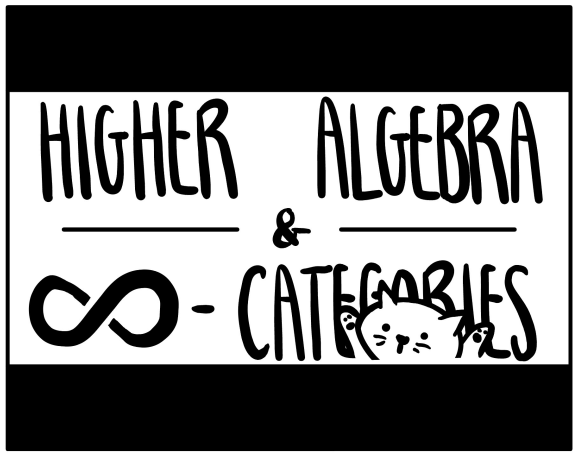
# HIGHER ALGEBRA

MAXIMILIEN PÉROUX




## Contents



*Date*: February 23, 2023.

## 1. Lecture 1: Thursday, January 12th

Today: the **homotopy hypothesis**

**Classical algebra**: sets, monoids, groups, abelian groups, rings. Each of these are built up on the other. In higher courses, we may see groupoids, which are types of categories. A category is a generalization of a monoid, in some sense. We also have monoidal categories, which in some sense are a generalization of rings.

For higher algebra: spaces, $\mathbb{E}_1$-spaces, spectra, $\mathbf{E}_1$-ring spectra. Underlying this we have $\infty$-groupoids, $\infty$-categories, and monoidal $\infty$-categories.

We study spaces, not up to homeomorphism, but up to *weak homotopy equivalence*. We will study this in a minute. "Spaces" in this class will always mean the study of topological spaces up to weak homotopy equivalence.

We'll give a synthetic definition of what an infinity category is, and circle back to a technical definition in about a month.

**What is an $\infty$-category**?

An $\infty$-category (or $(\infty, 1)$-category) $\mathscr{C}$ should consist of:

(1) a class of objects
(2) a class of morphisms so that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a space
(3) $n$-morphisms for $n \geq 2$, where for instance 2-morphisms are between 1-morphisms, 3-morphisms between 2-morphisms, etc.
(4) morphisms can be composed in a suitable way
(5) $n$-morphisms for $n \geq 2$ are invertible in some sense.

An $\infty$-groupoid (or $(\infty, 0)$-category) should be an $\infty$-category where all the 1-morphisms are also invertible in some sense.

**Why study spaces up to weak homotopy equivalence**?

Recall by the Yoneda lemma, we have that

$$X \cong Y \Leftrightarrow \mathrm{Hom}_{\mathtt{Top}}(A, X) \cong \mathrm{Hom}_{\mathtt{Top}}(A, Y)$$

for all $A \in \mathtt{Top}$. Figuring out $\mathrm{Hom}(A, X)$ up to bijection for all $A$ is very difficult, so we prefer to study continuous maps up to homotopy. For $X$ and $Y$ nice enough, we say that $f \simeq g$ in $\mathrm{Hom}(X, Y)$ if there exists some path $I \to \mathrm{Map}(X, Y)$ so that $0 \mapsto f$ and $1 \mapsto g$. We define $[X, Y] = \mathrm{Hom}_{\mathtt{Top}}(X, Y)/\simeq$.

We see then that $X \simeq Y$ if and only if $[A, X] \cong [A, Y]$ for all $A \in \mathtt{Top}$.

We may ask when $[A, -] : \mathtt{Top}_* \to \mathtt{Set}$ factors through $\mathtt{Grp}$ or $\mathtt{Ab}$. We have that $[A, -]$ factors through $\mathtt{Grp}$ if and only if $A$ is a co-H-group in $\mathtt{Top}$. That is, we have maps

$$\begin{aligned} A &\to A \vee A \\ A &\to *, \end{aligned}$$

which is coassociative, counital, coinvertible.

**Example 1.1.** $S^n$, when $n \geq 1$, is a co-H-space. The map $S^n \to S^n \vee S^n$ is the pinch map.

We say that $X$ is *weakly homotopy equivalent* to $Y$, we write $X \sim Y$, if and only if there is a map $X \to Y$ inducing an isomorphism

$$\pi_n(X) = [S^n, X]_* \cong [S^n, Y]_* = \pi_n(Y),$$

for all $n \geq 0$ (for $n \geq 1$ this is a group isomorphism).

If $X \sim Y$, then $H_n(X) \cong H_n(Y)$ for any $n$.

**Theorem 1.2.** (Cellular approximation) For any $X$ in $\mathtt{Top}$, there exists $\widetilde{X}$ a CW complex with a canonical map $\widetilde{X} \xrightarrow{\sim} X$ that is a weak equivalence.

**Theorem 1.3.** (Whitehead) If $X, Y$ are CW complexes, then $X \xrightarrow{\simeq} Y$ is a homotopy equivalence if and only if $X \xrightarrow{\sim} Y$ is a weak homotopy equivalence.

**Exercise 1.4.** Find spaces $X$ and $Y$ which are weakly homotopy equivalent but not homotopy equivalent.

We denote by $\Delta$ the simplex category. Its objects are ordered sets of the form $[n] = \{0, 1, \ldots, n\}$, and its morphisms are order-preserving maps. We have that $\Delta$ is generated by *cofaces* and *codegeneracies*. The cofaces are of the form

$$d^0, d^1 : [0] \to [1],$$

skipping 0 or 1 in [1], etc. The codegeneracies look like $s^0 : [1] \to [0]$ which "repeat" an element.

The cofaces and codegeneracies satisfy certain *cosimplicial identities*.

If $\mathscr{C}$ is a category, we denote by $s\mathscr{C} = \mathscr{C}^{\Delta^{\mathrm{op}}}$ the simplicial objects in $\mathscr{C}$. If $\mathscr{C} = \mathtt{Set}$, we write $\mathtt{sSet}$ as the category of simplicial sets. A simplicial set $X_\bullet \in \mathtt{sSet}$ consists of sets $X_0, X_1, \dots$ together with face and degeneracy maps satisfying the simplicial identities.

**Example 1.5.** The *nerve of a small category.* Let $\mathscr{C} \in \mathtt{Cat}$ a small category. We denote by $N_\bullet\mathscr{C}$ the simplicial set with $N_0\mathscr{C} = \mathrm{ob}\mathscr{C}$, $N_1\mathscr{C} = \mathrm{mor}\mathscr{C}$, and $N_n\mathscr{C}$ the set of $n$ composable morphisms in $\mathscr{C}$. That is,

$$N_n\mathscr{C} = N_1\mathscr{C} \times_{N_0\mathscr{C}} \cdots \times_{N_0\mathscr{C}} N_1\mathscr{C}.$$

The face maps are source/target/composition. The degeneracies insert an identity morphism.

**Example 1.6.** Via Yoneda, we get a functor

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathtt{Set}.$$

If $X_\bullet$ is a simplicial set, we get that the set of $n$-simplices $X_n$ is in bijection with $\mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, X_\bullet)$.

**Example 1.7.** (Dold–Kan) We have $\mathtt{Ch}_R^{\geq 0} \xrightarrow{\Gamma} s\mathtt{Mod}_R$ is an isomorphism, where $\Gamma_m C_\bullet = \oplus_{[n]\twoheadrightarrow[k]} C_k$, with faces and degeneracies left as an exercise.

**Example 1.8.** Let $\Delta^n_{\mathtt{Top}} \subseteq \mathbb{R}^{n+1}$ be defined by

$$\left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \colon 0 \leq t_i \leq 1,\ \sum t_i = 1\right\}.$$

We can view $[n] = \{v_0, \dots, v_n\}$, and $v_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the $i$th place. Then if $\alpha : [m] \to [n]$ in $\Delta$, we can define $\alpha(v_i) = v_{\alpha(i)}$. Extend linearly to get $\alpha_* : \Delta^m_{\mathtt{Top}} \to \Delta^n_{\mathtt{Top}}$. We get then that $\Delta^\bullet_{\mathtt{Top}}$ is a cosimplicial topological space.

**Example 1.9.** If $X \in \mathtt{Top}$, we have $\mathrm{Sing}_\bullet(X) \in \mathtt{sSet}$ defined by $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathtt{Top}}\left(\Delta^n_{\mathtt{Top}}, X\right)$.

**Definition 1.10.** If $X_\bullet \in \mathtt{sSet}$, we define its *geometric realization* to be

$$|X_\bullet| = \amalg_{n\geq 0} X_n \times \Delta^n_{\mathtt{Top}} / \sim,$$

where $(x, s) \sim (y, t)$ if and only if there is some $\alpha : [m] \to [n]$ so that $\alpha^* y = x$ and $\alpha_* s = t$.

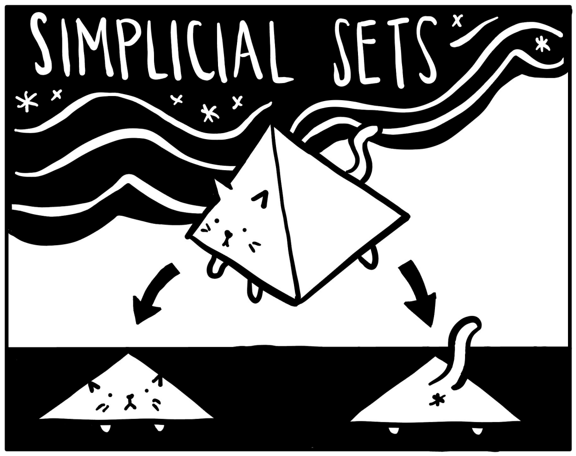
**Example 1.11.** $|\Delta^n_\bullet| \cong \Delta^n_{\mathtt{Top}}$.

**Exercise 1.12.** $|X_\bullet|$ is always a CW complex for any $X_\bullet \in \mathtt{sSet}$.

**Exercise 1.13.** We have an adjunction $|-| : \mathtt{sSet} \rightleftarrows \mathtt{Top} : \mathrm{Sing}(-)$

**Definition 1.14.** $X_\bullet \to Y_\bullet$ is a *weak homotopy equivalence* in `sSet` if $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ is a weak homotopy equivalence of spaces.

**Theorem 1.15.** (Quillen) Simplicial sets up to weak equivalence is equivalent to topological spaces up to weak homotopy equivalence. Moreover, for any $X \in$ `Top`, we have that $|\mathrm{Sing}(X)|$ is weakly equivalent to $X$.




## 2. Lecture 2: Tuesday, January 17th

**Today**: the homotopy hypothesis (continued).

Recall we are interested in studying `Top` up to weak homotopy equivalences. Equivalently, we are interested in studying `sSet` up to weak equivalence, and the relationship between the two was given by the geometric realization / singular complex adjunction.

Recall we've defined $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$. We will define the *kth horn* $\Lambda^n_k \subseteq \Delta^n$ as a coequalizer in `sSet`

$$\left( \coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{i \neq k} \Delta^{n-1} \right) \to \Lambda^n_k,$$

where the two maps are $\delta^{j-1}$ and $\delta^i$. The geometric realization of $\Lambda^n_k$ is the topological $n$-simplex, with the middle and the face opposite the $k$th edge removed.

**Definition 2.1.** We say that $Y \in \mathtt{sSet}$ is a *Kan complex* if for all $k \leq n$, and for every $\Lambda^n_k \to Y$, there exists a (not necessarily unique) lift:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Y \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

**Exercise 2.2.** $Y$ is a Kan complex if and only if for any $(n-1)$-simplices $y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$ such that $d_i y_j = d_{j-1} y_i$ for $i < j$, $i, j \neq k$, there exists an $n$-simplex $y$ such that $d_i y = y_i$ for all $i \neq k$.

**Exercise 2.3.** We have that $\mathrm{Sing}(X)$ is always a Kan complex for any $X \in \mathtt{Top}$.

**Exercise 2.4.** We have that $\Delta^n$ is not a Kan complex for $n \geq 1$.

**Exercise 2.5.** If $X \in s\mathtt{Grp}$, then the underlying simplicial set of $X$ is always a Kan complex.

Up to weak homotopy equivalence, every simplicial set is a Kan complex (will see this later).

Recall the Dold-Kan correspondence

$$s\mathtt{Mod}_{\mathbb{Z}} \cong \mathtt{Ch}^{\geq 0}_{\mathbb{Z}},$$

which sends weak homotopy equivalences to quasi-isomorphisms. Given a simplicial set $X_*$, we can take an associated simplicial abelian group $\mathbb{Z}[X_*]$ by taking the free group on $n$-simplices at level $n$. We can ask what $\mathbb{Z}[X_*]$ corresponds to as a chain complex. One answer is that

$$\mathbb{Z}[\mathrm{Sing}(X_*)] \leftrightarrow C_*(X;\mathbb{Z}).$$

This tells us that

$$\pi_*\left(\mathbb{Z}\left[\mathrm{Sing}(X)\right]\right) \cong H_*(X;\mathbb{Z}).$$

In some sense we can view $\mathbb{Z}[\mathrm{Sing}(X)]$ as being (equivalent to) the *free commutative monoid* on $X$. This is what is known as the *Dold-Thom theorem*.

**Homotopy hypothesis**: Spaces (up to weak equivalence) are $\infty$-groupoids. For us, spaces up to weak equivalences correspond to Kan complexes.

Given $X \in \mathtt{Kan}$, we can call $X_0$ the objects, and $X_1$ the morphisms. The horn filling conditions on horns tell you that you can *compose* and *invert* morphisms in $X_1$, witnessed by simplices in $X_2$.

**Definition 2.6.** A *quasi-category* (i.e. $\infty$-category) is a simplicial set with inner horn lifting property. That is, we can lift against horns $\Lambda^n_k$ for $0 < k < n$.

**Exercise 2.7.** A quasi-category has unique horn filling if and only if it is isomorphic to the nerve of a 1-category.

### Model categories

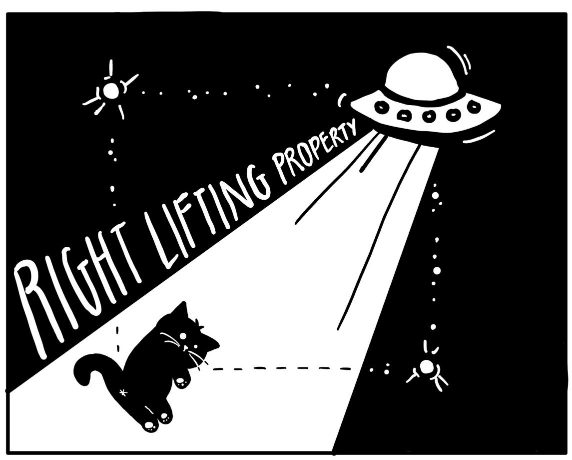
**Vista**: Every nice infinity category is equivalent in some sense to a model category. This will pretty much be the goal of this class.

**Notation 2.8.** Let $\mathcal{M}$ be a category, and $\chi \subseteq \mathcal{M}$ a class of morphisms. We define $\mathrm{LLP}(\chi)$ to be the class of morphisms in $\mathcal{M}$ so that $f$ has left lifting property with respect to all morphisms in $\chi$:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "\in \chi"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Similarly we can define $f \in \mathrm{RLP}(\chi)$ by

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\chi \ni"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$




**Definition 2.9.** A *weak factorization system* on a category $\mathcal{M}$ consists of a pair $(\mathscr{C}, \mathscr{F})$ of classes of morphisms such that

(1) Given any $f : X \to Y$ in $\mathcal{M}$, it factors (not necessarily uniquely) as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathscr{C}\ni"'] & & Y \\ & W \arrow[ur, "\in\mathscr{F}"'] & \end{tikzcd}$$

(2) $\mathscr{C} = \mathrm{LLP}(\mathscr{F})$ and $\mathscr{F} = \mathrm{RLP}(\mathscr{C})$.

**Example 2.10.** In `Set`, we have that mono and epimorphisms give a weak factorization system. A factorization is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{id}_X \times f"'] & & Y \\ & X \times Y \arrow[ur, "\pi_Y"'] & \end{tikzcd}$$

**Definition 2.11.** A *model structure* on $\mathcal{M}$ consists of three classes of morphisms:

| | |
|---|---|
| $W$ | weak equivalences |
| Cof | cofibrations |
| Fib | fibrations |

We denote by $\widetilde{\mathrm{Cof}} := \mathrm{Cof} \cap W$ and $\widetilde{\mathrm{Fib}} = \mathrm{Fib} \cap W$, and call these *trivial cofibrations* (resp. *trivial fibrations*). These are subject to the constraint that

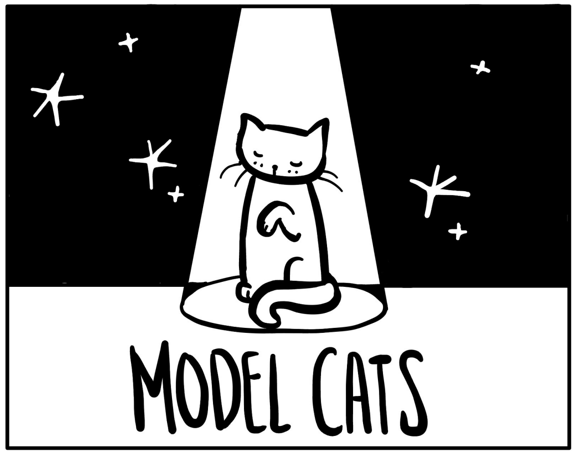
(1) $\mathcal{M}$ is bicomplete (all limits and colimits)[1]
(2) $W$ satisfies 2-out-of-3 property[2]
(3) $\left(\mathrm{Cof}, \widetilde{\mathrm{Fib}}\right)$ and $\left(\widetilde{\mathrm{Cof}}, \mathrm{Fib}\right)$ are weak factorization systems.

**Terminology 2.12.** A category with a model structure is referred to as a *model category*.

**Notation 2.13.** We will decorate each class of morphisms as

| | |
|---|---|
| $W$ | $\xrightarrow{\sim}$ |
| Cof | $\hookrightarrow$ |
| Fib | $\twoheadrightarrow$ |

[1]We might also require *finitely* bicomplete.
[2]If $f$ and $g$ are composable, and any two of $f$, $g$, $gf$ are in $W$ then so is the third.




**Exercise 2.14.** $W$, Cof, and Fib are closed under retracts: that is,

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[rr, bend right, equal] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

then if $g \in W$ (resp. Cof or Fib) then $f \in W$ (resp. Cof or Fib).

**Definition 2.15.** Let $\mathcal{M}$ be a model category, and let $\varnothing \in \mathcal{M}$ the initial object and $* \in \mathcal{M}$ the terminal object.

- We say that $X \in \mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to X$ is a cofibration.
- We say that $X \in \mathcal{M}$ is *fibrant* if the unique map $X \to *$ is a fibration.
- We say that $\widetilde{X}$ is a *cofibrant replacement* of $X$ if

$$\begin{tikzcd}
\varnothing \arrow[rr] \arrow[dr, hook] & & X \\
& \widetilde{X} \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

- We say that $\widetilde{X}$ is a *fibrant replacement* of $X$ if

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, hook, "\sim"'] & & * \\
& \widetilde{X} \arrow[ur, two heads] &
\end{tikzcd}$$

**Example 2.16.** $\mathcal{M} = \texttt{Top}$, $W$ = weak homotopy equivalences, Cof = relative CW complexes[3] The fibrations are determined by $\mathrm{Fib} = \mathrm{RLP}(\widetilde{\mathrm{Cof}})$. The fibrations are equivalently $\mathrm{RLP}(D^n \to D^n \times I)$. Every object here is fibrant, and the cofibrant objects are precisely the CW complexes. Cofibrant replacement is cellular approximation.

## 3. Lecture 3: Thursday, January 19th

**Proposition 3.1.** Identities and isomorphisms are weak equivalences in a model category.

*Proof.* For any $X \in \mathcal{M}$, we can fibrantly replace it to get $X \overset{\sim}{\hookrightarrow} \widetilde{X}$. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{id}"] \arrow[dr, hook, "\sim"'] & & X \arrow[dl, "\sim"] \\
& \widetilde{X}. &
\end{tikzcd}$$

By 2-out-of-3, we have that $\mathrm{id} : X \to X$ is also a weak equivalence.

More generally if $f : X \to Y$ is an isomorphism in $\mathcal{M}$, then by the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, equal] \arrow[d, "f"'] & Y \arrow[r, "f^{-1}"] \arrow[d, equal] & X \arrow[d, "f"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y,
\end{tikzcd}$$

we see that $f$ is contained in $W$. $\square$

If $(\mathscr{C}, \mathscr{F})$ is a weak factorization system, then both $\mathscr{C}$ and $\mathscr{F}$ are closed under retracts. Hence $\mathrm{Cof}, \widetilde{\mathrm{Cof}}, \mathrm{Fib}, \widetilde{\mathrm{Fib}}$ are closed under retracts. $W$ is also closed under retracts (exercise).

**Exercise 3.2.** We have that $\mathcal{M}$ is a model category if and only if $\mathcal{M}^{\mathrm{op}}$ is a model category.

**Theorem 3.3.** Cofibrations are closed under pushouts and coproducts.

[3] $A \hookrightarrow X$ is a *relative CW complex* if $X$ is built out of $A$ by attaching cells.

*Proof.* Given any test square, we can try to lift:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Z \arrow[r] \arrow[urr, dashed] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] & B. \end{tikzcd}$$

This map is constructed by universal property of the pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, dashed, bend right] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists!"] & \\ & & A. \end{tikzcd}$$

For coproducts, we can take $X_i \hookrightarrow Y_i$ for $i \in J$. Let's try to lift:

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, hook] & \amalg_i X_i \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Y_i \arrow[r] \arrow[urr, dashed] & \amalg_i Y_i \arrow[r] & B. \end{tikzcd}$$

We know that each $X_i \hookrightarrow Y_i$ is a cofibration hence it lifts against the big square. By universal property a map $\amalg_i Y_i \to A$ exists. $\square$

**Example 3.4.** If $\mathscr{C}$ is a bicomplete category, then $\mathscr{C}$ has a model structure where $W$ is the isomorphisms, and $\mathrm{Cof} = \mathrm{Fib} = \mathrm{mor}\mathscr{C}$.

**Example 3.5.** If $\mathcal{M} = \mathtt{Top}$, we have the *Quillen model structure*, with

- $W$ = weak homotopy equivalences
- Cof = retracts of relative CW complexes
- Fib = Serre fibrations ($\mathrm{RLP}(D^n \hookrightarrow D^n \times I)$).

**Example 3.6.** The Strøm (or Hurewicz) model structure on $\mathtt{Top}$:

- $W$ = homotopy equivalences
- Fib = Hurewicz fibrations ($\mathrm{RLP}(A \to A \times I)$ for all $A \in \mathtt{Top}$)
- Cof = closed cofibrations in $\mathtt{Top}$.

Fibrant replacement in the Strøm model structure looks like

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook] & & Y \\ & M_f \arrow[ur, two heads, "\simeq"'] & \end{tikzcd}$$

Where $M_f = (X \times I) \cup_X Y$ is the mapping cylinder.

**Example 3.7.** The *Kan model structure* on `sSet` with

- $W =$ weak homotopy equivalences
- $\mathrm{Cof} =$ monomorphisms (levelwise injections)
- $\mathrm{Fib} =$ Kan fibrations ($\mathrm{RLP}(\Lambda^n_k \to \Delta^n)$ for all $0 \le k \le n$).

Everything is cofibrant here (since the empty simplicial set injects into everything). Fibrant things are Kan complexes. This tells us that every simplicial set is weakly equivalent to a Kan complex!

**Theorem 3.8.** (Milnor) The natural map $X \to \mathrm{Sing}(|X|)$ is a weak homotopy equivalence for any simplicial set $X$. [Kerodon, 3.5.4.1]

**Definition 3.9.** Let $\mathscr{C}$ be a cat, and $W \subseteq \mathscr{C}$ a subcategory. A functor $F : \mathscr{C} \to \mathscr{D}$ is called the *localization of $\mathscr{C}$ with respect to $W$* if:

(1) $F(f) \in \mathrm{iso}\mathscr{D}$ if $f \in \mathrm{mor}W$
(2) For any other $F'$ satisfying (1), we have

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F'"] \arrow[d, "F"'] & \mathscr{D}' \\ \mathscr{C} \arrow[ur, "\exists!"'] & \end{tikzcd}$$

We denote by $\mathscr{C} \to \mathscr{C}[W^{-1}]$ the localization.

Here is a naive way to construct $\mathscr{C}[W^{-1}]$: we take the free category on $\mathscr{C}$ and "$W^{-1}$." That is, we take the same objects, but allow morphisms to be "zigzags" of morphisms forward in $\mathscr{C}$ and morphisms backwards in $W$, and we mod out by the relation that things in $W$ become isomorphisms. There are size issues here.

**Theorem 3.10.** If $\mathcal{M}$ is a model category, then localization $\mathcal{M} \to \mathcal{M}[W^{-1}]$ exists. We denote by $\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[W^{-1}]$ the homotopy category of $\mathcal{M}$.

Recall in `Top` that $f \simeq g : X \to Y$ if there is a map $H : X \times I \to Y$ so that $H(-,0) = f$ and $H(-,1) = g$.

**Definition 3.11.** Le t$\mathcal{M}$ be a model category. A *cylinder object* on $X \in \mathcal{M}$ is defined to be

$$\begin{tikzcd} X \amalg X \arrow[rr, "\nabla"] \arrow[dr, hook] & & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

The construction of cylinder objects is *not functorial*.

A *(left) homotopy* from $f$ to $g$ is a map $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. We denote this by $f \simeq g$.

**Proposition 3.12.** We have that $i_0 : X \to \mathrm{Cyl}(X)$ is a weak equivalence (and same for $i_1$).

*Proof.* We have

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, dashed, "i_0"'] & X \amalg X \arrow[r, "\nabla"] \arrow[d] & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

By 2-out-of-3 on the outside maps, the result follows. □

**Proposition 3.13.** If $X$ is cofibrant, then $i_0, i_1 : X \to \mathrm{Cyl}(X)$ are cofibrations.

*Proof.* Since cofibrations are preserved under pushouts, we have that $i_0$ and $i_1$ are cofibrations:

$$\begin{tikzcd} \varnothing \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "i_0"] \\ X \arrow[r, "i_1"'] \arrow[ur, phantom, "\ulcorner"] & X \amalg X \end{tikzcd}$$

□

**Theorem 3.14.** (Exercise) If $X$ is cofibrant, then homotopy $\simeq$ gives an equivalence relation on $\mathrm{Hom}(X, Y)$ for any $Y$.

We can think of a map

$$\mathrm{Hom}_{\mathcal{M}}(X, Y)/\simeq \times \mathrm{Hom}_{\mathcal{M}}(Y, Z)/\simeq \to \mathrm{Hom}_{\mathcal{M}}(X, Z)/\simeq$$
$$(f, g) \mapsto g \circ f.$$

In order for this to be well-defined, we need $Z$ to be fibrant.

**Lemma 3.15.** If $Z$ is fibrant, and $f \simeq g : X \to Z$, then if $h : X' \to X$, we have that $fh \simeq gh$.

*Proof.* We have $H : \mathrm{Cyl}(X) \to Y$ with $H_0 = f$ and $H_1 = g$. By lifting, we get

$$\begin{tikzcd} X' \amalg X' \arrow[r] \arrow[d, hook] & X \amalg X \arrow[r] & \mathrm{Cyl}(X) \arrow[d, two heads, "\sim"] \\ \mathrm{Cyl}(X') \arrow[r] \arrow[urr, dashed] & X' \arrow[r] & X. \end{tikzcd}$$

This gives the desired map. We used fibrancy of $Z$ to ensure that the map $\mathrm{Cyl}(X) \to X$ was a trivial fibration (or could be replaced with a better cylinder object using a map to $Z$). ☐

**Theorem 3.16.** In $\mathcal{M}$, given $f : X \to Y$ with $X$ cofibrant and $Y$ fibrant, then $f \in W$ if and only if $f$ is a homotopy equivalence.[4]

**Notation 3.17.** $\mathcal{M}_c$ = cofibrant objects in $\mathcal{M}$, and $\mathcal{M}_f$ = fibrant objects in $\mathcal{M}$. We denote by $\mathcal{M}_{cf}$ = objects which are *both* cofibrant and fibrant.

Concretely, we can define $\mathrm{Ho}(\mathcal{M})$ as the objects in $\mathcal{M}$, but where

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = \mathrm{Hom}_{\mathcal{M}_{cf}/\simeq}(RQX, RQY),$$

where $R$ is a fibrant replacement and $Q$ is a cofibrant replacement.

**Exercise 3.18.** Given $X \to Y$ in $\mathcal{M}$, there exists $QX \xrightarrow{\widetilde{f}} QY$ such that

$$\begin{tikzcd} QX \arrow[r, "\widetilde{f}"] \arrow[d, two heads, "\sim"] & QY \arrow[d, two heads, "\sim"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Here $\widetilde{f}$ is well-defined up to left homotopy.

Given some $\mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, we just need to check that $W \mapsto$ isos, and it is universal in that way.

## 4. Lecture 4: Tuesday, January 24th

**Definition 4.1.** Suppose $\mathcal{M}$ and $\mathcal{N}$ are model categories, and take a functor $F : \mathcal{M} \to \mathcal{N}$. A *left derived functor* of $F$ is an (absolute) right Kan extension of $F$ along

[4] Meaning that there is some $g : Y \to X$ with $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$.

$\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\
\mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed] \arrow[u, phantom, shift left=6, "\Leftarrow \ell"]
\end{tikzcd}$$

if $G : \mathrm{Ho}(\mathcal{M}) \to \mathcal{N}$ and $s : G \circ \gamma_{\mathcal{M}} \Rightarrow F$, then there exists a unique $s' : G \Rightarrow LF$ so that $\ell \circ (s' \circ \gamma_{\mathcal{M}}) = s$.

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\
\mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed] \arrow[ur, bend right] \arrow[u, phantom, shift left=6, "\Leftarrow \ell"] \arrow[ur, phantom, shift right=2, "\Uparrow s'"]
\end{tikzcd}$$

**Definition 4.2.** Let $F : \mathcal{M} \to \mathcal{N}$. A *total left derived functor* $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is the left derived functor of $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\gamma_{\mathcal{N}}} \mathrm{Ho}(\mathcal{N})$.

**Example 4.3.** If $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ where if $f \in W$ between cofibrant objects then $Ff$ is a weak equivalence in $\mathcal{N}$, then $\mathbb{L}F$ exists:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d] & \mathcal{N} \arrow[r] & \mathrm{Ho}(\mathcal{N}) \\
\mathrm{Ho}(\mathcal{M}) \arrow[urr, dashed]
\end{tikzcd}$$

We will have that $\mathbb{L}F(X) \xrightarrow{\sim} F(X)$ whenever $X$ is cofibrant. In general, $\mathbb{L}F(X) = F(Q(X))$.

**Definition 4.4.** Let $F : \mathcal{M} \to \mathcal{N}$. We say that $F$ is a *left Quillen functor* if

(i) $F$ is a left adjoint
(ii) $F$ preserves cofibrations and trivial cofibrations.

In this case if $G$ is a right adjoint, then we say the adjunction is a *Quillen adjunction / Quillen pair*.[5]

**Exercise 4.5.** Show that $L$ is left Quillen if and only if $G$ is right Quillen.

**Lemma 4.6.** (Ken Brown's Lemma) If $F : \mathcal{M} \to \mathcal{N}$ is any functor between model categories which sends trivial cofibrations between cofibrant objects to weak equivalences in $\mathcal{N}$, then $F$ sends any weak equivalence between cofibrant objects to weak equivalences.

[5]There is a dual notion of right Quillen functor, meaning it is a right adjoint which preserves fibrations and trivial fibrations.

*Proof.* Let $f : A \xrightarrow{\sim} B$, where $A, B \in \mathcal{M}_c$. We need $F(f)$ to be a weak equivalence. Consider the factorization of the coproduct of $f$ and the identity on $B$:

$$\begin{tikzcd}
A \amalg B \arrow[rr, "f \amalg \mathrm{id}_B"] \arrow[dr, hook, "q"'] & & B \\
& C \arrow[ur, two heads, "\sim", "p"'] &
\end{tikzcd}$$

Then consider the pushout:

$$\begin{tikzcd}
\varnothing \arrow[r, hook] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook, "i_A"] \arrow[ddr, "\sim"] & B \\
B \arrow[r, hook] \arrow[drr, "q"] \arrow[ddrr, equal] & A \amalg B \arrow[dr, "q"] & \\
& & C \arrow[uu, "p"] \arrow[d, "p"] \\
& & B
\end{tikzcd}$$

We have that

$$B \overset{i_B}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$

$$A \overset{i_A}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$

are both trivial cofibrations, hence their images under $F$ are weak equivalences. We see that

$$F(p) \circ F(q \circ \mathrm{id}_B) = F(p \circ q \circ \mathrm{id}_B) = F(\mathrm{id}_B).$$

Therefore $F(p)$ is a weak equivalence by 2-out-of-3. $\square$

**Theorem 4.7.** Suppose that $F : \mathcal{M} \to \mathcal{M}$ is left Quillen. Then $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ exists and can be defined as

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{Q} \mathrm{Ho}(\mathcal{M}_c) \xrightarrow{F} \mathrm{Ho}(\mathcal{N}).$$

Moreover, we obtain an adjunction on the homotopy categories:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) : \mathbb{R}G.$$

*Proof idea.* We have a natural iso

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y),$$

compatible with homotopy equivalence:

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y))/\simeq \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y)/\simeq$$

$\square$

**Theorem/Definition:** Take a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. Suppose that $f : X \xrightarrow{\sim} G(Y)$, with $X \in \mathcal{M}_c$ and $Y \in \mathcal{N}_f$ is a weak equivalence if and only if $f^\flat : F(X) \to Y$ is. Then $\mathbb{L}F$ and $\mathbb{R}G$ are equivalences of categories, we call this a *Quillen equivalence.*

**Example 4.8.** We have that

$$|-| : \mathtt{sSet}_{\mathrm{Kan}} \rightleftarrows \mathtt{Top}_{\mathrm{Quillen}} : \mathrm{Sing}(-)$$

is a Quillen equivalence.

**Example 4.9.** We have that

$$\mathrm{id} : \mathtt{Top}_{\mathrm{Quillen}} \rightleftarrows \mathtt{Top}_{\mathrm{Strøm}} : \mathrm{id}$$

is a Quillen adjunction but not a Quillen equivalence.

**Q**: If $\mathcal{M}$ and $\mathcal{N}$ are model categories such that there is an equivalence of categories $\mathrm{Ho}(\mathcal{M}) \cong \mathrm{Ho}(\mathcal{N})$, is this always coming from a Quillen equivalence?

**A**: No! Dugger–Shipley, 2009.

This indicates that Quillen equivalence is a good notion but it is not a *perfect* notion.

### Guided example: chain complexes

Let's take $\mathtt{Ch}_{\mathbb{Z}}$ to be homologically graded unbounded chain complexes. There are three model structures of interest. We first start with the projective one:

$(\mathtt{Ch}_{\mathbb{Z}})_{\mathrm{projective}}$:

- weak equivalences are quasi-isomorphisms
- fibrations are levelwise epimorphisms
- cofibrations are levelwise monomorphisms such that the cokernel of each $f_n : X_n \to Y_n$ is free.

If $M \in \mathtt{Ab}$, we define $S^n(M)$ to be the chain complex $M[n]$ which is concentrated in $M$ at degree $n$. If $M = \mathbb{Z}$, we call it $S^n$. We define $D^n(M)$ to be a chain complex

$$\cdots \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to \cdots$$

with two $M$'s concentrated in degrees $n$ and $n-1$. We call $D^n(\mathbb{Z}) =: D^n$.

**Exercise 4.10.** Show that fibrations are RLP$(0 \to D^n)$ for all $n$. That is,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ D^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

We claim this lifts iff $X \to Y$ is a levelwise epimorphism. We have that $\mathrm{Hom}_{\mathtt{Ch}}(D^n, Y) \cong Y_n$, so we are just asking if every element in $Y_n$ lifts to an element in $X_n$.

**Exercise 4.11.** Show that $\widetilde{\mathrm{Fib}} = \mathrm{RLP}\,(S^n \hookrightarrow D^{n+1})$ for all $n$. Consider $\mathrm{Hom}_{\mathtt{Ch}}(S^n, Y)$. A map looks like

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & Y_n \arrow[r] & Y_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

That is, it picks out a class in $Y_n$ which maps to zero under the differential. The data of a square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ D^n \arrow[r] & Y \end{tikzcd}$$

is the data of $(y, x) \in Y_n \oplus Z_{n-1}X$ so that $p(x) = dy$. Show that a lift exists if and only if $p$ is a trivial fibration.

Other model structures.

$(\mathtt{Ch}_R)_{\text{injective}}$:

- $W$ = quasi-isomorphisms
- Cof = fiberwise monomorphisms[6]
- Fib = fiberwise epimorphisms with fibrant kernel

We get a Quillen equivalence

$$\mathrm{id} : (\mathtt{Ch}_R)_{\text{projective}} \rightleftarrows (\mathtt{Ch}_R)_{\text{injective}} : \mathrm{id}.$$

We also have have a third one which is *not* Quillen equivalent.

$(\mathtt{Ch}_R)_{\text{Hurewicz}}$:

- $W$ = homotopy equivalences of chain complexes
- Cof = split levelwise monomorphisms

[6]Here we roughly have that $\mathrm{Cof} = \mathrm{LLP}(D^n \to 0)$ and $\widetilde{\mathrm{Fib}} = \mathrm{LLP}(D^{n+1} \to S^n)$.

- Fib = split levelwise epimorphisms

We denote by $\mathscr{D}(R) = \mathrm{Ho}\left((\mathtt{Ch}_R)_{\mathrm{proj}}\right)$ the *derived category* of a ring $R$.

We can also think about *connective* chain complexes (which are zero in negative degrees). We have an adjunction

$$\mathtt{Ch}_R \rightleftarrows \mathtt{Ch}_R^{\geq 0}.$$

This induces a model structure on $\mathtt{Ch}_R^{\geq 0}$ making it into a Quillen adjunction but not a Quillen equivalence. We denote by $\mathrm{Ho}(\mathtt{Ch}_R^{\geq 0}) = \mathscr{D}^{\geq 0}(R)$.

We get a model structure: $\left(\mathtt{Ch}_R^{\geq 0}\right)_{\mathrm{proj}}$

- $W$ = quasi-isomorphisms
- Fib = positive epimorphisms (may not be epi in degree 0)
- Cof = monomorphisms with projective cokernel. The cofibrant objects here are levelwise projective $R$-modules.

If we take $M \in \mathtt{Mod}_R$, we can view $S^0(M) \in \mathtt{Ch}_R^{\geq 0}$, and take a cofibrant replacement of it $P \overset{\sim}{\twoheadrightarrow} S^0(M)$. This is *exactly* a projective resolution of $M$!

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

**Example 4.12.** Let $M \in \mathtt{Mod}_R$. Then we can take

$$S^0(M) \otimes_R - : \mathtt{Ch}_R^{\geq 0} \to \mathtt{Ch}_R^{\geq 0}.$$

We can check that this is left Quillen. We can look at its total left derived functor $S^0(M) \otimes_R^{\mathbb{L}} -$. We can see that

$$M \otimes_R^{\mathbb{L}} N := S^0(M) \otimes_R^{\mathbb{L}} S^0(N) \simeq S^0(M) \otimes_R P_\bullet,$$

where $P_\bullet$ is a projective resolution of $N$. We have that

$$H_i(M \otimes_R^{\mathbb{L}} N) = \mathrm{Tor}_i^R(M, N).$$

**Exercise 4.13.** In the same way, if we want to derive hom, we can check that

$$\mathrm{Hom}_{\mathscr{D}^{\geq 0}(R)}(S^m(M), S^n(N)) \cong \mathrm{Ext}_R^{n-m}(M, N).$$

Via Dold-Kan, we have a Quillen adjunction

$$R[-] : \mathtt{sSet}_{\mathrm{Kan}} \rightleftarrows \mathtt{sMod}_R : U,$$

with the model structure on $\mathtt{sMod}_R$ given by weak homotopy equivalences as underlying simplicial sets, and fibrations as underlying Kan fibrations.

Then Dold-Kan takes the form of a Quillen equivalence

$$N : (\mathtt{sMod}_R)_{\mathrm{Kan}} \rightleftarrows (\mathtt{Ch}_R^{\geq 0})_{\mathrm{proj}} : \Gamma.$$

In general $N(X \otimes_R Y) \ncong N(X) \otimes_R N(Y)$, however $N(X \otimes Y) \cong N(X) \otimes_R N(Y)$. They both describe $\mathscr{D}^{\geq 0}(R)$ in a monoidal way.

## 5. Lecture 5: Thursday, January 26th

For Dold–Kan $\mathtt{Ch}_{\geq 0} \cong \mathtt{sMod}_R$, we have

$$M \otimes N \rightleftarrows M \otimes R \otimes N \rightleftarrows M \otimes R^{\otimes 2} N \cdots$$

we denote this by $B_\bullet(M, R, N)$ and call it the *bar construction.*

### Homotopy colimits

**Motivation**: Limits and colimits are not invariant under (weak) homotopy equivalence.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

However $\Sigma X \not\simeq *$.

Let $\mathcal{M}$ be a model category, and $\mathscr{C}$ a small category. Then we denote by $\mathrm{Fun}(\mathscr{C}, \mathcal{M}) = \mathcal{M}^{\mathscr{C}}$. Let $\mathscr{C}_0 \subseteq \mathscr{C}$ be the discrete subcategory spanned by $\mathrm{ob}(\mathscr{C})$. Let $\mathcal{M}^{\mathscr{C}_0} = \prod_{\mathscr{C}_0} \mathcal{M}$. This has a model structure where $W$, Fib, and Cof are determined objectwise.

Consider $\iota : \mathscr{C}_0 \hookrightarrow \mathscr{C}$. This induces a map

$$\begin{aligned} \iota^* : \mathcal{M}^{\mathscr{C}} &\to \mathcal{M}^{\mathscr{C}_0} \\ F &\mapsto F|_{\mathscr{C}_0}. \end{aligned}$$

This admits adjoints:

$$\iota_! \dashv i^* \dashv i_*.$$

We have that $\iota^*$ creates $W$ and Fib.

We have $\left(\mathcal{M}^{\mathscr{C}}\right)_{\mathrm{proj}}$:

- $W$ = objectwise weak equivalence

- Fib = objectwise fib
- Cof = ? induced by $\iota_!$Cof

We have that $\mathcal{M}$ is cocomplete, so we get a tensoring

$$\begin{aligned}\mathcal{M} \times \mathtt{Set}^{\mathscr{C}} &\to \mathcal{M}^{\mathscr{C}} \\ (X, F) &\mapsto X \otimes F = \amalg_{F(-)} X.\end{aligned}$$

We have $(X \times F)(c) = \amalg_{F(c)} X$.

There are representable functors

$$\begin{aligned}\mathscr{C}(c, -) : \mathscr{C} &\to \mathtt{Set} \\ d &\mapsto \mathscr{C}(c, d).\end{aligned}$$

By Yoneda, there is a natural iso

$$\mathtt{Set}^{\mathscr{C}}(\mathscr{C}(c, -), F) \cong F(c).$$

Tensoring with a representable functor gives

$$X \otimes \mathscr{C}(c, -) = \amalg_{\mathscr{C}(c,-)} X.$$

This is the *free diagram of $X$ generated at $c$.*

This gives an adjunction

$$- \otimes \mathscr{C}(c, -) : \mathcal{M} \rightleftarrows \mathcal{M}^{\mathscr{C}} : \mathrm{ev}_c.$$

In this case

$$\iota_!(F) = \amalg_c \amalg_{\mathscr{C}(c,-)} F(c),$$

which is the free diagram in $\mathcal{M}$ generated by $F$. Evaluating at $d$ gives

$$\iota_!(F)(d) = \amalg_{c \in \mathscr{C}} \amalg_{\mathscr{C}(c,d)} F(c).$$

This is the functor $\iota_! : \mathcal{M}^{\mathscr{C}_0} \to \mathcal{M}^{\mathscr{C}}$. We see that $\iota_! X$ is a left Kan extension

$$\begin{tikzcd} \mathscr{C}_0 \arrow[r, "X"] \arrow[d, hook, "\iota"'] & \mathcal{M} \\ \mathscr{C} \arrow[ur, dashed] & \end{tikzcd}$$

There is a diagonal functor

$$\begin{aligned}\mathcal{M} &\xrightarrow{\Delta} \mathcal{M}^{\mathscr{C}} \\ C &\mapsto \text{constant functor at } X.\end{aligned}$$

This admits adjoints

$$\operatorname{colim} \dashv \Delta \dashv \lim.$$

**Proposition 5.1.** The adjunction

$$\mathrm{colim} : \left(\mathcal{M}^{\mathscr{C}}\right)_{\mathrm{proj}} \rightleftarrows \mathcal{M} : \Delta$$

is Quillen.

We denote hocolim := $\mathbb{L}$colim. There is a map hocolim$(-) \to$ colim$(-)$, and

$$\mathrm{hocolim}(F) \simeq \mathrm{colim}(QF).$$

Here $QF$ denotes a cofibrant replacement in $\left(\mathcal{M}^{\mathscr{C}}\right)_{\mathrm{proj}}$. For a general $\mathscr{C}$, $QF$ is very difficult to determine.

Consider $\mathscr{C} = a \leftarrow b \to c$, and let $X \in \mathcal{M}^{\mathscr{C}_0}$. Then $\iota_! X$ is equal to

$$\begin{tikzcd} X(b) \arrow[r] \arrow[d] & X(b) \amalg X(c) \\ X(a) \amalg X(b) & \end{tikzcd}$$

Cofibrant objects in $\mathcal{M}^{\mathscr{C}}$ are of the form

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Z \\ Y & \end{tikzcd}$$

with $X$ cofibrant. Here cofibrant replacement is easy. We start with $Y \xleftarrow{f} X \xrightarrow{g} Z$, and we replace $X$ with $\widetilde{X} \xrightarrow{\sim} X$ to get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

If we cofibrantly replace $\widetilde{X} \to Z$, and similarly for $Y$, we get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Z} \\ \widetilde{Y} & \end{tikzcd}$$

The maps we used to fibrantly replace induces a fiberwise weak equivalence between this diagram and the one we started out with.

In $(\mathtt{Top})_{\text{Quillen}}$, we can take hocolim$(* \leftarrow X \to *)$. We cofibrantly replace $X$ if necessary, and replace $X \to *$ by $X \hookrightarrow CX$, which is a cofibration. In this case we see that

$$\operatorname{hocolim}\,(* \leftarrow X \to *) \simeq \operatorname{colim}(C\widetilde{X} \leftarrow \widetilde{X} \to C\widetilde{X}) = \Sigma\widetilde{X}.$$

More generally, hocolim$(Y \xleftarrow{f} X \xrightarrow{g} Z)$ is the double mapping cylinder $M(f,g)$.

**Theorem 5.2.** If $\mathcal{M}$ is a *left proper model category* then

$$\operatorname{hocolim}(Y \hookleftarrow X \to Z) \cong \operatorname{colim}(Y \hookleftarrow X \to Z).$$

*Proof.* In the easy case, $X$ is cofibrant, so we can factor the map to $Z$ to get

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & \widetilde{Z} \arrow[r, two heads, "\sim"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & H \arrow[r, dashed] & P. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The entire rectangle is a pushout, so $Z \to P$ is a cofibration, and the right square is a pushout by the pasting law, so $H \to P$ is a weak equivalence. □

**Example 5.3.** Let $\mathscr{C} = * \to * \to \cdots$. Show that $X_0 \to X_1 \to \cdots$ is cofibrant in $\mathcal{M}^{\mathscr{C}}$ if and only if $X_0$ is cofibrant and $X_i \hookrightarrow X_{i+1}$ is a cofibration for each $i$.

There is a third model structure on $\mathcal{M}^{\mathscr{C}}$ called the *Reedy model structure* (need $\mathscr{C}$ to be a Reedy cat). In this case, $\operatorname{hocolim}_{\Delta^{\mathrm{op}}}(X_\bullet) \cong \left|Q^{\text{Reedy}}X_\bullet\right|$, for $X$ a simplicial object in $\mathcal{M}$.

**Bar construction**: Let $\mathcal{M}$ a model cat, $\mathscr{C}$ a small cat, $F : \mathscr{C}^{\mathrm{op}} \to \mathcal{M}$, and $G : \mathscr{C} \to \mathcal{M}$. Then we define

$$B_\bullet\,(F, \mathscr{C}, G) := \amalg_{c_0 \in \mathscr{C}} F(c_0) \times G(c_0) \leftleftarrows \amalg_{c_0 \leftarrow c_1} F(c_0) \times G(c_1) \leftleftarrows \cdots$$

**Example 5.4.** If $F = * = G$, then

$$B_\bullet(*, \mathscr{C}, *) \cong N_\bullet(\mathscr{C}^{\mathrm{op}}).$$

**Pièce de résistance**:

**Theorem 5.5.** (Bousfield–Kan) If $F : \mathscr{C} \to \mathcal{M}$ is a functor, then

$$\operatorname{hocolim}_{\mathscr{C}}(F) \simeq |B_\bullet(*, \mathscr{C}, F)|\,.$$

## 6. Lecture 6: Tuesday, January 31st

### Combinatorial model categories

**Definition 6.1.** A model category is *combinatorial* if it is *presentable*[7] and *cofibrantly generated.*

To motivate presentability, let $X$ be a set. Then $X$ is determined by its elements, meaning that

$$\mathrm{Hom}_{\mathtt{Set}}(*, X) \cong X.$$

Then we can present $X$ as $X = \cup_{x \in X} \{*\}$.

**Definition 6.2.** A colimit is *filtered* if the diagram is filtered, meaning it is nonempty and every subdiagram has a cocone.

**Theorem 6.3.** (Exercise) In $\mathtt{Set}$, filtered colimits commute with finite limits. That is, if $F : I \times J \to \mathtt{Set}$ with $I$ finite and $J$ filtered, then

$$\mathrm{colim}_J \left( \lim_I F_I \right) \xrightarrow{\sim} \lim_I \left( \mathrm{colim}_J F_J \right)$$

is an isomorphism.

**Proposition 6.4.** A set $X$ is finite if and only if

$$\mathrm{Hom}_{\mathtt{Set}}(X, -) : \mathtt{Set} \to \mathtt{Set}$$

preserves filtered colimits.

*Proof.* For the backwards direction, let $I = \{X_i\}$ be the collection of finite subsets of $X$. Then $X = \mathrm{colim}_I X_i$. In particular, we have that

$$\mathrm{colim}_I \mathrm{Hom}(X, X_i) \cong \mathrm{Hom}(X, X)$$

$$(X \xrightarrow{f_i} X_i) \xrightarrow{\sim} \mathrm{id}_X?$$

For the forwards direction, $\mathrm{Hom}_{\mathtt{Set}}(*, -) \cong \mathrm{id}_{\mathtt{Set}}$ so it preserves colimits. Since $X$ is finite, we have that $X = \{x_1, \dots, x_n\}$, hence

$$\mathrm{Hom}(X, -) \cong \mathrm{Hom}(\cup_i \{x_i\}, -) \cong \lim_i \mathrm{Hom}(\{x_i\}, -).$$

Then we use finite limits commuting with filtered colimits. $\square$

**Definition 6.5.** An object $X \in \mathscr{C}$ is *compact* if $\mathrm{Hom}_{\mathscr{C}}(X, -) : \mathscr{C} \to \mathtt{Set}$ preserves filtered colimits.

[7] By this we mean "locally presentable."

Hence if $F : I \to \mathscr{C}$, with $I$ filtered, then a map $X \to \mathrm{colim}_I F$ factors through an $F(i)$.

**Examples 6.6.** Compact objects:

- `Set`, compact = finite set
- $\mathtt{Vect}_F$, compact = finite dimensional
- $\mathtt{Mod}_R$, compact = finitely presented
- `Grp`, compact = finitely presented
- `Top`, compact = finite sets with discrete topology
- `Ch`, compact = perfect chain complexes (bounded, levelwise finitely generated and projective)
- `sSet`, compact = finite simplicial sets ($X_n$ finite for each $n$, and there exists an $m$ so that all non-degenerate simplices have dimension $\leq m$).

A topological space is (topologically) compact if and only if $X \in \mathcal{O}(X)$ is (categorically) compact.

**Lemma 6.7.** Finite colimits of compact objects are compact.

**Definition 6.8.** A category $\mathscr{C}$ is *presentable* if

(1) $\mathscr{C}$ is cocomplete
(2) There exists a set $S$ of compact objects in $\mathscr{C}$ such that every object in $\mathscr{C}$ is a filtered colimit of objects in $S$.

We also say the "ind-completion" of $S$ is $\mathscr{C}$, denoted $\mathrm{Ind}(S) = \mathscr{C}$.

**Theorem 6.9.** $\mathscr{C}$ is presentable if and only if there is an adjunction of the form

$$\mathrm{Fun}(K^{\mathrm{op}}, \mathtt{Set}) \rightleftarrows \mathscr{C},$$

where $K$ is some small category, and the right adjoint is fully faithful and preserves filtered colimits.

We might take $K$ for example to to be isomorphism classes of compact objects in $\mathscr{C}$, then we have

$$\begin{aligned} \mathscr{C} &\to \mathrm{Fun}(K^{\mathrm{op}}, \mathtt{Set}) \\ X &\mapsto \left(K^{\mathrm{op}} \to \mathscr{C}\mathrm{op} \xrightarrow{\mathrm{Hom}(-,X)} \mathtt{Set}\right). \end{aligned}$$

**Theorem 6.10.** Suppose $\mathscr{C}$ and $\mathscr{D}$ presentable. Then $L : \mathscr{C} \to \mathscr{D}$ preserves colimits if and only if $L$ is a left adjoint.

### Cofibrantly generated model categories

**Definition 6.11.** Let $I$ be a set of maps in a cocomplete category, fix $\lambda$ to be an ordinal, and let $X : \lambda \to \mathscr{C}$ a functor, and suppose that $X(\alpha) \to X(\alpha+1)$ fits into

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & X(\alpha) \arrow[d] \\ B_\alpha \arrow[r] & X(\alpha+1), \end{tikzcd}$$

where $A_\alpha \to B_\alpha$ is in $I$. Then we say that $X(0) \to \mathrm{colim}_\lambda X$ is a *relative $I$-cell complex*. We say an object $Y \in \mathscr{C}$ is an *$I$-cell complex* if $\varnothing \to Y$ is a relative $I$-cell complex.

If $I = \{S^n \hookrightarrow D^{n+1}\}_{n \geq 0}$, then we are recovering the idea of CW complexes in spaces.

We denote by $\mathrm{Cell}_I(\mathscr{C})$ the class of relative $I$-cell complexes.

**Exercise 6.12.** We have that $\mathrm{Cell}_I(\mathscr{C})$ is the smallest class in $\mathscr{C}$ closed under composition, pushouts, and filtered colimits.

**Theorem 6.13.** *(Small object argument)* Let $\mathscr{C}$ be cocomplete, let $I$ a set of maps in $\mathscr{C}$, and suppose that for all $A \to B$ in $I$, we have that $A$ is compact with respect to the full subcategory of of $I$-cells in $\mathscr{C}$. Then there exists a functorial factorization of maps in $\mathscr{C}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & Y \\ & C \arrow[ur, "\delta"'] & \end{tikzcd}$$

with $\gamma \in \mathrm{Cell}_I(\mathscr{C})$ and $\delta \in \mathrm{RLP}(I)$.

*Proof idea.* Start with $X(0) = X$, and take a map $X(0) \to Y$. Suppose $X(\beta) = \mathrm{colim}_{\alpha<\beta} X(\alpha)$ is constructed with $X(\beta) \to Y$. Look at the set[8]

$$S = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X(\beta) \arrow[d] \\ B \arrow[r] & Y \end{tikzcd} : g \in I \right\}.$$

[8] Note this set is nonempty because we can take $g$ to be $\mathrm{id} : X(\beta) \to X(\beta)$.

Denote by $g_s$ the map $A \to B$ appearing in $s \in S$. Then we build

$$\begin{tikzcd} \amalg_{s\in S} A_s \arrow[r] \arrow[d, "\amalg_s g_s"'] \arrow[dr, phantom, "\ulcorner"] & X(\beta) \arrow[d, "\in \mathrm{Cell}_I(\mathscr{C})"] \\ \amalg_{s\in S} B_s \arrow[r] & X(\beta+1) \end{tikzcd}$$

By UP of the pushout, there is an induced map $X(\beta+1) \to Y$. Then we claim that

$$X(0) \to \mathrm{colim}_\beta X(\beta) =: C$$

is in $\mathrm{Cell}_I(\mathscr{C})$. The only thing left to show is that $C \to Y$ is in $\mathrm{RLP}(I)$. Take

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C = \mathrm{colim}_\beta X(\beta) \arrow[d] \\ B \arrow[r] & Y. \end{tikzcd}$$

Since $A$ is compact with respect to $I$-cells, the map $A \to C$ factors through some $X(\beta)$. Since $B \to Y$ factors through $X(\beta+1)$, we see that it lifts to $B \to C$. $\square$

**Definition 6.14.** A model category $\mathcal{M}$ is *cofibrantly generated* if there exist sets of maps $I, J$ in $\mathcal{M}$ so that

- Cof = retracts of $I$-cell complexes, denoted $\widehat{\mathrm{Cell}_I(\mathscr{C})}$[9]
- Cof = $\widehat{\mathrm{Cell}_J(\mathscr{C})}$

and "$I$ and $J$ permit the small object argument."

**Example 6.15.** For $\mathtt{Top}_{\text{Quillen}}$, we can take

$$\begin{aligned} I &= \left\{ S^n \hookrightarrow D^{n+1} \right\} \\ J &= \left\{ D^n \to D^n \times [0,1] \right\}. \end{aligned}$$

**Example 6.16.** For $\mathtt{sSet}_{\text{Kan}}$, we can take

$$\begin{aligned} I &= \left\{ \partial\Delta^n \to \Delta^n \right\} \\ J &= \left\{ \Lambda^k_n \to \Delta^n \right\}. \end{aligned}$$

**Example 6.17.** For $(\mathtt{Ch}_R)_{\text{proj}}$,

$$\begin{aligned} I &= \left\{ S^n \to D^{n+1} \right\} \\ J &= \left\{ 0 \to D^n \right\}. \end{aligned}$$

[9]The hat $\widehat{-}$ means "retracts of -"

**Example 6.18.** The Strøm model structure is not cofibrantly generated in the definition above.

**Theorem 6.19.** (Kan — Right transfer) Let $\mathcal{M}$ be a cofibrantly generated model category and $\mathscr{C}$ is any category where there is an adjunction

$$F : \mathcal{M} \rightleftarrows \mathscr{C} : G.$$

Then $\mathscr{C}$ has a model structure where $W$ and Fib are created by $G$. The model structure is cofibrantly generated by $F(I)$ and $F(J)$ if:

(1) $F(I)$ and $F(J)$ permit the small object argument
(2) $G\left(\mathrm{Cell}_{F(J)}\right)$ are weak equivalences in $\mathcal{M}$.

For combinatorial model categories, we get an inductive argument for building cofibrant replacements.

[Rezk-Schwede-Shipley] Combinatorial model categories are always simplicially enriched.

[Dugger] Any combinatorial model category $\mathcal{M}$ is Quillen equivalent to a localization of a projective Kan one:

$$L_\tau \mathrm{Fun}(K^{\mathrm{op}}, \mathtt{sSet}) \rightleftarrows \mathcal{M}.$$

## 7. Lecture 7: Thursday, February 2nd

[missed]

## 8. Lecture 8: Tuesday, February 7th

**Last time**: We had $\mathcal{M}$ a model category, and $\otimes$ a monoidal structure. We used this to give a monoidal structure on $\mathrm{Ho}(\mathcal{M})$, given by $\otimes^{\mathbb{L}}$, the *left derived tensor product*. We used this to give a homotopy theory on $\mathtt{Alg}(\mathcal{M})$, and $\mathtt{Mod}_R(\mathcal{M})$, etc.

**Q**: What are algebras in the homotopy category of a model structure $\mathcal{M}$? An example of interest is $\mathcal{M} = \mathtt{Top}$.

What are commutative algebras in $\mathtt{Top}$?

**Theorem 8.1.** (Moore) If $X \in \mathtt{CAlg}(\mathtt{Top})$, then there is a weak equivalence

$$\prod_{i=1}^{\infty} K(\pi_i(X), i) \to X.$$

*Proof.* Let $G_n = \pi_n(X)$. Then we take

$$0 \to F \to \mathbb{Z}[G_n] \to G_n \to 0.$$

Then we get that $\widetilde{H}_n(\vee_{g\in G_n} S^n) \cong \oplus_{g\in G_n}\widetilde{H}_n(S^n) = \mathbb{Z}[G_n]$. Using the Hurewicz theorem, there is an isomorphism

$$\pi_n(\vee S^n) \xrightarrow{\sim} \widetilde{H}_n(\vee S^n),$$

so we can pick $f_j \in \pi_n(S^n)$ for each $e_j$ in a basis of $F$. This gives us a pushout

$$\begin{tikzcd}
\vee_{j\in J} S^n \arrow[r] \arrow[d] & \vee_{g\in G_n} S^n \arrow[d] \\
* \arrow[r] & M(G_n, n) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

This gives a map $\vee_{n\geq 1} M(G_n, n) \to X$. By universal property, we get an algebra homomorphism[10][11]

$$\mathrm{SP}(\vee_{n\geq 1} M(G_n, n)) \to X$$

The Dold–Thom theorem states that $\pi_*\mathrm{SP}(Y) \cong \widetilde{H}_*(Y)$, given some connectedness hypothesis (path-connected?). We get that

$$\mathrm{SP}(\vee_{n\geq 1} M(G_n, n)) \cong \prod_n \mathrm{SP}(M(G_n, n)) = \prod_n K(G_n, n).$$

$\square$

**Definition 8.2.** We say that $X \in \mathtt{Alg}(\mathrm{Ho}(\mathtt{Top}))$ if and only if $X$ is a CW complex, with multiplication and unit

$$\begin{aligned} X \times X &\to X \\ * &\to X \end{aligned}$$

which are associative and unital *up to homotopy.*

These are also called *H-spaces*. The most prototypical example is a loop space.

**Example 8.3.** If $X$ is a based space, we can build $\Omega X$ as the homotopy pullback of the two maps from a point. Concatenation gives a map $\Omega X \times \Omega X \to \Omega X$.

[10] Here $\mathrm{SP}(-)$ denotes the infinite symmetric product, i.e. the free commutative algebra in $\mathtt{Top}$.

[11] The infinite symmetric product is left adjoint to the forgetful functor, i.e. $\mathrm{SP} : \mathtt{Top} \rightleftarrows \mathtt{CAlg}(\mathtt{Top}) : U$.

**Example 8.4.** Eilenberg-MacLane spaces $K(G, n)$ are uniquely determined up to homotopy. We have that

$$\pi_k\left(\Omega K(G, n)\right) \cong \pi_{k+1}(K(G, n))$$

therefore $\Omega K(G, n) = K(G, n-1)$.

**Q**: Given $X$ an $H$-space, such that $\pi_0 X$ is a group, is $X$ a loop space?

**A**: No, there are many grouplike $H$-spaces that are not equivalent to $\Omega X$. For example $S^7 \subseteq \mathbb{O}$ the unit octonians.

Loop spaces have an extra condition. Given $w, x, y, z \in \Omega X$, there is an association $(xy)z \simeq x(yz)$. There is a pentagon witnessing the different ways to associate four elements.

We can keep going with 5 loops, 6 loops... and we get the Stasheff associahedra $K(n)$, which tell us how to concatenate $n$ loops. These give maps

$$K(n) \times (\Omega X)^n \to \Omega X,$$

witnessing the higher associativities of concatenation. We call this an $A_\infty$-algebra structure.

**Theorem 8.5.** (Stasheff) Given $X$ connected, we have that $X \simeq \Omega Y$ for some $Y$ if and only if $X$ is an $A_\infty$-algebra in spaces that is grouplike.

**Rigidification**: We have that $\mathrm{Ho}(\mathtt{Alg}(\mathtt{sSet}, \times)) \simeq \mathtt{Alg}_{A_\infty}(\mathrm{Ho}(\mathtt{Top}))$.

## Operads

Let $\mathscr{C} = (\mathscr{C}, \otimes, I, [-,-])$ be a closed monoidal category.

**Definition 8.6.** An *operad* in $\mathscr{C}$ is a collection of objects $\{\mathcal{O}(j)\}_{j\geq 0}$ in $\mathscr{C}$ such that

(1) there is a right action of $\Sigma_j$ on $\mathcal{O}(j)$
(2) $\mathcal{O}(0) = I$
(3) $I \to \mathcal{O}(1)$ exists in $\mathscr{C}$
(4) composition

$$\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \xrightarrow{\gamma} \mathcal{O}(j_1 + \ldots + j_k)$$

for all $k \geq 0$ and $j_1, \ldots, j_k \geq 0$ such that they are equivariant, unital, and associative.

We think about $\mathcal{O}(j)$ as an abstract way to compose $j$-ary operations.

**Example 8.7.** We let Assoc be the operad defined by

$$\mathrm{Assoc}(j) = \amalg_{\sigma \in \Sigma_j} I.$$

We can define $\mathrm{Comm}(j) = I$.

**Example 8.8.** If $X \in \mathscr{C}$, the *endomorphism operad* is given by

$$\mathrm{End}_X(j) = [X^{\otimes j}, X].$$

**Definition 8.9.** A *morphism of operads* $\mathcal{O} \to \mathcal{O}'$ is a sequence of maps $\psi_j : \mathcal{O}(j) \to \mathcal{O}'(j)$ for $g \geq 0$ that are equivariant, associative, and unital.

**Definition 8.10.** Given $\mathcal{O}$ an operad in $\mathscr{C}$, an $\mathcal{O}$-algebra $(X, \theta)$ in $\mathscr{C}$ is $X \in \mathscr{C}$ together with a morphism of operads $\theta : \mathcal{O} \to \mathrm{End}_X$, sending $\mathcal{O}(j) \to \mathrm{End}_X(j)$. By adjointness, we think about this as $\mathcal{O}(j) \otimes X^{\otimes j} \to X$ which are associative and unital.

This gives us a category of $\mathcal{O}$-algebras, denoted $\mathtt{Alg}_{\mathcal{O}}(\mathscr{C})$.

**Example 8.11.** We have that

$$\mathtt{Alg}_{\mathrm{Assoc}}(\mathscr{C}) \cong \mathtt{Alg}(\mathscr{C})$$
$$\mathtt{Alg}_{\mathrm{Comm}}(\mathscr{C}) \cong \mathtt{CAlg}(\mathscr{C}).$$

We have that $\mathcal{M}$ is a monoidal model category if $\theta$ is nice enough, i.e. we get an adjunction

$$\mathcal{M} \rightleftarrows \mathtt{Alg}_{\mathcal{O}}(\mathcal{M}).$$

**Definition 8.12.** A *monad* in $\mathscr{C}$ is an algebra in $(\mathrm{Fun}(\mathscr{C}, \mathscr{C}), \circ, \mathrm{id}_{\mathscr{C}})$. That is, $M \in \mathtt{Alg}(\mathrm{Fun}(\mathscr{C}, \mathscr{C}))$ if we have $M : \mathscr{C} \to \mathscr{C}$ together with $\mu : M \circ M \Rightarrow M$, and $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathscr{C}$ that are associative and unital.

**Example 8.13.** Every adjunction $L : \mathscr{C} \rightleftarrows \mathscr{D} : R$ defines a monad $RL$.

**Definition 8.14.** An *algebra* $(X, \theta)$ over a monad $(M, \mu, \eta)$ in $\mathscr{C}$ is $X \in \mathscr{C}$ together with maps $\theta : M(X) \to X$ such that they are associative and unital, meaning that the diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & M(X) \arrow[d, "\theta"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} M(M(X)) \arrow[r, "\mu_{MX}"] \arrow[d, "M(\theta)"'] & M(X) \arrow[d, "\theta"] \\ M(X) \arrow[r, "\theta"'] & X. \end{tikzcd}$$

**Definition 8.15.** If $M$ is a monad, a *morphism of $M$-algebras* $(X, \theta) \to (X', \theta')$ is a map $f : X \to X'$ in $\mathscr{C}$ so that the diagram commutes

$$\begin{tikzcd} MX \arrow[r, "\theta"] \arrow[d, "Mf"'] & X \arrow[d, "f"] \\ MX' \arrow[r, "\theta'"'] & X'. \end{tikzcd}$$

**Example 8.16.** Consider $R$ a commutative ring, and the adjunction

$$- \otimes_{\mathbb{Z}} R : \mathtt{Ab} \rightleftarrows \mathtt{Mod}_R : U.$$

This forms a monad $M := - \otimes_{\mathbb{Z}} R : \mathtt{Ab} \to \mathtt{Ab}$. Then $\mathtt{Alg}_M(\mathtt{Ab})$ is equivalent to $\mathtt{Mod}_R$.

This is not always true! When this happens we say the adjunction is *monadic*.

Given a monadic adjunction

$$\mathscr{C} \rightleftarrows \mathscr{D} = \mathtt{Alg}_{RL}(\mathscr{C}),$$

we get a ton of things for free:

- $R$ will preserve colimits if $RL$ does
- get things like free monadic resolutions, bar constructions, etc.

## 9. Lecture 9: Thursday, February 9th

[missed]

## 10. Lecture 10: Thursday, February 16th

**Definition 10.1.** A simplicial set $\mathscr{C}$ is an *$\infty$-category* (or *quasi-category*) if it has inner horn filling — for all $0 < k < n$, we have

$$\begin{tikzcd} \Lambda^k_n \arrow[r] \arrow[d] & \mathscr{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

We shall see that $\infty$-categories are fibrant objects in $\mathtt{sSet}$ with the Joyal model structure.

**Example 10.2.**

(1) If $\mathscr{C}$ is a Kan complex, then it is an $\infty$-category
(2) If $\mathscr{C}$ is a category, then $N\mathscr{C}$ is an $\infty$-category.

**Definition 10.3.** Given an $\infty$-category $\mathscr{C}$, the *objects* of $\mathscr{C}$ are the vertices,[12] the *morphisms* are 1-simplices. We have *source* and *target* maps $d^1, d^0 : \mathscr{C}_1 \to \mathscr{C}_0$.[13] We define the *set of morphisms* from $X$ to $Y$ as the pullback

$$\begin{tikzcd} \hom_{\mathscr{C}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C}_1 \arrow[d, "(s,t)"] \\ \mathscr{C}_1 \arrow[r, "(X,Y)"'] & \mathscr{C}_0 \times \mathscr{C}_0. \end{tikzcd}$$

We have that $\hom_{\mathscr{C}}(X,Y)$ is the set of vertices of a simplicial set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$, which forms a Kan complex.

**Definition 10.4.** Given $X \in \mathscr{C}$ we define $\mathrm{id}_X \in \mathscr{C}_1$ by $s^0(X)$.

How do we compose? Composition won't be unique, but it will be unique *up to homotopy*.

Given $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, this determines a map of simplicial sets $\Lambda^2_1 \to \mathscr{C}$. By inner horn lifting, we have

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & \mathscr{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

We refer to the filling as a *composition*:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z. \end{tikzcd}$$

**Exercise 10.5.** Given an $\infty$-category $\mathscr{C}$, how can we define $\mathscr{C}^{\mathrm{op}}$? Would want that $N(\mathscr{C}^{\mathrm{op}}) \cong (N\mathscr{C})^{\mathrm{op}}$.[14]

**Detour**: Let $A \in \texttt{Cat}$, and let $\mathscr{C}$ be a cocomplete category. Recall that $\mathrm{Fun}(A^{\mathrm{op}}, \texttt{Set})$ is the free cocompletion. Given a functor $A \to \mathscr{C}$, by universal property there is a

[12] $X \in \mathscr{C}$ means $X \in \mathscr{C}_0$

[13] We write $f : X \to Y$ in $\mathscr{C}$ to mean $f \in \mathscr{C}_1$ with $s(f) = X$ and $t(f) = Y$.

[14] Every Kan complex has that $\mathscr{C}^{\mathrm{op}} \cong \mathscr{C}$.

map

$$\begin{tikzcd} A \arrow[r, "Q"] \arrow[d, hook] & \mathscr{C} \\ \mathrm{Fun}(A^{\mathrm{op}}, \mathtt{Set}) \arrow[ur, dashed, "|-|_Q"'] & \end{tikzcd}$$

This gives us an adjunction

$$| - |_Q : \mathrm{Fun}(A^{\mathrm{op}}, \mathtt{Set}) \rightleftarrows \mathscr{C} : \mathrm{Sing}_Q(-).$$

Here $\mathrm{Sing}_Q(-) = \mathrm{Hom}_{\mathscr{C}}(Q(-), X)$.

**Example 10.6.** If $\mathscr{C} = \mathtt{Top}$, then we can take $\Delta_{\mathtt{Top}} : \Delta \to \mathtt{Top}$, sending $[n]$ to $\Delta^n_{\mathtt{Top}}$. In this case, we recover the usual $| - |$ and $\mathrm{Sing}(-)$ adjunction.

**Example 10.7.** If $\mathscr{C} = \mathtt{Cat}$, there is a functor $\Delta \to \mathtt{Cat}$ sending $[n]$ to the associated poset category. We get an associated adjunction:

$$\tau : \mathtt{sSet} \rightleftarrows \mathtt{Cat} : N,$$

since $N = \mathrm{Hom}_{\mathtt{Cat}}([-], \mathscr{C})$.

**Exercise 10.8.** Describe $\tau : \mathtt{sSet} \to \mathtt{Cat}$ explicitly.

We call $\tau$ the fundamental category functor, essentially it will produce the homotopy category of an $\infty$-category.

**Definition 10.9.** Given an $\infty$-category $\mathscr{C}$, two morphisms $f : X \to Y$ and $g : Y \to Z$ are *homotopic*, written $f \simeq g$, if there exists a 2-simplex $\sigma : \Delta^2 \to \mathscr{C}$ with boundary $(g, f, \mathrm{id}_X)$:

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

**Example 10.10.** If $\mathscr{C}$ is an ordinary category, then in $N\mathscr{C}$, we have that $f \simeq g$ if and only if $f = g$.

**Proposition 10.11.** Given $\mathscr{C}$ an $\infty$-category, and $X, Y \in \mathscr{C}$, the homotopy relation provides an equivalence relation on $\mathrm{hom}_{\mathscr{C}}(X, Y)$.

**Definition 10.12.** We denote by $[f]$ the homotopy class of $f$.

*Sketch.* We first need to show reflexivity, so we want to find a 2-cell witnessing

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, equal] \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

We check that this is $s_0(f)$, where $f \in \mathscr{C}_1$, and $s_0 : \mathscr{C}_1 \to \mathscr{C}_2$.

For symmetry, suppose we have $f \simeq g$. We want to show $g \simeq f$. We can fill a $\Lambda^3_2$ witnessing this.

Transitivity is left as an exercise. □

**Definition 10.13.** Given $\mathscr{C}$ an $\infty$-category, define the 1-category $\mathrm{Ho}(\mathscr{C})$ to be the *homotopy category*, given by

$$\mathrm{obHo}(\mathscr{C}) = \mathscr{C}_0$$
$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(X, Y) = \mathrm{hom}_{\mathscr{C}}(X, Y)/\simeq .$$

In order to show this, we need to argue that composition is well-defined up to homotopy.

Suppose we have two compositions

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h_1"'] & & Z. \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h_2"'] & & Z. \end{tikzcd}$$

We want to argue that $h_1 \simeq h_2$. This can be done by filling the horn of a 3-simplex.

**Proposition 10.14.** When we restrict the adjunction $\tau \dashv N$ to $\infty$-categories, we get an adjunction

$$\mathrm{Ho}(-) : \mathtt{Cat}_\infty \rightleftarrows \mathtt{Cat} : N.$$

The way to compose arrows is contractible.

**Theorem 10.15.** The inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a map

$$\mathrm{Hom}_*(\Delta^2, \mathscr{C}) \to \mathrm{Hom}_*(\Lambda^2_1, \mathscr{C})$$

which is a trivial Kan fibration for any $\mathscr{C} \in \mathtt{Cat}_\infty$.

Here $\mathrm{Hom}_*$ is the *internal hom*, where $\mathrm{Hom}_*(X, Y) := \mathrm{Hom}_{\mathtt{sSet}}(\Delta^* \times X, Y)$.

*Proof.* Exercise □

As a consequence, we can take a pullback diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_*(\Delta^2, \mathscr{C}) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Hom}_*(\Lambda^2_1, \mathscr{C}). \end{tikzcd}$$

Then the pullback $P \to \Delta^0$ should be a trivial fibration, meaning that $P$ is a contractible Kan complex.

**Definition 10.16.** Given $\mathscr{C}$ an $\infty$-category and $X, Y \in \mathscr{C}$, recall that a map $f : X \to Y$ corresponds to $\Delta^1 \to \mathscr{C}$ whose faces are $X$ and $Y$. An *$n$-morphism* from $X$ to $Y$ is simply a map $\Delta^n \to \mathscr{C}$ such that $\Delta^{\{0,\dots,n-1\}} = X$ and $\Delta^{\{n\}} = Y$.

For $n \geq 2$, all $n$-morphisms are invertible in some sense.

**Definition 10.17.** Two objects $X$ and $Y$ in $\mathscr{C}$ are *equivalent*, written $X \simeq Y$, if there exists a 1-morphism $f : X \to Y$ in $\mathscr{C}$ such that $[f]$ in $\mathrm{Ho}(\mathscr{C})$ is an *isomorphism.*

**Definition 10.18.** An $\infty$-*groupoid* is an $\infty$-category for which $\mathrm{Ho}(\mathscr{C})$ is a groupoid, meaning all the 1-morphisms are equivalences.

**Theorem 10.19.** (Homotopy hypothesis) We get that $\mathscr{C}$ is an $\infty$-groupoid if and only if $\mathscr{C}$ is a Kan complex.

## 11. Lecture 11: Tuesday, February 21st

[missed]

## 12. Lecture 12: Thursday, February 23rd

### Adjoint functors and colimits

**Last time**: Recall that a 1-morphism in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$[15] is precisely a natural transformation $\eta : F \to G$, where $F, G : \mathscr{C} \to \mathscr{D}$. In other words, it is $\eta : \Delta^1 \times \mathscr{C} \to \mathscr{D}$.

We have $\mathtt{hQCat} = \mathrm{Ho}(\mathtt{Cat}_\infty)$, where objects are infinity categories, and the morphisms are

$$\mathrm{Hom}_{\mathtt{hQCat}}(\mathscr{C}, \mathscr{D}) = \Pi_0 \left(\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\simeq}\right).$$

That is, it is the set of equivalence classes of functors $\mathscr{C} \to \mathscr{D}$.

[15] The simplicial set $\mathrm{Fun}(\Delta^\bullet \times \mathscr{C}, \mathscr{D})$

If $\mathscr{C}$ is an $\infty$-category, and $X, Y \in \mathscr{C}$, we defined $\mathrm{Hom}_{\mathscr{C}}(X,Y)_\bullet$ to be the simplicial set given by the pullback

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X,Y)_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathscr{C}) \arrow[d] \\
\Delta^0 \arrow[r] & \mathrm{Fun}(\{0\}, \mathscr{C})_\bullet \times \mathrm{Fun}(\{1\}, \mathscr{C}).
\end{tikzcd}$$

**Proposition 12.1.** We have that $\mathrm{Hom}_{\mathscr{C}}(X,Y) \in \mathtt{Kan}$.

*Sketch.* This follows from a more general fact that for $A \hookrightarrow B$ a subsimplicial set with $A_0 = B_0$, and $\mathscr{C}$ an $\infty$-category, then $P$ is always a Kan complex

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(B, \mathscr{C}) \arrow[d] \\
\Delta^0 \arrow[r, "f"'] & \mathrm{Fun}(A, \mathscr{C}).
\end{tikzcd}$$

Need to show that every $u$ in $\mathrm{Fun}(B, \mathscr{C})_1$ *in the pullback* is a weak equivalence. We have an evaluation map for every $b \in B_0 = A_0$, given by $\mathrm{ev}_b : \mathrm{Fun}(B, \mathscr{C}) \to \mathrm{Fun}(\{b\}, \mathscr{C})$, mapping $u$ to $u_{f(b)}$. We claim that $u_{f(b)} = \mathrm{id}_{f(b)}$, since the diagram commutes

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathscr{C}) \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\{b\}, \mathscr{C}) \\
& \mathrm{Fun}(A, \mathscr{C}) \arrow[ur] &
\end{tikzcd}$$

□

## Adjoint functors

**Definition 12.2.** Let $F : \mathscr{C} \to \mathscr{D}$, and $G : \mathscr{D} \to \mathscr{C}$ be functors of $\infty$-categories. We say that $F \dashv G$ if there exist natural transformations $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ ad $\varepsilon : FG \to \mathrm{id}_{\mathscr{D}}$ so that:

(1) there exists $\Delta^2 \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ witnessing

$$\begin{tikzcd}
& FGF \arrow[dr, "\varepsilon \mathrm{id}_F"] & \\
F\mathrm{id}_{\mathscr{C}} \arrow[ur, "\mathrm{id}_{\mathscr{C}}\eta"] \arrow[rr, "\mathrm{id}"'] & & \mathrm{id}_{\mathscr{C}}F.
\end{tikzcd}$$

(2) there exists $\Delta^2 \to \mathrm{Fun}(\mathscr{D}, \mathscr{C})$ witnessing

$$\begin{tikzcd} & GFG \arrow[dr, "\mathrm{id}\varepsilon"] & \\ \mathrm{id}_{\mathscr{C}}G \arrow[ur, "\eta\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & & G\mathrm{id}_{\mathscr{C}}. \end{tikzcd}$$

**Remark 12.3.** We have that $\eta : \mathrm{id} \to GF$ depends only on $[\eta]$ in $\mathrm{Ho}(\mathrm{Fun}(\mathscr{C}, \mathscr{D}))$. If $\eta$ is given, then $\varepsilon$ is unique up to homotopy.

**Example 12.4.** If $\mathscr{C}$ and $\mathscr{D}$ are ordinary categories, then we have a 1-categorical adjunction

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : G$$

if and only if we have an $\infty$-categorical adjunction

$$NF : N\mathscr{C} \rightleftarrows N\mathscr{D} : NG.$$

**Example 12.5.** If $X, Y \in \mathtt{Kan}$, then $F : X \to Y$ is an adjoint if and only if $F$ is a homotopy equivalence of simplicial sets. The unit and counit become the witnesses of homotopy equivalence.

**Remark 12.6.** If we have an adjunction $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ of $\infty$-categories, then $F$ and $G$ are homotopy equivalences of simplicial sets. The converse is not true in general.

**Exercise 12.7.** If $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of $\infty$-categories, then it is both a left and right adjoint functor.

**Proposition 12.8.** Given $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ of $\infty$-categories, then

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathscr{C}) \rightleftarrows \mathrm{Ho}(\mathscr{D}) : \mathrm{Ho}(G)$$

is an adjunction of 1-categories. That is, **if** we know $F \dashv G$ in $\infty$-categories, then to check if $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ is a unit, it is enough to check that $\mathrm{Ho}(\eta)$ is the unit.

However the converse is not true!

**Warning**: Suppose we take $F : \Delta^0 \to X$ with $X \in \mathtt{Kan}$ simply connected, and $F$ picks $x \in X_0$. Then $\mathrm{Ho}(F) \dashv \mathrm{Ho}(G)$ because $\mathrm{Ho}(X)$ will be simply connected. But it does not imply that $F \dashv G$ unless $X$ is contractible.

There $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{D})}(FC, D) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(C, GD)$ for any $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

**Theorem 12.9.** Take $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ functors of $\infty$-categories. Then $F \dashv G$ with unit $\eta$ if and only if the composite

$$\mathrm{Hom}_{\mathscr{D}}(FC, D) \xrightarrow{G} \mathrm{Hom}_{\mathscr{C}}(GFC, GD) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathscr{C}}(C, GD)$$

is a weak homotopy equivalence between Kan complexes (aka a homotopy equivalence) for all $C, D$.

The forward direction is straightforward, but the backwards direction uses (co)cartesian fibration stuff.

### Limits and colimits

Recall that if $\mathscr{C}$ is an ordinary category, then $i \in \mathscr{C}$ is *initial* if for all $X \in \mathscr{C}$, there is a unique $i \xrightarrow{!} X$. That is, $\mathrm{Hom}_{\mathscr{C}}(i, X) = *$.

**Definition 12.10.** In an $\infty$-category $\mathscr{C}$, we have that $i \in \mathscr{C}$ is *initial* if $\mathrm{Hom}_{\mathscr{C}}(i, X) \simeq *$ is contractible for all $X \in \mathscr{C}$.

**Definition 12.11.** Let $\mathscr{C}$ be an $\infty$-category, and $K_\bullet \in \mathtt{sSet}$. Then for any $X \in \mathscr{C}$, denote by $\underline{X} \in \mathrm{Fun}(K, \mathscr{C})$ the constant functor valued at $X$. The assignment $X \mapsto \underline{X}$ defines a diagonal map

$$\Delta : \mathscr{C} \to \mathrm{Fun}(K, \mathscr{C}).$$

This is defined by precomposing with $K \to \Delta^0$, and looking at $\mathscr{C} \simeq \mathrm{Fun}(\Delta^0, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$.

**Definition 12.12.** Let $u : K \to \mathscr{C}$ be a diagram. We say a natural transformation $\alpha : \underline{L} \to u$ exhibits $L \in \mathscr{C}$ as *a limit of* $u$ if for all $X \in \mathscr{C}$, we have that the composite

$$\mathrm{Hom}_{\mathscr{C}}(X, L) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{X}, \underline{L}) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{X}, u)$$

is a (weak) homotopy equivalence of Kan complexes.

**Definition 12.13.** We say that $\beta : u \to \underline{C}$ exhibits $C$ as a *colimit of* $u$ if, for all $Y \in \mathscr{C}$, the composite

$$\mathrm{Hom}_{\mathscr{C}}(C, Y) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{C}, \underline{Y}) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(u, \underline{C})$$

is a (weak) homotopy equivalence.

Note that if $\alpha$ or $\beta$ exist, they are unique up to equivalence.

**Example 12.14.** If $\mathscr{C}$ is an ordinary category, then $u : K \to N\mathscr{C}$ is equivalent to a map $\tau(u) : \tau K \to \mathscr{C}$. We can check that $L \in \mathscr{C}$ is $\lim(\tau u)$ in a 1-categorical sense if and only if $L \in \mathscr{C}$ is a limit of $u$ in an $\infty$-categorical sense.

**Example 12.15.** Let $f : X \to Y$ in an $\infty$-cat $\mathscr{C}$. Then $f$ is an equivalence if and only if $f$ exhibits $Y$ as a colimit $\{X\} \to \mathscr{C}$, if and only if $f$ exhibits $X$ as a limit $\{Y\} \to \mathscr{C}$.

**Example 12.16.** Taking the identity diagram $\varnothing \to \mathscr{C}$, the notion of limit/colimit matches the notion of terminal/initial object.

**Proposition 12.17.** A limit $L \in \mathscr{C}$ is unique up to homotopy. Therefore we usually define it as $\lim_K(u)$.

**Proposition 12.18.** We have that $\mathscr{C}$ admits all $K$-indexed limits if and only if

$$\Delta : \mathscr{C} \to \mathrm{Fun}(K, \mathscr{C})$$

is a left adjoint. The right adjoint is given by $\lim_K(-)$.

Equalizers are limits along $\Delta^1 \amalg_{\partial\Delta^1} \Delta^1$, pullbacks are limits along $\Delta^1 \times \Delta^1 - (0,0)$, etc.

# References