SF1685: Calculus

The mean value theorem and implicit differentiation

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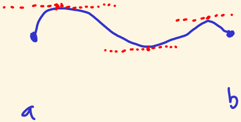
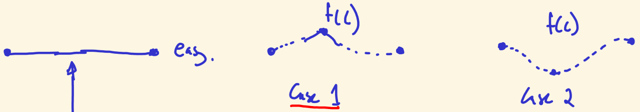
Rolle's theorem

Suppose f is continuous on $[a, b]$, and differentiable on (a, b). Moreover, suppose $f(a) = f(b)$. Then there is some $c \in (a, b)$ such that $f'(c) = 0$.



Rolle's theorem, proof



If f is a constant, then we are done.
Since f is continuous on $[a, b]$, it must attain its maximum (minimum[1]) somewhere, say at c, so that $f(a) < f(c) > f(b)$.
Now look at

$$\lim_{h\to 0^+} \frac{\overbrace{[f(c+h) - f(c)]}^{\text{non-positive}}}{h} = \underbrace{\lim_{h\to 0^-} \frac{\overbrace{[f(c+h) - f(c)]}^{\text{non-pos}}}{\underbrace{h}_{\text{negative}}}}_{\text{non-negative}}.$$

The first limit is non-positive, while the other is non-negative. Hence, the limit must be 0.

$$\boxed{f'(c) = 0} \quad \blacksquare$$

[1] change sign if minimum

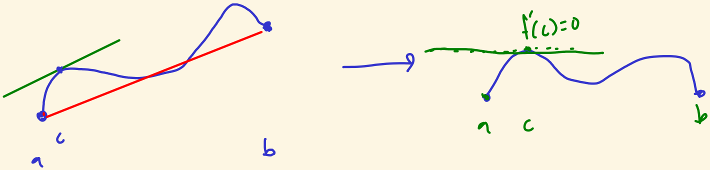
The mean value theorem

$f(a) \neq f(b)$ perhaps.

Suppose f is continuous on $[a, b]$, and differentiable on (a, b). Then there is some $c \in (a, b)$ such that

$$f'(c) = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{slope of line between } f(a) \text{ and } f(b).}.$$

Proof sketch: Let $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}x$. Then verify that $g(a) = g(b)$, and apply Rolle's theorem.



Increasing and decreasing functions

We say that $f(x)$ is **increasing** on (a, b) if

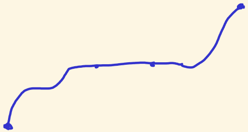
strictly

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

We say that $f(x)$ is **non-decreasing** on (a, b) if

(weakly increasing)

$$x_1 < x_2 \implies f(x_1) \leq f(x_2).$$



Using the mean value theorem, we can show that for differentiable functions f, if

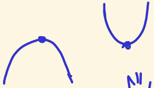
$$f'(x) > 0 \text{ on } (a, b), \text{ then } f \text{ is increasing on } (a, b).$$

Similar statements hold for $f'(x) \geq 0$ etc.

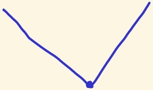
Derivative at a maxima/minima

If f is differentiable at (a, b) and attains its maximum at $c \in (a, b)$, then $f'(c) = 0$.

Note that if f is not differentiable at some point, this could be a minimum/maximum.

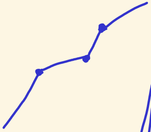


$f''(x) < 0$ $\quad$ $f''(x) > 0$



← minimum, but derivative doesn't exist

$f(x) = |x|$



Critical pt, where $f'(x) = 0$

local max

min $\quad$ saddle

2-variables

Exercise — discussion

Cases!
$[-3,0], [0,3]$

Observation:
$f(x) = f(-x)$ for all x.

Lesson: look for symmetries

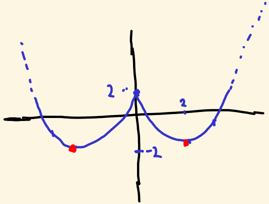
Question

Find minimum and maximum of $f(x) = x^2 - 4|x| + 2$ on the interval $[-3, 3]$.

→ Check critical pts. → $f(-2) = f(2) = -2$. minimum value

→ pts where derivative doesn't exist → $f(0) = 2$ maximum value

→ endpts of the interval. → $f(-3) = f(3) = -1$

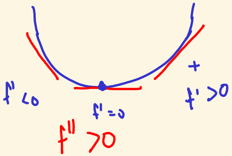


Suppose $x > 0$. Then $f(x) = x^2 - 4x + 2$

$f'(x) = 2x - 4$

so $f'(2) = 0$. critical pt

Same way $x < 0$, $f(x) = x^2 + 4x + 2 \Rightarrow f'(-2) = 0$



f'' tells us how f' changes

|| If $f'' > 0$, then the slope of the tangent increases as x increases.

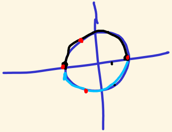
Implicit functions

Implicitly defined functions: We can define $y = y(x)$ via some relation

$$F(x, y) = 0.$$

The set of $(x, y) \in \mathbb{R}^2$ fulfilling this is in general a curve, and not a graph. Segments of this curve can be interpreted as functions.

Implicit function: example



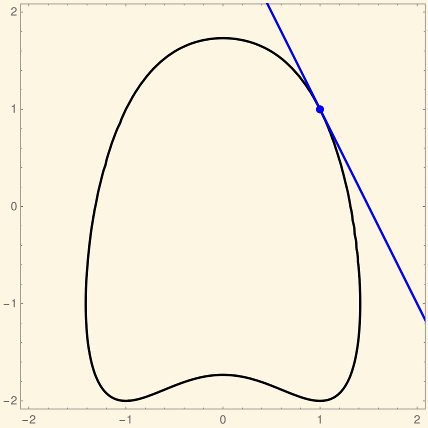
$x^2+y^2-1=0$

For example, $x^2 + y^2 = 1$ defines a set in $\mathbb{R}^2$. Solving for y, we have two segments,

$$y_1(x) = \sqrt{1 - x^2}, \qquad y_2(x) = -\sqrt{1 - x^2}.$$

Implicit function: example II

Let us define a curve via $x^4 + y^2 + x^2 y = 3$.



How do we find the slope of the tangent at $(x, y) = (1, 1)$?

Implicit function derivative

Suppose a curve is defined via

$$F(x, y) = C,$$

and that (x_0, y_0) is a point on the curve. Near this point[2], we can think of y as a function of x. So

$$F(x, y(x)) = C.$$

We then take $\frac{d}{dx}$ on both sides, and solve for $y'(x)$.

[2] Near most points

Implicit function derivative

Return to $x^4 + y^2 + x^2y = 3$. Around $(x, y) = (1, 1)$, we have $y = y(x)$.
Taking derivative (*remembering that y is a function of x!*) gives

by chain rule

$D[y^2] = 2y \cdot y'$

$$D[x^4 + y^2 + x^2y] = D[3]$$
$$4x^3 + 2yy' + 2xy + x^2y' = 0.$$

solve for y'.

Hence,

$$x^2y' + 2yy' = -4x^3 - 2xy \qquad \Longrightarrow \qquad y' = -\frac{4x^3 + 2xy}{x^2 + 2y}.$$

In particular, if $(x, y) = (1, 1)$, then $y' = -(4 + 2)/3 = -2$.
Hence, the slope we seek is -2.

Derivative of an inverse function

Let $f(x)$ be a differentiable function, and let $f^{-1}(x)$ be it's inverse. Hence,

$$f^{-1}(f(x)) = x.$$

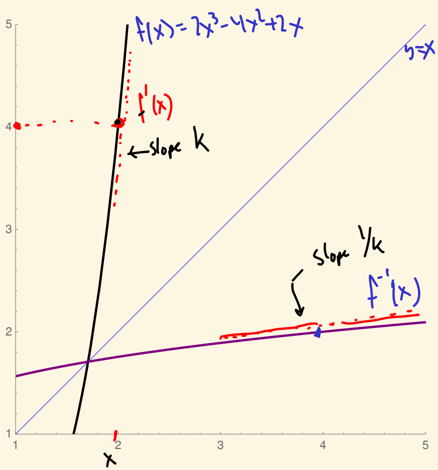
Take the derivative wrt x on both sides, and use the chain rule:

$$(f^{-1})'(f(x)) \cdot f'(x) = 1.$$

Conclusion: if $y = f(x)$, then $(f^{-1})'(y) = \frac{1}{f'(x)}$.

Example

Let $f(x) = 2x^3 - 4x^2 + 2x$, defined on $x \geq 1$ (black curve).
Find the derivative of $f^{-1}(x)$ at $x = 4$.



Need to solve $4 = 2x^3 - 4x^2 + 2x$.

$2 = x^3 - 2x^2 + x$

$x = 2$ is a solution

Thus $(f^{-1})'(4) = \dfrac{1}{f'(2)}$

$= \dfrac{1}{6\cdot 4 - 8\cdot 2 + 2} = \dfrac{1}{10}$

$f'(x) = 6x^2 - 8x + 2$

Answer: $1/10$

Question

Let $(x+y)^3 + 2\sqrt{1+x} = 12$, define a curve. Determine the tangent line to the curve at $(x, y) = (3, -1)$.

Question

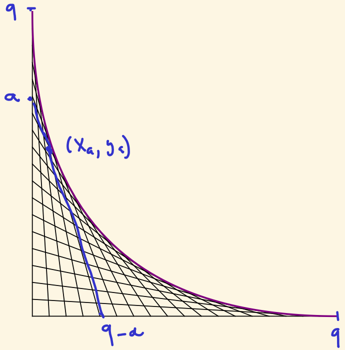
Let $f(x) = x^4 + 2x + 1$ be defined on $x \geq 0$. Find the slope of the tangent line, which tangents $f^{-1}(x)$ at $x = 21$.

Question

During a slow lecture, you start doodling in your notebook. You connect the point $a \in [0, 9]$ on the y-axis, with the point $9 - a$ on the x-axis. This seem to create some sort of curve. Your professor notices you doodling and exclaims

Ah, that curve is given by $(x - y)^2 - 18(x + y) + 81 = 0$.

Verify that the professor is correct, by making sure that each of the lines you have drawn is a tangent to the curve.



Hints for previous question

1. Determine the equation for the line corresponding to the parameter a.
2. Find the intersection between such a line, and the curve. You will get a coordinate, (x_a, y_a).
3. Verify that the slope of the curve in this point agrees with the slope of the line.