SF1685: Calculus

Series II

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Recall that a **series** is an infinite sum $\sum_{n=1}^{\infty} a_n$.

Last class, we saw that the geometric series

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

whenever $|r| < 1$. For $r \leq -1$ it does not converge, and if $r \geq 1$, it diverges to ∞.

Suppose $\sum_{n=1}^{\infty} a_n$ converges. Then $\lim_{n\to\infty} a_n = 0$.

From this, we can conclude that if a_n does not approach 0, then the series cannot converge to a finite number.

Properties of limits / series

Assuming all limits exist, we have

- $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty} (a_n + b_n) = \left(\sum_{n=1}^{\infty} a_n\right) + \left(\sum_{n=1}^{\infty} b_n\right)$

Moreover, if $a_n \leq b_n$ for all n, then

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

Compare w. integrals.

Integral inequality

Suppose $f(x)$ is a positive, decreasing function. Then

$$\int_1^{n+1} f(x)dx \le \sum_{k=1}^{n} f(k) \le \int_0^n f(x)dx$$

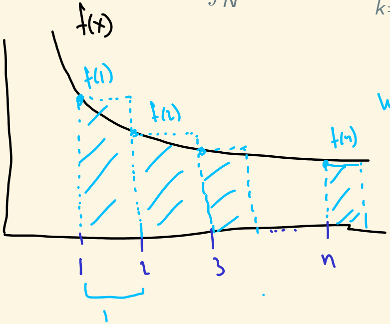
This can be used to prove

$$\int_N^{\infty} f(x)dx \le \sum_{k=N}^{\infty} f(k) \le f(N) + \int_N^{\infty} f(x)dx$$

want to estimate $\sum_{k=1}^{\infty} f(k)$. // choose N s.t. $f(N) < \varepsilon$.

Compute $\sum_{1}^{N-1} f(k)$ finite

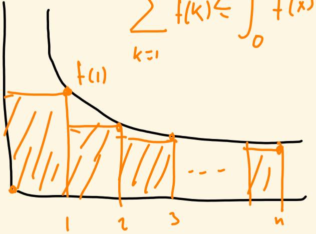
use this for the remaining terms.



We have:

$$\sum_{k=1}^{n} f(k) \le \int_1^{n+1} f(x)dx$$

$$\sum_{k=1}^{n} f(k) \le \int_0^n f(x)\,dx$$



Convergence tests

Suppose $a_n > 0$ for all n, and that $a_n = f(n)$ for some non-increasing function $f : \mathbb{R} \to \mathbb{R}$, whenever $n \geq N$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_N^{\infty} f(x)dx$ does.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$, Compare with $\displaystyle\int_1^{\infty} \frac{1}{x(x+2)}\,dx$, (*)

But $\displaystyle\int_1^{\infty} \frac{1}{x(x+2)}\,dx \leq \int_1^{\infty} \frac{1}{x^2}\,dx = \left[-1/x\right]_1^{\infty} = 1.$

no need to do partial fractions, we only care about convergence.

So, (*) converges, hence, the sum converges.

p-series

We have that

$$\sum_{n\geq 1}\frac{1}{n^p} \quad , \quad \text{Compare w. } \int_1^\infty \frac{1}{x^p}\,dx$$

converges if $p > 1$, and diverges if $p \leq 1$.

We can use integrals get upper and lower bounds of sums, but they are in general not equal.

Example $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_n}$, F_n Fibonacci numbers, $1, 1, 2, 3, 5, 8, 13, \ldots$

This sum converges, $F_n \sim C\cdot\left(\frac{\sqrt{5}+1}{2}\right)^n$ "geometric series"

but we have no exact expression for this sum. Is this algebraic? open problem.

Prime number reciprocals

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

p_n : n^{th} prime

$\frac{1}{p_n} \to 0$ as $n \to \infty$

"

What about

$$\sum_{p \text{ prime}} \frac{1}{p}. \quad = \infty$$

Does this converge or diverge?

This diverges, but very slowly.

Search for a proof, Wikipedia

Prime number thm $\boxed{p_n \sim n \cdot \log(n)}$ $\quad \sum_{n=2}^{\infty} \frac{1}{n \cdot \log(n)}$, does this converge?

Comparison questions

Which of the following series converge?

- $\sum_{n=2}^{\infty} \frac{1}{\log(n)}$ We know $\sum_{n=2}^{\infty} \frac{1}{n} = \infty$, and $\frac{1}{n} < \frac{1}{\ln(n)}$ So $\sum \frac{1}{n} < \sum \frac{1}{\ln(n)}$ ← diverges, also diverges.
- $\sum_{n=1}^{\infty} \frac{n}{2^n}$ There is some N, s.t $n < 1.5^n$, whenever $n > N$. So $\sum_{n=N}^{\infty} \frac{n}{2^n} < \sum_{n=N}^{\infty} \left(\frac{1.5}{2}\right)^n < \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4}$ finite. geom. series.
- $\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$, diverges.
- $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+2n^2+5n-1}$. Diverges.
- $\sum_{n=1}^{\infty} \underbrace{\cos(1/n)}_{\to 1}$ so diverges

. Compare with $\int_2^{\infty} \frac{1}{x \cdot \ln(x)}\, dx = \left[\begin{array}{l} u = \ln(x) \\ du = \frac{1}{x}\, dx \end{array}\right] = \int_{\ln(2)}^{\infty} \frac{1}{u}\, du = [\ln(u)]_{\ln(2)}^{\infty} = \infty$, diverges,

. $\frac{n^2+1}{n^3+2n^2+5n-1} \geq \frac{n^2}{n^3+n^3+n^3} = \frac{1}{3n}$ and $\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges.

↑ if $n \geq 5$

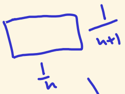
A specific series

$\boxed{\text{Open Question}}$

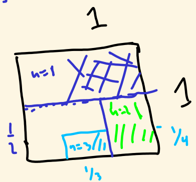
Geometric proof?

Can the rectangles tile a 1×1 square?

Consider the rectangles



area



Compute

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}. = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$$

Hint: Use partial fraction decomposition.

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$$

$$1 = (n+1)a + bn$$

$$\begin{cases} a = 1 \\ b = -1 \end{cases}$$

Telescoping series

$$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots \cdots$$

n=1 n=2

In fact,

$$\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{N+1}$$

so letting $N \to \infty$, the limit is 1.

Limit comparison tests

Suppose $\{a_n\}$, $\{b_n\}$ are two sequences, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \leq \infty$$

If L is finite, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

If $L > 0$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Comparison questions — again

Which of the following series converge? Use the limit comparison technique, and compare with some p-series.

- $\sum_{n=2}^{\infty} \frac{1}{\log(n)}$ skip
- $\sum_{n=1}^{\infty} \frac{n}{2^n}$ limit test $b_n = \frac{1}{1.5^n}$, $a_n = \frac{n}{2^n}$ $\frac{a_n}{b_n} = n \cdot \left(\frac{1.5}{2}\right)^n \to 0$ limit is 0. $\sum_{n=1}^{\infty} \frac{1}{1.5^n}$ converges
- $\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$, difficult with limit test
- $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+2n^2+5n-1}$ compare w. $\frac{1}{n}$, diverges, limit is 1
- $\sum_{n=1}^{\infty} \sin(n^{-2})$ compare with $\sum 1/n^2$, converges,

$$\lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = \lim_{t\to 0^+} \frac{\sin(t^2)}{t^2} = 1.$$

So $\sum_{n=1}^{\infty} \sin(n^{-2})$ has same behavior as $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges.

Ratio test

Suppose $\{a_n\}$ is a positive sequence, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

If $0 \leq r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

If $1 < r$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $r = 1$ we cannot make any conclusion.

Idea:

$a_n \sim C \cdot r^n$

So, same behavior as some geometric series.

Root test

Same idea,

$a_n \sim C \cdot r^n$

Suppose $\{a_n\}$ is a positive sequence, and

$$\lim_{n\to\infty} (a_n)^{1/n} = r.$$

If $0 \leq r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

If $1 < r$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $r = 1$ we cannot make any conclusion.

Example final question

Evaluate the integral $\int_1^\infty \frac{4}{x^2+4x}\,dx$, and determine if the series $\sum_{n=1}^\infty \frac{4}{n^2+4n}$ converges.

$$\int \frac{4}{x^2+4x}\,dx = \int \frac{1}{x} - \frac{1}{x+4}\,dx = \log|x| - \log|x+4|$$

$$= \log\left|\frac{x}{x+4}\right| + C$$

$$\frac{4}{x(x+4)} = \frac{a}{x} + \frac{b}{x+4}$$

$$4 = a(x+4) + bx$$

$a=1$, $b=-1$.

So $$\int_1^\infty \frac{4}{x^2+4x}\,dx = \left[\log\left|\frac{x}{x+4}\right|\right]_1^\infty = 0 - \log(1/5)$$

$$= \log(5).$$

Finite

By integral comparison test, the sum converges.

Example final question

Evaluate the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \underbrace{\sum_{k=1}^{n} \frac{1}{\sqrt{k}}}_{(*)}$$

$\frac{1}{\sqrt{x}}$ is a positive, decreasing function.

Lower Estimate:

$$\int_1^{n+1} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{n+1} = 2\sqrt{n+1} - 2.$$

Upper estimate

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \le \int_0^n \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_0^n = 2\sqrt{n}.$$

So

$$\underbrace{\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(2\sqrt{n+1} - 2\right)} \le \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\right) \le \underbrace{\lim_{n\to\infty} \frac{1}{\cancel{\sqrt{n}}}\left(2\cancel{\sqrt{n}}\right)}_{=2}$$

$$2 = \lim_{n\to\infty} 2\cdot\underbrace{\sqrt{\frac{n+1}{n}}}_{\to 1} - \underbrace{\frac{2}{\sqrt{n}}}_{\to 0}$$

By squeeze thm, the limit is 2 ☒

Example final question

Show that

$$\frac{\pi}{4} \le \underbrace{\sum_{k=0}^{\infty} \frac{1}{k^2+4}}_{S} \le \frac{\pi}{4} + \frac{1}{4}$$

We know, by integral calculus,

$$\int_0^\infty \frac{1}{x^2+4}\,dx \le S \le \underbrace{f(0)}_{=1/4} + \int_0^\infty \frac{1}{x^2+4}\,dx$$

It suffices to show that

$$\int_0^\infty \frac{1}{x^2+4}\,dx = \pi/4.$$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \quad \text{use } \frac{x}{2}=u, \quad dx = 2\,du$$

$$= \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\arctan(x/2) + C.$$

So

$$\frac{1}{2}\left[\arctan(x/2)\right]_0^\infty = \frac{1}{2}\left(\pi/2 - 0\right) = \pi/4 \quad \blacksquare$$