Linear Prediction

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Introduction

- The Fourier transform involves integration over the whole real line
- A finite cutoff in the time domain introduces artifacts into the frequency domain
- Linear prediction is an ansatz for extrapolating time domain data to remove these finite time artifacts

Preliminaries

Consider some signal $G(t)$ with Fourier transform $A(\omega)$

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t) e^{-i\omega t}$$

If we only know $G(t)$ in some finite time window $[-\frac{T}{2}, \frac{T}{2}]$ then our Fourier transform becomes

$$D(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t)\,\mathrm{rect}(t/T) e^{-i\omega t}$$

where

$$\mathrm{rect}(t) = \Theta\left(t + \frac{1}{2}\right) - \Theta\left(t - \frac{1}{2}\right)$$

Recalling that

$$\mathcal{F}\{f\} * \mathcal{F}\{g\} = \sqrt{2\pi}\mathcal{F}\{f \cdot g\}$$

we have

$$\begin{aligned} D(\omega) &= \frac{1}{\sqrt{2\pi}}\mathcal{F}\{G\} * \mathcal{F}\{\mathrm{rect}\,(t/T)\} \\ &= \int_{-\infty}^{\infty} d\omega' \left[\frac{T}{2\pi}\,\mathrm{sinc}\left(\frac{(\omega-\omega')\,T}{2}\right)\right] A(\omega') \\ &= KA \end{aligned}$$

So our data is given by the true result convolved with a singular kernel.

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Green's functions

Often the objects we are interested in are Green's functions which have the structure

$$G(k,\omega) \sim \frac{1}{(\omega - \epsilon_k - \Sigma(k,\omega))}$$

We can Fourier transform this object to obtain

$$\begin{aligned} G(k,t) &\sim \int_{-\infty}^{\infty} d\omega G(k,\omega) e^{+i\omega t} \\ &\sim \sum_j c_j e^{i\omega_j t} \end{aligned}$$

where c_j, ω_j are the residues and poles of $G(k,\omega)$ in the upper complex plane

Linear Prediction

- If the self energy is not too complicated $G(k, t)$ will be given by a superposition of a small number of complex exponential terms
- Take this as ansatz, fit and extrapolate
- This can be done efficiently using linear prediction

The linear prediction ansatz is given by

$$x_n = \sum_{j=1}^{N} c_j e^{i\omega_j t_n}$$

which can be recast as an autoregressive model

$$x_n = -\sum_{k=1}^{p} a_k x_{n-k}$$

To get a_k and p in terms of c_j, ω_j and N plug the first equation into the second

$$\sum_{j=1}^{N} c_j e^{i\omega_j t_n} = -\sum_{k=1}^{p} a_k \sum_{j=1}^{N} c_j e^{i\omega_j t_{n-k}}$$
$$= -\sum_{j=1}^{N} c_j e^{i\omega_j t_n} \underbrace{\left(\sum_{k=1}^{p} a_k e^{-i\omega_j t_k}\right)}_{=-1}$$

$$\Longrightarrow \sum_{k=0}^{p} e^{-i\omega_j t_k} a_k = 0 \qquad (a_0 = 1)$$

The previous equation can be written

$$M\boldsymbol{a} = \mathbf{0}$$

where $M \in \mathbb{C}^{N\times(p+1)}$ is given by

$$M_{jk} = e^{-i\omega_j t_k}$$

from this we see that when $p = N$ we have exactly one non-trivial solution. Note, the other N degrees of freedom associated with c_j are realized in terms of the initial conditions of the autoregressive model.

Fitting

In order to fit the model to some data $\boldsymbol{x}$ we write down the autoregressive form of the equations as $Q\boldsymbol{a} = -\boldsymbol{x}$ where $Q \in \mathbb{C}^{N \times p}$ is given by

$$Q_{nk} = x_{n-k}$$

The normal equations then read

$$R\boldsymbol{a} = -\boldsymbol{r}$$

where

$$r_j = (Q^\dagger x)_j = \sum_n x^*_{n-j} x_n$$

$$R_{ji} = (Q^\dagger Q)_{ji} = \sum_n x^*_{n-j} x_{n-i}$$

are the autocorrelations of the signal

The normal equations can be rewritten as

$$\begin{aligned}
\sum_k R_{jk} a_k &= \sum_k \sum_n x_{n-j}^* x_{n-k} a_k \\
&= \sum_n x_{n-j}^* \sum_k a_k x_{n-k} \\
&= -\sum_n x_{n-j}^* \tilde{x}_n \\
&= -r_j \\
&= -\sum_n x_{n-j}^* x_n
\end{aligned}$$

where $\tilde{x}_n$ is the prediction and x_n is the data.
$\implies$ correlation of the prediction and the signal is equal to the autocorrelation of the signal

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Applications

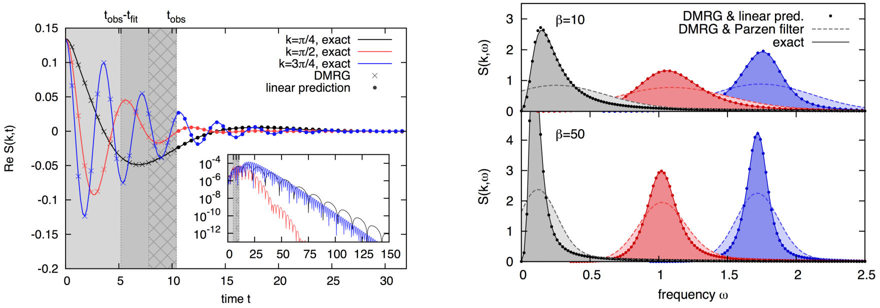


Figure: t-DMRG, Barthel (2009)

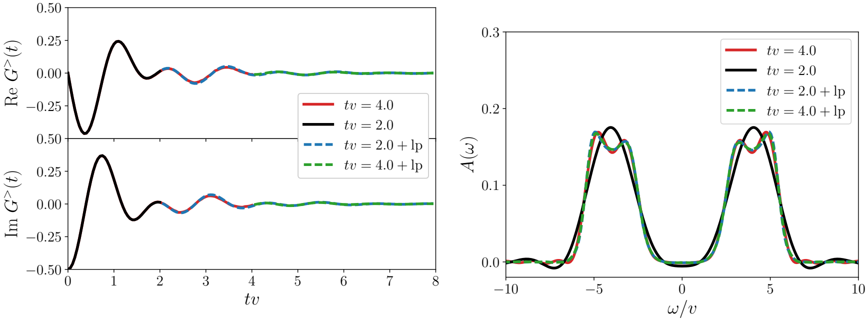


Figure: Realtime DMFT ($p = 9$, $t_{\mathrm{fit}} = 1.0$)

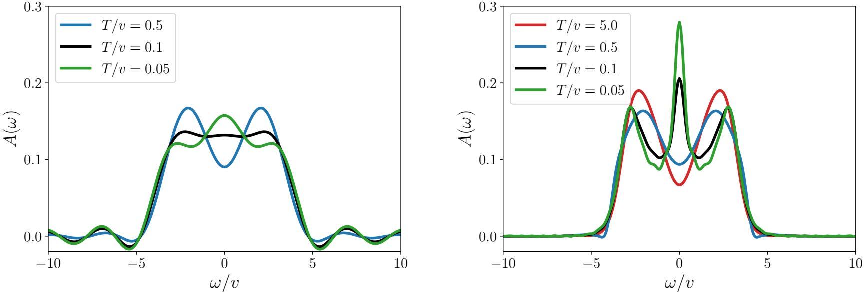
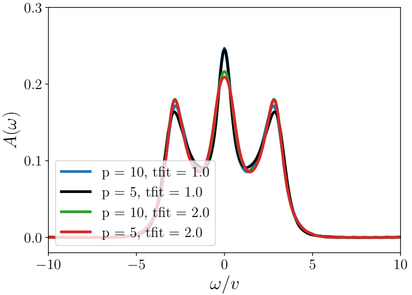


Figure: Realtime DMFT ($p = 9$, $t_{\mathrm{fit}} = 1.0$)

Benchmarks



0.3
0.2
0.1
0.0
$A(\omega)$
−10
−5
0
5
10
ω/v
p = 10, tfit = 1.0
p = 5, tfit = 1.0
p = 10, tfit = 2.0
p = 5, tfit = 2.0
0.3
0.2
0.1
0.0
$A(\omega)$
−10
−5
0
5
10
ω/v
p = 10, tfit = 1.0
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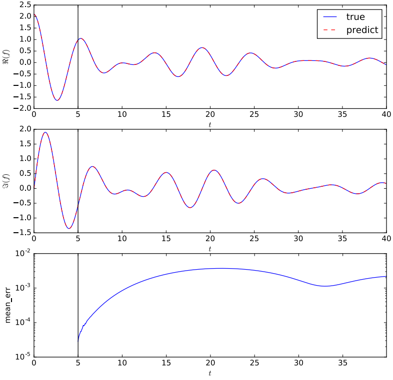

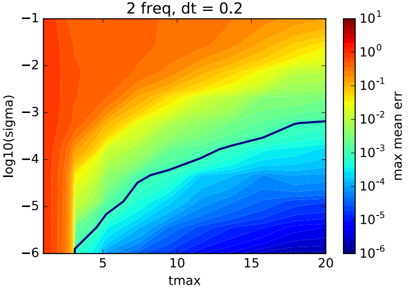

Figure: Noise tolerance

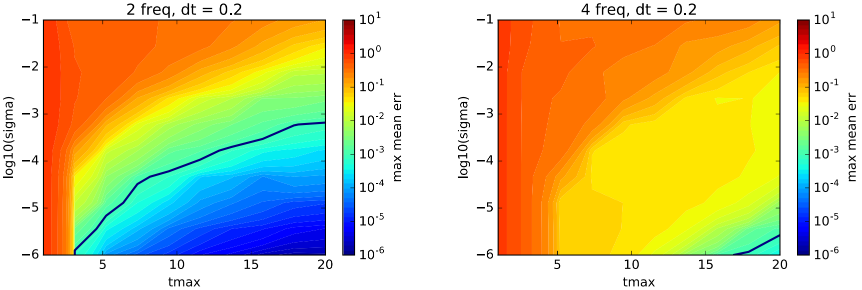

Figure: Number of Frequencies

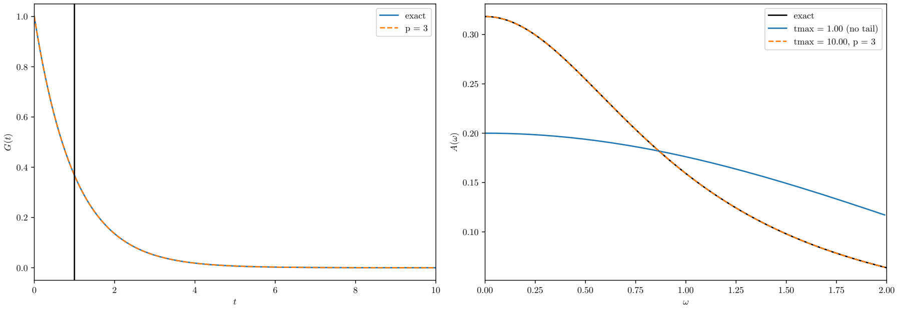

Figure: Lorentzian

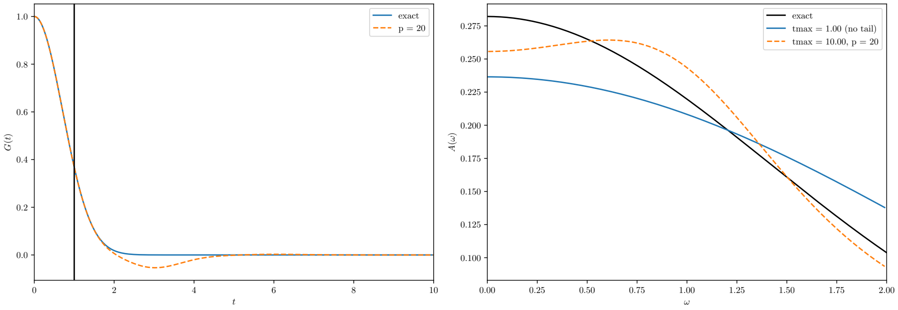

Figure: Gaussian

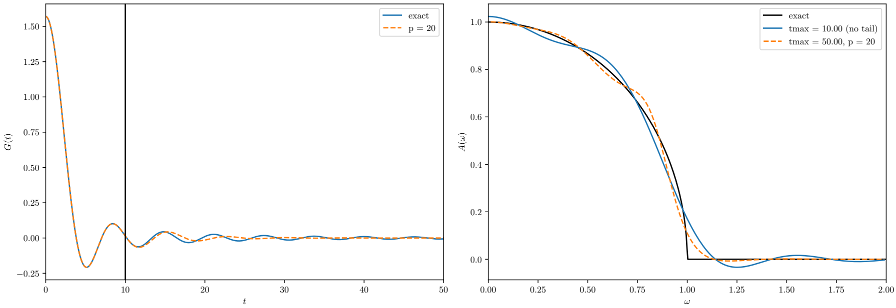

Figure: Semi-circular