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1) $x = TAG$

a)

$$P(x) = a_{0\pi_1} \prod_{i=1}^{3} e(x_i)\, a_{\pi_i \pi_{i-1}}$$

$$P(x = TAG) = \underbrace{(0.7)}_{a_{0\pi_1}} \underbrace{(0.5)}_{e(T_1)} \underbrace{(0.7)}_{a_{\pi_2\pi_1}} \underbrace{(0.4)}_{e(A_2)} \underbrace{(0.7)}_{a_{\pi_3\pi_2}} \underbrace{(0.4)}_{e(G_3)} \underbrace{(0.9)}_{a_{end\,\pi_3}}$$

$$\boxed{= 0.0247}$$

b)

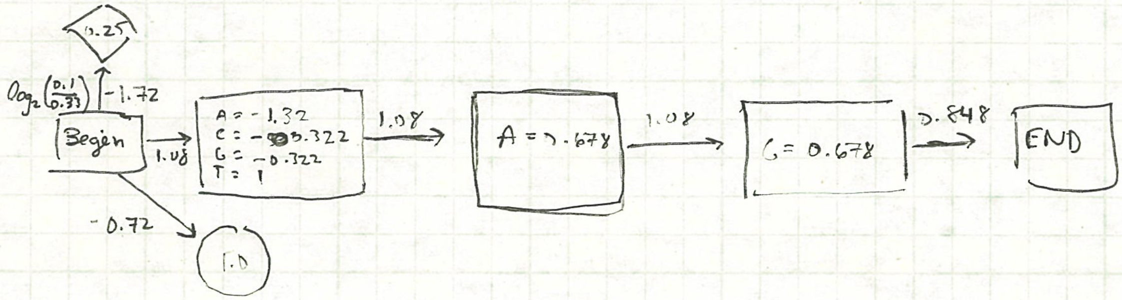
$$P(x = \diamond T \circ G) = (0.1)(0.25)(1)(0.5)(0.2)(1)(1)(0.4)(0.9)$$

$$\boxed{= 0.0009}$$

c) The original path (TAG) is the more likely path

$$\frac{P(a)}{P(b)} = \frac{0.0247}{0.0009} = 27.4$$

d)



$$P(x) = \log_2\left(\frac{a_{0\pi_1}}{0.33}\right) + \sum_{i=1}^{3} e(x_i)\, a_{\pi_i \pi_{i-1}}$$

$$= (1.08) + (1)(1.08) + (0.678)(1.08) + (0.678)(0.848)$$

$$\boxed{P(x) = 3.47}$$

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② 2 hidden states → H = high GC content ; L = low GC content

Transition Probabilities

$a_{start-H} = 0.5$
$a_{start-L} = 0.5$
$a_{HH} = 0.5$
$a_{HL} = 0.5$
$a_{LL} = 0.6$
$a_{LH} = 0.4$

Emission Probability

H: T = 0.2, C = 0.3, A = 0.2, G = 0.3

L: T = 0.3, C = 0.2, A = 0.3, G = 0.2

n = 9

x = GGCACTGAA

i = 1

$l = L: \ V_L(1) = e_L(G)\, a_{start-L} \cdot V_0 = (0.2)(0.5)(1) = 0.1$

$l = H: \ V_H(1) = e_H(G)\, a_{start-H} \cdot V_0 = (0.3)(0.5)(1) = 0.15$

i = 2

$l = L: \ V_L(2) = e_L(G) \max\begin{pmatrix} V_L(1) \cdot a_{LL} \\ V_H(1) \cdot a_{HL} \end{pmatrix} = (0.2)\max\begin{pmatrix} 0.06 \\ 0.075 \end{pmatrix} = 0.015$

$l = H: \ V_H(2) = e_H(G) \max\begin{pmatrix} V_L(1) \cdot a_{LH} \\ V_H(1) \cdot a_{HH} \end{pmatrix} = (0.3)\max\begin{pmatrix} 0.04 \\ 0.075 \end{pmatrix} = 0.0225$

i = 3

$l = L: \ V_L(3) = e_L(C) \max\begin{pmatrix} V_L(2) \cdot a_{LL} \\ V_H(2) \cdot a_{HL} \end{pmatrix} = (0.2)\max\begin{pmatrix} 0.009 \\ 0.01125 \end{pmatrix} = 0.00225$

$l = H: \ V_H(3) = e_H(C) \max\begin{pmatrix} V_L(2) \cdot a_{LH} \\ V_H(2) \cdot a_{HH} \end{pmatrix} = (0.3)\max\begin{pmatrix} 0.006 \\ 0.01125 \end{pmatrix} = 0.003375$

i = 4

$l = L: \ V_L(4) = e_L(A) \max\begin{pmatrix} V_L(3) \cdot a_{LL} \\ V_H(3) \cdot a_{HL} \end{pmatrix} = (0.3)\max\begin{pmatrix} 0.00135 \\ 0.00169 \end{pmatrix} = 0.000507$

$l = H: \ V_H(4) = e_H(A) \max\begin{pmatrix} V_L(3) \cdot a_{LH} \\ V_H(3) \cdot a_{HH} \end{pmatrix} = (0.2)\max\begin{pmatrix} 9e-4 \\ 0.00169 \end{pmatrix} = 0.000338$

i = 5

$l = L: \ V_L(5) = e_L(C) \max\begin{pmatrix} V_L(4)\, a_{LL} \\ V_H(4)\, a_{HL} \end{pmatrix} = (0.2)\max\begin{pmatrix} 0.0003042 \\ 0.000169 \end{pmatrix} = 6.084 \times 10^{-5}$

$l = H: \ V_H(5) = e_H(C) \max\begin{pmatrix} V_L(4)\, a_{LH} \\ V_L(4)\, a_{HH} \end{pmatrix} = (0.3)\max\begin{pmatrix} 0.0002028 \\ 0.000169 \end{pmatrix} = 6.084 \times 10^{-5}$

i = 6

$l = L: \ V_L(6) = e_L(T) \max\begin{pmatrix} V_L(5)\, a_{LL} \\ V_H(5)\, a_{HL} \end{pmatrix} = (0.3)\max\begin{pmatrix} 3.6504e-5 \\ 3.042e-5 \end{pmatrix} = 1.09512 \times 10^{-5}$

$l = H: \ V_L(6) = e_H(T) \max\begin{pmatrix} V_L(5)\, a_{LH} \\ V_H(5)\, a_{HH} \end{pmatrix} = (0.2)\max\begin{pmatrix} 2.4336e-5 \\ 3.042e-5 \end{pmatrix} = 6.084 \times 10^{-6}$

i = 7

$l = L: \ V_L(7) = e_L(G) \max\begin{pmatrix} V_L(6)\, a_{LL} \\ V_L(6)\, a_{HL} \end{pmatrix} = (0.2)\max\begin{pmatrix} 6.57072e-6 \\ 3.042e-6 \end{pmatrix} = 1.3141 \times 10^{-6}$

$l = H: \ V_H(7) = e_H(G) \max\begin{pmatrix} V_L(6)\, a_{LH} \\ V_H(6)\, a_{HH} \end{pmatrix} = (0.3)\max\begin{pmatrix} 4.3804e-6 \\ 3.042e-6 \end{pmatrix} = 1.3141 \times 10^{-6}$

i = 8

$l = L: \ V_L(8) = e_L(A) \max\begin{pmatrix} V_L(7)\, a_{LL} \\ V_H(7)\, a_{HL} \end{pmatrix} = (0.3)\max\begin{pmatrix} 7.8846e-7 \\ 6.5705e-7 \end{pmatrix} = 2.3654 \times 10^{-7}$

$l = H: \ V_H(8) = e_H(A) \max\begin{pmatrix} V_L(7)\, a_{LH} \\ V_H(7)\, a_{HH} \end{pmatrix} = (0.2)\max\begin{pmatrix} 5.2564e-7 \\ 6.5705e-7 \end{pmatrix} = 1.3141 \times 10^{-7}$

(2) Cont.

$i=9$

$l=L$: $v_L(9) = e_L(A) \max\begin{pmatrix} v_L(8)\, a_{LL} \\ v_H(8)\, a_{HL} \end{pmatrix} = (0.3) \max\begin{pmatrix} 1.4192\text{e-}7 \\ 6.5705\text{e-}8 \end{pmatrix} = 4.2577 \times 10^{-8}$

$l=H$: $v_H(9) = e_H(A) \max\begin{pmatrix} v_L(8)\, a_{LH} \\ v_H(8)\, a_{HH} \end{pmatrix} = (0.2) \max\begin{pmatrix} 9.4616\text{e-}8 \\ 6.5705\text{e-}8 \end{pmatrix} = 1.8923 \times 10^{-8}$

$i=10$

$$v_{end} = \max\begin{pmatrix} v_L(9) \cdot a_{L\,end} \\ v_H(9) \cdot a_{H\,end} \end{pmatrix} = \max\begin{pmatrix} 4.2577 \times 10^{-8} \\ 1.8923 \times 10^{-8} \end{pmatrix} = 4.2577 \times 10^{-8}$$

Trace back

$\boxed{\text{H H H L L L L L L}}$

G G C A C T G A A

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(3) 2 hidden states : H & L

Transition Probs

$a_{start-H} = 0.5$
$a_{start-L} = 0.5$
$a_{HH} = 0.5$
$a_{HL} = 0.5$
$a_{LL} = 0.6$
$a_{LH} = 0.4$

Emission Probs

H: T: 0.2, C: 0.3, A: 0.2, G: 0.3

L: T: 0.3, C: 0.2, A: 0.3, G: 0.2

X = GGCACTGAA

Forward

$f_s(0) = 1$

i=1

$l=L: \; f_L(1) = e_L(G)(f_L(0)\, a_{start\cdot L}) = (0.2)(1 \cdot 0.5) = 0.1$

$l=H: \; f_H(1) = e_H(G)(f_H(0)\, a_{start\cdot H}) = (0.3)(0.5) = 0.15$

$P(X_{1:1}) = 0.1 + 0.15 = 0.25$

i=2

$l=L: \; f_L(2) = e_L(G)(f_L(1)a_{LL} + f_H(1)a_{HL}) = (0.2)(0.06 + 0.75) = 0.162$

$l=H: \; f_H(2) = e_H(G)(f_L(1)a_{LH} + f_H(1)a_{HH}) = (0.3)(0.04 + 0.75) = 0.237$

$P(X_{1:2}) = 0.162 + 0.237 = 0.399$

i=3

$l=L: \; f_L(3) = e_L(C)(f_L(2)a_{LL} + f_H(2)a_{HL}) = (0.2)(0.0972 + 0.1185) = 0.0431$

$l=H: \; f_H(3) = e_H(C)(f_L(2)a_{LH} + f_H(2)a_{HH}) = (0.3)(0.0648 + 0.1185) = 0.0550$

$P(X_{1:3}) = 0.0431 + 0.0550 = 0.0981$

i=4

$l=L: \; f_L(4) = e_L(A)(f_L(3)a_{LL} + f_H(3)a_{HL}) = (0.3)(0.0259 + 0.0275) = 0.01602$

$l=H: \; f_H(4) = e_H(A)(f_L(3)a_{LH} + f_H(3)a_{HH}) = (0.2)(0.0173 + 0.0275) = 0.00896$

$P(X_{1:4}) = 0.01602 + 0.00896 = 0.02498$

i=5

$l=L: \; f_L(5) = e_L(C)(f_L(4)a_{LL} + f_H(4)a_{HL}) = (0.2)(9.612e{-3} + 4.48e{-3}) = 2.818 \times 10^{-3}$

$l=H: \; f_H(5) = e_H(C)(f_L(4)a_{LH} + f_H(4)a_{HH}) = (0.3)(6.408e{-3} + 4.48e{-3}) = 3.27 \times 10^{-3}$

$P(X_{1:5}) = 2.818 \times 10^{-3} + 3.27 \times 10^{-3} = 6.088 \times 10^{-3}$

i=6

$l=L: \; f_L(6) = e_L(T)(f_L(5)a_{LL} + f_H(5)a_{HL}) = (0.3)(1.691e{-3} + 1.635e{-3}) = 9.98 \times 10^{-4}$

$l=H: \; f_H(6) = e_H(T)(f_L(5)a_{LH} + f_H(5)a_{HH}) = (0.2)(1.127e{-3} + 1.635e{-3}) = 5.52 \times 10^{-4}$

$P(X_{1:6}) = 9.98 \times 10^{-4} + 5.52 \times 10^{-4} = 1.55 \times 10^{-3}$

i=7

$l=L: \; f_L(7) = e_L(G)(f_L(6)a_{LL} + f_H(6)a_{HL}) = (0.2)(5.99e{-4} + 2.76e{-4}) = 1.75 \times 10^{-4}$

$l=H: \; f_H(7) = e_H(G)(f_L(6)a_{LH} + f_H(6)a_{HH}) = (0.3)(3.99e{-4} + 2.76e{-4}) = 2.03 \times 10^{-4}$

$P(X_{1:7}) = 1.75 \times 10^{-4} + 2.03 \times 10^{-4} = 3.78 \times 10^{-4}$

i=8

$l=L: \; f_L(8) = e_L(A)(f_L(7)a_{LL} + f_H(7)a_{HL}) = (0.3)(1.05e{-4} + 1.02e{-4}) = 6.21 \times 10^{-5}$

$l=H: \; f_H(8) = e_H(A)(f_L(7)a_{LH} + f_H(7)a_{HH}) = (0.2)(7e{-5} + 1.02e{-4}) = 3.44 \times 10^{-5}$

$P(X_{1:8}) = 6.21 \times 10^{-5} + 3.44 \times 10^{-5} = 9.65 \times 10^{-5}$

i=9

$l=L: \; f_L(9) = e_L(A)(f_L(8)a_{LL} + f_H(8)a_{HL}) = (0.3)(3.726e{-5} + 1.72e{-5}) = 1.634 \times 10^{-5}$

$l=H: \; f_H(9) = e_H(A)(f_L(8)a_{LH} + f_H(8)a_{HH}) = (0.2)(2.484e{-5} + 1.72e{-5}) = 8.408 \times 10^{-6}$

$P(X_{1:9}) = P(X) = 1.634 \times 10^{-5} + 8.408 \times 10^{-6} = 2.475 \times 10^{-5}$

$b_L(end) = b_H(end) = 1$

<u>Backward</u>

<u>$i = 9$</u>

$l=L: \; b_L(9) = a_{LL}\, e_L(A)\, b_L(end) + a_{LH}\, e_H(A)\, b_H(end) = [(0.6)(0.3)(1) + (0.4)(0.2)(1)] = 0.26$

$l=H: \; b_H(9) = a_{HL}\, e_L(A)\, b_L(end) + a_{HH}\, e_H(A)\, b_H(end) = [(0.5)(0.3)(1) + (0.5)(0.2)(1)] = 0.25$

<u>$i = 8$</u>

$l=L: \; b_L(8) = a_{LL}\, e_L(A)\, b_L(9) + a_{LH}\, e_H(A)\, b_H(9) = (0.6)(0.3)(0.26) + (0.4)(0.2)(0.25) = 0.0668$

$l=H: \; b_H(8) = a_{HL}\, e_L(A)\, b_L(9) + a_{HH}\, e_H(A)\, b_H(9) = (0.5)(0.3)(0.26) + (0.5)(0.2)(0.25) = 0.064$

<u>$i = 7$</u>

$l=L: \; b_L(7) = a_{LL}\, e_L(G)\, b_L(8) + a_{LH}\, e_H(G)\, b_H(8) = (0.6)(0.2)(0.0668) + (0.4)(0.3)(0.064) = 1.570 \times 10^{-2}$

$l=H: \; b_H(7) = a_{HL}\, e_L(G)\, b_L(8) + a_{HH}\, e_H(G)\, b_H(8) = (0.5)(0.2)(0.0668) + (0.5)(0.3)(0.064) = 1.628 \times 10^{-2}$

<u>$i = 6$</u>

$l=L: \; b_L(6) = a_{LL}\, e_L(T)\, b_L(7) + a_{LH}\, e_H(T)\, b_H(7) = (0.6)(0.3)(1.57e{-2}) + (0.4)(0.2)(1.628e{-2}) = 4.128 \times 10^{-3}$

$l=H: \; b_H(6) = a_{HL}\, e_L(T)\, b_L(7) + a_{HH}\, e_H(T)\, b_H(7) = (0.5)(0.3)(1.57e{-2}) + (0.5)(0.2)(1.628e{-2}) = 3.98 \times 10^{-3}$

<u>$i = 5$</u>

$l=L: \; b_L(5) = a_{LL}\, e_L(C)\, b_L(6) + a_{LH}\, e_H(C)\, b_H(6) = (0.6)(0.2)(4.128e{-3}) + (0.4)(0.3)(3.98e{-3}) = 9.730 \times 10^{-4}$

$l=H: \; b_H(5) = a_{HL}\, e_L(C)\, b_L(6) + a_{HH}\, e_H(C)\, b_H(6) = (0.5)(0.2)(4.128e{-3}) + (0.5)(0.3)(3.98e{-3}) = 1.01 \times 10^{-3}$

<u>$i = 4$</u>

$l=L: \; b_L(4) = a_{LL}\, e_L(A)\, b_L(5) + a_{LH}\, e_H(A)\, b_H(5) = (0.6)(0.3)(9.730e{-4}) + (0.4)(0.2)(1.01e{-3}) = 2.56 \times 10^{-4}$

$l=H: \; b_H(4) = a_{HL}\, e_L(A)\, b_L(5) + a_{HH}\, e_H(A)\, b_H(5) = (0.5)(0.3)(9.730e{-4}) + (0.5)(0.2)(1.01e{-3}) = 2.47 \times 10^{-4}$

<u>$i = 3$</u>

$l=L: \; b_L(3) = a_{LL}\, e_L(C)\, b_L(4) + a_{LH}\, e_H(C)\, b_H(4) = (0.6)(0.2)(2.56e{-4}) + (0.4)(0.3)(2.47e{-4}) = 6.036 \times 10^{-5}$

$l=H: \; b_H(3) = a_{HL}\, e_L(C)\, b_L(4) + a_{HH}\, e_H(C)\, b_H(4) = (0.5)(0.2)(2.56e{-4}) + (0.5)(0.3)(2.47e{-4}) = 6.67 \times 10^{-5}$

<u>$i = 2$</u>

$l=L: \; b_L(2) = a_{LL}\, e_L(G)\, b_L(3) + a_{LH}\, e_H(G)\, b_H(3) = (0.6)(0.2)(6.036e{-5}) + (0.4)(0.3)(6.67e{-5}) = 1.525 \times 10^{-5}$

$l=H: \; b_H(2) = a_{HL}\, e_L(G)\, b_L(3) + a_{HH}\, e_H(G)\, b_H(3) = (0.5)(0.2)(6.036e{-5}) + (0.5)(0.3)(6.67e{-5}) = 1.604 \times 10^{-5}$

<u>$i = 1$</u>

$l=L: \; b_L(1) = a_{LL}\, e_L(G)\, b_L(2) + a_{LH}\, e_H(G)\, b_H(2) = (0.6)(0.2)(1.525e{-5}) + (0.4)(0.3)(1.604e{-5}) = 3.75 \times 10^{-6}$

$l=H: \; b_H(1) = a_{HL}\, e_L(G)\, b_L(2) + a_{HH}\, e_H(G)\, b_H(1) = (0.5)(0.2)(1.525e{-5}) + (0.5)(0.3)(1.604e{-5}) = 3.931 \times 10^{-6}$

$$P(x) = 3.75 \times 10^{-6} + 3.931 \times 10^{-6} = 7.68 \times 10^{-6}$$

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(4)

$$P(\pi_i = k \mid x) = \frac{f_k(x_i)\, b_k(x_i)}{P(x)}$$

x = GGCA

H:

$f_H(4) = 0.00896$
$b_H(4) = 1$

$$Pr(\pi_4 = H \mid x) = \frac{(0.00896)(1)}{0.02498} = \boxed{0.359}$$

L:

$f_L(4) = 0.01602$
$b_L(4) = 1$

$$Pr(\pi_4 = L \mid x) = \frac{(0.01602)(1)}{0.02498} = \boxed{0.641}$$

x = GGCACTGAA

H:

$f_H(9) = 8.408 \times 10^{-6}$
$b_H(9) = 1$

$$Pr(\pi_9 = H \mid x) = \frac{(8.408 \times 10^{-6})(1)}{2.475 \times 10^{-5}} = \boxed{0.340}$$

L:

$f_L(9) = 1.634 \times 10^{-5}$
$b_L(9) = 1$

$$Pr(\pi_9 = L \mid x) = \frac{(1.634 \times 10^{-5})(1)}{2.475 \times 10^{-5}} = \boxed{0.660}$$

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(5)

Viterbi Smooting and Decoding

Initialization $(i=0)$: $V_0(0) = 1$

Recursion $(i=1 \ldots L)$: $v_l(i) = e_l(x_i)\, \max_k (v_k(i-1) \cdot a_{kl})$

$ptr_i(l) = \operatorname{argmax}_k (v_k(i-1) a_{kl})$

Termination: $P(x, \pi^*) = \max_k (v_k(L) a_{k0})$

$\pi^*_L = \operatorname{argmax}_k (V_k(L) a_{k0})$

Traceback $(i = L \ldots 1)$: $\pi^*_{i-1} = ptr_i(\pi_i^*)$

Posterior Decoding

$$P(\pi_i = k \mid x) = \frac{f_k(i)\, b_k(i)}{P(x)}$$

These two methods for HMM models differ in what their output is telling you about a given sequence. In the viterbi method, the output tells you what state the sequence is currently in. However, it does not give you any confidence as to how probable it is in that state. Conversly, the posterior decoding approach tells you the probability that a sequence is in a given state k at the i^{th} position of sequence X.