SF1685: Calculus

Partial fraction decomposition

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Rational expressions

Today, we focus mainly on rational expressions, $P(x)/Q(x)$ where P, Q are polynomials. If $P(x)/Q(x)$ is a rational expression, we can write it as

$$\frac{P(x)}{Q(x)} = K(x) + \frac{R(x)}{Q(x)}$$

where $\deg(R) < \deg(Q)$. This is accomplished by using **polynomial division**.

The polynomial $K(x)$ is easy to integrate.

Partial fraction decomposition

PFD is a method for rewriting a rational function. Suppose first that $Q(x) = (x - x_1)\cdots(x - x_d)$, and all roots are simple, $\deg(P) < \deg(Q)$. Then we can find numbers $a_1, \ldots, a_d$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \cdots + \frac{A_d}{x - x_d}.$$

$A_i \in \mathbb{R}$

Examples

$$\frac{x+1}{(x-1)(x+2)} = \frac{1/3}{(x+2)} + \frac{2/3}{(x-1)}$$

$$\frac{10x^2}{(x+3)(x+6)(x-2)} = -\frac{6}{x+3} + \frac{15}{x+6} + \frac{1}{x-2}$$

PFD, multiple roots

What if there's a multiple root? Suppose $\deg(P) < k$. Then we can write

$r \in \mathbb{R}$.

$$\frac{P(x)}{(x-r)^k} = \frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_k}{(x-r)^k}.$$

$A_i \in \mathbb{R}$.

Example:

$$\frac{x^2-1}{(x-2)^3} = \frac{4}{(x-2)^2} + \frac{3}{(x-2)^3} + \frac{1}{(x-2)}$$

Combining the two cases

We can combine these two cases:

$$\frac{P(x)}{(x-a)(x-b)(x-c)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \underbrace{\frac{C_1}{x-c} + \frac{C_2}{(x-c)^2}}_{\text{c has multiplicity 2.}}$$

if $\deg(P) < 4$.

Complex roots

Q real coeffs, can be factored into degree 1 and deg. 2 expressions, w. real coeffs.

w. conj. pair of complex roots.

In case we have factors of Q with complex roots, we can do as follows:

$$\frac{P(x)}{(x^2+x+2)(x^2+x+3)} = \frac{A_1x+B_1}{x^2+x+2} + \frac{A_2x+B_2}{x^2+x+3}$$

$$\frac{P(x)}{(x^2+x+2)^2} = \frac{A_1x+B_1}{x^2+x+2} + \frac{A_2x+B_2}{(x^2+x+2)^2}$$

$$\frac{P(x)}{(x-1)\cdot(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

Steps

Dealing with $P(x)/Q(x)$:

1. Use polynomial division if needed, to reduce to a case where $\deg(P) < \deg(Q)$.
2. Factor $Q(x)$ into linear and quadratic factors.
3. Write down the right-hand side with unknown coefficients in the numerators.
4. Write everything on common denominator.
5. Equate coefficients, solve system of linear equations.

Many cases $\longmapsto$ Do lots of practice problems!

A useful formula

Compute $\int \frac{1}{x^2+a^2}\,dx$. $(*)$ If $a=1$, we get $\arctan(x)$

$$\frac{1}{x^2+a^2} = \frac{1/a^2}{\frac{x^2}{a^2}+\frac{a^2}{a^2}} = \frac{1}{a^2}\cdot\frac{1}{\left(\frac{x}{a}\right)^2+1} \quad \text{(Algebra)}$$

Thus $(*) = \frac{1}{a}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,\frac{dx}{a}$, $\left[\begin{array}{l} \frac{x}{a} = u \\ \frac{dx}{a} = du \end{array}\right]$. So $(*) = \frac{1}{a}\int \frac{1}{u^2+1}\,du$

$$= \frac{1}{a}\arctan(u) + C$$

$$= \frac{\arctan\left(\frac{x}{a}\right)}{a} + C$$

∎

Example

Compute $\int \frac{x+1}{x^2+1} dx$.

$$\int \underbrace{\frac{x}{x^2+1}}_{(A)} + \frac{1}{x^2+1} \, dx$$

(A). ux $\quad x^2 = u$

$2x\,dx = du$

$x\,dx = \frac{du}{2}$

$$(A) = \int \frac{1}{u+1} \frac{du}{2} = \frac{\ln|u+1|}{2} + C$$

$\Rightarrow$ By (A) computation

$$\frac{1}{2}\log|x^2+1| + \arctan(x) + C. \quad \blacksquare$$

Example

Compute $\int \frac{1}{\sin(x)}\,dx$.

Want some substitution:

$$\frac{1}{\sin(x)} = \frac{\sin(x)}{\sin^2(x)} = \frac{\sin(x)}{1-\cos^2(x)}$$

So w. subst. $u = \cos(x)$

$$du = -\sin(x)\,dx$$

We get

$$(*) \quad \int \frac{1}{u^2-1}\,du.$$

Want

$$\frac{1}{(u+1)(u-1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = (u+1)A + (u-1)B.$$

2 ways, insert values for u,
or equate coeffs.

$u=1$ gives $1 = 2A$
$u=-1$ $1 = -2B$

So $A = 1/2$, $B = -1/2$

$$(*) = \int \frac{1/2}{u-1} - \frac{1/2}{u+1}\,du$$

$$= \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$$

$$= \frac{1}{2}\ln|\cos(x)-1| - \frac{1}{2}\ln|\cos(x)+1| + C$$

Example

Compute $\int \frac{x}{x^2-4} dx$.

solve w.

$u = x^2 - 4$

easy,

Alt $x^2 - 4 = (x-2)(x+2)$

PFD . . .

Example

Compute $\int \frac{x+1}{x^2+5x+6}\,dx$.

Factor: $x^2+5x+6 = (x+2)(x+3)$

$$\frac{x+1}{(x+2)(x+3)} = \frac{-1}{x+2} + \frac{2}{x+3}$$

Prim. func: $-\ln|x+2| + 2\ln|x+3| + C$. $\square$

$$\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\Longrightarrow \quad x+1 = (x+3)A + (x+2)B$$

Coeffs:

$$\underline{x} + \underline{1} = \underline{Ax + Bx} + \underline{3A+2B}$$

$$\begin{cases} 1 = A+B \\ 1 = 3A+2B \end{cases} \qquad \begin{aligned} A &= -1 \\ B &= 2 \end{aligned}$$

Example

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

Show that $\int \frac{2x-1}{(x+1)^2}\,dx = \frac{3}{x+1} + 2\log(x+1) + C.$

PFD

$$\frac{2x-1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$2x-1 = (x+1)A + B$$

Put $x=-1$, this gives $B=-3$.

Coeff. of x must match: $A=2$.

Thus,

$$\int \frac{2}{x+1} - \frac{3}{(x+1)^2}\,dx =$$

$$2\log|x+1| + \frac{3}{x+1} + C$$

Example

Show that $\int \frac{4x^3+2x-2}{(x^2+1)^2}dx = -\frac{x}{x^2+1} + \frac{1}{x^2+1} + 2\log\left(x^2+1\right) - \arctan(x) + C.$

PFD:

$$\frac{4x^3+2x-2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$4x^3+2x-2 = (x^2+1)(Ax+B) + Cx+D.$$

Comparing x^3 - coeff: $\Rightarrow$ $A=4$.

x^2 - coeffs: $\Rightarrow$ $0=B$

x - coeffs $\Rightarrow$ $2 = A+C$, so $C=-2$.

const - coeff $\Rightarrow$ $-2 = B+D$ $\Rightarrow$ $D=-2$

Continuation

$$\int \frac{4}{x^2+1} - \frac{2x+2}{(x^2+1)^2}\,dx$$

$4 \cdot \arctan(x)$

$$\int \frac{2x+2}{(x^2+1)^2}\,dx = \int \frac{2x}{(x^2+1)^2} + \frac{2}{(x^2+1)^2}\,dx$$

subst

$u = x^2$

$du = 2x\,dx$

$$\int \frac{1}{(u+1)^2}\,du = \frac{-1}{x^2+1}$$

$$\int \frac{2}{(x^2+1)^2}\,dx$$

$$\shortparallel$$

$$\int \frac{2}{(1+x^2)}\frac{dx}{(1+x^2)}$$

$$\shortparallel$$

$\cancel{x} = \tan(u)$

$$dx = \frac{1}{\cos^2(u)}\,du$$

$$= \frac{\cos^2 u + \sin^2(u)}{\cos^2(u)}$$

$$= 1 + \tan^2(u)$$

$$\boxed{\frac{dx}{1+x^2} = du}$$

$$\int \frac{2}{1+\tan^2(u)}\,du$$

$$2\int \cos^2(u)\,du =$$

$$\int 1+\cos(2u)\,du$$

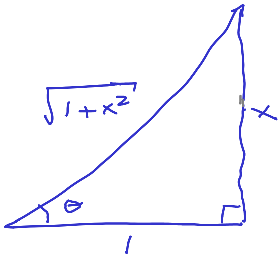
$$\cos^2(u) = \frac{1+\cos(2u)}{2}$$

$$\int 1+\cos 2u \, du = u + \frac{\sin(2u)}{2} + C \qquad [u = \arctan(x)]$$

$$= \arctan(x) + \frac{x}{1+x^2} + C$$

$$\frac{\sin(2u)}{2} = \underline{\sin(u)} \cdot \underline{\cos(u)}$$

- $\sin(\arctan(x))$?
- $\cos(\arctan(x))$?



$$\arctan(x) = \theta$$

$$\sin(\theta) = \frac{x}{\sqrt{1+x^2}}.$$

$$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}.$$

<u>Answer</u>: $3\arctan(x) + \frac{1}{1+x^2} - \frac{x}{1+x^2} + C$

Example — how to deal complex roots

Show that $\int \frac{4x+4}{x^2+4x+20} dx = 2\log\left(x^2+4x+20\right) - \arctan\left(\frac{x+2}{4}\right) + C$.

Continuation

Example

Show that $\int \cancel{\frac{4x^3+2x-2}{(x^2+1)^2}}\,dx = -\frac{x}{x^2+1} + \frac{1}{x^2+1} + 2\log\left(x^2+1\right) - \arctan(x) + C.$

Continuation

Example

Show that $\int \frac{3\sin(2x)}{\sin^2(x)+2\cos(x)+1}\,dx = 4\log(2-\cos(x)) + 2\log(\cos(x)+1) + C.$

Continuation

Compute the PFD of the following expressions

$$\frac{x^3-3x^2-2}{x^2-6x+8} \qquad \frac{x^4}{(x^2+2)^2} \qquad \frac{x^3}{x^3-5x^2+6x} \qquad \frac{x^3}{x^3+2x^2+5x}$$