SF1686 Calculus in several variables
Solutions to the exam 20 December 2021

1. Find an equation for the tangent plane at the point $(1, -1, 0)$ to the surface

$$z = \ln(1 + x^2 + y^3).$$

Lösning. If $f(x, y) = \ln(1 + x^2 + y^3)$, we have that $f(1, -1) = 0$ and

$$f'_x = \frac{2x}{1 + x^2 + y^3} \quad \text{and} \quad f'_y = \frac{3y^2}{1 + x^2 + y^3}$$

and hence $f'_x(1, -1) = 2$ and $f'_y(1, -1) = 3$. An equation for the tangent plane is

$$z = 2(x - 1) + 3(y + 1).$$

$\square$

Svar: $z = 2(x - 1) + 3(y + 1)$

2. Compute the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (e^{x+y} + y^2 + x + 1, e^{x+y} + x^2 + x + 2)$ and γ is the ellipse $2x^2 + 3y^2 = 6$.

Lösning. We see that $\mathbf{F} = (P, Q) = (e^{x+y} + y^2 + x + 1, e^{x+y} + x^2 + x + 2)$ is infinitely differentiable in the entire plane and that γ is the smooth positively oriented boundary of the ellipse D given by $2x^2 + 3y^2 \leq 6$. We may therefore use Green's formula to obtain

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \iint_D (2x - 2y + 1)\, dxdy = \iint_D dxdy = \pi\sqrt{6},$$

where at last equality we have used symmetry to find that the integral of $2x$ and the integral of $2y$ are both 0 and that the area of the ellipse is $\pi\sqrt{3}\sqrt{2}$. (Assuming the ellipse is traversed once in the positive direction.)

$\square$

Svar: $\pi\sqrt{6}$

3. Compute the integral
$$\iiint_K \frac{1}{1+x^2+y^2+z^2}\,dV,$$
where K is the region given by the inequalities $x^2+y^2+z^2 \leq 1$ and $z \leq 0$.

Lösning. Using spherical coordinates (R, φ, θ) in the usual fashion the region K can be described by $0 \leq R \leq 1$, $\pi/2 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$. Remembering the jacobian $R^2 \sin\varphi$ we get
$$\begin{aligned}\iiint_K \frac{1}{1+x^2+y^2+z^2}\,dV &= \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^1 \frac{R^2\sin\varphi}{1+R^2}\,dRd\varphi d\theta \\ &= \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin\varphi\,d\varphi \int_0^1 \frac{R^2}{1+R^2}\,dR \\ &= 2\pi\left(1-\frac{\pi}{4}\right)\end{aligned}$$

$\square$

Svar: $2\pi\left(1-\frac{\pi}{4}\right)$

4. Find the maximum and minimum values of $f(x,y,z) = x^2 + y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.

Lösning. The function f is continuous and the unit sphere is compact so the existence of a maximum value is guaranteed. There are no singular points on the sphere and so according to Lagrange's multiplier method the maximum value is obtained at a point on the sphere where ∇f and ∇g are parallell (if $g(x,y,z) = x^2+y^2+z^2$). We get the system of equations

$$\begin{cases} 2x = k2x \\ 1 = k2y \\ 1 = k2z \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

From the first equation we see that either $x = 0$ or $k = 1$. If $x = 0$ equations 2 and 3 yield $y = z$ and inserting this into equation 4 we obtain the points $\pm(0, 1/\sqrt{2}, 1/\sqrt{2})$, where the function f takes the values $\pm\sqrt{2}$. If, on the other hand, $k = 1$ then equations 2 and 3 yield $y = z = 1/2$ and inserting this in equation 4 we get $x = \pm 1/\sqrt{2}$. We obtain the points $\pm(1/\sqrt{2}, 1/2, 1/2)$ where the function f takes the value $3/2$. Comparing, we see that the maximum value of $f(x,y,z) = x^2 + y + z$ on the unit sphere is $3/2$ and the minimum value is $-\sqrt{2}$.

$\square$

Svar: Maximum value $3/2$, minimum value $-\sqrt{2}$

5. Use Stokes' theorem to compute $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (-x^3, -z^3, y^3)$ and γ is the intersection of the cylinder $y^2 + z^2 = 1$ and the plane $x + 2y + 2z = 3$, positively oriented when viewed from the top of the positive x-axis.

Lösning. Since $\mathbf{F}$ is infinitely differentiable and γ is the smooth oriented boundary of a smooth surface D which is the part of the plane inside the cylinder, Stokes' theorem yields that

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \iint_D \mathbf{curlF} \cdot \hat{\mathbf{N}}\, dS.$$

We compute $\mathbf{curlF} = (3y^2 + 3z^2, 0, 0)$ and parametrize D by letting $x = 3 - 2r\cos\theta - 2r\sin\theta$ and $y = r\cos\theta$ and $z = r\sin\theta$ where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ and use this to compute the flux integral. We obtain:

$$\iint_D \mathbf{curlF} \cdot \hat{\mathbf{N}}\, dS = \int_0^{2\pi} \int_0^1 3r^3\, dr\, d\theta = \frac{3\pi}{2}.$$

$\square$

Svar: $3\pi/2$

6. Give a precise formulation and a proof of the theorem that states that the gradient of a two-variable function at a point is normal to the level curve of the function passing through that point.

   Lösning. See the text book, Calculus by Adams and Essex, Theorem 6 of Chapter 12.7. □

   Svar: