# Finite Difference approximation
# for Asian option with fixed strike price

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January 10, 2017

## 1 Introduction

This report is our attempt to study the numerical approximation for the PDE of Asian option with fixed strike call. Considered to be an exotic option (i.e. more complicated than the Vanilla European option), the Asian option with fixed strike call $K$ and maturity $T$ has the payoff:

$$\mathbb{E}(A - K)_+$$

where A is defined as the arithmetic average price of underlying assets during time period $[0, T]$:

$$A \equiv \frac{1}{T} \int_0^T S_u du$$

Here S is a Geometric Brownian Motion following usual Black-Scholes assumption. Roger & Shi (1995) proposes an alternative PDE which is much more simpler and cost less time to solve than the original PDE of the option value, but Dubois & Lalièvre (2005) note that approximation results in the former paper is poor in term of accuracy. We will try to use both methods in same settings (similar parameters) and compare the results in order to justify this argument.

## 2 Derivation of PDE of Asian Fixed Strike Call

Assuming the price of stock follows Geometric Brownian Motion:

$$dS_t = rdt + \sigma dB_t$$

where $B_t$ is a standard Brownian motion, we construct a portfolio consisting of one option and an amount of $-\Delta$ of the underlying asset as the hedging amount. The value of the portfolio then is:

$$\pi = V - \Delta S$$

With $V := V(t, S, A)$. The change in value of this portfolio in one time step is:

$$d\pi = dV - \Delta dS \tag{2.1}$$

Here $\Delta$ is held fixed during the time step. Applying Ito's formula for $dV$:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial A}dA + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d\langle S, S\rangle_t$$

With: $d\langle S, S\rangle_t = \sigma^2 t^2 dt; dA = -\frac{1}{t^2}\int_0^t S_\tau d\tau dt + \frac{1}{t}S_t dt = \frac{S_t - A_t}{t}dt$
Replacing into (2.1) we have:

$$\begin{aligned} d\pi &= \frac{\partial V}{\partial t}dt + \frac{S_t - A_t}{t}\frac{\partial V}{\partial A}dt + \frac{\partial V}{\partial S}dS + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial S^2}dt - \Delta dS \\ &= \Big(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial A}\frac{S_t - A_t}{t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial S^2}\Big)dt + \Big(\frac{\partial V}{\partial S} - \Delta\Big)dS \end{aligned} \tag{2.2}$$

The random component in (2.2) can be eliminated by choosing a delta-hedging strategy:

$$\Delta = \frac{\partial V}{\partial S} \tag{2.3}$$

Using the concept of no arbitrage opportunity with the assumption of no transaction costs, the return on an amount of $\pi$ invested in riskless portforlio would see a growth of $r\pi\ dt$ in time $dt$ would be:

$$d\pi = r\pi dt \tag{2.4}$$

From (2.2), (2.3) and (2.4):

$$\begin{aligned} &\frac{\partial V}{\partial t}+\frac{\partial V}{\partial A}\frac{S_t-A_t}{t}+\frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial S^2}=r\Big(V-\frac{\partial V}{\partial S}S\Big)\\ \Leftrightarrow\ &\frac{\partial V}{\partial t}+\frac{\partial V}{\partial A}\frac{S_t-A_t}{t}+\frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial S^2}-rV+r\frac{\partial V}{\partial S}S=0 \end{aligned}$$

With fixed strike Asian Call option the payoff function is:

$$\begin{aligned} &\varphi(S_T,A_T)=(A_T-K)_+=\Big(\frac{1}{T}\int_0^T S_\tau d\tau-K\Big)_+\\ \Rightarrow\ &\varphi(S_T,A_T)=V(T,S,A) \end{aligned}$$

So:

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{\partial V}{\partial A}\dfrac{S_t-A_t}{t}+\dfrac{\sigma^2 S_t^2}{2}\dfrac{\partial^2 V}{\partial S^2}-rV+r\dfrac{\partial V}{\partial S}S=0\\ \varphi(S_T,A_T)=V(T,S,A) \end{cases} \tag{2.5}$$

This is the Black-Scholes partial differential equation for the case of Asian Call Option with fixed strike. As pointed out by Rogers & Shi (1995), the PDE (2.5) is hard to solve nummerically. They then proposed a method of changing variable in order for the PDE to be solved much more efficiently. Denote $x\equiv\frac{K-tA/T}{S}$, the value of the option is equivalent to:

$$V(t,S,A)=Sf(T-t,x)$$

Then we have:

$$\begin{aligned} \frac{\partial V}{\partial t}&=S\Big(-\frac{\partial f}{\partial t}-\frac{A}{TS}\frac{\partial f}{\partial x}\Big)\\ \frac{\partial V}{\partial A}&=-\frac{t}{T}\frac{\partial f}{\partial x}\\ \frac{\partial V}{\partial S}&=f+S\frac{\partial f}{\partial S}=f+S\frac{\partial x}{\partial S}\frac{\partial f}{\partial x}=f-x\frac{\partial f}{\partial x}\\ \frac{\partial^2 V}{\partial S^2}&=\frac{x^2}{S}\frac{\partial^2 f}{\partial x^2} \end{aligned}$$

and replacing into (2.5):

$$\begin{cases} S\Big(-\dfrac{\partial f}{\partial t}-\dfrac{A}{TS}\dfrac{\partial f}{\partial x}\Big)-\dfrac{t}{T}\dfrac{\partial f}{\partial x}\dfrac{S-A}{t}+\dfrac{\sigma^2 S^2}{2}\dfrac{x^2}{S}\dfrac{\partial^2 f}{\partial x^2}-rSx\dfrac{\partial f}{\partial x}=0\\ \varphi(S_T,A_T)=V(T,S,A)=Sf(0,x) \end{cases}$$

$$\Leftrightarrow\begin{cases} \dfrac{\partial f}{\partial t}+\Big(\dfrac{1}{T}+rx\Big)\dfrac{\partial f}{\partial x}-\dfrac{\sigma^2 x^2}{2}\dfrac{\partial^2 f}{\partial x^2}=0\\ f(0,x)=\dfrac{V(T,S,A)}{S}=\dfrac{(A_T-K)_+}{S}=\Big[-\Big(\dfrac{K-\frac{TA_T}{T}}{S}\Big)\Big]_+=x_- \end{cases} \tag{2.6}$$

The price of the option at time $t = 0$ is $V(0, S_0, S_0) = Sf(T, x = \frac{K}{S_0})$.

Regarding the boundary conditions of the scheme, one can guess that $\lim_{x \to +\infty} f(t, x) = 0$, and $\lim_{x \to -\infty} f(t, x) = g(t, x)$ where $g$ satifies the PDE in (2.6) and has the form $g(t, x) = xa(t) + b(t)$, and $f(0, x) = g(0, x) = -x$. So we can derive:

$$\begin{cases} x\dfrac{\partial a}{\partial t}(t) + \dfrac{\partial b}{\partial t}(t) + \left(\dfrac{1}{T} + rx\right)a(t) - \dfrac{\sigma^2 x^2}{2} \times 0 = 0 \\ f(0, x) = g(0, x) = -x \end{cases}$$

$$\Leftrightarrow \begin{cases} x\left(\dfrac{\partial a}{\partial t} + ra\right) + \dfrac{\partial b}{\partial t} + \dfrac{a}{T} = 0 \\ xa(0) + b(0) = -x \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{\partial a}{\partial t} + ra = 0 \\ \dfrac{\partial b}{\partial t} + \dfrac{a}{T} = 0 \\ b(0) = 0 \\ a(0) = -1 \end{cases}$$

Solving this system of equations, we get:

$$\begin{cases} a(t) = -e^{-rt} \\ b(t) = \dfrac{1}{rT} - \dfrac{1}{rT}e^{-rt} \end{cases}$$

Then $g(t, x) = -xe^{-rt} - \dfrac{1}{rT}e^{-rt} + \dfrac{1}{rT}$.

Instead of $\Omega = (-\infty, +\infty)$ we consider a truncated domain $\Omega = [X_{min}, X_{max}]$ in which $f(t, X_{min}) \approx g(t, X_{min})$ and $f(t, X_{max}) \approx 0$ Also since $f(0, x) = -x$ then $x \leq 0$ and in the calculation of current price of option we used current price $x = \frac{K}{S_0}$ which is generally equal arround 1. Hence we can choose $X_{min} \leq 0$ and $X_{max} > 1$. In our project, we take values $X_{min} = 0$ and $X_{max} = 2$

# 3 Finite Difference scheme

From previous session, we have the PDE on truncated domain $\Omega = [X_{min}, X_{max}]$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \Big(\dfrac{1}{T} + rx\Big)\dfrac{\partial f}{\partial x} - \dfrac{\sigma^2 x^2}{2}\dfrac{\partial^2 f}{\partial x^2} = 0, t \in [0,T] \\ f(t, X_{min}) = f_l(t) = X_{min}e^{-rt} - \dfrac{1}{rT}e^{-rt} + \dfrac{1}{rT} \\ f(t, X_{max}) = f_r(t) = 0 \\ f(0,x) = x_- \end{cases} \tag{3.1}$$

Let $h = \dfrac{X_{max} - X_{min}}{I+1}$ be a spatial mesh step, $\tau = T/N$ be the time step. Then $x_i = X_{min} + ih, i = 0, 1.., I+1$ is the mest point and $t_n = n\tau, n = 0, 1, ..N$. We are looking for $U_j^n$, an approximation of $f(t_n, x_j)$. Base on the framework we introduce 3 numerical finite difference schemes to solve the PDE (3.1) on specified domain.

## 3.1 Euler Explicit Scheme

Using forward difference for $\dfrac{\partial f}{\partial t}$ and centered difference approximation for $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial^2 f}{\partial x^2}$ we have:

$$\begin{cases} \dfrac{U_i^{n+1} - U_i^n}{\tau} + \dfrac{\sigma^2 x_i^2}{2}\Big(\dfrac{-U_{i-1}^n + 2U_j^n - U_{j+1}n}{h^2}\Big) + \Big(\dfrac{1}{T} + rx_i\Big)\dfrac{U_{i+1}^n - U_{i-1}^n}{2h} = 0 \\ U_0^n = f_l(t) = -X_{min}e^{-rt} - \dfrac{1}{T}e^{-rt} + \dfrac{1}{rT} \\ U_{I+1}^n = f_r(t) = 0 \\ U_i^0 = x_- \end{cases} \tag{3.2}$$

Let $\alpha_i = \dfrac{\sigma^2 x_i^2}{2h^2}, \beta_i = \dfrac{1/T + rx_i}{2h}$. We could write the above system under vector form as follows:

$$\frac{U^{n+1} - U^n}{\tau} + AU^n + q^n) = 0$$
$$\Leftrightarrow U_{n+1} = (I - \tau A)U^n - \tau q^n)$$

in which A is a square matrix of dimension I and $q^n$ is a column vector of $1 \times I$. These parameters satisfy:

$$-(\alpha_i + \beta_i)U_{i-1}^n + 2\alpha_i U_i^n + (-\alpha_i + \beta_i)U_{i+1}^n = (AU^n + q^n)_i$$

So:

$$A = \begin{bmatrix} 2\alpha_1 & -\alpha_1 + \beta_1 & 0 & 0 & \dots & 0 \\ -\alpha_2 - \beta_2 & 2\alpha_2 & -\alpha_2 + \beta_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -(\alpha_I + \beta_I) & 2\alpha_I \end{bmatrix}$$

$$q_n = \begin{bmatrix} -(\alpha_1 + \beta_1)U_0^n \\ 0 \\ \vdots \\ (-\alpha_I + \beta_I)U_{I+1}^n \end{bmatrix} = \begin{bmatrix} -(\alpha_1 + \beta_1)f_l(t_n) \\ 0 \\ \vdots \\ (-\alpha_I + \beta_I)f_r(t_n) \end{bmatrix}$$

*Error analysis*

Denote $f_j^n \equiv f(t_n, x_j)$ the exact solution of PDE (3.1), and:

$$f^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_I^n \end{bmatrix}$$

On the class, we already derived consistency error as follows:

$$\varepsilon^n = \frac{f^{n+1} - f^n}{\tau} + Af^n + q^n \leq C(\tau + h^2)$$

Also from EE scheme, we got:

$$0 = \frac{U^{n+1} - U^n}{\tau} + AU^n + q^n$$

Then, we can dereive the error of EE scheme as belows:

$$E^n = U^n - f^n \Rightarrow -\varepsilon^n = \frac{E^{n+1} - E^n}{\tau} + AE^n \Rightarrow E^{n+1} = (I - \tau A)E^n - \tau\varepsilon^n$$

Let $B = I - \tau A \Rightarrow E^{n+1} = BE^n - \tau\varepsilon^n$

$$\Leftrightarrow E^{n+1} = B(E^{(n-1)} - \tau\varepsilon^{(n-1)} - \tau\varepsilon^{(n-1)}) - \tau\varepsilon^{(n)}$$

$$\vdots$$

$$\Leftrightarrow E^n = B^n E^0 - \sum_{k=0}^{n-1} \tau B^{n-k+1}\varepsilon^k$$

$$\Leftrightarrow \|E^{(n)}\|_\infty \leq \|B\|_\infty^n \|E^{(0)}\|_\infty + \sum_{k=0}^{n-1} \tau \|B\|_\infty^{n-k+1} \|\varepsilon^{(k)}\|_\infty$$

$$\Leftrightarrow \|E^{(n)}\|_\infty \leq \|B\|_\infty^n \|E^{(0)}\|_\infty + \tau_n C(\tau + h^2)$$

$$\Leftrightarrow \|E^{(n)}\|_\infty \leq TC(\tau + h^2)$$

in which (.) denotes numerical order and $C$ denotes a constant.

Here we need conditions to make sure $\|B\|_\infty \leq 1$

$$B = I - \tau A = \begin{bmatrix} 1 - 2\tau\alpha_1 & \tau(\alpha_1 - \beta_1) & 0 & 0 & \dots & 0 \\ \tau(\alpha_2 + \beta_2) & 1 - 2\tau\alpha_2 & \tau(\alpha_2 - \beta_2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 - 2\alpha_I \end{bmatrix}$$

$$\Rightarrow \|B\|_\infty \leq Max(|1 - 2\tau\alpha_i| + |\tau(\alpha_i + \beta_i)| + \tau|\alpha_i - \beta_i|)$$

$$\leq 1 \text{ iff } \begin{cases} 1 - 2\tau\alpha_i \geq 0 \forall i & (CFL) \\ \alpha_i > \beta_i \forall j & (H) \end{cases}$$

$$(H) : \frac{\sigma^2 x_i^2}{2h^2} > \frac{\frac{1}{T} + rx_i}{2h} \forall i \Leftrightarrow h \leq \frac{\sigma^2 X_i^2}{\frac{1}{T} + rX_i} \forall i \Leftrightarrow h \leq \frac{\sigma^2 X_{min}^2}{\frac{1}{T} + rX_{min}}$$

$$(CFL) : 2\tau\alpha_i \leq 1 \forall i \Leftrightarrow \sigma^2 X_i^2 \frac{\tau}{h^2} \leq 1 \forall i \Leftrightarrow \sigma^2 X_{max}^2 \frac{\tau}{h^2} \leq 1$$

Then under (H) and (CFL):

$$\|E^{(n)}\| \leq TC(\tau + h^2) = C(\tau + h^2)$$

$$\Leftrightarrow Max|U_i^n - f_i^n| \leq C(\tau + h^2)$$

This gives us the error of order 1 in time and order 2 in space of EE scheme.

*Stability analysis*

From (3.2) we get: $U_i^{n+1} = U_i^n(1-2\tau\alpha_i)+\tau U_{i-1}^n(\alpha_i+\beta_i)+\tau U_{i+1}^n(\alpha_i-\beta_i)$.To be stable scheme, $\|U^n\|_\infty$ needs to be bounded above. Let $\bar{U}_n \equiv Max_{0\le i\le I+1}|U_i^n|$, then:

$$|U_i^{n+1}| \le \bar{U}_n[|1-2\tau\alpha_i| + \tau|\alpha_i+\beta_i| + \tau|\alpha_i-\beta_i|]$$

Under the (CFL), (H) conditions, we get $|U_i^{n+1}| \le \bar{U}_n$. So:

$$\bar{U}_{n+1} \le \bar{U}_n \le \cdots \le \|f_0\|_\infty < \infty$$

which means the EE scheme is stable under (CFL) and (H) conditions.

### 3.2 Euler Implicit Scheme

The Implicit Euler scheme with centered difference approximation for the first spatial derivative is:

$$\begin{cases} \dfrac{U_i^{n+1}-U_i^n}{\tau} + \dfrac{\sigma^2 x_i^2}{2}\Big(\dfrac{-U_{i-1}^{n+1}+2U_i^{n+1}-U_{i+1}^{n+1}}{h^2}\Big) + \Big(\dfrac{1}{T}+rx_i\Big)\dfrac{U_{i+1}^{n+1}-U_{i-1}^{n+1}}{2h} = 0 \\ U_0^{n+1} = f_l(t_{n+1}) = X_{min}e^{-rt_{n+1}} - \dfrac{1}{T}e^{-rt_{n+1}} + \dfrac{1}{rT} \\ U_{I+1}^{n+1} = f_r(t_{n+1}) = 0 \\ U_i^0 = x_- \end{cases} \tag{3.3}$$

Using the same idea of writing in vector form we can derive the following solution:

$$\begin{aligned} &\frac{U^{n+1}-U^n}{\tau} + AU^{n+1} + q^{n+1} = 0 \\ \Leftrightarrow\ & U^{n+1} = (I+\tau A)^{-1}(U^n - \tau q^{n+1}) \end{aligned}$$

*Error analysis*

$$\begin{aligned} \varepsilon^n &= \frac{f^{n+1}-f^n}{\tau} + Af^{n+1} + q^{n+1} \\ 0 &= \frac{U^{n+1}-U^n}{\tau} + AU^{n+1} + q^{n+1} \end{aligned}$$

$E^n = U^n - f^n$ then:

$$-\varepsilon^n = \frac{E^{n+1} - E^n}{\tau} + AE^{n+1}$$
$$\Leftrightarrow E^{n+1} = (I + \tau A)^{-1}(E^n - T\varepsilon^n)$$

Let $B = I + \tau A \Leftrightarrow E^{n+1} = B^{-1}(E^n - \tau E^n)$

So: $E^n = (B^{-1})^n E^0 - \sum_{k=0}^{n-1} \tau (B^{-1})^{n-k+1} \varepsilon^k$

Similar to previous scheme, we can derive $\|E^n\|_\infty \leq C(\tau + h^2)$. It is easy to prove $\|B^{-1}\|_\infty^n \leq 1$ since B is tridiagonal matrix with $B_{ii} = 1 + \tau 2\alpha_i, B_{ii+1} = \tau(-\alpha_i + \beta_i), B_{ii-1} = -\tau(\alpha_i + \beta_i)$.

$$|B_{ii}| = 1 + \tau 2\alpha_i$$
$$|B_{ii-1}| + |B_{ii+1}| = \tau(\alpha_i + \beta_i) + \tau(\alpha_i - \beta_i)$$

If $|B_{ii}| \geq \delta + \sum_{j \neq i} B_{ij}$ or B is $\delta$-diagonal determinant matrix then $\|B^{-1}\|_\infty \leq \frac{1}{\delta}$ with $\delta = 1$. This happens only when (H): $\alpha_i \geq \beta_i \Leftrightarrow h \leq X_min \frac{\sigma^2}{r}$

So $E^n \leq C(\tau + h^2)$ or $Max|U^n - f^n| \leq C(\tau + h^2)$ under (H) condition.

This gives the EI has the error of order 1 in time and 2 in space.

*Stability analysis*

From (3.3) setup we get:

$$U^{n+1} = (I + \tau A)^{-1}(U^n - \tau q^{n+1}) = B^{-1}(U^n - \tau q^{n+1})$$
$$\Leftrightarrow \|U^{n+1}\|_\infty \leq \|B^{-1}|_\infty(\|U^n\|_\infty + \tau\|q^{n+1}\|_\infty)$$

As shown before $\|B^{-1}|_\infty \leq 1$ then $\|U^{n+1}\|_\infty \leq \|U^n\|_\infty + \tau\|q^{n+1}\|_\infty$

Also:

$$q^{n+1} = \begin{bmatrix} -(\alpha_1 + \beta_1)U_0^{n+1} \\ 0 \\ \vdots \\ (-\alpha_I + \beta_I)U_{I+1}^{n+1} \end{bmatrix} = \begin{bmatrix} -(\alpha_1 + \beta_1)(-X_{min}e^{-rt_{n+1}} - \frac{1}{rT}e^{-rt_{n+1}} + \frac{1}{rT}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\|q^{n+1}\|_\infty = |\alpha_1 + \beta_1||-X_{min}e^{-rt_{n+1}} - \frac{1}{rT}e^{-rt_{n+1}} + \frac{1}{rT}|$ is contious with respect to $t$ in bounded interval $t \in [0, T] \Rightarrow \|q^{n+1}\|_\infty$ is bounded by a constant. So:

$$\|U^{n+1}\|_\infty \leq \|U^n\|_\infty + \tau C$$

Take recursion then

$$\Leftrightarrow \|U^n\|_\infty \leq \|f_0\|_\infty + n\tau C \leq \|f_0\|_\infty + TC$$

So for EI scheme, we do not need (CFL) condition for stability. Only (H) condition is enough for EI to be stable.

### 3.3 Crank-Nicholson Scheme

The Crank-Nicholson scheme with centered difference approximation for the first spatial derivative is:

$$\begin{cases} \dfrac{U_i^{n+1}-U_i^n}{\tau} + \phi\Big[\dfrac{\sigma^2 x_i^2}{2}\Big(\dfrac{-U_{i-1}^{n+1}+2U_i^{n+1}-U_{i+1}^{n+1}}{h^2}\Big) + \Big(\dfrac{1}{T}+rx_i\Big)\dfrac{U_{i+1}^{n+1}-U_{i-1}^{n+1}}{2h}\Big] \\ +(1-\phi)\Big[\dfrac{\sigma^2 x_i^2}{2}\Big(\dfrac{-U_{i-1}^{n}+2U_i^{n}-U_{i+1}^{n}}{h^2}\Big) + \Big(\dfrac{1}{T}+rx_i\Big)\dfrac{U_{i+1}^{n}-U_{i-1}^{n}}{2h}\Big] = 0 \\ U_0^n = f_l(t_n) = X_{min}e^{-rt_n} - \dfrac{1}{rT}e^{-rt_n} + \dfrac{1}{rT} \\ U_{I+1}^n = f_r(t_n) = 0 \\ U_i^0 = x_- \end{cases} \tag{3.4}$$

We obtain a vector form for CN scheme:

$$\frac{U^{n+1}-U^n}{\tau} + \phi(AU^{n+1}+q^{n+1}) + (1-\phi)(AU^n+q^n) = 0$$

In our study, we choose $\phi = \frac{1}{2}$

$$\Leftrightarrow U^{n+1} = \Big(I + \frac{\tau A}{2}\Big)^{-1}\Big[\Big(I - \frac{\tau A}{2}\Big)U^n - \frac{\tau}{2}\Big(q^n + q^{n+1}\Big)\Big]$$

<u>*Error Analysis*</u>

Consistency error:

$$\begin{aligned} |\varepsilon_i^n| = |\frac{f_i^{n+1}-f_i^n}{\tau} &+ \frac{1}{2}\Big[\frac{\sigma^2 x_i^2}{2}\Big(\frac{-f_{i-1}^{n+1}+2f_i^{n+1}-f_{i+1}^{n+1}}{h^2}\Big) + \Big(\frac{1}{T}+rx_i\Big)\frac{f_{i+1}^{n+1}-f_{i-1}^{n+1}}{2h}\Big]| \\ &+\frac{1}{2}\Big[\frac{\sigma^2 x_i^2}{2}\Big(\frac{-f_{i-1}^{n}+2f_i^{n}-f_{i+1}^{n}}{h^2}\Big) + \Big(\frac{1}{T}+rx_i\Big)\frac{f_{i+1}^{n}-f_{i-1}^{n}}{2h}\Big] \\ &-\frac{1}{2}\Big|f_t(t_n,x_i) - \frac{1}{2}\sigma^2 x_i^2 f_{XX}(t_n,x_i) + \Big(\frac{1}{T}+rx_i\Big)f_X(t_n,x_i)\Big| \\ &-\frac{1}{2}\Big|f_t(t_{n+1},x_i) - \frac{1}{2}\sigma^2 x_i^2 f_{XX}(t_{n+1},x_i) + \Big(\frac{1}{T}+rx_i\Big)f_X(t_{n+1},x_i)\Big| \end{aligned} \tag{3.5}$$

On the class we already proved:

$$\Big|\frac{f_{i-1}^{n+1}-2f_i^{n+1}+f_{i+1}^{n+1}}{h^2}-f_{XX}(t_{n+1},x_i)\Big|\le\frac{1}{12}\|f_{4X}\|_\infty h^2 \tag{3.6}$$

$$\Big|\frac{f_{i-1}^{n}-2f_i^{n}+f_{i+1}^{n}}{h^2}-f_{XX}(t_{n},x_i)\Big|\le\frac{1}{12}\|f_{4X}\|_\infty h^2 \tag{3.7}$$

$$\Big|\frac{f_{i+1}^{n+1}-f_{i-1}^{n+1}}{2h}-f_{X}(t_{n+1},x_i)\Big|\le\frac{1}{6}\|f_{3X}\|_\infty h^2 \tag{3.8}$$

$$\Big|\frac{f_{i+1}^{n}-f_{i-1}^{n}}{2h}-f_{X}(t_{n},x_i)\Big|\le\frac{1}{6}\|f_{3X}\|_\infty h^2 \tag{3.9}$$

Also, using Taylor expansion:

$$f_i^{n+1}=f_i^n+\tau f_t(t_n,x_i)+\frac{\tau^2}{2}f_{tt}(\varepsilon,x_i)$$
$$f_i^{n}=f_i^{n+1}-\tau f_t(t_{n+1},x_i)+\frac{\tau^2}{2}f_{tt}(\varepsilon',x_i)$$

with $\varepsilon,\varepsilon'\in(t_n,t_{n+1})$. Subtract 1st equation to 2nd equation we get:

$$\frac{f_i^{n+1}-f_i^n}{\tau}=\frac{1}{2}\Big[f_t(t_n,x_i)+f_t(t_{n+1},x_i)+\frac{\tau^2}{4}\Big(f_{tt}(\varepsilon,x_i)-f_{tt}(\varepsilon',x_i)\Big)\Big]$$
$$\Rightarrow\Big|\frac{f_i^{n+1}-f_i^n}{\tau}|-\frac{1}{2}\Big[f_t(t_n,x_i)+f_t(t_{n+1},x_i)\Big]\Big|\le\frac{\tau^2}{4}\Big(f_{tt}(\varepsilon,x_i)-f_{tt}(\varepsilon',x_i)\Big)$$

Substitute equations (3.6) - (3.9) into (3.5) we got:

$$|\varepsilon_j^n|\le\frac{\tau^2}{4}\|f_{tt}\|_\infty-\frac{1}{2}\sigma^2x_i^2\frac{h^2}{12}\|f_{4X}\|_\infty+\Big(\frac{1}{T}+rx_i\Big)\frac{h^2}{6}\|f_{3X}\|_\infty$$
$$\le C\Big(\|f_{tt}\|_\infty,\|f_{4X}\|_\infty,\|f_{3X}\|_\infty\Big)(\tau^2+h^2)\approx C(\tau^2+h^2)$$

So Crank-Nicholson scheme is (2,2) consistent with PDE (3.1)

$$\varepsilon^n=\frac{f^{n+1}-f^n}{\tau}+\frac{1}{2}(Af^n+q^n)+\frac{1}{2}(Af^{n+1}+q^{n+1})$$
$$0=\frac{U^{n+1}-U^n}{\tau}+\frac{1}{2}(AU^n+q^n)+\frac{1}{2}(AU^{n+1}+q^{n+1})$$

Consider error: $E^n=U^n-f^n$

$$\Rightarrow-\varepsilon^n=\frac{E^{n+1}-E^n}{\tau}+\frac{1}{2}AE^n+\frac{1}{2}AE^{n+1}$$
$$\Rightarrow-\tau E^n=E^{n+1}-E^n+\frac{1}{2}\tau AE^n+\frac{1}{2}\tau AE^{n+1}$$
$$\Rightarrow E^{n+1}=\Big(I+\frac{\tau A}{2}\Big)^{-1}\Big[\Big(I-\frac{1}{2}\tau A\Big)E^n-\tau\varepsilon^n\Big]$$
$$\Rightarrow\|E^{n+1}\|_\infty\le\Big\|I+\frac{\tau A}{2}^{-1}\Big\|_\infty\Big[\Big\|I-\frac{1}{2}\tau A\Big\|_\infty\|E^n\|_\infty+\tau\|\varepsilon^n\|_\infty\Big]$$
$$\Rightarrow\|E^{n+1}\|_\infty\le\|E^n\|_\infty+\tau\|\varepsilon^n\ \text{(see explanation in Stability Analysis below)}$$
$$\Rightarrow\|E^{n+1}\|\le\|E^n\|_\infty+\tau C(\tau^2+h^2)$$

or

$$\|E^{n+1}\|_\infty \le \|E^{n-1}\|_\infty + 2\tau C(\tau^2 + h^2)$$
$$\vdots$$
$$\|E^{n+1}\|_\infty \le \|E^0\|_\infty + TC(\tau^2 + h^2) = C(\tau^2 + h^2)$$

So the order of CN scheme is 2 in time and 2 in space, which is the most accurate of the 3 schemes.

*Stability Analysis*

For the CN scheme:

$$\begin{aligned}
U^{n+1} &= \Big(I + \frac{\tau A}{2}\Big)^{-1}\Big[\Big(I - \frac{\tau A}{2}\Big)U^n - \frac{\tau}{2}\Big(q^n + q^{n+1}\Big)\Big] \\
&= \frac{I - \tau A/2}{I + \tau A/2}U^n - \frac{\tau}{2(I + \tau A/2)}(q^n + q^{n+1}) \\
\Rightarrow \|U^{n+1}\|_\infty &\le \frac{\|I - \tau A/2\|_\infty}{\|I + \tau A/2\|_\infty}\|U^n\| + \frac{\tau}{2\|I + \tau A/2\|_\infty}(\|q^n\| + \|q^{n+1}\|)
\end{aligned}$$

As proved in EI scheme, $\|(I + \tau A/2)^{-1}\|_\infty \le 1$ with $\alpha_i \ge \beta_i$ (H). Also:
$\|I - \tau A/2\|_\infty \le max_\tau(|1 - \tau\alpha_i| + \tau/2|\alpha_i + \beta_i| + |\alpha_i - \beta_i|)$
We can put conditions above $1 - \tau\alpha_i \ge 0$ to simplify above inequality as $\|I - \tau A/2\|_\infty \le 1$. And as previous scheme, we already prove $\|q^n\|_\infty < C$, $\|q^{n+1}\|_\infty < C$. In the end we get:

$$\|U^{n+1}\|_\infty \le \|U^n\|_\infty + C$$
$$\text{or } \|U^{n+1}\|_\infty \le \|U_0\|_\infty + nC \le C$$

which gives us stability. Even above we put condition $1-\tau\alpha_i \ge 0 \Rightarrow \dfrac{\tau\sigma^2 x_i^2}{2h^2} \le 1$ so $\tau \approx h^2$. But indeed we do not need this condition for stability in CN scheme, which help reducing computation time.

## 4 Numerical Results

We use the code `Asian.m` to run finite difference scheme to get the approximation. The code is tweaked slightly to make it be able to find and return the correct value of the option at time 0 using interpolation. Also when running

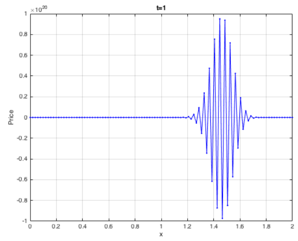

Figure 1: EE

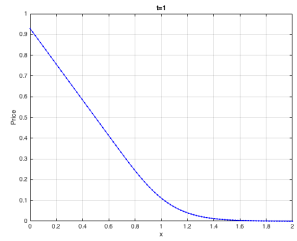

Figure 2: EI

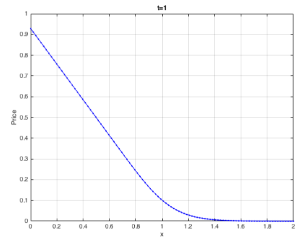

Figure 3: CN

the program will first ask for input of the parameters. We believe doing this way will reduce the risk of making mistakes when modifying parameters directly inside the code. Our default settings is $T = 1, X_{min} = 0, X_{max} = 2, K = S_0 = 100, \sigma = 0.3, r = 0.15$ We also use a reference value to calculate error and checking rate of convergence, which is $P = 10.209774$ obtained when running CN scheme with default settings and taking $N = 2000, I = 3000$.

### 4.1 Convergence & Stability

When taking $N = I = 100$, we can clearly see that the EE scheme is not stable. This is due to the fact that $CFL = 9.181 > 1$. The case for EI and CN is not the same when both scheme showing stability. This confirms the derivation above for the necessity of CFL condition for EE scheme. When using EE scheme, we take $N \equiv I^2$ to get rid of this problem. However, computation time can be greatly increased.

To test the order in time and space, we run 3 schemes and check with the reference value. Table below shows that it takes the EE and EI scheme large step size to converge to 1 and 2 accurate digits respectively, the CN scheme converge to 3 digit accurately using only small space and time step. The computation time for CN therefore is significantly smaller than EE and EI scheme.

| Scheme | P | Time | N | I |
|---|---|---|---|---|
| EE | 10.204784 | 25.4 | 25000 | 500 |
| EI | 10.211209 | 272.06 | 49000 | 700 |
| CN | 10.209612 | 7.93 | 600 | 900 |

(Reference Value: 10.209774)

## 4.2 Order of Scheme

To test the order of EE and EI, we take $N = \frac{I^2}{4}$, then with our defaut settings, $\tau \approx h^2 \Rightarrow E_k = C(h_k)^2$. Here $E_k$ is the absolute error at step k, caculated as the difference between the price obtained by doubling I each time and the reference price. Since $N = \frac{I^2}{4}$, the mesh step $h_k$ is also reduced about twice at each step. Hence we could expected that $e_{k-1} - e_k \approx 4$. The formula for order is $\alpha_k = \frac{ln(E_{k-1}/E_k)}{ln(h_{k-1}/h_k)} = \frac{ln(E_{k-1}/E_k)}{ln(2)}$. These tables below illustrate our argument:

**First order in time & Second order in space for EE**

| I | Error | $e_{k-1}/e_k$ | Order |
|---|---|---|---|
| 40 | 0.431121 | - | - |
| 80 | 0.108174 | 3.985440 | 1.994739 |
| 160 | 0.026569 | 4.071437 | 2.025538 |
| 320 | 0.006584 | 4.035389 | 2.012708 |
| 640 | 0.001604 | 4.104738 | 2.037290 |

(Reference Value: 10.209774, $\frac{I^2}{4}$)

**First order in time & Second order in space for EI**

| I | Error | $e_{k-1}/e_k$ | Order |
|---|---|---|---|
| 40 | 0.139082 | - | - |
| 80 | 0.018472 | 7.529342 | 2.912524 |
| 160 | 0.004014 | 4.601893 | 2.202228 |
| 320 | 0.000999 | 4.018018 | 2.006484 |
| 640 | 0.000288 | 3.468750 | 1.794416 |

(Reference Value: 10.209774, $\frac{I^2}{4}$)

From the tables, $\alpha_k \approx 2$ so $E_k \approx Ch^2 \approx C\tau$. We can conclude that both EE and EI schemes have the first order in time and the second order in space.

For the case of CN scheme, we use $X_{min} = -1$ and choose $I = 3N$ to get $\tau \approx h$. Since it is the order of 2 in both time and space, if we double the value of $N$ the error should be decreased about 4 times. Table below illustrates this. $\alpha_k = 2$ so $E_k \approx Ch^2 \approx C\tau$

**Second order in time & Second order in space for CN**

| N | Error | $e_{k-1}/e_k$ | Order |
|---|---|---|---|
| 50 | 0.079856 | - | - |
| 100 | 0.019623 | 4.069510 | 2.024855 |
| 200 | 0.004849 | 4.046814 | 2.016786 |
| 400 | 0.001170 | 4.144444 | 2.051179 |
| 800 | 0.000251 | 4.661355 | 2.220749 |

(Reference Value: 10.209774, $I = 3N$)

### 4.3 Results Comparison

We uses parameters $\sigma = \{0.3; 0.05\}, r = 0.15, S_0 = 100, T = 1$ which are common between the schemes of Dubois - Lelivre and Rogers - Shi for the interval $[X_{min} = 0, X_{max} = 2]$. The scheme we applied in comparision is Crank-Nicholson with $I = 1.5N$ which is the same setttings as Dubois - Lelièvre while Rogers - Shi used the mesh step h = 0.005.

**Results comparison with other papers**

| $\sigma$ | K | Our results | D& L | R& S | Low | Up |
|---|---|---|---|---|---|---|
| 0.05 | 95 | 11.094092 (N=600) | 11.09409 (N=300) | 11.090 | 11.094094 | 11.094096 |
| | 100 | 6.7941 (N=2000) | 6.7943 (N=1000) | 6.777 | 6.794354 | 6.794465 |
| | 105 | 2.742603 (N=2000) | 2.7444 (N=3000) | 2.639 | 2.744406 | 2.744581 |
| 0.3 | 90 | 16.512345 (N=600) | 16.512 (N=300) | 16.510 | 16.512024 | 16.523720 |
| | 100 | 10.209158 (N=500) | 10.209 (N=300) | 10.208 | 10.208724 | 10.214085 |
| | 110 | 5.730025 (N=400) | 5.7304 (N=1000) | 5.731 | 5.728161 | 5.735488 |

Our results achieved is quite accurate compare to the method of Dubois - Lelivre. They are precise up to 3 digits after decimal point. And when $\sigma = 0.3$ all the results are well between the two bounds proposed by Thompson. However when $\sigma = 0.05$, all the results are out of the boundaries. The results of Dubois - Lelivre and Rogers - Shi have the same problems. The reason could be when $\sigma$ small, the (H) condition $h \leq \frac{\sigma^2 X_i^2}{\frac{1}{T}+rX_i} \forall i$, the mesh step should be small enough, but our number of I is still large. And the capacity of our personal computers is not enough to run with larger I, we left here for further check.

## 4.4 Changing boundary

We come back to the default settings stated in the beginning of the section and use it to compare the result with reference value when using $X_{min} = 0$ and $X_{min} = -1$ for CN scheme and taking $I = 1.5N$. It is easily to notice that the results with $X_{min} = 0$ is better. This could be explained by the fact that both scenerios have the same time step, but the former has the smaller mesh step than the latter. It also could be possible that the condition (H) could be failed with $X_{min} \leq 0$ since there is double number of points around 0, hence likely occurs more error. Also the choice of $X_{min} = 0$ is far enough from $X = \frac{K}{S_0} = 1$ with $K \approx S_0$ mainly used in our project.

**Results comparison with different boundaries**

| | $X_{min}$ = 0 | $X_{min}$ = -1 |
|---|---|---|
| N = 250 | 10.207145 (0.002629) | 10.203563 (0.006211) |
| N = 500 | 10.209158 (0.000616) | 10.208262 (0.001512) |
| N = 1000 | 10.209662 (0.000112) | 10.209437 (0.000337) |
| N = 1500 | 10.209755 (0.000019) | 10.209655 (0.000119) |
| N = 2000 | 10.209774 (0.000000) | 10.209731 (0.000043) |

## 5 Conclusion

In this project we implemented finite difference approach with 3 different schemes: EI, EN, CN to approximate the value of Asian call option with fixed strike price. Generally, we can obtained accurate results for both small and large volatilities, especially with CN sheme. Also, we also confirmed the similiarity between the order of three schemes in numerical experiment and in theory. Comparing to the methods proposed by Dubois - Lelivre and Rogers - Shi, our methods seems to be relatively easy and accurate to be implemented. However, for more precise results, we would recommend the method of Dubois - Lelivre.

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