SF1685: Calculus

Introduction

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Main topics for the course

- Functions
- Limits and continuity
- Derivatives
- Applications of derivatives (finding minima and maxima)
- Integrals
- Applications of integrals
- Series (infinite sums)

Some notation

Sets of numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$, $\mathbb{R}$.

Intervals of real numbers: $[1, 5]$, $[\pi, 8)$.

Set membership symbols: $\in$ and $\notin$. $x \in \mathbb{R}.$

Set building notation:

such that

$$X = \{x \in \mathbb{R} : x < 0 \text{ or } x^2 = x\}.$$

What is a function?

A **function** $f : X \to Y$, associates a value, $f(x)$, to every $x \in X$.

The set X is called the **domain**, and Y is the **range**.

We are sometimes sloppy, and deduce the domain from the context — usually the largest one that makes sense.

Some questions to discuss

Question

$\pm\sqrt{4}$ $\sqrt{x}$

Which values of x solve $x^2 = 4$? What is $\sqrt{9}$?

pos. sol to $y^2 = x$

$\begin{cases} x_1 = 2 \\ x_2 = -2 \end{cases}$ 3 Not -3

Question

What is the natural range and domain for each of the functions $\sin(x) \neq 0$

$\Rightarrow x \neq k \cdot \pi$, $k \in \mathbb{Z}$.

$$f(x) = \sqrt{x-2}, \quad g(x) = \frac{x^2-4}{x-2}, \quad h(x) = \frac{1}{\sin(x)}?$$

$\{x \in \mathbb{R} : x \neq k\pi, k \in \mathbb{Z}\}$

$[2, \infty)$

$x \neq 2$

$\{x \in \mathbb{R} \mid x \neq 2\}$

$g(x) = \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = x+2.$

Question

What is the natural range and domain for the function $\log(x)$?

log denotes the natural logarithm.

$(0, \infty)$, Range: $\mathbb{R}$.

Solutions

Combining functions

If we have two functions, f and g, we can combine them in several ways to create new functions:

- $f + g$. This is the function $x \mapsto f(x) + g(x)$.
- $f \cdot g$. This is the function $x \mapsto f(x) \cdot g(x)$.
- $f \circ g$. This is the function $x \mapsto f(g(x))$. Called **composition**.

Combining functions — discussion

Let $f(x) = \sqrt{x}$, $g(x) = \log(1 - x)$ and $h(x) = 1 + x^2$.

Question

What are the domains of $f + g$, $f + h$ and $g + h$?

Question

What are the domains of the functions $f \circ h$, $h \circ f$ and $g \circ f$?

Combining functions — notes

$\sqrt{x}^2 = x$ if $x \geq 0$

Domains: f : $[0, \infty)$ $\sqrt{x}$

g : $(-\infty, 1)$ $\log(1-x)$

h : $\mathbb{R}$ $1+x^2$

$f+g$: $[0,1)$

$f+h$: $[0, \infty)$

$g+h$: $(-\infty, 1)$

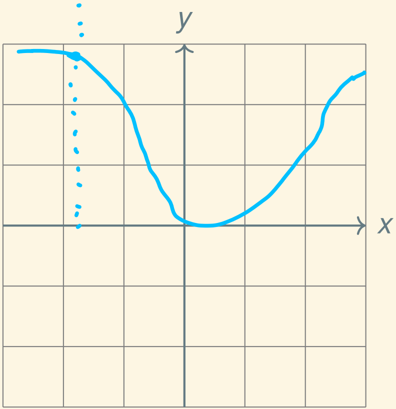
$f \circ h = \sqrt{1+x^2}$, $\mathbb{R}$

$h \circ f = 1+\sqrt{x}^2$, $[0,\infty)$

$g \circ f = \log(1-\sqrt{x})$, $[0,1)$.

The graph of a function

Formally, the **graph** of a function $f : \mathbb{R} \to \mathbb{R}$ is the set $\{(x, f(x)) : x \in \mathbb{R}\}$. This is a subset of $\mathbb{R}^2$, which we sometimes like to draw (in the xy-plane).



Polynomials

Polynomials are functions which can be expressed in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If $a_n \neq 0$, then the **degree** of P is n. The set of polynomials with coefficients in $\mathbb{R}$ is denoted $\mathbb{R}[x]$.

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What is the degree of the polynomial 1? Of the polynomial 0?

Degree is $-\infty$.

$\deg(P+Q) \leq \max(\deg P, \deg Q)$

$\deg(P \cdot Q) = \deg(P) \cdot \deg(Q)$

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*The number of zeros (**roots**) of a polynomial is at most as much as its degree.*

Rational functions

A **rational function** is a function of the form

$$\frac{P(x)}{Q(x)}, \qquad \text{where } P, Q \text{ are polynomials.}$$

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What is the domain for such a function? Is $f(x) = 2x + 3$ a rational function?

Yes: $\frac{2x+3}{1}$

Trigonometric functions

We have $\sin(x)$, $\cos(x)$, $\tan(x)$ and so on. We always use **radians**.

Identities you should know:

$$\sin^2(x) + \cos^2(x) = 1, \qquad \cos(2x) = \cos^2(x) - \sin^2(x), \qquad \sin(2x) = 2\sin(x)\cos(x).$$

$$\cos(2x) + 1 = 2\cos^2 x$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Piecewise functions, strange functions

Sometimes, it is convenient to define functions with cases, so-called **piecewise functions**. The **absolute value function** (with $\mathbb{R}$ as domain) is defined as

$$|x| \coloneqq \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Another example is

$$g(x) \coloneqq \begin{cases} 2x+3 & \text{if } x \geq 2 \\ 0 & \text{if } 0 < x < 2 \\ x^2 & \text{if } x < 0 \end{cases}.$$

$g(-7) = (-7)^2 = 49.$

$g(0) = \underset{?}{\underline{\qquad}}$ undefined!

About the equality sign (=)

Consider these mathematical expressions:

1. $x^2 + 5x + 6 = 0$ Equation, true for some x.
2. $f(x) := x^2 + \sin(x)$ $:=$ Definition, introduces notation.
3. $\cos^2(x) + \sin^2(x) = 1$ Identity, relation.
4. $a^2 + b^2 = c^2$ Relation, Identity:

5. $x^2/x = x$ "Identity" algebra,
6. $x^2 \geq 0 \iff x \neq 0$ Equivalence.
7. $h(x) = 0$ Definition, or equation, identity,

What is the *meaning* (or intention) of the equality sign in each case?

Find maximum value of

$f(x) := x^2 - 2x + 3,$

$f'(x) = 2x - 2 = 0$ Equation

identity

$x = 1$

We see

$f'(x) = 2x - 2.$

We now solve

$f'(x) = 0$

$\iff 2x - 2 = 0 \ldots$

Notes?

If $f(x) = 0$ for ~~all~~ all $x \in \mathbb{R}$,

we may write $f(x) \equiv 0$,

identically equal to.

$\sin^2(x) + \cos^2(x) \equiv 1$

$f(x)$

one variable

$f(x, y, z)$

multi-variate + L.A

calculus.

Exercises I

Question

Let $f(x) = 2x + \frac{1}{x}$ and $g(x) = x^2 - \frac{1}{x}$. What are the natural domains for f and g? What is the natural domain for $f + g$?

Question

Find all solutions to the equation $\frac{\sin(x)}{x} = 0$.

Question

Find numbers a and b so that $\frac{x+1}{x^2+5x+6} = \frac{a}{x+2} + \frac{b}{x+2}$.

Exercises II

Question

Verify that $|x| = \sqrt{x^2}$ is true for all $x \in \mathbb{R}$.

Question

Let $f(x)$ be defined as $\sqrt{1-x}$ if $x \leq 1$ and $-\sqrt{x-1}$ otherwise. What is the domain and range of f? Is f **invertible**?

Notes?