HW#6 SOLUTIONS

① $\int_0^1 kx^2(1-x^3)\,dx = 1 \Rightarrow k = 6$

② $f(x) = \frac{x}{2}$; $0 < x < 2$ $\quad P(X>1) = \int_1^2 \frac{x}{2}\,dx = \frac{3}{4}$

a) P(both observations >1) $= \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}$

b) P(exactly 2 out of 3 are greater than 1) $= \binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) = \frac{27}{64}$

③ a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 6x(1-x)\,dx = 1$ ✓

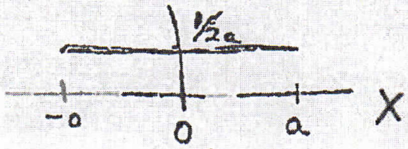
b) $P(X<b) = \int_0^b 6x(1-x)\,dx = b^2(3-2b)$

$2P(X>b) = 2[1-b^2(3-2b)]$

Setting them equal and solving for b gives $\doteq .615$

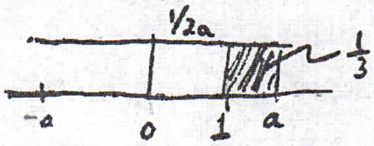
c) $P\left(X<\frac{1}{2} \,\middle|\, \frac{1}{3}<X<\frac{2}{3}\right) = \dfrac{P\left(\frac{1}{3}<X<\frac{1}{2}\right)}{P\left(\frac{1}{3}<X<\frac{2}{3}\right)} = \frac{1}{2}$

④



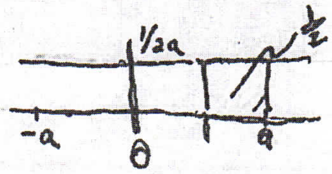
$\Rightarrow f(x) = \frac{1}{2a}$; $-a \le x \le a$

a)



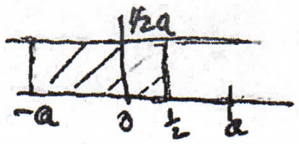
$\Rightarrow (a-1)\frac{1}{2a} = \frac{1}{3} \Rightarrow a = 3$

b)



$\Rightarrow (a-1)\frac{1}{2a} = \frac{1}{2} \Rightarrow$ IMPOSSIBLE

c)



$\Rightarrow \left(a+\frac{1}{2}\right)\frac{1}{2a} = .7 \Rightarrow a = 1.25$

Let T = rainfall duration (random variable)

(5.) Mean value of 2.725 for the exponential distribution implies

$$\lambda = \frac{1}{2.725} = .367$$

a) $P(T \geq 2) = \int_2^{\infty} .367 e^{-.367t}\,dt = e^{-.367(2)} \doteq .48$

b) $P(T \leq 3) = \int_0^{3} .367 e^{-.367t}\,dt = 1 - e^{-.367(3)} \doteq .67$

c) $P(2 \leq T \leq 3) = P(T \leq 3) - P(T \leq 2) = .67 - .52 \doteq .15$

(6) Let T = waiting time between particles
Then $T \sim$ exponential with $\lambda = 3$

a) $P(T \leq 2) = \int_0^{2} 3e^{-3t}\,dt = 1 - e^{-3(2)} \doteq .997$ METHOD 1

b) METHOD 1: Use T: exponential time between particles
Since the particles arrive independently, it is irrelevant that it has been at least 6 seconds since last particle.

$P(\text{nothing arrives in next 6 seconds}) = P(T \geq 6) = \int_6^{\infty} 3e^{-3t}\,dt$

$= e^{-3(6)} = e^{-18}$ = negligible 1.5×10^{-8}

METHOD 2: Use X = # of arrivals per second and t = 6 sec.
$\sim$ Poisson with λ = 3 arrivals / second

so: P(nothing arrives in next 6 seconds) = P(0 arrivals in next 6 seconds)

$= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-3(6)} = e^{-18}$ negligible 1.5×10^{-8} as in METHOD 1

a) or Method 2: Let X = # arrivals per second
X ~ Poisson (λ = 3 arr/sec.)

$P(T \leq 2) = P(X \geq 1 \text{ in 2 seconds}) = 1 - P(X = 0 \text{ arrivals in 2 seconds}) = 1 - \frac{e^{-3(2)}[(3)(2)]^0}{0!}$

$= .997$

t = 2 seconds

(7.) 2 accidents/week = .4 accidents/day = λ

a) Let T = time between accidents
then T ~ exponential with $\lambda = .4$
T ~ $.4e^{-.4t}$

$P(T>3) = \int_3^\infty .4e^{-.4t}\,dt = e^{-.4(3)} = e^{-1.2} = .301$

b) If Y = time to 3rd accident
then
$P(Y>5) = \underbrace{P(\text{fewer than 3 accidents in 5 days})}_{\text{a Poisson question}}$

Let X = # accidents $\longrightarrow$ X ~ Poisson(λ) with $\lambda = .4$

P(Y>5) = P(X = 0 accidents in 5 days)
+ P(X = 1 accident in 5 days)
+ P(X = 2 accidents in 5 days)

$$= \frac{[(.4)(5)]^0 e^{-.4(5)}}{0!} + \frac{[.4(5)]^1 e^{-.4(5)}}{1!} + \frac{[(.4)(5)]^2 e^{-.4(5)}}{2!}$$

= .135 + .27 + .27

$\doteq$ .675

8. a) since $\int_{\text{all } x} f(x)\,dx$ must equal 1,
we have

$$\int_1^\infty \frac{k}{x^4}\,dx = \frac{k\,x^{-3}}{-3}\Big|_{x=1}^{x=\infty} = 0 - \left(-\frac{k}{3}\right) = 1 \Rightarrow k = 3$$

b) $$F(x) = P(X \le x) = \int_1^x \frac{3}{s^4}\,ds = \frac{3s^{-3}}{-3}\Big|_{s=1}^{s=x} = -x^{-3} + 1 = 1 - x^{-3}$$

$$\text{or } = 1 - \frac{1}{x^3}$$

c) we can of course compute these probabilities using the pdf, but to use the cdf:

$$P(X > 2) = 1 - P(X \le 2) = 1 - F(2) = 1 - \left(1 - \frac{1}{2^3}\right) = \frac{1}{8} = .125$$

$$P(2 < X < 3) = P(X \le 3) - P(X \le 2) = F(3) - F(2)$$
$$= \left(1 - \frac{1}{3^3}\right) - \left(1 - \frac{1}{2^3}\right) = .088$$

d) ***NOTE: similar problem on pages 2 and 3 of Lecture 6 notes

standard deviation of $X = \sqrt{Var(X)}$

where $V(X) = E(X^2) - [E(X)]^2$

so: $$E(X^2) = \int x^2 f(x)\,dx = \int_1^\infty x^2 \cdot \frac{3}{x^4}\,dx = \int_1^\infty 3x^{-2}\,dx$$
$$= \frac{3x^{-1}}{-1}\Big|_{x=1}^{x=\infty} = -\frac{3}{x}\Big|_{x=1}^{x=\infty} = 3$$

and $$E(X) = \int x f(x)\,dx = \int_1^\infty x \frac{3}{x^4}\,dx = \int_1^\infty 3x^{-3}\,dx = \frac{3x^{-2}}{-2}\Big|_1^\infty$$
$$= -\frac{3}{2x^2}\Big|_1^\infty = \frac{3}{2}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

and standard deviation of $X = \sqrt{V(X)} = \sqrt{\frac{3}{4}} = .866$