$\int f(x)\,dx$

$\downarrow$ generalize by introducing an additional parameters

(*) $\int f(x,t)\,dx$

SF1685: Calculus

Computing volume and area when rotating a graph

$D_t \int f(x,t)\,dx$

$\parallel$

$\int D_t f(x,t)\,dx$

$\parallel$

$F(x,t)$

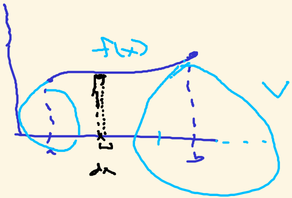
$\longrightarrow \int F(x,t)\,dx$ (*)

Lecturer: Per Alexandersson, perale@kth.se

Rotational volume around the x-axis

Rotating the curve $f(x)$ around the x-axis, creates a solid. It has volume

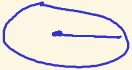
$$V = \int_a^b \pi f(x)^2 dx$$



volume: $f(x)^2 \pi \cdot dx$

add all pieces.

cross-cut

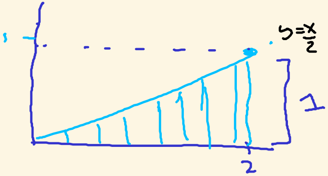


radius: $f(x)$

area: $\pi f(x)^2$

Example

Compute the of the cone created as the line $y = x/2$ is rotated around the x-axis, on the interval $[0, 2]$. Verify that this agrees with the formula you learned in high-school.



$$V = \pi \int_0^2 \left(\frac{x}{2}\right)^2 dx$$

$$= \frac{\pi}{4} \int_0^2 x^2 \, dx$$

$$= \frac{\pi}{4} \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{2\pi}{3}$$

High-school:

$$\frac{1}{3} Ah =$$

$$\frac{1}{3} \cdot \pi \cdot 1^2 \cdot 2$$

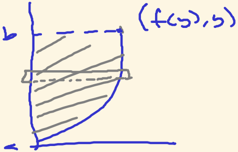
$$= \frac{2\pi}{3}$$

Rotational volume around the y-axis

Rotating the curve $x = f(\cancel{x}\,y)$ around the y-axis creates a solid between the curve and the y-axis. It has volume

$$V = \int_a^b \pi f(y)^2 \, dy$$

Here, a and b are points on the y-axis as well.



Example

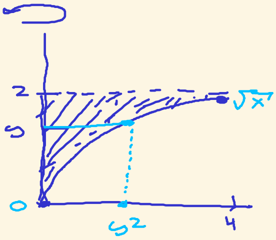
The function $y = \sqrt{x}$ is rotated around the y-axis, $0 \le x \le 4$, and determines a solid between the curve and the y-axis. Find its volume.

Volume is

$$\pi \int_0^2 (y^2)^2 \cdot dy$$

$$\pi \int_0^2 y^4 \, dy$$

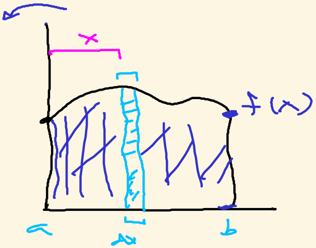
$$\pi \left[\frac{y^5}{5} \right]_0^2 = \frac{\pi}{5} \cdot 32 \quad \blacksquare$$



Rotational volume around the y-axis (shell method)

We can also compute volume created *under f*, when rotated around the y-axis, by cutting the shape into thin shells. The formula is

$$V = 2\pi \int_a^b x f(x) dx.$$

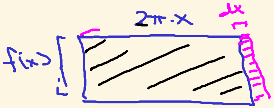


shell piece has

volume: $2\pi x \cdot f(x) \cdot dx$

height

surface area



Example

The function $y = \sin(x)$ is rotated around the y-axis, $0 \le x \le \pi$, and determines a solid under the curve. Find its volume.

$$V = 2\pi \int_0^{\pi} x \cdot \sin(x)\, dx$$

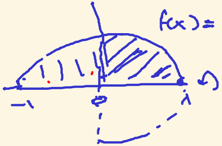
$$\int x \cdot \sin(x)\, dx = -x \cdot \cos x + \int \cos x\, dx$$
$$= -x \cdot \cos x + \sin(x) + C$$

So $V = 2\pi \Big[\sin(x) - x \cdot \cos x \Big]_0^{\pi} = 2\pi(\pi - 0) = 2\pi^2$ ∎

Volume of a sphere

Let's compute the volume of a sphere with radius 1. Compare the two methods (shell vs. rotation around x-axis).

We use $x^2+y^2=1$, $y^2 = 1-x^2$, $y=\sqrt{1-x^2}$ (upper part)



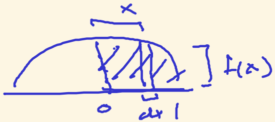
Volume is:

$$\pi \int_{-1}^{1} (1-x^2)\,dx$$

$$\pi \left[x - \frac{x^3}{3} \right]_{-1}^{1} = \pi\left(\frac{2}{3} - (-2/3)\right) = \frac{4\pi}{3}$$

(rotation around x-axis).

Volume of a sphere, continuation



Shell method: Rot. around y-axis.

$$\frac{V}{2} = 2\pi \int_0^1 x\cdot\sqrt{1-x^2}\,dx = \cancel{2}\pi \int_1^0 \sqrt{u}\cdot\left(\frac{-du}{\cancel{2}}\right) = \pi\left[\frac{2u^{3/2}}{3}\right]_0^1$$

$$\left[\begin{array}{l} 1-x^2 = u \\ -2x\,dx = du \\ x\cdot dx = -\dfrac{du}{2} \end{array}\right]$$

$$= \frac{2\pi}{3}$$

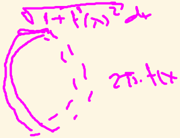
So volume is $\frac{4\pi}{3}$ $\blacksquare$

Surface area, when rotating around x-axis

The outer surface area of the solid created when f is rotated around the x-axis is

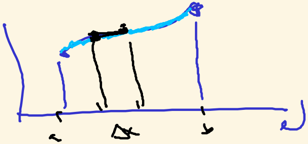
$$A = 2\pi \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2}\, dx.$$

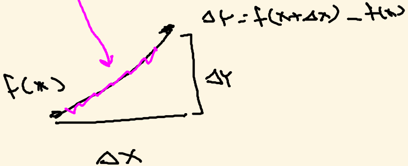
Trick: Consider a small segment on the x-axis, with length Δx, starting at x. The approximate length from $(x, f(x))$ to $(x + \Delta x, f(x + \Delta x))$ is



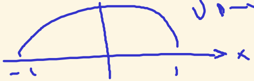
$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \underbrace{(\Delta x)}_{dx} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

approaches the derivative, $f'(x)$





Surface area of a sphere



$$\sqrt{1-x^2} = f(x)$$

$$f'(x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$$

$$(f'(x))^2 = \frac{x^2}{1-x^2}.$$

$$S.A = 2\pi \int_{-1}^{1} \sqrt{1-x^2}\cdot\sqrt{1+\frac{x^2}{1-x^2}}\,dx$$

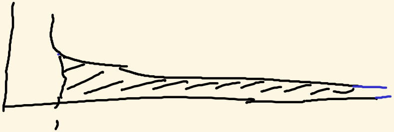
$$= 2\pi \int_{-1}^{1} \sqrt{(1-x^2)+x^2}\,dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{1}\,dx = 2\pi\,[x]_{-1}^{1} = 4\pi.$$

Gen. formula, sphere radius r, $4\pi \cdot r^2$.

Gabriel's horn — A pain to paint?

Gabriel's horn is the long hollow tube obtained by rotating the graph $\frac{1}{x}$ around the x-axis, on the interval $[1,\infty)$. Estimate its surface area and compute the volume it encloses. Can you paint its inside?



Fractals:

Infinite circumference

Finite area

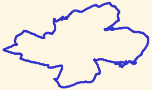
inside

Volume:

$$\pi\int_1^\infty \left(\frac{1}{x}\right)^2 dx$$

$$\pi\int_1^\infty x^{-2}\,dx = \pi\left[-1/x\right]_1^\infty = \pi(0-(-1)) = \pi.$$

Koch Snowflake



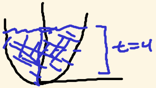
Surface area:

$$2\pi\int_1^\infty \frac{1}{x}\cdot\underbrace{\sqrt{1+\frac{1}{x^4}}}_{\geq 1}\,dx \geq 2\pi\int_1^\infty \frac{1}{x}\,dx = 2\pi\left[\ln x\right]_1^\infty = \infty$$

. We have infinite surface area.

Flowing water

A cup is determined by the curve $y = x^2$ rotated around the y-axis. Water rises in the cup at the rate 1cm/s, starting with an empty cup at $t = 0$. *How much water is in the cup at $t = 4$? How much water (cm^3) is entering the cup each second, at $t = 4$?*



Let's compute volume at time t

$$V(t) = \underline{\pi \int_0^t \sqrt{y}^2 \, dy} = \pi \left[\frac{y^2}{2} \right]_0^t = \frac{\pi t^2}{2}.$$

Volume at $t=4$ is $V(4) = 8\pi$

$\cdot\ V(t) = F(t) - F(0).$

$\cdot\ V'(t) = f(t).$

Rate, is $V'(t) = \pi \cdot t$,

So $V'(4) = 4\pi \ cm^3/s$

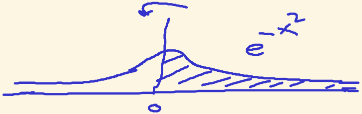
Exercise

Find the surface area of the volume when $y = 4 + 3x^2$ is rotated around the x-axis, with $0 \leq x \leq 1$.

Exercise

$$y = e^{-x^2} \to \sqrt{-\ln(y)} = x$$

Find the volume of the under the curve $f(x) = e^{-x^2}$ as it rotates around the y-axis.

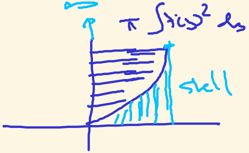


Shell method:

$$V = 2\pi \int_0^\infty x \cdot e^{-x^2}\, dx \qquad x^2 = u,\quad 2x\, dx = du$$

$$= \pi \int_0^\infty e^{-u}\, du$$

$$= \pi \left[-e^{-u}\right]_0^\infty = \pi(0 - (-1)) = \pi$$



Alt. method

$$\pi \int_0^1 \sqrt{-\ln(y)}^{\,2}\, dy = \pi$$

Improper, $y=0$

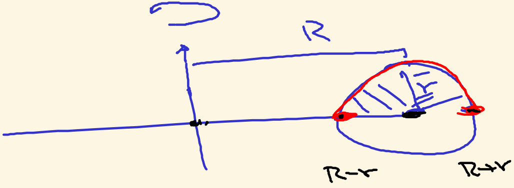
$$\pi \int_0^1 -\ln(y)\, dy =$$ Partial int

$$\pi = \left[-(y \cdot \ln(y) - y)\right]_0^1 = \pi(1 - 0) = \pi$$

$0 \cdot -\infty$

Volume of a torus

Compute the volume of torus, with "big" radius R, and "small" radius r.



$f(x) = \sqrt{r^2 - (x-R)^2}$

$\pi \int_{R-r}^{R+r} f(x)^2 \, dx$

....

Circle centered at $(0,0)$, with radius r:

$\sqrt{r^2 - x^2}$.

[Mult. by 2 in the end to get bottom part.]

Verify that you get $\pi R \cdot r$

Parametric curves — brief intro