APPROXIMATION FOR THE SHEAR STRESS IN THE FLUID APPROXIMATION (CLASS IV, 1104.2935)

We begin with (2.2d)

$$(\bar\rho+\bar p)\sigma = \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\int_0^\infty f_0(q)\,dq\,\frac{q^4}{\epsilon}\Psi_2$$

We derive both sides with respect to conformal time $\frac{d}{dz}$ (dot derivative):

$$(\dot\rho+\dot p)\sigma + (\rho+p)\dot\sigma = \underbrace{\frac{8\pi}{3}(-4H)\left(\frac{T_0}{a}\right)^4\int_0^\infty f_0(q)\,dq\,\frac{q^4}{\epsilon}\Psi_2}_{\equiv\,(\rho+p)\sigma} + \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\int_0^\infty f_0(q)\,dq\,q^4\frac{\partial}{\partial z}\left(\frac{\Psi_2}{\epsilon}\right) \qquad (1)$$

$$\left[\frac{d}{dz}\frac{1}{a^4} = -4\frac{1}{a^5}\dot a = -4H\frac{1}{a^4}\right]$$

We will use: $\dot\rho = -3H(1+w)\rho$; $\dot p = \frac{\partial}{\partial z}(w\rho) = \dot w\rho + \dot\rho w = \dot w\rho - 3H(1+w)w\rho$

so that $\dot\rho + \dot p = \dot w\rho - 3H(1+w)^2\rho$

Dividing (1) by $(\rho+p)$, we will have

$$\frac{\dot\rho+\dot p}{\rho+p} = \frac{\dot w\rho - 3H(1+w)^2\rho}{\rho(1+w)} = \frac{\dot w}{1+w} - 3H(1+w)$$

Recall that $c_g^2 \equiv \dot p/\dot\rho = \dfrac{\dot w\rho - 3H(1+w)w\rho}{-3H(1+w)\rho} = -\dfrac{1}{3H}\dfrac{\dot w}{1+w} + w$

So $-3H c_g^2 = \dfrac{\dot w}{1+w} - 3Hw$

Then: $$\frac{\dot\rho+\dot p}{\rho+p} = -3Hc_g^2 + 3Hw - 3H(1+w) = -3Hc_g^2 - 3H = -3H(1+c_g^2) \qquad (2)$$

$$\dot\sigma + \frac{\dot\rho+\dot p}{\rho+p}\sigma = -4H\sigma + \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\int_0^\infty f_0(q)\,dq\,q^4\frac{\partial}{\partial z}\left(\frac{\Psi_2}{\epsilon}\right) \times \frac{1}{\rho+p}$$

Plugging (2) in:

$$\dot\sigma + H(1-3c_g^2)\sigma = \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\,q^4\frac{\partial}{\partial z}\left(\frac{\Psi_2}{\epsilon}\right) \qquad (3.6)$$

Which corresponds to eq. (3.6) in CLASS IV.

⚠ Recall that we have defined $c_g^2 = \dfrac{\dot p}{\dot\rho}$, not $c_s^2 = \dfrac{\delta p}{\delta\rho}$.

Until now, everything has been exact. Now is time for the fluid approximation, which requires we truncate the Boltzmann hierarchy of diff. equations.

To do so, we follow Ma & Bertschinger truncation scheme, eq. (58) from astro-ph/9506072, using $l_{max} = 2$, and recover:

$$\Psi_3 = \frac{5\varepsilon}{qkz}\Psi_2 - \Psi_1 , \qquad (3.5)$$

which is eq. (3.5) in CLASS IV.

Now we retrieve the Boltzmann equation for Ψ_2, (2.4c) from CLASS IV:

$$\dot{\Psi}_2 = \frac{qk}{5\varepsilon}(2\Psi_1 - 3\Psi_3) - \frac{1}{15}(\dot{h}+6\dot{\eta})\frac{d\ln f_0}{d\ln q} \qquad (2.4c)$$

Plugging (3.5) into (2.4c),

$$\dot{\Psi}_2 = \frac{2qk}{5\varepsilon}\Psi_1 - 3\left(\frac{5\varepsilon}{qkz}\Psi_2 - \Psi_1\right) - \frac{1}{15}(\dot{h}+6\dot{\eta})\frac{d\ln f_0}{d\ln q} =$$

$$\left\{ \begin{array}{l} \alpha = \frac{1}{2k^2}(\dot{h}+6\dot{\eta}) \\ (\dot{h}+6\dot{\eta}) = 2k^2\alpha \end{array} \right.$$

$$= \left(\frac{2qk}{5\varepsilon} - 1\right)\Psi_1 - \frac{15\varepsilon}{qkz}\Psi_2 - \frac{2}{15}\alpha k^2 \frac{d\ln f_0}{d\ln q} \qquad (2)$$

Our plan is to use this to plug it into (3.6), however what happens to $\frac{\partial}{\partial z}\left(\frac{\Psi_2}{\varepsilon}\right)$?

$$\frac{\partial}{\partial z}\left(\frac{\Psi_2}{\varepsilon}\right) = \frac{1}{\varepsilon}\dot{\Psi}_2 - \Psi_2\frac{1}{\varepsilon^2}\dot{\varepsilon} = \frac{1}{\varepsilon}\dot{\Psi}_2 - \Psi_2\frac{1}{\varepsilon^2}\frac{\partial}{\partial z}\left(\sqrt{q^2+m^2a^2}\right)$$

$$= \frac{1}{\varepsilon}\dot{\Psi}_2 - \Psi_2\frac{1}{\varepsilon^2}\frac{1}{2\varepsilon}2a\underset{m^2}{\uparrow}\dot{a} = \frac{1}{\varepsilon}\dot{\Psi}_2 - \Psi_2\frac{a^2m^2}{\varepsilon^2}\frac{H}{\varepsilon} =$$

$$= \frac{1}{\varepsilon}\left(\frac{2qk}{5\varepsilon} - 1\right)\Psi_1 - \frac{15}{qkz}\Psi_2 - \frac{2}{15}\frac{\alpha k^2}{\varepsilon}\frac{d\ln f_0}{d\ln q} - \Psi_2\frac{a^2m^2}{\varepsilon^2}\frac{H}{\varepsilon} =$$

$$= \frac{1}{\varepsilon}\left(\frac{2qk}{5\varepsilon} - 1\right)\Psi_1 - \left(\frac{15}{qkz} + \frac{a^2m^2}{\varepsilon^2}\frac{H}{\varepsilon}\right)\Psi_2 - \frac{2}{15}\frac{\alpha k^2}{\varepsilon}\frac{d\ln f_0}{d\ln q}$$

* I'm really puzzled about this!

Recall (3.6):

$$\dot{\sigma} + H(1-3c_g^2)\sigma = \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4 \frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\, q^4 \frac{\partial}{\partial z}\left(\frac{\Psi_2}{\varepsilon}\right) \qquad (3.6)$$

We will use the definitions:

$$(\rho+p)\Theta = 4\pi k\left(\frac{T_0}{a}\right)^4\int_0^\infty f(q)\,dq\, q^3\frac{q^2}{\varepsilon^2}\Psi_1$$

$$(\rho+p)\Sigma = \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\int_0^\infty f_0(q)\,dq\,\frac{q^4}{\varepsilon}\frac{q^2}{\varepsilon^2}\Psi_2$$

so keep them in mind!

We are left with (get ready...):

$$\dot{\sigma} + H(1-3c_g^2)\sigma = \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\, q^4\frac{1}{\varepsilon}\left(\frac{2kq}{5\varepsilon} - 1\right)\Psi_1 \qquad ①$$

$$- \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\, q^4\left(\frac{15}{qkz} + \frac{a^2m^2}{\varepsilon^2}\frac{H}{\varepsilon}\right)\Psi_2 \qquad ②$$

$$- \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\, q^4\frac{2}{15}\alpha k^2\frac{d\ln f_0}{d\ln q} \qquad ③$$

We need to go step by step

(1) $\frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4 \frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\, q^3\left(\underbrace{\frac{2k}{5}\frac{q^2}{\varepsilon^2}\Psi_1}_{\sim\Theta} \;-\; \underbrace{\frac{q}{\varepsilon}\Psi_1}_{\sim \text{what the fuck is this}}\right) =$ → aquest terme no l'havia de tenir :(

$$= \frac{2}{3}\times\frac{2}{5}\Theta - \frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\, q^3\,\frac{q}{\varepsilon}\Psi_1$$

(2) $\underbrace{\frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4 \frac{1}{\rho+p}}_{A}\int_0^\infty f_0(q)\,dq\,\frac{q^4}{\varepsilon}\left(\left(\frac{15}{kz}\right)\frac{\varepsilon}{q}\Psi_2 + (a^2m^2H)\frac{1}{\varepsilon^2}\Psi_2\right) = \left\{ a^2m^2 = \varepsilon^2 - q^2 \right\} =$

$$A\left(\frac{15}{kz}\right)\int_0^\infty f_0(q)\,dq\,\frac{q^4}{\varepsilon}\left(\frac{\varepsilon}{q}\right)\Psi_2 + AH\int_0^\infty \underbrace{f_0(q)\,dq\,\frac{q^4}{\varepsilon}\Psi_2}_{\sim\sigma} - AH\int_0^\infty \underbrace{f_0(q)\,dq\,\frac{q^4}{\varepsilon}\frac{q^2}{\varepsilon^2}}_{\sim\Sigma}\Psi_2$$

↳ aixó em sobra

(3) $\frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\frac{1}{\rho+p}\int_0^\infty f_0(q)\,dq\,q^4\,\frac{2}{15}\frac{\alpha k^2}{\varepsilon}\frac{d\ln f_0}{d\ln q} =$

$$= \underbrace{\frac{8\pi}{3}\left(\frac{T_0}{a}\right)^4\frac{1}{\rho+p}\frac{2}{15}\alpha k^2}_{B}\int_0^\infty f_0(q)\,dq\,\frac{q^4}{\varepsilon}\frac{q}{f_0}\frac{df_0}{dq} = \ldots$$

$$= B\times\int_0^\infty \frac{q^5}{\varepsilon}\frac{df_0}{dq}\,dq = B\times\left[\underbrace{\frac{q^5}{\varepsilon}f_0(q)\Big|_0^\infty}_{0} - \int_0^\infty \underbrace{\frac{5q^4}{\varepsilon}f_0\,dq}_{\sim p} + \int_0^\infty \underbrace{\frac{q^6}{\varepsilon^3}f_0\,dq}_{\sim \mathfrak{P}\ (\text{pseudo-pressure})}\right]$$

$$\frac{d}{dq}\left(\frac{q^5}{\varepsilon}\right) = \frac{5q^4}{\varepsilon} - \frac{q^6}{\varepsilon^3}$$

$$\frac{d}{dq}\frac{1}{\sqrt{q^2+m^2}} = -\frac{1}{2}\frac{1}{\varepsilon^3}2q$$

[...] Leaving details (and mistakes) aside, one should retrieve (3.7) from CLASS IV:

$$\dot\sigma = -3\left(\frac{1}{z} + H\left[\frac{2}{3} - c_g^2 - \frac{1}{3}\frac{\Sigma}{\sigma}\right]\right)\sigma + \frac{2}{3}\left[\Theta + \alpha k^2\underbrace{\frac{w}{1+w}\left(5 - \frac{\mathfrak{P}}{p}\right)}_{3c_g^2}\right]$$

$$\dot\sigma = -3\left(\frac{1}{z} + H\left[\frac{2}{3} - c_g^2 - \frac{1}{3}\frac{\Sigma}{\sigma}\right)\sigma + \frac{2}{3}\left[\Theta + 3\alpha c_g^2 k^2\right]$$

We have three fluid parameters, in principle independent:

- c_g^2 → the adiabatic sound speed
- w_σ ; s.t. $\Sigma = 3w_\sigma\sigma$
- w_θ or c_{vis}^2, $[\Theta + 3c_g^2\alpha k^2] = 3w_\theta[\theta + \alpha k^2] = \frac{4c_{vis}^2}{1+w}[\theta + \alpha k^2]$

Of course, plus c_s^2 and w which enter the equations for δ and θ.

CLASS says: $c_s^2 \simeq c_g^2$; $w_\sigma \simeq \frac{1}{3}\frac{\mathfrak{P}}{p}$; $c_{vis}^2 \simeq 3wc_g^2$

we know this is improvable (Caio)

from assuming that Ψ_2 is q-indep., but this is an approx. too

varies between schemes.

If you have standard massive neutrinos

$m \Rightarrow$ c_s^2 c_g^2 w w_σ c_{vis}^2