HW #5
Math/CS 471, Fall 2021

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1 Task 1: Describe the system of ODEs

1.1 What

We were asked to interpret the system of ordinary differential equations (ODEs) that describe the movement of a flock of birds around a tasty piece of food. Movement on a 2-D grid was assumed. The following conditions were given:

1. A flock consists of N birds.
2. $B_k(t)$ is a length 2 row vector corresponding to the kth bird's x and y coordinates at time t.
3. $C(t)$ is a length 2 row vector corresponding to the position (x, y) of the food at time t.
4. $f(t, B(t))$ represents forcing terms on the system of ODEs, i.e. the flocking or repelling force on the birds. It is a length $N \times 2$ vector corresponding to the bird positioning adjustments made by each force for a given time step t.

The forcing terms will be described in detail in the following sections. The tuning parameters will be described in Section 1.3.

1.2 How

The first bird, $B_1(t)$, is designated the leader of the flock. He or she is only interested in obtaining food at the coordinates given by $C(t)$. This attraction force is defined by:

$$F_1^{food}(t, B(t)) = \gamma_1(C(t) - B_1(t)) \tag{1}$$

It should be noted that this force is only applied to the leader of the flock. As the leader approaches the food, $|C(t) - B_1(t)|$ will decrease, causing the bird not to sit "exactly" on top of the food.

The second force affecting the flock is the following force. The rest of the flock needs this force in order to follow their leader. Otherwise, they will be unable reach the delicious food and lose track of their leader. This attraction force is defined by:

$$F_k^{follow}(t, B(t)) = \gamma_2(B_1(t) - B_k(t)) \tag{2}$$

This following force is applied to each bird within the flock. It grows proportionally to each bird's distance from their leader, B_1. Thus, as the flock approaches the leader, they will approach at a slower rate relative to γ_2.

The third force affecting the flock is the flocking force. This force is the desire of the flock to move towards the center of the flock. As the time increments, the birds will continue to move towards the center until they are on top of one another. This force is defined by:

$$\overline{B}(t) = \sum_{k=1}^{N} \frac{B_k(t)}{N} \tag{3}$$

$$F_k^{fl}(t, B(t)) = \kappa(\overline{B}(t) - B_k(t)), k = 2, 3, ..., N \tag{4}$$

This force applied to all of the flock except for the leader. It was one of the primary forces that impacted the diameter of the flock.

The fourth and final force affecting the flock was the repelling force. This force represented the impulse of the birds to no sit right on top of one another. As the birds began to approach each other, this force would increase causing them to repel. It was defined by:

$$F_{k,j}^{rep}(t, B(t)) = \sum_{i=1}^{5} \rho \frac{(B_{k,j}(t)) - (B_{l_i^k j}(t))}{((B_{k,j}(t)) - (B_{l_i^k j}(t)))^2 + \delta} \tag{5}$$

This force was the counter force to the flocking force with it acting as a means to prevent birds from flying on top of one another. While the flocking force pulled the birds together, the repelling force would increase as the birds got within a certain distance of one another. It should be noted that the force only impacted the closest 5 birds which meant that force large flock numbers (like $N = 100$ birds, the visualization wouldn't scale as expected. In other words, the closest five birds would be repelled from a given bird at a given time stamp, but there still might be a large number that are still close and would feel the flocking force and as such, the flock might appear more clumped up the larger the number of birds.

1.3 Why

First, the attraction force to food, F^{food}, depends on a user-defined parameter, γ_1. γ_1 describes how delicious the food tastes to the leader bird. As γ_1 increases, the leader will approach the food at a faster rate. If it is set to 0, the leader (and, implicitly, the flock) will remain stationary as it will not have any need to reach the food.

The following force, F^{follow}, depends on a parameter, γ_2. γ_2 describes how charismatic the leader of the flock is. As γ_2 increases, the flock will follow the leader at a closer distance, eventually staying "on top" of the leader. When it is set to 0, the flock will completely disregard their leader and stay in place, never actually reaching the food to share.

The flocking force, F^{flock}, is partially impacted by the parameter κ, which impacts how strongly the birds want to be in the center of the flock. This is simply a scalar that impacts the overall magnitude of the force. A smaller value for kappa will cause the birds to flock together more slowly and to stay relatively less clumped up as the repelling force will dominate the interaction.

The repelling force, F^{repel}, is impacted by two different parameters, ρ, which impacts the overall magnitude of the repelling force, and then δ, which impacts the degree to which the birds repel. In other words, when ρ is zeroed out, the entire flocking force goes to zero and the flock will compress into a singular point and if ρ is increased. When δ is zeroed out, the degree with which each bird is repelled increases causing the flock to essentially scatter apart within a few time steps.

2 Task 2: Implementation and Experimentation

2.1 What

This section asked us to examine the behaviors of changing each of the parameters including the food flag.

2.2 How

The Runge-Kutta 4 (RK4) method was used to numerically evaluate the system of ODEs. In Python, numpy can be used to implement RK4 for our system as seen below:

```
k1 = f(y, t, food_flag, alpha, gamma_1, ...)
k2 = f((y+(k1*(dt/2.0))), (t+(dt/2.0)), food_flag, alpha, gamma_1, ...)
k3 = f((y+(k2*(dt/2.0))), (t+(dt/2.0)), food_flag, alpha, gamma_1, ...)
k4 = f((y+(dt*k3)), (t+dt), food_flag, alpha, gamma_1, ...)

y = y + (((dt/6.0))*(k1+2.0*k2+2.0*k3+k4))

return y
```

Next, the forcing terms were implemented using numpy's built-in argsort and sum routines, in addition to for loops, as follows:

```
# food
c = zeros((1,2))
if food_flag == 0:
    c[0,0] = 0.0
    c[0,1] = 0.0
else:
    c[0,0] = sin(alpha*t)
    c[0,1] = cos(alpha*t)
f_food = zeros_like(y)
f_food[0] = gamma_1 * (c - y[0])

# follow
```

```
f_follow = zeros_like(y)
f_follow = gamma_2 * (y[0] - y)

# flock
f_flock = zeros_like(y)
y_bar = sum(y / N, axis=0)
f_flock = kappa * (y_bar - y)
f_flock[0,0] = 0
f_flock[0,1] = 0

# repel
f_repel = zeros_like(y)
neighbors = zeros((N,5))
# Loop through flock and assemble list of birds by distance
flock_distance = zeros((N,N))
for i in range(N):
    #calculate distance for all birds around given bird
    for j in range(N):
        if (i != j):
            flock_distance[i,j] = distance(y[i], y[j])
        else:
            flock_distance[i,j] = Inf

# Sort the flock_distance array columns
sorted_distances_indexes = argsort(flock_distance, axis=1)
neighbors = sorted_distances_indexes[:,0:5]

#iterate over neighbors and generate sum of repel?
# Outer loop is each bird
for k in range(1,neighbors.shape[0]):
    # Inner loop is each five closest neighbors
    for i in range(neighbors.shape[1]):
        f_repel[k] += rho*( (y[k]-y[neighbors[k,i]])
                      / ((y[k] - y[neighbors[k,i]])**2 + delta) )
```

It should be noted that the leader bird was colored blue in our movie visualizations in order to make his or her location known.

2.3 Why

First, the food flag either causes the food to be a stationary object at the origin, or to move in a circle. It was clearly seen in the movies provided (the default parameter movie and the stationaryfood movie in the repository, that the flocking forces and subsequent behaviors were all functioning properly. When the food is stationary, the leader begins to move towards the food, followed by the remaining flock moving towards the leader. Both of these interactions are as expected indicating that the leader's charisma (γ_2) was strong and that the tastiness of the food was also strong (γ_1). When the food moves in a circular pattern, the flock follows the food around the path it takes.

The next parameter, α, describes the radius of the circular path (a parameterized function $(x, y) = (\sin(\alpha t), \cos(\alpha t)))$ that the food moves in. It is observed that this parameter only changes the path of the food, so as long as γ_1, the magnitude of how good the food tastes, is greater than 0, the leader (and hopefully the rest of the flock, depending on γ_2) will follow the larger or smaller circular path of the food. An example of a larger path can be seen in the repository.

The next parameter, γ_1, controls how close to the food the leader will follow. As seen in the repository, the leader bird will follow the food at a closer distance as γ_1 increases. This will visually appear as if the flock stays "right on top" of the food, assuming $\gamma_2 > 0$. This effectively controls how tasty the food is.

The parameter, γ_2, describes the flock's confidence in the leader bird. If γ_2 is set to 0, the flock will remain at its start position, completely disregarding their leader's intelligence. Otherwise, as γ_2 increases, each bird attempts to follow their leader at a closer distance. Both effects can be seen in our repository.

The next parameter, κ, relates to the safety of the flock. As κ increases, movies in our repository show that the overall distance between birds decreases, and this occurs at a faster rate. If it is 0, the shape of the flock diverges and does not resemble a flock.

It is noted that all three forces above involve at least 2 entities' distances', so their effect will decrease as they approach each other.

Finally, the repelling force depends on two parameters, ρ and δ. ρ controls the overall magnitude of the repelling force. Thus, it can be seen in certain movies of our repository that as ρ increases, the overall size of the flock grows because the birds are repelled quicker.

The δ parameter can be used to alleviate an inherent problem in the model for the repelling forcing term. Since this force is only applied to each bird's five closest neighbors at a given time step, as N grows, more birds will sit on top of each other yielding unrealistic behavior. Small decreases in δ can spread out a densely packed flock while keeping their repelling speed roughly constant.

3 Task 3: Flock Diameter

3.1 What

One interesting element of the visualization was the diameter of the flock and how it would change with the given parameters. We were asked to identify how the flock diameter changes over time with varying sets of input parameters.

3.2 How

Flock diameter was mainly impacted by the flocking force and the repelling force, although there were some interactions that came into place with the number of birds in the simulation. We defined the diameter of the flock to be the distance between the two further birds in the flock. To accomplish this, we used a simple Euclidean distance calculation (which was also used in the repelling force itself) and then created a two dimensional array that calculated the distances between all points and the returned the max value from the array. The following is our block of code defining the distance and the diameter:

```
# Function for calculating Euclidean distance between birds
# Passing in two bird locations, each bird is vector size (1,2)
# y is first bird, y1 is second bird
def distance(y, y1):
    distance = sqrt((y[0]-y1[0])**2 + (y[1]-y1[1])**2)
    return distance

# Calculates the flock diameter
def diameter(y):
    distances = zeros((N,N))
    for i in range(y.shape[0]):
        for j in range(y.shape[0]):
            distances[i,j] = distance(y[i],y[j])

    diameter = max(distances.reshape(-1,))
    return diameter
```

3.3 Why

As described in section 1.3, as the flocking strength increased, the smaller the diameter would get. The flocking force would slowly begin to dominate the repelling force preventing the force from pushing the birds away causing them to start bunching more on top of one another. As the repelling force began to increase, the diameter of the flock would increase. As the repelling term began to overtake the strength of the flocking term, the birds would not be able to move towards one another in a given time interval more than the distance that they'd be repelled causing the diameter to slowly increase throughout the simulation. It is important to note that the code provided had randomly generated locations for the birds to appear so each graph will look somewhat different simply because the birds may be initially more or less spread out.

Included below is a graph of the default settings of the skeleton code, with a population of birds $N = 30$. The diameter can be seen spiking initially primarily because the birds are randomly positioned far away from one another. The diameter decreases as the flocking force increases, pulling the birds together. Throughout the bulk of the plot, the diameter remains roughly consistent, oscillating between a given range:

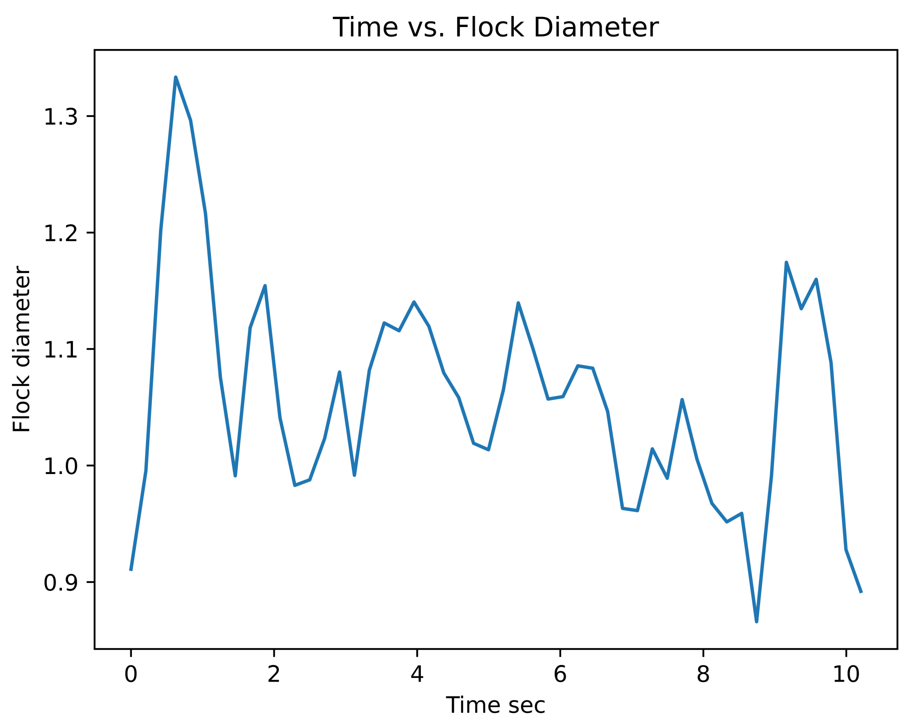

Figure 1: Plot for the default parameter setting with a N = 30 population

This default figure can be compared with two additional figures, one in which the repelling force is zeroed out and the flocking force dominates, and another in which the flocking force is zeroed out and the repelling force dominates. To get a repelling force of zero, we simply set the ρ parameter to zero, which zeroed out the entire term.

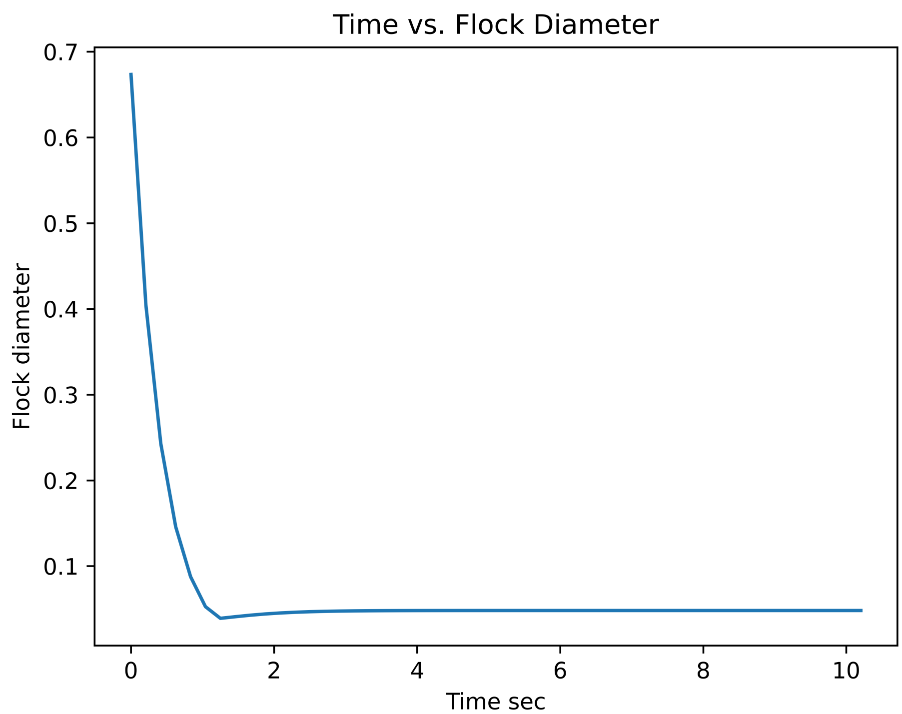

Figure 2: Plot for the default parameter settings with a N = 30 population and $\rho = 0.0$

As we can clearly see above, without a repelling force, the birds start initially spread apart and then rapidly converge as the time interval progresses. As the flocking force takes over and the birds are pulled on top of one another, we can see the diameter of the flock gets close to zero and then stays consistent over the rest of the simulation as there is no force to push the flock away.

We can examine the opposite effect with the following figure indicating a plot where the flocking force is zeroed out. This is achieved by setting the κ variable to zero:

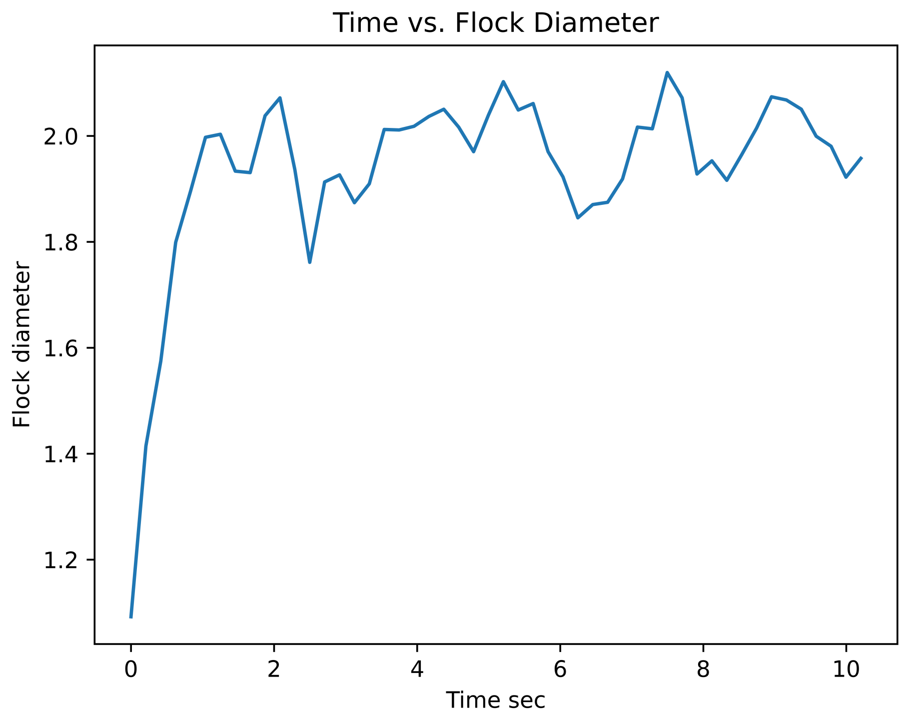

Figure 3: Plot for the default parameter settings with a N = 30 population and $\kappa = 0.0$

As we can clearly see in the figure, the flock starts off with a diameter of their randomly distributed starting locations and that rapidly increases as the repelling force begins to dominate the interactions. After a certain period of time, the flock beings to fluctuate within a certain range for the diameter, likely because the the food source was circling and the follow force was causing the paths of the flock to intersect to some degree.

Overall, we can clearly see that the flocking force and the repelling force have a profound impact on the diameter of the flock and the interaction of the two forces is the primary factor in determining the overall size of the flock. Supplementary plots of the flock diameter versus time can be found in our repository.

4 Notes

All referenced movies and plots can be found in the hw5/files subdirectory of our repository.

Files are named according to the parameters used:
[movie/diameter_plot]_N_gamma_1_gamma_2_alpha_kappa_rho_delta_.[mp4/png]