This is pretty loose and maybe I should improve it... I assume the proper choice of norm without specifying what it is. But, I think it gets the idea across. I use $O(\Delta t)$ notation very freely... I'm certain that there are some errors, but I don't think that they influence the final result.

I have the following system of differential equations:

$$\dot{\tau} = \nabla v - \frac{\mu}{\eta}\tau \tag{1}$$

$$\nabla \cdot \tau = 0 \tag{2}$$

where τ is the (vector) shear stress and v is the (scalar) velocity in an antiplane strain scenario.

I expand the velocity, v in a Taylor series around the initial velocity, v_n.

$$v = v_n + \dot{v}\Delta t + O(\Delta t^2) \tag{3}$$

I insert this into Equation 1.

$$\dot{\tau} = \nabla(v_n + \dot{v}\Delta t + O(\Delta t^2)) - \frac{\mu}{\eta}\tau \tag{4}$$

Now, I perform the splitting. The first equation is an initial value ODE to find the value of the tentative stress update, $\hat{\tau}(t_n + \Delta t)$. I ignore velocity evolution in this step.

$$\dot{\hat{\tau}} = \nabla(v_n + O(\Delta t)) - \frac{\mu}{\eta}\hat{\tau} \tag{5}$$

with $\hat{\tau}(t <= t_n) = \tau(t <= t_n)$ (they match exactly for all prior time). Using a second order Backward Differentiation Formula (BDF2) method for the ODE, I can write a discrete version of the time derivative:

$$\frac{1}{2\Delta t}(3\hat{\tau}_{n+1} - 4\hat{\tau}_n + \hat{\tau}_{n-1}) = \nabla(v_n + O(\Delta t)) - \frac{\mu}{\eta}\hat{\tau}_{n+1} \tag{6}$$

Rearranging the equation gives:

$$\hat{\tau}_{n+1} = \frac{2\Delta t}{3}(4\hat{\tau}_n - \hat{\tau}_{n-1} + \nabla v_n - \frac{\mu}{\eta}\hat{\tau}_{n+1}) + O(\Delta t^2) \tag{7}$$

From the derivation of the BDF2 formula, I know that:

$$\|\hat{\tau}_{n+1} - \hat{\tau}(t = t_n + \Delta t)\| \leq C_1 \Delta t^2 \tag{8}$$

I've carried through the error term that resulted from dropping velocity evolution, so that:

$$\|\hat{\tau}(t = t_n + \Delta t) - \tau(t = t_n + \Delta t)\| \leq C_2 \Delta t^2 \tag{9}$$

The total error in the ODE step is:

$$\|\hat{\tau}_{n+1} - \tau(t = t_n + \Delta t)\| \leq \|\hat{\tau}(t = t_n + \Delta t) - \tau(t = t_n + \Delta t)\| + \|\hat{\tau}_{n+1} - \hat{\tau}(t = t_n + \Delta t)\| \tag{10}$$

$$\|\hat{\tau}_{n+1} - \tau(t = t_n + \Delta t)\| \leq C_3 \Delta t^2 \tag{11}$$

Returning to the description of the second step as a projection onto a divergence free function space, this error bound confirms my intuition that the tentative stress update won't be very far from being divergence free, and thus the projection step is a small correction. In fact, as long as the projection step doesn't make the error in the stress *worse*, it's true "purpose" is to get a good updated velocity estimate.

The second step is:

$$\frac{3}{2\Delta t}(\tau_{n+1} - \hat{\tau}_{n+1}) = \mu\nabla\left(\frac{\partial v}{\partial t}\Delta t\right) = \mu\nabla(v(t = t_n + \Delta t) - v_n + O(\Delta t^2)) \tag{12}$$

To complete the error analysis, I need to analyze the Poisson update to the velocity. This will give the error between the estimated new velocity, v_{n+1}, and the true new velocity, $v(t = t_n + \Delta t)$.

In exact form, this is:

$$-\nabla^2(v_{n+1} - v_n + O(\Delta t^2)) = \frac{3}{2\mu\Delta t}(\hat{\tau}(t = t_n + \Delta t)) = \frac{3}{2\mu\Delta t}(\hat{\tau}_{n+1} + O(\Delta t^2)) \tag{13}$$

Moving both the $O(\Delta t^2)$ terms to the RHS, the error can be viewed as a difference in forcing.

Using a finite element discretization makes the weakest assumptions about smoothness. The Lax-Milgram Theorem holds for the Poisson equation in weak form so that I have a continuous dependence of the gradient of the solution on the forcing.

$$\|\nabla(v_{n+1} - v_n)\| \leq C_4\|\nabla \cdot \hat{\tau}_{n+1}\| \tag{14}$$

And so looking at the error:

$$\|\nabla(v_{n+1} - v(t = t_n + \Delta t))\| \leq C_5\Delta t^2 \implies \|v_{n+1} - v(t = t_n + \Delta t)\| \leq C_6\Delta t^2 \tag{15}$$

Using a finite difference formulation gives a similar bound, but requires more stringent smoothness assumptions (I think C^4 rather than C^1).

This closes our string of errors showing that all the errors are $O(\Delta t^2)$.

$$\|\tau(t = t_n + \Delta t) - \tau_{n+1}\| \leq C_7\Delta t^2 \tag{16}$$