

Semi-supervised Learning with Deep Generative Models

Kingma et. al. (2014)

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CS 7180

Motivating Question

How can we model data of increasing size when obtaining label information is difficult?

High-level Answer

We can estimate missing label information by using a probabilistic model.

Most Relevant Previous Work

Pitelis, N., Russell, C., and Agapito, L. (2014). Semi-supervised learning using an unsupervised atlas. *In Proceedings of the European Conference on Machine Learning (ECML)*, volume LNCS 8725, pages 565 – 580.

- ▶ Observing that high-dimensional datasets often lie on or near *manifolds* of locally low rank can help avoid the *curse of dimensionality*
- ▶ Experiments show how using unlabelled data to learn the underlying manifold improves classifier accuracy when trained on limited labelled data
 1. **Unsupervised learning of the underlying manifold:** Approximate the manifold of data on the original space by fitting an atlas of low-dimensional overlapping affine charts.
 2. **Supervised training of an SVM:** Proposed a new family of Mercer Kernels for SVM-based supervised learning which uses soft-assignment of datapoints to the underlying low-dimensional affine charts to generate the kernels

Specifying the Probabilistic Model for Missing Labels

Kingma et. al. (2014)

- ▶ Data appears as pairs $(\mathbf{X}, \mathbf{Y}) = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)\}$ with the i -th observation $x_i \in \mathbb{R}^D$ and a corresponding class label $y_i \in \{1, \dots, L\}$
 - ▶ Each pair of observations (x_i, y_i) has a corresponding latent variable z_i
 - ▶ Empirical distribution over the labelled and unlabelled subsets is referred to as $\tilde{p}_l(\mathbf{x}, y)$ and $\tilde{p}_u(\mathbf{x})$
- ▶ We can estimate y_i for x_i in distribution $\tilde{p}_u(\mathbf{x})$ by finding the maximum probability of $p(y_i)$ by using a set of features related to z_i and a predictive model
 1. **Latent-feature discriminative model (M1)**
 2. **Generative semi-supervised model (M2)**
 3. **Stacked generative semi-supervised model (M1+M2)**

Bayes Rule is used when specifying M1 & M2

$$\begin{aligned}p(x, y) &= p(x)p(y|x) \\ &= p(y)p(x|y) \\ p(x|y) &= \frac{p(x)p(y|x)}{p(y)}\end{aligned}$$

for models M1 ¹, $p(z|x)$, and M2 ²; $p(y|x)$

¹Kingma et. al. (2014) equation (1)

²Kingma et. al. (2014) equation (2)

(M1) Latent-feature discriminative model

$$y \Leftarrow p(z|x) = \frac{p(z)p(x|z)}{p(x)}$$

where

$p(z) = \mathcal{N}(z|0, I)$ Gaussian distribution of z given a missing label y

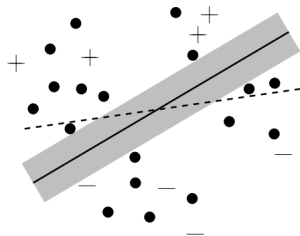
$p(x|z) = f(x; z, \theta)$ likelihood function, parameters θ of a set of z

$p(x) = \tilde{p}_u(x)$ unlabelled subset of $x_i \in \mathbb{R}^D$

Kingma et. al. (2014) eq. (1)

(M1) Predicting Class Labels y

Approximate samples from the posterior distribution over the latent variables $p(z|x)$ are used as features to train a classifier that predicts class labels y



(transductive) SVM³ finds the largest margin w.r.t. the training **and** the test vectors

³See Figure 6.2 from Chapelle, O., B. Schölkopf, and A. Zien.
"Semi-Supervised Learning." (2006).

(M2) Generative semi-supervised model

$$p(y|\mathbf{x}) = \frac{p(\mathbf{x}|y)p(y)}{p(\mathbf{x})} \approx \frac{p_{\theta}(\mathbf{x}|y, \mathbf{z})p(y)}{p(\mathbf{x})}$$

where

$$p(y) = \text{Cat}(y|\pi)$$

multinomial distribution, y can be latent

$$p(\mathbf{z}) = \mathcal{N}(\mathbf{z}|\mathbf{0}, \mathbf{I})$$

Gaussian distribution of z when missing y

$$p_{\theta}(\mathbf{x}|y, \mathbf{z}) = f(\mathbf{x}; y, \mathbf{z}, \theta)$$

likelihood function, nonlinear parameters

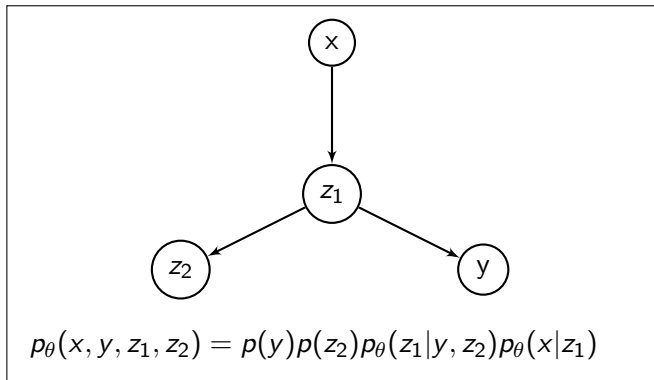
$$p(\mathbf{x})$$

all \mathbf{x} in dist. of real numbers; $\mathbf{x} \in \mathbb{R}^D$

Stacked generative semi-supervised model (M1 + M2)

Combine M1 and M2

1. Learn a new latent representation z_1 from M1
2. Use embeddings from z_1 instead of raw data x , to create a generative semi-supervised model M2



Scaling Up: Lower Bound Objective

Lower Bound Objective⁴: computation of the exact posterior distribution is intractable for models M1 and M2

$$\text{M1: } q_{\phi}(z|x) = \mathcal{N}(z|u_{\phi}(x), \text{diag}(\sigma_{\phi}^2(\mathbf{x}))) \quad (3)$$

$$\begin{aligned} \text{M2: } q_{\phi}(\mathbf{z}|y, \mathbf{x}) &= \mathcal{N}(\mathbf{z}|\mu_{\phi}(y, \mathbf{x}), \text{diag}(\sigma_{\phi}^2(\mathbf{x}))); \\ q_{\phi}(y|\mathbf{z}) &= \text{Cat}(y|\pi_{\phi}(x)), \end{aligned} \quad (4)$$

where

$\sigma_{\phi}(x)$	vector of standard deviations
$\pi_{\phi}(x)$	probability vector
$\mu_{\phi}(x), \sigma_{\phi}(x), \pi_{\phi}(x)$	Maximum likelihood Priors (MLPs)

⁴Kingma et. al. (2014) equations (3), (4)

Scaling Up: M1 Model Objective

Variational bound $\mathcal{J}(x)$ on the marginal likelihood of a single data point is⁵

$$\log p_{\theta}(x) \geq \mathbb{E}_{q_{\phi}(z|x)}[\log p_{\theta}(x|z)] - KL[q_{\phi}(z|x)||p_{\theta}(z)] = -\mathcal{J}(x)$$

Approximate posterior is used as a feature extractor for the labelled data set, and the features used for training the classifier

⁵Kingma et. al. (2014) Equation 5

Scaling Up: M2 Model Objective

When y_i is observed for the (x_i, y_i) data pair, extend from M1

$$\begin{aligned}\log p_{\theta}(x) &\geq \mathbb{E}_{q_{\phi}(z|x,y)}[\log p_{\theta}(x|y,z) + \log p_{\theta}(y) + \log p(z) - q_{\theta}(z|x,y)] \\ &= -\mathcal{L}(x, y)\end{aligned}$$

In the case where y_i is missing,

$$\begin{aligned}\log p_{\theta}(x) &\geq \mathbb{E}_{q_{\phi}(y,z|x)}[\log p_{\theta}(x|y,z) + \log p_{\theta}(y) \\ &\quad + \log p(z) - \log q_{\phi}(y,z|x)] \\ &= \sum_y q_{\phi}(y|x)(-\mathcal{L}(x, y)) + \mathcal{H}(q_{\phi}(y|x)) \\ &= -\mathcal{U}(x)\end{aligned}$$

The bound on the marginal likelihood for the entire dataset is now⁶

$$\mathcal{J} = \sum_{(x,y) \sim \tilde{p}_l} \mathcal{L}(x, y) + \sum_{x \sim \tilde{p}_u} \mathcal{U}(x)$$

⁶See Kingma et. al. (2014) equations 6-9

Optimization Techniques

- ▶ Bounds from M1 and M2 objective function equations provides for optimization of both θ and ϕ parameters
 - ▶ Optimization can be done *jointly* without using the EM Algorithm
- ▶ Use deterministic reparameterizations of the expectations in the objective function and *Monte Carlo* approximation
- ▶ Previous work refers to this as *stochastic gradient variational Bayes*⁷ and *stochastic backpropagation*⁸

⁷Kingma, D. P. and Welling, M. (2014). Auto-encoding variational Bayes. *In Proceedings of the International Conference on Learning Representations (ICLR)*.

⁸Rezende, D. J., Mohamed, S., and Wierstra, D. (2014). Stochastic backpropagation and approximate inference in deep generative models. *In Proceedings of the International Conference on Machine Learning (ICML)*, volume 32 of JMLR WCP.

Optimization Algorithms

Algorithm 1 Learning in model M1

```
while generativeTraining() do
   $\mathcal{D} \leftarrow \text{getRandomMiniBatch}()$ 
   $\mathbf{z}_i \sim q_\phi(\mathbf{z}_i | \mathbf{x}_i) \quad \forall \mathbf{x}_i \in \mathcal{D}$ 
   $\mathcal{J} \leftarrow \sum_n \mathcal{J}(\mathbf{x}_i)$ 
   $(\mathbf{g}_\theta, \mathbf{g}_\phi) \leftarrow (\frac{\partial \mathcal{J}}{\partial \theta}, \frac{\partial \mathcal{J}}{\partial \phi})$ 
   $(\theta, \phi) \leftarrow (\theta, \phi) + \Gamma(\mathbf{g}_\theta, \mathbf{g}_\phi)$ 
end while
while discriminativeTraining() do
   $\mathcal{D} \leftarrow \text{getLabeledRandomMiniBatch}()$ 
   $\mathbf{z}_i \sim q_\phi(\mathbf{z}_i | \mathbf{x}_i) \quad \forall \{\mathbf{x}_i, y_i\} \in \mathcal{D}$ 
   $\text{trainClassifier}(\{\mathbf{z}_i, y_i\})$ 
end while
```

Algorithm 2 Learning in model M2

```
while training() do
   $\mathcal{D} \leftarrow \text{getRandomMiniBatch}()$ 
   $y_i \sim q_\phi(y_i | \mathbf{x}_i) \quad \forall \{\mathbf{x}_i, y_i\} \notin \mathcal{O}$ 
   $\mathbf{z}_i \sim q_\phi(\mathbf{z}_i | y_i, \mathbf{x}_i)$ 
   $\mathcal{J}^\alpha \leftarrow \text{eq. (9)}$ 
   $(\mathbf{g}_\theta, \mathbf{g}_\phi) \leftarrow (\frac{\partial \mathcal{L}^\alpha}{\partial \theta}, \frac{\partial \mathcal{L}^\alpha}{\partial \phi})$ 
   $(\theta, \phi) \leftarrow (\theta, \phi) + \Gamma(\mathbf{g}_\theta, \mathbf{g}_\phi)$ 
end while
```

$$\nabla_{\{\theta, \phi\}} \mathbb{E}_{q_\phi(z|x)} [\log p_\theta(x|z)] = \mathbb{E}_{\mathcal{N}(\epsilon|0, I)} [\nabla_{\{\theta, \phi\}} \log p_\theta(x | \mu_\theta(x) + \sigma_\phi(x) \odot \epsilon)].$$

Kingma et. al. (2014) equation 11

Results: Benchmark Classification

Table 1: Benchmark results of semi-supervised classification on MNIST with few labels.

N	NN	CNN	TSVM	CAE	MTC	AtlasRBF	M1+TSVM	M2	M1+M2
100	25.81	22.98	16.81	13.47	12.03	8.10 (\pm 0.95)	11.82 (\pm 0.25)	11.97 (\pm 1.71)	3.33 (\pm 0.14)
600	11.44	7.68	6.16	6.3	5.13	–	5.72 (\pm 0.049)	4.94 (\pm 0.13)	2.59 (\pm 0.05)
1000	10.7	6.45	5.38	4.77	3.64	3.68 (\pm 0.12)	4.24 (\pm 0.07)	3.60 (\pm 0.56)	2.40 (\pm 0.02)
3000	6.04	3.35	3.45	3.22	2.57	–	3.49 (\pm 0.04)	3.92 (\pm 0.63)	2.18 (\pm 0.04)

Results: Image Classification

Table 2: Semi-supervised classification on the SVHN dataset with 1000 labels.

KNN	TSVM	M1+KNN	M1+TSVM	M1+M2
77.93 (± 0.08)	66.55 (± 0.10)	65.63 (± 0.15)	54.33 (± 0.11)	36.02 (± 0.10)

Table 3: Semi-supervised classification on the NORB dataset with 1000 labels.

KNN	TSVM	M1+KNN	M1+TSVM
78.71 (± 0.02)	26.00 (± 0.06)	65.39 (± 0.09)	18.79 (± 0.05)

Discussion

You can use this model for a few things
TODO: just summarize their discussion

Any questions? Thanks!