1 Changing the equation of state

Following [1], the background energy and pressure for massive neutrinos are given by

$$\rho_\nu = \rho_{\rm ini}\, a^{-4} \int q^2\, {\rm d}q\, {\rm d}\Omega\, \varepsilon\, f_0(q)\,, \qquad P_\nu = \frac{\rho_{\rm ini}}{3} a^{-4} \int q^2\, {\rm d}q\, {\rm d}\Omega\, \frac{q^2}{\varepsilon}\, f_0(q)\,, \tag{1}$$

with $\varepsilon = \sqrt{q^2 + m_\nu^2 a^2}$ and $\rho_{\rm ini}$ a normalization constant that . In the ultra-relativistic limit $\varepsilon \sim q$, $P_\nu = \frac{1}{3}\rho_\nu$, i.e. $w = 1/3$. Also, both ρ and p evolve as a^{-4}. In the non-relativistic limit, $a^2 m_\nu^2 \gg q^2$, and then $P_\nu \to 0$, such that $w = 0$. Now, ρ evolves with a^{-3} due to the a factor in ε. The mass naturally introduces a phase transition between $w = 1/3$ and $w = 0$, and the ρ evolution varies smoothly from a^{-4} to a^{-3}.

Massless neutrinos with $m_\nu = 0$ fulfill $\varepsilon = q$ and then (1) becomes

$$\rho_\nu = \rho_{\rm ini}\, a^{-4} \int q^2\, {\rm d}q\, {\rm d}\Omega\, q\, f_0(q)\,, \qquad P_\nu = \frac{\rho_{\rm ini}}{3}\, a^{-4} \int q^2\, {\rm d}q\, {\rm d}\Omega\, q\, f_0(q)\,. \tag{2}$$

That is, massless neutrinos always have $w = 1/3$.

1.1 Arbitrary equation of state

From the energy-momentum tensor conservation, we can retrieve

$$\frac{{\rm d}\rho}{\rho} = -3\, \frac{{\rm d}a}{a}\, (1 + w(a)) \tag{3}$$

for an arbitrary equation of state $w(a)$. The solution is trivial,

$$\rho(a) = \rho_{\rm ini} \exp\left\{-3 \int_{a_{\rm ini}}^{a} \frac{da'}{a'}(1 + w(a'))\right\}\,. \tag{4}$$

Then, we modify the massless neutrino energy density and pressure as follows:

$$\rho_\nu = \rho_{\rm ini}\, \exp\left\{-3 \int_{a_{\rm ini}}^{a} \frac{da'}{a'}(1 + w(a'))\right\} \int q^2\, {\rm d}q\, {\rm d}\Omega\, q\, f_0(q)\,, \tag{5}$$

$$P_\nu = \rho_{\rm ini}\, w(a) \exp\left\{-3 \int_{a_{\rm ini}}^{a} \frac{da'}{a'}(1 + w(a'))\right\} \int q^2\, {\rm d}q\, {\rm d}\Omega\, q\, f_0(q)\,. \tag{6}$$

This way, we necessarily have $P_\nu/\rho_\nu = w(a)$ and the correct evolution with a during all history.

1.2 Different parametrisations for $w(a)$

A first possibility is to try to fit if neutrinos go non-relativistic. We introduce an equation of state which goes from UR to NR in a single step at $a = a_{\rm tr}$,

$$w(a) = \begin{cases} 1/3 & a < a_{\rm tr} \\ 0 & a > a_{\rm tr} \end{cases} . \tag{7}$$

We expect cosmology to put a bound on $a_{\rm tr}$ in the same manner that it can put a bound on m_ν.

However, we can do an arbitrary number of steps n,

$$w(a) = \begin{cases} 1/3 & a < a_0 \\ w_i & a_{i-1} < a < a_i \\ w_n & a > a_{n-1} \end{cases} , \tag{8}$$

with $i = 1, \ldots, n$.

2 CLASS results

In the following examples, we fix $\omega_b, \omega_{\rm cdm}, \theta_s, A_s, n_s, \tau_{\rm reio}$ to their ΛCDM values, and vary the amount of neutrinos and their equation of state.

We want to compare a single massive neutrino with $m_\nu = 0.33\,\mathrm{eV}$ to a signle massless neutrino with an equation of state which tries to mimic the mass. The three different equations of state for this neutrino are shown in figure 1. The other two neutrinos are massless with $w = 1/3$.

First of all, in figure 2 we can see that the energy density and pressure of this neutrino species both evolve as expected. The energy density and pressure defined in (5) and (6) correctly match the mass evolution.

We expect the background contribution from three cases to be, at least, similar. As shown in figure 3, we compute the CMB anisotropies and find that, well, they are not. Adding additional steps to the equation of state helps to reproduce the oscillations at large multipoles, but the matching is far from good, specially at small multipoles. Note, however, that the perturbations have not been modified and, in the three cases, they are the ones from massive neutrinos.

Then, what happens if we turn off the contribution of the "massive" neutrino (or that with an arbitrary equation of state) to perturbations? Well, in figure 4 we can see that the matching to the CMB anisotropies is almost exact, specially for the 20-step equation of state case.

If this is true, it would mean that perturbations are important and not negligible with respect to the background. There is something in the effect of neutrinos that is not

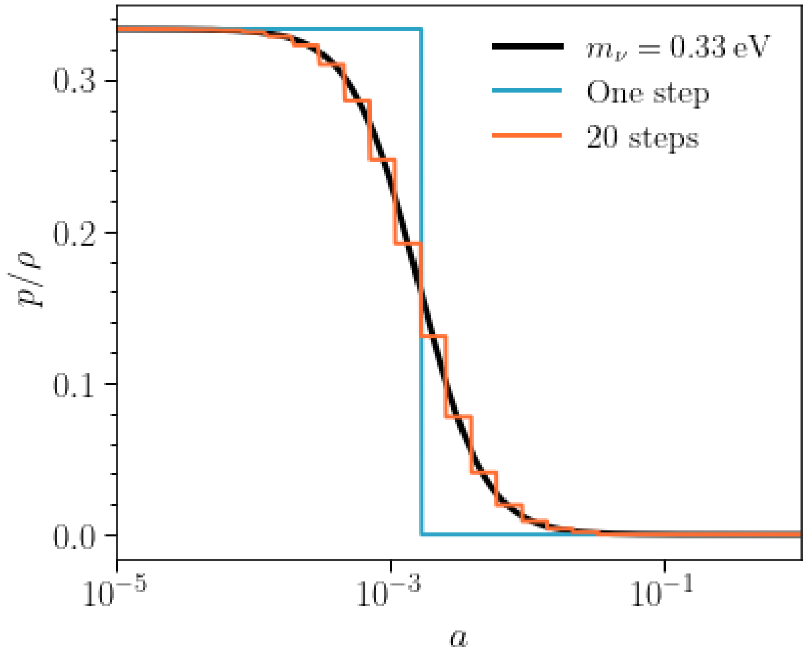

Figure 1: Evolution of the equation of state with the scale factor. In black, the equation of state corresponding to a $m_\nu = 0.33\,\mathrm{eV}$. In blue, a single-step equation of state which transitions at $a_{\mathrm{tr}} = 1.5 \times 10^{-3}$. In orange, a 20-step equation of state which mimics the behaviour of its mass analogous.

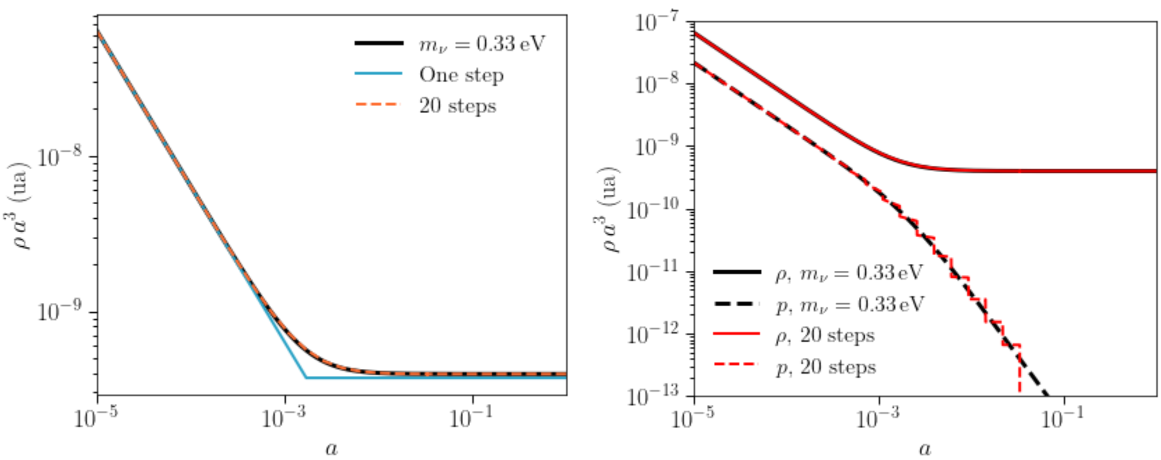

Figure 2: Evolution of the energy density and pressure. In black, a massive neutrino with $m_\nu = 0.33\,\mathrm{eV}$. In blue and orange (and red), a massless neutrino with the equation of state from figure 1.

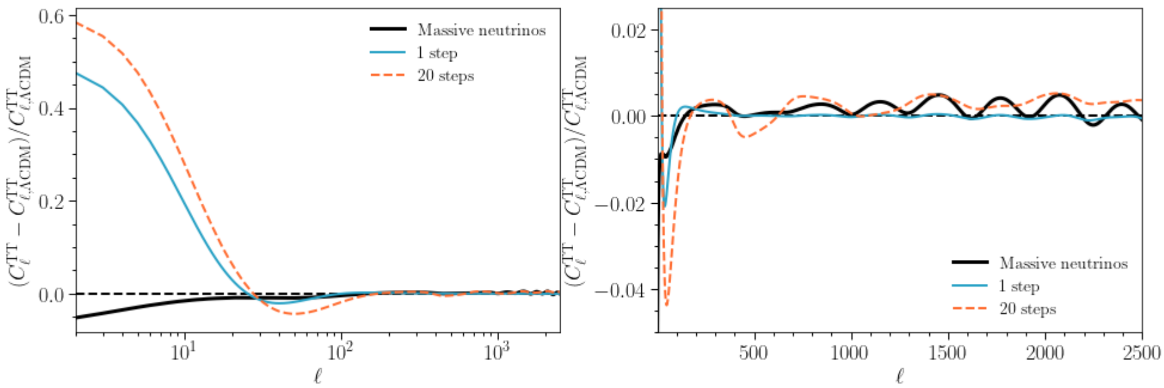

Figure 3: Difference of CMB anisotropies with respect to the ΛCDM model with three massless neutrinos. NCDM perturbations are on as if the neutrino were massive. Left, multipoles in log scale. Right, multipoles in linear scale. In black, a massive neutrino with $m_\nu = 0.33\,\mathrm{eV}$. In blue and orange, a massless neutrino with the equation of state from figure 1.

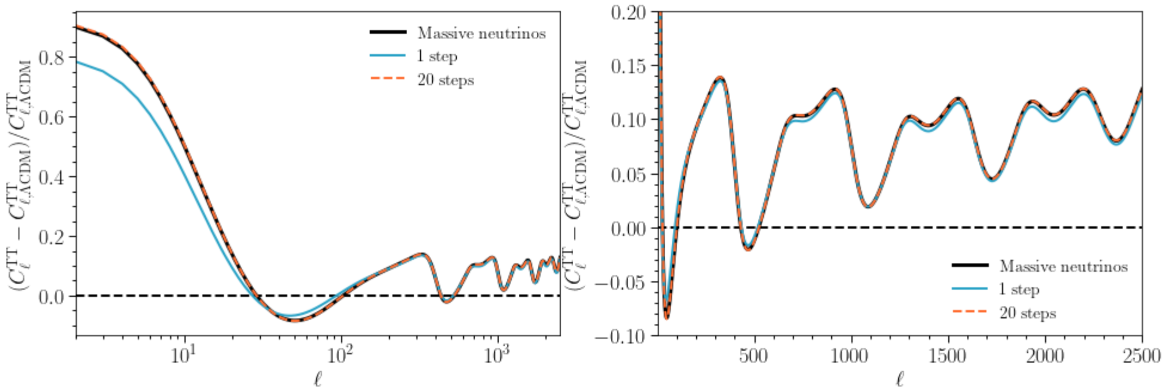

Figure 4: Difference of CMB anisotropies with respect to the ΛCDM model with three massless neutrinos. NCDM perturbations are off. Left, multipoles in log scale. Right, multipoles in linear scale. In black, a massive neutrino with $m_\nu = 0.33\,\mathrm{eV}$. In blue and orange, a massless neutrino with the equation of state from figure 1.

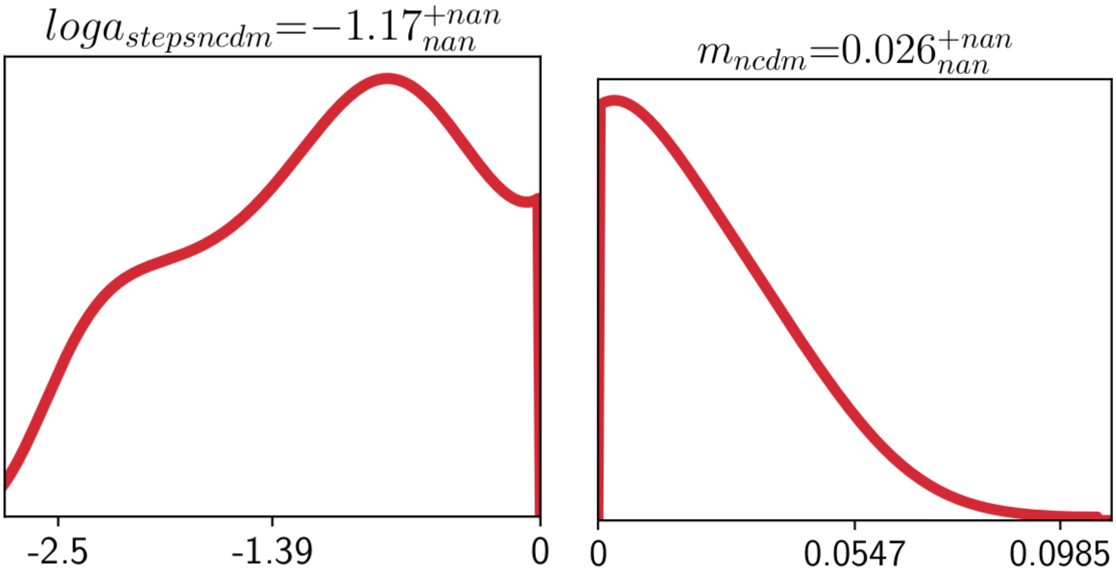

Figure 5: Posterior distributions for $\log a_{\rm tr}$ and m_ν.

directly reproduced by the equation of state on the background. We must now discover what.

3 A fit to the mass of the neutrino

Now we repeat the same exercise and try to see which bound does the Planck data put on our models. We compare one massive neutrino, with one massless neutrino described by a one-step equation of state like (7). The free parameters are m_ν and $a_{\rm tr}$, respectively. The prior distribution is flat in $\log a_{\rm tr} \in (-4.2, 0)$. The posterior distributions from a relatively-short chain are shown in figure 5. By absolutely rough ocular inspection, we see that the data prefers $a_{\rm tr} \gtrsim 10^{-2.5} \sim 3 \times 10^{-3}$, which roughly corresponds to a mass bound $m_\nu \lesssim 0.17\,\mathrm{eV}$. Contrarily, the fit to m_ν gives $m_\nu \lesssim 0.1\,\mathrm{eV}$, which is twice as constringent. Maybe the $\log a_{\rm tr}$ does not fall as steeply as it should, but the order of magnitude is, at least, correct.

If it's of any interest, I attach in figure 6 the values of m_ν and $a_{\rm tr}$ in the MontePython MCMC. As one can see, the MCMC avoids sampling points with small scale factor or large mass.

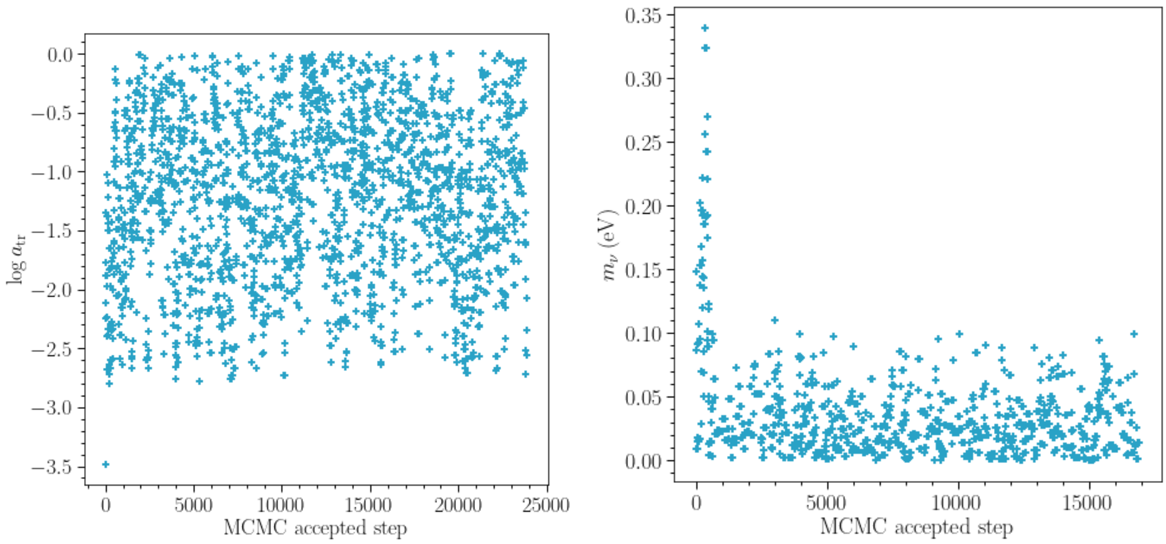

Figure 6: Sampled points during the MCMC for $\log a_{\rm tr}$ and m_ν.

4 Perturbative neutrino continuity equations

Our goal is to obtain the equations for the evolution of the neutrino perturbations in the synchronous gauge,

$$\begin{aligned}
\dot{\delta} &= -(1+w)\left(\theta + \frac{\dot{h}}{2}\right) - 3\,\frac{\dot{a}}{a}\left(\frac{\delta P}{\delta \rho} - w\right)\delta, \\
\dot{\theta} &= -\frac{\dot{a}}{a}(1-3w)\,\theta - \frac{\dot{w}}{1+w}\,\theta + \frac{\delta P/\delta\rho}{1+w}\,k^2\delta - k^2\sigma\,.
\end{aligned} \tag{9}$$

These equations do not come from the Boltzmann equations, but are just consequence of the conservation of the tensor-energy momentum at first order in perturbations, that is,

$$T^{\mu\nu}_{;\mu} = \partial_\mu T^{\mu\nu} + \Gamma^{\nu}{}_{\alpha\beta}T^{\alpha\beta} + \Gamma^{\alpha}{}_{\alpha\beta}T^{\nu\beta} = 0\,. \tag{10}$$

Here we will try to prove that (9) indeed can be derived from (10).

4.1 Christoffel symbols

In the synchronous gauge, the metric is given by

$$\mathrm{d}s^2 = a^2(\tau)\left(-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\right)\,, \tag{11}$$

where τ is the conformal time, and h_{ij} the perturbation to the spatial metric. The non-zero elements of the metric are then

$$g_{00} = -a^2(\tau)\,, \quad g_{ij} = a^2(\tau)\,(\delta_{ij} + h_{ij})\,. \tag{12}$$

And the elements of the inverse matrix are

$$g^{00} = -\frac{1}{a^2(\tau)}\,, \quad g^{ij} = \frac{1}{a^2(\tau)}\left(\delta^{ij} - h^{ij}\right)\,. \tag{13}$$

It is easy to check that $g^{ij}g_{jk} = \delta^i{}_k$ at the first order in perturbation theory. Note that δ^{ij}, h^{ij} live in Euclidean 3D space, so we don't really care about indices being upper or lower.

Now, we can compute the Christoffel symbols from

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\nu g_{\beta\mu} + \partial_k g_{\beta\nu} - \partial_m g_{\mu\beta}\right)\,. \tag{14}$$

The only non-vanishing symbols are

$$\Gamma^0{}_{00} = H(\tau)\,, \tag{15}$$

$$\Gamma^0{}_{ij} = H(\tau)\left(\delta_{ij} + h_{ij}\right) + \frac{1}{2}\dot{h}_{ij}\,, \tag{16}$$

$$\Gamma^k{}_{0i} = H(\tau)\delta^k{}_i + \frac{1}{2}\dot{h}_{ki}\,, \tag{17}$$

$$\Gamma^k{}_{ij} = \frac{1}{2}\delta^{kl}\left(\partial_j h_{li} + \partial_i h_{lj} - \partial_l h_{ij}\right)\,. \tag{18}$$

The dot stands for a derivative with respect to conformal time.

4.2 Energy-momentum tensor

From [1], we have that

$$T^0{}_0 = -(\bar{\rho} + \delta\rho) \tag{19}$$

$$T^0{}_i = (\bar{\rho} + \bar{P})v_i = -T^i{}_0 \tag{20}$$

$$T^i{}_j = (\bar{P} + \delta P)\delta^i{}_j + \Sigma^i{}_j\,, \quad \Sigma^i{}_i = 0\,. \tag{21}$$

Also from [1], we have $\delta \equiv \delta\rho/\rho$ and

$$(\bar{\rho} + \bar{P})\theta \equiv ik^j\delta T^0{}_j, \quad (\bar{\rho} + \bar{P})\sigma \equiv -\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\Sigma^i{}_j\,. \tag{22}$$

In position space, this is

$$(\bar{\rho} + \bar{P})\theta = \partial^j\delta T^0{}_j = \partial^j\left[(\bar{\rho} + \bar{P})v_j\right], \quad (\bar{\rho} + \bar{P})\sigma = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\right)\Sigma^i{}_j\,. \tag{23}$$

We will need to move the indices up, as

$$T^{00} = g^{00}T^0{}_0 = \frac{1}{a^2}(\bar{\rho} + \delta\rho)\,, \tag{24}$$

$$T^{0i} = g^{ij}T^0{}_j = \frac{1}{a^2}\left(\delta^{ij} - h^{ij}\right)(\bar{\rho} + \bar{P})v_j\,, \tag{25}$$

$$T^{ij} = g^{jk}T^i{}_k = \frac{1}{a^2}\left(\delta^{jk} - h^{jk}\right)\left[(\bar{P} + \delta P)\delta^i{}_k + \Sigma^i{}_k\right]\,. \tag{26}$$

4.3 Continuity equations

We now want to compute (10). We begin with $\nu = 0$, that is

$$\partial_\mu T^{\mu 0} + \Gamma^0{}_{\alpha\beta}T^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\beta}T^{0\beta} = 0\,. \tag{27}$$

Term by term,

$$\begin{aligned}
\partial_\mu T^{\mu 0} &= \partial_0 T^{00} + \partial_i T^{0i} = \partial_0 \left(\frac{1}{a^2} [\bar\rho + \delta\rho] \right) + \partial^i i \left([\bar\rho + \bar P] v_i \right) = \\
&= -2\, \frac{H}{a^2} \left(\bar\rho + \delta\rho \right) + \frac{1}{a^2} \left(\dot{\bar\rho} + \delta\dot\rho \right) + \frac{1}{a^2} (\bar\rho + \bar P)\theta = \\
&= -2\, \frac{H}{a^2} \left(\bar\rho + \delta\rho \right) - \frac{3H}{a^2} \left(\bar P + \bar\rho \right) + \frac{1}{a^2}\, \delta\dot\rho + \frac{1}{a^2} (\bar\rho + \bar P)\theta \,.
\end{aligned} \tag{28}$$

$$\begin{aligned}
{\Gamma^0}_{\alpha\beta} T^{\alpha\beta} &= {\Gamma^0}_{00} T^{00} + {\Gamma^0}_{ij} T^{ij} = \\
&= \frac{H}{a^2} (\bar\rho + \delta\rho) + \left[H(\delta_{ij} + h_{ij}) + \frac{1}{2} h'_{ij} \right] \frac{1}{a^2} (\delta^{il} - h^{il}) \left[(\bar P + \delta P) {\delta^j}_l + {\Sigma^j}_l \right] = \\
&= \frac{H}{a^2} (\bar\rho + \delta\rho) + \frac{3H}{a^2} (\bar P + \delta P) + \frac{\dot h}{2a^2} (\bar P + \delta P) \,.
\end{aligned} \tag{29}$$

References

[1] Chung-Pei Ma and Edmund Bertschinger. Cosmological perturbation theory in the synchronous and conformal newtonian gauges. *Astrophys.J. 455 (1995) 7-25*, June 1995.