Saaty

kevin herly

European Journal of Operational Research 48 (1990) 9–26
North-Holland

How to make a decision: The Analytic Hierarchy Process

Thomas L. Saaty
Joseph M. Katz Graduate School of Business, University of Pittsburgh, Pittsburgh, PA 15260, USA

Abstract: This paper serves as an introduction to the Analytic Hierarchy Process – A multicriteria decision making approach in which factors are arranged in a hierarchic structure. The principles and the philosophy of the theory are summarized giving general background information of the type of measurement utilized, its properties and applications.

Keywords: Decision, priority, rank, cost–benefit, scales, ratios

1. How to structure a decision problem

Perhaps the most creative task in making a decision is to choose the factors that are important for that decision. In the Analytic Hierarchy Process we arrange these factors, once selected, in a hierarchic structure descending from an overall goal to criteria, subcriteria and alternatives in successive levels.

To a person unfamiliar with the subject there may be some concern about what to include and where to include it. When constructing hierarchies one must include enough relevant detail to:

represent the problem as thoroughly as possible, but not so thoroughly as to lose sensitivity to change in the elements;

consider the environment surrounding the problem;

identify the issues or attributes that contribute to the solution; and

identify the participants associated with the problem.

Arranging the goals, attributes, issues, and stakeholders in a hierarchy serves two purposes. It provides an overall view of the complex relationships inherent in the situation; and helps the decision maker assess whether the issues in each level are of the same order of magnitude, so he can compare such homogeneous elements accurately. One certainly cannot compare according to size a football with Mt. Everest and have any hope of getting a meaningful answer. The football and Mt. Everest must be compared in sets of objects of their class. Later we give a fundamental scale of use in making the comparison. It consists of verbal judgments ranging from equal to extreme (equal, moderately more, strongly more, very strongly more, extremely more) corresponding to the verbal judgments are the numerical judgments (1, 3, 5, 7, 9) and compromises between these values. We have completed compiling a dictionary of hierarchies pertaining to all sorts of problems, from personal to corporate to public.

A hierarchy does not need to be complete, that is, an element in a given level does not have to function as an attribute (or criterion) for *all* the elements in the level below. A hierarchy is *not* the traditional decision tree. Each level may represent a different cut at the problem. One level may represent social factors and another political factor to be evaluated in terms of the social factors or vice versa. Further, a decision maker can insert or eliminate levels and elements as necessary to clarify the task of setting priorities or to sharpen the focus on one or more parts of the system. Elements that have a global character can be represented at the higher levels of the hierarchy, others that specifically characterize the problem at hand can be developed in greater depth. The task of setting priorities requires that the criteria, the

Received November 1989

properties or features of the alternatives being compared, and the alternatives themselves are gradually layered in the hierarchy so that it is meaningful to compare them among themselves in relation to the elements of the next higher level.

Finally, after judgments have been made on the impact of all the elements and priorities have been computed for the hierarchy as a whole, sometimes, and with care, the less important elements can be dropped from further consideration because of their relatively small impact on the overall objective. The priorities can then be recomputed throughout, either with or without changing the remaining judgments.

2. Scales of measurement – Avoiding mere number crunching

One might argue the whole process of decision making is so unstructured and so amorphous that it is no use trying to be precise. One is then tempted to go further and conclude that "participant satisfaction" is the main objective of decision making. Were this the case, then multicriteria decision making would be a simple matter of using ingenuity, supported with mathematical terminology, to improve numbers that please people. But different sets of arbitrary numbers are likely to result, producing different decisions, and we are right back where we started. One set of numbers pleases a group of people who might be equally pleased with another set of numbers that contradicts the recommendations of the first set. This is mere number crunching. If a decision support theory is to be trustworthy there must be uniqueness in the representation of judgments, the scales derived from these judgments, and the scales synthesized from the derived scales.

Let us consider for a moment group interaction that often leads to certain expectations, sometimes arising from the very heat of the debate. Such a debate cannot always incorporate in its arguments the refinements resulting from the mathematical tradeoffs to ensure drawing valid conclusions. Sometimes participants accept a process and its outcome because the situation is so complex and the arguments so convoluted that whatever surfaces in the end appears plausible. Although convincing a group about its qualitative preferences involves the politics of persuasion and of wheeling and dealing, it is essential that the decision theory itself used to assist the group in arriving at a decision be invariant to politics and behavior. It should be a science of scaling based on mathematics, philosophy and psychology.

Among the various number crunching practices, the most objectionable one is to assign any set of numbers to judgments on alternatives under a particular criterion, and then normalize these numbers (by multiplying them by a constant that is the reciprocal of their sum). Generally, different sets of numbers are used to scale the judgments for the alternatives under different criteria. All the new normalized sets now lie in the interval [0, 1], no matter what scale they originally came from, and can be passed off to the uninitiated as comparable. The appealing part of this practice is that the numbers have an apparently uniform underlying structure and look not unlike probabilities. Thus they go unchallenged by the decision maker and are then manipulated by the consultant or facilitator who weights and adds them to find the most preferred alternative.

Note that when the initial numbers assigned before normalization are ordinals, arbitrary numbers that preserve order but carry no information about differences or ratios of relative magnitudes, the resulting transformation produces a new set of ordinals lying between zero and one and you can be sure only that it preserves the same order. The operations of weighting and adding cannot then be meaningful because it is very likely that different results will be produced for different choices of the ordinal numbers. By a judicious choice of ordinals one can make an alternative that is dominant on even one criterion, no matter how unimportant that criterion may be, have the largest value after weighting and adding and thus turn out to be the most preferred. Any method such as this is not a decision theory but an approach that can mislead people.

The Analytic Hierarchy Process is rigorously concerned with the scaling problem and what sort of numbers to use, and how to correctly combine the priorities resulting from them. A scale of measurement consists of three elements: A set of objects, a set of numbers, and a mapping of the objects to the numbers. In a standard scale a unit is used to construct the rest of the numbers of the scale. Examples of such a unit are the inch, the pound, the angstrom, and the dollar. A standard scale can be used to measure object or events with respect to the property for which a scale is designed to measure. Since the unit is arbitrary, one can have different numbers to which the objects are mapped. Because a standard scale is not unique, it is important to interpret the meaning of the numbers used in the scale. Thus, in general, the numbers obtained from such a scale are merely stimuli for the memory (what it felt like the last time the temperature was $-15°$C) and have no intrinsic significance. However, most carefully designed standard scales are helpful in that they preserve certain numerical relations in the measurements (the mapping) of the objects, giving us a better way to interpret the stimuli they are measuring than arbitrary scales.

A scale may or may not have zero for an origin. For example, a scale of ordinal numbers can begin with any number. A ratio scale, such as absolute temperature preserves origin. Interval scales, which measure the same phenomena like temperature, preserve linear relations, but may have different origins. Zero on the Fahrenheit scale is a different temperature than zero on the Celsius scale. Both are interval scales. Again, the numbers on these scales mean nothing unless one can recall situations associated with the numerical readings being considered. They are just a convenient means of communicating characteristics of objects or situations without everybody having to experience them.

It often happens that the interpretations of numerical stimuli from a given standard scale differ depending on the circumstances. There is no simple rule that can be applied to interpret readings from even a single scale when it is applied to a natural phenomenon. Intensity of sun light has a different significance for different purposes. It may be useful for sunbathing, but too bright for reading. Similarly, a monotonic relation between successive readings from a standard scale do not assure us that even higher readings will be better (or worse). More (or less) temperature does not necessarily correspond to more (or less) usefulness. Low temperatures are uncomfortable. As the readings rise, they become more comfortable and as they rise higher they can again become uncomfortable. On the other hand, to preserve some foods, low temperatures are very desirable and as the readings rise, they become undesirable, and as they rise still higher they could become desirable again. Our values of comfort and desirability and other social effects have to be at the bottom of every interpretation and depend on higher goals that we may have in mind.

For a large number of scales used in physics, it is implicitly assumed that the scale can be extended out to infinity and applied to every imaginable circumstance. In other words, interpretation in physics assumes events as homogeneous, no matter how near or far from the origin they may fall. What is most astonishing is the assumption in physics that objects yet unknown but with the same dimensional characteristics of the known objects being measured can in fact be measured in the same way. Realization of the different interpretations we can make of the same number in physics would indicate that when numbers fall outside the realm of experience it is logical to suspend the extension of the truth we construct from experience to a domain for which we have no knowledge and feeling. It would be mostly fictive speculation.

In economics, the arithmetic value of a dollar is assumed to be the same no matter whether a person has only one or a million dollars. But in reality it is not. To buy a new Mercedes, ten dollars and one hundred dollars are nearly equally inadequate or useless as down payments. On the other hand, for buying groceries, a hundred dollars is much more useful, practically ten times more useful than ten dollars. The first thousand dollars earned is much more important than the first thousand dollars earned after a million.

We must be constantly and carefully attentive to how we interpret data from scales. Standard scales force on us a way of thinking that is not in complete harmony with the way we really feel about what they are measuring.

There is a more general method of measurement that does not make use of standard scales. It is the method of relative measurements useful for properties for which there is no standard scale of measurement (love, political clout, straightness). These are known as intangible properties. The number of such properties is extremely large. We can scarcely hope to device standard scales for them all. We are driven to relative scales, and a surprising thing is that they can serve as a standard for how to handle the very few standard scales we have, and not the other way round. A remarkable aspect of relative scales is that they can use information from standard scales when there is a particular need to do so. Measurements in a standard ratio scale are transformed to measurements in a relative ratio scale by normalizing them.

This conversion process gives us a hint about the difference between the two kinds of scales. A relative scale for a property is generated for a specific set of entities or objects. A standard scale for a property is always out there ready to be called into use. More significantly, a relative scale is essential to represent priority or importance if one is generating the scale by making direct observations and judgments about the property under study. It is also useful when one is interpreting what the data from a standard scale really signify. Relative scales are always needed to represent subjective understanding. More is said about arithmetic relations between the two types of scales in Sections 8 and 12.

3. Paired comparisons as ratios

When we measure something with respect to a property, we usually use some known scale for that purpose. A basic contribution to the subject of this paper, the Analytic Hierarchy Process (AHP) is how to derive relative scales using judgment or data from a standard scale, and how to perform the subsequent arithmetic operation on such scales avoiding useless number crunching. The judgments are given in the form of paired comparisons [6,7,8]. One of the uses of a hierarchy is that it allows us to focus judgment separately on each of several properties essential for making a sound decision. The most effective way to concentrate judgement is to take a pair of elements and compare them on a single property without concern for other properties or other elements. This is why paired comparisons in combination with the hierarchical structure are so useful in deriving measurement. We also note that sometimes comparisons are made on the basis of standards established in memory through experience or training.

Assume that we are given n stones, $A_1, \ldots, A_n$, whose weights $w_1, \ldots, w_n$, respectively, are known to us. Let us form the matrix of pairwise ratios whose rows give the ratios of the weights of each stone with respect to all others. Here the smaller of each pair of stones is used as the unit, and the larger one is measured in terms of multiples of that unit. It is difficult to do the inverse comparison without again using the smaller stone as the unit. This is a sort of bias in human thinking, which leads to considering a nonsymmetrical outcome and the inclination not to force symmetry on it. We have the matrix equation:

$$
\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array}
\begin{array}{c}
\begin{array}{cccc} A_1 & A_2 & \ldots & A_n \end{array} \\
\begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix}
\end{array}
\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}
= n \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.
$$

The foregoing formulation has the advantage of giving us the solution. But it also gives rise to a far reaching theoretical interpretation. We have multiplied A on the right by the vector of weights $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$. The result of this multiplication is nw. If n is an eigenvalue of A, then w is the eigenvector associated with it. Now A has rank one because every row is a constant multiple of the first row. Thus all its eigenvalues except one are zero. The sum of the eigenvalues of a matrix is equal to its trace, the sum of the diagonal elements, and in this case, the trace of A is equal to n. Therefore, n is the largest, or principal, eigenvalue of A.

The solution of $Aw = nw$, called the principal right eigenvector of A, consists of positive entries and is unique to within a multiplicative constant. To make w unique, we normalize its entries by dividing by their sum. It is clear that if we are given the comparison matrix A, we can recover the scale. In this case the solution is the normalized version of any column of A.

The matrix $A = (a_{ij})$, $a_{ij} = w_i/w_j$, $i, j = 1, \ldots, n$, has positive entries everywhere and satisfies the reciprocal property $a_{ji} = 1/a_{ij}$. Any matrix with this property is called a reciprocal matrix. In addition, A is *consistent* because the following condition is satisfied:

$$a_{jk} = a_{ik}/a_{ij}, \quad i, j, k = 1, \ldots, n. \tag{1}$$

We see that the entire matrix can be constructed from a set of n elements which form a chain (or more generally, a spanning tree, in graph-theoretic terminology) across the rows and columns.

It is easy to prove that a consistent matrix must have the ratio form $A = (w_i/w_j)$, $i, j = 1, \ldots, n$. A necessary condition for consistency is that A be reciprocal. We show below that a necessary and sufficient condition for consistency is that the principal eigenvalue of A be equal to n, the order of A. When A is inconsistent, these two observations serve to help us derive a ratio scale whose ratios are close to those of an underlying scale $w = (w_1, \ldots, w_n)$. The reciprocal axiom of the AHP ensures that perturbations of a ratio scale are themselves reciprocal. The homogeneity axiom ensures for the inconsistent case that the perturbations would be small, and hence that the two principal eigenvalues are close, from which it would follow by an argument given in [14, p. 67] that the derived scale is close to an underlying ratio scale. In addition, this second axiom enables us to explore the improvement of some of the judgments, thus also the improvement of inconsistency and the scale approximation. But there still remains the question of order preservation. The method we use to derive the scale must not only yield a ratio scale, but also capture the order inherent in the judgments, a very strong requirement indeed.

In a general decision-making environment, we cannot give the precise values of the w_i/w_j but only estimates of them. Let us consider estimates of these values given by an expert who may make small errors in judgment. It is known from eigenvalue theory [14], that a small perturbation around a simple eigenvalue, as we have in n when A is consistent, leads to an eigenvalue problem of the form $Aw = \lambda_{\max} w$ where $\lambda_{\max}$ is the principal eigenvalue of A where A may no longer be consistent but is still reciprocal. The problem now is: to what extent does w reflect the expert's actual opinion? Note that if we obtain w by solving this problem, and then form a matrix with the entries (w_i/w_j), we obtain an approximation to A by a consistent matrix.

We now show the interesting, and perhaps surprising result that inconsistency throughout the matrix can be captured by a single number $\lambda_{\max} - n$, which measures the deviation of the judgments from the consistent approximation.

Let $a_{ij} = (1 + \delta_{ij}) w_i/w_j$, $\delta_{ij} > -1$, be a perturbation of w_i/w_j, where w is the principal eigenvector of A.

Theorem 1. $\lambda_{\max} \geqslant n$.

Proof: Using $a_{ji} = 1/a_{ji}$, and $Aw = \lambda_{\max} w$, we have

$$\lambda_{\max} - n = \frac{1}{n} \sum_{1 \leqslant i < j \leqslant n} \frac{\delta_{ij}^2}{1 + \delta_{ij}} \geqslant 0. \qquad \square \tag{2}$$

Theorem 2. *A is consistent if and only if* $\lambda_{\max} = n$.

Proof. If A is consistent, then because of (1), each row of A is a constant multiple of a given row. This implies that the rank of A is one, and all but one of its eigenvalues λ_i, $i = 1, 2, \ldots, n$, are zero. However, it follows from our earlier argument that $\sum_{i=1}^{n} \lambda_i = \text{Trace}(A) = n$. Therefore, $\lambda_{\max} = n$. Conversely, $\lambda_{\max} = n$ implies $\delta_{ij} = 0$, and $a_{ij} = w_i/w_j$. $\square$

For the consistency index (CI), we adopt the value $(\lambda_{\max} - n)/(n - 1)$. It is the negative average of the other roots of the characteristic polynomial of A. This value is compared with the same index obtained as an average over a large number of reciprocal matrices of the same order whose entries are random. If the ratio (called the consistency ratio CR) of CI to that from random matrices is significantly small (carefully specified to be about 10% or less), we accept the estimate of w. Otherwise, we attempt to improve consistency.

The reader may know about the experimental findings of the psychologist George Miller in the 1950's [4]. He found that in general, people (such as chess experts) could deal with information involving simultaneously only a few facts, seven plus or minus two, he wrote. With more, they become confused and cannot handle the information. This is in harmony with the stability of the principal eigenvalue to small perturbations when n is small [7,14], and its central role in the measurement of consistency.

Vargas [12] studied the case where the coefficients of the matrix are random variables. He focused his attention on gamma distributed coefficients and derived a Dirichlet distribution for the components of the eigenvector when the matrix is consistent. When the matrix is inconsistent, the 10% consistency bound is a sufficient measure to ensure that the eigenvector follows the Dirichlet distribution with given parameters which can be computed from the corresponding consistent matrix. The gamma assumption is a powerful one because of the inherent density of linear combinations of these distributions.

4. Two examples

The AHP is used with two types of measurement, relative and absolute, the latter having to do with memory standards mentioned above. In both, paired comparisons are performed to derive priorities for criteria with respect to the goal. In relative measurement, paired comparisons are performed throughout the hierarchy including on the alternatives in the lowest level of the hierarchy with respect to the criteria in the level above. In absolute measurement, paired comparisons are also performed through the hierarchy with the exceptions of the alternatives themselves. The level just above the alternatives consists of intensities or grades which are refinements of the criteria or subcriteria governing the alternatives. One pairwise compares the grades themselves under each criterion by answering questions such as: How much better is a student applicant with excellent grades than one with very good grades? and how much better is a student applicant with average letters of recommendation than one with poor ones? and so on. The alternatives are not pairwise compared, but simply rated as to what category in which they fall under each criterion. A weighting and summing process yields their overall ranks. This will become clear in the second example below. There is no reason why forcing standards on a problem should produce the same outcome obtained through relative measurement. These are two different descriptive (what can be) and normative (what should be) settings.

4.1. *Relative measurement: Choosing the best house to buy*

When advising a family of average income to buy a house, the family identified eight criteria which they thought they had to look for in a house. These criteria fall into three categories: economic, geographic and physical. Although one may have begun by examining the relative importance of these clusters, the family felt they wanted to prioritize the relative importance of all the criteria without working with clusters. The problem was to decide which of three candidate houses to choose. *The first step* is the structuring of the problem as a hierarchy.

In the first (or top) level is the overall goal of 'Satisfaction with house'. In the second level are the eight criteria which contribute to the goal, and the third (or bottom) level are the three candidate houses which are to be evaluated in terms of the criteria in the second level. The definitions of the criteria follow and the hierarchy is shown in Figure 1.

The criteria important to the individual family were:

(1) *Size of house*: Storage space; size of rooms, number of rooms; total area of house.

(2) *Location to bus lines*: Convenient, close bus service.

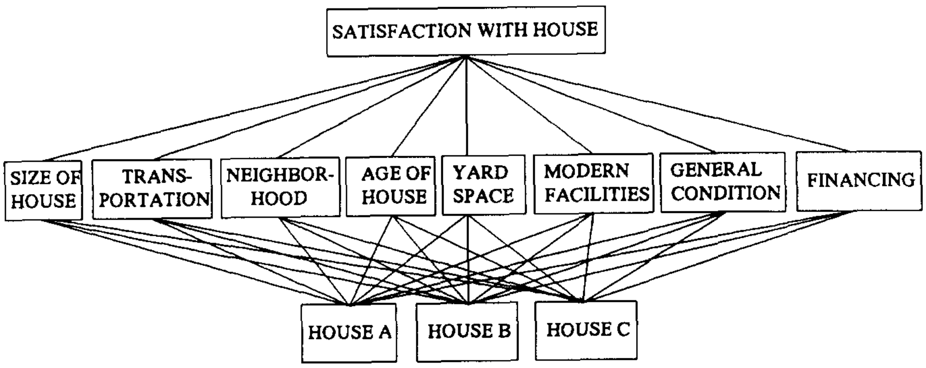

Figure 1. Decomposition of the problem into a hierarchy

Table 1
The fundamental scale

Intensity of importance on an absolute scale	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Moderate importance of one over another	Experience and judgment strongly favor one activity over another
5	Essential or strong importance	Experience and judgement strongly favor one activity over another
7	Very strong importance	An activity is strongly favored and its dominance demonstrated in practice
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2, 4, 6, 8	Intermediate values between the two adjacent judgments	When compromise is needed
Reciprocals	If activity i has one of the above numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i	
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining n numerical values to span the matrix

(3) *Neighborhood*: Little traffic, secure, nice view, low taxes, good condition of neighborhood.

(4) *Age of house*: Self-explanatory.

(5) *Yard space*: Includes front, back and side, and space from neighbors.

(6) *Modern facilities*: Dishwashers, garbage disposals, air conditioning, alarm system, and other such items possessed by a house.

(7) *General condition*: Repairs needed, walls, carpet, drapes, cleanliness, wiring.

(8) *Financing available*: Assumable mortgage; seller financing available, or bank financing.

The second step is the elicitation of pairwise comparison judgments. Arrange the elements in the second level into a matrix and elicit judgments from the people who have the problem about the relative importance of the elements with respect to the overall goal, Satisfaction with House. The scale to use in making the judgments is given in Table 1. This scale has been validated for effectiveness, not only in many applications by a number of people, but also through theoretical comparisons with a large number of other scales.

The questions to ask when comparing two

Table 2
Pairwise comparison matrix for level 1

	1	2	3	4	5	6	7	8	Priority vector
1	1	5	3	7	6	6	$\frac{1}{3}$	$\frac{1}{4}$	0.173
2	$\frac{1}{5}$	1	$\frac{1}{3}$	5	3	3	$\frac{1}{5}$	$\frac{1}{7}$	0.054
3	$\frac{1}{3}$	3	1	6	3	4	6	$\frac{1}{5}$	0.188
4	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{6}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{7}$	$\frac{1}{8}$	0.018
5	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	3	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{6}$	0.031
6	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	4	2	1	$\frac{1}{5}$	$\frac{1}{6}$	0.036
7	3	5	$\frac{1}{6}$	7	5	5	1	$\frac{1}{2}$	0.167
8	4	7	5	8	6	6	2	1	0.333

$\lambda_{max} = 9.669$, $CI = 0.238$, $CR = 0.169$

criteria are of the following kind: of the two criteria being compared, which is considered more important by the family buying the house with respect to the overall goal of family satisfaction with the house?

When the elements being compared are closer together than indicated by the scale, one can use the scale 1.1, 1.2, ..., 1.9. If still finer, one can use the appropriate percentage refinement.

The matrix of pairwise comparisons of the criteria given by the homebuyers in this case is shown in Table 2, along with the resulting vector of priorities. The vector of priorities is the principal eigenvector of the matrix. It gives the relative priority of the criteria measured on a ratio scale. In this case financing has the highest priority with 33% of the influence.

In Table 2, instead of naming the criteria, we use the number previously associated with each. Next we move to the pairwise comparisons of the elements in the lowest level. The elements to be compared pairwise are the houses with respect to how much better one is than the other is satisfying each criterion in level 2. Thus there will be eight 3×3 matrices of judgments since there are eight elements in level 2, and 3 houses to be pairwise compared for each element. Again, the matrices contain the judgments of the family involved. To understand the judgments, a brief description of the houses follows.

House A. This house is the largest of them all. It is located in a neighborhood with little traffic and low taxes. Its yard space is comparably larger than houses B and C. However, the general condition is not very good and it needs cleaning and painting. Also, the financing is unsatisfactory because it would have to be bank-financed at high interest.

House B. This house is a little smaller than House A and is not close to a bus route. The neighborhood gives one the feeling of insecurity because of traffic conditions. The yard space is

Table 3
Comparison matrices and local priorities

Size of house	A	B	C	Priority vector	Yard space	A	B	C	Priority vector
A	1	6	8	0.754	A	1	5	4	0.674
B	$\frac{1}{6}$	1	4	0.181	B	$\frac{1}{5}$	1	$\frac{1}{3}$	0.101
C	$\frac{1}{8}$	$\frac{1}{4}$	1	0.065	C	$\frac{1}{4}$	3	1	0.226
	$\lambda_{max} = 3.136$,	CI = 0.068,	CR = 0.117			$\lambda_{max} = 3.086$,	CI = 0.043,	CR = 0.074	
Transportation	A	B	C	Priority vector	Modern facilities	A	B	C	Priority vector
A	1	7	$\frac{1}{5}$	0.233	A	1	8	6	0.747
B	$\frac{1}{7}$	1	$\frac{1}{8}$	0.005	B	$\frac{1}{8}$	1	$\frac{1}{5}$	0.060
C	5	8	1	0.713	C	$\frac{1}{6}$	5	1	0.193
	$\lambda_{max} = 3.247$,	CI = 0.124,	CR = 0.213			$\lambda_{max} = 3.197$,	CI = 0.099,	CR = 0.170	
Neighborhood	A	B	C	Priority vector	General condition	A	B	C	Priority vector
A	1	8	6	0.745	A	1	$\frac{1}{2}$	$\frac{1}{2}$	0.200
B	$\frac{1}{8}$	1	$\frac{1}{4}$	0.065	B	2	1	1	0.400
C	$\frac{1}{6}$	4	1	0.181	C	2	1	1	0.400
	$\lambda_{max} = 3.130$,	CI = 0.068,	CR = 0.117			$\lambda_{max} = 3.000$,	CI = 0.000,	CR = 0.000	
Age of house	A	B	C	Priority vector	Financing	A	B	C	Priority vector
A	1	1	1	0.333	A	1	$\frac{1}{7}$	$\frac{1}{5}$	0.072
B	1	1	1	0.333	B	7	1	3	0.650
C	1	1	1	0.333	C	5	$\frac{1}{3}$	1	0.278
	$\lambda_{max} = 3.000$,	CI = 0.000,	CR = 0.000			$\lambda_{max} = 3.065$,	CI = 0.032,	CR = 0.056	

Table 4
Local and global priorities

	1 (0.173)	2 (0.054)	3 (0.188)	4 (0.018)	5 (0.031)	6 (0.036)	7 (0.167)	8 (0.333)	
A	0.754	0.233	0.754	0.333	0.674	0.747	0.200	0.072	0.396
B	0.181	0.055	0.065	0.333	0.101	0.060	0.400	0.650	0.341
C	0.065	0.713	0.181	0.333	0.226	0.193	0.400	0.278	0.263

fairly small and the house lacks the basic modern facilities. On the other hand, the general condition is very good. Also, an assumable mortgage is obtainable which means the financing is good with a rather low interest rate.

House C. House C is very small and has few modern facilities. The neighborhood has high taxes, but is in good condition and seems secure. The yard space is bigger than that of House B, but is not comparable to House A's spacious surroundings. The general condition of the house is good and it has a pretty carpet and drapes.

The matrices of comparisons of the houses with respect to the criteria and their local priorities are given in Table 3.

The third step is to establish the composite or global priorities of the houses. We lay out the local priorities of the house with respect to each criterion in a matrix and multiply each column of vectors by the priority of the corresponding criterion and add across each row which results in the desired vector of the houses in Table 4. House A which was the least desirable with respect to financing (the highest priority criterion), contrary to expectation, had the largest priority. It was the house that was bought.

4.2. *Absolute measurement: Employee evaluation*

Absolute measurement is applied to rank alternatives in terms of ratings, intensities or grades of the criteria. These grades may take the form: excellent, very good, good, average, below average, poor and very poor. After establishing a scale of priorities for the criteria (or subcriteria, if there are some) through paired comparisons, the grades which may be different for each criterion or subcriterion, are in turn pairwise compared according

Table 5
The hierarchy of employee evaluation

Goal:	Employee performance evaluation					
Criteria:	Technical (0.061)	Maturity (0.196)	Writing skills (0.043)	Verbal skills (0.071)	Timely work (0.162)	Potential (personal) (0.466)
Intensities:	Excell. (0.604)	Very (0.731)	Excell. (0.733)	Excell. (0.750)	Nofollup (0.731)	Great (0.750)
	Abv. avg. (0.245)	Accep. (0.188)	Avg. (0.199)	Avg. (0.171)	On time (0.188)	Averag. (0.171)
	Avg. (0.105)	Immat. (0.181)	Poor (0.068)	Poor (0.078)	Remind (0.081)	Bel. avg. (0.078)
	Bel. avg. (0.046)					
Alternatives:						
(1) Mr. *X*	Excell.	Very	Avg.	Excell.	On time	Great
(2) Ms. *Y*	Avg.	Very	Avg.	Avg.	Nofollup	Avg.
(3) Mr. *Z*	Excell.	Immat.	Avg.	Excell.	Remind	Great

to their parent criterion. An alternative is evaluated, for each criterion or subcriterion, by identifying the grade which best describes it. Finally, the weighted or global priorities of the grades are added to produce a ratio scale for the alternative. Absolute measurement needs standard to make it possible to judge whether the alternative is acceptable or not. Absolute measurement is useful in student admission, faculty tenure and promotion, employee evaluation, and in other areas where there is fairly good agreement on standard which are then used to rate alternatives one at a time.

Let us consider an abbreviated version of the problem of evaluating employee performance. The hierarchy for the evaluation and the priorities derived through paired comparisons are shown below. It is then followed by a rating of each employee for the quality of performance under each criterion and summing the resulting scores of obtain his overall rating. The hierarchy in Table 5 can be more elaborate, including subcriteria, followed by the intensities for expressing quality.

Let us now show how to obtain the total score for Mr. X (see Table 5):

$$0.061 \times 0.604 + 0.196 \times 0.731 + 0.043 \times 0.199 + 0.071 \times 0.750 + 0.162 \times 0.188 + 0.466 \times 0.750 = 0.623.$$

Similarly the score for Ms. Y and Mr. Z can be shown to be 0.369 an 0.478 respectively. It is clear that we can rank any number of candidates along these lines. Here the vector of priorities of the criteria has been weighted by the vector of relative number of intensities under each criterion and then renormalized. We call this a structural rescaling of the priorities.

5. Theoretical considerations [7]

There is a well known principle in mathematics that is widely practiced, but seldom enunciated with sufficient forcefulness to impress its importance. A necessary condition that a procedure for solving a problem be a good one is that if it produces desired results, and we perturb the variables of the problem in some small sense, it gives us results that are 'close' to the original ones. This is precisely the use of continuity and uniform continuity—to assure that after transforming a variable, originally nearby values go over to nearby values. An extension of this philosophy in problems where order relations between the variables are important, is that on small perturbations of the variables, the procedure produces close, order preserving results. The procedure described here has this characteristic.

Because of the natural way in which a matrix of ratios and small perturbations of that matrix lead to a principal eigenvalue problem, one may start with this and generalize to positive matrices that need not be reciprocal. The German mathematician Oskar Perron proved in 1907 that, if $A = (a_{ij})$, $a_{ij} > 0$, $i, j = 1, \ldots, n$, then A has a simple positive eigenvalue $\lambda_{\max}$ (called the principal eigenvalue of A) and $\lambda_{\max} > |\lambda_k|$ for the remaining eigenvalues of A. Furthermore, the principal eigenvector $w = (w_1, \ldots, w_n)^{\mathrm{T}}$ that is a solution of $Aw = \lambda_{\max} w$ has $w_i > 0$, $i = 1, \ldots, n$. We can write the norm of the vector w as $\|w\| = e^{\mathrm{T}} w$ where $e = (1, 1, \ldots, 1)^{\mathrm{T}}$ and we can normalize w by dividing it by its norm. For uniqueness, when we refer to w we mean its normalized form. Our purpose here is to show how important the principal eigenvector is in determining the rank of the alternatives through dominance walks.

We have seen that ratio scale estimation has a natural setting in principal eigenvalue formulation. We will now show that in addition, the principal eigenvector also has the order preserving properties we seek.

When A is consistent, one way to define the order of the alternatives is to require that one row of the matrix dominate elementwise another row. But when A is inconsistent it is no longer possible to define dominance in this manner. Instead, we borrow the concept of dominance from graph theory where the sum of the coefficients in each row of A is used. This concept carries over in a natural way to the inconsistent case. But we must look at a different way to capture dominance by considering further possibilities not simply from the matrix itself or some arbitrary power of it, but from all its powers.

The matrix A captures only the dominance of one alternative over every other in one step. But an alternative can dominate a second by first dominating a third alternative and then the third dominates the second. Thus, the first alternative dominates the second in two steps. It is known that the result for dominance in two steps is obtained by squaring the pairwise comparison matrix. Similarly, dominance can occur in three steps, four steps and so on, the value of each is obtained by raising the matrix to the corresponding power. The rank order of an alternative is the sum of the relative values for dominance in its row, in one step, two steps and so on averaged over the number of steps. The question is whether this average tends to a meaningful limit. It is easy to see that it does when A is consistent because $A^k = n^{k-1}A$.

We can think of the alternatives as the nodes of a directed graph. With every directed arc from node i to node j (which need not be distinct), is associated a nonnegative number a_{ij} of the dominance matrix. In graph-theoretic terms this is the intensity of the arc. Define a k-walk to be a sequence of k arcs such that the terminating node of each arc except the last is the source node of the arc which succeeds it. The *intensity of a k-walk* is the product of the intensities of the arcs in the walk. With these ideas, we can interpret the matrix A^k: the (i, j) entry of A^k is the sum of the intensities of all k-walks from node i to node j.

Definition. The dominance of an alternative along all walks of length $k \leqslant m$ is given by

$$\frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^{\mathrm{T}} A^k e}. \tag{3}$$

Observe that the entries of $A^k e$ are the row sums of A^k and that $e^{\mathrm{T}} A^k e$ is the sum of all the entries of A.

Theorem 3. *The dominance of each alternative along all walks k, is given by the solution of the eigenvalue problem* $Aw = \lambda_{\max} w$.

Proof. Let

$$s_k = \frac{A^k e}{e^{\mathrm{T}} A^k e} \tag{4}$$

and

$$t_m = \frac{1}{m} \sum_{k=1}^{m} s_k. \tag{5}$$

The convergence of the components of t_m to the same limit as the components of s_m is the standard Cesaro summability. Since

$$s_k = \frac{A^k e}{e^{\mathrm{T}} A^k e} \to w \quad \text{as } k \to \infty, \tag{6}$$

where w is the normalized principal right eigenvector of A, we have

$$t_m = \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^{\mathrm{T}} A^k e} \to w \quad \text{as } m \to \infty. \tag{7}$$

The solution of the eigenvalue problem is obtained by raising the matrix A to a sufficiently large power then summing over the rows and normalizing to obtain the priority vector $w = (w_1, \ldots, w_n)^{\mathrm{T}}$. The process is stopped when the difference between components of the priority vector obtained at the k-th power and at the $(k+1)$st power is less than some predetermined small value.

In reference [7] we gave at least five different ways of deriving the priorities from the matrix of paired comparisons. Besides the eigenvector solution, these include the direct row sum average, the normalized column average, and methods which minimize the sum of the errors of the differences between the judgments and their derived values such as the methods of least squares and logarithmic least squares. We pointed out that the logarithmic least squares solution coincides with the principal right eigenvector solution for matrices of order $n = 3$, which is the first value of n for which inconsistency is possible and left and right eigenvectors are reciprocals of each other which is not always the case for larger values of n. Since the appearance of the Analytic Hierarchy Process in the literature, other methods have been proposed [7,15]. All methods yield the same answer when the matrix is consistent. The combined use of a measure of inconsistency which can be derived in terms of both left and right eigenvectors, along with the right eigenvector solution which captures the dominance expressed in the judgments, is an effective way to look at the problem. We argue that so long as inconsistency is tolerated, dominance is the basic theoretical concept for deriving a scale and no other method qualifies. In addition, a counterexample has been provided in which the method not only does not generate a good approximation, but also reverses rank. Some have even resorted to artificial axiomatization thinking that it gives a method the appearance of rigor, although axioms are assumptions not proofs. It may be that the arithmetic of a method is simpler than that used to obtain the eigenvector, but that no longer matters, because of the widespread use of the computer. It is reasonable to argue that a theory for making sound decisions must stand on clearly justifiable grounds.

The software package Expert Choice, useful in teaching and in real applications, can handle both relative and absolute measurement, as well as having special capabilities such as structural rescaling, combining group judgments, sensitivity analysis and dependence among the decision alternatives [2]. The reader interested in pursuing the subject further should consult references [3,7,8,13,15].

6. Normalization – Scarcity and abundance

Normalization in the AHP is not just a mechanical operation. It contains information on the total dominance of the alternatives being compared which enables us to apportion the priority of the criterion to each alternative according to the relative dominance of the alternative.

Normalization can also be associated with the idea of scarcity and abundance of the presence of a criterion in the alternatives such as redness in fruit. Too many fruits that are red make red abundant and unimportant in differentiating between individual fruits. Conversely, if redness occurs intensely in some fruits but not in others, it is thus scarce and can be used as a criterion to differentiate in making a decision among fruits. Thus, the greater the contrast among the alternatives, the more useful is the priority value of the criterion allotted to each. Conversely, when for example, the contrast of the alternatives is smallest (when the total dominance is equal to n^2, see below) and hence all the alternatives are alike, the portion of the criterion priority assigned to each is equal. Thus a criterion with respect to which there is greater contrast and dominance among the alternatives is 'scarce' and consequently more influential in determining rank than another criterion on which there is no distinction among the alternatives, i.e., their matrix of paired comparisons has more 1's in it. In being scarce, more of the criterion is allotted to the more dominant alternative and hence it affects the final ranking of that alternative more.

An 'abundant' criterion contributes less to rank determination because it contributes an equal or nearly equal priority to each alternative and when the sum is taken over the criteria it has little effect in determining rank (recall that if $a > b$ then $a + c > b + c$ and c does not reverse order). An alternative that is a copy of another can dilute the priority of a decisive criterion so that it is no longer the controlling one in determining the final rank. We have:

Theorem 4. $\Sigma_{ij} a_{ij} = n^2$, *if and only if all alternatives have equal dominance, i.e.,* $a_{ij} = 1$, i, $j = 1, \ldots, n$, *with respect to a criterion.*

It is easy to see that n^2 is the minimum value of possible total dominance by a paired comparison of n alternatives. The maximum value is, using 9 as the upper range of the scale of paired comparisons, $[\frac{1}{2}n(n-1)](9+\frac{1}{9}) + n = \frac{1}{9}(41n^2 - 32n)$.

Our discussion of absolute and relative measurement may be framed in economic terms. In absolute measurement, *value* and *need* are identical; 'the more value, the better the need is satisfied'. In relative measurement, *value* is assessed in terms of *need*. Here surplus value may or may not satisfy more need. In fact, there are instances where satiation takes place and abundance can lead to a decline in the satisfaction of need.

7. Clustering

Comparisons of elements in pairs requires that they be homogeneous or close with respect to the common attribute; otherwise significant errors may be introduced into the process of measurement. In addition, the number of elements being compared must be small (not more than 9) to improve consistency and the corresponding accuracy of measurement. For example, we may cluster apples in one way according to size, in another way according to color and in still another way according to age. The question then is how to perform clustering of homogeneous elements in an efficient way to facilitate paired comparisons. Clustering is a process of grouping elements with respect to a common property. One can then decompose the set of ordered elements with respect to an attribute into clusters of, for example, seven elements each, from largest to smallest. The smallest element of the largest cluster is included as one of the seven elements of the next cluster. The relative weights of all the elements in this second cluster are divided by the weight of the common element and then multiplied by its weight in the first cluster in this manner both clusters become commensurate and are pooled together. The process is then repeated to the remaining clusters. Sometimes one may need to introduce hypothetical elements in order to preserve the gradual descent from large to small.

We discuss three ways as to how to perform clustering on the alternatives of a decision problem whose number may be very large and needs a different sorting for each of several attributes. They are ordered in the following discussion from the least to the most efficient way.

7.1. The elementary approach

Given n elements in a level of the hierarchy, one may first make a pass through them by comparing one element with another, dropping it and picking another if that one is perceived to be larger and continuing the comparison. Thus, the largest element is selected in $n-1$ such comparisons. The process is repeated for the remaining $n-1$ elements to identify the second largest element and so on. In the end, the elements would be arranged in descending order of size or intensity according to an attribute and are sequentially clustered into groupings of a few elements each from the largest to the smallest. This process is highly inefficient and requires the astronomical number of $(n-1)!$ comparisons.

7.2. Trial and error clustering

The alternatives may be put into groups of large, medium and small. Then the elements in each group are put into several clusters of a few elements each, and a first pass at comparisons is used to identify misfits which are then taken out and put into the appropriate one of the other two categories. Reclustering is then performed and comparisons are carried out. If elements are found not to fit, they are again moved to the appropriate category. This process is repeated for each attribute.

7.3. Clustering by absolute measurement

Each alternative is evaluated by absolute measurement for an attribute, and thus in descending order on that attribute. They can then be clustered into small groups as described above and pairwise compared.

One reason why absolute measurement may not be desirable is that it is strongly subjective. In paired comparisons, measurement is based on observation of the relative intensity of a property between two elements. Absolute measurement is based on observations stored in memory which depend on experience and on the ability to recall it. For many problems, it is useful to first carry out absolute measurement to sort and cluster the elements, and then follow that with relative measurement for greater accuracy. This is particularly relevant in predicting most likely outcomes which involve synergy among the alternatives.

8. Combining relative and absolute measurement – Cost–benefit analysis

An easy pitfall for an individual who has just learned about the AHP is to take two or more criteria on which alternatives are measured on the same existing standard scale, such as dollars or kilograms, normalize each set and then compose with respect to the criteria. One quickly discovers that the answer is not the same as that obtained through the usual arithmetic and hastily concludes that the AHP is at fault. To avoid this problem, one must exercise caution in converting measurements on a standard scale to relative values when several such criteria are involved [9].

For the arithmetic to conform with what one ordinarily does, assign each criterion a priority that is the sum of the measurements of the alternatives with respect to it, divided by the sum of the measurements of all the alternatives under all the criteria measured with that unit. To find the composite priority for each alternative multiply the criterion weight by its corresponding normalized weight under that criterion and sum over the criteria. The result may be considered as the weight of the alternative with respect to one super criterion composed of all the criteria with the unit of measurement. That super criterion may then be compared with other intangible criteria and other super criteria with different units of measurement. Alternatively, one can perform priority comparisons on all the data available without ascribing linearity to them as one ordinarily does with numbers. That is to say, one interprets what the numbers mean and uses judgment rather than cranking the number mechanically. With rare exception, the judgement approach is by far the more effective procedure using the AHP to deal with the underlying complexity of a decision problem, and one need not be afraid to do it. It is known that scales are invented to facilitate communication, and must be experienced for a long time before they can be associated with our value system, during which time they almost always are modified.

The AHP is a descriptive theory. Therefore, it is not an automatic set-up for accommodating any normative approach such as utility maximization. It needs to be interpreted and adapted for that purpose. It is even more difficult when a normative theory has many exceptions so the AHP interpretation would not be universal. In utility maximization it is assumed that it is always the case that the more utility or more money the better, but there are many instances when this is not true. For example, a government agency that is left with more money at the end of the year, gets a smaller allocation in a future budget. A rich individual in a poor country under revolt by the poor is likely to get killed. The availability of money is often an incentive to use money where other alternatives could be more effective. Thus, AHP adaptation for utility calculation purposes needs to take such caveats into consideration. One idea to keep in mind with the AHP is that the montonicity of utilities need not be preserved and may be contradicted.

The foregoing has bearing on benefit–cost analyses. Expected utilities are used for repeated decision making. In that case, benefit cost analysis from two hierarchies is applied, and from it one can also calculate marginal benefit to cost ratios. Resource allocation may be made by using benefit to cost ratios thus derived. Short range decisions often include low costs as benefits in a single hierarchy. This is a useful approach when a decision to spend money on one of several options has already been made. It is one way to combine benefits and costs as one does with dollars by taking differences. In the AHP one does not use differences. In such one-time decisions, costs may be regarded as inverse benefits so that benefits, low costs and other inconveniences are used to establish priorities for the alternatives.

If, on the other hand, benefits and costs are measured in dollars along with intangible factors, they must first be composed according to the dollars and then combined with the intangible criteria as described earlier. Marginal benefit cost analysis along traditional lines can be carried out by arranging the costs in increasing order, and then forming ratios of successive differences of benefits and costs in that order. The very first ratio is that of the alternative with the smallest cost. Then one forms the ratio of differences with that between the next highest cost and the smallest one in the denominator, and the corresponding difference in benefits in the numerator. Whenever a difference in a numerator is negative, that succeeding alternative is dropped from consideration. In this manner, the alternative yielding the highest marginal ratio is chosen.

The next question is how to find a reasonable way to combine benefits (B) and cost (C), when a ratio rather than a single hierarchy are used [10]. This is useful in considering conflict problems involving more than one value system. Since the hierarchy of costs leads to a vector of values indicating relative maximum costs, the reciprocals of the entries of this vector could be thought of as the minimum costs incurred in jointly maximizing benefits and minimizing costs. In general, the outcomes from a hierarchy of benefits and a hierarchy of costs are two ratio scales whose corresponding ratios lead to a meaningful ratio scale. Their differences would not be meaningful.

To maximize B/C is equivalent to maximizing $\log B/C$, which if B and C are close, may be approximated by $(B/C) - 1$ or $(B - C)/C$ known as return on investment, ROI. If B and C are not comparable, it would be initially clear that only the benefits or only the costs determine whether the allocation should be made or not.

Some people have attempted to compare the foregoing with what is traditionally done in a single criterion choice problem. An example where $B - C$ alone gives rise to misleading results is given by my colleague L.G. Vargas: You have $1 million to invest to get $1.101 million or you have $500000 to invest to get $600000. Which is a better investment?

$$B - C \text{ analysis gives:}\quad 1.101 - 1 = 0.101,$$

$$0.6 - 0.5 = 0.1,$$

and the first alternative would be chosen. ROI analysis yields 10.1% and 20%, respectively, and the second alternative which minimizes risk would be correctly chosen.

9. The semiotic connection

The Analytic Hierarchy Process is a tool of information communication and signification. In a general sense, it belongs to the study of language or semiotics. Semiotics or semiology is a coding in which one considers signals related by a set of rules or syntax; a set of states or contents called a semantic system, a set of possible behavioral responses independent of the content system, and a rule associating signals with contents or with behavioral responses. The idea of a code covers all four phenomena mentioned above. Eco [1] uses *s*-code for the first three and code for the fourth.

According to Morris [5], the creator of semiotics, semiosis is the process in which something functions as a sign. He mentions that both man and animals do respond to certain things as signs of something else, but their complexity in man is found in speech, art, writing, and even medical diagnoses. Science and signs are inseparably interconnected. Semiotics, a step in the unification of science, supplies the foundations for any science of signs; linguistics, logic, mathematics, rhetoric and aesthetics.

In the Analytic Hierarchy Process, words are used for concepts involved in decisions. In a sense, the purpose of all information is to decide on something—even if it is an imaginary hypothesis. The hierarchy is the syntax, the subject of a decision is the semantic, prioritization is the rule associating signals with content, and behavior comprises the possible decision alternatives. This way of looking at the AHP needs to be highlighted and emphasized to drive home the universality of the decision process.

10. Catastrophe and the AHP

One of the most burning quests we undertake using the knowledge we acquire is explaining and forecasting, and in particular, forecasting catastrophes, or, mathematically speaking, sudden discontinuities that affect the order of the alternatives. This is a situation that is at odds with the earlier mathematical principle of small perturbations we gave in which small changes in input lead to small changes in outcomes. In catastrophes, there is a part of the problem in which small changes in conditions at some critical value can give rise to very large changes in outcome. All we can do is either to anticipate and take strong preventive action or prepare ourselves in advance with contingency plans together with emotional acceptance for how to deal with the emergency to pay a smaller penalty if it occurs, or be prepared to pay the full penalty having accepted that it could happen. Because catastrophes are usually unseen surprises, it gives us a feeling of control to think we can account for catastrophes in our daily lives. According to the dictionary, a catastrophe is a sudden disaster or happening that causes great harm or damage; a calamity. Even the idea of what constitutes a catastrophe is relative. Someone whose life is tormented by an opponent may regard it as a blessing if a catastrophe befalls the tormentor. In other terms, a catastrophe may be regarded as a very strong discontinuity in thinking. How do we represent a catastrophe in terms of the AHP?

In a catastrophe the element of surprise arises from a shift in ranking of the outcomes. Thus, if a situation is ongoing a certain choice of alternatives may be indicated, but a slight change in the situation may cause a sudden shift to another choice that would have been previously very undesirable. How can we allow for catastrophic occurrences in the AHP?

One way to allow for a catastrophe is to always include a criterion for the unknown that represents a cluster of unforseen threats. It may itself have subcriteria. The alternatives are carefully prioritized with respect to this criterion for the unexpected and its descendants. The criterion itself is assigned a low priority. Now we can imagine that at each instant the judgments in the criteria are flashed in their totality on a screen followed by the best choice of alternatives. With some changes in judgments, in general, choice is stable and the changes in relative priority of the alternative slight. After a certain lapse of time, there is a sudden change in judgment in the direction of this criterion making it more important. The corresponding choice of alternatives is also suddenly changed surfacing a very undesirable alternative. Thus, one might include among the

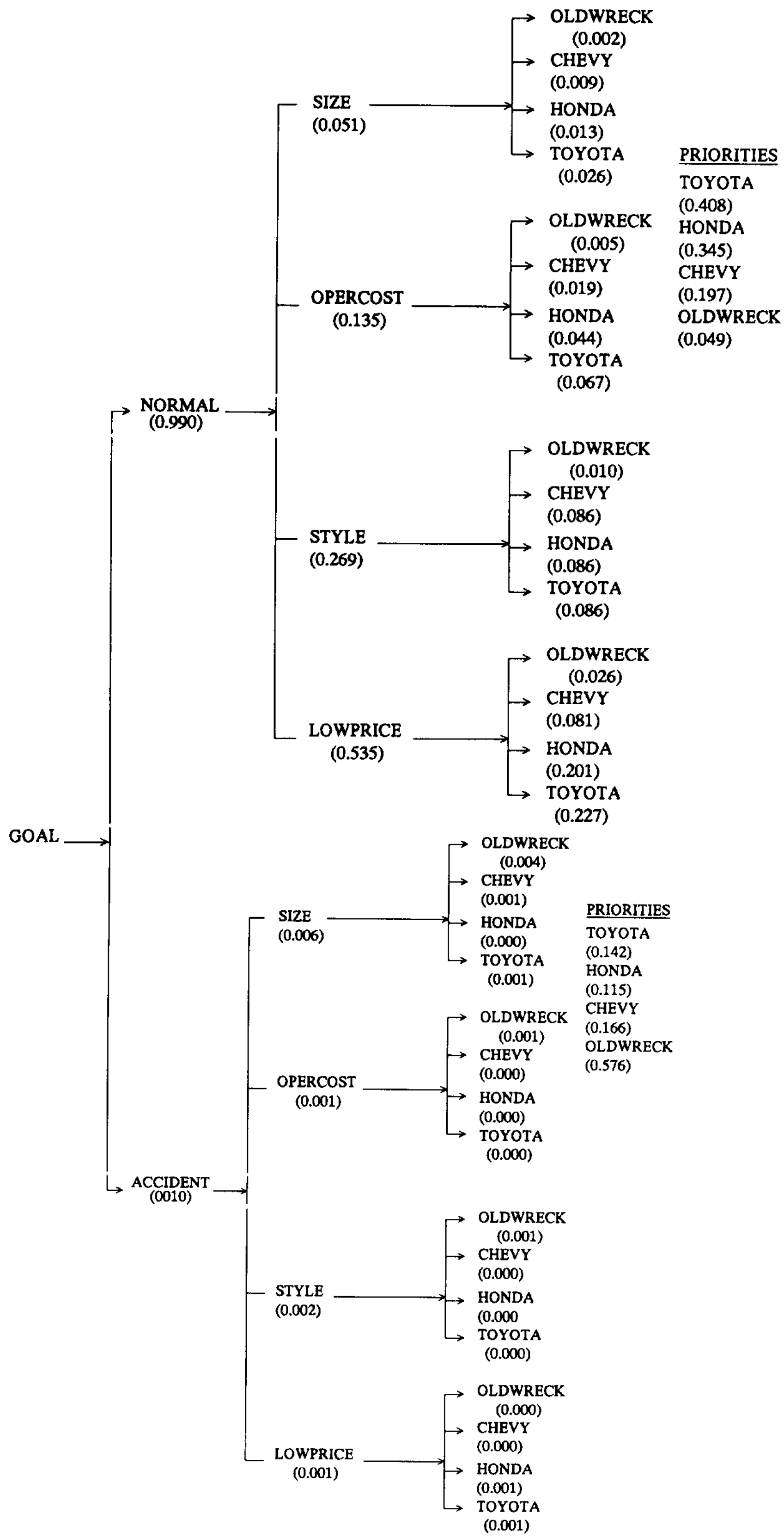

Figure 2. The catastrophe hierarchy

alternatives some catastrophic ones. This type of thinking would apply in hierarchies where a real wrong choice is extremely undesirable and costly.

We can also speak of chaos in the AHP: a situation where the possibility of exerting control by creating a hierarchy and setting judgment no longer exists, for it is not clear what takes precedence over what. Usually catastrophe produces chaos. It is possible to represent both catastrophe and chaos in the continuous setting of the AHP.

Both catastrophe and chaos are relative. A revolution may be regarded as an undesirable catastrophe that uproots a society or as a necessary kind of action to produce a desired end. In planning, strong out-of-the-ordinary action may be crucial for affecting change.

A simple illustration of how to represent a catastrophe is given in Figure 2. An individual buying a car makes an innocent choice by assuming that normal conditions will prevail over accidents in the ratio of 0.99 to 0.01. He chooses a nice new Toyota, although he knows that there is a chance that an accident can happen. If he were to buy the car under the assumption of a high likelihood of an accident, 0.99 to 0.01, and if he were to act in his best self interest, he would choose an old wreck to buy. One answer to this dilemma is neither to buy the old wreck nor the Toyota, but to buy a fairly good used car. But most of us are idealists who do not think that accidents will happen to us, and so we buy new cars.

11. What affects rank [11]

Although in catastrophes there is a sudden shift in rank, due to a change in the importance of the criteria, what happens to the rank of the alternatives in the more mundane event that their number is changed by adding new ones or deleting old ones? The traditional rule is that rank reversal is not acceptable if, given that the alternatives themselves are independent of each other, a new alternative does not introduce a new criterion or change the weights of the existing criteria. We tend to treat rank in a possessive manner by sometimes insisting that it stay the same no matter what logic says. The absolute mode of measurement of the AHP complies with this normative inclination, but the relative mode of measurement does not, because of the dependence of the measurements of the alternatives on each other. However, relative measurement *will* preserve rank with respect to a *single criterion* when the comparisons are consistent. Most people understand the dependence of alternatives in light of the notions of scarcity and abundance discussed earlier. There is no need to improvise notions of relevant and irrelevant alternatives, as is done in utility theory, because with relative measurement everything being compared is by definition relevant. Utility theory is obsessed with the reversal of rank because in that theory it can happen even with respect to a single criterion. This is a phenomenon that is strongly counterintuitive and can never be made mathematically right. Utility theorists, however, try to make it right by philosophical arguments about what is or is not a relevant alternative.

12. Summary of principles

The AHP generates relative ratio scales of measurement. The measurements of a set of objects on a standard scale can be converted to relative scale measurements through normalization. Only in a very localized way can a relative set of measurements have a unit, obtained by dividing the entire set by the smallest measurement. The normalization and composition of weights of alternative with respect to more than a single criterion measured on the same standard scale leads to nonsensical numbers, because normalizing separate sets of numbers destroys the linear relation among them. The weights must first be composed with respect to all such criteria and then normalized for AHP use. We can interpret such composition as we did in Section 8 as a special kind of weighting of the particular criteria. Thus, the AHP, with its relative measurement offers no guide on the outcome of manipulations based on combining different measurements from a standard scale such as a criterion of benefits and a criterion of costs, both measured in dollars, and used to select a best alternative.

If we do not insist that the linearity of a scale needs to be preserved (an old habit from when we did not have an effective way to interpret the information content of readings from a standard scale), we can then treat every criterion as an intangible. In that case we must bear in mind that the weights of the criteria do not derive from some underlying standard scale. If they must depend on such a scale, we are back to the need to compose before normalization.

The moral is that we are sometimes led into developing blind expectations for that to which we are accustomed out of habit, and not necessarily because its truth is something written in granite. We believe that our own tempered understanding should produce closer results to experience than simply following tradition, which has possibly rutted our thinking, and induced us to forego change in search of better ways that give better answers.

References

[1] Eco, U., *A Theory of Semiotics*, First Midland Book Edition, Indiana University Press, IN, 1976.

[2] Forman, E., and Saaty, T., *Expert Choice Software Package for IBM PC*, Expert Choice, Inc., Pittsburgh, PA 1983–1990.

[3] Harker, P.T. (ed.), "Special Issue on The Analytic Hierarchy Process", *Socio-Economic Planning Sciences* 20/6 (1986).

[4] Miller, G.A., "The magical number seven, plus or minus two: some limits on our capacity for processing information", *The Psychological Review* 63 (1956) 81–97.

[5] Morris, C.W., *Foundation of the Theory of Signs*, International Encyclopedia of Unified Science, Vols. I and II: Foundations of the Unity of Science, Vol. 1, No. 2, University of Chicago Press, 1938.

[6] Saaty, T.L., *Decision Making for Leaders: The Analytic Hierarchy Process for Decisions in a Complex World*, RWS Publications, Pittsburgh, PA, 1986; Original version published by Lifetime Learning Publications, 1982.

[7] Saaty, T.L., *Multicriteria Decision Making: The Analytic Hierarchy Process*, 1988; Revised and published by the author; Original version published by McGraw-Hill, New York, 1980.

[8] Saaty, T.L., "Scaling method for priorities in hierarchical structures", *Journal of Mathematical Psychology* 15/3 (1977) 234–281.

[9] Saaty, T.L., "A note on the AHP and expected value theory", *Socio-Economic Planning Sciences* 20/6 (1986) 397–398.

[10] Saaty, T.L., and Kearns, K.P., *Analytical Planning: The Organization of Systems*, International Series in Modern Applied Mathematics and Computer Science, Vol. 7, Pergamon Press, New York, 1985.

[11] Saaty, T.L., and Vargas, L.G., "Inconsistency and rank preservation", *Journal of Mathematical Psychology* 28/2 (1984).

[12] Vargas, L.G., "Reciprocal matrices with random coefficients", *Mathematical Modelling* 3 (1982) 69–81.

[13] Vargas, L.G. (ed.), "Special Issue on The Analytic Hierarchy Process", *Mathematical Modelling* 9/3–5 (1987).

[14] Wilkinson, J.H., *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.

[15] Zahedi, F., "The analytic hierarchy process – A survey of the method and its applications", *Interfaces* 16/4 (1986).