SF1685: Calculus

Taylor series

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Taylor series

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n.$

Let $f(x)$ be a function such that one can compute its $(n+1)$th derivative. Then **the Taylor polynomial of degree n, for f, at a** is

$$P_n(x) := f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Moreover, we have that

$$|f(x) - P_n(x)| \leq \frac{1}{(n+1)!} \max_{d \in [a,x]} |f^{(n+1)}(d)| \cdot (x-a)^{n+1}.$$

Want larger n to give smaller error

Maclaurin series

When $a = 0$, we have the Maclaurin series:

$$f(x) \approx f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}$$

We can always shift a function:

$g(x)$ near $x = a$, $\longrightarrow$ $f(x) := g(x+a)$ near $x = 0$.

Some Maclaurin series , x near 0,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

for any $x \in \mathbb{R}$.

$e^0 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$e \approx 2.71 = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \cdots$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{\overbrace{x \cdot x \cdot x \cdots x}^{n}}{\underbrace{1 \cdot 2 \cdot 3 \cdots n}_{n!}}$$

In this case, the series converge everywhere, as the error,

$e^5 = 1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \cdots \frac{5^n}{n!}$

$$\frac{1}{(n+1)!} \overbrace{\max_{d\in[a,x]} \left|f^{(n+1)}(d)\right|}^{\leq 1} \cdot x^{n+1} = \frac{x^{n+1}}{(n+1)!}$$

approach 0 for any fixed x, when we let ~~x~~ $n \to \infty$.

Complex numbers again

$t \in \mathbb{R}$ $\quad i^2 = -1$

We can now define e^{it} and motivate Euler's formula $e^{it} = \cos(t) + i\sin(t)$.

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots$$

$$= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} + \cdots$$

$$= \underbrace{\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right)}_{\cos(t)} + i\underbrace{\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)}_{\sin(t)}$$

$$= \cos(t) + i\sin(t)$$

Fibonacci #'s

$1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$$G(x) = 1 + 2x + 3x^2 + 5x^3 + \cdots = \frac{x+1}{1-x-x^2}$$

roots say how quickly the numbers grow.

Two other important series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad |x|<1$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{\cancel{4}\,5}x^{\cancel{4}\,5} + \cdots \qquad |x|<1$$

Question

What does $1 - 1/2 + 1/3 - 1/4 + 1/5 + \cdots$ approach?

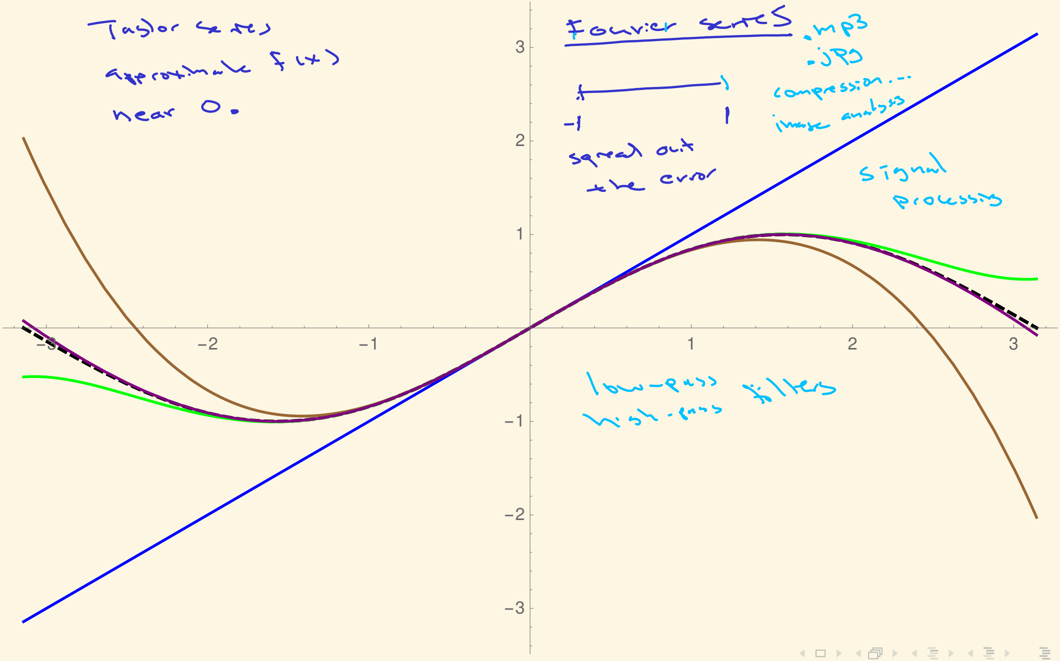
$\log(2)$, by plugging in $x = 1$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \xrightarrow{?} \text{diverges! (later)}$$

$$\sum_{k=1}^{n} \frac{1}{k} \approx \log(n)$$

$$\zeta(2) = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \longrightarrow \frac{\pi^2}{6}$$

Basel problem, solved by Euler.


Taylor series
approximate f(x)
near 0.
Fourier series
.mp3
.jpg
compression...
image analysis
-1
1
spread out
the error
signal
processing
low-pass
high-pass
filters
3
2
1
−1
−2
−3
−3
−2
−1
1
2
3

Example

Compute the Taylor polynomial for e^{2x} at $x = 0$ of degree 2.

$f(x) = e^{2x}$.

$f'(x) = 2e^{2x}$.

$f''(x) = 4e^{2x}$.

Taylor poly:

$$1 + 2 \cdot \frac{x}{1!} + 4 \cdot \frac{x^2}{2!}$$

Alternatively: $e^x = 1 + x + \frac{x^2}{2!} + \ldots$

so $e^{2x} = \underbrace{1 + 2x + \frac{(2x)^2}{2!}} + \ldots$

So second deg Taylor is $\underbrace{1 + 2x + \frac{4x^2}{2!}}$.

Example

Compute the Taylor polynomial for $e^{\sin(x)}$ at $x = 0$ of degree $\underline{2}$.

Alt 1: Compute derivatives...

Alt 2: $e^x = 1 + x + \frac{x^2}{2} + O(x^3)$

$\sin(x) = \underbrace{x + O(x^3)}$

Big O's notation

$O(x^4)$ anything of the form $a x^4 + b x^{4+1} + c x^{4+2} + \dots$

So $e^{\sin(x)} = 1 + (x + O(x^3)) + \frac{(x + O(x^3))^2}{2!}$

$= \underline{1} + \underline{x} + O(x^3) + \underline{\frac{x^2}{2!}} + O(x^3)$

Answer: $1 + x + \frac{x^2}{2!}$

Example

Compute the Taylor polynomial for $e^{\sin(x)}$ at $x = 0$ of degree 2.

Sam .

Example

Compute the Taylor polynomial for e^x at $x = 1$ of degree 2.

$$f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!}$$

$$e + e(x-1) + e\frac{(x-1)^2}{2!}$$

Example

Approximate cos(1) to an error of $1/5000$.

We use expansion at $x = 0$. The general size of the error is

$$\frac{1}{(n+1)!} \max_{d\in[a,x]} \underbrace{\left|f^{(n+1)}(d)\right|}_{\leq 1} \cdot x^{n+1}$$

so in our case, we have $x = 1$, and $\left|f^{(n+1)}(d)\right| \leq 1$, so the error is

$$\frac{1}{(n+1)!} 1 \cdot 1^{n+1}.$$

If we pick $n = 6$, we have that $(n+1)! = 7! = 5040$, so we should use a degree-6 Taylor series. Hence,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos(1) = \underbrace{1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}}_{} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.5403\ldots$$

$$\cos(1) = 0.5403 \pm \frac{1}{5000}$$

Maclaurin series for arctan(x)

The Maclaurin series for arctan(x) is

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \qquad \underline{|x| \le 1}$$

with error term $E_n(x) = x^{n+1}/(n+1)$.

One way to compute digits of π (not very efficient)

So what does

$$4\left(\underbrace{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots}_{\arctan(1) = \frac{\pi}{4}}\right) = \pi$$

converge to?

Can we use the Maclaurin series to estimate arctan(2)**?** No!

$E_n(2) = \dfrac{2^{n+1}}{n+1}$, this grows with n.

Trick: $\arctan(x) = \frac{\pi}{2} - \arctan(1/x)$

Computing limits

Show that the limit below is equal to 2.

$$\lim_{x \to 0} \frac{1 - \cos(2x)}{\underbrace{x^2 - 3x + \sin(3x)}_{(*)}}$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + O(x^4)$$

so

$$\sin(3x) = 3x - \frac{(3x)^3}{3!} + O(x^5)$$

$$(*) = \frac{\cancel{1} - \left(\cancel{1} - \frac{4x^2}{2} + O(x^4)\right)}{x^2 - \cancel{3x} + \cancel{3x} - \frac{27x^3}{6} + O(x^5)} = \frac{\left(2x^2 + O(x^4)\right) \cdot 1/x^2}{\left(x^2 - \frac{9}{2}x^3 + O(x^4)\right) \cdot 1/x^2}$$

$$= \frac{2 + O(x^2)}{1 + O(x)}$$

This approaches 2 as $x \to 0$.

Computing limits

Show that the limit below is equal to $3/2$.

$$\lim_{x\to 0} \frac{e^{\sin(x)} - \frac{\cos(x)}{1-x}}{\arctan(x) - x}$$

- $\sin(x) = x - \frac{x^3}{3!} + o(x^5)$
- $e^{\sin(x)} = 1 + \left(x - \frac{x^3}{3!} + o(x^5)\right) + \left(x - \frac{x^3}{3!} + o(x^5)\right)^2 \frac{1}{2!} + \underbrace{\left(x - \frac{x^3}{3!} + o(x^5)\right)^3}_{3!} + o(x^4)$

$$= 1 + x + \frac{x^2}{2!} + \cancel{\frac{-x^3}{6}} + \cancel{\frac{x^3}{6}} + o(x^4)$$

- $\cos(x) = 1 - \frac{x^2}{2!} + o(x^4)$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + o(x^4)$

$$\to \left(1 - \frac{x^2}{2!} + o(x^4)\right)\left(1 + x + x^2 + x^3 + o(x^4)\right)$$

$$= 1 + x + x^2 - \frac{x^2}{2} + x^3 - \frac{x^3}{2} + o(x^4)$$

$$= \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{2} + o(x^4)}_{\frac{\cos x}{1-x}}$$

$$\arctan(x) = x - \frac{1}{3}x^3 + o(x^5)$$

$$\arctan(x) - x = -\frac{x^3}{3} + o(x^5)$$

Thus we have

$$\lim_{x\to 0} \frac{\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2!}} + o(x^4) - \left(\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + \frac{x^3}{2} + o(x^4)\right)}{-\frac{x^3}{3} + o(x^5)}$$

$$\lim_{x\to 0} \frac{-\frac{x^3}{2} + o(x^4)}{-\frac{x^3}{3} + o(x^5)} = \frac{-1/2}{-1/3} = \frac{3}{2} \quad \blacksquare$$

Question

Find the Taylor polynomial for $\sqrt{1+x}$ at $x = 0$, of degree 3.