On the General BER Expression of One- and Two-Dimensional Amplitude Modulations

Kyongkuk Cho and Dongweon Yoon, *Member, IEEE*

***Abstract*—Quadrature amplitude modulation (QAM) is an attractive technique to achieve high rate transmission without increasing the bandwidth. A great deal of recent attention has been devoted to the study of bit error rate (BER) performance of QAM, and approximate expressions for the bit error probability of QAM have been developed in many places in the literature. However, the exact and general BER expression of QAM with an arbitrary constellation size has not been derived yet. In this paper we provide an exact and general closed-form expression of the BER for one-dimensional and two-dimensional amplitude modulations, i.e., PAM and QAM, under an additive white Gaussian noise (AWGN) channel when Gray code bit mapping is employed. The provided BER expressions offer a convenient way to evaluate the performance of PAM and QAM systems for various cases of practical interest. Moreover, simple approximations can be found from our expressions, which are the same as the well-known approximations, if only the dominant terms are considered.**

***Index Terms*—Bit error probability, PAM, QAM, gray coding.**

I. Introduction

The emergence of multimedia applications and services demands an ever-increasing bandwidth in communication systems. Reliable high-speed data communications over insufficient channel bandwidth is one of the major challenges of harsh wireless environments that push the achievable spectral efficiency far below its theoretical limits. A QAM scheme is a useful modulation technique for achieving high data rate transmission without increasing the bandwidth of wireless communication systems. QAM, combined with other schemes, has gained great attention in overcoming detrimental channel impairments. For example, an adaptive modulation scheme can maximize the throughput of wireless communication systems when combined with the QAM scheme [1]–[3].

Although an exact evaluation of bit error probability for M-ary square QAM can be obtained for arbitrary signal constellation size, indeed, it is quite tedious to express in a closed form [4]. As an example, the exact BER expressions for 16-QAM and 64-QAM are presented in [5] and [6]. However, the exact and general BER expression of arbitrary M-ary square QAM has not yet been derived.

Conventionally, the BER approximation of M-ary square QAM has been performed by either calculating the symbol error probability [4], [7] or by simply estimating it using lower/upper bounds [7]. Tighter approximate expressions for the bit error probability of M-ary square QAM have recently been developed in [8] and [9], based on signal-space concepts and recursive algorithms, respectively. Although approximate expressions in the literature may provide accurate values of the BER in high signal-to-noise ratio (SNR), the evaluation of the BER using those expressions tends to deviate from its exact values when SNR is low.

The objective of this paper is to present an exact and general closed-form expression of the BER performance for one-dimensional and two-dimensional amplitude modulations, i.e., PAM and QAM, under an AWGN channel when Gray code mapping is employed. First we analyze the BER performance of an I-ary PAM scheme. Regular patterns in the kth bit error probability are observed while developing the BER expression. From these patterns we provide the exact and general closed-form BER expression of an I-ary PAM. Then we extend the regular patterns found in the BER analysis to an arbitrary M-ary square QAM scheme and present a general closed-form expression for BER of M-ary square QAM. We also analyze the BER performance of an arbitrary $I \times J$ rectangular QAM by considering that this signaling format consists of two PAM schemes, i.e., I-ary and J-ary PAM. A simple approximate BER expression for rectangular QAM is given as well. Indeed, this expression is the exact and general closed-form BER expression for arbitrary one– and two-dimensional amplitude modulation schemes.

The rest of this paper is organized as follows. Section II contains an analysis of the BER performance of a one-dimensional signaling format, PAM. The regular patterns for developing BER expressions are tabulated and an exact and general bit error probability of I-ary PAM is presented for an arbitrary amplitude level. In Section III we extend the BER performance analysis to a two-dimensional signaling format, M-ary square QAM, and we analyze the BER performance of an $I \times J$ rectangular QAM signal. Finally, in Section IV, we conclude with a brief summary of our work.

II. General BER Analysis of PAM

In this section we derive the exact bit error probability of an I-ary PAM. We determine and illustrate the regular patterns shown in the kth bit error probability due to the characteristics of Gray code bit mapping. From these regularities we formulate the exact BER expression.

Paper approved by G. M. Vitetta, the Editor for Equalization and Fading Channels of the IEEE Communications Society. Manuscript received March 26, 2001; revised July 25, 2001, October 9, 2001, and January 10, 2002.

K. Cho is with the LG Electronics, Mobile Communication Tech. Gr., CDMA System Research Laboratory, LG R&D Complex, Dongan-gu, Anyang-shi, Kyongki-do 431-749, Korea (e-mail: kyongkuk@lge.com).

D. Yoon is with the Division of Computer and Communications Engineering, Daejeon University, Daejeon 300-716, Korea (e-mail: dwyoon@dju.ac.kr).

Publisher Item Identifier 10.1109/TCOMM.2002.800818.

A. System Model

Signal waveforms of one-dimensional amplitude modulation I-ary PAM can be expressed as

$$s(t) = A_I \cos 2\pi f_c t, \quad 0 \le t < T \tag{1}$$

where A_I is the signal amplitude of the in-phase components, f_c is the carrier frequency, and T is the symbol time duration. In the I-ary PAM scheme, a serial data sequence is converted to $\log_2 I$ bits of data. In (1), the amplitude A_I is selected independently from the set $\{\pm d, \pm 3d, \ldots, \pm(I-1)d\}$, where $2d$ is the minimum distance between signal points. The relation between the minimum signal distance $2d$ and the bit energy E_b can be expressed as follows:

$$d = \sqrt{\frac{3\log_2 I \cdot E_b}{(I^2-1)}}. \tag{2}$$

In this modulation scheme, a perfect Gray code is assumed [10], and all the symbols are assumed to be transmitted equally likely.

In the presence of noise, the received waveform can be expressed as follows:

$$r(t) = s(t) + n(t) \tag{3}$$

where $n(t)$ is a zero-mean AWGN with a two-sided power spectrum density $N_0/2$. The received PAM signal may be demodulated coherently and, for simplicity, we assume perfect carrier recovery and symbol synchronization. Note that the positions of the bits in the PAM symbols have an effect on the probability of them being in error.

B. Regular Patterns in Developing BER Expression for I-ary PAM

The average bit error probability of an I-ary PAM scheme can be obtained by averaging the error probability of the kth bit, $P_b(k)$, where $k \in \{1, 2, \ldots, \log_2 I\}$ [5]. Thus, $P_b(k)$ characterizes the BER of an I-ary PAM scheme. First we present the BER expression for 4-PAM, and tabulate the regular patterns for developing BER expressions. Then we generalize the BER expression of I-ary PAM for an arbitrary amplitude level I.

1) BER of 4-PAM: Let us begin with the simple case of 4-PAM. Fig. 1 shows the 4-PAM signal constellation and its decision regions where each signal point is represented by a 2-bit symbol, constituted by i_1 and i_2 bits.

In Fig. 1, $\underline{i_1}$ and $\underline{i_2}$ denote the regions in which $i_1 = 1$ and $i_2 = 1$, respectively. We will show that the position of the bits in the 2-bit symbol has an effect on the bit error probability. For 4-PAM there are two possible cases (class I, class II) of the probabilities, $P_b(k)$, that the kth bit ($k = 1, 2$) is in error.

For class I, we consider only the i_1 bit. Then the constellation can be separated into two regions based on the decision boundary represented by the solid line at the origin (see Fig. 1). In this case, zero is the decision boundary and the signal point can be at a distance d or $3d$ from the decision boundary. A bit error will occur if the noise exceeds d or $3d$. In this case, the probability that the first bit, i_1, is in error is

$$P_b(1) = \frac{1}{4}\left[\operatorname{erfc}\left(\frac{d}{\sqrt{N_0}}\right) + \operatorname{erfc}\left(\frac{3d}{\sqrt{N_0}}\right)\right] \tag{4}$$

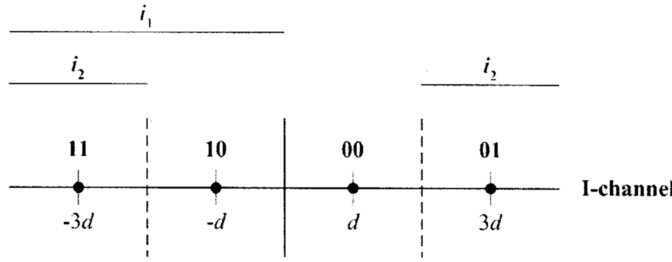

Fig. 1. Signal space diagram for 4-PAM.

where $\operatorname{erfc}(\cdot)$ is the complementary error function, which is defined as [11]

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\, du. \tag{5}$$

For class II, we consider the i_2 bit while ignoring the i_1 bit. The dashed lines crossing $-2d$ and $2d$ are given as the decision boundaries of the decision regions in Fig. 1. From the regions separated by these boundaries, we can define the individual cases of bit error as related to the magnitude of the noise. In the left half side of the plane, when $i_2 = 1$, a bit error will occur if the noise exceeds d but will not occur if the noise exceeds $5d$. In a similar way, a bit error will occur if the noise exceeds $-d$ or $3d$ for the case of $i_2 = 0$. The same analysis also applies for the right half side of the plane. Therefore, the error probability of the second bit, i_2, can be expressed as

$$P_b(2) = \frac{1}{4}\left[2\operatorname{erfc}\left(\frac{d}{\sqrt{N_0}}\right) + \operatorname{erfc}\left(\frac{3d}{\sqrt{N_0}}\right) - \operatorname{erfc}\left(\frac{5d}{\sqrt{N_0}}\right)\right]. \tag{6}$$

Finally, the exact average bit error probability for 4-PAM is obtained by averaging the bit error probabilities given by (4) and (6), that is

$$P_b = \frac{1}{2}\sum_{k=1}^{2} P_b(k). \tag{7}$$

By following steps similar to those used above, bit error probabilities can be easily obtained for higher-order PAM (for example, 8-PAM, 16-PAM, 32-PAM, and so on).

2) Description of the Regular Pattern in BER Expression for I-ary PAM: From the above results, the bit error probability, $P_b(k)$, that the kth bit is in error can be represented as follows:

$$P_b(k) = \frac{1}{I}\sum X_I(k) \cdot \operatorname{erfc}\left(\frac{Y_I(k)d}{\sqrt{N_0}}\right) \tag{8}$$

From this expression we can find regular patterns in the BER derivation. We summarize the derivation regularities in Table I, where the sequences $[X_I(k)]$ and $[Y_I(k)]$ are tabulated for various I-ary PAM schemes. These patterns provide valuable information for deriving the BER of coherently demodulated Gray coded I-ary PAM such that the signal constellation of the kth bits for the case of I-PAM is similar to the case of $I/2$-PAM.

As shown in Table I, a regular pattern can be observed for the BER analysis. When $k = \log_2 I$, the 1st term of $[X_I(k)]$ begins with a value of $I/2$. The value for the second term is the first decremented by 1, and the third term is the second with a change of sign. The value of the fourth term of $[X_I(k)]$ is

TABLE I
REGULAR PATTERNS OF $[X_I(k)]$ AND $[Y_I(k)]$ FOR BER OF I-ARY PAM

I	k	$[X_I(k)]$	$[Y_I(k)]$
2	1	1	1
4	1	1,1	1, 3
	2	2, 1, -1	1, 3, 5
8	1	1,1,1,1	1, 3, 5, 7
	2	2,2, 1,1, -1,-1	1, 3, 5, 7, 9, 11
	3	4, 3, -3, -2, 2, 1, -1	1, 3, 5, 7, 9, 11, 13
16	1	1,1,1,1,1,1,1,1	1, 3, 5, 7, 9, 11, 13, 15
	2	2,2,2,2, 1,1,1,1, -1,-1,-1,-1	1, 3, 5, 7, 9, 11, …,19, 21, 23
	3	4,4, 3,3, -3,-3, -2,-2, 2,2, 1,1, -1,-1	1, 3, 5, 7, 9, 11, …, 23, 25, 27
	4	8, 7, -7, -6, 6, 5, -5, -4, 4, 3, -3, -2, 2, 1, -1	1, 3, 5, 7, 9, 11, …, 25, 27, 29
I	1	1,1,1,…,1,1	1, 3, 5, 7, …, $2(I/2-1)+1$
	2	2,2,2,…,2,2, 1,1,1,…,1,1, -1,-1,-1,…,-1,-1	1, 3, 5, 7, …, $2(3I/4-1)+1$
	3	4,4,4,…,4,4, 3,3,3,…,3,3, -3,-3,-3,…,-3,-3, …, 1,1,1,…,1,1, -1,-1,-1,…,-1,-1	1, 3, 5, 7, …, $2(7I/8-1)+1$
	4	8,8,8,…,8,8, 7,7,7,…,7,7, -7,-7,-7,…,-7,-7, …, 1,1,1,…,1,1, -1,-1,-1,…,-1,-1	1, 3, 5, 7, …, $2(15I/16-1)+1$
	…	…	…
	$\log_2 I$	$I/2$, $(I/2-1)$, $-(I/2-1)$, $-(I/2-2)$, …, 5, -5, -4, 4, 3 ,-3,-2, 2, 1, -1	1, 3, 5, 7, …, $2(I-1-1)+1$

TABLE II
GRAY CODED BIT SEQUENCE FOR AN I-ARY PAM SIGNAL CONSTELLATION.

	Left half plane								Right half plane							
2-PAM								1	0							
4-PAM							11	10	00	01						
8-PAM					111	110	100	101	001	000	010	011				
16-PAM	1111	1110	1100	1101	1001	1000	1010	1011	0011	0010	0000	0001	0101	0100	0110	0111
:	…								…							

the third incremented by 1 and the fifth value is the fourth with a change of sign. This pattern repeats for the remaining terms until the value of $[X_I(k)]$ becomes -1. For the other values of k, the total number of $[X_I(k)]$ terms doubles as I doubles due to a twofold increase in the run-length of each term where $\underline{*, *, \ldots, *}$ denotes the run (Table I). When $k = 1$, all of the coefficients of the complementary error functions $[X_I(k)]$ are equal to 1, and the number of $[X_I(k)]$ terms is equal to $I/2$. $[Y_I(k)]$ is a finite arithmetic sequence with a common difference of 2, whose first term is 1 and last term is $2[(1-2^k)I-1]+1$.

The above results are based on the consistency of the bit format for a Gray coded signal constellation. Table II shows a Gray coded bit sequence for an I-ary PAM signal constellation.

For the left half side of the plane, the first bits are

$$
\begin{array}{ll}
\text{2-PAM} & 1 \\
\text{4-PAM} & 11 \\
\text{8-PAM} & 1111 \\
\text{16-PAM} & 11\,111\,111
\end{array}
$$

the second bits are

$$
\begin{array}{ll}
\text{4-PAM} & 10 \\
\text{8-PAM} & 1100 \\
\text{16-PAM} & 11\,110\,000
\end{array}
$$

the third bits are

$$
\begin{array}{ll}
\text{8-PAM} & 1001 \\
\text{16-PAM} & 11\,000\,011 \\
\text{32-PAM} & 1\,111\,000\,000\,001\,111
\end{array}
$$

and so on. It is obvious that the number of $[X_I(k)]$ terms doubles as I doubles. This pattern is observable in all of the bit sequences except for the bit sequence corresponding to the least significant bit (LSB). The LSB formats in the signal constellation are shown in Table III.

For the LSB$(k = \log_2 I)$, the particular pattern [1001] is repeated once as I doubles (except when $I = 2$), i.e., the sequence [1001] is repeated $I/4$ times in the sequence corresponding

TABLE III
LEAST SIGNIFICANT BIT OF AN I-ARY PAM SIGNAL CONSTELLATION.

	Left half plane								Right half plane							
2-PAM								1	0							
4-PAM							1	0	0	1						
8-PAM					1	0	0	1	1	0	0	1				
16-PAM	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
:	...								...							

to the LSB for I-ary PAM. Without considering the LSB of 2-PAM, the points in each half plane for the I-ary PAM LSB actually constitute an $I/2$-PAM LSB constellation.

Table I can be derived by observing the Gray code bit mapping described in Tables II and III.

C. General BER Expression of I-ary PAM

From the regularities shown in Table I and (8), the probability that the kth bit is in error can be formulated as follows:

$$P_b(k)=\frac{1}{I}\sum_{i=0}^{(1-2^{-k})I-1}\left\{(-1)^{\left\lfloor\frac{i\cdot 2^{k-1}}{I}\right\rfloor}\left(2^{k-1}-\left\lfloor\frac{i\cdot 2^{k-1}}{I}+\frac{1}{2}\right\rfloor\right)\cdot \operatorname{erfc}\left((2i+1)\sqrt{\frac{3\log_2 I\cdot\gamma}{(I^2-1)}}\right)\right\} \quad (9)$$

where $\gamma=(E_b/N_0)$ denotes the SNR per bit, and $\lfloor x\rfloor$ denotes the largest integer to x.

Finally, the exact average bit error probability of arbitrary I-ary PAM can be obtained by averaging the bit error probability given by (9), that is

$$P_b=\frac{1}{\log_2 I}\sum_{k=1}^{\log_2 I}P_b(k) \quad (10)$$

Note that for $I=2$, (10) reduces to the well-known bit error probability of binary phase-shift keying (BPSK).

Some numerical examples of (10) are presented in Fig. 2 which shows the BER performance of I-ary PAM signals as a function of the SNR per bit for $I=2,4,8,16,32,64$ and 128 under an AWGN channel. As shown in Fig. 2, an additional 4–5 dB of SNR is required to transmit an extra bit to maintain an average BER of 10^{-3}.

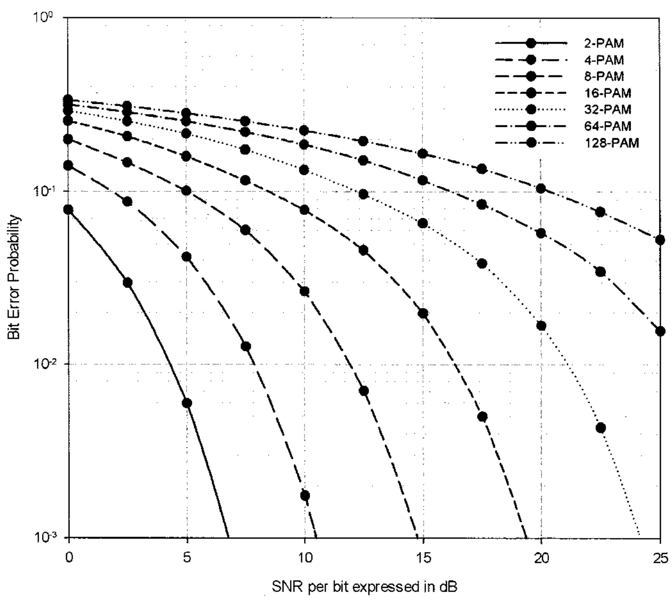

Fig. 2. Exact BER of I-ary PAM.

III. GENERAL BER ANALYSIS OF QAM

In this section, we derive the exact and general BER expression for M-ary QAM. We first analyze the bit error probability of square QAM and then provide the BER expression of rectangular QAM.

A. System Model

M-ary square QAM signal waveforms consist of two independently amplitude-modulated carriers in quadrature, which can be expressed as

$$s(t)=A_I\cos 2\pi f_c t-A_J\sin 2\pi f_c t,\quad 0\le t<T \quad (11)$$

where A_I and A_J are respectively the amplitudes of the in-phase and quadrature components, f_c is the carrier frequency, and T is the symbol time duration. In M-ary square QAM, $\log_2 M$ bits of the serial information stream are mapped onto a two-dimensional signal constellation using Gray coding. In (11), A_I and A_J are selected independently over the set $\{\pm d,\pm 3d,\ldots,\pm(\sqrt{M}-1)d\}$ where $2d$ is the Euclidean distance between two adjacent signal points and is given by

$$d=\sqrt{\frac{3\log_2 M\cdot E_b}{2(M-1)}} \quad (12)$$

where E_b is the bit energy. Fig. 3 illustrates the square 16-QAM signal constellation and its decision regions. In this case, the signal constellation is assigned a perfect two-dimensional Gray code [10]. Furthermore, we assume that all the symbols are equally likely to be transmitted.

The demodulation of the received QAM signal is achieved by performing two quadrature PAM demodulations. For simplicity, we assume perfect carrier recovery and symbol synchronization. Note that the positions of the bits in the QAM symbols have an effect on the probability of them being in error.

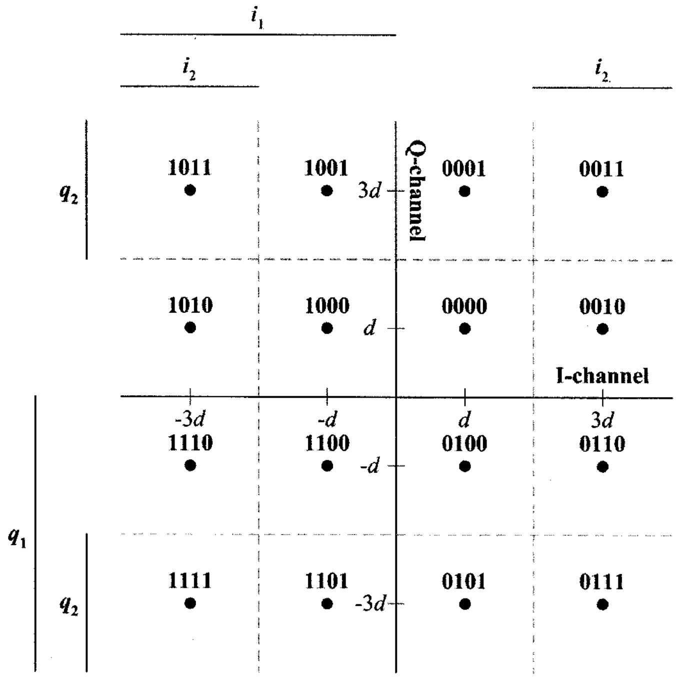

Fig. 3. Signal-space diagram for square 16-QAM.

B. General BER Expression of M-ary Square QAM

The BER analysis of arbitrary M-ary square QAM can be obtained through steps similar to those followed in the analysis of I-ary PAM. That is, the bit error probability, $P_b(k)$, that the kth bit is in error can be expressed as

$$P_b(k) = \frac{1}{\sqrt{M}} \sum X_M(k) \mathrm{erfc}\left(Y_M(k)\sqrt{\frac{3\log_2 M \cdot \gamma}{2(M-1)}}\right). \tag{13}$$

From this expression we can find regular patterns of $[X_M(k)]$ and $[Y_M(k)]$ in the BER derivation for various M-ary square QAM schemes. From these regular patterns, the kth bit error probability of M-ary square QAM is expressed as

$$P_b(k) = \frac{1}{\sqrt{M}} \sum_{i=0}^{(1-2^{-k})\sqrt{M}-1} \left\{ (-1)^{\left\lfloor \frac{i\cdot 2^{k-1}}{\sqrt{M}} \right\rfloor} \times \left(2^{k-1} - \left\lfloor \frac{i\cdot 2^{k-1}}{\sqrt{M}} + \frac{1}{2} \right\rfloor\right) \cdot \mathrm{erfc}\left((2i+1)\sqrt{\frac{3\log_2 M \cdot \gamma}{2(M-1)}}\right) \right\}. \tag{14}$$

When $k = 1$, (14) reduces to the familiar result ([8], (16)), that is

$$P_b(1) = \frac{1}{\sqrt{M}} \sum_{i=0}^{\frac{\sqrt{M}}{2}-1} \mathrm{erfc}\left((2i+1)\sqrt{\frac{3\log_2 M \cdot \gamma}{2(M-1)}}\right). \tag{15}$$

Since the demodulation of an M-ary square QAM signal is equivalent to the demodulation of two independent $\sqrt{M}$-ary PAM signals in quadrature, a bit error occurs when a received vector resides in the wrong decision region for both in-phase and quadrature PAM signals. In other words, M-ary square QAM can be considered as a combination of two independent PAM signals, I-ary PAM of the in-phase signal and J-ary PAM of the quadrature signal, where $I = J = \sqrt{M}$. Not surprisingly, (14) is identical to (9) if $\sqrt{M}$ is substituted for I in (9) [4].

Finally, the exact expression of average bit error probability of M-ary square QAM can be obtained by averaging the bit error probability given by (14), yielding

$$P_b = \frac{1}{\log_2 \sqrt{M}} \sum_{k=1}^{\log_2 \sqrt{M}} P_b(k) \tag{16}$$

Note that for $M = 4$, (16) reduces to the well-known BER expression of Quadrature Phase Shift Keying (QPSK).

If only the first and the second terms $(i = 0, 1)$ in (16) are considered, an approximate BER expression for M-ary square QAM can be obtained from (16) by neglecting the higher order terms, i.e.

$$P_b \cong \frac{\sqrt{M}-1}{\sqrt{M}\log_2\sqrt{M}} \mathrm{erfc}\left(\sqrt{\frac{3\log_2 M \cdot \gamma}{2(M-1)}}\right) + \frac{\sqrt{M}-2}{\sqrt{M}\log_2\sqrt{M}} \mathrm{erfc}\left(3\sqrt{\frac{3\log_2 M \cdot \gamma}{2(M-1)}}\right). \tag{17}$$

Note that (17) is identical to the result presented in [9, eq. (16)].

For high SNR, the first term $(i = 0)$ is dominant in (16). Thus, for high SNR the BER of M-ary square QAM can be approximated to a certain degree of accuracy by neglecting some of the higher order terms in (16), i.e.

$$P_b \cong \frac{\sqrt{M}-1}{\sqrt{M}\log_2\sqrt{M}} \mathrm{erfc}\left[\sqrt{\frac{3\log_2 M \cdot \gamma}{2(M-1)}}\right]. \tag{18}$$

Note that (18) is the same result as [9, eq. (13)], [12, eq. (7.24)], [13, eq. (11.46a)], and [14].

C. Extension to Rectangular QAM

As shown in the specific case of square QAM, QAM consists of two independent PAM signals. QAM with a rectangular signal constellation can also be analyzed with a similar method. Fig. 4 illustrates an 8×4 rectangular QAM signal constellation and its decision regions.

From the geometry of a signal constellation for $I \times J$ rectangular QAM, the relation between the Euclidean distance of the nearest signal points and the average bit energy E_b is expressed as

$$d = \sqrt{\frac{3\log_2(I \cdot J) \cdot E_b}{I^2 + J^2 - 2}}. \tag{19}$$

Now, the probability that the kth bit in in-phase components and the lth bit in quadrature components are in error in terms of SNR can be expressed as follows:

$$P_I(k) = \frac{1}{I} \sum_{i=0}^{(1-2^{-k})I-1} \left\{ (-1)^{\left\lfloor \frac{i\cdot 2^{k-1}}{I} \right\rfloor} \left[2^{k-1} - \left\lfloor \frac{i \cdot 2^{k-1}}{I} + \frac{1}{2} \right\rfloor\right] \cdot \mathrm{erfc}\left((2i+1)\sqrt{\frac{3\log_2(I\cdot J)\cdot \gamma}{I^2+J^2-2}}\right) \right\}. \tag{20}$$

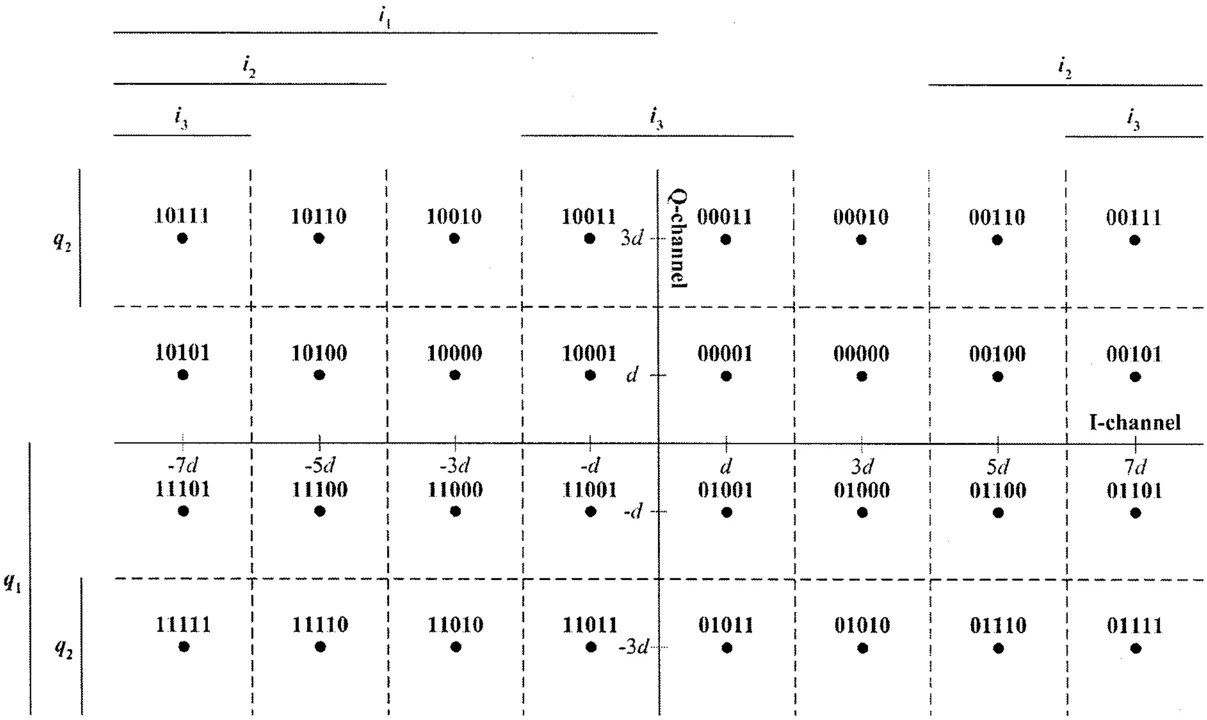

Fig. 4. Signal-space diagram for 8×4 rectangular QAM.

$$P_J(l) = \frac{1}{J} \sum_{j=0}^{(1-2^{-l})J-1} \left\{ (-1)^{\left\lfloor \frac{j \cdot 2^{l-1}}{J} \right\rfloor} \left[2^{l-1} - \left\lfloor \frac{j \cdot 2^{l-1}}{J} + \frac{1}{2} \right\rfloor \right] \cdot \operatorname{erfc}\left((2j+1)\sqrt{\frac{3\log_2(I \cdot J) \cdot \gamma}{I^2 + J^2 - 2}} \right) \right\}. \quad (21)$$

Finally, the average bit probability of $I \times J$ rectangular QAM under an AWGN channel can be obtained by averaging the bit error probabilities given by (20) and (21)

$$P_b = \frac{1}{\log_2(I \cdot J)} \left(\sum_{k=1}^{\log_2 I} P_I(k) + \sum_{l=1}^{\log_2 J} P_J(l) \right). \quad (22)$$

Note that for $I = 2$ and $J = 1$, (22) reduces to the BER of a BPSK signal. Also, when $I = J = \sqrt{M}$, (22) reduces to (16), the average bit error probability of M-ary square QAM, and when $J = 1$, (22) leads to (10), the average bit error probability of I-ary PAM.

For high SNR vales, the terms with $i = 0$ and $j = 0$ in (20) and (21), respectively, will be dominant, thus an approximate BER expression for rectangular QAM can be obtained from (22) by neglecting the higher order terms

$$P_b \cong \frac{1}{\log_2(I \cdot J)} \left[\frac{I-1}{I} \operatorname{erfc}\left(\sqrt{\frac{3\log_2(I \cdot J) \cdot \gamma}{I^2 + J^2 - 2}} \right) + \frac{J-1}{J} \operatorname{erfc}\left(\sqrt{\frac{3\log_2(I \cdot J) \cdot \gamma}{I^2 + J^2 - 2}} \right) \right]. \quad (23)$$

Note that when $I = J = \sqrt{M}$, (23) reduces to (18).

Fig. 5 shows the BER performance of M-ary QAM signals as a function of the SNR per bit for $M = 4, 8, 16, 32, 64, 128, 256$ and 1024 under an AWGN channel. As shown in Fig. 5, an additional 4–5 dB of SNR is required to transmit an extra bit per dimension to maintain an average BER of 10^{-3}. In Fig. 6, the exact BER expression obtained from (16) is compared with the approximations obtained from ([9], (13)) and ([9], (16)) for $M = 256, 1024$, and 4096 square QAM. As shown in Fig. 6, there are noticeable differences in BER performance between the exact expression and the approximations when M is large and SNR is low. Furthermore, there is a performance offset between them for a severe fading channel even when SNR is high. In [15] it is shown that there is a significant performance difference between the exact and approximate expressions under a severe fading case even when SNR is high.

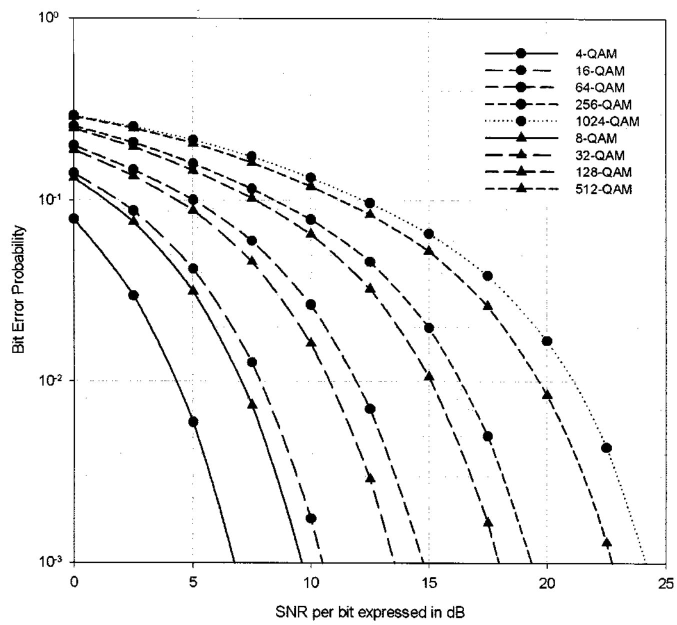

Fig. 5. Exact BER of M-ary QAM.

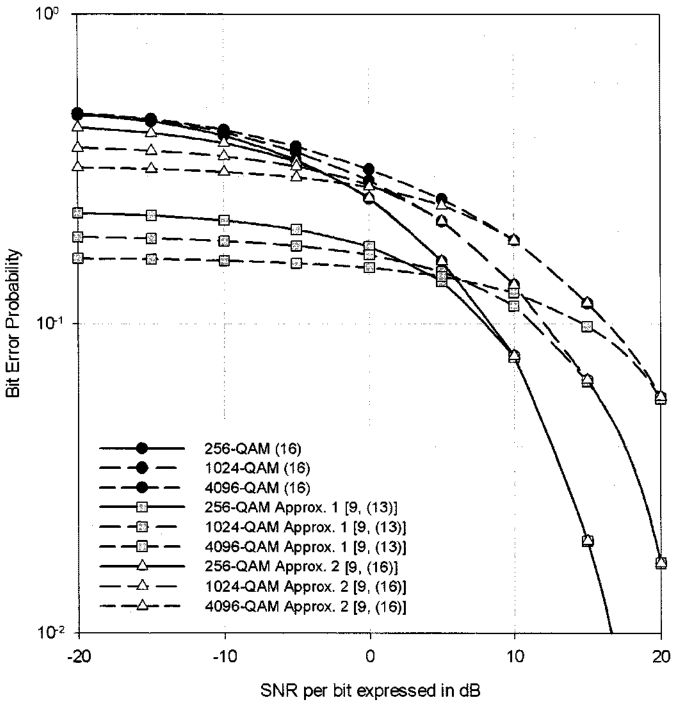

Fig. 6. BER performance comparisons.

IV. CONCLUSION

In this paper we have presented and analyzed an exact and general closed-form expression for the BER of the coherent demodulation of Gray coded one-dimensional and two-dimensional amplitude modulations, i.e., PAM and QAM, under an AWGN channel. We have tabulated the regular patterns for developing a BER expression and formulated the general BER expression from the regular patterns. This derivation is based on the consistency of the bit mapping of a Gray coded signal constellation. The BER performance between the exact expression and the approximations shows significant differences when the modulation level is high and SNR is low.

As the final result is expressed in the form of a weighted sum of complementary error functions, it can also be easily extended to the exact BER expressions in the various fading cases with or without diversity reception. The result presented is sufficiently general to offer a convenient way to evaluate the performance of arbitrary PAM and QAM systems for various cases of practical interest. Moreover, simple approximations can be found from our BER expressions if only the dominant terms are considered.

ACKNOWLEDGMENT

The authors would like to thank Dr. Hanzo and Dr. Yang of University of Southampton, Southhampton, U.K., and Dr. Simon of Jet Propulsion Laboratory (JPL), Pasadena, CA, for their useful comments. They would also like to express their gratitude to the anonymous reviewers and editor for valuable comments and suggestions that enhanced the quality of the paper.

REFERENCES

[1] W. Webb and R. Steele, "Variable rate QAM for mobile radio," *IEEE Trans. Commun.*, vol. 43, pp. 2223–2230, July 1995.

[2] A. J. Goldsmith and S. G. Chua, "Variable-rate variable-power MQAM for fading channels," *IEEE Trans. Commun.*, vol. 45, pp. 1218–1230, Oct. 1997.

[3] M. S. Alouini, X. Tang, and A. J. Goldsmith, "An adaptive modulation scheme for simultaneous voice and data transmission over fading channels," *IEEE J. Select. Areas Commun.*, vol. 17, pp. 837–850, May 1999.

[4] M. K. Simon, S. M. Hinedi, and W. C. Lindsey, *Digital Communication Techniques*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[5] W. T. Webb and L. Hanzo, *Modern Quadrature Amplitude Modulation*. New York: IEEE Press, 1994.

[6] M. P. Fitz and J. P. Seymour, "On the bit error probability of QAM modulation," *Int. J. Wireless Inform. Networks*, vol. 1, no. 2, pp. 131–139, 1994.

[7] J. G. Proakis, *Digital Communications*. New York: McGraw-Hill, 1995.

[8] J. Lu, K. B. Letaief, J. C.-I. Chuang, and M. L. Liou, "M-PSK and M-QAM BER computation using signal-space concepts," *IEEE Trans. Commun.*, vol. 47, pp. 181–184, Feb. 1999.

[9] L. Yang and L. Hanzo, "A recursive algorithm for the error probability evaluation of M-QAM," *IEEE Commun. Lett.*, vol. 4, pp. 304–306, Oct. 2000.

[10] W. J. Weber III, "Differential encoding for multiple amplitude and phase shift keying systems," *IEEE Trans. Commun.*, vol. 26, pp. 385–391, Mar. 1978.

[11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*. New York: Academic, 1980.

[12] B. Sklar, *Digital Communications*. Englewood Cliffs, NJ: Prentice-Hall, 1988.

[13] I. A. Glover and P. M. Grant, *Digital Communications*: Prentice-Hall, 1998.

[14] I. Korn, *Digital Communications*: Van Nostrand Reinhold, 1985.

[15] K. Cho, D. Yoon, W. Jeong, and M. Kavehrad, "BER analysis of arbitrary rectangular QAM," in *Proc. IEEE 35th Asilomar Conf.*, vol. 2, Pacific Grove, CA, Nov. 2001, pp. 1056–1059.

[16] D. Yoon, K. Cho, and J. Lee, "Bit error probability of M-ary quadrature amplitude modulation," in *Proc. IEEE VTC Fall 2000*, vol. 5, Boston, MA, Sept. 2000, pp. 2422–2427.

Kyongkuk Cho received the B.Sc. and M.Sc. degrees from Hanyang University, Seoul, Korea, in 1995 and 1997, respectively.

Currently, he is with LG Electronics Inc., Seoul, Korea, as a Senior Engineer. His research interests are in modulation, wireless communication, and third-generation wireless CDMA systems.

Dongweon Yoon was born in Seoul, Korea, in 1966. He received the B.S.(summa cum laude), M.S., and Ph.D. degrees in electronic communications engineering from Hanyang University, Seoul, Korea, in 1989, 1992, and 1995, respectively.

From March 1995 to August 1997, he was an Assistant Professor in the Division of Electronic and Information Engineering of Dongseo University, Pusan, Korea. Since September 1997, he has been on the faculty of Daejeon University, Daejeon, Korea, where he is now an Associate Professor in the Division of Computer and Communications Engineering. From February 1997 to December 1997, he was an Invited Researcher at the Electronics and Telecommunications Research Institute (ETRI), Daejeon, Korea. During 2001–2002, he was a Visiting Professor at the Pennsylvania State University, University Park, PA. He has served as a consultant for a number of companies and given many lectures on the topics of wireless communications. His research interests include coding, modulation, communication theory, and wireless communications.

Dr. Yoon is a member of the Korean Institute of Communication Sciences (KICS), the Institute of Electronics Engineers of Korea (IEEK), and the Korea Information Processing Society (KIPS).