HIGHER ALGEBRA

MAXIMILIEN PÉROUX

1. Lecture 1: Thursday, January 12th

Today: the **homotopy hypothesis**

Classical algebra: sets, monoids, groups, abelian groups, rings. Each of these are built up on the other. In higher courses, we may see groupoids, which are types of categories. A category is a generalization of a monoid, in some sense. We also have monoidal categories, which in some sense are a generalization of rings.

For higher algebra: spaces, $\mathbb{E}_1$-spaces, spectra, $\mathbf{E}_1$-ring spectra. Underlying this we have ∞-groupoids, ∞-categories, and monoidal ∞-categories.

We study spaces, not up to homeomorphism, but up to *weak homotopy equivalence*. We will study this in a minute. "Spaces" in this class will always mean the study of topological spaces up to weak homotopy equivalence.

We'll give a synthetic definition of what an infinity category is, and circle back to a technical definition in about a month.

What is an ∞-category?

An ∞-category (or $(\infty, 1)$-category) $\mathscr{C}$ should consist of:

(1) a class of objects
(2) a class of morphisms so that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a space
(3) n-morphisms for $n \geq 2$, where for instance 2-morphisms are between 1-morphisms, 3-morphisms between 2-morphisms, etc.
(4) morphisms can be composed in a suitable way
(5) n-morphisms for $n \geq 2$ are invertible in some sense.

An ∞-groupoid (or $(\infty, 0)$-category) should be an ∞-category where all the 1-morphisms are also invertible in some sense.

Why study spaces up to weak homotopy equivalence?

Date: January 12, 2023.

Recall by the Yoneda lemma, we have that

$$X \cong Y \Leftrightarrow \mathrm{Hom}_{\mathtt{Top}}(A, X) \cong \mathrm{Hom}_{\mathtt{Top}}(A, Y)$$

for all $A \in \mathtt{Top}$. Figuring out $\mathrm{Hom}(A, X)$ up to bijection for all A is very difficult, so we prefer to study continuous maps up to homotopy. For X and Y nice enough, we say that $f \simeq g$ in $\mathrm{Hom}(X, Y)$ if there exists some path $I \to \mathrm{Map}(X, Y)$ so that $0 \mapsto f$ and $1 \mapsto g$. We define $[X, Y] = \mathrm{Hom}_{\mathtt{Top}}(X, Y)/\simeq$.

We see then that $X \simeq Y$ if and only if $[A, X] \cong [A, Y]$ for all $A \in \mathtt{Top}$.

We may ask when $[A, -] : \mathtt{Top}_* \to \mathtt{Set}$ factors through $\mathtt{Grp}$ or $\mathtt{Ab}$. We have that $[A, -]$ factors through $\mathtt{Grp}$ if and only if A is a co-H-group in $\mathtt{Top}$. That is, we have maps

$$A \to A \vee A$$
$$A \to *,$$

which is coassociative, counital, coinvertible.

Example 1.1. S^n, when $n \geq 1$, is a co-H-space. The map $S^n \to S^n \vee S^n$ is the pinch map.

We say that X is *weakly homotopy equivalent* to Y, we write $X \sim Y$, if and only if there is a map $X \to Y$ inducing an isomorphism

$$\pi_n(X) = [S^n, X]_* \cong [S^n, Y]_* = \pi_n(Y),$$

for all $n \geq 0$ (for $n \geq 1$ this is a group isomorphism).

If $X \sim Y$, then $H_n(X) \cong H_n(Y)$ for any n.

Theorem 1.2. (Cellular approximation) For any X in $\mathtt{Top}$, there exists $\widetilde{X}$ a CW complex with a canonical map $\widetilde{X} \xrightarrow{\sim} X$ that is a weak equivalence.

Theorem 1.3. (Whitehead) If X, Y are CW complexes, then $X \xrightarrow{\simeq} Y$ is a homotopy equivalence if and only if $X \xrightarrow{\sim} Y$ is a weak homotopy equivalence.

Exercise 1.4. Find spaces X and Y which are weakly homotopy equivalent but not homotopy equivalent.

We denote by Δ the simplex category. Its objects are ordered sets of the form $[n] = \{0, 1, \ldots, n\}$, and its morphisms are order-preserving maps. We have that Δ is generated by *cofaces* and *codegeneracies*. The cofaces are of the form

$$d^0, d^1 : [0] \to [1],$$

skipping 0 or 1 in [1], etc. The codegeneracies look like $s^0 : [1] \to [0]$ which "repeat" an element.

The cofaces and codegeneracies satisfy certain *cosimplicial identities.*

If $\mathscr{C}$ is a category, we denote by $s\mathscr{C} = \mathscr{C}^{\Delta^{\mathrm{op}}}$ the simplicial objects in $\mathscr{C}$. If $\mathscr{C} = \mathtt{Set}$, we write $\mathtt{sSet}$ as the category of simplicial sets. A simplicial set $X_\bullet \in \mathtt{sSet}$ consists of sets $X_0, X_1, \ldots$ together with face and degeneracy maps satisfying the simplicial identities.

Example 1.5. The *nerve of a small category.* Let $\mathscr{C} \in \mathtt{Cat}$ a small category. We denote by $N_\bullet\mathscr{C}$ the simplicial set with $N_0\mathscr{C} = \mathrm{ob}\mathscr{C}$, $N_1\mathscr{C} = \mathrm{mor}\mathscr{C}$, and $N_n\mathscr{C}$ the set of n composable morphisms in $\mathscr{C}$. That is,

$$N_n\mathscr{C} = N_1\mathscr{C} \times_{N_0\mathscr{C}} \cdots \times_{N_0\mathscr{C}} N_1\mathscr{C}.$$

The face maps are source/target/composition. The degeneracies insert an identity morphism.

Example 1.6. Via Yoneda, we get a functor

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathtt{Set}.$$

If $X_\bullet$ is a simplicial set, we get that the set of n-simplices X_n is in bijection with $\mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, X_\bullet)$.

Example 1.7. (Dold–Kan) We have $\mathtt{Ch}_R^{\geq 0} \xrightarrow{\Gamma} s\mathtt{Mod}_R$ is an isomorphism, where $\Gamma_m C_\bullet = \oplus_{[n] \twoheadrightarrow [k]} C_k$, with faces and degeneracies left as an exercise.

Example 1.8. Let $\Delta^n_{\mathtt{Top}} \subseteq \mathbb{R}^{n+1}$ be defined by

$$\left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \colon 0 \leq t_i \leq 1,\ \sum t_i = 1 \right\}.$$

We can view $[n] = \{v_0, \ldots, v_n\}$, and $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the ith place. Then if $\alpha : [m] \to [n]$ in Δ, we can define $\alpha(v_i) = v_{\alpha(i)}$. Extend linearly to get $\alpha_* : \Delta^m_{\mathtt{Top}} \to \Delta^n_{\mathtt{Top}}$. We get then that $\Delta^\bullet_{\mathtt{Top}}$ is a cosimplicial topological space.

Example 1.9. If $X \in \mathtt{Top}$, we have $\mathrm{Sing}_\bullet(X) \in \mathtt{sSet}$ defined by $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathtt{Top}}\left(\Delta^n_{\mathtt{Top}}, X\right)$.

Definition 1.10. If $X_\bullet \in \mathtt{sSet}$, we define its *geometric realization* to be

$$|X_\bullet| = \amalg_{n \geq 0} X_n \times \Delta^n_{\mathtt{Top}} / \sim,$$

where $(x, s) \sim (y, t)$ if and only if there is some $\alpha : [m] \to [n]$ so that $\alpha^* y = x$ and $\alpha_* s = t$.

Example 1.11. $|\Delta^n_\bullet| \cong \Delta^n_{\mathtt{Top}}$.

Exercise 1.12. $|X_\bullet|$ is always a CW complex for any $X_\bullet \in \mathtt{sSet}$.

Exercise 1.13. We have an adjunction $|-| : \mathtt{sSet} \leftrightarrows \mathtt{Top} : \mathrm{Sing}(-)$

Definition 1.14. $X_\bullet \to Y_\bullet$ is a *weak homotopy equivalence* in $\mathtt{sSet}$ if $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ is a weak homotopy equivalence of spaces.

Theorem 1.15. (Quillen) Simplicial sets up to weak equivalence is equivalent to topological spaces up to weak homotopy equivalence. Moreover, for any $X \in \mathtt{Top}$, we have that $|\mathrm{Sing}(X)|$ is weakly equivalent to X.

It is not true that $Y \sim \mathrm{Sing}(|Y|)$ for all $Y \in \mathtt{sSet}$. We need Y to be a *Kan complex*.

References