HIGHER ALGEBRA

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1. Lecture 1: Thursday, January 12th

Today: the **homotopy hypothesis**

Classical algebra: sets, monoids, groups, abelian groups, rings. Each of these are built up on the other. In higher courses, we may see groupoids, which are types of categories. A category is a generalization of a monoid, in some sense. We also have monoidal categories, which in some sense are a generalization of rings.

For higher algebra: spaces, $\mathbb{E}_1$-spaces, spectra, $\mathbf{E}_1$-ring spectra. Underlying this we have ∞-groupoids, ∞-categories, and monoidal ∞-categories.

We study spaces, not up to homeomorphism, but up to *weak homotopy equivalence*. We will study this in a minute. "Spaces" in this class will always mean the study of topological spaces up to weak homotopy equivalence.

We'll give a synthetic definition of what an infinity category is, and circle back to a technical definition in about a month.

What is an ∞-category?

An ∞-category (or $(\infty, 1)$-category) $\mathscr{C}$ should consist of:

(1) a class of objects
(2) a class of morphisms so that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a space
(3) n-morphisms for $n \geq 2$, where for instance 2-morphisms are between 1-morphisms, 3-morphisms between 2-morphisms, etc.
(4) morphisms can be composed in a suitable way
(5) n-morphisms for $n \geq 2$ are invertible in some sense.

An ∞-groupoid (or $(\infty, 0)$-category) should be an ∞-category where all the 1-morphisms are also invertible in some sense.

Why study spaces up to weak homotopy equivalence?

Date: January 17, 2023.

Recall by the Yoneda lemma, we have that

$$X \cong Y \Leftrightarrow \mathrm{Hom}_{\mathtt{Top}}(A, X) \cong \mathrm{Hom}_{\mathtt{Top}}(A, Y)$$

for all $A \in \mathtt{Top}$. Figuring out $\mathrm{Hom}(A, X)$ up to bijection for all A is very difficult, so we prefer to study continuous maps up to homotopy. For X and Y nice enough, we say that $f \simeq g$ in $\mathrm{Hom}(X, Y)$ if there exists some path $I \to \mathrm{Map}(X, Y)$ so that $0 \mapsto f$ and $1 \mapsto g$. We define $[X, Y] = \mathrm{Hom}_{\mathtt{Top}}(X, Y)/\simeq$.

We see then that $X \simeq Y$ if and only if $[A, X] \cong [A, Y]$ for all $A \in \mathtt{Top}$.

We may ask when $[A, -] : \mathtt{Top}_* \to \mathtt{Set}$ factors through $\mathtt{Grp}$ or $\mathtt{Ab}$. We have that $[A, -]$ factors through $\mathtt{Grp}$ if and only if A is a co-H-group in $\mathtt{Top}$. That is, we have maps

$$\begin{gathered} A \to A \vee A \\ A \to *, \end{gathered}$$

which is coassociative, counital, coinvertible.

Example 1.1. S^n, when $n \geq 1$, is a co-H-space. The map $S^n \to S^n \vee S^n$ is the pinch map.

We say that X is *weakly homotopy equivalent* to Y, we write $X \sim Y$, if and only if there is a map $X \to Y$ inducing an isomorphism

$$\pi_n(X) = [S^n, X]_* \cong [S^n, Y]_* = \pi_n(Y),$$

for all $n \geq 0$ (for $n \geq 1$ this is a group isomorphism).

If $X \sim Y$, then $H_n(X) \cong H_n(Y)$ for any n.

Theorem 1.2. (Cellular approximation) For any X in $\mathtt{Top}$, there exists $\widetilde{X}$ a CW complex with a canonical map $\widetilde{X} \xrightarrow{\sim} X$ that is a weak equivalence.

Theorem 1.3. (Whitehead) If X, Y are CW complexes, then $X \xrightarrow{\simeq} Y$ is a homotopy equivalence if and only if $X \xrightarrow{\sim} Y$ is a weak homotopy equivalence.

Exercise 1.4. Find spaces X and Y which are weakly homotopy equivalent but not homotopy equivalent.

We denote by Δ the simplex category. Its objects are ordered sets of the form $[n] = \{0, 1, \ldots, n\}$, and its morphisms are order-preserving maps. We have that Δ is generated by *cofaces* and *codegeneracies*. The cofaces are of the form

$$d^0, d^1 : [0] \to [1],$$

skipping 0 or 1 in $[1]$, etc. The codegeneracies look like $s^0 : [1] \to [0]$ which "repeat" an element.

The cofaces and codegeneracies satisfy certain *cosimplicial identities.*

If $\mathscr{C}$ is a category, we denote by $s\mathscr{C} = \mathscr{C}^{\Delta^{\mathrm{op}}}$ the simplicial objects in $\mathscr{C}$. If $\mathscr{C} = \mathtt{Set}$, we write $\mathtt{sSet}$ as the category of simplicial sets. A simplicial set $X_\bullet \in \mathtt{sSet}$ consists of sets $X_0, X_1, \dots$ together with face and degeneracy maps satisfying the simplicial identities.

Example 1.5. The *nerve of a small category.* Let $\mathscr{C} \in \mathtt{Cat}$ a small category. We denote by $N_\bullet\mathscr{C}$ the simplicial set with $N_0\mathscr{C} = \mathrm{ob}\mathscr{C}$, $N_1\mathscr{C} = \mathrm{mor}\mathscr{C}$, and $N_n\mathscr{C}$ the set of n composable morphisms in $\mathscr{C}$. That is,

$$N_n\mathscr{C} = N_1\mathscr{C} \times_{N_0\mathscr{C}} \cdots \times_{N_0\mathscr{C}} N_1\mathscr{C}.$$

The face maps are source/target/composition. The degeneracies insert an identity morphism.

Example 1.6. Via Yoneda, we get a functor

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathtt{Set}.$$

If $X_\bullet$ is a simplicial set, we get that the set of n-simplices X_n is in bijection with $\mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, X_\bullet)$.

Example 1.7. (Dold–Kan) We have $\mathtt{Ch}_R^{\geq 0} \xrightarrow{\Gamma} s\mathtt{Mod}_R$ is an isomorphism, where $\Gamma_m C_\bullet = \oplus_{[n]\twoheadrightarrow[k]} C_k$, with faces and degeneracies left as an exercise.

Example 1.8. Let $\Delta^n_{\mathtt{Top}} \subseteq \mathbb{R}^{n+1}$ be defined by

$$\left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \colon 0 \leq t_i \leq 1,\ \sum t_i = 1\right\}.$$

We can view $[n] = \{v_0, \dots, v_n\}$, and $v_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the ith place. Then if $\alpha : [m] \to [n]$ in Δ, we can define $\alpha(v_i) = v_{\alpha(i)}$. Extend linearly to get $\alpha_* : \Delta^m_{\mathtt{Top}} \to \Delta^n_{\mathtt{Top}}$. We get then that $\Delta^\bullet_{\mathtt{Top}}$ is a cosimplicial topological space.

Example 1.9. If $X \in \mathtt{Top}$, we have $\mathrm{Sing}_\bullet(X) \in \mathtt{sSet}$ defined by $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathtt{Top}}\left(\Delta^n_{\mathtt{Top}}, X\right)$.

Definition 1.10. If $X_\bullet \in \mathtt{sSet}$, we define its *geometric realization* to be

$$|X_\bullet| = \amalg_{n\geq 0} X_n \times \Delta^n_{\mathtt{Top}} / \sim,$$

where $(x, s) \sim (y, t)$ if and only if there is some $\alpha : [m] \to [n]$ so that $\alpha^* y = x$ and $\alpha_* s = t$.

Example 1.11. $|\Delta^n_\bullet| \cong \Delta^n_{\mathtt{Top}}$.

Exercise 1.12. $|X_\bullet|$ is always a CW complex for any $X_\bullet \in \texttt{sSet}$.

Exercise 1.13. We have an adjunction $|-|:\texttt{sSet} \leftrightarrows \texttt{Top} : \mathrm{Sing}(-)$

Definition 1.14. $X_\bullet \to Y_\bullet$ is a *weak homotopy equivalence* in $\texttt{sSet}$ if $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ is a weak homotopy equivalence of spaces.

Theorem 1.15. (Quillen) Simplicial sets up to weak equivalence is equivalent to topological spaces up to weak homotopy equivalence. Moreover, for any $X \in \texttt{Top}$, we have that $|\mathrm{Sing}(X)|$ is weakly equivalent to X.

It is not true that $Y \sim \mathrm{Sing}(|Y|)$ for all $Y \in \texttt{sSet}$. We need Y to be a *Kan complex*.

2. Lecture 2: Tuesday, January 17th

Today: the homotopy hypothesis (continued).

Recall we are interested in studying $\texttt{Top}$ up to weak homotopy equivalences. Equivalently, we are interested in studying $\texttt{sSet}$ up to weak equivalence, and the relationship between the two was given by the geometric realization / singular complex adjunction.

Recall we've defined $\Delta^n = \mathrm{Hom}_\Delta(-,[n])$. We will define the *kth horn* $\Lambda^n_k \subseteq \Delta^n$ as a coequalizer in $\texttt{sSet}$

$$\left(\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{i\neq k} \Delta^{n-1}\right) \to \Lambda^n_k,$$

where the two maps are δ^{j-1} and δ^i. The geometric realization of Λ^n_k is the topological n-simplex, with the middle and the face opposite the kth edge removed.

Definition 2.1. We say that $Y \in \texttt{sSet}$ is a *Kan complex* if for all $k \le n$, and for every $\Lambda^n_k \to Y$, there exists a (not necessarily unique) lift:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Y \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Exercise 2.2. Y is a Kan complex if and only if for any $(n-1)$-simplices $y_1,\dots,y_{k-1},y_{k+1},\dots,y_n$ such that $d_iy_j = d_{j-1}y_i$ for $i<j$, $i,j\neq k$, there exists an n-simplex y such that $d_iy = y_i$ for all $i \neq k$.

Exercise 2.3. We have that $\mathrm{Sing}(X)$ is always a Kan complex for any $X \in \texttt{Top}$.

Exercise 2.4. We have that Δ^n is not a Kan complex for $n \ge 1$.

Exercise 2.5. If $X \in s\mathtt{Grp}$, then the underlying simplicial set of X is always a Kan complex.

Up to weak homotopy equivalence, every simplicial set is a Kan complex (will see this later).

Recall the Dold-Kan correspondence

$$s\mathtt{Mod}_{\mathbb{Z}} \cong \mathtt{Ch}_{\mathbb{Z}}^{\geq 0},$$

which sends weak homotopy equivalences to quasi-isomorphisms. Given a simplicial set X_*, we can take an associated simplicial abelian group $\mathbb{Z}[X_*]$ by taking the free group on n-simplices at level n. We can ask what $\mathbb{Z}[X_*]$ corresponds to as a chain complex. One answer is that

$$\mathbb{Z}[\mathrm{Sing}(X_*)] \leftrightarrow C_*(X; \mathbb{Z}).$$

This tells us that

$$\pi_* \left(\mathbb{Z}\left[\mathrm{Sing}(X)\right]\right) \cong H_*(X; \mathbb{Z}).$$

In some sense we can view $\mathbb{Z}[\mathrm{Sing}(X)]$ as being (equivalent to) the *free commutative monoid* on X. This is what is known as the *Dold-Thom theorem*.

Homotopy hypothesis: Spaces (up to weak equivalence) are ∞-groupoids. For us, spaces up to weak equivalences correspond to Kan complexes.

Given $X \in \mathtt{Kan}$, we can call X_0 the objects, and X_1 the morphisms. The horn filling conditions on horns tell you that you can *compose* and *invert* morphisms in X_1, witnessed by simplices in X_2.

Definition 2.6. A *quasi-category* (i.e. ∞-category) is a simplicial set with inner horn lifting property. That is, we can lift against horns Λ_k^n for $0 < k < n$.

Exercise 2.7. A quasi-category has unique horn filling if and only if it is isomorphic to the nerve of a 1-category.

Model categories

Vista: Every nice infinity category is equivalent in some sense to a model category. This will pretty much be the goal of this class.

Notation 2.8. Let $\mathcal{M}$ be a category, and $\chi \subseteq \mathcal{M}$ a class of morphisms. We define $\mathrm{LLP}(\chi)$ to be the class of morphisms in $\mathcal{M}$ so that f has left lifting property with respect to all morphisms in χ:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "\in \chi"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Similarly we can define $f \in \mathrm{RLP}(\chi)$ by

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\chi \ni"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Definition 2.9. A *weak factorization system* on a category $\mathcal{M}$ consists of a pair $(\mathscr{C}, \mathscr{F})$ of classes of morphisms such that

(1) Given any $f : X \to Y$ in $\mathcal{M}$, it factors (not necessarily uniquely) as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathscr{C}\ni"'] & & Y \\ & W \arrow[ur, "\in\mathscr{F}"'] & \end{tikzcd}$$

(2) $\mathscr{C} = \mathrm{LLP}(\mathscr{F})$ and $\mathscr{F} = \mathrm{RLP}(\mathscr{C})$.

Example 2.10. In `Set`, we have that mono and epimorphisms give a weak factorization system. A factorization is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{id}_X \times f"'] & & Y \\ & X \times Y \arrow[ur, "\pi_Y"'] & \end{tikzcd}$$

Definition 2.11. A *model structure* on $\mathcal{M}$ consists of three classes of morphisms:

W	weak equivalences
Cof	cofibrations
Fib	fibrations

We denote by $\widetilde{\mathrm{Cof}} := \mathrm{Cof} \cap W$ and $\widetilde{\mathrm{Fib}} = \mathrm{Fib} \cap W$, and call these *trivial cofibrations* (resp. *trivial fibrations*). These are subject to the constraint that

(1) $\mathcal{M}$ is bicomplete (all limits and colimits)
(2) W contains identities and it satisfies 2-out-of-3 property[1]
(3) $\left(\mathrm{Cof}, \widetilde{\mathrm{Fib}}\right)$ and $\left(\widetilde{\mathrm{Cof}}, \mathrm{Fib}\right)$ are weak factorization systems.

Terminology 2.12. A category with a model structure is referred to as a *model category*.

Notation 2.13. We will decorate each class of morphisms as

[1]If f and g are composable, and any two of f, g, gf are in W then so is the third.

$$\begin{array}{c|c} W & \xrightarrow{\sim} \\ \mathrm{Cof} & \hookrightarrow \\ \mathrm{Fib} & \twoheadrightarrow \end{array}$$

Exercise 2.14. W, Cof, and Fib are closed under retracts: that is,

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[rr, bend right, equal] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

then if $g \in W$ (resp. Cof or Fib) then $f \in W$ (resp. Cof or Fib).

Definition 2.15. Let $\mathcal{M}$ be a model category, and let $\varnothing \in \mathcal{M}$ the initial object and $* \in \mathcal{M}$ the terminal object.

- We say that $X \in \mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to X$ is a cofibration.
- We say that $X \in \mathcal{M}$ is *fibrant* if the unique map $X \to *$ is a fibration.
- We say that $\widetilde{X}$ is a *cofibrant replacement* of X if

$$\begin{tikzcd} \varnothing \arrow[rr] \arrow[dr, hook] & & X \\ & \widetilde{X} \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

- We say that $\widetilde{X}$ is a *fibrant replacement* of X if

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim"'] & & * \\ & \widetilde{X} \arrow[ur, two heads] & \end{tikzcd}$$

Example 2.16. $\mathcal{M} = \texttt{Top}$, W = weak homotopy equivalences, Cof = relative CW complexes[2] The fibrations are determined by Fib = RLP($\widetilde{\mathrm{Cof}}$). The fibrations are equivalently RLP($D^n \to D^n \times I$). Every object here is fibrant, and the cofibrant objects are precisely the CW complexes. Cofibrant replacement is cellular approximation.

[2] $A \hookrightarrow X$ is a *relative CW complex* if X is built out of A by attaching cells.

References