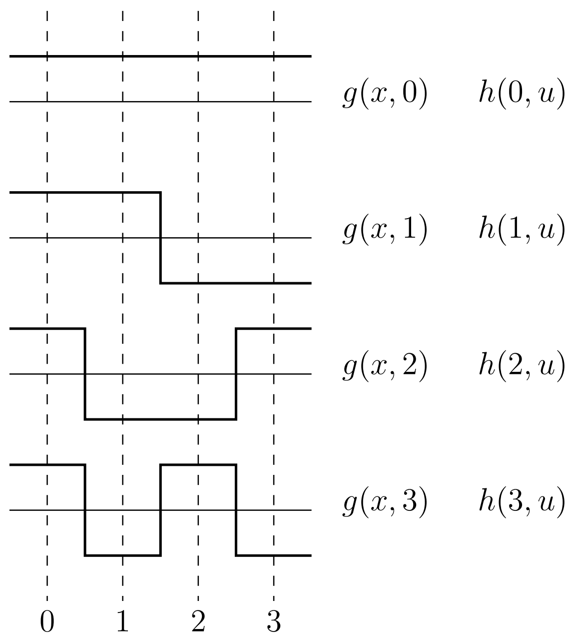
EEL 6562
Image Processing & Computer Vision
Walsh-Hadamard Transform Example



The forward and reverse transforms:

$$T(u) = \sum_{x=0}^{3} f(x)\, g(x,u)$$

$$f(x) = \sum_{u=0}^{3} T(u)\, h(x,u)$$

The g's (and also the h's) must satisfy an orthogonality condition:

$$\sum_{x=0}^{3} g(x,u_1)\, g(x,u_2) = \begin{cases} 1 & \text{if } u_1 = u_2; \\ 0 & \text{otherwise.} \end{cases}$$

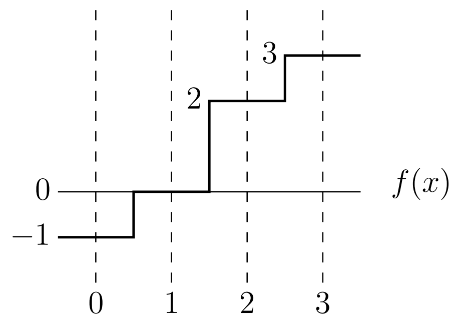
Confirm orthogonality for $u_1 = 1$:

$$\sum_{x=0}^{3} g(x,1)\, g(x,0) = (1/2)(1/2) + (1/2)(1/2) + (-1/2)(1/2) + (-1/2)(1/2) = 0$$

$$\sum_{x=0}^{3} g(x,1)\, g(x,1) = (1/2)(1/2) + (1/2)(1/2) + (-1/2)(-1/2) + (-1/2)(-1/2) = 1$$

$$\sum_{x=0}^{3} g(x,1)\, g(x,2) = (1/2)(1/2) + (1/2)(-1/2) + (-1/2)(-1/2) + (-1/2)(1/2) = 0$$

$$\sum_{x=0}^{3} g(x,1)\, g(x,3) = (1/2)(1/2) + (1/2)(-1/2) + (-1/2)(1/2) + (-1/2)(-1/2) = 0$$



Find the WHT of $f(x)$:

$$T(0) = \sum_{x=0}^{3} f(x)\, g(x,0) = (-1)(1/2) + (0)(1/2) + (2)(1/2) + (3)(1/2) = 2$$

$$T(1) = \sum_{x=0}^{3} f(x)\, g(x,1) = (-1)(1/2) + (0)(1/2) + (2)(-1/2) + (3)(-1/2) = -3$$

$$T(2) = \sum_{x=0}^{3} f(x)\, g(x,2) = (-1)(1/2) + (0)(-1/2) + (2)(-1/2) + (3)(1/2) = 0$$

$$T(3) = \sum_{x=0}^{3} f(x)\, g(x,3) = (-1)(1/2) + (0)(-1/2) + (2)(1/2) + (3)(-1/2) = -1$$

This means that $f(x)$ can be expressed as a weighted sum of the basis functions $h(x,u)$:

$$f(x) = (2)h(x,0) + (-3)h(x,1) + (0)h(x,2) + (-1)h(x,3)$$

We could have found the WHT of $f(x)$ using matrix multiplication:

$$\begin{bmatrix} 2 \\ -3 \\ 0 \\ -1 \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

Notice that the matrix A is symmetric; i.e. $A = A^T$.

We can confirm the orthogonality of all the basis functions at once by showing that $A^T A = I$:

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

At the same time, this shows that A is its own inverse; i.e. $A = A^{-1}$.