SF1685: Calculus

Drawing graphs

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Drawing graphs

Features we want to consider:

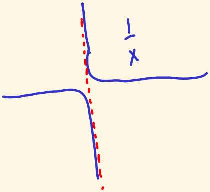
- Domain
- Zeros
- Critical points and singular points
- Regions of increase/decrease.
- y-intercept
- Vertical asymptotes
- Asymptotes at infinity
- Convexity and inflection points (where second derivative changes sign)
- Special features — symmetry

Vertical asymptotes

We say that $f(x)$ has a vertical asymptote at $x = a$, if

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} f(x) = \pm\infty.$$

Typical example : $\frac{1}{x}$ has a vertical asymptote at $x=0$.



Asymptote at ∞

Alt: Do polynomial division

$$\frac{x^2+2}{x+1} = \frac{x^2+x-x+2}{x+1}$$

$$= x + \frac{-x+2}{x+1}$$

$$= x + \frac{-x-1+3}{x+1}$$

$$= x - 1 + \frac{3}{x+1} \approx x-1$$

Idea: $\underset{x}{\underline{f(x)}} \approx \underset{x}{\underline{kx+m}}$ when $x \gg 0$. $\to k$

We look at

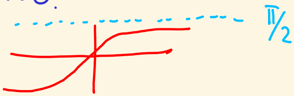
$$\lim_{x\to\infty} \frac{f(x)}{x}.$$

If this is finite, then this is the *slope*, k, of the asymptote. We then find m by

$$\lim_{x\to\infty} f(x) - kx = m.$$

Thus, $f(x)$ has the asymptote $kx+m$ as $x \to \infty$.

Example: $\arctan(x) \approx \frac{\pi}{2}$ for large x



Asymptote at $-\infty$

Idea: $f(x) \approx kx + m$ when $x \ll 0$.

We look at

$$\lim_{x \to -\infty} \frac{f(x)}{x}.$$

If this is finite, then this is the *slope*, k, of the asymptote. We then find m by

$$\lim_{x \to -\infty} f(x) - kx = m.$$

Sketch $f(x) = (2x + 4)(x + 10)/(x + 6)$. $\quad \frac{2(x+2)(x+10)}{x+6}$

We have that $f(-10) = f(-2) = 0$.

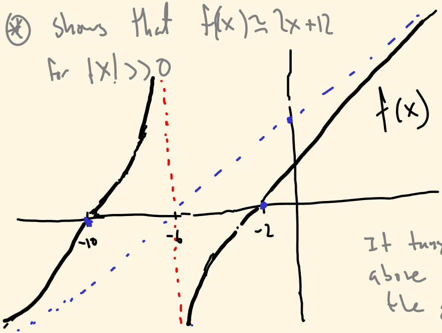
Also, $f(-6)$ is undefined, $\lim_{x \to -6^+} f(x) = -\infty$, $\lim_{x \to -6^-} f(x) = +\infty$. vertical asymptote at $x = -6$.

$$f(x) = \frac{2x^2 + 20x + 4x + 40}{x+6},$$

$$\begin{array}{r|l} 2x + 12 & \\ \hline 2x^2 + 24x + 40 & x+6 \\ -2x^2 - 12x & \\ \hline 12x + 40 & \\ -12x - 72 & \\ \hline -32 & \end{array}$$

so $\left[f(x) = \underbrace{2x + 12} - \frac{32}{x+6} \right]$ (*)

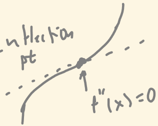
(*) shows that $f(x) \simeq 2x + 12$ for $|x| >> 0$



Derivative $f'(x) = 2 + \frac{32}{(x+6)^2}$.

$f'(x) > 0$ for all $x \in \mathbb{R} \setminus \{-6\}$

No critical pts.



$$f''(x) = \frac{-64}{(x+6)^3}$$

If tangents are above or below the graph ⟷ $f''(x) < 0$ if $x > -6$, pos otherwise

Sketch $f(x) = \sqrt{x-2} + \sqrt{11-x}$.

Domain for $f(x)$ is $[2, 11]$.

$$f'(x) = \frac{1}{2\sqrt{x-2}} - \frac{1}{2\sqrt{11-x}} = \frac{1}{2}\,\frac{\sqrt{11-x} - \sqrt{x-2}}{\sqrt{x-2}\,\sqrt{11-x}}$$

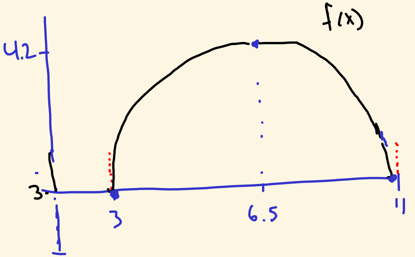
$$f'(x) = 0 \iff \sqrt{11-x} - \sqrt{x-2} = 0 \iff \sqrt{11-x} = \sqrt{x-2} \quad \text{so} \quad 11-x = x-2 \iff x = \frac{13}{2} = 6.5.$$

So $x = 13/2$ is a critical pt.

We have $f(2) = 3$, $f(11) = 3$.

$$f(6.5) = \sqrt{9/2} + \sqrt{9/2} = 3\sqrt{2} \approx 3 \cdot 1.42 \approx 4.2$$

	2		6.5		11
$f(x)$	3		4.2		3
$f'(x)$	$+\infty$	$+$	0	$-$	$-\infty$



Sketch $f(x) = \arctan(x)/x$.

$\lim_{x\to 0} \frac{\arctan(x)}{x} = [x=\tan(y)] = \lim_{y\to 0} \frac{y}{\tan(y)} = \lim_{y\to 0} \frac{y\cos(y)}{\sin(y)} = 1$. No vertical asymptote at 0.

Recall, $\arctan(-x) = -\arctan(x)$, so $f(-x) = f(x)$, so $f(x)$ is even. Symmetric wrt y-axis.
(Enough to consider $x \geq 0$).

Now, $\lim_{x\to\infty} \frac{\arctan(x)}{x} = 0$. Moreover, $f(x) > 0$ if $x > 0$.

Now

$$f'(x) = -\frac{\arctan(x)}{x^2} + \frac{1}{x(x^2+1)} = \frac{x - (x^2+1)\cdot\arctan(x)}{x^2(x^2+1)}$$

What is $\lim_{x\to 0} f'(x)$? $\arctan(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + O(x^7)$ (Taylor expansion) (Later)

$$\lim_{x\to 0} \frac{x - (x^2+1)\left(x - \frac{x^3}{3} + O(x^5)\right)}{x^2(x^2+1)} = \lim_{x\to 0} \frac{-\cancel{x^3} + \frac{x^3}{3} + O(x^5)}{\cancel{x^2}(x^2+1)} = 0$$

Numerator $\to 0$

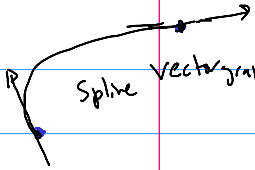
Denom $\to 1$

Conclusion: $\lim_{x\to 0} f'(x) = 0$.

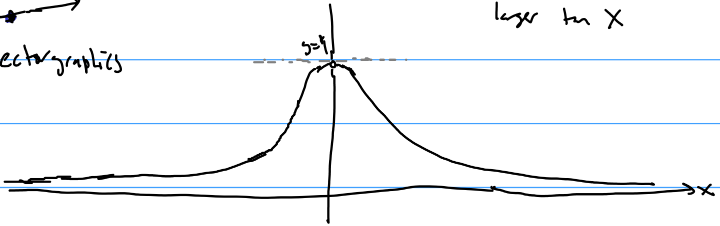
More crit. pts?

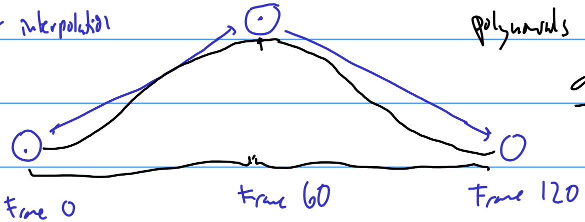
$$x - (x^2+1)\arctan(x) = 0 \iff x = \underbrace{(x^2+1)}_{\text{larger than } x}\underbrace{\arctan(x)}_{\approx \frac{\pi}{2} \text{ for large } x}$$

No solutions for large x.



Spline vectorgraphics





polynomials of degree 3.

$$\underline{a}x^3 + \underline{b}x^2 + \underline{c}x + \underline{d}$$

Splines vector graphics.

Adobe illustrator.

Sketch $f(x) = (x^2 - 1)^{2/3}$.

Sketch $f(x) = \frac{2x}{|x-4|-x}$.

Sketch $f(x) = \log(x^2) + 2x - 1$.

Sketch $y^2 = x^2 + 1$ and $y^2 = x^3 + 1$