Example 2.17

4

Figure 2.48 shows an electrical circuit consisting of two inductors, two resistors, and a capacitor. A voltage V_a is applied to the circuit. Derive an expression for the mathematical model for the circuit.

Solution

The circuit consists of two nodes and two loops. We can apply Kirchhoff's current law to the nodes. For node 1,

$$i_1 + i_2 + i_3 = 0$$

or

$$\frac{V_a - v_1}{R_1} + \frac{1}{L_1}\int (0 - v_1)\mathrm{d}t + \frac{1}{L_2}\int (v_2 - v_1)\mathrm{d}t = 0 \tag{2.86}$$

Differentiating Equation 2.86 with respect to time, we get

$$\frac{\dot{V}_a}{R_1} - \frac{\dot{v}_1}{R_1} - \frac{v_1}{L_1} + \frac{v_2}{L_2} - \frac{v_1}{L_2} = 0$$

$$\frac{\dot{V}_a}{R_1} = \frac{\dot{v}_1}{R_1} + \left(\frac{1}{L_1} + \frac{1}{L_2}\right)v_1 - \frac{v_2}{L_2} \tag{2.87}$$

For node 2,

$$i_4 + i_5 + i_6 = 0$$

or

$$\frac{1}{L_2}\int (v_1 - v_2)\mathrm{d}t + C\frac{\mathrm{d}(0 - v_2)}{\mathrm{d}t} + \frac{0 - v_2}{R_2} = 0 \tag{2.88}$$

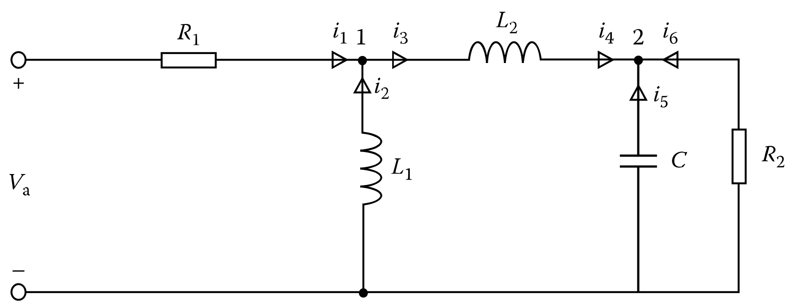

FIGURE 2.48
Electrical circuit.

Differentiating Equation 2.88 with respect to time, we obtain

$$\frac{(v_1 - v_2)}{L_2} - C\ddot{v}_2 - \frac{v_2}{R_2} = 0$$

which can be written as

$$C\ddot{v}_2 + \frac{\dot{v}_2}{R_2} - \frac{v_1}{L_2} + \frac{v_2}{L_2} = 0 \tag{2.89}$$

Equations 2.87 and 2.89 describe the state of the system. These two equations can be represented in matrix form as in equation

$$\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} + \frac{1}{L_2} & -\frac{1}{L_2} \\ -\frac{1}{L_2} & \frac{1}{L_2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{\dot{V}_a}{R_1} \\ 0 \end{bmatrix} \tag{2.90}$$

2.7.3 Modeling of Electromechanical Systems

Electromechanical systems such as electric motors and electric pumps are used in most industrial and commercial applications. Figure 2.49 shows a simple dc motor circuit. The torque produced by the motor is proportional to the applied current and is given by

$$T = k_t i \tag{2.91}$$

where
- T is the torque produced
- k_t is the torque constant
- i is the motor current

Assuming there is no load connected to the motor, the motor torque can be expressed as given in Equation 2.92.

$$T = I\frac{d\omega}{dt} \quad \text{or} \quad I\frac{d\omega}{dt} = k_t i \tag{2.92}$$

As the motor armature coil is rotating in a magnetic field there will be a back emf induced in the coil in such a way as to oppose the change producing it. This emf is proportional to the angular speed of the motor and is given by Equation 2.93.

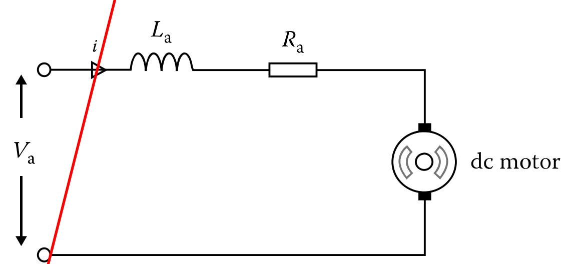

FIGURE 2.49
Simple dc motor.