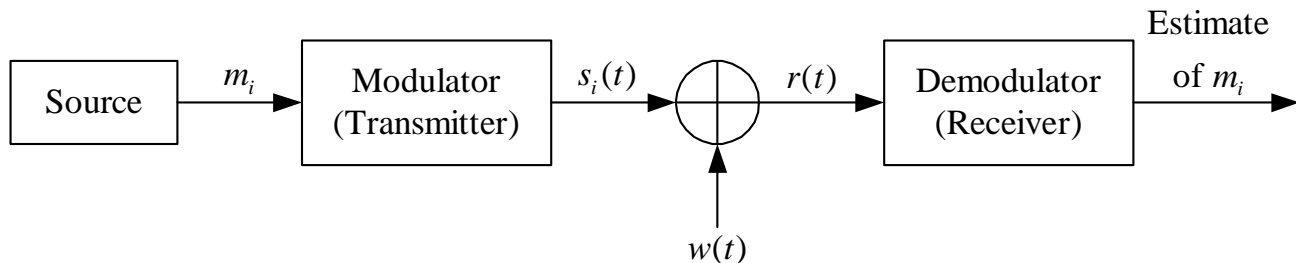# Signal Constellations, Optimum Receivers and Error Probabilities
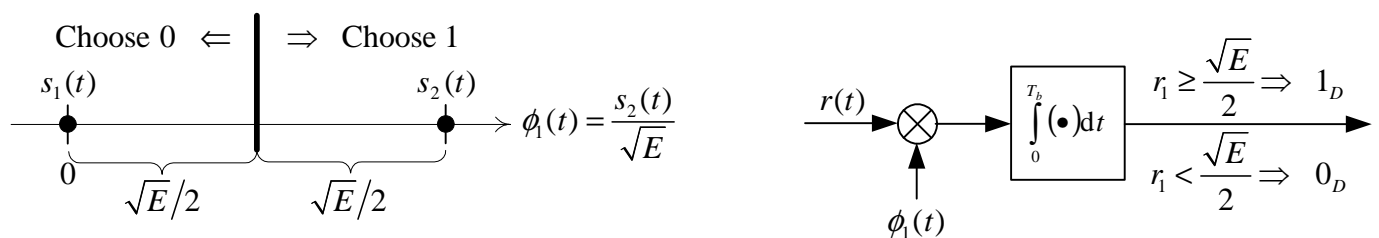

- There are M messages. For example, each message may carry $\lambda = \log_2 M$ bits. If the bit duration is T_b, then the message (or symbol) duration is $T_s = \lambda T_b$. The messages in two different time slots are statistically independent.

- The M signals $s_i(t)$, $i = 1, 2, \ldots, M$, can be arbitrary but of finite energies E_i. The a priori message probabilities are P_i. Unless stated otherwise, it is assumed that the messages are equally probable.

- The noise $w(t)$ is modeled as a *stationary*, zero-mean, and white Gaussian process with power spectral density of $N_0/2$.

- The receiver observes the received signal $r(t) = s_i(t) + w(t)$ over one symbol duration and makes a decision to what message was transmitted. The criterion for the optimum receiver is to minimize the probability of making an error.

- Optimum receivers and their error probabilities shall be considered for various signal constellations.
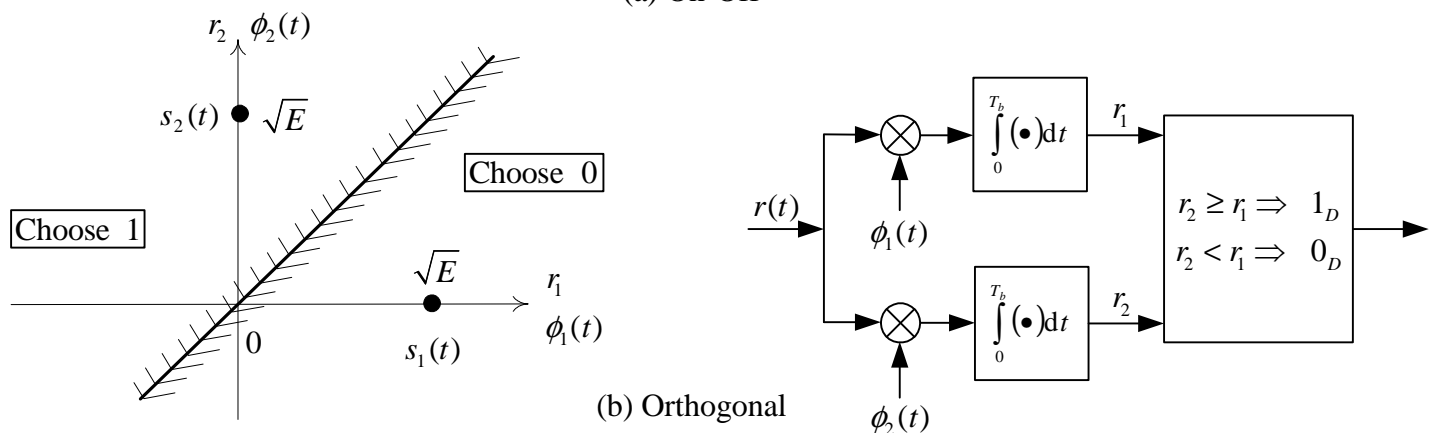
Binary Signalling

Received signal:

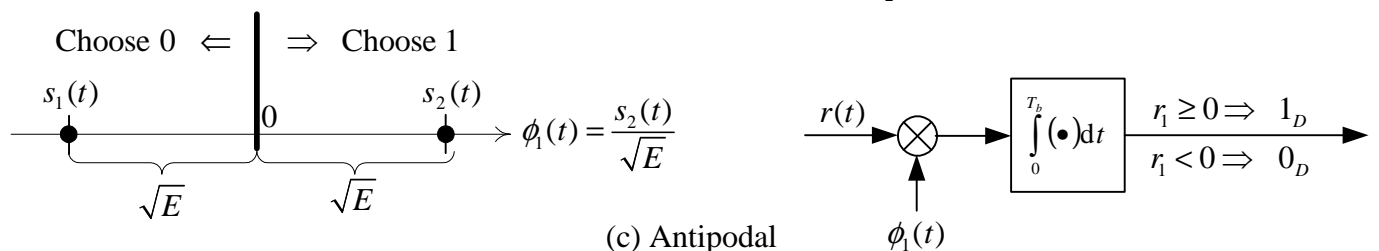|	On-Off	Orthogonal	Antipodal
H_1 (0):	$r(t) = w(t)$	$r(t) = s_1(t) + w(t)$	$r(t) = -s_2(t) + w(t)$
H_2 (1):	$r(t) = s_2(t) + w(t)$	$r(t) = s_2(t) + w(t)$	$r(t) = s_2(t) + w(t)$
		$\int_0^{T_s} s_1(t)s_2(t)\mathrm{d}t = 0$	

Decision space and receiver implementation:

Choose 0 $\Leftarrow$ | $\Rightarrow$ Choose 1

$s_1(t)$ $s_2(t)$

$\phi_1(t) = \dfrac{s_2(t)}{\sqrt{E}}$

0 $\sqrt{E}/2$ $\sqrt{E}/2$

$r(t)$ $\otimes$ $\int_0^{T_b}(\bullet)\mathrm{d}t$ $\quad r_1 \geq \dfrac{\sqrt{E}}{2} \Rightarrow 1_D$

$\phi_1(t)$ $\qquad r_1 < \dfrac{\sqrt{E}}{2} \Rightarrow 0_D$

(a) On-Off

r_2 $\phi_2(t)$

$s_2(t) \bullet \sqrt{E}$

Choose 0

Choose 1

$\sqrt{E}$ $\qquad r_1$

0 $\qquad \phi_1(t)$

$s_1(t)$

$r(t)$ $\otimes$ $\int_0^{T_b}(\bullet)\mathrm{d}t$ $\quad r_1$

$\phi_1(t)$

$r_2 \geq r_1 \Rightarrow 1_D$
$r_2 < r_1 \Rightarrow 0_D$

$\otimes$ $\int_0^{T_b}(\bullet)\mathrm{d}t$ $\quad r_2$

(b) Orthogonal $\qquad \phi_2(t)$

Choose 0 $\Leftarrow$ | $\Rightarrow$ Choose 1

$s_1(t)$ $s_2(t)$

0

$\phi_1(t) = \dfrac{s_2(t)}{\sqrt{E}}$

$\sqrt{E}$ $\sqrt{E}$

$r(t)$ $\otimes$ $\int_0^{T_b}(\bullet)\mathrm{d}t$ $\quad r_1 \geq 0 \Rightarrow 1_D$

$r_1 < 0 \Rightarrow 0_D$

(c) Antipodal $\qquad \phi_1(t)$

Error performance:

|	On-Off	Orthogonal	Antipodal
From decision space:	$Q\left(\frac{E}{2N_0}\right)$	$Q\left(\frac{E}{N_0}\right)$	$Q\left(\frac{2E}{N_0}\right)$
With the same E_b:	$Q\left(\frac{E_b}{N_0}\right)$	$Q\left(\frac{E_b}{N_0}\right)$	$Q\left(\frac{2E_b}{N_0}\right)$

The above shows that, with the same *average energy per bit* $E_b = \frac{1}{2}(E_1 + E_2)$, antipodal signalling is 3 dB more efficient than orthognal signalling, which has the same performance as that of on-off signalling. This is graphically shown below.
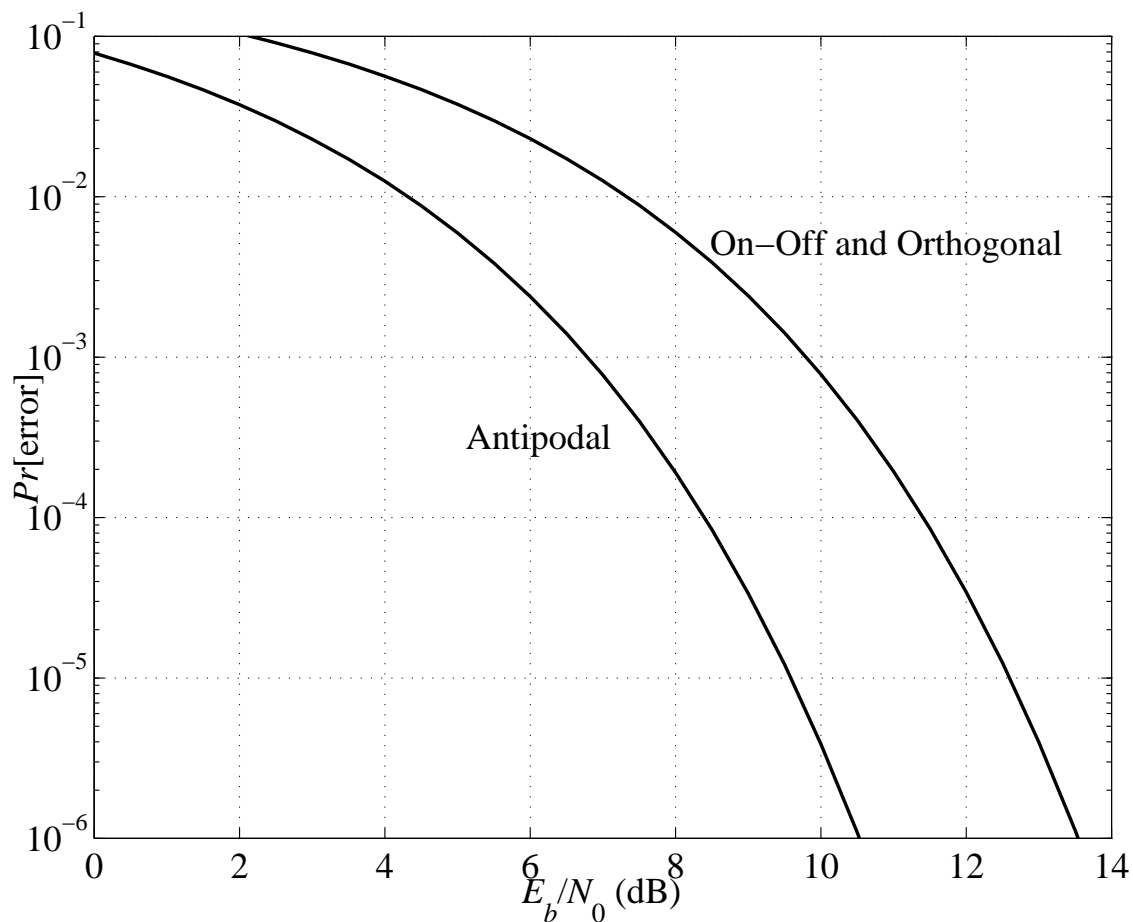
Example 1: In passband transmission, the digital information is encoded as a variation of the amplitude, frequency and phase (or their combinations) of a sinusoidal carrier signal. The carrier frequency is much *higher* than the highest frequency of the modulating signals (messages). Binary amplitude-shift keying (BASK), binary phase-shift keying (BPSK) and binary frequency-shift keying (BFSK) are examples of on-off, orthogonal and antipodal signallings, respectively.


(a) Binary Data

(b) Modulating Signal

(c) ASK Signal

(d) PSK Signal

(e) FSK Signal
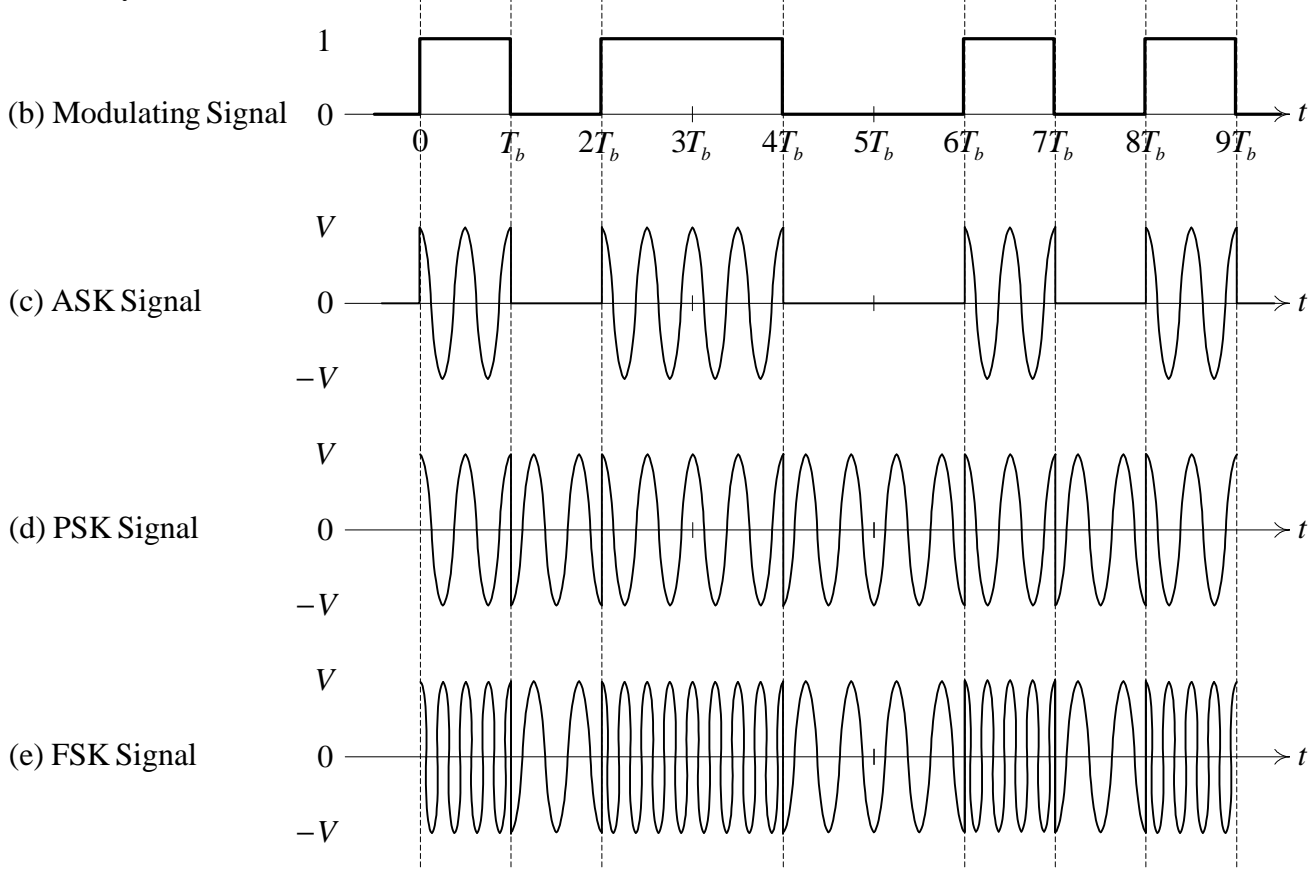
	BASK	BFSK	BPSK
$s_1(t)$:	0	$V\cos(2\pi f_1 t)$	$-V\cos(2\pi f_c t)$
$s_2(t)$:	$V\cos(2\pi f_c t)$	$V\cos(2\pi f_2 t)$	$V\cos(2\pi f_c t)$
$0 \le t \le T_b$	$f_c = \frac{n}{T_b}$	$f_2 + f_1 = \frac{n}{2T_b}$	$f_c = \frac{n}{T_b}$
	n is an integer	$f_2 - f_1 = \frac{m}{2T_b}$	n is an integer

$a_0 = 1 \qquad a_2 = 0 \qquad a_4 = 0 \qquad a_6 = 1$

$a_1 = 1 \qquad a_3 = 0 \qquad a_5 = 1 \qquad a_7 = 1$

(a) QPSK Signal

(b) OQPSK Signal

Example 2: Various binary baseband signaling schemes are shown below. The optimum receiver and its error performance follows easily once the two signals $s_1(t)$ and $s_2(t)$ used in each scheme are identified.
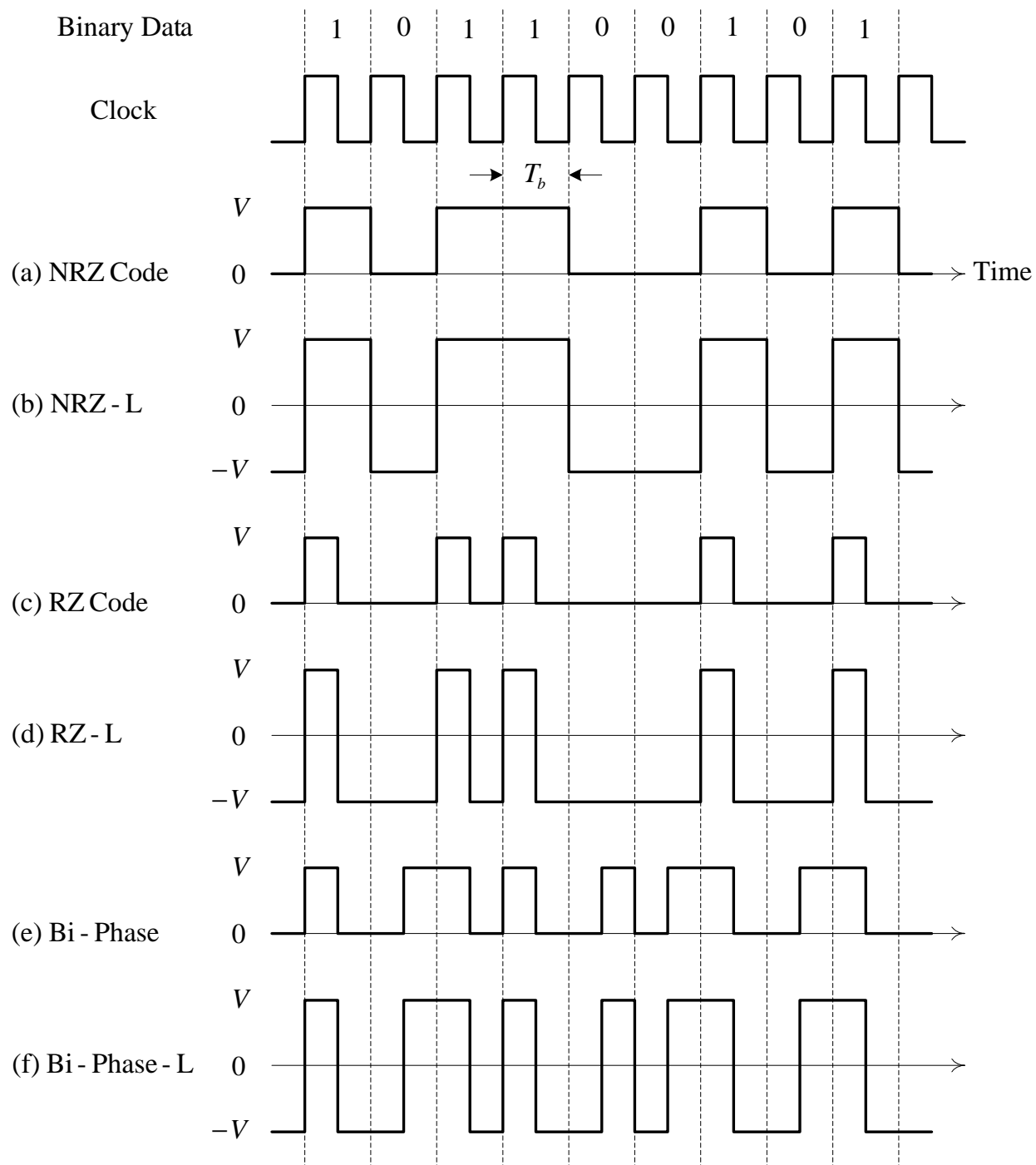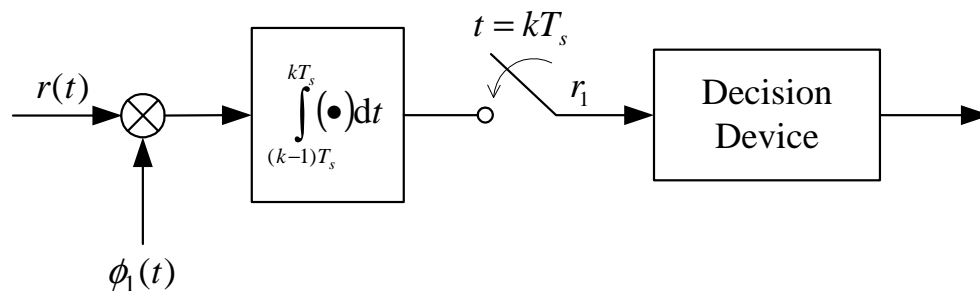
M-ary Pulse Amplitude Modulation

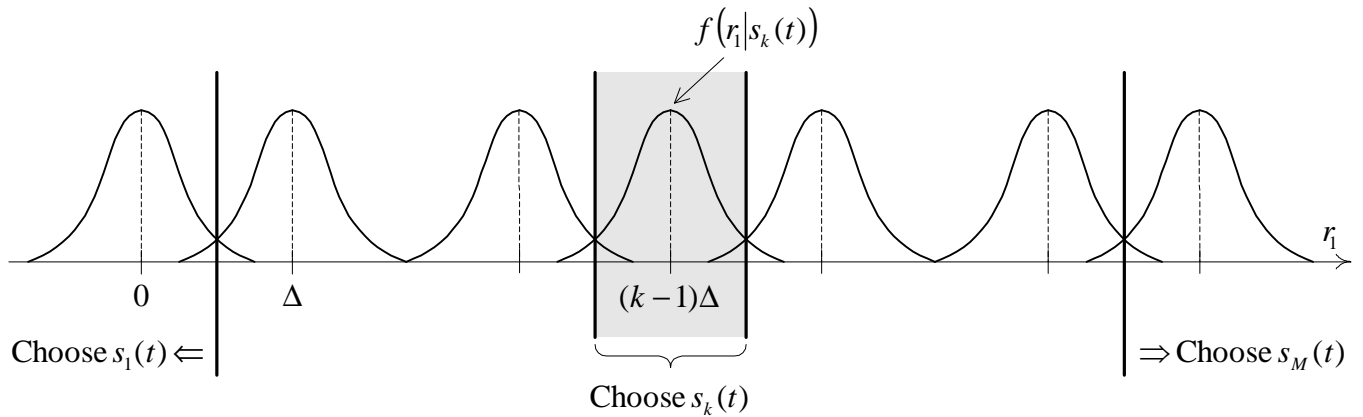$$s_i(t) = (i-1)\Delta\phi_1(t), \quad i = 1, 2, \ldots, M \tag{1}$$
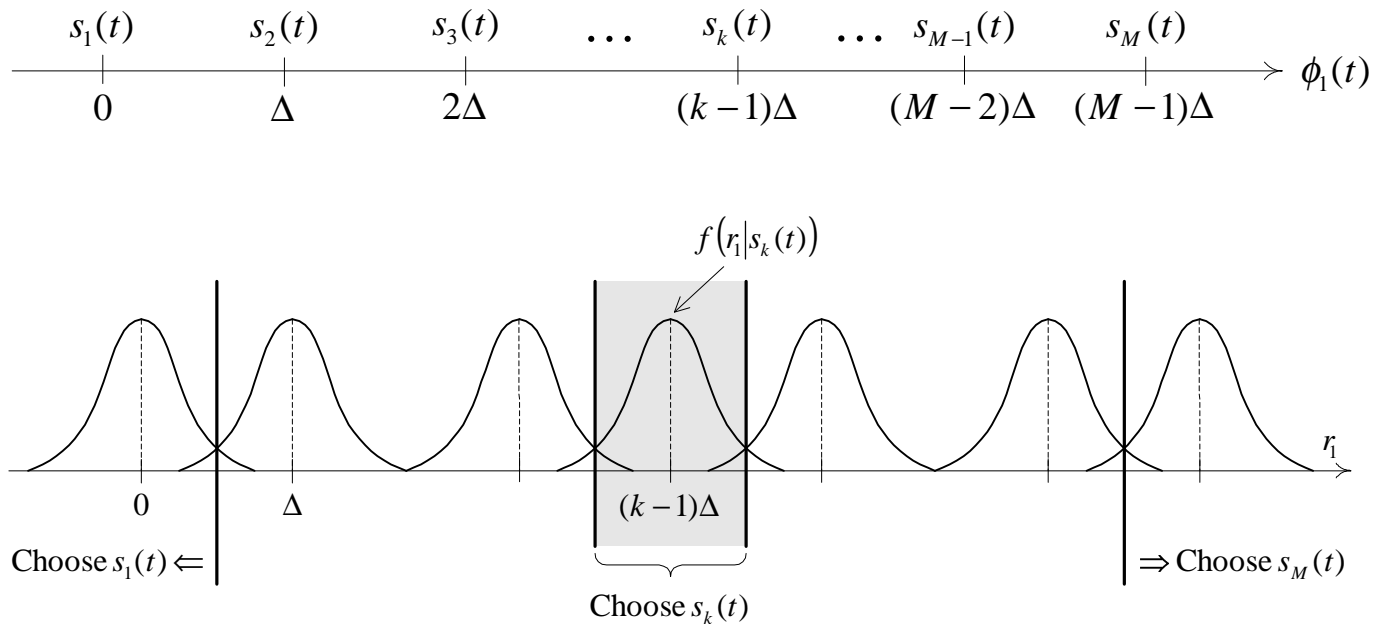
Decision rule:

choose $s_{k-1}(t)$ if $\left(k - \frac{3}{2}\right)\Delta < r_1 < \left(k - \frac{1}{2}\right)\Delta, \; k = 2, 3, \ldots, M-1$

choose $s_1(t)$ if $r_1 < \frac{\Delta}{2}$ and choose $s_M(t)$ if $r_1 > \left(M - \frac{3}{2}\right)\Delta$

$\Rightarrow$ The optimum receiver computes $r_1 = \int_{(k-1)T_s}^{kT_s} r(t)\phi_1(t)\mathrm{d}t$ and determines which signal r_1 is closest to:

Probability of error: $($ $)$

- For the $M-2$ inner signals: $\Pr[\text{error}|s_i(t)] = 2Q\left(\Delta/\sqrt{2N_0}\right)$.

- For the two end signals: $\Pr[\text{error}|s_i(t)] = Q\left(\Delta/\sqrt{2N_0}\right)$

- Since $\Pr[s_i(t) = 1/M$, then

$$\Pr[\text{error}] = \sum_{i=1}^{M} \Pr[s_i(t)]\Pr[\text{error}|s_i(t)] = \frac{2(M-1)}{M}Q\left(\Delta/\sqrt{2N_0}\right)$$

- The above is symbol (or message) error probability. If each message carries $\lambda = \log_2 M$ bits, the *bit error probability* depends on the mapping and it is often tedious to compute exactly. If the mapping is a Gray mapping (i.e., the mappings of the nearest signals differ in only one bit), a good approximation is $\Pr[\text{bit error} = \Pr[\text{error}]/\lambda M$.

- A modified signal set to minimize the average transmitted energy (M is even):
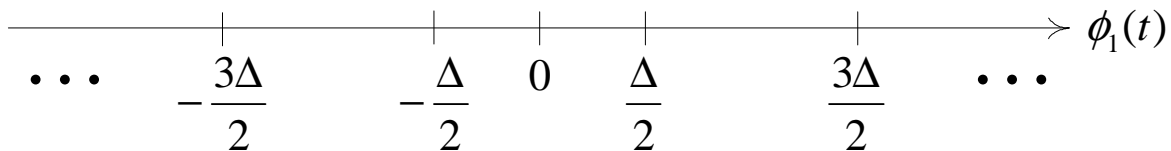
- If $\phi_1(t)$ is a sinusoidal carrier, i.e., $\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)$, where $0 \le t \le T_s$; $f_c = k/T_s$; k is an integer, the scheme is also known as M-ary amplitude shift keying (M-ASK).
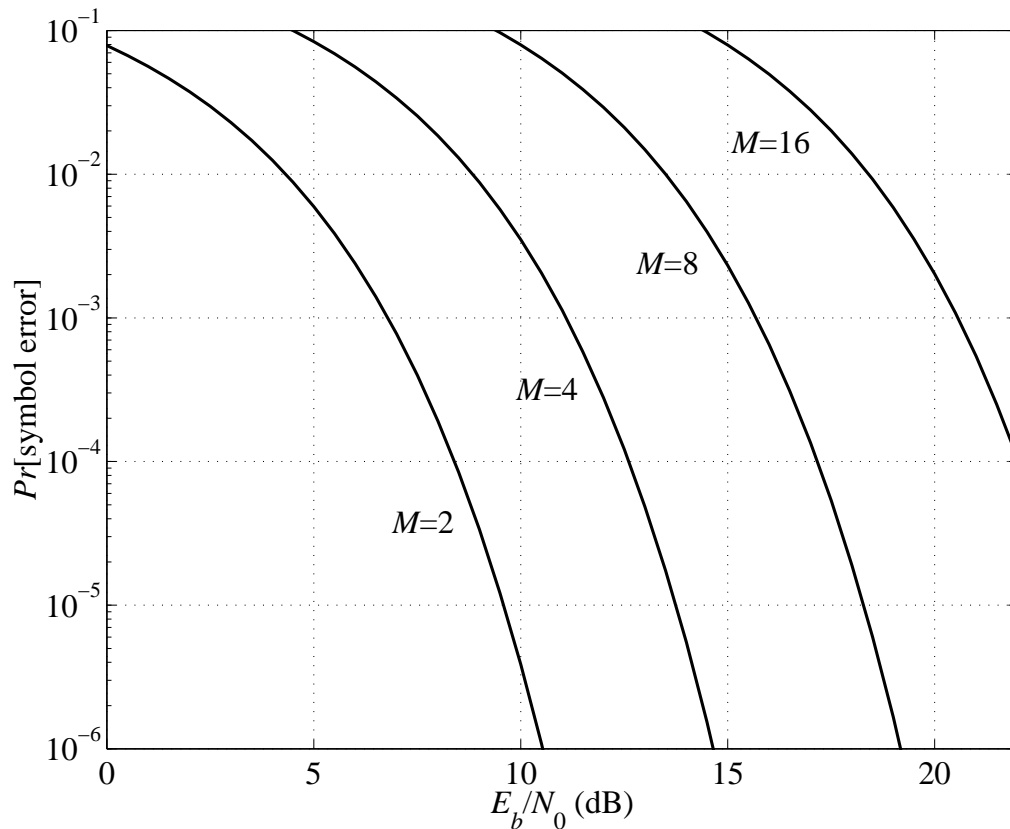
Compute

To express the probability in terms of E_b/N_0, compute the average transmitted energy per message (or symbol) as follows:

$$
\begin{aligned}
E_s &= \frac{\sum_{i=1}^{M} E_i}{M} = \frac{\Delta^2}{4M} \sum_{i=1}^{M} (2i - 1 - M)^2 \\
&= \frac{\Delta^2}{4M} \left[\frac{1}{3} M(M^2 - 1) \right] = \frac{(M^2 - 1)\Delta^2}{12}
\end{aligned}
\tag{2}
$$

Thus the the average transmitted energy per bit is

$$
E_b = \frac{E_s}{\log_2 M} = \frac{(M^2 - 1)\Delta^2}{12 \log_2 M} \Rightarrow \Delta = \sqrt{\frac{(12 \log_2 M) E_b}{M^2 - 1}}
\tag{3}
$$

$$
\Rightarrow \Pr[\text{error}] = \frac{2(M - 1)}{M} Q \left(\sqrt{\frac{(6 \log_2 M) E_b}{(M^2 - 1) N_0}} \right)
\tag{4}
$$

M-ary Phase Shift Keying (M-PSK)

The messages are distinguished by M phases of a sinusoidal carrier:

$$
\begin{aligned}
s_i(t) &= V \cos\left[2\pi f_c t - \frac{(i-1)2\pi}{M}\right], \ 0 < t < T_s; \ i = 1, 2, \ldots, M \\
&= \underbrace{\sqrt{E}\cos\left[\frac{(i-1)2\pi}{M}\right]}_{s_{i1}}\phi_1(t) + \underbrace{\sqrt{E}\sin\left[\frac{(i-1)2\pi}{M}\right]}_{s_{i2}}\phi_2(t) \qquad (5)
\end{aligned}
$$

where $E = V^2 T_s/2$ and the two orthonormal basis functions are:

$$
\phi_1(t) = \frac{V\cos(2\pi f_c t)}{\sqrt{E}}; \ \phi_2(t) = \frac{V\sin(2\pi f_c t)}{\sqrt{E}} \qquad (6)
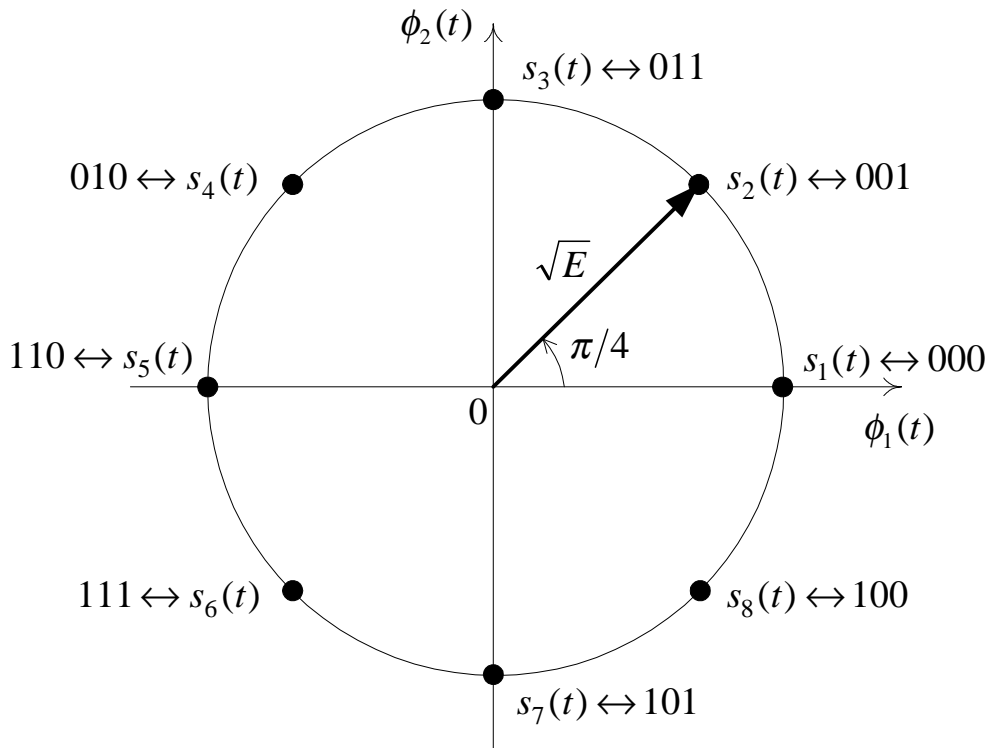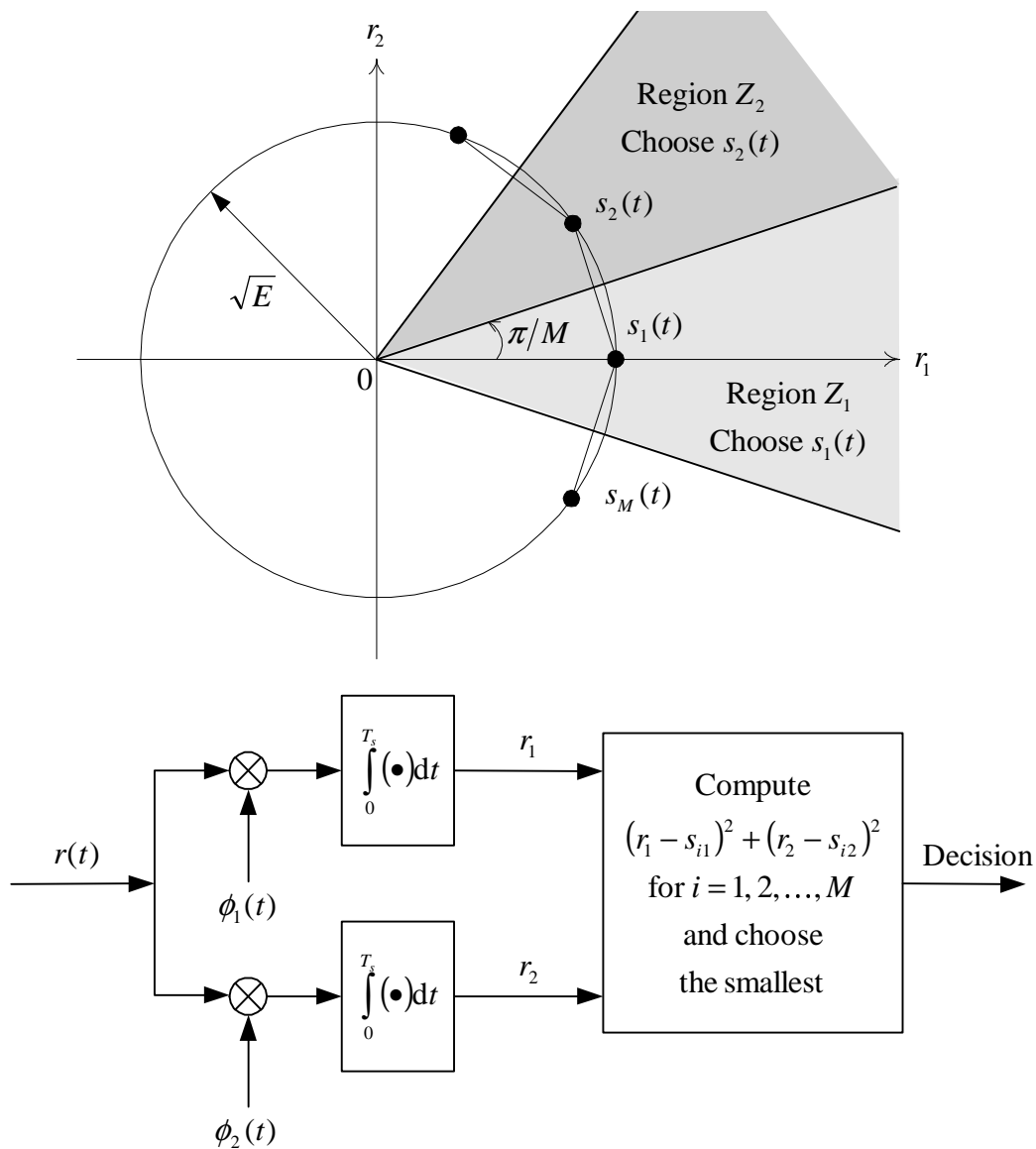$$

$\underline{M = 8}$:

Figure 1: Signal space for 8-PSK with Gray mapping.

Decision regions and receiver implementation:

Probability of error:

$$
\begin{aligned}
\Pr[\text{error}] &= \frac{1}{M} \sum_{i=1}^{M} \Pr[\text{error}|s_i(t)] = \Pr[\text{error}|s_1(t)] \\
&= 1 - \Pr[(r_1, r_2) \in Z_1 | s_1(t)] \\
&= 1 - \int\int_{(r_1, r_2) \in Z_1} p(r_1, r_2 | s_1(t)) \mathrm{d}r_1 \mathrm{d}r_2
\end{aligned}
\tag{7}
$$

where $p(r_1, r_2|s_1(t)) = \dfrac{1}{\pi N_0} \exp\left[-\dfrac{(r_1 - \sqrt{E})^2 + r_2^2}{N_0} \right]$.

Changing the variables $V = \sqrt{r_1^2 + r_2^2}$ and $\Theta = \tan^{-1}\frac{r_2}{r_1}$ (polar coordinate system), the join pdf of V and Θ is

$$p(V, \Theta) = \frac{V}{\pi N_0} \exp\left(-\frac{V^2 + E - 2\sqrt{E}V\cos\Theta}{N_0}\right) \tag{8}$$

Integration of $p(V, \Theta)$ over the range of V yields the pdf of Θ:

$$\begin{aligned} p(\Theta) &= \int_0^\infty p(V, \Theta)\mathrm{d}V \\ &= \frac{1}{2\pi}e^{-\frac{E}{N_0}\sin^2\Theta}\int_0^\infty V e^{-\left(V - \sqrt{2E/N_0}\cos\Theta\right)^2/2}\mathrm{d}V \end{aligned} \tag{9}$$

With the above pdf of Θ, the error probability can be computed as:

$$\begin{aligned} \Pr[\text{error}] &= 1 - \Pr[-\pi/M \le \Theta \le \pi/M|s_1(t)] \\ &= 1 - \int_{\pi/M}^{\pi/M} p(\Theta)\mathrm{d}\Theta \end{aligned} \tag{10}$$

In general, the integral of $p(\Theta)$ as above does not reduce to a simple form and must be evaluated numerically, except for $M = 2$ and $M = 4$.

An approximation to the error probability for large values of M and for large symbol signal-to-noise ratio (SNR) $\gamma_s = E/N_0$ can be obtained as follows. First the error probability is lower bounded by

$$\begin{aligned} \Pr[\text{error}] &= \Pr[\text{error}|s_1(t)] \\ &> \Pr[(r_1, r_2) \text{ is closer to } s_2(t) \text{ than } s_1(t)|s_1(t)] \\ &= Q\left\{\sin\left(\frac{\pi}{M}\right)\sqrt{E/N_0}\right\} \end{aligned} \tag{11}$$
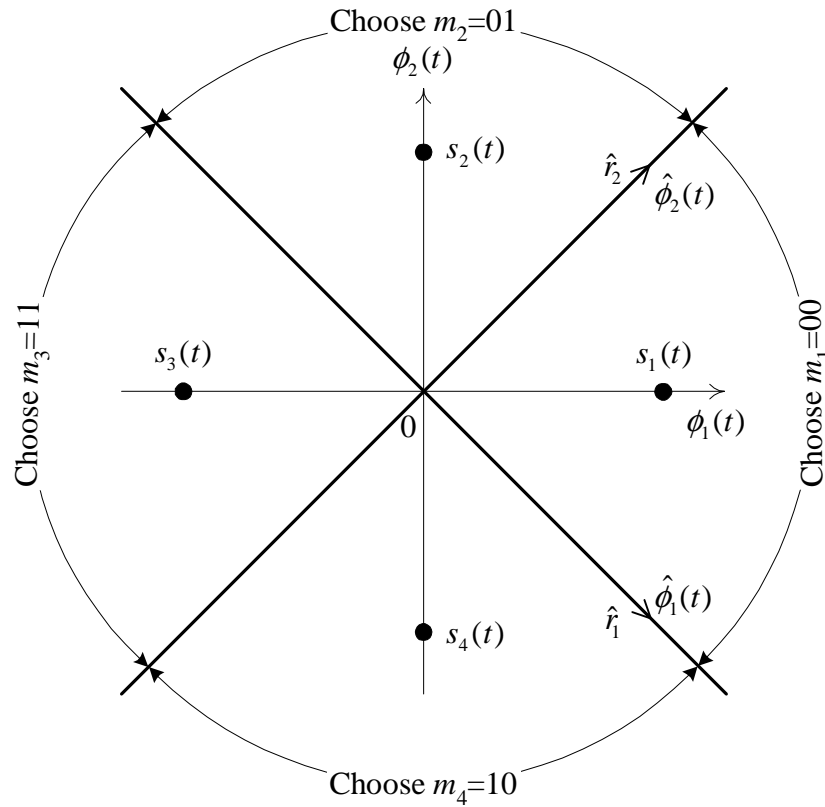
The upper bound is obtained by the following union bound:

$$\begin{aligned} \Pr[\text{error}] &= \Pr[\text{error}|s_1(t)] \\ &< \Pr[(r_1, r_2) \text{ is closer to } s_2(t) \text{ OR } s_{M-1}(t) \text{ than } s_1(t)|s_1(t)] \\ &< 2Q\left(\sin\left(\frac{\pi}{M}\right)\sqrt{E/N_0}\right) \end{aligned} \tag{12}$$

Since the lower and upper bounds only differ by a factor of two, they are very tight for high SNR. Thus a good approximation to the error probability of M-PSK is:

$$\begin{aligned} \Pr[\text{error}] &\approx 2Q\left(\sin\left(\frac{\pi}{M}\right)\sqrt{E/N_0}\right) \\ &= 2Q\left(\sqrt{\lambda(2E_b/N_0)}\sin\left(\frac{\pi}{M}\right)\right) \end{aligned} \tag{13}$$

Quadrature Phase Shift Keying (QPSK):

$$\text{Pr[correct]} = \text{Pr}[\hat{r}_1 \geq 0 \text{ and } \hat{r}_2 \geq 0 | s_1(t)] = \left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right]^2 \tag{14}$$

$$\Rightarrow \text{Pr[error]} = 1 - \text{Pr[correct]} = 1 - \left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right]^2$$

$$= 2Q\left(\sqrt{\frac{E}{N_0}}\right) - Q^2\left(\sqrt{\frac{E}{N_0}}\right) \tag{15}$$

The bit error probability of QPSK with Gray mapping: Can be obtained by considering different conditional message error probabilities:

$$\Pr[m_2|m_1] = Q\left(\sqrt{\frac{E}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right] \tag{16}$$

$$\Pr[m_3|m_1] = Q^2\left(\sqrt{\frac{E}{N_0}}\right) \tag{17}$$

$$\Pr[m_4|m_1] = Q\left(\sqrt{\frac{E}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right] \tag{18}$$

The bit error probability is therefore

$$\begin{aligned}\Pr[\text{bit error}] &= 0.5\Pr[m_2|m_1] + 0.5\Pr[m_4|m_1] + 1.0\Pr[m_3|m_1] \\ &= Q\left(\sqrt{\frac{E}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\end{aligned} \tag{19}$$

where the viewpoint is taken that one of the two bits is chosen at random, i.e., with a probability of 0.5. The above shows that QPSK with Gray mapping has exactly the same bit-error-rate (BER) performance with BPSK, while its bit rate can be double for the same bandwidth.

In general, the bit error probability of M-PSK is difficult to obtain for an arbitrary mapping. For Gray mapping, again a good approximation is $\Pr[\text{bit error}] = \Pr[\text{sumbol error}]/\lambda$. The exact calculation of the bit error probability can be found in the following paper:
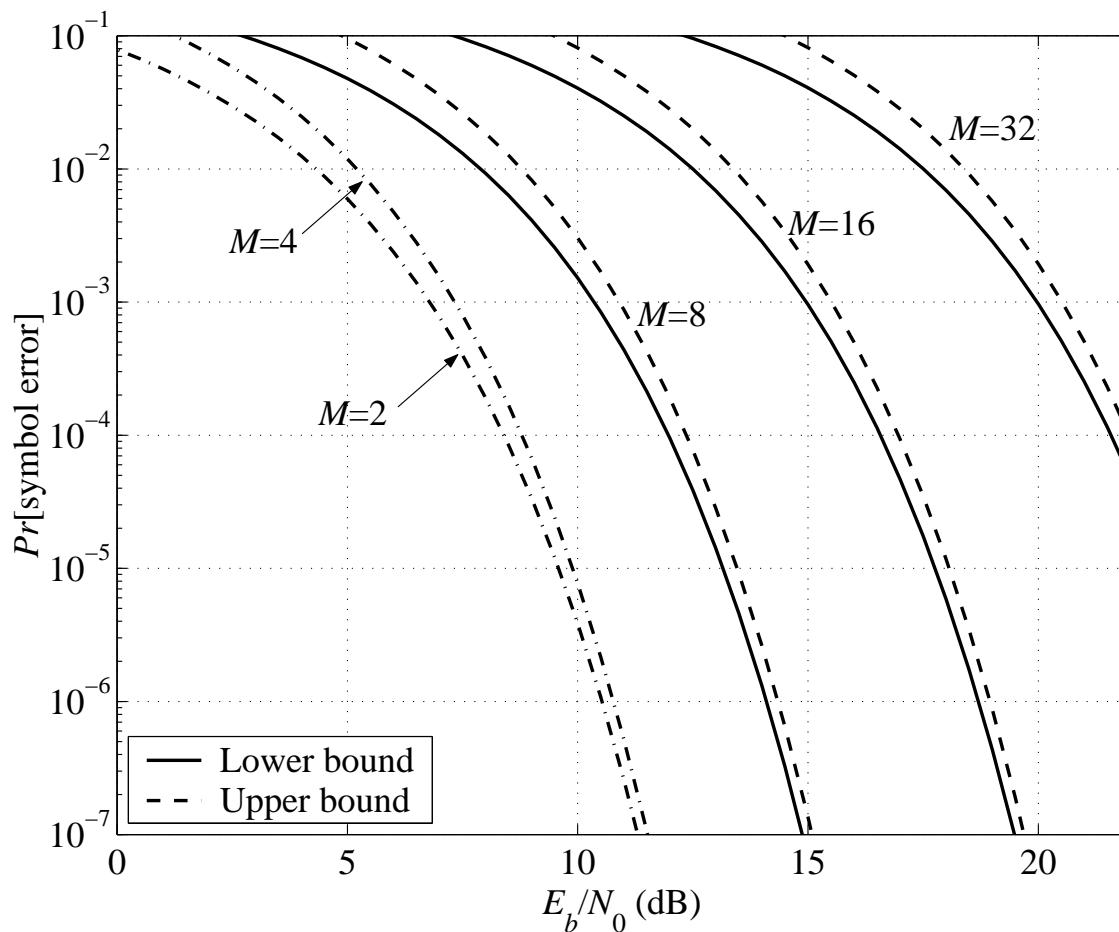
J. Lassing, E. G. Ström, E. Agrell and T. Ottosson, "Computation of the Exact Bit-Error Rate of Coherent M-ary PSK With Gray Code Bit Mapping", *IEEE Trans. on Communications*, vol. 51, Nov. 2003, pp. 1758–1760.

Comparison of M-PSK and BPSK

$$\Pr[\text{error}]_{M\text{-ary}} \simeq 2Q\left(\sqrt{\lambda(2E_b/N_0)}\sin\left(\frac{\pi}{M}\right)\right), \quad (M > 4)$$

$$\Pr[\text{error}]_{\text{QPSK}} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\Pr[\text{error}]_{\text{BPSK}} = Q(\sqrt{2E_b/N_0})$$

| λ	M	M-ary BW/binary BW	$\lambda\sin^2(\pi/M)$	M-ary energy/binary energy
3	8	1/3	0.44	3.6 dB
4	16	1/4	0.15	8.2 dB
5	32	1/5	0.05	13.0 dB
6	64	1/6	0.44	17.0 dB

M-ary Quadrature Amplitude Modulation

- In QAM modulation, the messages are encoded into both the amplitudes and phases of the carrier:

$$
s_i(t) = V_{\mathsf{c},i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) + V_{\mathsf{s},i}\sqrt{\frac{2}{T_s}}\sin(2\pi f_c t) \tag{20}
$$

$$
= \sqrt{E_i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t - \theta_i), \quad \begin{array}{l} 0 \leq t \leq T_s \\ i = 1, 2, \ldots, M \end{array} \tag{21}
$$

where $E_i = V_{\mathsf{c},i}^2 + V_{\mathsf{s},i}^2$ and $\theta_i = \tan^{-1}(V_{\mathsf{s},i}/V_{\mathsf{c},i})$.

- Examples of QAM constellations are shown on the next page.

- Rectangular QAM constellations, shown below, are the most popular. For rectangular QAM, $V_{\mathsf{c},i}, V_{\mathsf{s},i} \in \{(2i - 1 - M)\Delta/2\}$.
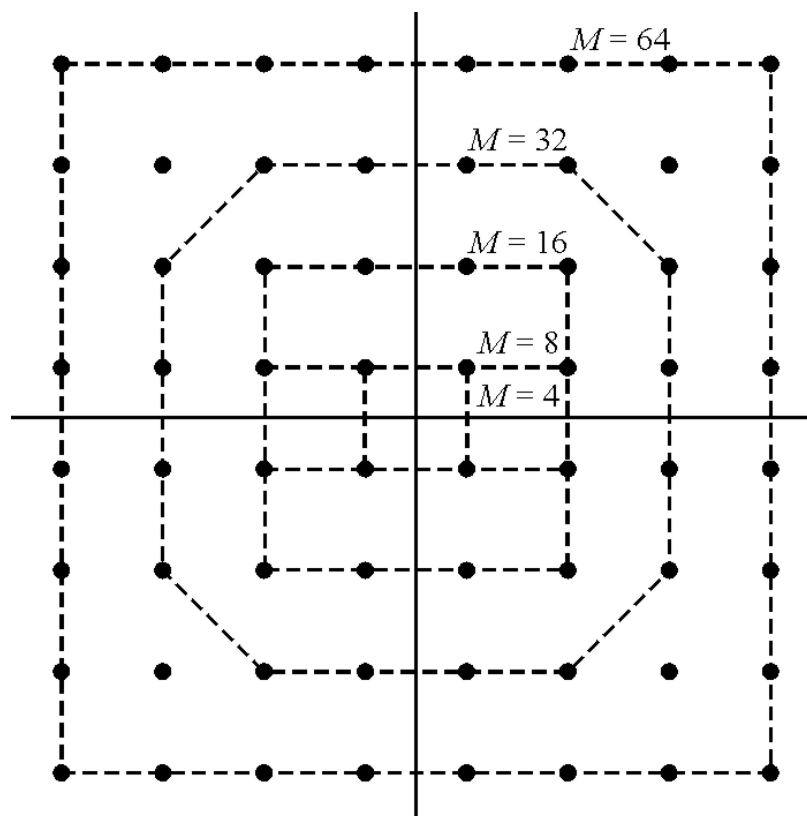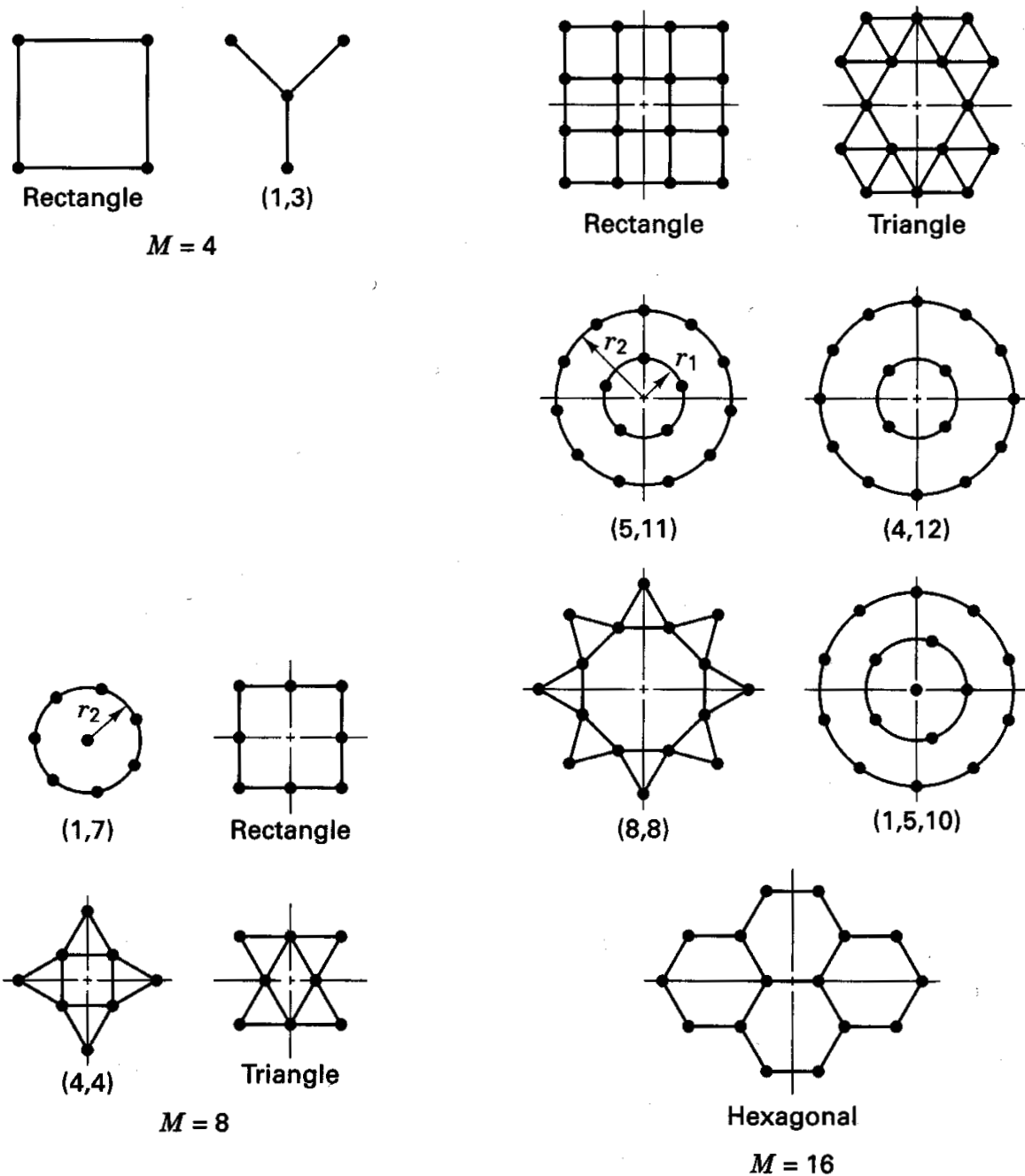
Examples of QAM Constellations:

Figure 9.17 *M*-ary symbol constellations. (Reprinted with permission from C. M. Thomas, M. Y. Weidner, and S. H. Durrani, "Digital Amplitude-Phase Keying with *M*-ary Alphabets," *IEEE Trans. Commun.*, vol. COM22, no. 2, Feb. 1974, Figs. 2 and 3, p. 170. © 1974 IEEE

Another example: The 16-QAM signal constellation shown below is an international standard for telephone-line modems (called V.29). The decision regions of the minimum distance receiver are also drawn.
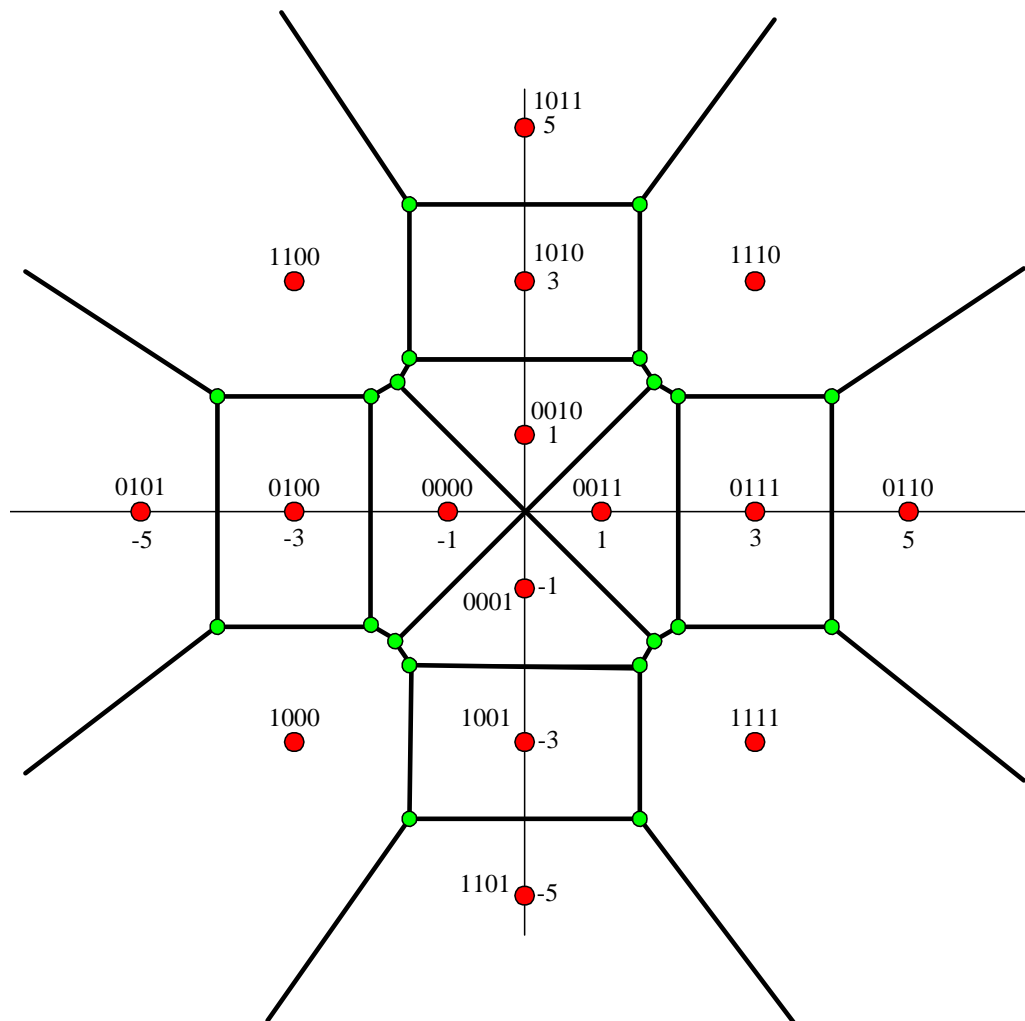
Receiver implementation (same as for M-PSK):
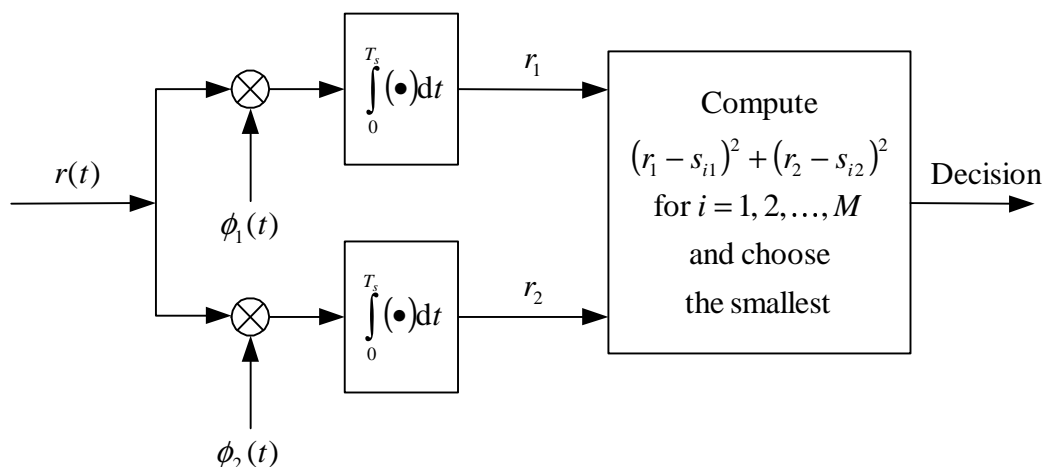
Error performance of rectangular M-QAM: For $M = 2^\lambda$, where λ is even, QAM consists of two ASK signals on quadrature carriers, each having $\sqrt{M} = 2^{\lambda/2}$ signal points. Thus the probability of symbol error can be computed as

$$\Pr[\text{error}] = 1 - \Pr[\text{correct}] = 1 - \left(1 - \Pr_{\sqrt{M}}[\text{error}]\right)^2 \tag{22}$$

where $\Pr_{\sqrt{M}}$ is the probability of error of $\sqrt{M}$-ary ASK:

$$\Pr_{\sqrt{M}}[\text{error}] = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right) \tag{23}$$

Note that in the above equation, E_s is the average energy per symbol of the QAM constellation.
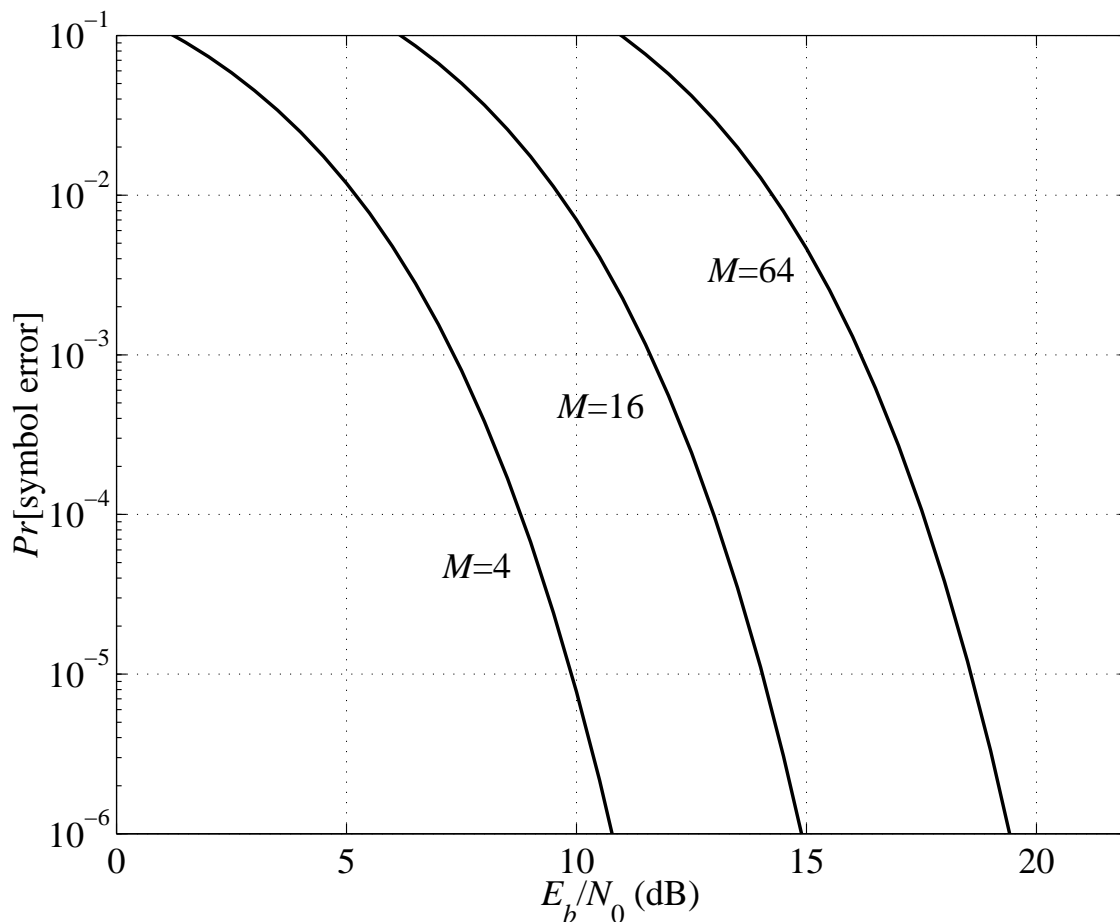
Comparison of M-QAM and M-PSK:

$$\Pr[\text{error}]_{\mathsf{PSK}} \approx Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\frac{\pi}{M}\right) \tag{24}$$

$$\Pr[\text{error}]_{\mathsf{PSK}} \approx \Pr_{\sqrt{M}-\mathsf{ASK}}[\text{error}] = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right) \tag{25}$$

Since the error probability is dominated by the argument of the Q-function, one may simply compares the arguments of Q for the two modulation schemes. The ratio of the arguments is

$$\Rightarrow \mathcal{R}_M = \frac{3/(M-1)}{2\sin^2(\pi/M)} \tag{26}$$

M	$10\log_{10}\mathcal{R}_M$
8	1.65 dB
16	4.20 dB
32	7.02 dB
64	9.95 dB

Thus M-ary QAM yields a better performance than M-ary PSK for the same bit rate (i.e., same M) and the same transmitted energy (E_b/N_0).

M-ary Orthognal Signals

$$s_i(t) = \sqrt{E}\phi_i(t), \quad i = 1, 2, \ldots, M \tag{27}$$

(a) M=2

(b) M=3
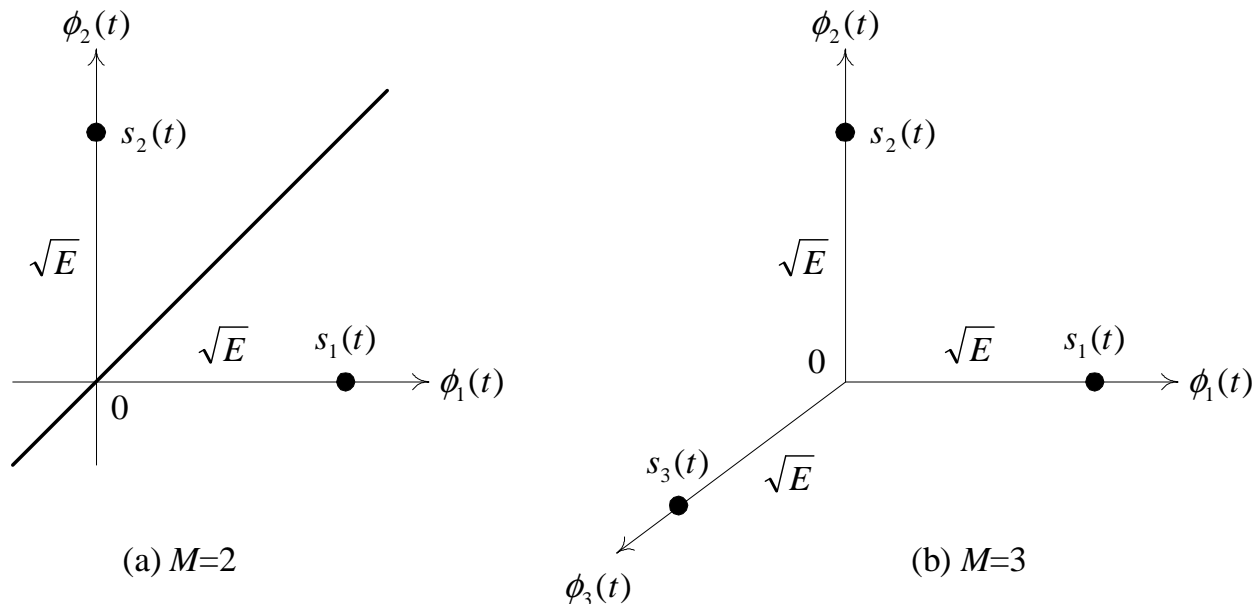
The decision rule of the minimum distance receiver follows easily:

$$\boxed{\text{Choose } s_i(t) \text{ if } r_i > r_j, \quad j = 1, 2, 3, \ldots, M; \ j \neq i} \tag{28}$$

<u>Minimum distance receiver</u>:
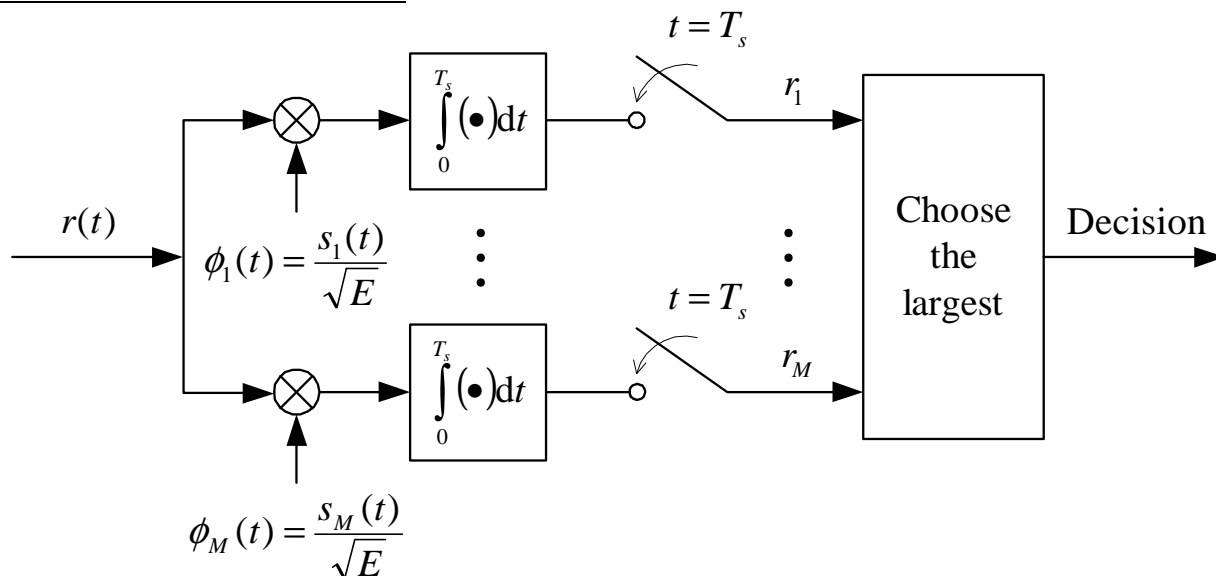
(*Note*: Instead of $\phi_i(t)$ one may use $s_i(t)$)

When the M orthonormal functions are chosen to be orthogonal sinusoidal carriers, the signal set is known as M-ary frequency shift keying (M-FSK):

$$s_i(t) = V \cos(2\pi f_i t), \ 0 \le t \le T_s; \ i = 1, 2, \ldots, M \tag{29}$$

where the frequencies are chosen so that the signals are orthogonal over the interval $[0, T_s]$ seconds:

$$f_i = (k \pm (i-1)) \left(\frac{1}{2T_s} \right), \quad i = 1, 2, \ldots, M \tag{30}$$

Error performance of M orthogonal signals:

To determine the message error probability consider that message $s_1(t)$ was transmitted. Due to the symmetry of the signal space and because the messages are equally likely

$$
\begin{aligned}
\mathrm{Pr[error]} &= \mathrm{Pr[error}|s_1(t)] = 1 - \mathrm{Pr[correct}|s_1(t)] \\
&= 1 - \mathrm{Pr[all} \ r_j < r_1) : j \ne 1|s_1(t)] \\
&= 1 - \prod_{j=2}^{M} \mathrm{Pr}[r_j < r_1) : j \ne 1|s_1(t)] \\
&= 1 - \int_{R_1=-\infty}^{\infty} \prod_{j=2}^{M} \mathrm{Pr}[(r_j < R_1)|s_1(t)]p(R_1|s_1(t))\mathrm{d}R_1 \\
&= 1 - \int_{R_1=-\infty}^{\infty} \left[\int_{R_2=-\infty}^{R_1} (\pi N_0)^{-0.5} \exp\left\{ -\frac{R_2^2}{N_0} \right\} \mathrm{d}R_2 \right]^{M-1} \\
&\quad \times (\pi N_0)^{-0.5} \exp\left\{ -\frac{(R_1 - \sqrt{E})^2}{N_0} \right\} \mathrm{d}R_1
\end{aligned}
\tag{31}
$$

The above integral can only be evaluated numerically. It can be normalized so that only two parameters, namely M (the number of messages) and E/N_0 (the signal-to-noise) enter into the numerical integration as given below:

$$
\begin{aligned}
\mathrm{Pr[error]} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\mathrm{d}x \right)^{M-1} \right] \\
&\quad \cdot \exp\left[-\frac{1}{2} \left(y - \sqrt{\frac{2E}{N_0}} \right)^2 \right] \mathrm{d}y
\end{aligned}
\tag{32}
$$

The relationship between probability of bit error and probability of symbol error for an M-ary orthogonal signal set can be found as follows. For equiprobable orthogonal signals, all symbol errors are equiprobable and occur with probability

$$\frac{\Pr[\text{symbol error}]}{M-1} = \frac{\Pr[\text{symbol error}]}{2^\lambda - 1}$$

There are $\binom{\lambda}{k}$ ways in which k bits out of λ may be in error. Hence the average number of bit errors per λ-bit symbol is

$$\sum_{k=1}^{\lambda} k \binom{\lambda}{k} \frac{\Pr[\text{symbol error}]}{2^\lambda - 1} = \lambda \frac{2^{\lambda-1}}{2^\lambda - 1} \Pr[\text{symbol error}] \qquad (33)$$

The probability of bit error is the the above result divided by λ. Thus

$$\frac{\Pr[\text{bit error}]}{\Pr[\text{symbol error}]} = \frac{2^{\lambda-1}}{2^\lambda - 1} \qquad (34)$$

Note that the above ratio approaches $1/2$ as $\lambda \to \infty$.
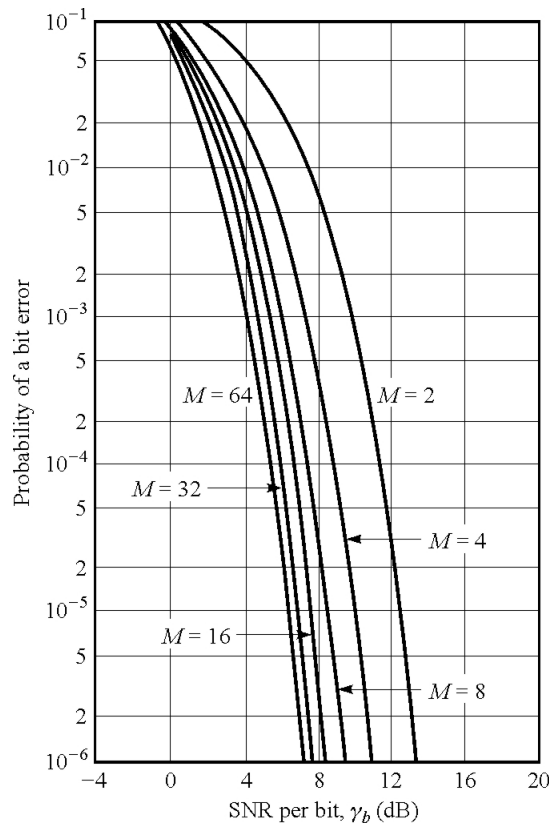

Figure 2: Probability of *bit error* for M-orthogonal signals versus $\gamma_b = E_b/N_0$.

Union Bound: To provide insight into the behavior of $\Pr[\text{error}]$, consider upper bounding (called the union bound) the error probability as follows.

$$
\begin{aligned}
\Pr[\text{error}] &= \Pr[(r_1 < r_2) \text{ or } (r_1 < r_3) \text{ or}, \ldots, \text{ or } (r_1 < r_M)|s_1(t)] \\
&< \Pr[(r_1 < r_2)|s_1(t)] + \ldots + \Pr[(r_1 < r_M)|s_1(t)] \\
&= (M-1)Q\left(\sqrt{E/N_0}\right) < MQ(\sqrt{E/N_0}) \\
&< Me^{-(E/2N_0)}
\end{aligned}
\tag{35}
$$

where a simple bound on $Q(x)$, namely $Q(x) < \exp\left\{-\frac{x^2}{2}\right\}$ has been used. Let $M = 2^\lambda$, then

$$
\Pr[\text{error}] < e^{\lambda \ln 2} e^{-(\lambda E_b/2N_0)} = e^{-\lambda(E_b/N_0 - 2\ln 2)/2}
\tag{36}
$$

The above equation shows that there is a definite threshold effect with orthogonal signalling. As $M \to \infty$, the probability of error **approaches zero** exponentially, provided that:

$$
\frac{E_b}{N_0} > 2\ln 2 = 1.39 = 1.42\text{dB}
\tag{37}
$$

A different interpretation of the upper bound in (36) can be obtained as follows. Since $E = \lambda E_b = P_s T_s$ (P_s is the transmitted power) and the bit rate $r_b = \lambda/T_s$, (36) can be rewritten as:

$$
\Pr[\text{error}] < e^{\lambda \ln 2} e^{-(P_s T_s/2N_0)} = e^{-T_s[-r_b \ln 2 + P_s/2N_0]}
\tag{38}
$$

The above implies that if $-r_b \ln 2 + P_s/2N_0 > 0$, or $\boxed{r_b < \dfrac{P_s}{2\ln 2 N_0}}$ the probability or error tends to zero as T_s or M become larger and larger. This behaviour of error probability is quite surprising since what it shows is that: provided the bit rate is small enough, the error probability can be made arbitrarily small even though the transmitter power can be finite. The obvious disadvantage, however, of the approach is that the bandwidth requirement increases with M. As $M \to \infty$, the transmitted bandwidth goes to infinity.

Since the union bound is not a very tight upper bound at sufficiently low SNR due to the fact that the upper bound $Q(x) < \exp\left\{-\frac{x^2}{2}\right\}$ for the Q function is loose. Using a more elaborate bounding techniques it can be shown that (see texts by Proakis, Wozencraft and Jacobs) $\Pr[\text{error}] \to 0$ as $\lambda \to \infty$, provided that $E_b/N_0 > \ln 2 = 0.693 \ (-1.6\text{dB})$.

Biorthognal Signals

(a) *M*=2   (b) *M*=3
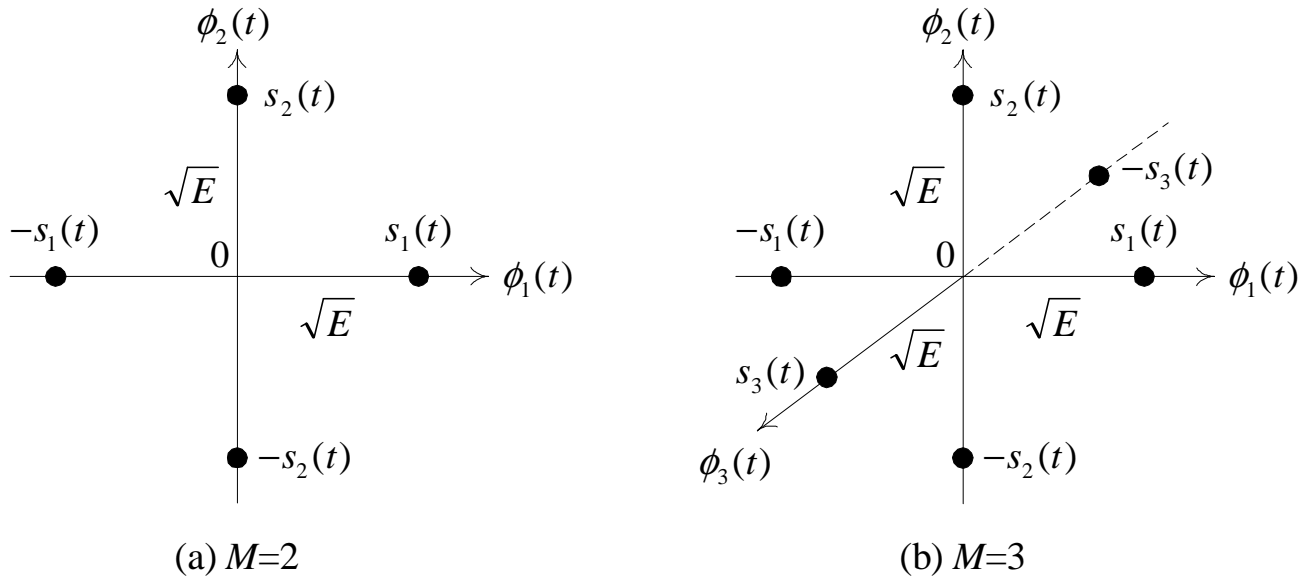
A biorthogonal signal set can be obtained from an original orthogonal set of N signals by augmenting it with the negative of each signal. Obviously, for the biorthogonal set $M = 2N$. Denote the additional signals by $-s_i(t)$, $i = 1, 2, \ldots, N$ and assume that each signal has energy E.

The received signal $r(t)$ is closer than $s_i(t)$ than $-s_i(t)$ if and only if (iff) $\int_0^{T_s} [r(t) - \sqrt{E}\phi_i(t)]^2 \mathrm{d}t < \int_0^{T_s} [r(t) + \sqrt{E}\phi_i(t)]^2 \mathrm{d}t$. This happens iff $r_i = \int_0^{T_s} r(t)\phi_i(t) \mathrm{d}t > 0$. Similarly, $r(t)$ is closer to $s_i(t)$ than to $s_j(t)$ iff $r_i > r_j$, $j \neq 1$ and $r(t)$ is closer to $s_i(t)$ than to $-s_j(t)$ iff $r_i > -r_j$, $j \neq 1$. It follows that the decision rule of the minimum-distance receiver for biorthogonal signalling can be implemented as

$$\boxed{\text{Choose } \pm s_i(t) \text{ if } |r_i| > |r_j| \text{ and } \pm r_i > 0, \quad \forall j \neq i} \tag{39}$$

The conditional probability of a correct decision for equally likely messages, given that $s_1(t)$ is transmitted and that

$$r_1 = \sqrt{E} + w_1 = R_1 > 0 \tag{40}$$

is just

$$
\begin{aligned}
\Pr[\text{correct}|s_1(t), R_1 > 0] &= \Pr[-R_1 < \text{all } r_j < R_1) : j \neq 1 | s_1(t), R_1 > 0] \\
&= (\Pr[-R_1 < r_j < R_1 | s_1(t), R_1 > 0])^{N-1} \\
&= \left[\int_{R_2=-R_1}^{R_1} (\pi N_0)^{-0.5} \exp\left\{ -\frac{R_2^2}{N_0} \right\} \mathrm{d}R_2 \right]^{N-1}
\end{aligned}
\tag{41}
$$

Averaging over the pdf of R_1 we have

$$\Pr[\text{correct}|s_1(t)] = \int_{R_1=0}^{\infty} \left[\int_{R_2=-\infty}^{R_1} (\pi N_0)^{-0.5} \exp\left\{ -\frac{R_2^2}{N_0} \right\} dR_2 \right]^{N-1}$$
$$\times (\pi N_0)^{-0.5} \exp\left\{ -\frac{(R_1 - \sqrt{E})^2}{N_0} \right\} dR_1 \qquad (42)$$

Again, by virtue of symmetry and the equal a priori probability of the messages the above equation is also the $\Pr[\text{correct}]$. Finally, by noting that $N = M/2$ we obtain

$$\Pr[\text{error}] = 1 - \int_{R_1=0}^{\infty} \left[\int_{R_2=-\infty}^{R_1} (\pi N_0)^{-0.5} \exp\left\{ -\frac{R_2^2}{N_0} \right\} dR_2 \right]^{\frac{M}{2}-1}$$
$$\times (\pi N_0)^{-0.5} \exp\left\{ -\frac{(R_1 - \sqrt{E})^2}{N_0} \right\} dR_1 \qquad (43)$$

The difference in error performance for M biorthogonal and M orthogonal signals is negligible when M and E/N_0 are large, but the number of dimensions (i.e., bandwidth) required is reduced by one half in the biorthogonal case.

Hypercube Signals (Vertices of a Hypercube)

(a) $M = 2$ (b) $M = 2^2 = 4$ (c) $M = 2^3 = 8$
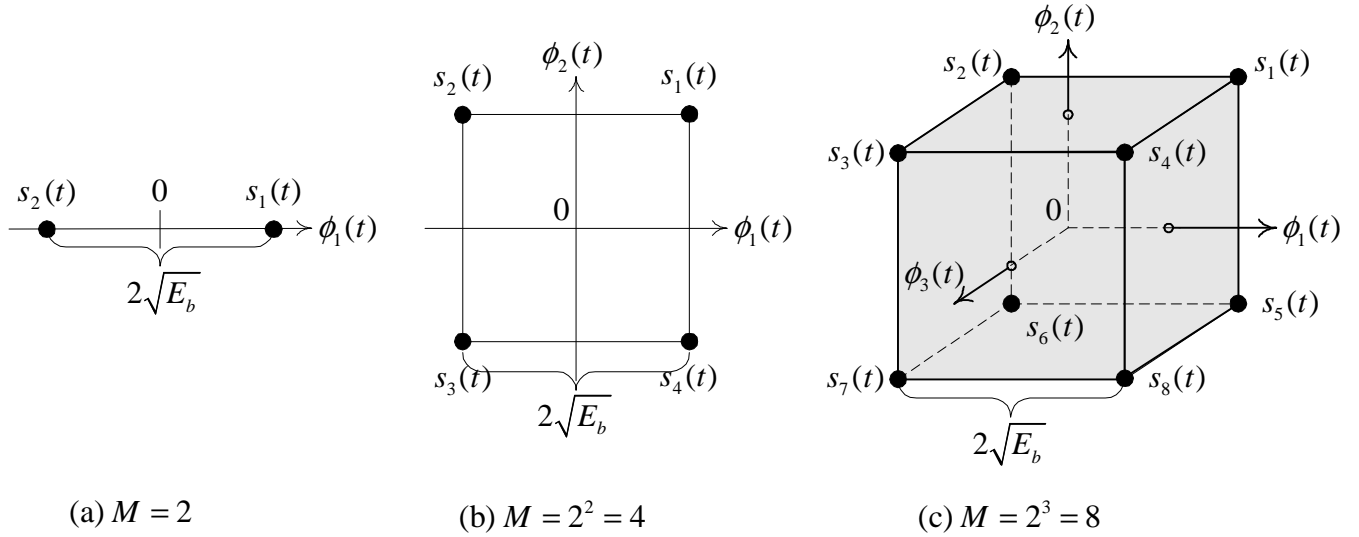
Here the $M = 2^\lambda$ signals are located on the vertices of an λ-dimensional hypercube centered at the origin. This configuration is shown geometrically above for $\lambda = 1, 2, 3$. The hypercube signals can be formed as follows:

$$s_i(t) = \sqrt{E_b} \sum_{j=1}^{\lambda} b_{ij} \phi_j(t), \quad b_{ij} \in \{\pm 1\}, i = 1, 2, \ldots, M = 2^\lambda \tag{44}$$

Thus the components of the signal vector $\mathbf{s}_i = [s_{i1}, s_{i2}, \ldots, s_{i\lambda}]^\top$ are $\pm\sqrt{E_b}$.

To evaluate the error probability, assume that the signal

$$\mathbf{s}_1 \stackrel{\triangle}{=} \left(-\sqrt{E_b}, -\sqrt{E_b}, \ldots, -\sqrt{E_b} \right) \tag{45}$$

is transmitted. First claim that no error is made if the noise components along $\phi_j(t)$ satisfy

$$w_j < \sqrt{E_b}, \text{ for all } j = 1, 2, \ldots, \lambda \tag{46}$$

The proof is immediate. When $\mathbf{x} = \mathbf{s}_1 + \mathbf{w}$ is received, the jth component of $\mathbf{x} - \mathbf{s}_i$ is:

$$(x_j - s_{ij}) = \begin{cases} w_j, & \text{if } s_{ij} = -\sqrt{E_b} \\ 2\sqrt{E_b} - w_j, & \text{if } s_{ij} = +\sqrt{E_b} \end{cases} \tag{47}$$

Since Equation (46) implies:

$$2\sqrt{E_b} - w_j > w_j \text{ for all } j, \tag{48}$$

it follows that

$$|\mathbf{x} - \mathbf{s}_i|^2 = \sum_{j=1}^{\lambda} (x_j - s_{ij})^2 > \sum_{j=1}^{\lambda} w_j^2 = |\mathbf{x} - \mathbf{s}_1|^2 \qquad (49)$$

for all $\mathbf{s}_i \neq \mathbf{s}_1$ whenever (46) is satisfied.

Next claim that an error is made if, for at least one j,

$$w_j > \sqrt{E_b} \qquad (50)$$

This follows from the fact that $\mathbf{x}$ is closer to $\mathbf{s}_j$ than to $\mathbf{s}_1$ whenever (50) is satisfied, where $\mathbf{s}_j$ denotes that signal with components $\sqrt{E_b}$ along the jth direction and $-\sqrt{E_b}$ in all other directions. (Of course, $\mathbf{x}$ may be still closer to some signal other than $\mathbf{s}_j$, but it cannot be closest to $\mathbf{s}_1$).

Equations (49) and (50) together imply that a correct decision is made if and only if (46) is satisfied. The probability of this event, given that $m = m_1$, is therefore:

$$
\begin{aligned}
\Pr[\text{correct}|m_1] &= \Pr\left[\text{all } w_j < \sqrt{E_b}; j = 1, 2, \ldots, \lambda\right] \\
&= \prod_{j=1}^{\lambda} \Pr\left[w_j < \sqrt{E_b}\right] = \left(1 - \Pr\left[w_j \geq \sqrt{E_b}\right]\right)^{\lambda} \\
&= (1 - p)^{\lambda} \qquad (51)
\end{aligned}
$$

in which,

$$p = \Pr\left[w_j \geq \sqrt{E_b}\right] = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

is again the probability of error for two equally likely signals separated by distance $2\sqrt{E_b}$. Finally, from symmetry:

$$\Pr[\text{correct}|m_i] = \Pr[\text{correct}|m_1] \text{ for all } i, \qquad (52)$$

hence,

$$\Pr[\text{correct}] = (1 - p)^{\lambda} = \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^{\lambda} \qquad (53)$$

In order to express this result in terms of signal energy, we again recognize that the distance squared from the origin to each signal $\mathbf{s}_i$ is the same. The transmitted energy is therefore independent of i, hence can be designated by E. Clearly

$$|\mathbf{s}_i|^2 = \sum_{j=1}^{\lambda} s_{ij}^2 = \lambda E_b = E \qquad (54)$$

Thus

$$p = Q\left(\sqrt{\frac{2E}{\lambda N_0}}\right) \tag{55}$$

The simple form of the result $\Pr[\text{correct}] = (1 - p)^\lambda$ suggests that a more immediate derivation may exist. Indeed one does. Note that the jth coordinate of the random signal s is a priori equally likely to be $+\sqrt{E_b}$ or $-\sqrt{E_b}$, independent of all other coordinates. Moreover, the noise w_j disturbing the jth coordinate is independent of the noise in all other coordinates. Hence, by the theory of sufficient statistics, a decision may be made on the jth coordinate without examining any other coordinate. This single-coordinate decision corresponds to the problem of binary signals separated by distance $2\sqrt{E_b}$, for which the probability of correct decision is $1 - p$. Since in the original hypercube problem a correct decision is made if only if a correct decision is make on every coordinate, an since these decisions are independent, it follows immediately that $\Pr[\text{correct}] = (1 - p)^\lambda$.