SF1685: Calculus

Introduction of log and exp

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Rules for exponents

I assume you are familiar with rules, such as

$$e^0 = 1, \qquad e^x e^y = e^{x+y}, \qquad e^{-x} = \frac{1}{e^x}, \qquad (e^x)^y = e^{xy},$$

and similar for other bases.

$\log(1) = 0 \qquad \log(xy) = \log(x) + \log(y) \qquad \log(1/x) = -\log(x)$

$\log(x^y) = y \cdot \log(x)$

Other logarithms

We have a function a^x, for $a > 0$, which is invertible if $a \neq 1$. The inverse is denoted $\log_a(x)$. In other words,

$$\log_a a^x = x \qquad \text{and } a^{\log_a(x)} = x,$$

whenever $a > 0$ and $a \neq 1$.

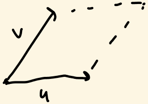
We have the relation

$$\boxed{\log_a(x) = \frac{\log_b(x)}{\log_b(a)},}$$

so it is enough to be able to compute one particular logarithm.

Definition of natural logarithm

One possible definition:

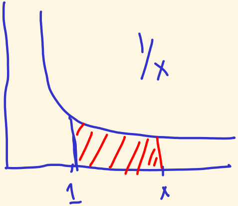


$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

Natural log: $\log(x) = \begin{cases} \text{area under } 1/x \text{ from } 1 \text{ to } x, \text{ if } x \geq 1 \\ \text{negative area under } 1/x \text{ from } x \text{ to } 1, \text{ if } x < 1. \end{cases}$

With this definition, we can easily show that $D[\log(x)] = 1/x$.

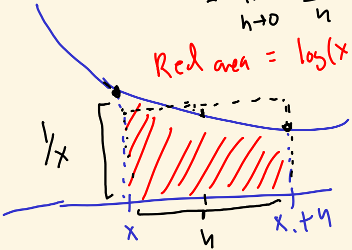
$x > 1$



$$D[\log x] = \lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h}$$

$$= \lim_{h \to 0} \frac{h \cdot 1/x}{h} = 1/x.$$

Red area $= \log(x+h) - \log(x) \approx h \cdot \frac{1}{x}$



Derivative of $\log(x)$ — using standard limits

Here we assume $x > 0$.

$\left[e^{\log(x)} = x \right]$

$$
\begin{aligned}
D[\log(x)] &= \lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \log\left(\frac{x+h}{x}\right) \\
&= \lim_{h \to 0} \log\left(1 + \frac{h}{x}\right)^{\frac{1}{h}} \\
&= \lim_{n \to \infty} \log \underbrace{\left(1 + \frac{1/x}{n}\right)^{n}} \\
&= \log e^{1/x}
\end{aligned}
$$

$h = 1/n$

So, $D[\log(x)] = 1/x$.

Remember: $\left[\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{n} = e^{x} \right]$

Derivative of e^x

$$\begin{aligned}
D[\exp(x)] &= \lim_{h\to 0} \frac{\exp(x+h) - \exp(x)}{h} \\
&= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h\to 0} e^x \boxed{\frac{e^h - 1}{h}} \\
&= e^x
\end{aligned}$$

Getting familiar logarithms

Question

Solve

$$\log_9(4) + \log_3(5) = \frac{\log_4(x)}{\log_4(3)}.$$

We have $\log_9(4) = \frac{\log_3(4)}{\log_3(3^2)} = \frac{\log_3(4)}{2} = \frac{1}{2}\log_3 4 = \log_3 4^{1/2} = \log_3(2)$.

We also have $\frac{\log_4(x)}{\log_4(3)} = \log_3(x)$.

Thus, we want to solve $\log_3(2) + \log_3(5) = \log_3(x)$

$\Rightarrow \log_3(10) = \log_3(x)$

$\Rightarrow x = 10$.

Practical problem

Alice has written $k^1, k^2, \ldots, k^{14}$ for some integer k in a text file, one number on each line. After a computer crash, the file is corrupt, and the digits have been replaced by random characters:

```
         Æ    k^1
        ¡‖    k^2
       *††
      ⊠¡‖
     ⊠⊠ø¶
    ε¶□□Æ
    ‖¡ø¶¡
   ¡□†ε¡□
  †Æø□‖⊠¶
  ø¡ø□†Æ¶
 ¡Æ¡ø¡⊠¡⊠
 øø¡εε‖Æ¶¶
‖ø¶¶ε†‖ε¡†
‖¡¡□†□¡‖ø*    k^14    10 digits
```

$\log_{10}(10) = 1$

$\log_{10}(100) = 2$

$\log_{10}(1000) = 3$

If $\log_{10}(n) = 2.7$ then n has 3 digits

k^{14} has 10 digits

$\Rightarrow 8 < \log_{10}(k^{14}) < 9$

$\frac{8}{14} < \log_{10}(k) < \frac{9}{14}$

$\Downarrow$

$[10^{4/7} < k < 10^{9/14}]$

$3.7 < k < 4.4$

$\Rightarrow \underline{k=4}$

With this information only, can we recover the value of k?

$[\log_{10}(n) \approx (\#\text{digits in } n) - 1]$

$32, \ldots, 2^{32}$

$64, \ldots\ 2^{64}$

Practical problem — notes

Linux time = # seconds
since Jan 1st 1970.

32 bits to store this.

Issue if # seconds > 2^{32}.

From Wikipedia

At 03:14:08 UTC on Tuesday, 19 January 2038, 32-bit versions of the Unix timestamp will cease to work

Y2k-bug

Writing mathematics

See separate set of slides.

More problems

Question

Where is $f(x) := x^2 + 2\log(x)$ increasing? Is $f(x)$ invertible?

Question

Find the minimum and maximum of $f(x) = \log(x^2) + x^2 - 5x + 2$ on the interval $[\frac{1}{4}, 4]$.

Question

Let $g(x) = \arctan(x^2) + \log(x^2 + 1) - x^2$. How many solutions does $g(x) = 1/4$ have?
Hint: Find out where $g(x)$ is increasing and decreasing. Make a rough sketch of the graph.

Question

Consider the curve $y^2 + x + y + xy = 3$. Determine for which values of a, the line $y = a(x + 1) - 1$ is a tangent to the curve. *This needs some algebra.*