The scaling discussed here will be for a given timestep T. N denotes the number of atomic orbitals and N_τ denotes the number of imaginary time points. At each timestep, we need to calculate the self-energy and then the Greens function for three Keldysh indices, $R, <, \rceil$. We calculate R on the lower triangle, $<$ on the upper triangle and $\rceil$ everywhere. Stated more clearly we calculate

$$G^R(T,t') \quad 0 \le t' \le T \tag{1}$$
$$G^<(t,T) \quad 0 \le t \le T \tag{2}$$
$$G^\rceil(T,\tau) \quad 0 \le \tau \le N_\tau, \tag{3}$$

for each timestep T, similarly for the self energy.

1 Hubbard Self-Energy

The Self-Energy is given by

$$\Sigma_{ij\sigma}(t,t') = U(t)U(t')G_{ij\sigma}(t,t')G_{ij\bar{\sigma}}(t,t')G_{ji\bar{\sigma}}(t',t). \tag{4}$$

We need to calculate all Keldysh components of the self-energy, that is, $\Sigma^R, \Sigma^<, \Sigma^\rceil$. To do this, we first calculate the Polarization bubble $P_{ij\bar{\sigma}}(t,t') = G_{ij\bar{\sigma}}(t,t')G_{ji\bar{\sigma}}(t',t)$.

$$P^R_{ij}(T,t') = G^R_{ij}(T,t')G^<_{ji}(t',T) + G^<_{ij}(T,t')G^A_{ji}(t',T) \qquad \mathcal{O}(TN^2) \tag{5}$$
$$P^<_{ij}(t',T) = G^<_{ij}(t',T)G^>_{ji}(T,t') \qquad \mathcal{O}(TN^2) \tag{6}$$
$$P^\rceil_{ij}(T,\tau) = G^\rceil_{ij}(T,\tau)G^\lceil_{ji}(\tau,T) \qquad \mathcal{O}(N_\tau N^2) \tag{7}$$

Next we multiply P by the interactions to get $U(t)P(t,t')U(t')$

$$P^R(T,t') \leftarrow U(T)P^R(T,t')U(t') \qquad \mathcal{O}(TN^2) \tag{8}$$
$$P^<(t',T) \leftarrow U(t')P^<(t',T)U(T) \qquad \mathcal{O}(TN^2) \tag{9}$$
$$P^\rceil(T,\tau) \leftarrow U(T)P^\rceil(T,\tau)U(0^-) \qquad \mathcal{O}(N_\tau N^2) \tag{10}$$

To finalize everything, we contract the final Greens function with the Polarization bubble

$$\Sigma^R_{ij}(T,t') = G^R_{ij}(T,t')P^R_{ij}(T,t') + G^<_{ij}(T,t')P^R_{ij}(T,t') + G^R_{ij}(T,t')P^<_{ij}(T,t') \qquad \mathcal{O}(TN^2) \tag{11}$$
$$\Sigma^<_{ij}(T,t') = G^<_{ij}(T,t')P^<_{ij}(T,t') \qquad \mathcal{O}(TN^2) \tag{12}$$
$$\Sigma^\rceil_{ij}(T,\tau) = G^\rceil_{ij}(T,\tau)P^\rceil_{ij}(T,\tau) \qquad \mathcal{O}(N_\tau N^2) \tag{13}$$

2 Decomposed Interaction HF

The HF portion of the code does the contractions

$$\Sigma^{HF}_{\sigma ij} = \sum_{kl}\sum_{\alpha} V^\alpha_{ij}V^\alpha_{kl}\sum_{\sigma'}\rho_{\sigma' kl} - \sum_{kl}\sum_{\alpha} V^\alpha_{il}V^\alpha_{kj}\rho_{\sigma kl}. \tag{14}$$

The Hartree diagram can be done in two steps

$$X^\alpha \leftarrow \sum_{kl} V^\alpha_{kl}\sum_{\sigma'}\rho_{\sigma' kl} \quad \mathcal{O}(N^2 N_\alpha) \tag{15}$$
$$\Sigma^H_{\sigma ij} \leftarrow \sum_{\alpha} V^\alpha_{ij}X^\alpha \quad \mathcal{O}(N^2 N_\alpha) \tag{16}$$

Similarly the Fock diagram is done in two steps

$$X^{\alpha}_{\sigma lj} \leftarrow \sum_k V^{\alpha}_{kj} \rho_{\sigma kl} \quad \mathcal{O}(N^3 N_\alpha) \tag{17}$$

$$\Sigma^F_{\sigma ij} \leftarrow \sum_l \sum_\alpha V^{\alpha}_{il} X^{\alpha}_{\sigma lj} \quad \mathcal{O}(N^3 N_\alpha) \tag{18}$$

3 Decomposed Interaction GF2

For the decomposed interaction, we solve the bubble and exchange diagram separately.

3.1 Bubble Diagram

The bubble diagram is given by

$$\Sigma^{\text{bubble}}_{\sigma,ij}(t,t') = \sum_{lnp} \sum_{kqm} \sum_{\alpha\beta} V^{\alpha}_{il} V^{\alpha}_{np} V^{\beta}_{kj} V^{\beta}_{qm} G_{\sigma lk}(t,t') \sum_{\sigma'} G_{\sigma' pq}(t,t') G_{\sigma' mn}(t',t). \tag{19}$$

We need to solve for each of the three Keldysh indices on a given time-slice

$$\Sigma^{\text{bubble},R}_{\sigma,ij}(T,t') \quad 0 \le t' \le T \tag{20}$$

$$\Sigma^{\text{bubble},<}_{\sigma,ij}(t,T) \quad 0 \le t \le T \tag{21}$$

$$\Sigma^{\text{bubble},\rceil}_{\sigma,ij}(T,\tau) \quad 0 \le \tau \le N_\tau \tag{22}$$

3.1.1 Retarded Bubble

First let us look at the retarded term. We start by defining $X^{\beta}_{\sigma' nq}(t',T) \equiv \sum_m V^{\beta}_{qm} G_{\sigma' mn}(t',T)$. We will need $X^A, X^<, X^>$ so we do the contractions

$$X^{\beta,A}_{\sigma' nq}(t',T) \leftarrow \sum_m V^{\beta}_{qm} G^{A}_{\sigma' mn}(t',T) \quad \mathcal{O}(TN^3 N_\alpha) \tag{23}$$

$$X^{\beta,<}_{\sigma' nq}(t',T) \leftarrow \sum_m V^{\beta}_{qm} G^{<}_{\sigma' mn}(t',T) \quad \mathcal{O}(TN^3 N_\alpha) \tag{24}$$

$$X^{\beta,>}_{\sigma' nq}(t',T) \leftarrow \sum_m V^{\beta}_{qm} G^{>}_{\sigma' mn}(t',T) \quad \mathcal{O}(TN^3 N_\alpha) \tag{25}$$

Next we define $Y^{\alpha}_{\sigma' nq}(T,t') \equiv \sum_p V^{\alpha}_{np} G_{\sigma' pq}(T,t')$, we will need $Y^<, Y^R$ so we do the contractions

$$Y^{\alpha,<}_{\sigma' nq}(T,t') \leftarrow \sum_p V^{\alpha}_{np} G^{<}_{\sigma' pq}(T,t') \quad \mathcal{O}(TN^3 N_\alpha) \tag{26}$$

$$Y^{\alpha,R}_{\sigma' nq}(T,t') \leftarrow \sum_p V^{\alpha}_{np} G^{R}_{\sigma' pq}(T,t') \quad \mathcal{O}(TN^3 N_\alpha) \tag{27}$$

Now we define the quantity $\Pi^{\alpha\beta}(T,t') \equiv \sum_{nq} \sum_{\sigma'} Y^{\alpha}_{\sigma' nq}(T,t') X^{\beta}_{\sigma' nq}(t',T)$. We will need the components $\Pi^R, \Pi^<$

$$\Pi^{\alpha\beta,R}(T,t') \leftarrow \sum_{\sigma' nq} (Y^{\alpha,R}_{\sigma' nq}(T,t') X^{\beta,<}_{\sigma' nq}(t',T) + Y^{\alpha,<}_{\sigma' nq}(T,t') X^{\beta,A}_{\sigma' nq}(t',T)) \quad \mathcal{O}(TN_\alpha^2 N^2) \tag{28}$$

$$\Pi^{\alpha\beta,<}(T,t') \leftarrow \sum_{\sigma' nq} Y^{\alpha,<}_{\sigma' nq}(T,t') X^{\beta,>}_{\sigma' nq}(t',T) \quad \mathcal{O}(TN_\alpha^2 N^2) \tag{29}$$

We now contract Π with one of the remaining interaction terms $Z^{\alpha}_{kj}(T,t') \equiv \sum_\beta V^{\beta}_{kj}\Pi^{\alpha\beta}(T,t')$.

$$Z^{\alpha,R}_{kj}(T,t') \leftarrow \sum_{\beta} V^{\beta}_{kj}\Pi^{\alpha\beta,R}(T,t') \quad \mathcal{O}(TN^2N_\alpha^2) \tag{30}$$

$$Z^{\alpha,<}_{kj}(T,t') \leftarrow \sum_{\beta} V^{\beta}_{kj}\Pi^{\alpha\beta,<}(T,t') \quad \mathcal{O}(TN^2N_\alpha^2) \tag{31}$$

Finally, we do the final contractions into $\Sigma^R_{ij}(T,t') = \sum_{\alpha k} Y^{\alpha}_{\sigma ik}(T,t')Z^{\alpha}_{kj}(T,t')$.

$$\Sigma^{\text{bubble},R}_{ij}(T,t') \leftarrow \sum_{\alpha k}(Y^{\alpha,R}_{\sigma ik}(T,t')Z^{\alpha,R}_{kj}(T,t') + Y^{\alpha,<}_{\sigma ik}(T,t')Z^{\alpha,R}_{kj}(T,t') + Y^{\alpha,R}_{\sigma ik}(T,t')Z^{\alpha,<}_{kj}(T,t')) \quad \mathcal{O}(TN^3N_\alpha) \tag{32}$$

3.1.2 Lesser Bubble

Next let us look at the lesser term. We start by defining $X^{\beta}_{\sigma' qn}(T,t) \equiv \sum_m V^{\beta}_{qm}G_{\sigma' mn}(T,t)$. We will need $X^>$ so we do the contraction

$$X^{\beta,>}_{\sigma' nq}(T,t) \leftarrow \sum_{m} V^{\beta}_{qm}G^>_{\sigma' mn}(T,t) \quad \mathcal{O}(TN^3N_\alpha) \tag{33}$$

Next we define $Y^{\alpha}_{\sigma' nq}(t,T) \equiv \sum_p V^{\alpha}_{np}G_{\sigma' pq}(t,T)$, we will need $Y^<$ so we do the contraction

$$Y^{\alpha,<}_{\sigma' nq}(t,T) \leftarrow \sum_{p} V^{\alpha}_{np}G^<_{\sigma' pq}(t,T) \quad \mathcal{O}(TN^3N_\alpha) \tag{34}$$

Now we define the quantity $\Pi^{\alpha\beta}(t,T) \equiv \sum_{nq}\sum_{\sigma'} Y^{\alpha}_{\sigma' nq}(t,T)X^{\beta}_{\sigma' nq}(T,t)$. We will need the component $\Pi^<$

$$\Pi^{\alpha\beta,<}(t,T) \leftarrow \sum_{\sigma' nq} Y^{\alpha,<}_{\sigma' nq}(t,T)X^{\beta,>}_{\sigma' nq}(T,t) \quad \mathcal{O}(TN_\alpha^2N^2) \tag{35}$$

We now contract Π with one of the remaining interaction terms $Z^{\alpha}_{kj}(t,T) \equiv \sum_\beta V^{\beta}_{kj}\Pi^{\alpha\beta}(t,T)$.

$$Z^{\alpha,<}_{kj}(t,T) \leftarrow \sum_{\beta} V^{\beta}_{kj}\Pi^{\alpha\beta,<}(t,T) \quad \mathcal{O}(TN^2N_\alpha^2) \tag{36}$$

Finally, we do the final contractions into $\Sigma^<_{ij}(t,T) = \sum_{\alpha k} Y^{\alpha}_{\sigma ik}(t,T)Z^{\alpha}_{kj}(t,T)$.

$$\Sigma^{\text{bubble},<}_{ij}(t,T) \leftarrow \sum_{\alpha k} Y^{\alpha,<}_{\sigma ik}(t,T)Z^{\alpha,<}_{kj}(t,T) \quad \mathcal{O}(TN^3N_\alpha) \tag{37}$$

3.1.3 Right Mixing Bubble

Lastly let us look at the Right Mixing term. We start by defining $X^{\beta}_{\sigma' qn}(\tau,T) \equiv \sum_m V^{\beta}_{qm}G_{\sigma' mn}(\tau,T)$. We will need $X^{\lceil}$ so we do the contraction

$$X^{\beta,\lceil}_{\sigma' nq}(\tau,T) \leftarrow \sum_{m} V^{\beta}_{qm}G^{\lceil}_{\sigma' mn}(\tau,T) \quad \mathcal{O}(N_\tau N^3N_\alpha) \tag{38}$$

Next we define $Y^{\alpha}_{\sigma' nq}(T,\tau) \equiv \sum_p V^{\alpha}_{np}G_{\sigma' pq}(T,\tau)$, we will need $Y^{\rceil}$ so we do the contraction

$$Y^{\alpha,\rceil}_{\sigma' nq}(T,\tau) \leftarrow \sum_{p} V^{\alpha}_{np}G^{\rceil}_{\sigma' pq}(T,\tau) \quad \mathcal{O}(N_\tau N^3N_\alpha) \tag{39}$$

Now we define the quantity $\Pi^{\alpha\beta}(T,\tau) \equiv \sum_{nq}\sum_{\sigma'} Y^{\alpha}_{\sigma' nq}(T,\tau)X^{\beta}_{\sigma' nq}(\tau,T)$. We will need the component $\Pi^{\rceil}$

$$\Pi^{\alpha\beta,\rceil}(T,\tau) \leftarrow \sum_{\sigma' nq} Y^{\alpha,\rceil}_{\sigma' nq}(T,\tau)X^{\beta,\lceil}_{\sigma' nq}(\tau,T) \quad \mathcal{O}(N_\tau N_\alpha^2 N^2) \tag{40}$$

We now contract Π with one of the remaining interaction terms $Z^{\alpha}_{kj}(T,\tau) \equiv \sum_\beta V^{\beta}_{kj}\Pi^{\alpha\beta}(T,\tau)$.

$$Z^{\alpha,\rceil}_{kj}(T,\tau) \leftarrow \sum_{\beta} V^{\beta}_{kj}\Pi^{\alpha\beta,\rceil}(T,\tau) \quad \mathcal{O}(N_\tau N^2 N_\alpha^2) \tag{41}$$

Finally, we do the final contractions into $\Sigma^{\rceil}_{ij}(T,\tau) = \sum_{\alpha k} Y^{\alpha}_{\sigma ik}(T,\tau)Z^{\alpha}_{kj}(T,\tau)$.

$$\Sigma^{\text{bubble},\rceil}_{ij}(T,\tau) \leftarrow \sum_{\alpha k} Y^{\alpha,\rceil}_{\sigma ik}(T,\tau)Z^{\alpha,\rceil}_{kj}(T,\tau) \quad \mathcal{O}(N_\tau N^3 N_\alpha) \tag{42}$$

3.2 Exchange Diagram

The Exchange diagram is given by

$$\Sigma^{\text{exch}}_{\sigma,ij}(t,t') = -\sum_{lnp}\sum_{kqm}\sum_{\alpha\beta} V^{\alpha}_{il}V^{\alpha}_{np}V^{\beta}_{qj}V^{\beta}_{km}G_{\sigma lk}(t,t')G_{\sigma pq}(t,t')G_{\sigma mn}(t',t). \tag{43}$$

We need to solve for each of the three Keldysh indices on a given time-slice

$$\Sigma^{\text{exch},R}_{\sigma,ij}(T,t') \quad 0 \le t' \le T \tag{44}$$

$$\Sigma^{\text{exch},<}_{\sigma,ij}(t,T) \quad 0 \le t \le T \tag{45}$$

$$\Sigma^{\text{exch},\rceil}_{\sigma,ij}(T,\tau) \quad 0 \le \tau \le N_\tau \tag{46}$$

3.2.1 Retarded Exchange

First let us calculate the retarded component of the exchange diagram. We start by defining $X^{\alpha}_{nq}(T,t') \equiv \sum_p V^{\alpha}_{np}G_{pq}(T,t')$. We will need the $X^R, X^>, X^<$ components.

$$X^{\alpha,R}_{nq}(T,t') \leftarrow \sum_p V^{\alpha}_{np}G^{R}_{pq}(T,t') \quad \mathcal{O}(TN^3N_\alpha) \tag{47}$$

$$X^{\alpha,<}_{nq}(T,t') \leftarrow \sum_p V^{\alpha}_{np}G^{<}_{pq}(T,t') \quad \mathcal{O}(TN^3N_\alpha) \tag{48}$$

$$X^{\alpha,>}_{nq}(T,t') \leftarrow \sum_p V^{\alpha}_{np}G^{>}_{pq}(T,t') \quad \mathcal{O}(TN^3N_\alpha) \tag{49}$$

Next we define $Y^{\alpha}_{mq}(t',T) = \sum_n G_{mn}(t',T)X^{\alpha}_{nq}(T,t')$, we need the $Y^<, Y^A$ components

$$Y^{\alpha,<}_{mq}(t',T) = \sum_n G^{<}_{mn}(t',T)X^{\alpha,>}_{nq}(T,t') \quad \mathcal{O}(TN^3N_\alpha) \tag{50}$$

$$Y^{\alpha,A}_{mq}(t',T) = \sum_n (G^{A}_{mn}(t',T)X^{\alpha,<}_{nq}(T,t') + G^{<}_{mn}(t',T)X^{\alpha,R}_{nq}(T,t')) \quad \mathcal{O}(TN^3N_\alpha) \tag{51}$$

Next we calculate $\Pi^{\alpha\beta}_{mj}(t',T) = \sum_q Y^{\alpha}_{mq}(t',T)V^{\beta}_{qj}$, and get the $<, A$ components

$$\Pi^{\alpha\beta,<}_{mj}(t',T) = \sum_q Y^{\alpha,<}_{mq}(t',T)V^{\beta}_{qj} \quad \mathcal{O}(TN^3N_\alpha^2) \tag{52}$$

$$\Pi^{\alpha\beta,A}_{mj}(t',T) = \sum_q Y^{\alpha,A}_{mq}(t',T)V^{\beta}_{qj} \quad \mathcal{O}(TN^3N_\alpha^2) \tag{53}$$

Next we define $Z^{\alpha}_{kj}(t',T) = \sum_{m\beta} V^{\beta}_{km}\Pi^{\alpha\beta}_{mj}(t',T)$. We need the $Z^<, Z^A$ components.

$$Z^{\alpha,<}_{kj}(t',T) = \sum_{m\beta} V^{\beta}_{km}\Pi^{\alpha\beta,<}_{mj}(t',T) \quad \mathcal{O}(TN^3N_\alpha^2) \tag{54}$$

$$Z^{\alpha,A}_{kj}(t',T) = \sum_{m\beta} V^{\beta}_{km}\Pi^{\alpha\beta,A}_{mj}(t',T) \quad \mathcal{O}(TN^3N_\alpha^2) \tag{55}$$

To finish the calculation we contract Z with X, $\Sigma_{ij}(T,t') = \sum_{k\alpha} X^{\alpha}_{ik}(T,t')Z^{\alpha}_{kj}(t',T)$

$$\Sigma^R_{ij}(T,t') = \sum_{k\alpha}(X^{\alpha,R}_{ik}(T,t')Z^{\alpha,<}_{kj}(t',T) + X^{\alpha,<}_{ik}(T,t')Z^{\alpha,A}_{kj}(t',T)) \quad \mathcal{O}(TN^3N_\alpha) \tag{56}$$

3.2.2 Lesser Exchange

Next we do the lesser component of the exchange diagram. We start by defining $X^{\alpha}_{nq}(T,t) \equiv \sum_p V^{\alpha}_{np}G_{pq}(T,t)$. We will need the $X^>$ component.

$$X^{\alpha,>}_{nq}(T,t) \leftarrow \sum_p V^{\alpha}_{np}G^>_{pq}(T,t) \quad \mathcal{O}(TN^3N_\alpha) \tag{57}$$

Next we define $Y^{\alpha}_{mq}(t,T) = \sum_n G_{mn}(t,T)X^{\alpha}_{nq}(T,t)$, we need the $Y^<$ component

$$Y^{\alpha,<}_{mq}(t,T) = \sum_n G^<_{mn}(t,T)X^{\alpha,>}_{nq}(T,t) \quad \mathcal{O}(TN^3N_\alpha) \tag{58}$$

Next we calculate $\Pi^{\alpha\beta}_{mj}(t,T) = \sum_q Y^{\alpha}_{mq}(t,T)V^{\beta}_{qj}$, and get the $<$ component

$$\Pi^{\alpha\beta,<}_{mj}(t,T) = \sum_q Y^{\alpha,<}_{mq}(t,T)V^{\beta}_{qj} \quad \mathcal{O}(TN^3N_\alpha^2) \tag{59}$$

Next we define $Z^{\alpha}_{kj}(t,T) = \sum_{m\beta} V^{\beta}_{km}\Pi^{\alpha\beta}_{mj}(t,T)$. We need the $Z^<$ components.

$$Z^{\alpha,<}_{kj}(t,T) = \sum_{m\beta} V^{\beta}_{km}\Pi^{\alpha\beta,<}_{mj}(t,T) \quad \mathcal{O}(TN^3N_\alpha^2) \tag{60}$$

To finish the calculation we contract Z with X, $\Sigma_{ij}(t,T) = \sum_{k\alpha} Z^{\alpha}_{kj}(t,T)X^{\alpha}_{ik}(T,t)$

$$\Sigma^<_{ij}(t,T) = \sum_{k\alpha} Z^{\alpha,<}_{kj}(t,T)X^{\alpha,>}_{ik}(T,t) \quad \mathcal{O}(TN^3N_\alpha) \tag{61}$$

3.2.3 Right Mixing Exchange

Lastly we do the right mixing component of the exchange diagram. We start by defining $X^{\alpha}_{nq}(\tau,T) \equiv \sum_p V^{\alpha}_{np}G_{pq}(\tau,T)$. We will need the $X^{\lceil}$ component.

$$X^{\alpha,\lceil}_{nq}(\tau,T) \leftarrow \sum_p V^{\alpha}_{np}G^{\lceil}_{pq}(\tau,T) \quad \mathcal{O}(N_\tau N^3N_\alpha) \tag{62}$$

Next we define $Y^{\alpha}_{mq}(T,\tau) = \sum_n G_{mn}(T,\tau)X^{\alpha}_{nq}(\tau,T)$, we need the $Y^{\rceil}$ component

$$Y^{\alpha,\rceil}_{mq}(T,\tau) = \sum_n G^{\rceil}_{mn}(T,\tau)X^{\alpha,\lceil}_{nq}(\tau,T) \quad \mathcal{O}(N_\tau N^3N_\alpha) \tag{63}$$

Next we calculate $\Pi^{\alpha\beta}_{mj}(T,\tau) = \sum_q Y^{\alpha}_{mq}(T,\tau)V^{\beta}_{qj}$, and get the $\rceil$ component

$$\Pi^{\alpha\beta,\rceil}_{mj}(T,\tau) = \sum_q Y^{\alpha,\rceil}_{mq}(T,\tau)V^{\beta}_{qj} \quad \mathcal{O}(N_\tau N^3 N_\alpha^2) \tag{64}$$

Next we define $Z^{\alpha}_{kj}(T,\tau) = \sum_{m\beta} V^{\beta}_{km}\Pi^{\alpha\beta}_{mj}(T,\tau)$. We need the $Z^{\rceil}$ components.

$$Z^{\alpha,\rceil}_{kj}(T,\tau) = \sum_{m\beta} V^{\beta}_{km}\Pi^{\alpha\beta,\rceil}_{mj}(T,\tau) \quad \mathcal{O}(N_\tau N^3 N_\alpha^2) \tag{65}$$

To finish the calculation we contract Z with X, $\Sigma_{ij}(T,\tau) = \sum_{k\alpha} Z^{\alpha}_{kj}(T,\tau)X^{\alpha}_{ik}(T,\tau)$

$$\Sigma^{\rceil}_{ij}(T,\tau) = \sum_{k\alpha} Z^{\alpha,\rceil}_{kj}(T,\tau)X^{\alpha,\lceil}_{ik}(\tau,T) \quad \mathcal{O}(N_\tau N^3 N_\alpha) \tag{66}$$

4 Retarded Timestep

For the time-stepping of the Retarded component of the Greens function, we solve the equation

$$-i\partial_{t'}G^R(T,t') - G^R(T,t')h^{MF}(t') - \int_{t'}^{T} d\bar{t}G^R(T,\bar{t})\Sigma^R(\bar{t},t') = 0 \tag{67}$$

$$i\partial_{t'}G^R(T,T-t') - G^R(T,T-t')h^{MF}(T-t') - \int_{0}^{t'} d\bar{t}G^R(T,T-\bar{t})\Sigma^R(T-\bar{t},T-t') = 0 \tag{68}$$

$$\tag{69}$$

for every $0 \le t' \le T$.

4.1 Start bootstrap

To solve for $1 \le t' \le k$, we solve a linear system

$$GM = Q \quad \mathcal{O}(N^4k^3). \tag{70}$$

Here, M is a $kN \times kN$ matrix and G is a $N \times kN$ matrix. The matrices M and Q are given by

$$M_{ln} = \frac{iI_{pd(n,l)}}{dt} - \delta_{ln}h^{MF}(T-l) - dtI_{gw(n,l)}\Sigma^R(T-l,T-n) \tag{71}$$

$$Q_n = \frac{-iI_{pd(n,0)}}{dt}G^R(T,T) + dtI_{gw(n,0)}G^R(T,T)\Sigma^R(T,T-n) \tag{72}$$

4.2 Stepping

Next to solve for $k < t' \le T$, we solve at each timestep a linear system

$$GM = Q \quad \mathcal{O}(T(N^4 + TN^3)). \tag{73}$$

The first T comes from the fact that we have to solve this equation for each t', the N^4 comes from solving the linear system and the TN^3 comes from doing the integral for each timestep.

where M, G, Q are now all $N \times N$ matrices. The matrices M and Q are given by

$$M = \frac{iI_{bd(0)}}{dt} - h^{MF}(T-t') - dtI_{\omega(0)}\Sigma^R(T-t',T-t') \tag{74}$$

$$Q = \sum_{i=1}^{k+1} \frac{-iI_{bd(i)}}{dt}G^R(T,T-t'+i) + dt\sum_{i=0}^{t'-1} I_{P(t',i)}G^R(T,T-i)\Sigma^R(T-i,T-t'). \tag{75}$$

5 Right Mixing Timestep

For the time-stepping of the $\rceil$ component of the Greens function, we solve the equation

$$i\partial_T G^\rceil(T,\tau) - h^{MF}(T)G^\rceil(T,\tau) - \int_0^T d\bar{t}\Sigma^R(T,\bar{t})G^\rceil(\bar{t},\tau) = \int_0^\beta d\bar{\tau}\Sigma^\rceil(T,\bar{\tau})G^M(\bar{\tau}-\tau) \tag{76}$$

for every $0 \le \tau \le N_\tau$.

5.1 Stepping

To solve for $0 \le \tau \le N_{tau}$, we solve at each τstep a linear system

$$MG = Q \quad \mathcal{O}(N_\tau(N^4 + N_\tau N^3 + TN^3)). \tag{77}$$

The first N_τ comes from the fact that we have to solve this equation for each τ, the N^4 comes from solving the linear system, the $N_\tau N^3$ comes from doing the $\rceil, M$ integral for each τstep and the TN^3 comes from the other integral.

M, G, Q are all $N \times N$ matrices and are given by

$$M = \frac{iI_{bd(0)}}{dt} - h^{MF}(T) - dtI_{\omega(0)}\Sigma^R(T,T) \tag{78}$$

$$Q = \sum_{i=1}^{k+1} \frac{-iI_{bd(i)}}{dt}G^\rceil(T-i,\tau) + dt\sum_{i=0}^{T-1} I_{P(T,i)}\Sigma^R(T,i)G^\rceil(i,\tau) + d\tau\sum_{i=0}^{N_\tau} I_{P(N_\tau,i)}\Sigma^\rceil(T,i)G^M(i-\tau). \tag{79}$$

6 Lesser Timestep

For the time-stepping of the Lesser component of the Greens function, we solve the equation

$$i\partial_t G^<(t,T) - h^{MF}(T)G^R(T,t') - \int_0^t d\bar{t}\Sigma^R(t,\bar{t})G^<(\bar{t},T) = \tag{80}$$

$$\int_0^T d\bar{t}\Sigma^<(t,\bar{t})G^A(\bar{t},T) - i\int_0^\beta d\tau\Sigma^\rceil(t,\tau)G^\lceil(\tau,T) \tag{81}$$

$$\tag{82}$$

for every $0 \le t \le T$.

6.1 Start bootstrap

To solve for $1 \le t \le k$, we solve a linear system

$$MG = Q \quad \mathcal{O}(N^4k^3). \tag{83}$$

Here, M is a $kN \times kN$ matrix and G is a $Nk \times N$ matrix. The matrices M and Q are given by

$$M_{ln} = \frac{iI_{pd(l,n)}}{dt} - \delta_{ln}h^{MF}(l) - dtI_{gw(n,l)}\Sigma^R(m,l) \tag{84}$$

$$Q_n = \frac{-iI_{pd(n,0)}}{dt}G^<(0,T) + dtI_{gw(n,0)}\Sigma^R(n,0)G^<(0,T) + dt\sum_{i=0}^{T}\Sigma^<(n,i)G^A(i,T) - id\tau\sum_{i=0}^{N_\tau}\Sigma^\rceil(n,i)G^\lceil(i,T) \tag{85}$$

6.2 Stepping

Next to solve for $k < t \leq T$, we solve at each timestep a linear system

$$MG = Q \quad \mathcal{O}(T(N^4 + TN^3 + TN^3 + N_\tau N^3)). \tag{86}$$

The first T comes from the fact that we have to solve this equation for each t, the N^4 comes from solving the linear system and the remaining terms come from the integrals.

M, G, Q are now all $N \times N$ matrices and are given by

$$M = \frac{iI_{bd(0)}}{dt} - h^{MF}(t) - dtI_{\omega(0)}\Sigma^R(t,t) \tag{87}$$

$$Q_n = \sum_{i=1}^{k+1} \frac{-iI_{bd(i)}}{dt} G^<(t-i,T) + dt\sum_{i=0}^{t-1} I_{gw(t,i)}\Sigma^R(t,i)G^<(i,T) + dt\sum_{i=0}^{T}\Sigma^<(t,i)G^A(i,T) - id\tau\sum_{i=0}^{N_\tau}\Sigma^\rceil(t,i)G^\lceil(i,T) \tag{88}$$