does this make sense?

What went wrong?



We expect positive answer!

$\frac{1}{x^2}$ is not defined at 0!

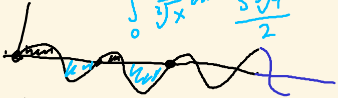
We should instead compute

$$\int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{2} \frac{1}{x^2}\,dx$$

Here, we should do limits!

But, $\int_0^2 \frac{1}{x^2}\,dx$ diverges!

So the integral diverges!



$$\int_0^\infty \sin(x)\,dx$$ Doesn't converge!

$$\int_0^t \sin(x)\,dx = 1 - \cos(t).$$

Does not have a limit as $t \to \infty$!

Comparison theorem

Suppose $0 \leq f(x) \leq g(x)$ on the interval $[a, b]$ (a and/or b may be $\pm\infty$). Then we have that

important!

$$0 \leq \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

In particular,

$$\int_a^b g(x)dx \text{ converges} \implies \int_a^b f(x)dx \text{ converges}.$$

Equivalently,

$$\int_a^b f(x)dx \text{ diverges} \implies \int_a^b g(x)dx \text{ diverges}.$$

diverges!

$$\int_1^\infty \frac{1}{x}dx \leq \int_1^\infty \frac{1}{x^2}dx$$

converges

Show that the integral below converge:

We use comparison thm.

$$\circledast \int_0^\infty e^{-x^2}\,dx$$

So $\circledast$ converges.

First integral:

$$\circledast = \underbrace{\int_0^1 e^{-x^2}\,dx}_{\text{converges}} + \underbrace{\int_1^\infty e^{-x^2}\,dx}$$

continuous, defined on [0,1] so the integral exists!

$$\le \int_1^\infty e^{-x}\,dx = \left[-e^{-x}\right]_1^\infty = \lim_{t\to\infty} -e^{-t} + e^{-1} = e^{-1}$$

$$-e^{-t} \to 0$$

because $x^2 \geq x$ on $[1,\infty)$

so $\frac{1}{e^{x^2}} \le \frac{1}{e^x}$ on $[1,\infty)$

any $t>0$ works, $t=1$ typical choice.

$$\int_0^\infty \frac{1}{x^4+1}\,dx = \underbrace{\int_0^t \frac{1}{x^4+1}\,dx}_{\text{converges}} + \underbrace{\int_t^\infty \frac{1}{x^4+1}\,dx}_{\circledast}$$

$\frac{1}{x^4+1}$ defined on $[0,\infty)$

$$\circledast \le \int_t^\infty \frac{1}{x^4}\,dx = \text{converges by p-integral.}$$

Show that $\int_0^\infty \frac{1}{2+\sin(x)}\,dx$ diverges.

$1 \le 2+\sin(x) \le 3$ for all x.

$1 \geq \frac{1}{2+\sin(x)} \geq \frac{1}{3}$ for all x.

So by comparison $\underbrace{\int_0^\infty \frac{1}{3}\,dx}_{=\lim_{t\to\infty}\left[\frac{x}{3}\right]_0^t = \infty} \le \int_0^\infty \frac{1}{2+\sin(x)}\,dx$

diverges!

So RHS diverges also. ∎

Show that $\int_0^\infty \frac{|\sin(x)|}{x^2}\,dx$ converges. ~~Note that we need to examine both endpoints!~~

Show that $\int_0^1 \frac{|\sin(x)|}{x^2}\,dx$ diverges.

First one, use comparison thm, compare with $\int_1^\infty \frac{1}{x^2}\,dx$, p-integral, converges.

Can we argue that the integral $\int_0^\infty \frac{\sin(x)}{x}\,dx$ converges?

For the second case, $\int_0^1 \frac{|\sin(x)|}{x^2}\,dx$ more tricky....

Enough to consider $\int_0^t \frac{\sin(x)}{x^2}\,dx$ for some small $t>0$.

Compare with $\int_0^t \underbrace{\frac{x}{x^2}}_{\frac{1}{x}} - \frac{x}{3!}\,dx$.

p-integral, diverges!

$\sin(x) \approx x - \frac{x^3}{3!} + \frac{\sin(\theta x)}{5!}x^5$ for some $\theta \in [0,1]$.

So, for sure $\sin(x) \geq x - \frac{x^3}{3!}$ on some small interval $[0,t]$, $t>0$.

R. Feynman

I had learned to do integrals by various methods shown in a book that my highschool physics teacher Mr. Bader had given me. [It] showed how to differentiate parameters under the integral sign — it's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. [If] guys at MIT or Princeton had trouble doing a certain integral, [then] I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me

Feynman's technique

Lets compute $\int_0^\infty \frac{\sin(x)}{x} dx$. First,

$$I(t) := \int_0^\infty \underline{e^{-xt}} \frac{\sin(x)}{x} dx$$

We see $\lim_{t \to \infty} I(t) = 0$,

Note, $I(0)$ is the value we seek. Now, $I(t)$ is a function which depends on t. We can differentiate it wrt t.

$$I'(t) = -\int_0^\infty \cancel{x} e^{-xt} \frac{\sin(x)}{\cancel{x}} dx = -\int_0^\infty e^{xt} \sin(x) dx.$$

We can now compute this integral using partial integration, and it is equal $-\frac{1}{t^2+1}$.

From this, we can find $I(t)$, and also compute $I(0)$.

$I(t) = C - \arctan(t)$

Since $\underline{I(\infty) = 0}$, $\quad I(t) = \frac{\pi}{2} - \arctan(t)$

So $I(0) = \frac{\pi}{2}$ $\blacksquare$

Problem

Find the area between the graphs $f(x) = e^{2-x}$ and $g(x) = e^{3-2x}$, on the interval where $f(x) \geq g(x)$.

Final, 24 Oct, 2017

Show that the integral $\int_0^\infty \frac{x}{(1+x^p)^2}\,dx$ diverges if $0 < p \leq 1$.

Final, 8 Jan, 2018

Determine if the integral $\int_1^\infty \frac{1}{\sqrt{x^3-1}}\,dx$ converges or diverges.

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Final, 8 Jan, 2018

Let $F(x)$ be defined as $F(x) := \int_0^x \frac{1-t}{1+t^{7/2}}\,dt$. Find for what x this attains its maximum, and determine if the limit $\underline{\lim_{x\to\infty} F(x)}$ exists.

indefinite

Final, 7 Jan, 2020, modified

Let $F(x)$ be defined as $F(x) := \int_0^x e^{t^2}\,dt$. Determine the tangent line to $F(x)$ at $x = 0$.

Misc. problems

Compute $\int_0^1 \arcsin(x)dx$.