Lớp 23CTT1A

Nhóm 02

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2.2 Tính các định thức cấp ba sau:

a/

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 2.(-1)^{1+1}.\begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix} + 1.(-1)^{3+1}.\begin{vmatrix} 1 & 1 \\ 5 & -2 \end{vmatrix} = 2.14 + 1.(-7) = 21$$

b/

$$\begin{vmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{vmatrix} \overset{dòng\ 1}{=} 3.(-1)^{1+1}.\begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} + (-2).(-1)^{1+2}.\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + (-4).(-1)^{1+3}.\begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix}$$
$$= 3.11 + (-2).(-2) + (-4).12 = -11$$

c) $\begin{bmatrix} -2 & -1 & 4 \\ 6 & -3 & -2 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 4 \\ 0 & -6 & 10 \\ 0 & -1 & 10 \end{bmatrix} = -2 \times \begin{bmatrix} -6 & 10 \\ -1 & 10 \end{bmatrix} = 100$

d)

$$\begin{vmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 7 & 6 & 5 \end{vmatrix} = \begin{vmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

2.4 Tính các định thức cấp bốn sau:

a/

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 3 & 1 & -4 \\ -2 & 3 & 3 & 1 \\ 3 & 3 & 1 & -1 \end{vmatrix} \overset{\substack{d_2+d_1 \\ d_3+2d_1}}{\underset{d_4-3d_1}{=}} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & -3 \\ 0 & 5 & 5 & 3 \\ 0 & 0 & -2 & -4 \end{vmatrix}$$

$$= 1.(-1)^{1+1}.\begin{vmatrix} 4 & 2 & -3 \\ 5 & 5 & 3 \\ 0 & -2 & -4 \end{vmatrix}$$

$$\overset{d_2-d_1}{\underset{d_1-4d_2}{=}} \begin{vmatrix} 0 & -10 & -27 \\ 1 & 3 & 6 \\ 0 & -2 & -4 \end{vmatrix}$$

$$= 1.(-1)^{2+1}.\begin{vmatrix} -10 & -27 \\ -2 & -4 \end{vmatrix}$$

$$= \ -(40-54)$$

$$= 14$$

b/

$$\begin{vmatrix} 1 & 3 & 2 & 4 \\ -2 & 2 & 4 & 1 \\ 2 & 2 & 5 & 4 \\ -3 & -1 & 3 & -2 \end{vmatrix} \overset{\substack{d_2+2d_1 \\ d_3-2d_1}}{\underset{d_4+3d_1}{=}} \begin{vmatrix} 1 & 3 & 2 & 4 \\ 0 & 8 & 8 & 9 \\ 0 & -4 & 1 & -4 \\ 0 & 8 & 9 & 10 \end{vmatrix}$$

$$= \ 1.(-1)^{1+1}.\begin{vmatrix} 8 & 8 & 9 \\ -4 & 1 & -4 \\ 8 & 9 & 10 \end{vmatrix}$$

$$\overset{d_2+\frac{1}{2}d_1}{\underset{d_3-d_1}{=}} \begin{vmatrix} 8 & 8 & 9 \\ 0 & 5 & 0.5 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 8.(-1)^{1+1}.\begin{vmatrix} 5 & 0.5 \\ 1 & 1 \end{vmatrix}$$

$$= 36$$

c)

$$\begin{bmatrix} 8 & -4 & 4 & -3 \\ 4 & 0 & 2 & 0 \\ 1 & -1 & 1 & -5 \\ 2 & 7 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -4 & 37 \\ 0 & 4 & -2 & 20 \\ 1 & -1 & 1 & -5 \\ 0 & 9 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 37 \\ 4 & -2 & 20 \\ 9 & 0 & 12 \end{bmatrix} = \begin{bmatrix} -4 & 0 & -3 \\ 4 & -2 & 20 \\ 9 & 0 & 12 \end{bmatrix}$$

$$= -2 \times \begin{bmatrix} -4 & -3 \\ 9 & 12 \end{bmatrix} = 42$$

d)

$$\begin{vmatrix} 6 & 4 & 1 & -8 \\ 7 & 0 & 3 & -3 \\ 2 & -2 & 3 & 2 \\ -9 & 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 10 & 0 & 7 & -4 \\ 7 & 0 & 3 & -3 \\ -16 & 0 & 7 & 6 \\ -9 & 1 & 2 & 2 \end{vmatrix} = 1x(-1)^{4+2} \begin{vmatrix} 10 & 7 & -4 \\ 7 & 3 & -3 \\ -16 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 10 & 7 & -4 \\ 7 & 3 & -3 \\ -16 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 10 & 7 & -4 \\ 7 & 3 & -3 \\ -16 & 7 & 6 \end{vmatrix}$$

$$= \frac{1}{3} \begin{vmatrix} 30 & 21 & -12 \\ 7 & 3 & -3 \\ -16 & 7 & 6 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} 2 & 9 & 0 \\ 7 & 3 & -3 \\ -2 & 13 & 0 \end{vmatrix} = \frac{1}{3}(-1)^{2+3}(-3) \begin{vmatrix} 2 & 9 \\ -2 & 13 \end{vmatrix} = 44$$

2.6 Tính các định thức cấp n sau:

a/

$$|A| = \begin{vmatrix} \mathbf{1} & 2 & 2 & \dots & 2 \\ 2 & \mathbf{2} & 2 & \dots & 2 \\ 2 & 2 & \mathbf{3} & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & \boldsymbol{n} \end{vmatrix} \underset{\substack{\dots \\ d_n - 2d_2}}{\overset{\substack{d_3 - 2d_2 \\ d_4 - 2d_2}}{=}} \begin{vmatrix} \mathbf{1} & 2 & 2 & \dots & 2 \\ 2 & \mathbf{2} & 2 & \dots & 2 \\ 0 & 0 & \mathbf{1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \boldsymbol{n-2} \end{vmatrix}$$

$$\overset{d_2 - 2d_1}{=} \begin{vmatrix} \mathbf{1} & 2 & 2 & \dots & 2 \\ 0 & \mathbf{-2} & -2 & \dots & -2 \\ 0 & 0 & \mathbf{1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \boldsymbol{n-2} \end{vmatrix} = 1.(-2).(n-2)! = (-2).(n-2)!$$

b) Đặt $A = \begin{pmatrix} a_1+1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2+1 & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3+1 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n+1 \end{pmatrix}$

+ $c_1 \coloneqq c_1 + c_2 + c_3 + \cdots + c_n$:

$$A \sim \begin{pmatrix} a_1+a_2+a_3+\cdots+1 & a_2 & a_3 & \cdots & a_n \\ a_1+a_2+a_3+\cdots+1 & a_2+1 & a_3 & \cdots & a_n \\ a_1+a_2+a_3+\cdots+1 & a_2 & a_3+1 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1+a_2+a_3+\cdots+1 & a_2 & a_3 & \cdots & a_n+1 \end{pmatrix}$$

+ $d_2 - d_1, d_3 - d_1, \dots d_n - d_1$:

$$A \sim \begin{pmatrix} a_1+a_2+a_3+\cdots+a_n+1 & a_2 & a_3 & \cdots & a_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\Rightarrow |A| = a_1 + a_2 + a_3 + \cdots + a_n + 1$$

c) $\begin{vmatrix} 1 & 2 & \cdots & \mathrm{n}-1 & n \\ 2 & 3 & \cdots & \mathrm{n} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathrm{n}-1 & \mathrm{n} & \cdots & \mathrm{n}-3 & \mathrm{n}-2 \\ n & 1 & \cdots & \mathrm{n}-2 & n-1 \end{vmatrix}$

Đặt det(A) = $|A| = \begin{vmatrix} 1 & 2 & \cdots & \mathrm{n}-1 & n \\ 2 & 3 & \cdots & \mathrm{n} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathrm{n}-1 & \mathrm{n} & \cdots & \mathrm{n}-3 & \mathrm{n}-2 \\ n & 1 & \cdots & \mathrm{n}-2 & n-1 \end{vmatrix}$

$$\xRightarrow{c_1 := c_1 + c_2 + \ldots + c_n} \begin{vmatrix} \frac{1}{2}n(n+1) & 2 & \cdots & \mathrm{n}-1 & n \\ \frac{1}{2}n(n+1) & 3 & \cdots & \mathrm{n} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}n(n+1) & \mathrm{n} & \cdots & \mathrm{n}-3 & \mathrm{n}-2 \\ \frac{1}{2}n(n+1) & 1 & \cdots & \mathrm{n}-2 & n-1 \end{vmatrix}$$

$$= \frac{1}{2}n(n+1).(-1)^{1+1}.\begin{vmatrix} 1 & 2 & \cdots & \mathrm{n}-1 & n \\ 1 & 3 & \cdots & \mathrm{n} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \mathrm{n} & \cdots & \mathrm{n}-3 & \mathrm{n}-2 \\ 1 & 1 & \cdots & \mathrm{n}-2 & n-1 \end{vmatrix}$$

$$\xrightarrow[\begin{subarray}{c} d_2 := d_2 - d_1 \\ d_3 := d_3 - d_1 \\ \vdots \\ d_n := d_n - d_1 \end{subarray}]{} \frac{1}{2}n(n+1).\begin{vmatrix} 1 & 2 & \cdots & \mathrm{n}-1 & n \\ 0 & 1 & \cdots & 1 & -(\mathrm{n}-1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \mathrm{n}-2 & \cdots & -2 & -2 \\ 0 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$= \frac{1}{2}n(n+1).1.(-1)^{1+1}.\begin{vmatrix} 1 & 1 & \cdots & 1 & -(n-1) \\ 2 & 2 & \cdots & -(\mathrm{n}-2) & -(\mathrm{n}-2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & -2 & \cdots & -2 & -2 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$\xrightarrow{d_1 := d_1 + d_{n-1}} \frac{1}{2}n(n+1).\begin{vmatrix} 0 & 0 & \cdots & 0 & -n \\ 2 & 2 & \cdots & -(\mathrm{n}-2) & -(\mathrm{n}-2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & -2 & \cdots & -2 & -2 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$= \frac{1}{2} n(n+1).(-\mathrm{n}).(-1)^n.\begin{vmatrix} 2 & 2 & \cdots & 2 & -(n-2) \\ 3 & 3 & \cdots & -(\mathrm{n}-3) & -(n-3) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & -2 & \cdots & -2 & -2 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$\xRightarrow{d_1 := d_1 + 2d_{n-2}} \frac{1}{2} n(n+1).(-\mathrm{n}).(-1)^n.\begin{vmatrix} 0 & 0 & \cdots & 0 & -n \\ 3 & 3 & \cdots & -(\mathrm{n}-3) & -(n-3) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & -2 & \cdots & -2 & -2 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$= \frac{1}{2} n(n+1).(-\mathrm{n}).(-1)^n.(-\mathrm{n}).(-1)^{n-1}\begin{vmatrix} 3 & 3 & \cdots & 3 & -(n-3) \\ 4 & 4 & \cdots & -(\mathrm{n}-4) & -(n-4) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & -2 & \cdots & -2 & -2 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$\xRightarrow{d_1 := d_1 + 3d_{n-3}} \frac{1}{2} n(n+1).(-\mathrm{n})^2.(-1)^{n+(n-1)}.\begin{vmatrix} 0 & 0 & \cdots & 0 & -n \\ 4 & 4 & \cdots & -(\mathrm{n}-4) & -(n-4) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & -2 & \cdots & -2 & -2 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$=\frac{1}{2}n(n+1).(-\mathrm{n})^2.(-1)^{n+(n-1)}.(-\mathrm{n}).(-1)^{n-2}.\begin{vmatrix} 4 & 4 & \cdots & 4 & -(n-4) \\ 5 & 5 & \cdots & -(\mathrm{n}-5) & -(n-5) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & -2 & \cdots & -2 & -2 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$\vdots$$

$$\vdots$$

$$=\frac{1}{2}n(n+1).(-\mathrm{n})^{n-4}.(-1)^{n+(n-1)+\ldots+5}.\begin{vmatrix} n-3 & \mathrm{n}-3 & -3 \\ n-2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix}$$

$$\xRightarrow{d_1 := d_1 + (n-3)d_3} \frac{1}{2}n(n+1).(-\mathrm{n})^{n-4}.(-1)^{n+(n-1)+\ldots+5}.\begin{vmatrix} 0 & 0 & -\mathrm{n} \\ n-2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix}$$

$$=\frac{1}{2}n(n+1).(-\mathrm{n})^{n-4}.(-1)^{n+(n-1)+\ldots+5}.(-\mathrm{n}).(-1)^4.\begin{vmatrix} n-2 & -2 \\ -1 & -1 \end{vmatrix}$$

$$\xRightarrow{d_1 := d_1 + (n-2)d_2} \frac{1}{2}n(n+1).(-\mathrm{n})^{n-3}.(-1)^{n+(n-1)+\ldots+4}.\begin{vmatrix} 0 & -\mathrm{n} \\ -1 & -1 \end{vmatrix}$$

$$= \frac{1}{2}n(n+1).(-\mathrm{n})^{n-3}.(-1)^{n+(n-1)+\ldots+4}.(-\mathrm{n}).(-1)^{3}.|-1|$$

$$= \frac{1}{2}n(n+1).(-\mathrm{n})^{n-2}.(-1)^{n+(n-1)+\ldots+4+3}.(-1)$$

$$= \frac{1}{2}n(n+1).(-\mathrm{n})^{n-2}.(-1)^{n+(n-1)+\ldots+4+3}.(-1)^{2+1}$$

$$= \frac{1}{2}n(n+1).(-\mathrm{n})^{n-2}.(-1)^{n+(n-1)+\ldots+4+3+2+1}$$

$$= \frac{1}{2}n(n+1).(-\mathrm{n})^{n-2}.(-1)^{\frac{1}{2}n(n+1)}$$

$$= \frac{1}{2}\mathrm{n}^{n-1}.(n+1).(-1)^{\frac{1}{2}n(n+1)+n-2}$$

$$= \frac{1}{2}.(-1)^{\frac{1}{2}(n+4)(n-1)}.\mathrm{n}^{n-1}.(n+1)$$

d) $\begin{vmatrix} x_1y_1+1 & x_1y_2+1 & \cdots & x_1y_n+1 \\ x_2y_1+1 & x_2y_2+1 & \cdots & x_2y_n+1 \\ \cdots & \cdots & \cdots & \cdots \\ x_ny_1+1 & x_ny_2+1 & \cdots & x_ny_n+1 \end{vmatrix}$

Đặt det(A) = $|A| = \begin{vmatrix} x_1y_1+1 & x_1y_2+1 & \cdots & x_1y_n+1 \\ x_2y_1+1 & x_2y_2+1 & \cdots & x_2y_n+1 \\ \cdots & \cdots & \cdots & \cdots \\ x_ny_1+1 & x_ny_2+1 & \cdots & x_ny_n+1 \end{vmatrix}$

$$\xrightarrow[d_2 := d_2 - d_3]{d_1 := d_1 - d_2} \begin{vmatrix} (x_1-x_2).y_1 & (x_1-x_2).y_2 & \cdots & (x_1-x_2).y_n \\ (x_2-x_3).y_1 & (x_2-x_3).y_2 & \cdots & (x_2-x_3).y_n \\ \cdots & \cdots & \cdots & \cdots \\ x_ny_1+1 & x_ny_2+1 & \cdots & x_ny_n+1 \end{vmatrix}$$

$$= (x_1-x_2).(x_2-x_3).\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_n \\ \cdots & \cdots & \cdots & \cdots \\ x_ny_1+1 & x_ny_2+1 & \cdots & x_ny_n+1 \end{vmatrix}$$

$$\xrightarrow{d_1 := d_1 - d_2} (x_1-x_2).(x_2-x_3).\begin{vmatrix} 0 & 0 & \cdots & 0 \\ y_1 & y_2 & \cdots & y_n \\ \cdots & \cdots & \cdots & \cdots \\ x_ny_1+1 & x_ny_2+1 & \cdots & x_ny_n+1 \end{vmatrix} (*)$$

$= (x_1-x_2).(x_2-x_3).0 = 0$ (do ma trận (*) có dòng 1 bằng 0)

2.8 A, B ∈ M_n(R). Chứng minh rằng:

a/ Chứng minh rằng: det(AB) = det(BA).

Ta có: A, B ∈ $M_n(\mathbb{R})$

$\Rightarrow \det(AB) = \det(A).\det(B) = \det(B).\det(A) = \det(BA)$

Vậy det(AB) = det(BA) (đpcm).

b/ Nếu B khả nghịch thì det(B^{-1}AB) = detA.

Vì A, B $\in M_n(\mathbb{R})$ $nên$ $AB \in M_n(\mathbb{R})$ (1)

Vì B $khả$ $nghịch$ $nên$ $B^{-1} \in M_n(\mathbb{R})$ (2)

Từ (1)(2) ta $có$:

$\det(B^{-1}\mathrm{AB}) = \det(B^{-1})\det(AB) = \det(AB)\det(B^{-1}) = \det(ABB^{-1}) = \det(AI_n) = \det(A)$(dpcm)

2.10 Tìm nghịch đảo của các ma trận trong Bài tập 2.9 bằng cách áp dụng công thức định thức.

a/

$A = \begin{pmatrix} 7 & 3 \\ 5 & 8 \end{pmatrix}$

$adj(A) = \begin{pmatrix} 8 & -3 \\ -5 & 7 \end{pmatrix}$

$|A| = \begin{vmatrix} 7 & 3 \\ 5 & 8 \end{vmatrix} = 7.8 - 5.3 = 41$

$$\Rightarrow A^{-1} = \frac{1}{|A|} adj(A) = \begin{pmatrix} \frac{8}{41} & -\frac{3}{41} \\ -\frac{5}{41} & \frac{7}{41} \end{pmatrix}$$

b)

$$\text{Đặt } A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$$

$$adj(A) = \begin{pmatrix} -57 & 33 & -3 \\ 51 & -30 & 6 \\ -3 & 6 & -3 \end{pmatrix}$$

$$|A| = \left| \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & -1 \\ 2 & 0 & -11 \end{pmatrix} \right| = 3(-1)^{2+1}\big(1.(-11) - 2(-1)\big) = 27$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} adj(A) = \begin{pmatrix} -\frac{19}{9} & \frac{11}{9} & -\frac{1}{9} \\ \frac{17}{9} & -\frac{10}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{2}{9} & -\frac{1}{9} \end{pmatrix}$$

c) $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

Đặt $A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

Ta xét:

$$det(A) = |A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 2 & 1 & 4 \end{vmatrix}$$

$$\xRightarrow[\substack{d_1 := d_1 - d_3 \\ d_2 := d_2 - 2d_3 \\ d_3 := d_3 - 2d_1}]{} \begin{vmatrix} 1 & 1 & -3 \\ 0 & 3 & -6 \\ 0 & -1 & 10 \end{vmatrix}$$

$$= 1.(-1)^{1+1} \begin{vmatrix} 3 & -6 \\ -1 & 10 \end{vmatrix}$$

$$= (3.10 - (-1).(-6)) = 24$$

Do $det(A) = 24 \neq 0$ nên A khả nghịch đồng thời ta cũng có ma trận phụ hợp của A:

$$adj(A) = \begin{pmatrix} 18 & -7 & -1 \\ -12 & 10 & -2 \\ -6 & 1 & 7 \end{pmatrix}$$

Do đó ma trận khả nghịch của A được tính bởi:

$$A^{-1} = \frac{1}{det(A)}.adj(A)$$

$$= \frac{1}{24} \cdot \begin{pmatrix} 18 & -7 & -1 \\ -12 & 10 & -2 \\ -6 & 1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & -\frac{7}{24} & -\frac{1}{24} \\ -\frac{1}{2} & \frac{5}{12} & -\frac{1}{12} \\ -\frac{1}{4} & \frac{1}{24} & \frac{7}{24} \end{pmatrix}$$

d) $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 3 & 2 & 3 \\ -1 & 2 & 2 & 1 \\ 1 & 3 & 2 & -1 \end{pmatrix}$

Đặt $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 3 & 2 & 3 \\ -1 & 2 & 2 & 1 \\ 1 & 3 & 2 & -1 \end{pmatrix}$

Ta xét:

$$det(A) = |A| = \begin{vmatrix} 1 & 1 & 2 & 2 \\ 1 & 3 & 2 & 3 \\ -1 & 2 & 2 & 1 \\ 1 & 3 & 2 & -1 \end{vmatrix}$$

$$\xrightarrow[d_4 := d_4 - d_1]{\substack{d_2 := d_2 - d_1 \\ d_3 := d_3 + d_1}} \begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 2 & 0 & -3 \end{vmatrix}$$

$$= 1.(-1)^{1+1} \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 0 & -3 \end{vmatrix}$$

$$\xrightarrow{c_1 := c_1 - 2c_3} \begin{vmatrix} 0 & 0 & 1 \\ -7 & 4 & 5 \\ 8 & 0 & -3 \end{vmatrix}$$

$$= 1.(-1)^{1+3} \begin{vmatrix} -7 & 4 \\ 8 & 0 \end{vmatrix}$$

$$= ((-7).0 - 8.4) = -32$$

Do $det(A) = -32 \neq 0$ nên A khả nghịch đồng thời ta cũng có ma trận phụ hợp của A:

$$adj(A) = \begin{pmatrix} -8 & -2 & 16 & -6 \\ 16 & -12 & 0 & -4 \\ -20 & 15 & -8 & -3 \\ 0 & -8 & 0 & 8 \end{pmatrix}$$

Do đó ma trận khả nghịch của A được tính bởi:

$$A^{-1} = \frac{1}{det(A)}.adj(A)$$

$$= \frac{1}{-32}.\begin{pmatrix} -8 & -2 & 16 & -6 \\ 16 & -12 & 0 & -4 \\ -20 & 15 & -8 & -3 \\ 0 & -8 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{1}{16} & -\frac{1}{2} & \frac{3}{16} \\ -\frac{1}{2} & \frac{3}{8} & 0 & \frac{1}{8} \\ \frac{5}{8} & -\frac{15}{32} & \frac{1}{4} & \frac{3}{32} \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} \end{pmatrix}$$

2.12/

$$Chứng\ minh: với\ A \in M_n(Z)\ thì\ detA \in Z$$

$Với\ k = 1\ thì\ detA = det(a_1) = a_1 \in Z$

$Giả\ sử\ mệnh\ đề\ đúng\ với\ k = n, hay\ detA_n = det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in Z$

$Ta\ cần\ chứng\ minh\ mệnh\ đề\ đúng\ với\ k = n+1, hay\ detA_{n+1} = det\begin{pmatrix} a_{11} & \cdots & a_{1(n+1)} \\ \vdots & \ddots & \vdots \\ a_{(n+1)1} & \cdots & a_{(n+1)(n+1)} \end{pmatrix}$

$$= \sum_{j=1}^{n+1} a_{(n+1)j}(-1)^{n+1+j} det(A\,(n+1|j))$$

$mà\ det(A(n+1|j)) \in Z\ theo\ giả\ sử\ (với\ 1 \le j \le n+1)$

$$=> \sum_{j=1}^{n+1} a_{(n+1)j}(-1)^{n+1+j} det(A\,(n+1|j)) \ \in Z$$

$Vậy\ nếu\ A\ \in\ M_n(Z)\ thì\ detA \in Z\ (đpcm)$

$$Chứng\ minh\ nếu\ A\ khả\ nghịch\ thì\ A^{-1} \in M_n(Z)\ \leftrightarrow\ |detA|\ =\ 1$$

$Ta\ có: A.A^{-1} = I_n \rightarrow det(AA^{-1}) = det(I_n) \rightarrow (detA)(detA^{-1}) = 1 \rightarrow detA = \dfrac{1}{detA^{-1}}$

$Do\ detA \in Z\ và\ detA^{-1}\ cũng \in Z\ nên\ detA^{-1}\ chỉ\ có\ thể\ là\ 1\ hoặc\ -1. Khi\ đó\ detA = \pm 1 \rightarrow |detA| = 1$

2.14 Giải các hệ phương trình sau bằng cách áp dụng quy tắc Cramer.

a/

$$\begin{cases} 4x_1 + 3x_2 = 4 \\ 2x_1 + 7x_2 = 10 \end{cases}$$

$Ta\ có: \Delta = \ |A| = \begin{vmatrix} 4 & 3 \\ 2 & 7 \end{vmatrix} = 22$

$\Delta_1 = |A_1| = \begin{vmatrix} 4 & 3 \\ 10 & 7 \end{vmatrix} = -2$

$\Delta_2 = |A_2| = \begin{vmatrix} 4 & 4 \\ 2 & 10 \end{vmatrix} = 32$

$V\text{ì } \Delta \neq 0$ *nên phương trình có nghiệm duy nhất*:

$x_1 = \dfrac{\Delta_1}{\Delta} = -\dfrac{1}{11}$

$x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{16}{11}$

b)

$$\begin{cases} x_1 + 6x_2 = 8 \\ -3x_1 + 9x_2 = 12 \end{cases}$$

Ta *có*: $\Delta = |A| = \left|\begin{pmatrix} 1 & 6 \\ -3 & 9 \end{pmatrix}\right| = 27$

$\Delta_1 = |A_1| = \left|\begin{pmatrix} 8 & 6 \\ 12 & 9 \end{pmatrix}\right| = 0$

$\Delta_2 = |A_2| = \left|\begin{pmatrix} 1 & 8 \\ -3 & 12 \end{pmatrix}\right| = 36$

$V\text{ì } \Delta \neq 0$ *nên phương trình có nghiệm duy nhất*:

$x_1 = \dfrac{\Delta_1}{\Delta} = 0$

$x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{4}{3}$

c) $\begin{cases} x_1 + x_2 - 2x_3 = 3 \\ 2x_1 + 3x_2 - 7x_3 = 10 \\ 5x_1 + 2x_2 + x_3 = 1 \end{cases}$

Ta giải hệ phương trình trên theo quy tắc Cramer:

Ta có:

$$\Delta = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -7 \\ 5 & 2 & 1 \end{vmatrix}$$

$$\xrightarrow[d_3 := d_3 - 5d_1]{d_2 := d_2 - 2d_1} \begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & -3 & 11 \end{vmatrix}$$

$$= 1.(-1)^{1+1} \begin{vmatrix} 1 & -3 \\ -3 & 11 \end{vmatrix}$$

$$= (1.11 - (-3).(-3)) = 2$$

$$\Delta_1 = \begin{vmatrix} 3 & 1 & -2 \\ 10 & 3 & -7 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\xrightarrow[d_2 := d_2 - 10d_3]{d_1 := d_1 - 3d_3} \begin{vmatrix} 0 & -5 & -5 \\ 0 & -17 & -17 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 1.(-1)^{3+1} \begin{vmatrix} -5 & -5 \\ -17 & -17 \end{vmatrix}$$

$$= ((-5).(-17) - (-5).(-17)) = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & -2 \\ 2 & 10 & -7 \\ 5 & 1 & 1 \end{vmatrix}$$

$$\xrightarrow[d_2 := d_2 - 10d_3]{d_1 := d_1 - 3d_3} \begin{vmatrix} -14 & 0 & -5 \\ -48 & 0 & -17 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= 1.(-1)^{3+2} \begin{vmatrix} -14 & -5 \\ -48 & -17 \end{vmatrix}$$

$$= -((-14).(-17) - (-48).(-5)) = 2$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 10 \\ 5 & 2 & 1 \end{vmatrix}$$

$$\xrightarrow[d_2 := d_2 - 10d_3]{d_1 := d_1 - 3d_3} \begin{vmatrix} -14 & -5 & 0 \\ -48 & -17 & 0 \\ 5 & 2 & 1 \end{vmatrix}$$

$$= 1.(-1)^{3+3} \begin{vmatrix} -14 & -5 \\ -48 & -17 \end{vmatrix}$$

$= ((-14).(-17) - (-48).(-5)) = -2$

Do $\Delta = 2 \neq 0$ nên hệ phương trình đã cho có nghiệm duy nhất:

$$\begin{cases} x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{0}{2} = 0 \\ x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{2}{2} = 1 \\ x_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{-2}{2} = -1 \end{cases}$$

d) $\begin{cases} 7x_1 + 2x_2 + 3x_3 = 1 \\ 5x_1 - 3x_2 + 2x_3 = 5 \\ 5x_1 - 8x_2 + 3x_3 = 11 \end{cases}$

Ta giải hệ phương trình trên theo quy tắc Cramer:

Ta có:

$$\Delta = \begin{vmatrix} 7 & 2 & 3 \\ 5 & -3 & 2 \\ 5 & -8 & 3 \end{vmatrix}$$

$$\xRightarrow[]{\substack{d_1 := d_1 - d_2 \\ d_2 := d_2 - 2d_1 \\ d_3 := d_3 - 3d_1}} \begin{vmatrix} 2 & 5 & 1 \\ 1 & -13 & 0 \\ -1 & -23 & 0 \end{vmatrix}$$

$$= 1.(-1)^{1+3} \begin{vmatrix} 1 & -13 \\ -1 & -23 \end{vmatrix}$$

$$= (1.(-23) - (-1).(-13)) = -36$$

$$\Delta_1 = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -3 & 2 \\ 11 & -8 & 3 \end{vmatrix}$$

$$\xRightarrow[]{\substack{d_2 := d_2 - 5d_1 \\ d_3 := d_3 - 11d_1}} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -13 & -13 \\ 0 & -30 & -30 \end{vmatrix}$$

$$= 1.(-1)^{1+1} \begin{vmatrix} -13 & -13 \\ -30 & -30 \end{vmatrix}$$

$$= ((-13).(-30) - (-30).(-13)) = 0$$

$$\Delta_2 = \begin{vmatrix} 7 & 1 & 3 \\ 5 & 5 & 2 \\ 5 & 11 & 3 \end{vmatrix}$$

$$\xrightarrow[d_3 := d_3 - 11d_1]{d_2 := d_2 - 5d_1} \begin{vmatrix} 7 & 1 & 3 \\ -30 & 0 & -13 \\ -72 & 0 & -30 \end{vmatrix}$$

$$= 1.(-1)^{1+2} \begin{vmatrix} -30 & -13 \\ -72 & -30 \end{vmatrix}$$

$$= -((-30).(-30) - (-72).(-13)) = 36$$

$$\Delta_3 = \begin{vmatrix} 7 & 2 & 1 \\ 5 & -3 & 5 \\ 5 & -8 & 11 \end{vmatrix}$$

$$\xrightarrow[d_3 := d_3 - 11d_1]{d_2 := d_2 - 5d_1} \begin{vmatrix} 7 & 2 & 1 \\ -30 & -13 & 0 \\ -72 & -30 & 0 \end{vmatrix}$$

$$= 1.(-1)^{1+3} \begin{vmatrix} -30 & -13 \\ -72 & -30 \end{vmatrix}$$

$$= ((-30).(-30) - (-72).(-13)) = -36$$

Do $\Delta = -36 \neq 0$ nên hệ phương trình đã cho có nghiệm duy nhất:

$$\begin{cases} x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{0}{-36} = 0 \\ x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{36}{-36} = -1 \\ x_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{-36}{-36} = 1 \end{cases}$$

2.16 Cho hệ phương trình phụ thuộc vào các tham số a, b

$$\begin{cases} x_1 + 2x_2 + ax_3 = 3 \\ 3x_1 - x_2 - ax_3 = 2 \text{ (*)} \\ 2x_1 + x_2 + 3x_3 = b \end{cases}$$

$$Ta\ có: \Delta = |A| = \left|\begin{pmatrix} 1 & 2 & a \\ 3 & -1 & -a \\ 2 & 1 & 3 \end{pmatrix}\right| = 2a - 21$$

$$\Delta_1 = |A_1| = \left|\begin{pmatrix} 3 & 2 & a \\ 2 & -1 & -a \\ b & 1 & 3 \end{pmatrix}\right| = 5a - ab - 21$$

$$\Delta_2 = |A_2| = \left|\begin{pmatrix} 1 & 3 & a \\ 3 & 2 & -a \\ 2 & b & 3 \end{pmatrix}\right| = -10a + 4ab - 21$$

$$\Delta_3 = |A_3| = \left|\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 2 & 1 & b \end{pmatrix}\right| = -7b + 21$$

a) (*) có nghiệm duy nhất $\Leftrightarrow \Delta \neq 0 \Leftrightarrow 2a - 21 \neq 0 \Leftrightarrow a \neq \frac{21}{2}$

b) (*) vô số nghiệm $\Rightarrow \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$$\Leftrightarrow \begin{cases} 2a - 21 = 0 \\ 5a - ab - 21 = 0 \\ -10a + 4ab - 21 = 0 \\ -7b + 21 = 0 \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{21}{2} \\ b = 3 \end{cases}$$

$\Rightarrow$ *Hệ phương trình* $(*)$:

$$\begin{cases} x_1 + 2x_2 + \dfrac{21}{2}x_3 = 3 \\ 3x_1 - x_2 - \dfrac{21}{2}x_3 = 2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$$

Ma trận hoá hệ phương trình:

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & 2 & \dfrac{21}{2} & 3 \\ 3 & -1 & -\dfrac{21}{2} & 2 \\ 2 & 1 & 3 & 3 \end{array}\right) \xrightarrow[2d_1]{\substack{d_2-3d_1 \\ d_3-2d_1}} \left(\begin{array}{ccc|c} 2 & 4 & 21 & 6 \\ 0 & -7 & -42 & -7 \\ 0 & -3 & -18 & -3 \end{array}\right) \xrightarrow[-\frac{1}{7}d_2]{\substack{7d_3 \\ d_3-3d_2}} \left(\begin{array}{ccc|c} 2 & 4 & 21 & 6 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} 2x_1 + 4x_2 + 21x_3 = 6 \\ x_2 + 6x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 1 + \dfrac{3}{2}t \\ x_2 = 1 - 6t \\ x_3 = t \in \mathbb{R} \end{cases}$$