Depth Penetration of Earthquake Rupture
Progress on Two Numerical Methods

Seismology and Geomechanics Group Meeting
17 December 2013

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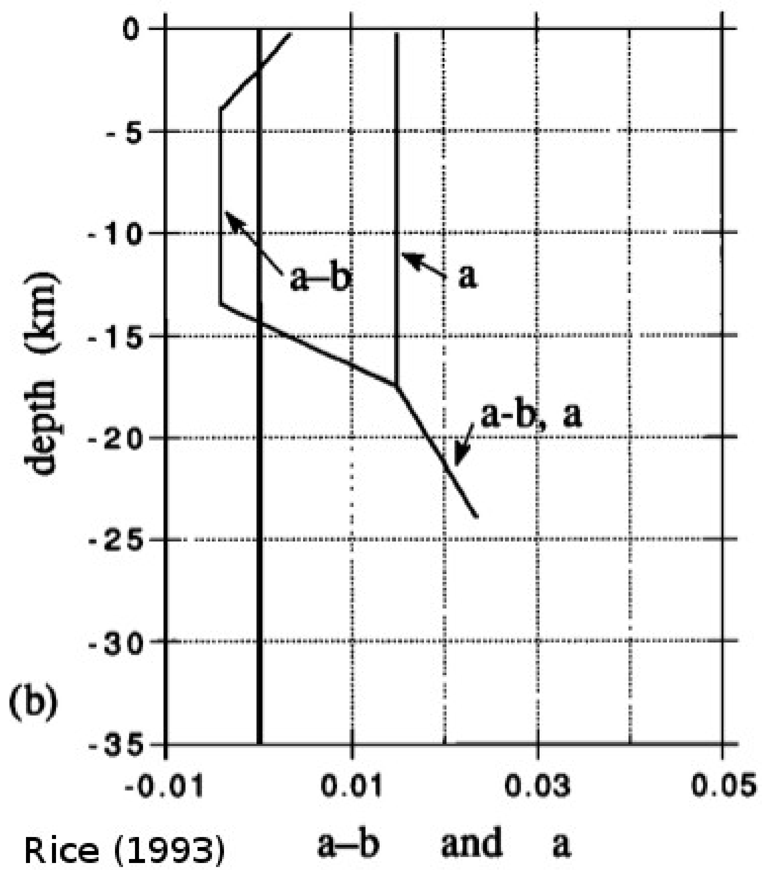
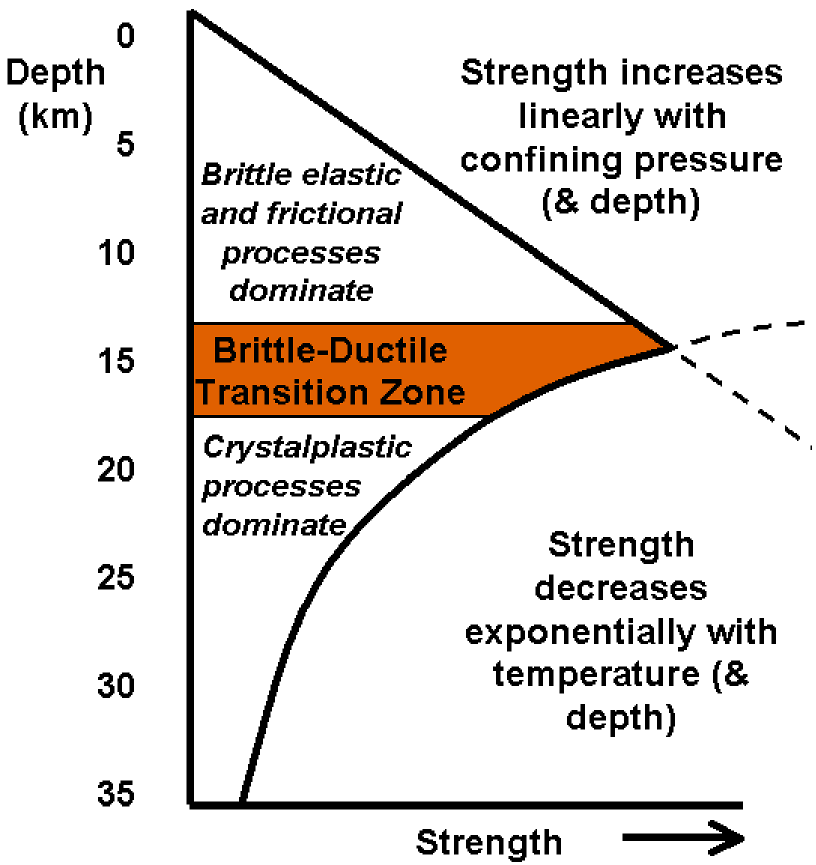
Rupture Depth Penetration

- Two simple paradigms for the depth termination of rupture:
  - Changes in frictional properties – velocity strengthening
  - Brittle-ductile transition – viscoplasticity, ductile shear zones



Rice (1993)



Computational Problems

- Near fault physics are very important
  - Temperature anomaly, ductile shear zones (many more!)
  - Requires very fine spatial resolution for rupture – meters.
- For wave propagation, the Courant-Friedrich-Lewy (CFL) condition causes big problems!

$$\frac{c\Delta t}{\Delta x} \leq 1$$

- Normally, CFL is a global stability condition.
- Idea: Larger elements take longer time steps (less compute time!)
  - Reduces the CFL condition to a local stability condition

Wave Equation

- Idea: Seismic waves are a coupled system of first order advection equations.

$$\boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$$

$$\rho\ddot{\vec{u}} = \nabla \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\vec{u}) + (\nabla\vec{u})^T$$

$$\text{advection: } \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

- In velocity-stress formulation:

$$\dot{\boldsymbol{\sigma}} = \lambda \mathrm{tr}(\nabla\vec{v})\boldsymbol{I} + \mu(\nabla\vec{v}) + (\nabla\vec{v})^T)$$

$$\dot{\vec{v}} = \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma}$$

- 1st Time derivative = 1st space derivative

Finite Volume Methods

$$\text{advection: } \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \quad \text{solutions: } u(x,t) = f(x+t)$$

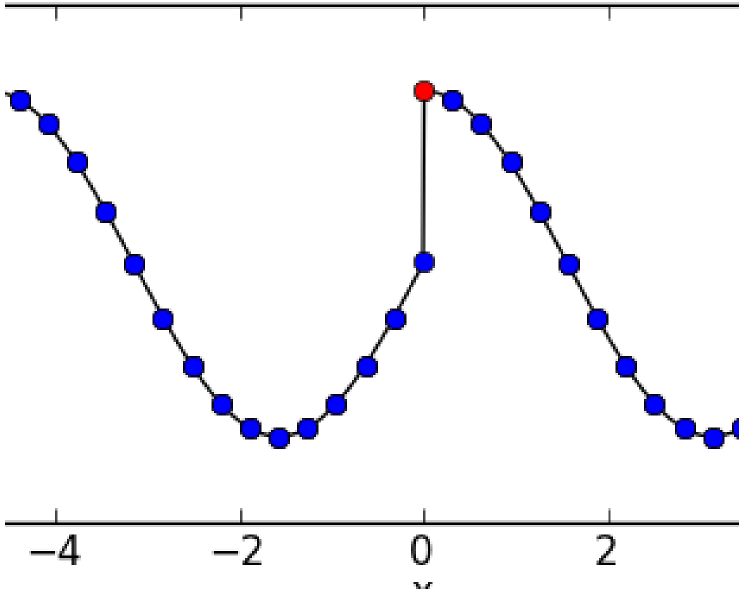
- Integrate over one “cell” and one time step
  - Use divergence theorem to consider flux at the boundaries.
  - Consider cell average, $\bar{u}_i^n$, in semi-discrete update:

$$\frac{\partial \bar{u}_i}{\partial t} = \frac{1}{\Delta x}\left(u(x_i + \frac{\Delta x}{2}, t) - u(x_i - \frac{\Delta x}{2}, t)\right)$$

- Need a method for computing $u(x,t)$

WENO Reconstruction

- Compute $u(x,t)$ as a shock-preserving local interpolation of grid averages, $\bar{u}_i^n$.
  - Preferentially weight smooth parts of the solution
- Example of $O(\Delta x^5)$ WENO reconstruction, note the preserved shock!



ADER Scheme

- Taylor expand $u(x,t)$ in time around t_n:

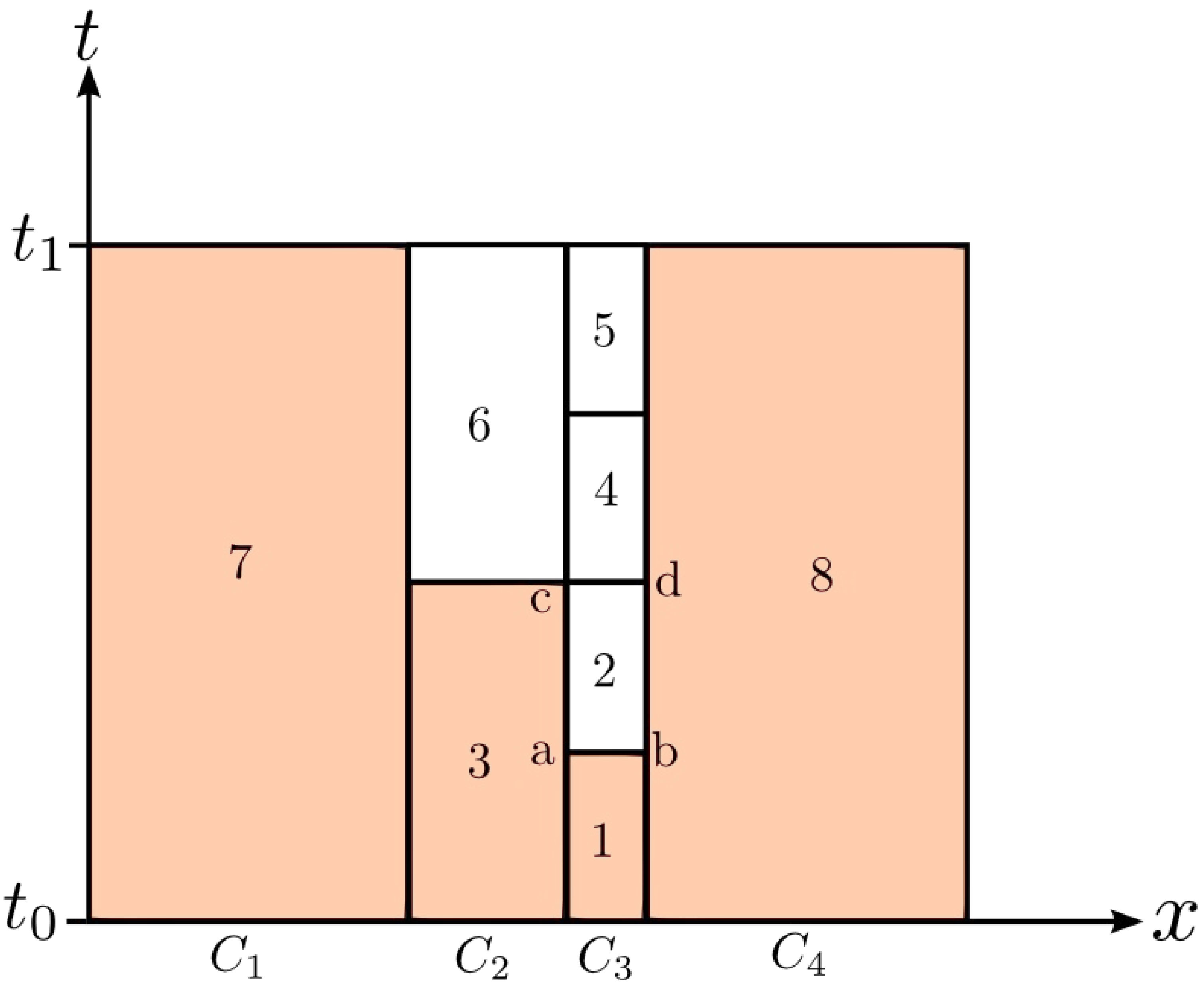
$$u(x,t) = u(x,t_n) + \frac{\partial u}{\partial t}(x,t_n)(t-t_n) + O((t-t_n)^2)$$

- Use governing equation to replace time derivatives with space derivatives. $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}$

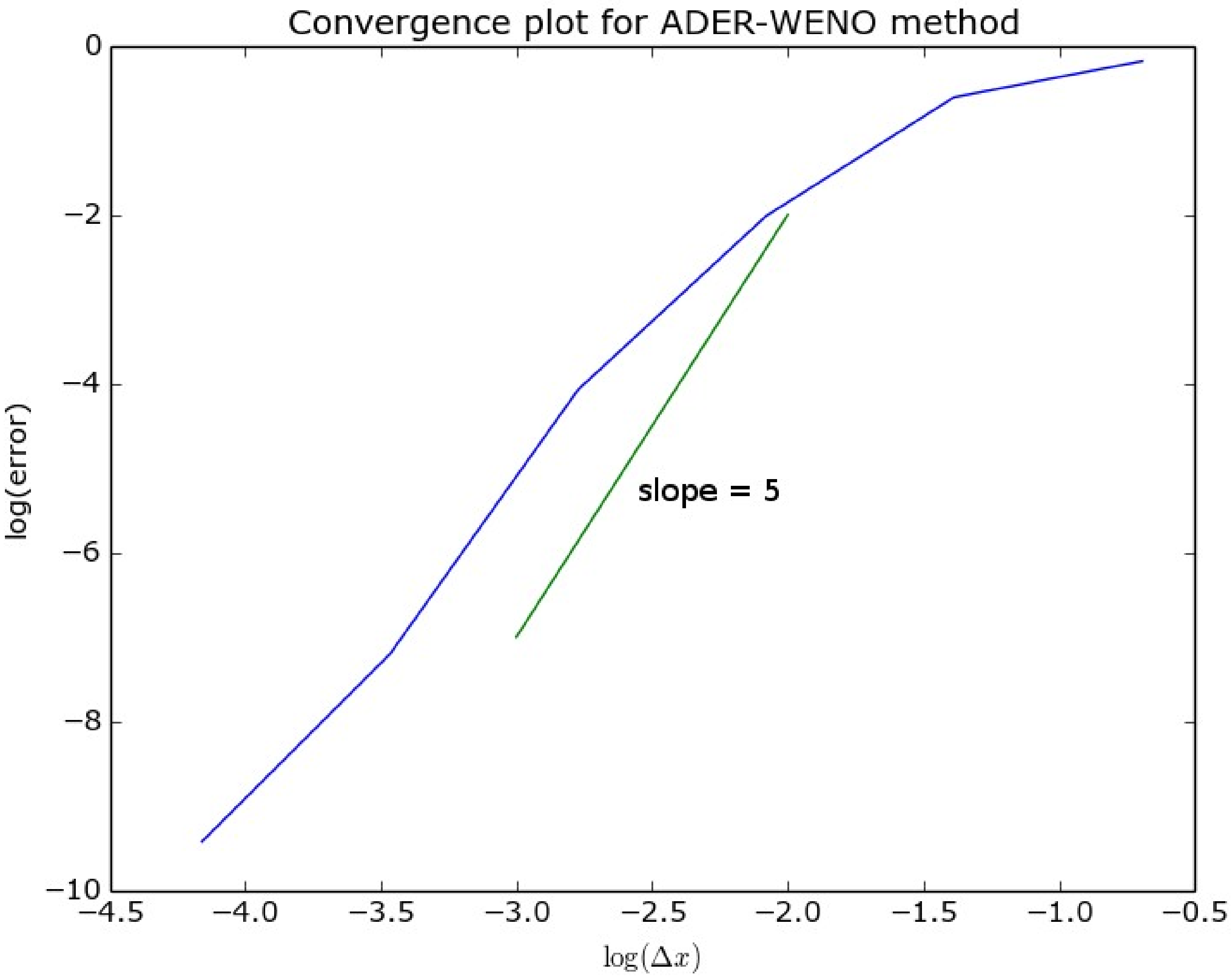
$$u(x,t) = u(x,t_n) + \frac{\partial u}{\partial x}(x,t_n)(t-t_n) + O((t-t_n)^2)$$

- We know $\dfrac{\partial u}{\partial x}$ from the polynomial reconstruction.
- Works for arbitrary number of Taylor series terms.
- Titarev and Toro (2002), Pelties and others (2012)

Local Time Stepping



Basic code is working



Interseismic Deformation

- Just to be clear: Different problem!
- Initial rupture stresses are driven/affected by:
  - Tectonic loading
  - Stable sliding on the fault
  - Deep, viscoplastic creep – especially ductile shear zones
- Time evolution depends on specific viscosity function.
- Goal: efficiently compute creep and temperature over many earthquake cycles for arbitrary viscosity functions.

Quasistatic Viscoplasticity

Governing equations:

$$\dot{\boldsymbol{\sigma}} = \lambda \mathrm{tr}(\dot{\boldsymbol{\epsilon}})\boldsymbol{I} + 2\mu\dot{\boldsymbol{\epsilon}} - \frac{\mu}{\eta_{eff}}(\boldsymbol{\sigma} - \frac{\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}}{3})$$

$$\nabla \cdot \boldsymbol{\sigma} = 0$$

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\nabla\vec{v}) + (\nabla\vec{v})^T$$

$$\text{Maxwell:} \frac{1}{\eta_{eff}} = \text{constant}$$

$$\text{Power law:} \frac{1}{\eta_{eff}} = A\sigma_{eff} \exp(\frac{-Q}{RT})$$

Simplified Equations

Antiplane geometry. No loss of generality!

Infinite strike-slip fault.

$$\vec{v} = (0, 0, v_z(x, y))$$

σ_{zx}, σ_{zy} are non-zero.

$$\dot{\boldsymbol{\epsilon}} = \nabla \vec{v}$$

$$\vec{\tau} = (\sigma_{xz}, \sigma_{yz}, 0)$$

$$\dot{\boldsymbol{\sigma}} = \lambda \mathrm{tr}(\dot{\boldsymbol{\epsilon}})\boldsymbol{I} + 2\mu\dot{\boldsymbol{\epsilon}} - \frac{\mu}{\eta_{eff}}(\boldsymbol{\sigma} - \frac{\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}}{3})$$

$$\boxed{\begin{array}{l} \text{simplifies to: } \dfrac{\partial \vec{\tau}}{\partial t} = \mu \nabla v_z - \dfrac{\mu}{\eta}\vec{\tau} \\ \text{still have: } \nabla \cdot \vec{\tau} = 0 \end{array}}$$

Numerical Difficulties

- Velocity evolution, but no time derivative of velocity!
- In general, nonlinear viscosity.
  - Requires Newton's method iterations.
- Non-symmetric linearized operator.
  - Rules out fast matrix solver methods like the conjugate gradient method.

Projection Method

- Idea: Expand velocity as a Taylor Series.

$$\frac{\partial \vec{\tau}}{\partial t} = \mu \nabla \left(v_z^n + \frac{\partial v_z}{\partial t}\Delta t + O(\Delta t^2)\right) - \frac{\mu}{\eta}\vec{\tau}$$

- Split-step update:
  - Step 1: Update the stress ignoring changes in the velocity.
  - Step 2: Evolve the velocity and force a divergence free stress.
- “Projection” – stress is explicitly projected onto a divergence free function space.

2nd Order Projection

Given: $\vec{\tau}_n, \vec{\tau}_{n-1}, v_n$

Step 1: $\dfrac{1}{2\Delta t}(3\hat{\tau}_{n+1} - 4\vec{\tau}_n + \vec{\tau}_{n-1}) = \mu\nabla(v_n) - \dfrac{\mu}{\eta}\hat{\tau}_{n+1}$

initial value problem for $\vec{\tau}$

solution is: $\hat{\tau}_{n+1}$

Step 2: $\dfrac{1}{2\mu\Delta t}(3\vec{\tau}_{n+1} - 3\hat{\tau}_{n+1}) = \nabla(\dfrac{\partial v}{\partial t}\Delta t) \approx \nabla(v_{n+1} - v_n)$

but $\nabla \cdot \vec{\tau_{n+1}} = 0$

take div: $\dfrac{-3}{2\mu\Delta t}(\nabla \cdot \hat{\tau}_{n+1}) = \nabla^2(v_{n+1} - v_n)$

Poisson equation!

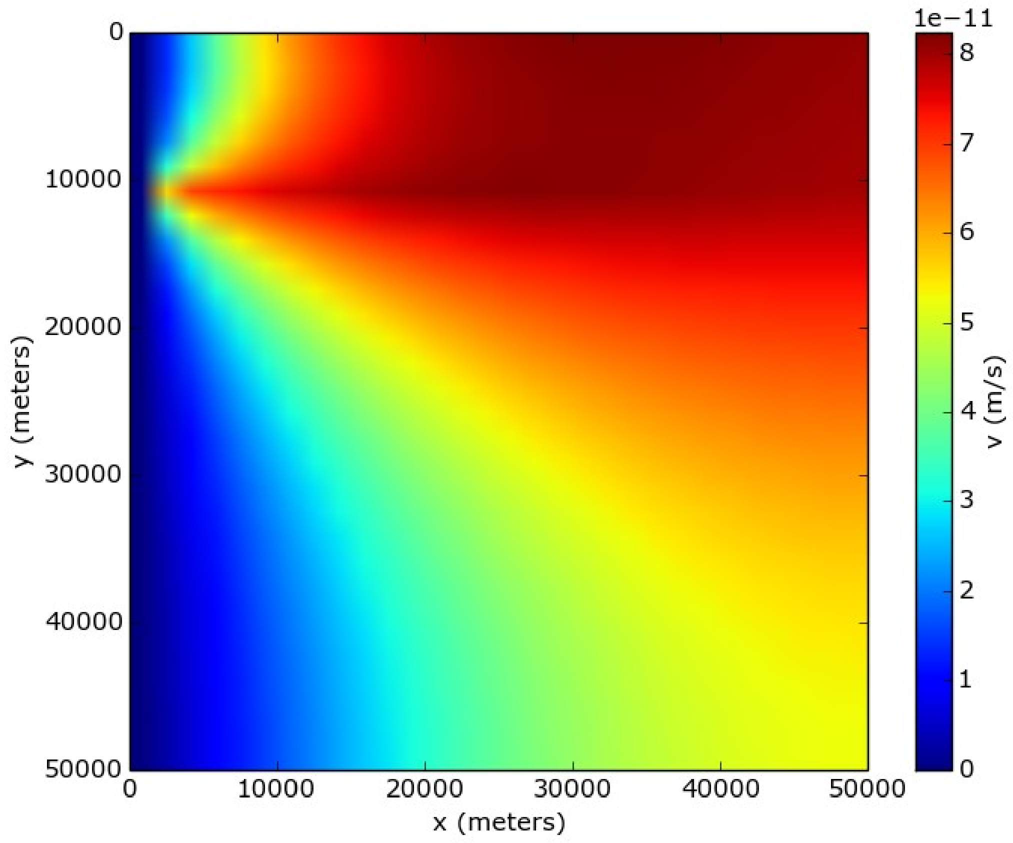
Adding the two steps gives the original equation

Projection Method – Benefits

- Both steps are provably $O(\Delta t^2)$
- ODE is easy to solve.
- Poisson is easy to solve very quickly – FFT, Multigrid
  - Note: Boundary Conditions are applied in the Poisson step.
- Nonlinearity is restricted to the ODE.
- Improves on Zienkiewicz and Cormeau (1974) and Hughes and Taylor (1978)

Example

100 years after a 2 meter slip, plate rate far field in upper layer. 10km elastic layer over viscoelastic half space, fault is locked.



I need to do much more verification and convergence testing

Conclusions

- Very efficient elastic wave propagation is possible with existing techniques
  - Local time stepping (ADER)
- Projection method gives a much easier quasistatic viscoplasticity problem
  - ODE + Poisson vs. Nonlinear, asymmetric elliptic
- I have a lot of work to do.
  - Convergence tests!