

## Graphical Abstract

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## Highlights

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# Multi-period and Multi-product Batch Processing Time Maximization Problem

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## Abstract

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*Keywords:* multiproduct batch, processing time maximization, C++, LINGO

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## 1. Introduction

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As scientific contributions,....

In the next sections, a presentation of problem is made along with an application example. Section 3 presents an interger linear mathematical model for the problem and section ?? presents an analytical method for its solution. Section 5 presents the tests and results obtained by a solver in C++ applying the analytical solution and a solver developed in LINGO to new proposed benchmarks. In sections 7 and 8, the contributions to this work are informed and acknowledgments are given, respectively. Finally, section 6 presents the paper's conclusions and suggestions for future works.

## 2. Multi-period and multi-product batch processing time maximization problem

In MPMPBPTM problem we consider a planning horizon with ND days, ( $ND \geq 1$ ), and a daily demand ( $UD_{id} \geq 0$ ) during this period for each product of the batch. Additionally, there is already a prior production plan,

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so that it is possible that there is already a previous production program for each product of the batch. Thus the availability for factory storage and shipping to outlets will be subject to change according to prior production planning. The problem is to determine a maximum processing time for the batch, taking into account the restrictions considered. For a better understanding of the problem, an example is presented below.

*Example:* A certain machine must process a batch containing 2 different products: A and B. The production rate of A is 60 g/min while the production rate of B is 40 g/min. The factory has free stock for a maximum of 3000 g of any product, and, according to the company's inventory policy, a maximum of 3000 g of product A and 2000 g of product B can be stocked at the factory. For the first day, there is a demand for 1000 g of product A and 500 g of product B, and for the second day, there is a demand for 500 g of product A and 300 g of product B. The factory has an outlet that has free space in stock of 1000 g, which can receive a maximum of 600 g of each product. There is a previous production planning for 1000 units of product A for the first day and 1500 units of product B for the second day. A maximum time of 100 minutes of this machine can be allocated for processing this batch. What is the maximum possible time for processing this batch ?

The following section presents a mathematical model for this problem.

### 3. Mathematical model

Given that:

$UD_{id}$  is the demand for the product  $i$  on day  $d$ ;  
 $PO_{id}$  is the planned production of the product  $i$  on day  $d$ ;  
 $O$  is the maximum quantity allowed for shipment of all products to outlets;  
 $UO_i$  is the maximum amount of product  $i$  that can be shipped to outlets;  
 $p_i$  is the production rate of product  $i$ ;  
 $Z$  is the timeout for batch processing;

$UI_{id}$  is the maximum quantity allowed for stocking in the factory the product  $i$  on day  $d$ ;  
being  $UI_{i0}$  the maximum quantity allowed for  $P_i$ ;  
 $I_d$  is the free factory storage of all products on the end of the day  $d$ ;

being  $I_0$  the initial free inventory for all products on factory on the start of the (*first*) planning day ( $d = 1$ );

$P_i$  is the amount of product  $i$  to be produced on batch on the current date (first planning day);

$D_{id}$  is the amount of product  $i$  delivered for the demand on day  $d$ ;

$O_{id}$  amount of product  $i$  shipped to factory outlets;

$T$  is the batch processing time.

We have the problem:

$$\max \quad T \quad (1)$$

*s.t.*

$$P_i - p_i * T = 0 \quad \forall i \quad (2)$$

$$P_i + UI_{i1} - D_{i1} - O_{i1} = UI_{i0} - PO_{i1} \quad (3)$$

$$UI_{id} - UI_{i(d-1)} - D_{id} - O_{id} = -PO_{id} \quad \forall i, \forall (d > 1) \quad (4)$$

$$I_1 + \sum_i \{P_i - D_{i1} - O_{i1}\} = I_0 - \sum_i PO_{i1} \quad (5)$$

$$I_d - I_{d-1} + \sum_i \{P_i - D_{id} - O_{id}\} = - \sum_i PO_{id} \quad \forall (d > 1) \quad (6)$$

$$\sum_d O_{id} \leq UO_i \quad \forall i \quad (7)$$

$$\sum_i \sum_d O_{id} \leq O \quad (8)$$

$$D_{i1} \leq UD_{i1} \quad \forall i \quad (9)$$

$$D_{id} \leq \sum_{k=1}^d UD_{ik} - \sum_{k=1}^{d-1} D_{ik} \quad \forall i, \forall d > 1 \quad (10)$$

$$T \leq Z \quad (11)$$

$$UI_{id}, I_d, O_{id}, D_{id} \in \mathbb{Z}^+ \quad \forall i, d \quad (12)$$

$$I_d \in \mathbb{Z}^+ \quad \forall d \quad (13)$$

where:

Constraints in (2) relate the quantity produced,  $P_i$ , to batch processing time  $T$ . Constraints in (3) calculate the quantity produced,  $P_i$ , as a function of the primary variables,  $D_i$ ,  $O_i$  and  $I_i$ . Constraints in (4), (5), and (7) state that the quantity delivered to demand, the quantity shipped to the outlets, and the factory-stocked quantity of each product must be less than their respective known limits. Constraints (6) and (8) state that both the sum of product quantities sent to the outlets and the sum of product quantities stocked in the factory must be less than their respective maximum allowed values. The restriction in (9) establishes that there is a batch processing time limit,  $Z$ , that must be respected. And finally, the constraints in (10) inform the nature of the decision variables.

#### 4. Exact optimizatin method

Uma solução ótima para este problema pode ser construída através de duas etapas, conforme explicado a seguir:

*etapa 1: distribuição da produção previamente planejada*

passo 1: Desconsidere a produção a ser planejada:

$$P_i = 0, \forall i \quad (14)$$

passo 2:

Calcule  $D_{i1}$  usando as equações em (15):

$$D_{i1} = \min\{UD_{i1}, PO_{i1}\} \quad \forall i \quad (15)$$

Calcule o valor da produção não ainda não alocada,  $S_{id}, \forall i$ , pelas equações em (16):

$$S_{i1} = PO_{i1} - D_{i1}, \quad \forall i \quad (16)$$

Calcule os valores  $D_{id}^+$ ,  $S_{id}^+$ ,  $D_{id}^-$  e  $S_{id}^-$ ,  $\forall i$ , usando as equações de (17) à (21).

$$D_{i1}^+ = D_{i1}^- = D_{i1} \quad (17)$$

$$D_{id}^+ = \min\left\{\left(\sum_{k=1}^d UD_{ik} - \sum_{k=1}^{d-1} D_{ik}\right), \left(\sum_{k=1}^{d-1} (PO_{ik} - D_{ik}^+) + PO_{id}\right)\right\} \quad \forall i, \forall (d > 1) \quad (18)$$

$$S_{id}^+ = PO_{id} - D_{id}^+, \quad \forall i, \forall (d > 1) \quad (19)$$

$$D_{id}^- = \min\left\{\left(\sum_{k=2}^d UD_{ik} - \sum_{k=2}^{d-1} D_{ik}\right), \left(\sum_{k=1}^{d-1} (PO_{ik} - D_{ik}^-) + PO_{id}\right)\right\} \quad \forall i, \forall (d > 1) \quad (20)$$

$$S_{id}^- = PO_{id} - D_{id}^-, \quad \forall i, \forall (d > 1) \quad (21)$$

Queremos encontrar  $D_{id}$ ,  $D_{id}^- \leq D_{id} \leq D_{id}^+$ ,  $\forall i$ , ( $d > 1$ ), tal que as restrições em (22) sejam respeitadas:

$$D_{id} = \min x \in \{D_{id}^-, D_{id}^+ \mid UI_{id} \geq 0\} \quad \forall i, \forall (d > 1) \quad (22)$$

Contudo, para solução do problema, iremos adotar:

$$D_{id} = D_{id}^- \quad \forall i, \forall (d > 1) \quad (23)$$

$$S_{id} = S_{id}^- \quad \forall i, \forall (d > 1) \quad (24)$$

passo 3: Calcule os valores  $O_{id}$ ,  $\forall i, d$ , usando as equações em (25):

$$O_{id} = \min\left\{S_{id}, \left(VO_i - \sum_{k=1}^{d-1} O_{ik}\right), \left(O - \sum_{j=1}^{i-1} \sum_{k=1}^{d-1} O_{jk}\right)\right\} \quad \forall i, d \quad (25)$$

passo 4:

calcule os valores para  $UI_{id}, \forall i, d$ , usando as equações em (3) e (4).  
calcule os valores para  $I_i, \forall i$ , usando as equações em (5) e (6).

*Step 2: MPBPTM problem construction*

So, based on calculated values, we built the MPBPTM problem using the equations:

$$UD_i = UD_{i1} - D_{i1} \quad \forall i \quad (26)$$

$$I = \min_d \{I_d\} \quad (27)$$

$$UI_i = \min_d \{UI_{id}\}, \quad \forall i \quad (28)$$

$$O = O - \sum_i \sum_d O_{id} \quad (29)$$

$$UO_i = UO_i - \sum_d O_{id}, \quad \forall i \quad (30)$$

Being  $p_i$  and  $Z$  defined according to the MPMPBPTM problem presented.

Obs: Note that some values of  $UI_{id}$  and  $I_d$  can be negative. This means that, in prior planning, some demand of the first day will be met by some previous planned production performed in a future period and that cannot be modified, i.e.  $D_{id} > D_{id}^-$  for at least one ordained pair  $(i, d)$ . In this case, considering that prior planning was a viable planning, the negative values of the  $UI_{id}$  and  $I_d$  will be offset by the positive surplus values of  $UD_i$ , so this will not result in change in the optimal solution of the problem through the method used in the next step.

The possible negative maximum values for variables are defined by the following equations. If this equations are not met, it means that initial planning was not a viable planning.

$$\max_d \{UI_{id}\} \geq UD_{i1} - D_{i1}, \quad \forall i \quad (31)$$



$$\max_d \{I_d\} \geq \sum_i \{UD_{i1} - D_{i1}\} \quad (32)$$

*Step 3: MPBPTM solution by Fraga's analytical method*

Once the MPBPTM problem is built, it can be solved by the analytical method proposed by Fraga (Fraga et al., 2023). This method can be resumed into two steps, as follows:

step 1: find  $T^*$ , where:

$$T^* = \lfloor \min\{\min_{\forall i} \{(UO_i + UI_i + UD_i)/p_i\}, (O + I + \sum_i UD_i) / \sum_i p_i\} \rfloor \quad (33)$$

step 2: calculate  $T$ , where:

$$T = \min\{T^*, Z\} \quad (34)$$

*Solution for the example presented before*

Applying the method for solving the example presented in section 2, we have:

Como planejamento pr vio de produ  o, temos:

$$P_{11} = 1000, P_{12} = 0, P_{21} = 0, P_{22} = 1500.$$

Como demanda, temos:

$$UD_{11} = 1000, UD_{12} = 500, UD_{21} = 500, UD_{22} = 300.$$

Distribuindo a produ  o para atender   demanda de acordo com as restri  es em (9) e (10), temos:

$$D_{11} = 1000, D_{12} = 0, D_{21} = 0, D_{22} = 300.$$

N o h  nenhuma sobra relativa a produ  o de A. E h  uma sobra de produ  o de 1200 unidades de B no segundo dia.

Como estoque livre nos outlets, temos:

$$UO_1 = 600; UO_2 = 600; O = 1000;$$

Assim, distribuindo a sobra da produção para os outlets de acordo com as restrições em (7) e (8), temos:

$$O_{11} = 0, O_{12} = 0, O_{21} = 0, O_{22} = 600.$$

Com relação ao estoque livre em fábrica, temos:

$$UI_{10} = 3000, UI_{20} = 2000, I_0 = 3000.$$

Portanto, calculando os valores disponíveis para produção de acordo com as equações em (3), (4) e (5), temos:

$$UI_{11} = 3000 - 1000 + 1000 + 0 = 3000, UI_{12} = 3000 - 0 + 0 + 0 = 3000, \\ UI_{21} = 2000 - 0 + 0 + 0 = 2000, UI_{22} = 2000 - 1500 + 300 + 600 = 1400.$$

$$I_1 = 3000 - 1000 + 1000 + 0 = 3000, I_2 = 3000 - 1500 + 300 + 600 = 2400.$$

Portanto temos agora as informações referentes aos limites estocagem em fábrica diário para cada produto assim como o limite de estocagem em fábrica diário para todos os produtos.

Definindo o problema MPBPTM, temos:

$$UD_1 = 1000 - 1000 = 0, UD_2 = 0 .$$

$$I = \min_d\{3000, 2400\} = 2400.$$

$$UI_1 = \min_d\{3000, 3000\} = 3000, UI_2 = \min_d\{2000, 1400\} = 1400.$$

$$O = 1000 - 600 = 400.$$

$$UO_1 = 600 - 0 = 600, UO_2 = 600 - 600 = 0.$$

$$p_1 = 60, p_2 = 40.$$

$$Z = 100.$$

Então, aplicando o método de partição do tempo proposto pro Fraga, temos:

$$T' = 0 \text{ min}$$

Then:

$$S_A = 0 \text{ g}$$

$$S_B = 0 \text{ g}$$

And so we have:

$$T'' \leq \lfloor (3000 \text{ g} + 600 \text{ g} + 0 \text{ g}) / (60 \text{ g/min}) \rfloor$$

$$T'' \leq 60 \text{ min} \quad \text{for A}$$

$$T'' \leq \lfloor (1400 \text{ g} + 0 \text{ g} + 0 \text{ g}) / (40 \text{ g/min}) \rfloor$$

$$T'' \leq 35 \text{ min} \quad \text{for B}$$

$$T'' \leq \lfloor (400 \text{ g} + 2400 \text{ g} + 0 \text{ g}) / (100 \text{ g/min}) \rfloor$$

$$T'' \leq 28 \text{ min} \quad \text{for A and B}$$

So that:

$$T'' = 28 \text{ min}$$

So we have:

$$T = 28 \text{ min}$$

As  $T < Z$ , then  $T = 28 \text{ min}$  will certainly be the optimal solution.

In this case we will have:

$$P_A = 1,680 \text{ g e } P_B = 1,120 \text{ g}$$

$$D_A = 0 \text{ g e } D_B = 0 \text{ g}$$

$$E_A = 1,680 \text{ g e } E_B = 1,120 \text{ g}$$

## 5. Tests and results

To test the developed model and analytical solution method, we created a solver in C++ and a solver in LINGO (<https://github.com/tbfraga/COPSolver>). The tests were performed on a notebook with an Intel i7 processor. We tested the solvers developed for the benchmarks presented on Tables 1, 2 and 3, and for random benchmarks generated by the following functions:

$$p_i = \text{rand()} \% 30 + 10 \quad (35)$$

$$UD_i = \text{rand()} \% 3000 + 800; \quad (36)$$

$$\text{seed1} = \text{rand()} \% 3000 + 500; \quad (37)$$

$$\text{seed2} = \text{rand()} \% 5000 + 1000; \quad (38)$$

$$O = N/2 * \text{seed1}; \quad (39)$$

$$UO_i = \text{rand()} \% (\text{seed1} - 500) + 500; \quad (40)$$

$$I = N/2 * \text{seed2}; \quad (41)$$

$$UI_i = \text{rand()} \% (\text{seed2} - 1000) + 1000; \quad (42)$$

$$Z = 100; \quad (43)$$

$i$	1	2	total
$p_i$	60	40	
$UD_i$	1000	500	
$UO_i$	600	600	1000
$UI_i$	3000	2000	3000

Z 100

Table 1: Benchmark MPBPTMP 001

$i$	1	2	3	total
$p_i$	60	40	50	
$UD_i$	1000	500	800	
$UO_i$	600	600	600	1500
$UI_i$	3000	2000	1000	3500

Z 100

Table 2: Benchmark MPBPTMP 002

Randomly generated benchmarks were named RMPBPTMP N, being N the number of products. Seeking to enable the reproduction of the results, in the computational construction of the benchmarks we used the function `srand((unsigned) source)`, where `source` is a defined value. To build the results presented in this work, we used `source=0`. The benchmarks used for the tests performed can be consulted at [github.com/blinded](https://github.com/blinded).

Table 4 presents the results obtained by applying the analytical method (C++ solver) and the LINGO solver for the solution of the previously presented benchmarks.

It is important to point out that the LINGO solver was developed with the purpose of validating the results found by the proposed analytical method. As the analytical method is a polynomial time complexity method, we did not intend to compare computational costs, however it is possible to verify that both solvers are capable of finding optimal solutions for the proposed benchmarks very quickly, even for very large problems.

$i$	1	2	3	4	5	6	7	8	9	10	total
$p_i$	60	40	50	40	30	50	60	10	20	40	
$UD_i$	1000	500	800	500	400	500	2000	300	500	1000	
$UO_i$	600	600	600	1500	300	200	500	800	0	200	3000
$UI_i$	3000	2000	1000	800	3000	1000	400	300	200	0	5000
										Z	100

Table 3: Benchmark MPBPTMP 003

problem	LINGO solver	time (s)	analytical method	time (s)
MPBPTMP 1	55	0.04	55	0.001
MPBPTMP 2	48	0.06	48	0.001
MPBPTMP 3	30	0.09	30	0.001
RMPBPTMP 20	100	0.08	100	0.001
RMPBPTMP 50	98	0.12	98	0.001
RMPBPTMP 100	98	0.12	98	0.002
RMPBPTMP 1,000	78	3.10	78	0.002
RMPBPTMP 10,000	70	168.87	70	0.006

Table 4: Results obtained with the LINGO solver and the analytical method

We have considered here a one-day period problem, so in a future work we will study the complexities of solving a multi-period problem and try to propose new solution methods if needed.

## 6. Conclusions and suggestions for future works

In this paper we presented the Multi-product Batch Processing Time Maximization problem, as well as a mathematical model and an analytical solution method for this problem. The mathematical model and analytical method were tested, respectively, by a solver developed with the LINGO software, from LINDO Systems, and with a solver developed in C++ language. Optimum results were found for all proposed benchmarks, very quickly, even in the case of very large problems, which demonstrates the efficiency of the proposed method. As in this paper we considered a planning period of one

day, in a future work we will study the complexities that arise when considering the same problem in a multi-period scenario. We will also verify if there is the possibility of extending the proposed analytical method to solve a class of linear integer programming problems with similar characteristics to the studied problem.

## 7. CRediT authorship contribution statement

T.B. Fraga: Conceptualization, Project administration, Supervision, Software, Methodology, Validation, Formal analysis, Writing – original draft, Writing – review & editing. Í.R.B. Aquino: Data curation.

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