Advanced Macroeconomics: Het.

Assignment 1

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1 Setup

First I define the **stationary equilibrium** for the model:

For a given Γ_{ss}, ϕ^0_{ss}, and ϕ^1_{ss} the steady state is

1. Quantities K_{ss}, L^0_{ss}, and L^1_{ss},
2. Prices r_{ss}, w^0_{ss}, and w^1_{ss},
3. A distribution D_{ss} over z_{t-1} and a_{t-1}
4. And policy functions $a^*_{ss}(z_t, a_{t-1})$, $\ell^{0*}_{ss}(z_t, a_{t-1})$, $\ell^{1*}_{ss}(z_t, a_{t-1})$, and $c^*_{ss}(z_t, a_{t-1})$

Such that

1. Firms maximize profits:

$$
\begin{aligned}
r_{ss} &= \alpha \Gamma_{ss} K_{ss}^{\alpha-1} (L^0_{ss})^{\frac{1-\alpha}{2}} (L^1_{ss})^{\frac{1-\alpha}{2}} - \delta \\
w^0_{ss} &= \frac{1-\alpha}{2} \Gamma_{ss} K^\alpha_{ss} (L^0_{ss})^{-\frac{1+\alpha}{2}} (L^1_{ss})^{\frac{1-\alpha}{2}} \\
w^1_{ss} &= \frac{1-\alpha}{2} \Gamma_{ss} K^\alpha_{ss} (L^0_{ss})^{\frac{1-\alpha}{2}} (L^1_{ss})^{-\frac{1+\alpha}{2}}
\end{aligned}
$$

2. $a^*_{ss}(\cdot)$, $\ell^{0*}_{ss}(\cdot)$, $\ell^{1*}_{ss}(\cdot)$, and $c^*_{ss}(\cdot)$ solves the household problem with $\{r_{ss}, w^0_{ss}, w^1_{ss}\}_{t=0}^\infty$
3. $\mathbf{D_{ss}} = \mathbf{\Lambda}'_{\mathbf{ss}} \mathbf{\Pi}'_{\mathbf{ss}} \mathbf{D_{ss}}$ is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfies with $A_{ss} = K_{ss}$
5. The capital market clears $A_{ss} = \int a^*_{ss}(\beta_i, \phi^0, \phi^1, z_t, a_{t-1}) d\mathbf{D_{ss}}$
6. Labor market 0 clears $L^0_{ss} = \int \ell^{0*}_{ss}(\beta_i, \phi^0, z_t, a_{t-1}) d\mathbf{D_{ss}}$
7. Labor market 1 clears $L^1_{ss} = \int \ell^{1*}_{ss}(\beta_i, \phi^1, z_t, a_{t-1}) d\mathbf{D_{ss}}$
8. The goods market clears $Y_{ss} = \int c^*_{ss}(\beta_i, \phi^0, \phi^1, z_t, a_{t-1}) d\mathbf{D_{ss}}$

Next I define the **transition path** for the model:

1. Shocks $\mathbf{Z} = \{\mathbf{\Gamma}, \boldsymbol{,} \boldsymbol{\phi^0}, \boldsymbol{\phi^1}\}$
2. Unknowns $\mathbf{U} = \{\mathbf{K}, \mathbf{L^0}, \mathbf{L^1}\}$
3. Targets: $\left\{\mathbf{A_t} - \mathbf{A^{hh}_t}\right\}$ (asset market clearing), $\left\{\mathbf{L^0_t} - \mathbf{L^{0,hh}_t}\right\}$, and $\left\{\mathbf{L^1_t} - \mathbf{L^{1,hh}_t}\right\}$ (clearing of both labor markets).
4. Aggregate variables: $\mathbf{X} = \left\{\mathbf{\Gamma}, \mathbf{K}, \mathbf{r}, \mathbf{w^0}, \mathbf{w^1}, \mathbf{L^0}, \mathbf{L^1}, \mathbf{C}, \mathbf{Y}, \mathbf{A}, \mathbf{A^{hh}}, \mathbf{C^{hh}}, \mathbf{L^{0,hh}}, \mathbf{L^{1,hh}}\right\}$

5. Household inputs: $\mathbf{X_t^{hh}} = \{\mathbf{r}, \mathbf{w^0}, \mathbf{w^1}, \boldsymbol{\phi^0}, \boldsymbol{\phi^1}\}$

6. Household outputs $\mathbf{Y_t^{hh}} = \left\{\boldsymbol{A^{hh}}, \boldsymbol{C^{hh}}, \boldsymbol{L^{0,hh}}, \boldsymbol{L^{1,hh}}\right\}$

Giving the equation system

$$\begin{bmatrix} A_t - A_t^{hh} \\ L_t^0 - L_t^{0,hh} \\ L_t^1 - L_t^{1,hh} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \end{bmatrix} \forall t \in \{0, 1, ...T-1\} \tag{1}$$

and

$$\begin{aligned}
r_t &= \alpha \Gamma_t K_{t-1}^{\alpha-1} (L_t^0)^{\frac{1-\alpha}{2}} (L_t^1)^{\frac{1-\alpha}{2}} - \delta \\
w_t^0 &= \frac{1-\alpha}{2} \Gamma_t K_{t-1}^{\alpha} (L_t^0)^{-\frac{1+\alpha}{2}} (L_t^1)^{\frac{1-\alpha}{2}} \\
w_t^1 &= \frac{1-\alpha}{2} \Gamma_t K_{t-1}^{\alpha} (L_t^0)^{\frac{1-\alpha}{2}} (L_t^1)^{-\frac{1+\alpha}{2}} \\
A_t &= K_t \\
A_t^{hh} &= \mathbf{a_t^{*'}} \mathbf{D_t} \\
L_t^{0,hh} &= \ell^{\mathbf{0},*'} \mathbf{D_t} \\
L_t^{1,hh} &= \ell^{\mathbf{1},*'} \mathbf{D_t} \\
\mathbf{D_t} &= \Pi_z' \underline{\mathbf{D}}_\mathbf{t} \\
\underline{\mathbf{D}}_\mathbf{t+1} &= \Lambda_t \mathbf{D_t} \\
&\underline{\mathbf{D}}_\mathbf{0} \textit{is given}
\end{aligned}$$

Figure 1 below, shows the DAG for the model.

Figure 1: Directed Acyclic Graph

Notes: Shows the Directed Acyclic Graph for the model. "phi_low" and "phi_high" is ϕ_t^0 and ϕ_t^1, respectively.

2 Solve for the stationary equilibrium

Below I show some of the values of interest as well as two figures that show what affects wealth inequality. Table 1 suggests that the model was solved correctly, as all the markets clear. It is

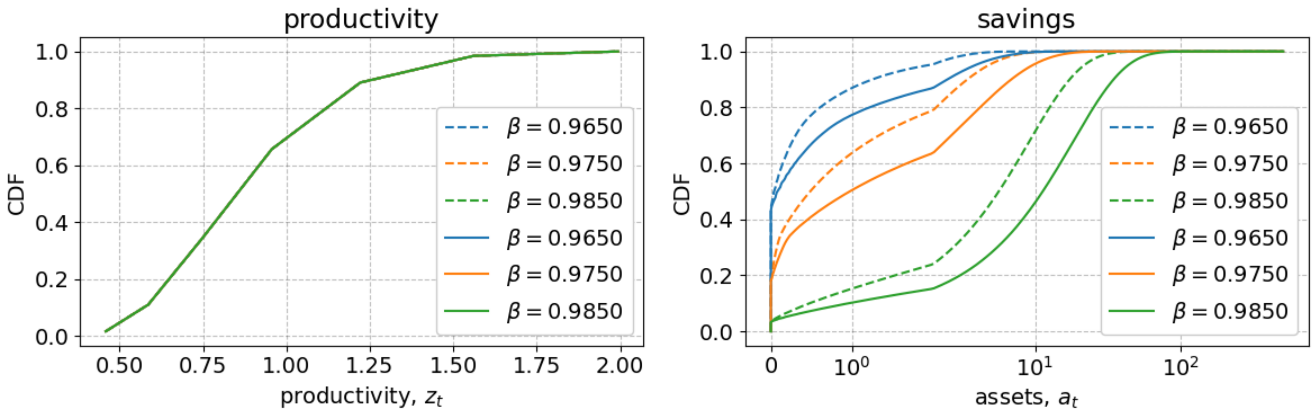
Table 1: Some variables of interest

	A_{ss}	L^0_{ss}	L^1_{ss}	Y_{ss}
Market Clearing	0.0000	0.0000	0.0000	0.0000
	Y_{ss}	I_{ss}	K_{ss}	Γ_{ss}
National Accounts	1.2831	0.4110	4.1098	1.0000
	w^0_{ss}	w^1_{ss}	L^0_{ss}	L^1_{ss}
Wage and Labour	0.6159	0.6159	0.6667	0.6667

perhaps worth noticing, that the wage and labour of both types are the same. This is because, while the most productive type is more productive, there are less of this type. And conversely for the least productive type, therefore the effective amount of labour is the same, leading to the same wage. If we glance at the wage equation for one of the household types in section 1, we see that if both types supply the same amount of labour, they effectively cancel each other out - leading to the same wage.

Figure 2 shows the distribution of savings by discount factor. We see that a low discount factor, meaning that the future is not highly valued, result in lower savings in general. If we compare the two types of households in the economy, the high productivity and the low productivity, we see that for a given beta, say 0.9850, the high productivity households have higher savings. For example, about 10% of high productivity households have $10^0 = 1$ asset or less, while about 18% of low productivity households have 1 asset or less. The same holds true for the two other betas. This means that the high productivity households save more for a given beta.

Figure 2: Distribution of savings

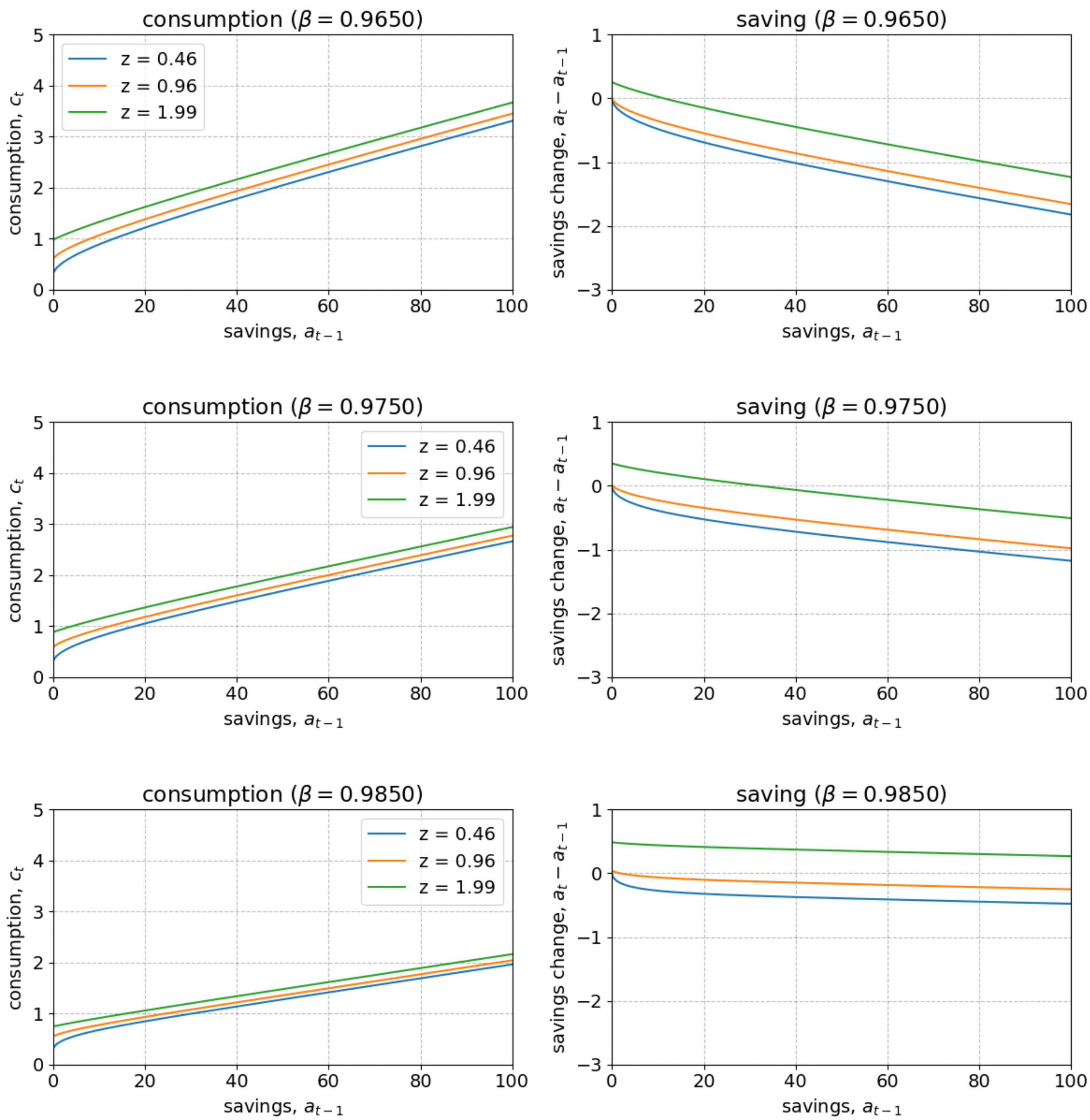


Notes: Dashed line shows betas belonging to low productivity households, while solid line shows betas belonging to high productivity households

Figure 3 shows consumption and savings for different income shocks for a given beta for low productivity households. Figure 4 shows the same, but for high productivity households. From figure 3, it is apparent that for any given beta, a positive income shock implies higher savings and higher consumption. For a high beta, say 0.9850, and a high income shock of 1.99, the change in savings is positive for all savings last period (or for all the plotted savings at least). This is because a high beta, means that the future is not discounted as high, as with a low beta. The opposite is true for a low beta and a low income shock (except for the special case where savings are zero). Overall this means that stochastic income and different discount rates are drivers of wealth inequality.

If we compare figure 3 and figure 4, we see that both savings and consumption is higher for given betas and given income shocks for the high productivity type. Meaning that productivity is a driver of inequality.

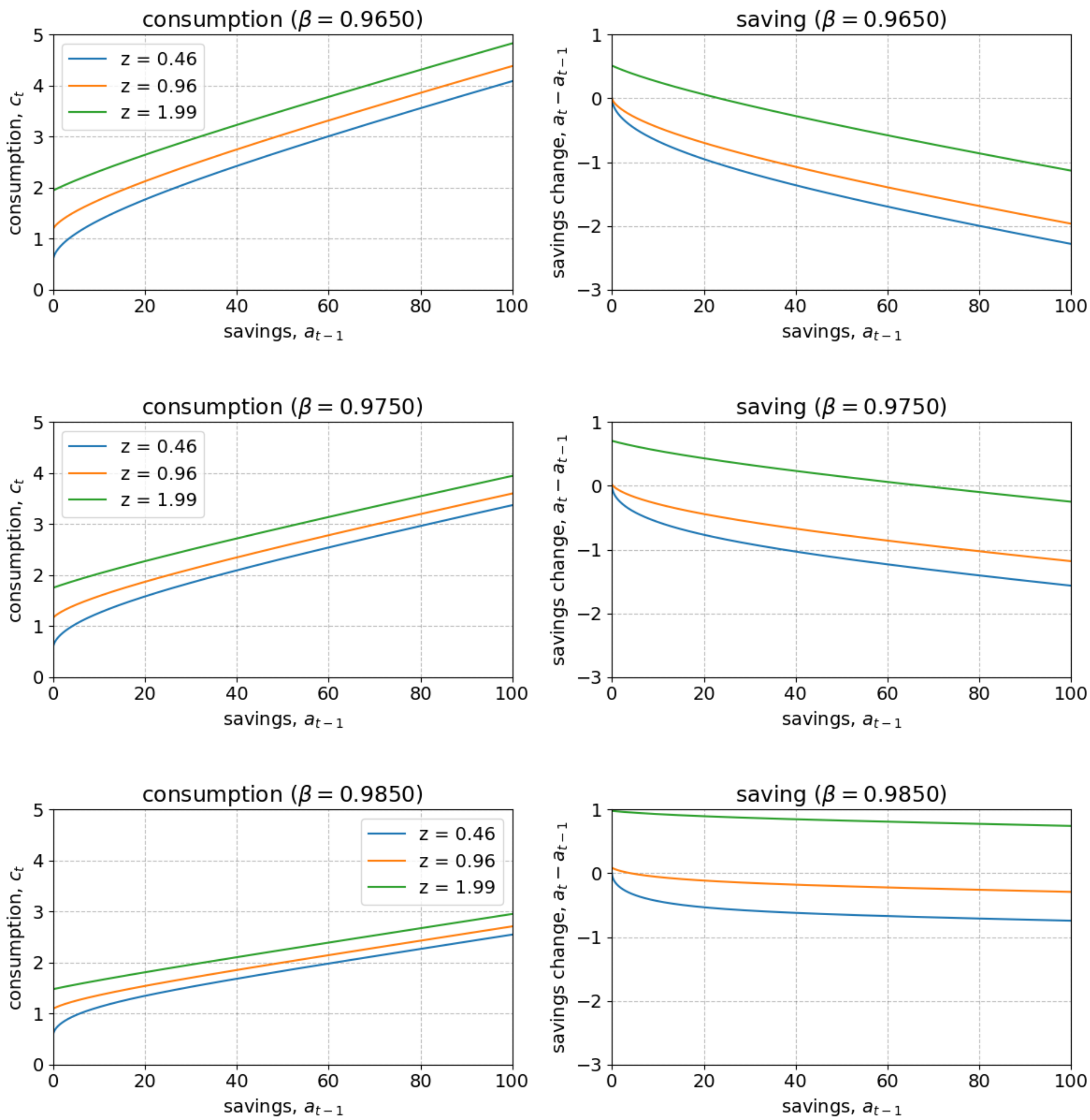
Figure 3: Consumption and savings for given betas for low productivity households



3 Computing Jacobians

Figure 5 shows the Jacobians of the household, where "phi_low" and phi_high is ϕ^0 and ϕ^1, respectively. The left column shows how savings react to changes in the interest rate, wages, and productivity. The right column shows how consumption reacts to the same variables. For example, if the interest rate increases in period 0, meaning an unexpected increase, the savings

Figure 4: Consumption and savings for given betas for high productivity households

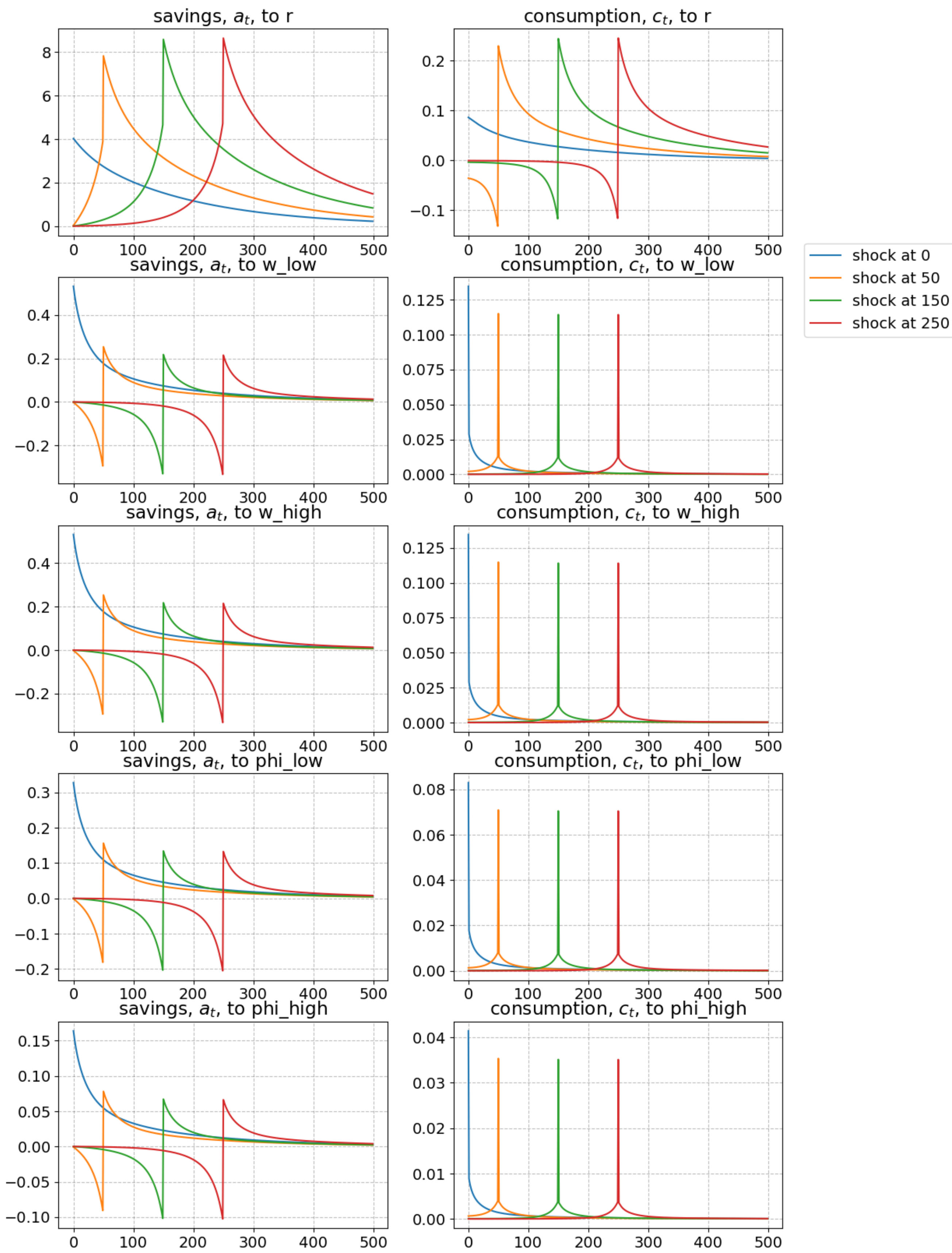


increase in the same period and asymptotically moves back to the baseline. An expected increase in the interest rate in, say, period 50 will result in an increase in savings before the interest rate increases. Looking at the right column, an unexpected increase in the interest rate, will result in increased consumption in almost all periods. If an unexpected shock happens, the consumption will decrease before the shock, this is likely because households want to save more to take full advantage of the higher interest rate. Why not just wait until the shock happens? If households did this, then they would have to have a period where they consume relatively less than if they slowly built up a stock of savings, i.e. households want to smooth their consumption.

Looking at the right column, an increase in ϕ_1 will result in increased consumption. If the shock is unexpected, then consumption will be higher in period 0 and move back towards the baseline. If the shock is expected, then consumption will increase before the shock happens. This is because an increase in ϕ is really an increase in the labour supply. If the labour supply increases, and we hold all other things equal, then they must earn more in total, meaning their lifetime income is higher, this means that they can consume more even before the shock happens.

Looking at the left column, an unexpected shock to ϕ_1, will increase the savings in period 0 and then move back to the baseline. For an expected shock, the household will use some of their savings before the shock happens. Again, this is to smooth their consumption.

Figure 5: Jacobians



4 Transition path

4.1 Transition path for temporary shock to labor productivity

Next, I solve for the transition path when ϕ_1 is 10% higher for 10 periods. In figure 6, the impulse response functions for wages and labor are shown, as well as the variables, Γ, ϕ_0, and ϕ_1. An impulse is made to ϕ_1 in period 10, which then lasts for 10 periods. When the shock happens is a bit arbitrary, but choosing a place that is not exactly at the beginning seems like a good way to be better able to see if there is an effect before the shock happens. Looking at how wages react, we can see that the wage of the low productivity type rises (mostly) and vice versa for the high productivity type. This might seem a bit counter intuitive. However, we should remember that a shock to the productivity is really a shock to the labor supply. That is, a positive shock to labor is really an increase in labor. Note how the labor of the high productivity type increases exactly by 10%. The labor supply of the low productivity type unchanged [1]. Since the labor supply of the high productivity type increases, the wage falls temporarily.

We can also calculate the discounted expected utility of the households, which is shown in table 2. The table shows the difference in utility between the path and the steady state by betas for both types. We see that the utility is higher in all cases, meaning everyone is better off. It not really possible to say who is the best off, due to different utility functions. I.e the level of utility is hard to compare.

Table 2: Change in utility from steady state

	$\beta = 0.965$	$\beta = 0.975$	$\beta = 0.9685$
Low productivity	0.488	0.573	0.544
	$\beta = 0.965$	$\beta = 0.975$	$\beta = 0.985$
High productivity	0.244	0.286	0.272

4.2 Transition path for permanent shock to productivity

Next, we make a permanent shock to the economy, by making ϕ^1 permanently 10% larger. The impulse response functions are showns in figure 7. We need to be a bit careful when interpreting the IRFs. We start off with the same productivity levels as in the baseline model, but in the steady state here, ϕ^1 is 10% larger. In the IRFs, the wages start of being about 1.7% lower than than in the new steady state and then converges to the new steady state level.

In table 3, the utility in the steady state less the path utility is shown. Again we see that everyone is better of.

[1] Not exactly, but note the scale on the axis for that type of labor

Table 3: Change in utility from the path to the new steady state

	$\beta = 0.965$	$\beta = 0.975$	$\beta = 0.9685$
Low productivity	0.272	0.301	0.348
	$\beta = 0.965$	$\beta = 0.975$	$\beta = 0.985$
High productivity	0.136	0.151	0.174

Figure 6: Impulse response functions for temporery shock

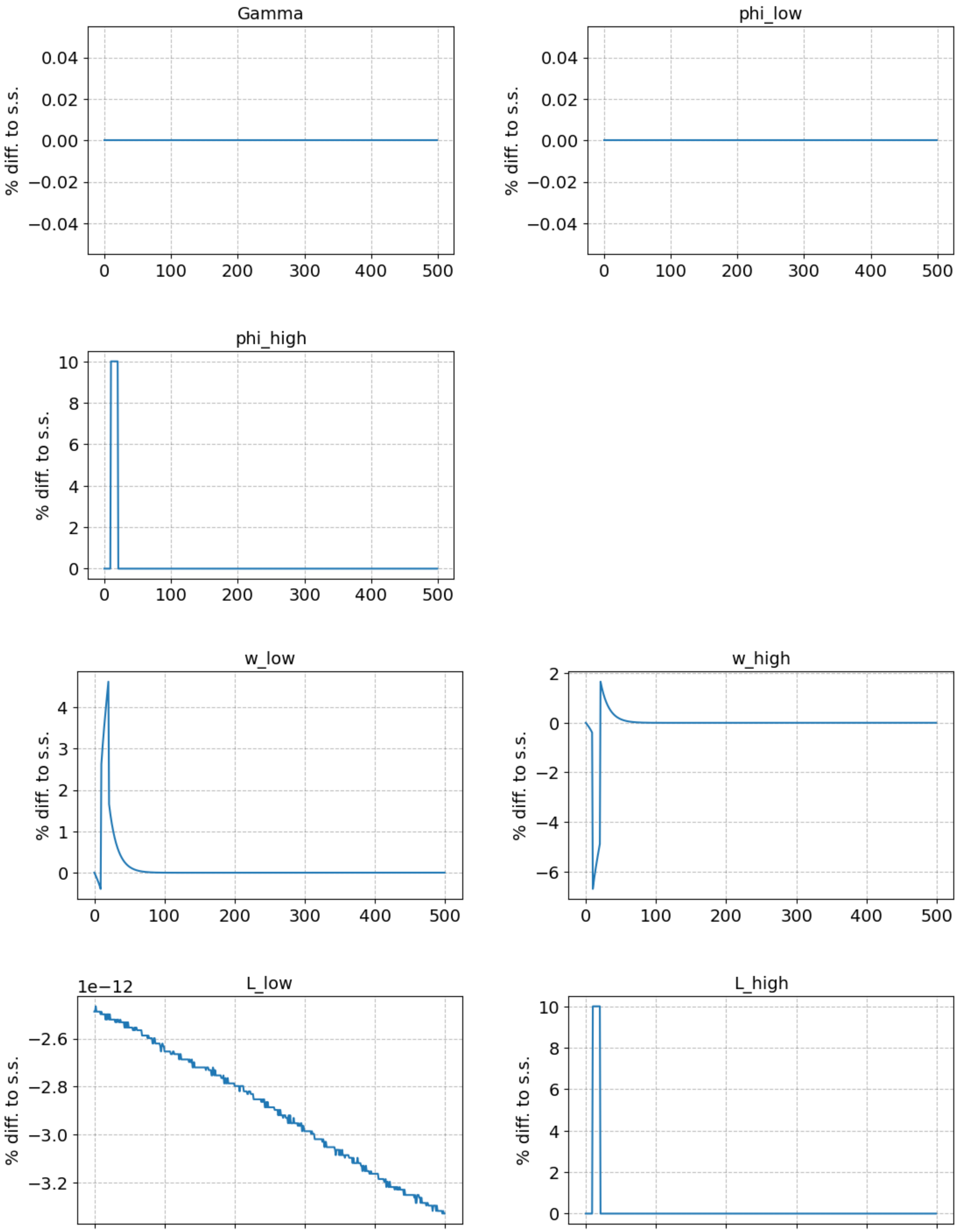


Figure 7: Impulse response functions for permantent shock