Optimal Lifetime Asset Allocation Glide Paths via CMA-ES and Bernstein Polynomials

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https://github.com/tbenacci/Evolutionary-Bernstein-Polynomial-Glide-Path

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Abstract

Target-date funds typically prescribe a monotone declining equity glide path over an investor's lifetime. We present a framework that discovers near-optimal lifetime allocation strategies by combining Bernstein polynomial parameterization with the Covariance Matrix Adaptation Evolution Strategy (CMA-ES). Each of four asset classes is assigned an independent smooth curve whose shape is optimized to minimize the probability of financial ruin, evaluated over 50,000 Latin Hypercube sampled Monte Carlo lifecycle simulations incorporating mortality, income, spending, Social Security, and bequest objectives. The framework converged in 300 generations (11.4 minutes) to a glide path achieving a 4.33% ruin probability with a 74.9% probability of leaving a $1.5M bequest and a median terminal wealth of $3.49M. The discovered path is non-monotone. Equity exposure declines into retirement, then rises again after age 85 as the probability of surviving to spend the money drops but the upside for heirs remains.

1 Introduction

A fair amount of academic literature has examined whether the conventional retirement glide path (monotone declining equity exposure over the investor's lifetime) is actually optimal. The standard target-date fund answer is a schedule of portfolio weights that slopes downward from high equity to low equity as retirement approaches and stays conservative thereafter. This shape reflects a single objective (reducing volatility near retirement) and assumes a single shape (monotone decline). Most investors, however, have two goals that pull in opposite directions. They want to fund retirement spending without running out of money and they want to leave something behind for heirs. Safety favors bonds. Growth favors equities.

The difficulty with searching for a better shape is dimensionality. An investor living from age 30 to a maximum of 103 has 73 years of life. With four asset classes, that is 292 free weights. Optimizing 292 noisy parameters is impractical. The insight that makes this tractable is to optimize the shape of a curve rather than the weights directly. If each asset's allocation is a smooth polynomial with 20 control points, the search space drops to 80 dimensions. Bernstein polynomials are used for this purpose. They guarantee that the resulting curves are smooth, bounded between 0 and 1, and can take any shape the optimizer discovers. The optimizer, CMA-ES [1], is an evolutionary algorithm designed for noisy, gradient-free problems in moderate dimensionality.

We ran this framework on a concrete scenario. A 30-year-old with $100k in savings, earning $85k/year, saving 12%, planning to retire at 65, spend $80k/year in retirement, collect $24k/year in Social Security starting at 67, and hoping to leave $1.5M to heirs. The optimizer converged in 300 generations and 11.4 minutes. The resulting glide path achieves a 4.33% probability of ruin with a 74.9% probability of meeting the bequest target and a median terminal wealth of $3.49M. The discovered path is non-monotone, confirming that a late-life equity increase can be optimal under realistic assumptions.

2 Literature Review

A substantial body of research has investigated whether the monotone-declining equity glide path is actually optimal. The studies differ in methodology but converge on a common finding. The optimal shape is more complex than a straight line down.

2.1 Rising Equity Glide Paths

Kitces and Pfau [5] showed that a rising equity glide path during retirement, starting conservative and increasing equity exposure by roughly 1% per year, reduces both the probability of failure and the magnitude of failure compared to a fixed or declining allocation. Their explanation centers on sequence-of-returns risk, as retirees are most vulnerable to equity losses in the first few years of retirement, when the portfolio is at its largest relative to remaining liabilities.

Kitces [6] later extended this into the "bond tent" concept, arguing that the optimal lifetime equity allocation is V-shaped. It is high during accumulation, lowest around the retirement date, and rising again in the first decade of retirement. The bond allocation forms an inverted V that peaks at retirement and is drawn down in the early withdrawal years. Both papers tested simple parametric forms. The rising glide path increased equity by a fixed increment each year. The V-shape was described conceptually rather than optimized numerically. The question of what the exact best shape looks like was left open.

2.2 Blanchett on Distribution Glide Paths

Blanchett [4] directly compared three glide path shapes during the distribution phase. The decreasing path started at 60% equity and declined by 1% per year, the constant path held 45% equity throughout, and the increasing path started at 30% and rose by 1% per year. His 2015 update incorporated current market conditions including low bond yields and elevated CAPE ratios. He found that the optimal shape depends heavily on return assumptions and initial market valuations. A rising path works best in overvalued environments while a declining path may be preferable when valuations are low. The shapes tested were all linear, and only two asset classes (stocks and bonds) were considered.

2.3 Rook on Optimal Static Glide Paths

The closest methodological predecessor to this paper is Rook [7], who sought to derive the mathematically optimal static glide path with respect to ruin probability. Rather than comparing pre-specified shapes, Rook formulated the problem as a continuous optimization and used gradient ascent with metaheuristics to search the space of all possible equity allocations at each year of retirement. His approach is notable for its mathematical rigor, including analysis of the conditions under which the optimal path is concave and a full C++ implementation.

Rook's framework has two limitations that the present work addresses. He optimized the equity ratio directly at each time step with no smoothness constraint, so the optimal path can in principle be jagged. Additionally, his model includes only two asset classes (stocks and bonds), so the allocation at each age is a single number rather than a vector.

2.4 How This Paper Differs

This paper differs in three respects. The search space is unconstrained in shape but constrained in smoothness. Bernstein polynomials [2] with 20 control points per asset guarantee infinitely differentiable curves regardless of control point values. The optimizer can discover any smooth shape but cannot produce a jagged path. Smoothness is a structural property of the basis, not a penalty term. The optimization also runs across four asset classes simultaneously (cash, international bonds, domestic equity, international equity) rather than a single equity/bond split. Lastly, CMA-ES [1] handles the noise inherent in Monte Carlo fitness evaluation without requiring gradients.

3 Data

3.1 Asset Class Parameters

Four asset classes are modeled with real (inflation-adjusted) return parameters calibrated to 1927 to 2006 US market history. Returns follow a multivariate lognormal: $\log(1 + R) \sim \mathcal{N}(\boldsymbol{\mu}_{\log}, \boldsymbol{\Sigma})$, where $\mu_{\log} = \log(1 + \mu_{\text{arith}}) - \sigma^2/2$.

Table 1: Asset Class Parameters (Real Returns) Reported are the arithmetic mean return and standard deviation for each asset class, calibrated to 1927 to 2006 US market data.

Asset Class	μ_{arith}	σ
Cash	0.81%	4.10%
Int. Bonds	2.52%	6.14%
Domestic Eq.	9.48%	20.99%
Int'l Eq.	7.25%	21.59%

Domestic and international equities are correlated at 0.60. Bonds and equities are nearly uncorrelated. Cash and international equities are slightly negatively correlated (-0.05). The covariance matrix is constructed as $\boldsymbol{\Sigma} = \text{diag}(\boldsymbol{\sigma}) \cdot \mathbf{C} \cdot \text{diag}(\boldsymbol{\sigma})$. The LHS return generator verified that sample means matched target means to three decimal places across all four asset classes. The number of asset classes is derived from the length of the asset name vector, so adding or removing asset classes propagates automatically through the return generator, Bernstein engine, fitness function, and plotting code.

3.2 Investor Parameters

Table 2: Investor Parameters Reported are the personal financial assumptions used in the lifecycle simulation.

Parameter	Value
Current age	30
Retirement age	65
Current savings	$100,000
Current income	$85,000
Savings rate	12%
Real income growth	1%/yr
Annual spending (ret.)	$80,000
Social Security benefit	$24,000/yr
SS start age	67
Bequest target	$1,500,000

The bequest target enters the model only as a threshold for computing P(bequest). The optimizer does not see it and does not try to maximize it. This is done to observe whether strategies that minimize ruin also produce favorable bequest outcomes without distorting the fitness landscape. The companion sensitivity analysis varies each parameter individually and in pairwise combinations to show how the optimized path's performance changes under different conditions.

3.3 Mortality

Mortality is modeled using a personalized table from an external prediction model. The resulting CDF (Figure 1) implies a median death age of 83 and a lifespan horizon of 73 years (age 31 to 103). Death ages for each Monte Carlo path are sampled by drawing a uniform random number and inverting this CDF. If the external mortality file is unavailable, a Gompertz hazard fallback is used: $h(x) = 0.00003 \cdot e^{0.085(x-30)}$.

The shape of the mortality curve directly affects the optimal glide path. The long right tail past age 90 is the region where the bequest motive dominates. Only about 20% of simulation paths survive past 90, but those paths carry disproportionate bequest potential. If the mortality distribution were truncated at age 85, there would be no late-life equity increase because there would be no late-life phase where spending probability is low.

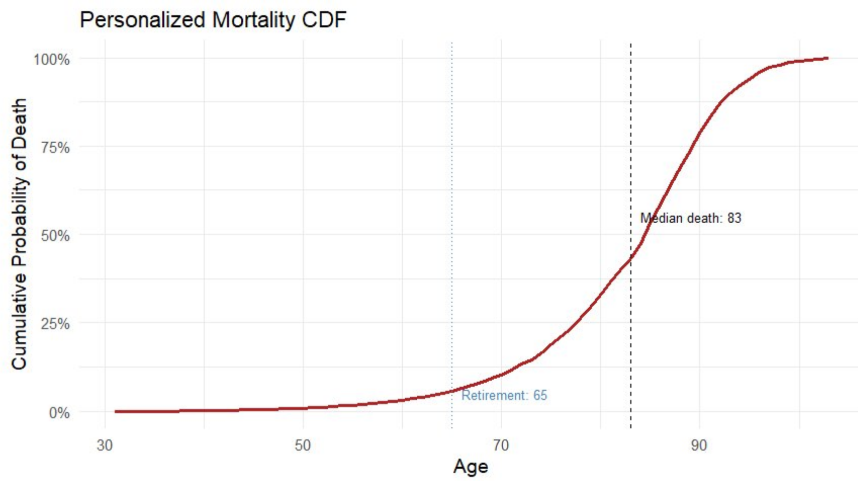

Figure 1: Personalized Mortality CDF Displayed are the cumulative probability of death by age, with retirement age (65) and median death age (83) marked.

4 Methodology

4.1 Bernstein Polynomial Parameterization

Each asset class gets its own Bernstein curve with $N_{\mathrm{CP}} = 20$ control points, defining a degree-19 polynomial over the investor's lifespan. The parameter τ maps linearly from 0 (current age) to 1 (maximum age). At each age, the curve value is a weighted average of the control points, with weights following a binomial distribution. The first few control points dominate early in life and the last few dominate late in life.

Twenty control points per asset provides enough flexibility to represent any reasonable allocation shape while keeping the total search space at 80 dimensions. For numerical stability, the polynomial is evaluated using the de Casteljau recurrence [2] rather than computing binomial coefficients directly.

A candidate solution is a vector of $4 \times 20 = 80$ control point values, each in $[0, 1]$. Each group of 20 is expanded into a full-lifespan curve, and the four raw curves are normalized at each age so the weights sum to one:

$$w_k(\tau) = \frac{\tilde{w}_k(\tau)}{\sum_{j=1}^{4} \tilde{w}_j(\tau)} \tag{1}$$

The convex hull property guarantees non-negative raw values, so the normalized weights are automatically valid portfolio weights at every age. The result is a matrix $\mathbf{W}$ of dimension 73×4, where row t is the portfolio weight vector at age $30 + t$.

4.2 Pre-generated Stochastic Environment

Before optimization begins, 50,000 LHS return paths (a 3D array of dimension 50,000 $\times$ 73 years $\times$ 4 assets) and 50,000 death ages are generated once. A Latin Hypercube of uniform samples is generated, transformed to standard normal via the inverse CDF, correlated using the Cholesky factor of $\mathbf{\Sigma}$, shifted by log-space means, exponentiated, and shifted to produce arithmetic returns.

This is held fixed for every fitness evaluation in every generation. If fresh random draws were used per evaluation, two candidates in the same generation would be compared on different luck.

The fitness ranking would be contaminated by noise and the optimizer would waste generations chasing phantom improvements. This is sometimes called the "common random numbers" approach in simulation optimization literature.

4.3 Fitness Function

The fitness function takes 80 control points, expands them into a weight matrix via the Bernstein engine, and simulates the full lifecycle for all 50,000 paths. During working years, savings contributions are added. The portfolio return each year is:

$$r_t^{\text{port}} = \sum_{k=1}^{4} w_{k,t} \cdot R_{k,t} \tag{2}$$

After retirement, spending net of Social Security is withdrawn. If wealth drops to zero, the path is marked as ruined. The primary fitness metric is:

$$P(\text{ruin}) = \frac{\#\text{ retired paths that ran out of money}}{\#\text{ paths that survived to retirement}} \tag{3}$$

P(bequest), median terminal wealth, and path roughness are tracked but do not influence the optimizer.

4.4 CMA-ES Optimizer

Each generation, 40 candidates are sampled from a multivariate normal, clamped to $[0, 1]$, and evaluated in parallel on the fixed Monte Carlo paths. The best 20 are recombined with log-spaced weights to update the mean. Two evolution paths track cumulative search progress and adapt the diagonal covariance and global step size sigma. Sigma shrinks as the algorithm converges. When it collapses below 10^{-5}, the run terminates. A stall detector halts the run if the best fitness has not improved for 40 consecutive generations.

Table 3: CMA-ES Configuration Reported are the optimizer hyperparameters and Monte Carlo budget.

Parameter	Value
Control points per asset	20
Total dimensions	80
Population (λ) / Parents (μ)	40 / 20
Max generations	300
Initial σ_0	0.25
Stall limit	40 gens
MC paths (fixed)	50,000

The initial mean is set to raw values of 0.20 (cash), 0.30 (bonds), 0.80 (domestic equity), 0.50 (international equity), replicated across all 20 control points per asset. After normalization, this produces a starting allocation of roughly 11/17/44/28%. The initial step size $\sigma_0 = 0.25$ covers a substantial fraction of the $[0, 1]$ feasible range. All fitness evaluations are performed in parallel on a cluster of worker processes. Each generation involves $40 \times 50{,}000 = 2{,}000{,}000$ individual lifecycle path simulations.

5 Results

5.1 Convergence

The optimizer ran for 300 generations in 11.4 minutes. The final step size sigma collapsed to 0.003 (Figure 3). P(ruin) dropped rapidly in the first 50 generations and then refined slowly over the remaining 250 (Figure 2). The generation mean converged toward the generation best, confirming population concentration rather than fragmentation.

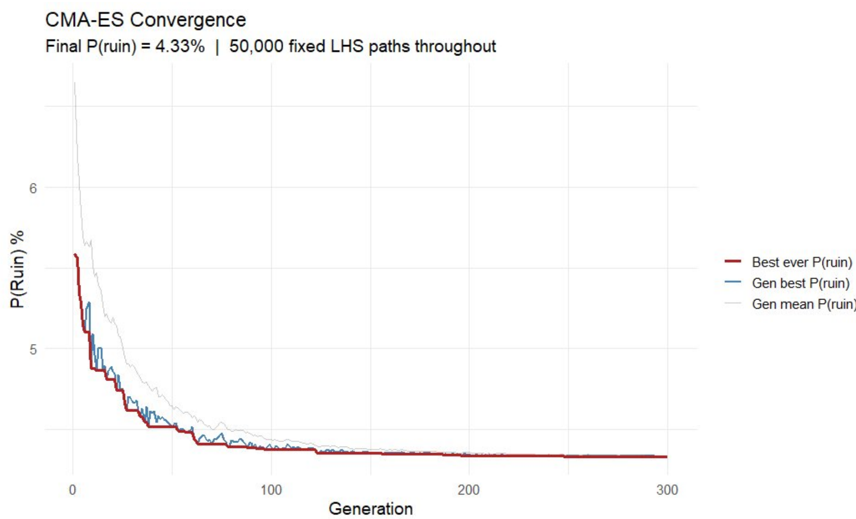

Figure 2: CMA-ES Convergence Displayed are P(Ruin) traces for the best candidate per generation, the population mean, and the best-ever value over 300 generations.

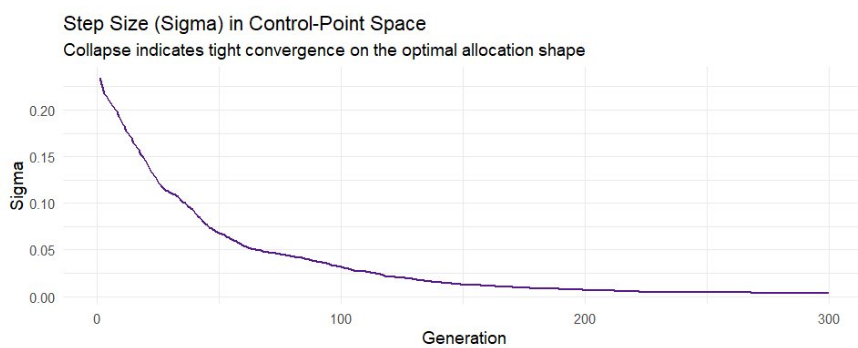

Figure 3: Step Size (Sigma) Evolution Displayed is the global step size in control-point space over 300 generations. Collapse from 0.25 to 0.003 indicates tight convergence.

Path roughness remained low and stable from the first generation onward (Figure 4), confirming that the Bernstein parameterization produces smooth paths without a roughness penalty. P(bequest) remained near 75% throughout the optimization despite not being part of the fitness function (Figure 5).

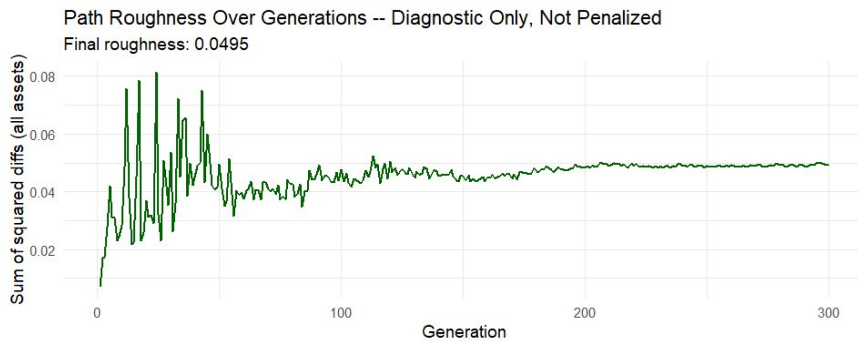

Figure 4: Path Roughness Over Generations Displayed is the sum of squared year-over-year weight changes across all assets. This metric is diagnostic only and not penalized.

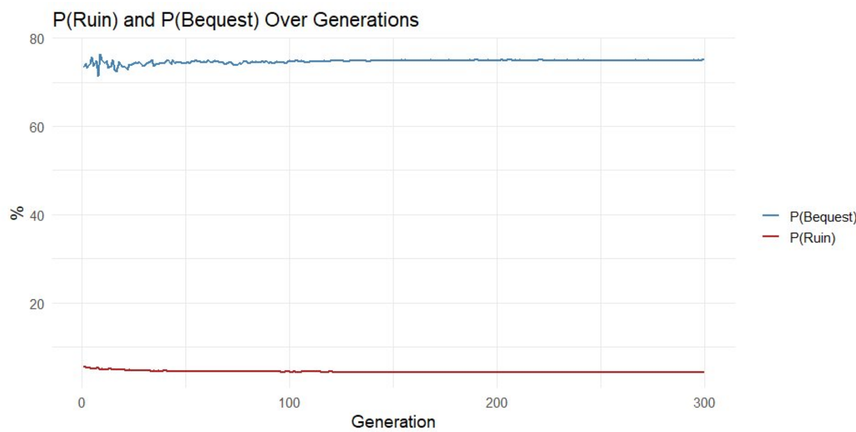

Figure 5: P(Ruin) and P(Bequest) Co-evolution Displayed are P(Ruin) and P(Bequest) over 300 generations. The optimizer targets only P(Ruin), but P(Bequest) remains high throughout.

5.2 Optimized Glide Path

Table 4: Optimization Results Reported are the final metrics after 300 generations.

Metric	Value
Generations	300
Final σ	0.003268
Runtime	11.4 min
P(Ruin)	4.33%
P(Bequest $\geq$ \$1.5M)	74.9%
Median terminal wealth	\$3,493,296
Path roughness	0.0488

The discovered glide path is shown in Figure 6. During the accumulation phase (ages 30 to 65), the portfolio is heavily weighted toward domestic equity. As retirement approaches, bonds and cash increase to build a buffer for the early withdrawal years when sequence-of-returns risk is highest.

International equity holds a substantial allocation (roughly 15 to 25%) throughout the lifespan, reflecting the diversification benefit of its moderate 0.60 correlation with domestic equity. Cash stays near 10 to 15% and serves as a liquidity buffer.

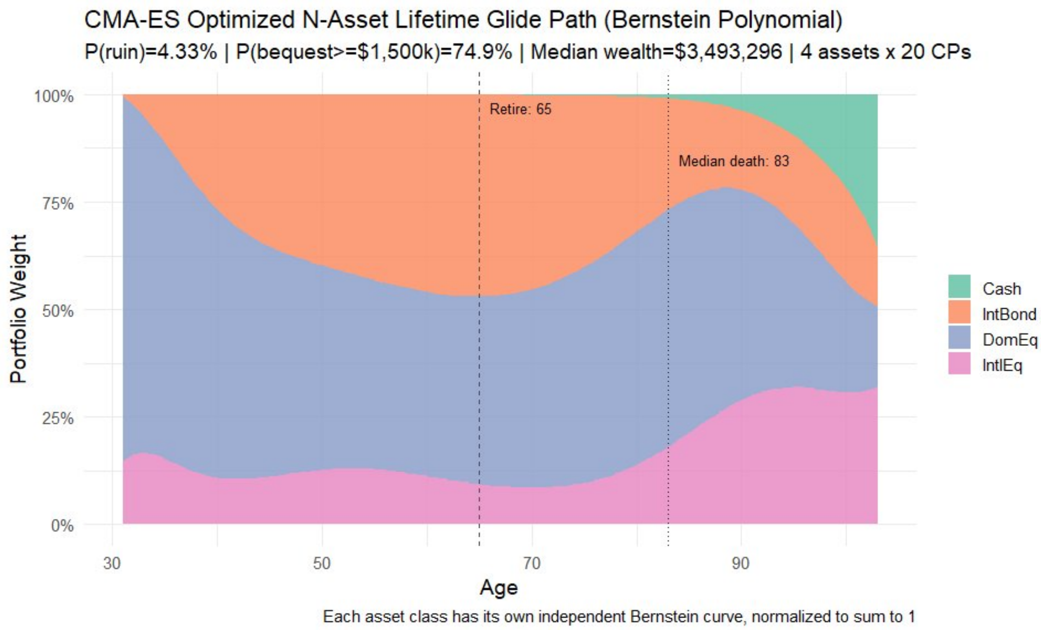

Figure 6: CMA-ES Optimized N-Asset Lifetime Glide Path Displayed is the four-asset allocation over the full lifespan. Total equity declines from about 85% at age 30 to roughly 55% at retirement, then rises again after age 85. Cash and bonds peak around retirement.

The ratio of total equity to total bond allocation traces a U-shape over the lifespan, declining into retirement and rising again thereafter. This is broadly consistent with the V-shaped lifetime equity allocation that Kitces [6] proposed conceptually as the "bond tent." The difference is that Kitces described a fixed linear increase in equity during the first decade of retirement, while the optimizer here discovered the shape freely and found the trough slightly after retirement with the equity rebound beginning closer to age 85.

The non-monotone feature appears after age 85. Total equity allocation rises from its post-retirement low back toward 75%. By age 85, the investor has about a 20% cumulative probability of having already died (Figure 1). Among those still alive, the portfolio has survived the dangerous early-retirement years. The withdrawal rate relative to remaining wealth has declined, and Social Security covers a larger fraction of the remaining need. The wealth that remains is increasingly likely to become a bequest rather than to be spent. The optimizer found that accepting more equity volatility at this stage reduces the overall ruin probability because the downside risk of equity is less threatening than at age 70 (fewer remaining years of spending to fund), while the upside contributes to both the bequest and to the small-probability scenarios where the investor lives past 95.

It is worth comparing this to a typical target-date fund. A Vanguard Target Retirement 2060 fund starts at approximately 90% equities and transitions to roughly 30% equities by age 72, holding steady thereafter. The conventional path includes only two effective asset classes, while ours allocates across four. The conventional path stops changing roughly seven years after retirement, while ours continues to evolve for the full remaining lifespan. Most importantly, the conventional path never reverses. Ours does.

6 Discussion

The Bernstein basis does three things that matter for this problem. It makes smoothness free, as any set of 20 control points in $[0, 1]$ produces an infinitely differentiable curve, so no roughness penalty is needed. This is confirmed by the path roughness of 0.0488, which is low and stable from the first generation. It compresses the search space from 292 per-year weights to 80 control points, a 73% reduction and imposes no shape restriction. The optimizer found a shape with a late-life equity increase that a grid search over simple parametric families would likely miss.

With 50,000 paths, the standard error of P(ruin) near 4% is about 0.09 percentage points. This is small enough for the optimizer to reliably distinguish between candidates. LHS further improves precision by ensuring the tails of the return distribution are well represented, which matters because ruin is a tail event. The total computational burden is $300 \times 40 \times 50{,}000 = 600{,}000{,}000$ path simulations, completed in 11.4 minutes with parallel evaluation.

Several limitations are worth noting. First, the return model is stationary multivariate lognormal, ignoring regime shifts, fat tails, and time-varying correlations. Second, spending is deterministic. Third, healthcare shocks would raise ruin probabilities and might change the optimal shape. Fourth, the mortality model does not account for correlations between health events and market conditions. Fifth, the optimizer minimizes P(ruin) only, and a multi-objective formulation could reveal Pareto-optimal trade-offs on the ruin-bequest frontier. Finally, the framework ignores taxes, transaction costs, and the practical difficulty of annual rebalancing.

With this knowledge, future works may wish to consider stochastic spending (healthcare costs that accelerate with age), correlated mortality and market risk, taxes and transaction costs, model selection over the number of control points, and additional asset classes such as real estate, commodities, or TIPS.

7 Conclusion

Upon optimizing lifetime asset allocation glide paths using Bernstein polynomial parameterization and CMA-ES across four asset classes and 50,000 Monte Carlo lifecycle simulations, there are several noteworthy results. The discovered path is non-monotone, with equity declining into retirement and then rising again after age 85, a shape that no standard target-date fund can express. The optimizer achieves a 4.33% ruin probability while simultaneously producing a 74.9% bequest probability and $3.49M median terminal wealth, despite optimizing only for ruin. The Bernstein polynomial parameterization guarantees smooth paths without a roughness penalty while allowing the optimizer to discover any shape the data supports. The specific shape depends on the specific inputs. An investor with no bequest motive, different mortality, or lower risk tolerance would see a different optimal path. The contribution of this work is the framework itself, a computationally efficient method for discovering optimal smooth allocation paths under any set of lifecycle assumptions.

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