

# CONVEX OPTIMIZATION

1 ONE-DIMENSIONAL

2 MULTI-DIMENSIONAL CASE

Consider a smooth function  $f : \Omega \rightarrow \mathbb{R}$ , where  $\Omega$  is a nonempty open subset of  $\mathbb{R}^n$ . We focus on the problem of finding  $\mathbf{x}_0 \in \Omega$  so that  $f(\mathbf{x}_0) = \max_{\mathbf{x} \in \Omega} f(\mathbf{x})$  or  $f(\mathbf{x}_0) = \min_{\mathbf{x} \in \Omega} f(\mathbf{x})$ , denoted by

$$\mathbf{x}_0 = \arg \max_{\mathbf{x} \in \Omega} f(\mathbf{x}) \text{ or } \mathbf{x}_0 = \arg \min_{\mathbf{x} \in \Omega} f(\mathbf{x})$$

Such an  $\mathbf{x}_0$  is called a **global maximum** or **global minimum** of  $f$  on  $\Omega$ . When  $f$  is a differentiable function, we only have the necessary condition for  $\mathbf{x}_0$  :  $\nabla f(\mathbf{x}_0) = \mathbf{0}$ . This condition is also sufficient in convex optimization.

# Outlines

- 1 ONE-DIMENSIONAL
- 2 MULTI-DIMENSIONAL CASE

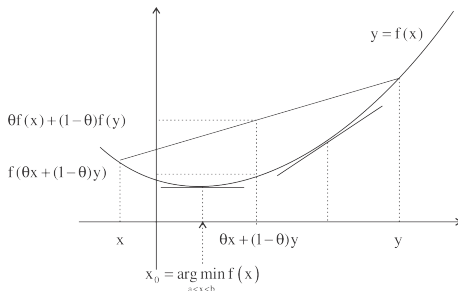
Let  $f : (a, b) \rightarrow \mathbb{R}$ ,  $-\infty \leq a < b \leq \infty$ .

**Definition 1.** The function  $f : (a, b) \rightarrow \mathbb{R}$  is **convex** if for all  $x, y \in (a, b)$  and  $0 \leq \theta \leq 1$ , we have

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y). \quad (1)$$

The function  $f$  is **concave** if  $-f$  is convex.

Geometrically, the inequality (1) means that the line segment between  $(x, f(x))$  and  $(y, f(y))$ , which is a **chord** from  $x$  to  $y$ , lies above the graph of  $f$ .



**Theorem 2.** Let  $f : (a, b) \rightarrow \mathbb{R}$ , where  $-\infty \leq a < b \leq \infty$ , be of class  $C^2$ , i.e. the second derivative  $f''$  exists and is a continuous function. The following statements are equivalent

- (i)  $f$  is a convex function.
- (ii)  $f(y) \geq f(x) + f'(x)(y - x)$ , for all  $x, y \in (a, b)$ .
- (iii)  $f''(x) \geq 0$ , for all  $x \in (a, b)$ .

In order to prove it, we need the following result, which is a particular case of Taylor's formula.

**Proposition 3.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be of class  $C^2$ . For each  $x, y \in (a, b)$ ,  $x < y$ , there exists  $x \leq z \leq y$  such that

$$f(y) = f(x) + f'(x)(y - x) + f''(z) \frac{(y - x)^2}{2!}.$$

**Proof of Theorem 2.** We shall prove  $(i) \Leftrightarrow (ii)$  and  $(ii) \Leftrightarrow (iii)$ . When (i) satisfies, with  $x, y \in (a, b)$  and  $0 \leq \theta \leq 1$ , (1) gives

$$f(y + \theta(x - y)) \leq f(y) + \theta(f(x) - f(y))$$

so with  $0 < \theta \leq 1$ , since

$$f(x) - f(y) \geq \frac{f(y + \theta(x - y)) - f(y)}{\theta} \rightarrow f'(y)(x - y)$$

whenever  $\theta \downarrow 0$ , (ii) is satisfied. Conversely, when (ii) holds, then with  $z = \theta x + (1 - \theta)y$ , we have

$$f(x) \geq f(z) + f'(z)(x - z), \quad (2)$$

and

$$f(y) \geq f(z) + f'(z)(y - z). \quad (3)$$

Multiply both sides of (2) by  $\theta$ , (3) by  $1 - \theta$ , and add, side by side, we get

$$\theta f(x) + (1 - \theta)f(y) \geq f(z) = f(\theta x + (1 - \theta)y),$$

and (i) is proved.

Now, when (ii) holds, we have, with  $x, y \in (a, b)$ ,  $x < y$ ,  
 $f(y) \geq f(x) + f'(x)(y - x)$  and  $f(x) \geq f(y) + f'(y)(x - y)$ .  
 Therefore

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x).$$

Divide both sides by  $(y - x)^2$ , we get

$$\frac{f'(y) - f'(x)}{y - x} \geq 0,$$

for all  $x, y \in (a, b)$ ,  $x < y$ . Letting  $y \rightarrow x$ , we get (iii). Conversely, by proposition 3, there exists  $z \in [x, y]$  so that

$$f(y) = f(x) + f'(x)(y - x) + \frac{1}{2}f''(z)(y - x)^2.$$

Since  $f''(z) \geq 0$ , we must have  $f(y) \geq f(x) + f'(x)(y - x)$ , i.e., (ii) holds.

Using Theorem 2 with  $-f$  when  $f$  is a concave function, we have



**Corollary 4.** Let  $f : (a, b) \rightarrow \mathbb{R}$ , where  $-\infty \leq a < b \leq \infty$ , be of class  $C^2$ . The following statements are equivalent

- (i)  $f$  is a concave function.
- (ii)  $f(y) \leq f(x) + f'(x)(y - x)$ , for all  $x, y \in (a, b)$ .
- (iii)  $f''(x) \leq 0$ , for all  $x \in (a, b)$ .

**Example 1.** The functions  $y = x^n$ , for even  $n \in \mathbb{N}$ , and  $y = e^x$  are convex. The function  $y = \ln x$  is concave.

Let  $f$  be a convex/concave function on  $(a, b)$  and let  $x_0 \in (a, b)$  that satisfies  $f'(x_0) = 0$ . The inequality (ii) of Theorem 2 or Corollary 3, with  $x = x_0$ , yields  $f(x) \geq f(x_0)$  or  $f(x) \leq f(x_0)$ , for all  $x \in (a, b)$ . Hence,  $x_0$  is a global minimum or a global maximum of  $f$  on  $(a, b)$  which shows that  $f'(x_0) = 0$  is also a sufficient condition for global extremum.

**Corollary 5.** Let  $f : (a, b) \rightarrow \mathbb{R}$ , where  $-\infty \leq a < b \leq \infty$ , be of class  $C^1$ , i.e., the first derivative  $f'$  exists and is continuous and let  $x_0 \in (a, b)$  such that  $f'(x_0) = 0$ .

(i) If  $f$  is convex then  $x_0 = \arg \min_{x \in (a, b)} f(x)$ .

(ii) If  $f$  is concave then  $x_0 = \arg \max_{x \in (a, b)} f(x)$ .

**Example 2.** (i) The function  $y = x^4 + 6x^2$ , defined on  $\mathbb{R}$ , has  $y' = 4x^3 + 12x$ ,  $y'' = 12x^2$ . Since  $y'' \geq 0$ , for all  $x \in \mathbb{R}$ ,  $y = x^4 + 6x^2$  is convex on  $\mathbb{R}$ . Now,  $y' = 0 \Leftrightarrow x = 3$  implies that  $\arg \min_{x \in \mathbb{R}} (x^4 + 6x^2) = 3$ .

(ii) The function  $y = -x \ln x$ , defined on  $(0, +\infty)$ , has  $y' = -\ln x + 1$ ,  $y'' = -\frac{1}{x}$ . Since  $y'' < 0$ , for all  $x \in (0, +\infty)$ ,  $y = -x \ln x$  is concave on  $(0, +\infty)$ . Now,  $y' = 0 \Leftrightarrow x = e$  implies that  $\arg \max_{x \in (0, +\infty)} (-x \ln x) = e$ .

# Outlines

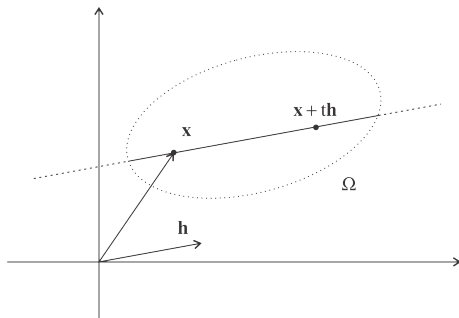
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# MULTI-DIMENSIONAL CASE

Let  $f : \Omega \rightarrow \mathbb{R}$ , where  $\Omega$  is a nonempty subset of  $\mathbb{R}^n$ .

First, we set some notations and terminologies.

**Definition 6.** A nonempty subset  $\Omega \subset \mathbb{R}^n$  is said to be an **open convex** set in  $\mathbb{R}^n$  if for all  $\mathbf{x} \in \Omega$  and  $\mathbf{h} \in \mathbb{R}^n$ ,  $\{t \in \mathbb{R} \mid \mathbf{x} + t\mathbf{h} \in \Omega\}$  is an open interval  $(a, b)$  in  $\mathbb{R}$ , for some  $-\infty \leq a < b \leq \infty$ .



**Example 3.** Any open convex set  $\Omega$  in  $\mathbb{R}$  must be an open interval.  $\mathbb{R}^n$  is itself an open convex subset.

$\mathbb{R} \times (0, \infty) = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  and  $(0, \infty) \times (0, \infty) = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\}$  are two convex open sets in  $\mathbb{R}^2$ .

**Definition 7.** A square matrix  $A \in \mathbb{R}^{n \times n}$  is said to be **positive semidefinite** (**negative semidefinite**, resp), noted  $A \geq 0$  ( $A \leq 0$ , resp.), if  $\mathbf{h}^T A \mathbf{h} \geq 0$  ( $\mathbf{h}^T A \mathbf{h} \leq 0$ , resp.) for all  $\mathbf{h} \in \mathbb{R}^n$ .

**Example 4.** Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

For all  $\mathbf{h} = (h, k) \in \mathbb{R}^2$ ,

$$\begin{aligned} \mathbf{h}^T A \mathbf{h} &= \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} h + 2k \\ 2h + 5k \end{bmatrix} \\ &= h(h + 2k) + k(2h + 5k) = h^2 + 4hk + 5k^2 \end{aligned}$$

By writing  $h^2 + 4hk = (h + 2k)^2 - 4k^2$ , we have

$$h^2 + 4hk + 5k^2 = (h + 2k)^2 - 4k^2 + 5k^2 = (h + 2k)^2 + k^2 \geq 0,$$

for all  $\mathbf{h} = (h, k) \in \mathbb{R}^2$ . Therefore,  $A$  is positive semidefinite.

Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix with (repeated) eigenvalues  $\lambda_i$ ,  $i = 1, \dots, n$ . We have

**Theorem 8.** The symmetric matrix  $A$  is positive semidefinite if  $\lambda_i \geq 0$ , for all  $i = 1, \dots, n$ , and is negative semidefinite if  $\lambda_i \leq 0$ , for all  $i = 1, \dots, n$ .

**Proof.** Let  $Q$  be an orthogonal matrix that orthogonally diagonalizes  $A$ ,  $Q^T A Q = D$  with  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ . Since and  $Q D Q^T = A$ , it follows that  $Q^T = Q^{-1}$  and for each  $\mathbf{h} \in \mathbb{R}^n$ , we have

$$\mathbf{h}^T A \mathbf{h} = \mathbf{h}^T (Q D Q^T) \mathbf{h} = (Q^T \mathbf{h})^T D (Q^T \mathbf{h}).$$

Let  $Q^T \mathbf{h} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ , the right hand side becomes

$$\begin{aligned}
 (Q^T \mathbf{h})^T D (Q^T \mathbf{h}) &= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\
 &= \sum_{i=1}^n \lambda_i y_i^2
 \end{aligned}$$

which shows that  $\mathbf{h}^T \mathbf{A} \mathbf{h} \geq 0$  when  $\lambda_i \geq 0$ , for all  $i = 1, \dots, n$ , and it is  $\leq 0$  when  $\lambda_i \leq 0$ , for all  $i = 1, \dots, n$ .

**Example 5.** Consider again the matrix  $\mathbf{A}$  of example 4. The characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}_2) = \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 5 - \lambda \end{vmatrix} = (1 - \lambda)(5 - \lambda) - 4 = \lambda^2 - 6\lambda + 5 = 0$$

shows that  $\mathbf{A}$  has two positive eigenvalues, 1 and 5. Therefore,  $\mathbf{A}$  is positive semidefinite.

**Definition 9.** Let  $f : \Omega \rightarrow \mathbb{R}$ , where  $\Omega$  is a nonempty open convex set in  $\mathbb{R}^n$ .

(i)  $f$  is said to be a **convex function** if for all  $\mathbf{x}, \mathbf{y} \in \Omega$  and  $0 \leq \theta \leq 1$ , we have

$$f(\theta \mathbf{x} + (1 - \theta) \mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta) f(\mathbf{y}).$$

(ii)  $f$  is said to be a **concave function** if  $-f$  is a convex function.

$$f(\theta \mathbf{x} + (1 - \theta) \mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta) f(\mathbf{y}).$$

Given a function  $f : \Omega \rightarrow \mathbb{R}$ , where  $\Omega$  is a nonempty open convex set in  $\mathbb{R}^n$ , for each  $\mathbf{x} \in \Omega$  and  $\mathbf{h} \in \mathbb{R}^n$ , let  $I = \{t \in \mathbb{R} \mid \mathbf{x} + t\mathbf{h} \in \Omega\}$  which is an open interval in  $\mathbb{R}$ , and let  $g_{\mathbf{x}, \mathbf{h}} : I \rightarrow \mathbb{R}$  be defined by  $g_{\mathbf{x}, \mathbf{h}}(t) = f(\mathbf{x} + t\mathbf{h})$ , for each  $t \in I$ .

It is easy to have the following important connection between the multivariate function  $f$  and the one-dimensional function  $g_{\mathbf{x}, \mathbf{h}}$  as follows.



**Proposition 10.** Let  $f : \Omega \rightarrow \mathbb{R}$  be a function defined on an open convex set  $\Omega$  in  $\mathbb{R}^n$  and let  $\{g_{\mathbf{x},\mathbf{h}}\}$  be the family of functions of one variable, defined by (??). Then

$f$  is a convex function if and only if  $g_{\mathbf{x},\mathbf{h}}$  are convex functions, for all  $\mathbf{x} \in \Omega$  and  $\mathbf{h} \in \mathbb{R}^n$ .

and

$f$  is a concave function if and only if  $g_{\mathbf{x},\mathbf{h}}$  are concave functions, for all  $\mathbf{x} \in \Omega$  and  $\mathbf{h} \in \mathbb{R}^n$ .

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be of class  $C^2$ . The gradient and the Hessian of  $f$  at  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , noted  $\nabla f(\mathbf{x})$  and  $\nabla^2 f(\mathbf{x})$ , respectively,

are defined by  $\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}$  and

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{x}) \end{pmatrix}. \text{ When } f \text{ is of}$$

class  $C^2$ ,  $\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})$ , for all  $i, j = 1, \dots, n$ . It follows that

$\nabla^2 f(\mathbf{x})$  is a symmetric matrix,  $\nabla^2 f(\mathbf{x})^T = \nabla^2 f(\mathbf{x})$ .

**Example 6.** (a) Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$f(x_1, x_2, x_3) = 2x_1 - x_2 + 3x_3$ . We have

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \text{ and } \nabla^2 f(x_1, x_2, x_3) = \mathbf{0}, \text{ the zero matrix in}$$

$\mathbb{R}^{3 \times 3}$ .

(b) Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3$ . We have

$$\nabla f(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 - x_2 + x_3 \\ 2x_2 - x_1 \\ 2x_3 + x_1 \end{pmatrix} \text{ and}$$

$$\nabla^2 f(x_1, x_2, x_3) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

**Proposition 11.** Let  $f : \Omega \rightarrow \mathbb{R}$  be of class  $C^2$ , where  $\Omega$  is a nonempty open convex set in  $\mathbb{R}^n$ . For each  $\mathbf{x} \in \Omega$  and  $\mathbf{h} = (h_1, h_2, \dots, h_n) \in \mathbb{R}^n$ , let  $I = \{t \in \mathbb{R} \mid \mathbf{x} + t\mathbf{h} \in \Omega\}$  and let  $g_{\mathbf{x}, \mathbf{h}} : I \rightarrow \mathbb{R}$  be defined. We have

$$g'_{\mathbf{x}, \mathbf{h}}(t) = \nabla f(\mathbf{x} + t\mathbf{h}) \mathbf{h} \text{ and } g''_{\mathbf{x}, \mathbf{h}}(t) = \mathbf{h}^T \nabla^2 f(\mathbf{x} + t\mathbf{h}) \mathbf{h}.$$

**Proof.** The differentiability of  $f$  gives the existence of a function  $\varepsilon(\mathbf{h})$ , defined on  $\|\mathbf{h}\| < r$ , such that  $\varepsilon(\mathbf{h}) \rightarrow 0$  as  $\mathbf{h} \rightarrow 0$ , and

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \nabla f(\mathbf{x}) \mathbf{h} + \|\mathbf{h}\| \varepsilon(\mathbf{h}).$$

Now, let  $t \in I$  be fixed. For all  $k \in \mathbb{R}$  so that  $t + k \in I$ , we have

$$\begin{aligned} g_{\mathbf{x},\mathbf{h}}(t+k) - g_{\mathbf{x},\mathbf{h}}(t) &= f(\mathbf{x} + (t+k)\mathbf{h}) - f(\mathbf{x} + t\mathbf{h}) = \nabla f(\mathbf{x} + t\mathbf{h})^\top (\mathbf{h}) \\ &= k \nabla f(\mathbf{x} + t\mathbf{h})^\top \mathbf{h} + |k| \|\mathbf{h}\| \varepsilon(k\mathbf{h}) \end{aligned}$$

and then

$$\frac{g_{\mathbf{x},\mathbf{h}}(t+k) - g_{\mathbf{x},\mathbf{h}}(t)}{k} = \nabla f(\mathbf{x} + t\mathbf{h})^\top \mathbf{h} + \frac{|k|}{k} \|\mathbf{h}\| \varepsilon(k\mathbf{h}).$$

Letting  $k \rightarrow 0$ , we have  $\|k\mathbf{h}\| \rightarrow 0$  which gives

$$\left| \frac{|k|}{k} \|\mathbf{h}\| \varepsilon(k\mathbf{h}) \right| = \|\mathbf{h}\| |\varepsilon(k\mathbf{h})| \rightarrow 0.$$

Therefore

$$\begin{aligned} g'_{\mathbf{x},\mathbf{h}}(t) &= \lim_{k \rightarrow 0} \frac{g_{\mathbf{x},\mathbf{h}}(t+k) - g_{\mathbf{x},\mathbf{h}}(t)}{k} \\ &= \nabla f(\mathbf{x} + t\mathbf{h})^\top \mathbf{h} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{h}) h_i \end{aligned}$$

Now, for each  $i = 1, 2, \dots, n$ , let  $g_i(t) = \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{h})$ . Apply the above result for  $\frac{\partial f}{\partial x_i}$  in place of  $f$ , we get

$$g'_i(t) = \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_i} \right) (\mathbf{x} + t\mathbf{h}) h_j = \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial x_i} (\mathbf{x} + t\mathbf{h}) h_j$$

and then

$$\begin{aligned} g''_{\mathbf{x}, \mathbf{h}}(t) &= \sum_{i=1}^n h_i g'_i(t) = \sum_{i=1}^n h_i \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial x_i} (\mathbf{x} + t\mathbf{h}) h_j \\ &= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial x_1} (\mathbf{x} + t\mathbf{h}) h_j \\ \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial x_2} (\mathbf{x} + t\mathbf{h}) h_j \\ \vdots \\ \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial x_n} (\mathbf{x} + t\mathbf{h}) h_j \end{bmatrix} \end{aligned}$$

as desired.

$$\begin{aligned}
&= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} (\mathbf{x} + t\mathbf{h}) & \frac{\partial^2}{\partial x_2 \partial x_1} (\mathbf{x} + t\mathbf{h}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} (\mathbf{x} + t\mathbf{h}) \\ \frac{\partial^2}{\partial x_1 \partial x_2} (\mathbf{x} + t\mathbf{h}) & \frac{\partial^2}{\partial x_2^2} (\mathbf{x} + t\mathbf{h}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} (\mathbf{x} + t\mathbf{h}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_1 \partial x_n} (\mathbf{x} + t\mathbf{h}) & \frac{\partial^2}{\partial x_2 \partial x_n} (\mathbf{x} + t\mathbf{h}) & \cdots & \frac{\partial^2}{\partial x_n^2} (\mathbf{x} + t\mathbf{h}) \end{bmatrix} \\
&= \mathbf{h}^T \nabla^2 f(\mathbf{x} + t\mathbf{h}) \mathbf{h}.
\end{aligned}$$

as desired.

We have the following multidimensional version of Theorem 2.

**Theorem 12.** Let  $f : \Omega \rightarrow \mathbb{R}$  be a function of class  $C^2$  on an open convex set  $\Omega$  in  $\mathbb{R}^n$ . The following statements are equivalent.

- (i)  $f$  is a convex function.
- (ii)  $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x})$ , for all  $\mathbf{x}, \mathbf{y} \in \Omega$ .
- (iii)  $\nabla^2 f(\mathbf{x}) \geq 0$ , for all  $\mathbf{x} \in \Omega$ .

**Proof.** For each  $\mathbf{z} \in \Omega$  and  $\mathbf{h} \in \mathbb{R}^n$ ,  $I = \{t \in \mathbb{R} \mid \mathbf{z} + t\mathbf{h} \in \Omega\}$ , and let  $g_{\mathbf{z},\mathbf{h}} : I \rightarrow \mathbb{R}$  be defined by  $g_{\mathbf{z},\mathbf{h}}(t) = f(\mathbf{z} + t\mathbf{h})$ , for all  $t \in I$ . By proposition 10, (i) is satisfied if and only if all functions  $g_{\mathbf{z},\mathbf{h}}$  are convex. Using Theorem 2, the statement that  $g_{\mathbf{z},\mathbf{h}}$  is convex is equivalent with the two following statements

- (a)  $g_{\mathbf{z},\mathbf{h}}(s) \geq g_{\mathbf{z},\mathbf{h}}(t) + g'_{\mathbf{z},\mathbf{h}}(t)(s - t)$ , for all  $s, t \in I$ .
- (b)  $g''_{\mathbf{z},\mathbf{h}}(t) \geq 0$ , for all  $t \in I$ .

By proposition 11,

$$g'_{\mathbf{z},\mathbf{h}}(t) = \nabla f(\mathbf{z} + t\mathbf{h}) \mathbf{h} \text{ và } g''_{\mathbf{z},\mathbf{h}}(t) = \mathbf{h}^T \nabla^2 f(\mathbf{z} + t\mathbf{h}) \mathbf{h}.$$

Letting  $\mathbf{z} = \mathbf{x}$ ,  $\mathbf{h} = \mathbf{y} - \mathbf{x}$ ,  $s = 1$ , and  $t = 0$  in (a), we have

$$f(\mathbf{y}) = g_{\mathbf{x},\mathbf{h}}(1) \geq g_{\mathbf{x},\mathbf{h}}(0) + g'_{\mathbf{x},\mathbf{h}}(0) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \mathbf{h}$$

which is (ii). Letting  $\mathbf{z} = \mathbf{x}$ , we have

$$\mathbf{h}^T \nabla^2 f(\mathbf{x}) \mathbf{h} = g''_{\mathbf{x},\mathbf{h}}(0) \geq 0, \text{ for all } \mathbf{h} \in \mathbb{R}^n,$$

which implies that  $\nabla^2 f(\mathbf{x}) \geq 0$ , i.e., (ii) is proved. Conversely, consider the function  $g_{\mathbf{z},\mathbf{h}}(t) = f(\mathbf{z} + t\mathbf{h})$ ,  $\mathbf{z} \in \Omega$  and  $\mathbf{h} \in \mathbb{R}^n$ . For  $s, t \in I$  by letting  $\mathbf{x} = \mathbf{z} + t\mathbf{h}$  and  $\mathbf{y} = \mathbf{z} + s\mathbf{h}$ , we get

$$g_{\mathbf{z},\mathbf{h}}(s) = f(\mathbf{z} + s\mathbf{h}) \geq f(\mathbf{z} + t\mathbf{h}) + \nabla f(\mathbf{z} + t\mathbf{h})^T ((s - t)\mathbf{h}) = g_{\mathbf{z},\mathbf{h}}(t)$$

whenever (ii) holds, and  $g''_{\mathbf{z},\mathbf{h}}(t) = \mathbf{h}^T \nabla^2 f(\mathbf{z} + t\mathbf{h}) \mathbf{h} \geq 0$  whenever (iii) holds. Therefore, if (ii) or (iii) hold, then, by Theorem 2,  $g_{\mathbf{z},\mathbf{h}}$  is a convex function and the Theorem is proved.

Condition (iii) is used to check if  $f$  is a convex function and condition (ii) shows that a solution of the equation  $\nabla f(\mathbf{x}) = \mathbf{0}$  is a global minimum of  $f$  on  $\Omega$ .

As in the one-dimensional case, letting  $-f$  in Theorem 12, we get the following result for concave functions.

**Corollary 13.** Let  $f : \Omega \rightarrow \mathbb{R}$  be a function of class  $C^2$  on an open convex set  $\Omega$  in  $\mathbb{R}^n$ . The following statements are equivalent.



- (i)  $f$  is a concave function.
- (ii)  $f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x})$ , for all  $\mathbf{x}, \mathbf{y} \in \Omega$ .
- (iii)  $\nabla^2 f(\mathbf{x}) \leq 0$ , for all  $\mathbf{x} \in \Omega$ .

By combining these results, we have the following important result for applications.

**Corollary 14.** Let  $\Omega$  be an open convex set of  $\mathbb{R}^n$ , let  $f : \Omega \rightarrow \mathbb{R}$  be of class  $C^1$ , i.e., all partial derivatives of  $f$  exist and are continuous functions, and let  $\mathbf{x}_0 \in \Omega$  such that  $\nabla f(\mathbf{x}_0) = \mathbf{0}$ .

- (i) If  $f$  is a convex function then  $\mathbf{x}_0 = \arg \min_{\mathbf{x} \in \Omega} f(\mathbf{x})$ .
- (ii) If  $f$  is a concave function then  $\mathbf{x}_0 = \arg \max_{\mathbf{x} \in \Omega} f(\mathbf{x})$ .

**Example 7.** Consider the quadratic form of example 6 (b), with

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3, (x_1, x_2, x_3) \in \mathbb{R}^3,$$

$$\nabla f(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 - x_2 + x_3 \\ 2x_2 - x_1 \\ 2x_3 + x_1 \end{pmatrix} \text{ and}$$

$$\nabla^2 f(x_1, x_2, x_3) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

The matrix  $\nabla^2 f(x_1, x_2, x_3)$  has positive eigenvalues  $2$ ,  $2 + \sqrt{2}$ , and  $2 - \sqrt{2}$  which shows that  $\nabla^2 f(x_1, x_2, x_3) \geq 0$  and then  $f$  is a convex function. Therefore, the unique solution of the equation  $\nabla f(x_1, x_2, x_3) = (0, 0, 0)$  is the global minimum of  $f$  on  $\mathbb{R}^3$ .

**Definition 15.** Let  $\mathbf{q} = (q_1, q_2, \dots, q_n) \in \mathbb{R}^n$  and let  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ .

(i) The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$f_{\mathbf{q}}(\mathbf{x}) = \langle \mathbf{q}, \mathbf{x} \rangle = \sum_{i=1}^n q_i x_i \equiv \mathbf{q}^T \mathbf{x},$$

for  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , is called a **linear map**.

(ii) The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$f_A(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j = \mathbf{x}^T A \mathbf{x},$$

for  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , is called a **quadratic form**.

**Example 8.** (a) The function  $f(x_1, x_2, x_3) = 2x_1 - x_2 + 3x_3$  of example 6 (a) is a linear map,  $f = f_{\mathbf{q}}$ , with  $\mathbf{q} = (2, -1, 3)$ .  
 (b) The function  $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3$  of example 6 (b) is a quadratic form,  $f = f_A$ , with

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

**Remark.** Let  $f_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  be a quadratic form on  $\mathbb{R}^n$  and let  $B = \frac{1}{2}(A + A^T)$ . Then  $B$  is symmetric and

$$\mathbf{x}^T B \mathbf{x} = \frac{1}{2} (\mathbf{x}^T A \mathbf{x} + \mathbf{x}^T A^T \mathbf{x}) = \mathbf{x}^T A \mathbf{x},$$

since  $\mathbf{x}^T A^T \mathbf{x} = (\mathbf{x}^T A^T \mathbf{x})^T = \mathbf{x}^T (A^T)^T \mathbf{x} = \mathbf{x}^T A \mathbf{x} \in \mathbb{R}^{1 \times 1}$ . So we can assume that the matrix  $A$  that defines the quadratic form  $f_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  is symmetric.

**Proposition 16.** Let  $f_{\mathbf{q}}(\mathbf{x})$  be a linear map with  $\mathbf{q} = (q_1, q_2, \dots, q_n) \in \mathbb{R}^n$  and let  $f_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  be a quadratic form with symmetric matrix  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ . We have

(i)  $\nabla f_{\mathbf{q}}(\mathbf{x}) = \mathbf{q}$  and  $\nabla^2 f_{\mathbf{q}}(\mathbf{x}) = \mathbf{0}$ , for all  $\mathbf{x} \in \mathbb{R}^n$ .

(ii)  $\nabla f_A(\mathbf{x}) = 2A\mathbf{x}$  and  $\nabla^2 f_A(\mathbf{x}) = 2A$ , for all  $\mathbf{x} \in \mathbb{R}^n$ .

**Proof.** (i) The identity  $f_{\mathbf{q}}(\mathbf{x}) = q_1 x_1 + \dots + q_i x_i + \dots + q_n x_n$  gives

$\frac{\partial f_{\mathbf{q}}}{\partial x_i}(\mathbf{x}) = q_i$ , for all  $i = 1, 2, \dots, n$ , and  $\frac{\partial^2 f_{\mathbf{q}}}{\partial x_j \partial x_i}(\mathbf{x}) = 0$ , for all  $i, j = 1, 2, \dots, n$ .

(ii) Since  $\frac{\partial}{\partial x_k}(x_i x_j) = 0$ , when  $k \neq i, j$ ,  $\frac{\partial}{\partial x_k}(x_k x_j) = x_j$ , when  $i = k \neq j$ , and  $\frac{\partial}{\partial x_k}(x_k x_k) = 2x_k$ , it follows from the identity  $f_A(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$  that, for each  $k = 1, 2, \dots, n$ ,

$$\begin{aligned}
\frac{\partial f_A}{\partial x_k}(\mathbf{x}) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_k} (x_i x_j) = \sum_{i=1}^n \left( a_{ik} \frac{\partial}{\partial x_k} (x_i x_k) + \sum_{j \neq k} a_{ij} \frac{\partial}{\partial x_k} (x_i x_j) \right) \\
&= \sum_{i=1}^n a_{ik} \frac{\partial}{\partial x_k} (x_i x_k) + \sum_{i=1}^n \left( \sum_{j \neq k} a_{ij} \frac{\partial}{\partial x_k} (x_i x_j) \right) = 2a_{kk}x_k + \sum_{i \neq k} a_{ik}x_i \\
&= 2a_{kk}x_k + \sum_{i \neq k} a_{ik}x_i + \sum_{j \neq k} a_{kj}x_j = 2 \sum_{i=1}^n a_{ki}x_i
\end{aligned} \tag{4}$$

which is the  $k$ -th row of  $2A\mathbf{x}$ . Hence,  $\nabla f_A(\mathbf{x}) = 2A\mathbf{x}$ . Now, by (4),  $\frac{\partial f_A}{\partial x_k}(\mathbf{x})$  is a linear function with  $\mathbf{q} = (2a_{k1}, 2a_{k2}, \dots, 2a_{kn})$ , the  $k$ -th row of  $2A$ . By (i),  $\frac{\partial^2 f_A}{\partial x_l \partial x_k}(\mathbf{x}) = a_{lk}$ , for each  $l, k = 1, 2, \dots, n$ , which gives  $\nabla^2 f_A(\mathbf{x}) = 2A$  as desired.

Finally, consider a matrix  $A \in \mathbb{R}^{N \times n}$ , with  $n \leq N$ ,  $\text{rank } A = n$ , and a vector  $\mathbf{b} \in \mathbb{R}^N$ . Let the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined by  $f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|^2$ . We have

$$\begin{aligned} f(\mathbf{x}) &= \langle A\mathbf{x} - \mathbf{b}, A\mathbf{x} - \mathbf{b} \rangle = (A\mathbf{x} - \mathbf{b})^T (A\mathbf{x} - \mathbf{b}) \\ &= (\mathbf{x}^T A^T - \mathbf{b}^T) (A\mathbf{x} - \mathbf{b}) \\ &= \mathbf{x}^T A^T A \mathbf{x} - \mathbf{b}^T A \mathbf{x} - \mathbf{x}^T A^T \mathbf{b} + \mathbf{b}^T \mathbf{b} \\ &= \mathbf{x}^T (A^T A) \mathbf{x} - (2\mathbf{b}^T A) \mathbf{x} + \mathbf{b}^T \mathbf{b}, \end{aligned}$$

since  $\mathbf{b}^T A \mathbf{x} = \mathbf{x}^T A^T \mathbf{b}$ . Therefore

$$\nabla f(\mathbf{x}) = 2(A^T A) \mathbf{x} - 2A^T \mathbf{b} \text{ and } \nabla^2 f(\mathbf{x}) = 2(A^T A).$$

Since

$\mathbf{h}^T \nabla^2 f(\mathbf{x}) \mathbf{h} = 2[\mathbf{h}^T (A^T A) \mathbf{h}] = 2[(\mathbf{h}^T A^T)(A\mathbf{h})] = 2\|A\mathbf{h}\|^2 \geq 0$ ,  
for all  $\mathbf{h} \in \mathbb{R}^n$ , the Hessian  $\nabla^2 f(\mathbf{x})$  is a positive semidefinite matrix.  
Moreover,  $\text{rank}(A^T A) = n$  implies that the matrix  $A^T A$  is invertible.

Hence, the solution  $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$  of the equation  $\nabla f(\mathbf{x}) = \mathbf{0}$  is the global minimum of  $f$ .

We state this result in the following theorem for future applications.

**Theorem 17.** Let a matrix  $A \in \mathbb{R}^{N \times n}$ , with  $n \leq N$ , so that  $\text{rank } A = n = \min(N, n)$ , and let  $\mathbf{b} \in \mathbb{R}^N$ . We have

$$(A^T A)^{-1} A^T \mathbf{b} = \arg \min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|^2.$$

**Example 9.** Let  $\mathbf{b} = (2, 2, 5, 8) \in \mathbb{R}^4$  and let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \in \mathbb{R}^{4 \times 2}.$$

It is easy to see that  $\dim(\text{col}(A)) = 2 = \text{rank}(A) = \min(4, 2)$ ,

$A^T A = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$  is an invertible matrix, and



$$(A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 17 \\ 53 \end{bmatrix} = \begin{bmatrix} -1 \\ 2.1 \end{bmatrix}. \text{ Hence}$$

$$(-1, 2.1) = \arg \min_{\mathbf{x} \in \mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|^2.$$

Now, with

$$A = \begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix} \in \mathbb{R}^{4 \times 3}.$$

We have  $\dim(\text{col}(A)) = 3 = \text{rank}(A) = \min(4, 3)$ ,

$$A^T A = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \text{ is an invertible matrix, and}$$

$$(A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}^{-1} \begin{bmatrix} 17 \\ 53 \\ 183 \end{bmatrix} = \begin{bmatrix} 2.75 \\ -1.65 \\ 0.75 \end{bmatrix}.$$

Hence

$$(2.75, -1.65, 0.75) = \arg \min_{\mathbf{x} \in \mathbb{R}^3} \|A\mathbf{x} - \mathbf{b}\|^2.$$