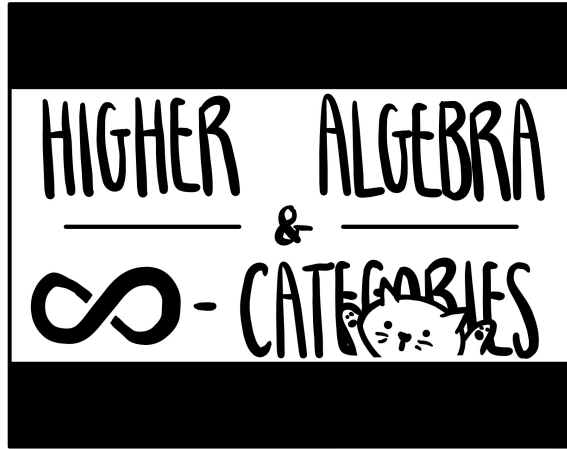


HIGHER ALGEBRA

MAXIMILIEN PÉROUX



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1. LECTURE 1: THURSDAY, JANUARY 12TH

Today: the **homotopy hypothesis**

Classical algebra: sets, monoids, groups, abelian groups, rings. Each of these are built up on the other. In higher courses, we may see groupoids, which are types of categories. A category is a generalization of a monoid, in some sense. We also have monoidal categories, which in some sense are a generalization of rings.

For higher algebra: spaces, \mathbb{E}_1 -spaces, spectra, \mathbf{E}_1 -ring spectra. Underlying this we have ∞ -groupoids, ∞ -categories, and monoidal ∞ -categories.

We study spaces, not up to homeomorphism, but up to *weak homotopy equivalence*. We will study this in a minute. “Spaces” in this class will always mean the study of topological spaces up to weak homotopy equivalence.

We’ll give a synthetic definition of what an infinity category is, and circle back to a technical definition in about a month.

What is an ∞ -category?

An ∞ -category (or $(\infty, 1)$ -category) \mathcal{C} should consist of:

- (1) a class of objects
- (2) a class of morphisms so that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a space
- (3) n -morphisms for $n \geq 2$, where for instance 2-morphisms are between 1-morphisms, 3-morphisms between 2-morphisms, etc.
- (4) morphisms can be composed in a suitable way

(5) n -morphisms for $n \geq 2$ are invertible in some sense.

An ∞ -groupoid (or $(\infty, 0)$ -category) should be an ∞ -category where all the 1-morphisms are also invertible in some sense.

Why study spaces up to weak homotopy equivalence?

Recall by the Yoneda lemma, we have that

$$X \cong Y \Leftrightarrow \operatorname{Hom}_{\mathbf{Top}}(A, X) \cong \operatorname{Hom}_{\mathbf{Top}}(A, Y)$$

for all $A \in \mathbf{Top}$. Figuring out $\operatorname{Hom}(A, X)$ up to bijection for all A is very difficult, so we prefer to study continuous maps up to homotopy. For X and Y nice enough, we say that $f \simeq g$ in $\operatorname{Hom}(X, Y)$ if there exists some path $I \rightarrow \operatorname{Map}(X, Y)$ so that $0 \mapsto f$ and $1 \mapsto g$. We define $[X, Y] = \operatorname{Hom}_{\mathbf{Top}}(X, Y) / \simeq$.

We see then that $X \simeq Y$ if and only if $[A, X] \cong [A, Y]$ for all $A \in \mathbf{Top}$.

We may ask when $[A, -] : \mathbf{Top}_* \rightarrow \mathbf{Set}$ factors through \mathbf{Grp} or \mathbf{Ab} . We have that $[A, -]$ factors through \mathbf{Grp} if and only if A is a co-H-group in \mathbf{Top} . That is, we have maps

$$\begin{aligned} A &\rightarrow A \vee A \\ A &\rightarrow *, \end{aligned}$$

which is coassociative, counital, coinvertible.

Example 1.1. S^n , when $n \geq 1$, is a co-H-space. The map $S^n \rightarrow S^n \vee S^n$ is the pinch map.

We say that X is *weakly homotopy equivalent* to Y , we write $X \sim Y$, if and only if there is a map $X \rightarrow Y$ inducing an isomorphism

$$\pi_n(X) = [S^n, X]_* \cong [S^n, Y]_* = \pi_n(Y),$$

for all $n \geq 0$ (for $n \geq 1$ this is a group isomorphism).

If $X \sim Y$, then $H_n(X) \cong H_n(Y)$ for any n .

Theorem 1.2. (Cellular approximation) For any X in \mathbf{Top} , there exists \tilde{X} a CW complex with a canonical map $\tilde{X} \xrightarrow{\sim} X$ that is a weak equivalence.

Theorem 1.3. (Whitehead) If X, Y are CW complexes, then $X \xrightarrow{\sim} Y$ is a homotopy equivalence if and only if $X \xrightarrow{\sim} Y$ is a weak homotopy equivalence.

Exercise 1.4. Find spaces X and Y which are weakly homotopy equivalent but not homotopy equivalent.

We denote by Δ the simplex category. Its objects are ordered sets of the form $[n] = \{0, 1, \dots, n\}$, and its morphisms are order-preserving maps. We have that Δ is generated by *cofaces* and *codegeneracies*. The cofaces are of the form

$$d^0, d^1 : [0] \rightarrow [1],$$

skipping 0 or 1 in $[1]$, etc. The codegeneracies look like $s^0 : [1] \rightarrow [0]$ which “repeat” an element.

The cofaces and codegeneracies satisfy certain *cosimplicial identities*.

If \mathcal{C} is a category, we denote by $s\mathcal{C} = \mathcal{C}^{\Delta^{\text{op}}}$ the simplicial objects in \mathcal{C} . If $\mathcal{C} = \mathbf{Set}$, we write \mathbf{sSet} as the category of simplicial sets. A simplicial set $X_\bullet \in \mathbf{sSet}$ consists of sets X_0, X_1, \dots together with face and degeneracy maps satisfying the simplicial identities.

Example 1.5. The *nerve of a small category*. Let $\mathcal{C} \in \mathbf{Cat}$ a small category. We denote by $N_\bullet \mathcal{C}$ the simplicial set with $N_0 \mathcal{C} = \text{ob} \mathcal{C}$, $N_1 \mathcal{C} = \text{mor} \mathcal{C}$, and $N_n \mathcal{C}$ the set of n composable morphisms in \mathcal{C} . That is,

$$N_n \mathcal{C} = N_1 \mathcal{C} \times_{N_0 \mathcal{C}} \cdots \times_{N_0 \mathcal{C}} N_1 \mathcal{C}.$$

The face maps are source/target/composition. The degeneracies insert an identity morphism.

Example 1.6. Via Yoneda, we get a functor

$$\Delta^n := \text{Hom}_\Delta(-, [n]) : \Delta^{\text{op}} \rightarrow \mathbf{Set}.$$

If X_\bullet is a simplicial set, we get that the set of n -simplices X_n is in bijection with $\text{Hom}_{\mathbf{sSet}}(\Delta^n, X_\bullet)$.

Example 1.7. (Dold–Kan) We have $\text{Ch}_R^{\geq 0} \xrightarrow{\Gamma} \mathbf{sMod}_R$ is an isomorphism, where $\Gamma_m C_\bullet = \bigoplus_{[n] \rightarrow [k]} C_k$, with faces and degeneracies left as an exercise.

Example 1.8. Let $\Delta_{\text{Top}}^n \subseteq \mathbb{R}^{n+1}$ be defined by

$$\left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : 0 \leq t_i \leq 1, \sum t_i = 1 \right\}.$$

We can view $[n] = \{v_0, \dots, v_n\}$, and $v_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the i th place. Then if $\alpha : [m] \rightarrow [n]$ in Δ , we can define $\alpha(v_i) = v_{\alpha(i)}$. Extend linearly to get $\alpha_* : \Delta_{\text{Top}}^m \rightarrow \Delta_{\text{Top}}^n$. We get then that $\Delta_{\text{Top}}^\bullet$ is a cosimplicial topological space.

Example 1.9. If $X \in \mathbf{Top}$, we have $\text{Sing}_\bullet(X) \in \mathbf{sSet}$ defined by $\text{Sing}_n(X) = \text{Hom}_{\mathbf{Top}}(\Delta_{\text{Top}}^n, X)$.

Definition 1.10. If $X_\bullet \in \mathbf{sSet}$, we define its *geometric realization* to be

$$|X_\bullet| = \coprod_{n \geq 0} X_n \times \Delta_{\mathbf{Top}}^n / \sim,$$

where $(x, s) \sim (y, t)$ if and only if there is some $\alpha : [m] \rightarrow [n]$ so that $\alpha^* y = x$ and $\alpha_* s = t$.

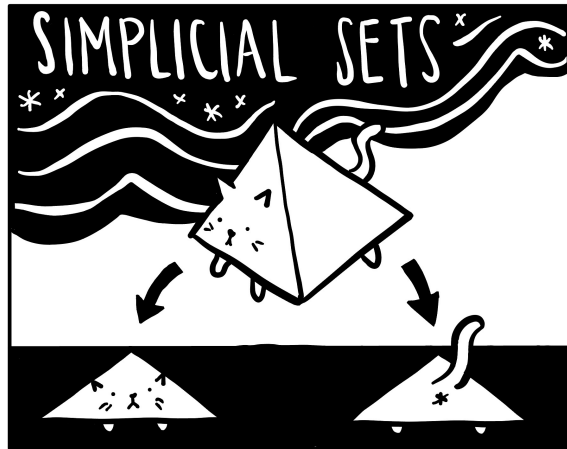
Example 1.11. $|\Delta_\bullet^n| \cong \Delta_{\mathbf{Top}}^n$.

Exercise 1.12. $|X_\bullet|$ is always a CW complex for any $X_\bullet \in \mathbf{sSet}$.

Exercise 1.13. We have an adjunction $|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathbf{Sing}(-)$

Definition 1.14. $X_\bullet \rightarrow Y_\bullet$ is a *weak homotopy equivalence* in \mathbf{sSet} if $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ is a weak homotopy equivalence of spaces.

Theorem 1.15. (Quillen) Simplicial sets up to weak equivalence is equivalent to topological spaces up to weak homotopy equivalence. Moreover, for any $X \in \mathbf{Top}$, we have that $|\mathbf{Sing}(X)|$ is weakly equivalent to X .



2. LECTURE 2: TUESDAY, JANUARY 17TH

Today: the homotopy hypothesis (continued).

Recall we are interested in studying \mathbf{Top} up to weak homotopy equivalences. Equivalently, we are interested in studying \mathbf{sSet} up to weak equivalence, and the relationship between the two was given by the geometric realization / singular complex adjunction.

Recall we've defined $\Delta^n = \text{Hom}_\Delta(-, [n])$. We will define the k th horn $\Lambda_k^n \subseteq \Delta^n$ as a coequalizer in **sSet**

$$\left(\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{i \neq k} \Delta^{n-1} \right) \rightarrow \Lambda_k^n,$$

where the two maps are δ^{j-1} and δ^i . The geometric realization of Λ_k^n is the topological n -simplex, with the middle and the face opposite the k th edge removed.

Definition 2.1. We say that $Y \in \mathbf{sSet}$ is a *Kan complex* if for all $k \leq n$, and for every $\Lambda_k^n \rightarrow Y$, there exists a (not necessarily unique) lift:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & Y \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array}$$

Exercise 2.2. Y is a Kan complex if and only if for any $(n-1)$ -simplices $y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n$ such that $d_i y_j = d_{j-1} y_i$ for $i < j$, $i, j \neq k$, there exists an n -simplex y such that $d_i y = y_i$ for all $i \neq k$.

Exercise 2.3. We have that $\text{Sing}(X)$ is always a Kan complex for any $X \in \mathbf{Top}$.

Exercise 2.4. We have that Δ^n is not a Kan complex for $n \geq 1$.

Exercise 2.5. If $X \in \mathbf{sGrp}$, then the underlying simplicial set of X is always a Kan complex.

Up to weak homotopy equivalence, every simplicial set is a Kan complex (will see this later).

Recall the Dold-Kan correspondence

$$\mathbf{sMod}_{\mathbb{Z}} \cong \mathbf{Ch}_{\mathbb{Z}}^{\geq 0},$$

which sends weak homotopy equivalences to quasi-isomorphisms. Given a simplicial set X_* , we can take an associated simplicial abelian group $\mathbb{Z}[X_*]$ by taking the free group on n -simplices at level n . We can ask what $\mathbb{Z}[X_*]$ corresponds to as a chain complex. One answer is that

$$\mathbb{Z}[\text{Sing}(X_*)] \leftrightarrow C_*(X; \mathbb{Z}).$$

This tells us that

$$\pi_* (\mathbb{Z} [\text{Sing}(X)]) \cong H_*(X; \mathbb{Z}).$$

In some sense we can view $\mathbb{Z}[\text{Sing}(X)]$ as being (equivalent to) the *free commutative monoid* on X . This is what is known as the *Dold-Thom theorem*.

Homotopy hypothesis: Spaces (up to weak equivalence) are ∞ -groupoids. For us, spaces up to weak equivalences correspond to Kan complexes.

Given $X \in \mathbf{Kan}$, we can call X_0 the objects, and X_1 the morphisms. The horn filling conditions on horns tell you that you can *compose* and *invert* morphisms in X_1 , witnessed by simplices in X_2 .

Definition 2.6. A *quasi-category* (i.e. ∞ -category) is a simplicial set with inner horn lifting property. That is, we can lift against horns Λ_k^n for $0 < k < n$.

Exercise 2.7. A quasi-category has unique horn filling if and only if it is isomorphic to the nerve of a 1-category.

Model categories

Vista: Every nice infinity category is equivalent in some sense to a model category. This will pretty much be the goal of this class.

Notation 2.8. Let \mathcal{M} be a category, and $\chi \subseteq \mathcal{M}$ a class of morphisms. We define $\text{LLP}(\chi)$ to be the class of morphisms in \mathcal{M} so that f has left lifting property with respect to all morphisms in χ :

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ f \downarrow & \nearrow \text{dashed} & \downarrow \in \chi \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

Similarly we can define $f \in \text{RLP}(\chi)$ by

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \chi \ni \downarrow & \nearrow \text{dashed} & \downarrow f \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

Definition 2.9. A *weak factorization system* on a category \mathcal{M} consists of a pair $(\mathcal{C}, \mathcal{F})$ of classes of morphisms such that

- (1) Given any $f : X \rightarrow Y$ in \mathcal{M} , it factors (not necessarily uniquely) as

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \mathcal{C} \ni & \nearrow \in \mathcal{F} \\ & W & \end{array}$$

- (2) $\mathcal{C} = \text{LLP}(\mathcal{F})$ and $\mathcal{F} = \text{RLP}(\mathcal{C})$.



Example 2.10. In \mathbf{Set} , we have that mono and epimorphisms give a weak factorization system. A factorization is

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \text{id}_X \times f & \nearrow \pi_Y \\ & X \times Y & \end{array}$$

Definition 2.11. A *model structure* on \mathcal{M} consists of three classes of morphisms:

W	weak equivalences
Cof	cofibrations
Fib	fibrations

We denote by $\widetilde{\text{Cof}} := \text{Cof} \cap W$ and $\widetilde{\text{Fib}} = \text{Fib} \cap W$, and call these *trivial cofibrations* (resp. *trivial fibrations*). These are subject to the constraint that

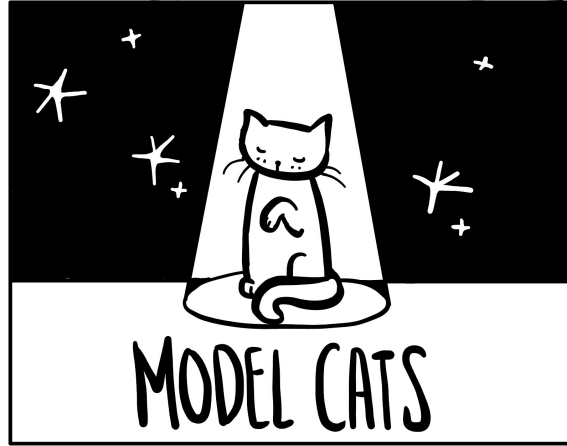
- (1) \mathcal{M} is bicomplete (all limits and colimits)¹
- (2) W satisfies 2-out-of-3 property²
- (3) $(\text{Cof}, \widetilde{\text{Fib}})$ and $(\widetilde{\text{Cof}}, \text{Fib})$ are weak factorization systems.

Terminology 2.12. A category with a model structure is referred to as a *model category*.

Notation 2.13. We will decorate each class of morphisms as

¹We might also require *finitely* bicomplete.

²If f and g are composable, and any two of f, g, gf are in W then so is the third.



$$\begin{array}{c|l} W & \xrightarrow{\sim} \\ \text{Cof} & \hookrightarrow \\ \text{Fib} & \twoheadrightarrow \end{array}$$

Exercise 2.14. W , Cof , and Fib are closed under retracts: that is,

$$\begin{array}{ccccc} \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet \\ f \downarrow & & \downarrow g & & \downarrow f \\ \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet \end{array}$$

then if $g \in W$ (resp. Cof or Fib) then $f \in W$ (resp. Cof or Fib).

Definition 2.15. Let \mathcal{M} be a model category, and let $\emptyset \in \mathcal{M}$ the initial object and $*$ $\in \mathcal{M}$ the terminal object.

- We say that $X \in \mathcal{M}$ is *cofibrant* if the unique map $\emptyset \rightarrow X$ is a cofibration.
- We say that $X \in \mathcal{M}$ is *fibrant* if the unique map $X \rightarrow *$ is a fibration.
- We say that \tilde{X} is a *cofibrant replacement* of X if

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & X \\ & \searrow & \nearrow \sim \\ & \tilde{X} & \end{array}$$

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- We say that \tilde{X} is a *fibrant replacement* of X if

$$\begin{array}{ccc} X & \xrightarrow{\quad} & * \\ & \searrow \sim & \nearrow \\ & \tilde{X} & \end{array}$$

Example 2.16. $\mathcal{M} = \mathbf{Top}$, $W =$ weak homotopy equivalences, $\mathbf{Cof} =$ relative CW complexes³ The fibrations are determined by $\mathbf{Fib} = \mathbf{RLP}(\widetilde{\mathbf{Cof}})$. The fibrations are equivalently $\mathbf{RLP}(D^n \rightarrow D^n \times I)$. Every object here is fibrant, and the cofibrant objects are precisely the CW complexes. Cofibrant replacement is cellular approximation.

3. LECTURE 3: THURSDAY, JANUARY 19TH

Proposition 3.1. Identities and isomorphisms are weak equivalences in a model category.

Proof. For any $X \in \mathcal{M}$, we can fibrantly replace it to get $X \xrightarrow{\sim} \tilde{X}$. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ & \searrow \sim & \swarrow \sim \\ & \tilde{X} & \end{array}$$

By 2-out-of-3, we have that $\text{id} : X \rightarrow X$ is also a weak equivalence.

More generally if $f : X \rightarrow Y$ is an isomorphism in \mathcal{M} , then by the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad f^{-1} \quad} & X \\ \downarrow f & & \parallel & & \downarrow f \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y, \end{array}$$

we see that f is contained in W . □

If $(\mathcal{C}, \mathcal{F})$ is a weak factorization system, then both \mathcal{C} and \mathcal{F} are closed under retracts. Hence $\mathbf{Cof}, \widetilde{\mathbf{Cof}}, \mathbf{Fib}, \widetilde{\mathbf{Fib}}$ are closed under retracts. W is also closed under retracts (exercise).

Exercise 3.2. We have that \mathcal{M} is a model category if and only if \mathcal{M}^{op} is a model category.

³ $A \hookrightarrow X$ is a *relative CW complex* if X is built out of A by attaching cells.

Fibrant replacement in the Strøm model structure looks like

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \nearrow \simeq \\ & M_f & \end{array}$$

Where $M_f = (X \times I) \cup_X Y$ is the mapping cylinder.

Example 3.7. The *Kan model structure* on **sSet** with

- W = weak homotopy equivalences
- Cof = monomorphisms (levelwise injections)
- Fib = Kan fibrations ($\text{RLP}(\Lambda_k^n \rightarrow \Delta^n)$ for all $0 \leq k \leq n$).

Everything is cofibrant here (since the empty simplicial set injects into everything). Fibrant things are Kan complexes. This tells us that every simplicial set is weakly equivalent to a Kan complex!

Theorem 3.8. (Milnor) The natural map $X \rightarrow \text{Sing}(|X|)$ is a weak homotopy equivalence for any simplicial set X . [Kerodon, 3.5.4.1]

Definition 3.9. Let \mathcal{C} be a cat, and $W \subseteq \mathcal{C}$ a subcategory. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called the *localization of \mathcal{C} with respect to W* if:

- (1) $F(f) \in \text{iso}\mathcal{D}$ if $f \in \text{mor}W$
- (2) For any other F' satisfying (1), we have

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F'} & \mathcal{D}' \\ F \downarrow & \nearrow \exists! & \\ \mathcal{C} & & \end{array}$$

We denote by $\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ the localization.

Here is a naive way to construct $\mathcal{C}[W^{-1}]$: we take the free category on \mathcal{C} and “ W^{-1} .” That is, we take the same objects, but allow morphisms to be “zigzags” of morphisms forward in \mathcal{C} and morphisms backwards in W , and we mod out by the relation that things in W become isomorphisms. There are size issues here.

Theorem 3.10. If \mathcal{M} is a model category, then localization $\mathcal{M} \rightarrow \mathcal{M}[W^{-1}]$ exists. We denote by $\text{Ho}(\mathcal{M}) = \mathcal{M}[W^{-1}]$ the homotopy category of \mathcal{M} .

Recall in **Top** that $f \simeq g : X \rightarrow Y$ if there is a map $H : X \times I \rightarrow Y$ so that $H(-, 0) = f$ and $H(-, 1) = g$.

Definition 3.11. Let \mathbf{tM} be a model category. A *cylinder object* on $X \in \mathcal{M}$ is defined to be

$$\begin{array}{ccc} X \amalg X & \xrightarrow{\nabla} & Y \\ & \searrow & \nearrow \sim \\ & \text{Cyl}(X) & \end{array}$$

The construction of cylinder objects is *not functorial*.

A *(left) homotopy* from f to g is a map $H : \text{Cyl}(X) \rightarrow Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. We denote this by $f \simeq g$.

Proposition 3.12. We have that $i_0 : X \rightarrow \text{Cyl}(X)$ is a weak equivalence (and same for i_1).

Proof. We have

$$\begin{array}{ccccc} & & \text{id} & & \\ & \curvearrowright & & \curvearrowleft & \\ X & \xrightarrow{\quad} & X \amalg X & \xrightarrow{\nabla} & Y \\ & \searrow \text{dashed } i_0 & \downarrow & \nearrow \sim & \\ & & \text{Cyl}(X) & & \end{array}$$

By 2-out-of-3 on the outside maps, the result follows. □

Proposition 3.13. If X is cofibrant, then $i_0, i_1 : X \rightarrow \text{Cyl}(X)$ are cofibrations.

Proof. Since cofibrations are preserved under pushouts, we have that i_0 and i_1 are cofibrations:

$$\begin{array}{ccc} \emptyset & \hookrightarrow & X \\ \downarrow & \lrcorner & \downarrow i_0 \\ X & \xrightarrow{i_1} & X \amalg X \end{array}$$

□

Theorem 3.14. (Exercise) If X is cofibrant, then homotopy \simeq gives an equivalence relation on $\text{Hom}(X, Y)$ for any Y .

We can think of a map

$$\begin{aligned} \text{Hom}_{\mathcal{M}}(X, Y) / \simeq \times \text{Hom}_{\mathcal{M}}(Y, Z) / \simeq &\rightarrow \text{Hom}_{\mathcal{M}}(X, Z) / \simeq \\ (f, g) &\mapsto g \circ f. \end{aligned}$$

In order for this to be well-defined, we need Z to be fibrant.

Lemma 3.15. If Z is fibrant, and $f \simeq g : X \rightarrow Z$, then if $h : X' \rightarrow X$, we have that $fh \simeq gh$.

Proof. We have $H : \text{Cyl}(X) \rightarrow Y$ with $H_0 = f$ and $H_1 = g$. By lifting, we get

$$\begin{array}{ccccc} X' \amalg X' & \longrightarrow & X \amalg X & \longrightarrow & \text{Cyl}(X) \\ \downarrow & & & \nearrow & \downarrow \sim \\ \text{Cyl}(X') & \longrightarrow & X' & \longrightarrow & X. \end{array}$$

This gives the desired map. We used fibrancy of Z to ensure that the map $\text{Cyl}(X) \rightarrow X$ was a trivial fibration (or could be replaced with a better cylinder object using a map to Z). \square

Theorem 3.16. In \mathcal{M} , given $f : X \rightarrow Y$ with X cofibrant and Y fibrant, then $f \in W$ if and only if f is a homotopy equivalence.⁴

Notation 3.17. \mathcal{M}_c = cofibrant objects in \mathcal{M} , and \mathcal{M}_f = fibrant objects in \mathcal{M} . We denote by \mathcal{M}_{cf} = objects which are *both* cofibrant and fibrant.

Concretely, we can define $\text{Ho}(\mathcal{M})$ as the objects in \mathcal{M} , but where

$$\text{Hom}_{\text{Ho}(\mathcal{M})}(X, Y) = \text{Hom}_{\mathcal{M}_{cf}/\simeq}(RQX, RQY),$$

where R is a fibrant replacement and Q is a cofibrant replacement.

Exercise 3.18. Given $X \rightarrow Y$ in \mathcal{M} , there exists $QX \xrightarrow{\tilde{f}} QY$ such that

$$\begin{array}{ccc} QX & \xrightarrow{\tilde{f}} & QY \\ \downarrow \sim & & \downarrow \sim \\ X & \xrightarrow{f} & Y. \end{array}$$

Here \tilde{f} is well-defined up to left homotopy.

Given some $\mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$, we just need to check that $W \mapsto \text{isos}$, and it is universal in that way.

⁴Meaning that there is some $g : Y \rightarrow X$ with $fg \simeq \text{id}$ and $gf \simeq \text{id}$.

4. LECTURE 4: TUESDAY, JANUARY 24TH

Definition 4.1. Suppose \mathcal{M} and \mathcal{N} are model categories, and take a functor $F : \mathcal{M} \rightarrow \mathcal{N}$. A *left derived functor* of F is an (absolute) right Kan extension of F along $\gamma_{\mathcal{M}} : \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \gamma_{\mathcal{M}} \downarrow & \swarrow \ell & \nearrow \\ \text{Ho}(\mathcal{M}) & & \end{array}$$

if $G : \text{Ho}(\mathcal{M}) \rightarrow \mathcal{N}$ and $s : G \circ \gamma_{\mathcal{M}} \Rightarrow F$, then there exists a unique $s' : G \Rightarrow LF$ so that $\ell \circ (s' \circ \gamma_{\mathcal{M}}) = s$.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \gamma_{\mathcal{M}} \downarrow & \swarrow \ell & \nearrow \\ \text{Ho}(\mathcal{M}) & & \end{array}$$

Definition 4.2. Let $F : \mathcal{M} \rightarrow \mathcal{N}$. A *total left derived functor* $\mathbb{L}F : \text{Ho}(\mathcal{M}) \rightarrow \text{Ho}(\mathcal{N})$ is the left derived functor of $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\gamma_{\mathcal{N}}} \text{Ho}(\mathcal{N})$.

Example 4.3. If $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{N}$ where if $f \in W$ between cofibrant objects then Ff is a weak equivalence in \mathcal{N} , then $\mathbb{L}F$ exists:

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} & \longrightarrow & \text{Ho}(\mathcal{N}) \\ \downarrow & & & \nearrow & \\ \text{Ho}(\mathcal{M}) & & & & \end{array}$$

We will have that $\mathbb{L}F(X) \xrightarrow{\sim} F(X)$ whenever X is cofibrant. In general, $\mathbb{L}F(X) = F(Q(X))$.

Definition 4.4. Let $F : \mathcal{M} \rightarrow \mathcal{N}$. We say that F is a *left Quillen functor* if

- (i) F is a left adjoint
- (ii) F preserves cofibrations and trivial cofibrations.

In this case if G is a right adjoint, then we say the adjunction is a *Quillen adjunction* / *Quillen pair*.⁵

Exercise 4.5. Show that L is left Quillen if and only if G is right Quillen.

⁵There is a dual notion of right Quillen functor, meaning it is a right adjoint which preserves fibrations and trivial fibrations.

Lemma 4.6. (Ken Brown's Lemma) If $F : \mathcal{M} \rightarrow \mathcal{N}$ is any functor between model categories which sends trivial cofibrations between cofibrant objects to weak equivalences in \mathcal{N} , then F sends any weak equivalence between cofibrant objects to weak equivalences.

Proof. Let $f : A \xrightarrow{\sim} B$, where $A, B \in \mathcal{M}_c$. We need $F(f)$ to be a weak equivalence. Consider the factorization of the coproduct of f and the identity on B :

$$\begin{array}{ccc} A \amalg B & \xrightarrow{f \amalg \text{id}_B} & B \\ & \searrow q \quad \nearrow p & \\ & C & \end{array}$$

Then consider the pushout:

$$\begin{array}{ccccc} \emptyset & \hookrightarrow & A & \xrightarrow{f} & B \\ \downarrow & & \downarrow i_A & \searrow \sim & \uparrow p \\ B & \hookrightarrow & A \amalg B & \xrightarrow{q} & C \\ & \searrow q & & \nearrow q & \\ & & & & B \end{array}$$

We have that

$$\begin{aligned} B &\xrightarrow{i_B} A \amalg B \xrightarrow{q} C \\ A &\xrightarrow{i_A} A \amalg B \xrightarrow{q} C \end{aligned}$$

are both trivial cofibrations, hence their images under F are weak equivalences. We see that

$$F(p) \circ F(q \circ \text{id}_B) = F(p \circ q \circ \text{id}_B) = F(\text{id}_B).$$

Therefore $F(p)$ is a weak equivalence by 2-out-of-3. □

Theorem 4.7. Suppose that $F : \mathcal{M} \rightarrow \mathcal{M}$ is left Quillen. Then $\mathbb{L}F : \text{Ho}(\mathcal{M}) \rightarrow \text{Ho}(\mathcal{N})$ exists and can be defined as

$$\text{Ho}(\mathcal{M}) \xrightarrow{Q} \text{Ho}(\mathcal{M}_c) \xrightarrow{F} \text{Ho}(\mathcal{N}).$$

Moreover, we obtain an adjunction on the homotopy categories:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) : \mathbb{R}G.$$

Proof idea. We have a natural iso

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y),$$

compatible with homotopy equivalence:

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y)) / \simeq \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y) / \simeq$$

□

Theorem/Definition: Take a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. Suppose that $f : X \xrightarrow{\sim} G(Y)$, with $X \in \mathcal{M}_c$ and $Y \in \mathcal{N}_f$ is a weak equivalence if and only if $f^\flat : F(X) \rightarrow Y$ is. Then $\mathbb{L}F$ and $\mathbb{R}G$ are equivalences of categories, we call this a *Quillen equivalence*.

Example 4.8. We have that

$$|-| : \mathbf{sSet}_{\mathrm{Kan}} \rightleftarrows \mathbf{Top}_{\mathrm{Quillen}} : \mathrm{Sing}(-)$$

is a Quillen equivalence.

Example 4.9. We have that

$$\mathrm{id} : \mathbf{Top}_{\mathrm{Quillen}} \rightleftarrows \mathbf{Top}_{\mathrm{Strøm}} : \mathrm{id}$$

is a Quillen adjunction but not a Quillen equivalence.

Q: If \mathcal{M} and \mathcal{N} are model categories such that there is an equivalence of categories $\mathrm{Ho}(\mathcal{M}) \cong \mathrm{Ho}(\mathcal{N})$, is this always coming from a Quillen equivalence?

A: No! Dugger–Shipley, 2009.

This indicates that Quillen equivalence is a good notion but it is not a *perfect* notion.

Guided example: chain complexes

Let's take $\mathbf{Ch}_{\mathbb{Z}}$ to be homologically graded unbounded chain complexes. There are three model structures of interest. We first start with the projective one:

$(\mathbf{Ch}_{\mathbb{Z}})_{\mathrm{projective}} :$

- weak equivalences are quasi-isomorphisms
- fibrations are levelwise epimorphisms
- cofibrations are levelwise monomorphisms such that the cokernel of each $f_n : X_n \rightarrow Y_n$ is free.

If $M \in \mathbf{Ab}$, we define $S^n(M)$ to be the chain complex $M[n]$ which is concentrated in M at degree n . If $M = \mathbb{Z}$, we call it S^n . We define $D^n(M)$ to be a chain complex

$$\cdots \rightarrow 0 \rightarrow M \xrightarrow{\text{id}} M \rightarrow 0 \rightarrow \cdots$$

with two M 's concentrated in degrees n and $n - 1$. We call $D^n(\mathbb{Z}) =: D^n$.

Exercise 4.10. Show that fibrations are $\text{RLP}(0 \rightarrow D^n)$ for all n . That is,

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ D^n & \longrightarrow & Y. \end{array}$$

We claim this lifts iff $X \rightarrow Y$ is a levelwise epimorphism. We have that $\text{Hom}_{\mathbf{ch}}(D^n, Y) \cong Y_n$, so we are just asking if every element in Y_n lifts to an element in X_n .

Exercise 4.11. Show that $\widetilde{\text{Fib}} = \text{RLP}(S^n \hookrightarrow D^{n+1})$ for all n . Consider $\text{Hom}_{\mathbf{ch}}(S^n, Y)$. A map looks like

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & Y_n & \longrightarrow & Y_{n-1} & \longrightarrow & \cdots \end{array}$$

That is, it picks out a class in Y_n which maps to zero under the differential. The data of a square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X \\ \downarrow & & \downarrow p \\ D^n & \longrightarrow & Y \end{array}$$

is the data of $(y, x) \in Y_n \oplus Z_{n-1}X$ so that $p(x) = dy$. Show that a lift exists if and only if p is a trivial fibration.

Other model structures.

$(\mathbf{Ch}_R)_{\text{injective}}$:

- W = quasi-isomorphisms
- Cof = fiberwise monomorphisms⁶
- Fib = fiberwise epimorphisms with fibrant kernel

⁶Here we roughly have that $\text{Cof} = \text{LLP}(D^n \rightarrow 0)$ and $\widetilde{\text{Fib}} = \text{LLP}(D^{n+1} \rightarrow S^n)$.

We get a Quillen equivalence

$$\mathrm{id} : (\mathbf{Ch}_R)_{\mathrm{projective}} \rightleftarrows (\mathbf{Ch}_R)_{\mathrm{injective}} : \mathrm{id}.$$

We also have have a third one which is *not* Quillen equivalent.

$(\mathbf{Ch}_R)_{\mathrm{Hurewicz}}$:

- W = homotopy equivalences of chain complexes
- Cof = split levelwise monomorphisms
- Fib = split levelwise epimorphisms

We denote by $\mathcal{D}(R) = \mathrm{Ho}\left((\mathbf{Ch}_R)_{\mathrm{proj}}\right)$ the *derived category* of a ring R .

We can also think about *connective* chain complexes (which are zero in negative degrees). We have an adjunction

$$\mathbf{Ch}_R \rightleftarrows \mathbf{Ch}_R^{\geq 0}.$$

This induces a model structure on $\mathbf{Ch}_R^{\geq 0}$ making it into a Quillen adjunction but not a Quillen equivalence. We denote by $\mathrm{Ho}(\mathbf{Ch}_R^{\geq 0}) = \mathcal{D}^{\geq 0}(R)$.

We get a model structure: $(\mathbf{Ch}_R^{\geq 0})_{\mathrm{proj}}$

- W = quasi-isomorphisms
- Fib = positive epimorphisms (may not be epi in degree 0)
- Cof = monomorphisms with projective cokernel. The cofibrant objects here are levelwise projective R -modules.

If we take $M \in \mathbf{Mod}_R$, we can view $S^0(M) \in \mathbf{Ch}_R^{\geq 0}$, and take a cofibrant replacement of it $P \xrightarrow{\sim} S^0(M)$. This is *exactly* a projective resolution of M !

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M \longrightarrow 0. \end{array}$$

Example 4.12. Let $M \in \mathbf{Mod}_R$. Then we can take

$$S^0(M) \otimes_R - : \mathbf{Ch}_R^{\geq 0} \rightarrow \mathbf{Ch}_R^{\geq 0}.$$

We can check that this is left Quillen. We can look at its total left derived functor $S^0(M) \otimes_R^{\mathbb{L}} -$. We can see that

$$M \otimes_R^{\mathbb{L}} N := S^0(M) \otimes_R^{\mathbb{L}} S^0(N) \simeq S^0(M) \otimes_R P_{\bullet},$$

where P_\bullet is a projective resolution of N . We have that

$$H_i(M \otimes_R^{\mathbb{L}} N) = \mathrm{Tor}_i^R(M, N).$$

Exercise 4.13. In the same way, if we want to derive hom, we can check that

$$\mathrm{Hom}_{\mathcal{D}^{\geq 0}(R)}(S^m(M), S^n(N)) \cong \mathrm{Ext}_R^{n-m}(M, N).$$

Via Dold-Kan, we have a Quillen adjunction

$$R[-] : \mathbf{sSet}_{\mathrm{Kan}} \rightleftarrows \mathbf{sMod}_R : U,$$

with the model structure on \mathbf{sMod}_R given by weak homotopy equivalences as underlying simplicial sets, and fibrations as underlying Kan fibrations.

Then Dold-Kan takes the form of a Quillen equivalence

$$N : (\mathbf{sMod}_R)_{\mathrm{Kan}} \rightleftarrows (\mathbf{Ch}_R^{\geq 0})_{\mathrm{proj}} : \Gamma.$$

In general $N(X \otimes_R Y) \not\cong N(X) \otimes_R N(Y)$, however $N(X \otimes Y) \cong N(X) \otimes_R N(Y)$. They both describe $\mathcal{D}^{\geq 0}(R)$ in a monoidal way.

5. LECTURE 5: THURSDAY, JANUARY 26TH

For Dold-Kan $\mathbf{Ch}_{\geq 0} \cong \mathbf{sMod}_R$, we have

$$M \otimes N \rightleftarrows M \otimes R \otimes N \rightleftarrows M \otimes R^{\otimes 2} N \dots$$

we denote this by $B_\bullet(M, R, N)$ and call it the *bar construction*.

Homotopy colimits

Motivation: Limits and colimits are not invariant under (weak) homotopy equivalence.

$$\begin{array}{ccc} X & \hookrightarrow & CX \\ \downarrow & \lrcorner & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & * \end{array}$$

However $\Sigma X \not\cong *$.

Let \mathcal{M} be a model category, and \mathcal{C} a small category. Then we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{M}) = \mathcal{M}^{\mathcal{C}}$. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the discrete subcategory spanned by $\mathrm{ob}(\mathcal{C})$. Let $\mathcal{M}^{\mathcal{C}_0} = \prod_{\mathcal{C}_0} \mathcal{M}$. This has a model structure where W , Fib , and Cof are determined object-wise.

Consider $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$. This induces a map

$$\begin{aligned} \iota^* : \mathcal{M}^{\mathcal{C}} &\rightarrow \mathcal{M}^{\mathcal{C}_0} \\ F &\mapsto F|_{\mathcal{C}_0}. \end{aligned}$$

This admits adjoints:

$$\iota_! \dashv \iota^* \dashv \iota_*.$$

We have that ι^* creates W and Fib .

We have $(\mathcal{M}^{\mathcal{C}})_{\text{proj}}$:

- W = objectwise weak equivalence
- Fib = objectwise fib
- $\text{Cof} = ?$ induced by $\iota_! \text{Cof}$

We have that \mathcal{M} is cocomplete, so we get a tensoring

$$\begin{aligned} \mathcal{M} \times \mathbf{Set}^{\mathcal{C}} &\rightarrow \mathcal{M}^{\mathcal{C}} \\ (X, F) &\mapsto X \otimes F = \coprod_{F(-)} X. \end{aligned}$$

We have $(X \times F)(c) = \coprod_{F(c)} X$.

There are representable functors

$$\begin{aligned} \mathcal{C}(c, -) : \mathcal{C} &\rightarrow \mathbf{Set} \\ d &\mapsto \mathcal{C}(c, d). \end{aligned}$$

By Yoneda, there is a natural iso

$$\mathbf{Set}^{\mathcal{C}}(\mathcal{C}(c, -), F) \cong F(c).$$

Tensoring with a representable functor gives

$$X \otimes \mathcal{C}(c, -) = \coprod_{\mathcal{C}(c, -)} X.$$

This is the *free diagram of X generated at c* .

This gives an adjunction

$$- \otimes \mathcal{C}(c, -) : \mathcal{M} \rightleftarrows \mathcal{M}^{\mathcal{C}} : \text{ev}_c.$$

In this case

$$\iota_!(F) = \coprod_c \coprod_{\mathcal{C}(c, -)} F(c),$$

which is the free diagram in \mathcal{M} generated by F . Evaluating at d gives

$$\iota_!(F)(d) = \coprod_{c \in \mathcal{C}} \coprod_{\mathcal{C}(c, d)} F(c).$$

This is the functor $\iota_! : \mathcal{M}^{\mathcal{C}_0} \rightarrow \mathcal{M}^{\mathcal{C}}$. We see that $\iota_! X$ is a left Kan extension

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{X} & \mathcal{M} \\ \downarrow \iota & \nearrow & \\ \mathcal{C} & & \end{array}$$

There is a diagonal functor

$$\begin{aligned} \mathcal{M} &\xrightarrow{\Delta} \mathcal{M}^{\mathcal{C}} \\ C &\mapsto \text{constant functor at } X. \end{aligned}$$

This admits adjoints

$$\text{colim} \dashv \Delta \dashv \text{lim}.$$

Proposition 5.1. The adjunction

$$\text{colim} : (\mathcal{M}^{\mathcal{C}})_{\text{proj}} \rightleftarrows \mathcal{M} : \Delta$$

is Quillen.

We denote $\text{hocolim} := \mathbb{L}\text{colim}$. There is a map $\text{hocolim}(-) \rightarrow \text{colim}(-)$, and

$$\text{hocolim}(F) \simeq \text{colim}(QF).$$

Here QF denotes a cofibrant replacement in $(\mathcal{M}^{\mathcal{C}})_{\text{proj}}$. For a general \mathcal{C} , QF is very difficult to determine.

Consider $\mathcal{C} = a \leftarrow b \rightarrow c$, and let $X \in \mathcal{M}^{\mathcal{C}_0}$. Then $\iota_! X$ is equal to

$$\begin{array}{ccc} X(b) & \longrightarrow & X(b) \amalg X(c) \\ \downarrow & & \\ X(a) \amalg X(b) & & \end{array}$$

Cofibrant objects in $\mathcal{M}^{\mathcal{C}}$ are of the form

$$\begin{array}{ccc} X & \hookrightarrow & Z \\ \downarrow & & \\ Y & & \end{array}$$

with X cofibrant. Here cofibrant replacement is easy. We start with $Y \xleftarrow{f} X \xrightarrow{g} Z$, and we replace X with $\tilde{X} \xrightarrow{\sim} X$ to get

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & Y \\ \downarrow & & \\ & & Z \end{array}$$

If we cofibrantly replace $\tilde{X} \rightarrow Z$, and similarly for Y , we get

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{Z} \\ \downarrow & & \\ \tilde{Y} & & \end{array}$$

The maps we used to fibrantly replace induces a fiberwise weak equivalence between this diagram and the one we started out with.

In $(\mathbf{Top})_{\text{Quillen}}$, we can take $\text{hocolim}(* \leftarrow X \rightarrow *)$. We cofibrantly replace X if necessary, and replace $X \rightarrow *$ by $X \hookrightarrow CX$, which is a cofibration. In this case we see that

$$\text{hocolim}(* \leftarrow X \rightarrow *) \simeq \text{colim}(C\tilde{X} \leftarrow \tilde{X} \rightarrow C\tilde{X}) = \Sigma\tilde{X}.$$

More generally, $\text{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z)$ is the double mapping cylinder $M(f, g)$.

Theorem 5.2. If \mathcal{M} is a *left proper model category* then

$$\text{hocolim}(Y \hookleftarrow X \rightarrow Z) \cong \text{colim}(Y \hookleftarrow X \rightarrow Z).$$

Proof. In the easy case, X is cofibrant, so we can factor the map to Z to get

$$\begin{array}{ccccc} X & \hookrightarrow & \tilde{Z} & \xrightarrow{\sim} & Z \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & H & \dashrightarrow & P. \end{array}$$

The entire rectangle is a pushout, so $Z \rightarrow P$ is a cofibration, and the right square is a pushout by the pasting law, so $H \rightarrow P$ is a weak equivalence. \square

Example 5.3. Let $\mathcal{C} = * \rightarrow * \rightarrow \dots$. Show that $X_0 \rightarrow X_1 \rightarrow \dots$ is cofibrant in $\mathcal{M}^{\mathcal{C}}$ if and only if X_0 is cofibrant and $X_i \hookrightarrow X_{i+1}$ is a cofibration for each i .

There is a third model structure on $\mathcal{M}^{\mathcal{C}}$ called the *Reedy model structure* (need \mathcal{C} to be a Reedy cat). In this case, $\text{hocolim}_{\Delta^{\text{op}}}(X_{\bullet}) \cong |Q^{\text{Reedy}} X_{\bullet}|$, for X a simplicial object in \mathcal{M} .

Bar construction: Let \mathcal{M} a model cat, \mathcal{C} a small cat, $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{M}$, and $G : \mathcal{C} \rightarrow \mathcal{M}$. Then we define

$$B_{\bullet}(F, \mathcal{C}, G) := \coprod_{c_0 \in \mathcal{C}} F(c_0) \times G(c_0) \rightrightarrows \coprod_{c_0 \leftarrow c_1} F(c_0) \times G(c_1) \rightrightarrows \cdots$$

Example 5.4. If $F = * = G$, then

$$B_{\bullet}(*, \mathcal{C}, *) \cong N_{\bullet}(\mathcal{C}^{\text{op}}).$$

Pièce de résistance:

Theorem 5.5. (Bousfield–Kan) If $F : \mathcal{C} \rightarrow \mathcal{M}$ is a functor, then

$$\text{hocolim}_{\mathcal{C}}(F) \simeq |B_{\bullet}(*, \mathcal{C}, F)|.$$

6. LECTURE 6: TUESDAY, JANUARY 31ST

Combinatorial model categories

Definition 6.1. A model category is *combinatorial* if it is *presentable*⁷ and *cofibrantly generated*.

To motivate presentability, let X be a set. Then X is determined by its elements, meaning that

$$\text{Hom}_{\mathbf{Set}}(*, X) \cong X.$$

Then we can present X as $X = \cup_{x \in X} \{*\}$.

Definition 6.2. A colimit is *filtered* if the diagram is filtered, meaning it is nonempty and every subdiagram has a cocone.

Theorem 6.3. (Exercise) In \mathbf{Set} , filtered colimits commute with finite limits. That is, if $F : I \times J \rightarrow \mathbf{Set}$ with I finite and J filtered, then

$$\text{colim}_J \left(\lim_I F_I \right) \xrightarrow{\sim} \lim_I (\text{colim}_J F_J)$$

is an isomorphism.

Proposition 6.4. A set X is finite if and only if

$$\text{Hom}_{\mathbf{Set}}(X, -) : \mathbf{Set} \rightarrow \mathbf{Set}$$

preserves filtered colimits.

⁷By this we mean “locally presentable.”

Proof. For the backwards direction, let $I = \{X_i\}$ be the collection of finite subsets of X . Then $X = \operatorname{colim}_I X_i$. In particular, we have that

$$\begin{aligned} \operatorname{colim}_I \operatorname{Hom}(X, X_i) &\cong \operatorname{Hom}(X, X) \\ (X \xrightarrow{f_i} X_i) &\xrightarrow{\sim} \operatorname{id}_X? \end{aligned}$$

For the forwards direction, $\operatorname{Hom}_{\mathbf{Set}}(*, -) \cong \operatorname{id}_{\mathbf{Set}}$ so it preserves colimits. Since X is finite, we have that $X = \{x_1, \dots, x_n\}$, hence

$$\operatorname{Hom}(X, -) \cong \operatorname{Hom}(\cup_i \{x_i\}, -) \cong \lim_i \operatorname{Hom}(\{x_i\}, -).$$

Then we use finite limits commuting with filtered colimits. \square

Definition 6.5. An object $X \in \mathcal{C}$ is *compact* if $\operatorname{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves filtered colimits.

Hence if $F : I \rightarrow \mathcal{C}$, with I filtered, then a map $X \rightarrow \operatorname{colim}_I F$ factors through an $F(i)$.

Examples 6.6. Compact objects:

- \mathbf{Set} , compact = finite set
- \mathbf{Vect}_F , compact = finite dimensional
- \mathbf{Mod}_R , compact = finitely presented
- \mathbf{Grp} , compact = finitely presented
- \mathbf{Top} , compact = finite sets with discrete topology
- \mathbf{Ch} , compact = perfect chain complexes (bounded, levelwise finitely generated and projective)
- \mathbf{sSet} , compact = finite simplicial sets (X_n finite for each n , and there exists an m so that all non-degenerate simplices have dimension $\leq m$).

A topological space is (topologically) compact if and only if $X \in \mathcal{O}(X)$ is (categorically) compact.

Lemma 6.7. Finite colimits of compact objects are compact.

Definition 6.8. A category \mathcal{C} is *presentable* if

- (1) \mathcal{C} is cocomplete
- (2) There exists a set S of compact objects in \mathcal{C} such that every object in \mathcal{C} is a filtered colimit of objects in S .

We also say the “ind-completion” of S is \mathcal{C} , denoted $\operatorname{Ind}(S) = \mathcal{C}$.

Theorem 6.9. \mathcal{C} is presentable if and only if there is an adjunction of the form

$$\mathrm{Fun}(K^{\mathrm{op}}, \mathbf{Set}) \rightleftarrows \mathcal{C},$$

where K is some small category, and the right adjoint is fully faithful and preserves filtered colimits.

We might take K for example to be isomorphism classes of compact objects in \mathcal{C} , then we have

$$\begin{aligned} \mathcal{C} &\rightarrow \mathrm{Fun}(K^{\mathrm{op}}, \mathbf{Set}) \\ X &\mapsto \left(K^{\mathrm{op}} \rightarrow \mathcal{C}_{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-, X)} \mathbf{Set} \right). \end{aligned}$$

Theorem 6.10. Suppose \mathcal{C} and \mathcal{D} presentable. Then $L : \mathcal{C} \rightarrow \mathcal{D}$ preserves colimits if and only if L is a left adjoint.

Cofibrantly generated model categories

Definition 6.11. Let I be a set of maps in a cocomplete category, fix λ to be an ordinal, and let $X : \lambda \rightarrow \mathcal{C}$ a functor, and suppose that $X(\alpha) \rightarrow X(\alpha + 1)$ fits into

$$\begin{array}{ccc} A_\alpha & \longrightarrow & X(\alpha) \\ \downarrow & & \downarrow \\ B_\alpha & \longrightarrow & X(\alpha + 1), \end{array}$$

where $A_\alpha \rightarrow B_\alpha$ is in I . Then we say that $X(0) \rightarrow \mathrm{colim}_\lambda X$ is a *relative I -cell complex*. We say an object $Y \in \mathcal{C}$ is an *I -cell complex* if $\emptyset \rightarrow Y$ is a relative I -cell complex.

If $I = \{S^n \hookrightarrow D^{n+1}\}_{n \geq 0}$, then we are recovering the idea of CW complexes in spaces.

We denote by $\mathrm{Cell}_I(\mathcal{C})$ the class of relative I -cell complexes.

Exercise 6.12. We have that $\mathrm{Cell}_I(\mathcal{C})$ is the smallest class in \mathcal{C} closed under composition, pushouts, and filtered colimits.

Theorem 6.13. (*Small object argument*) Let \mathcal{C} be cocomplete, let I a set of maps in \mathcal{C} , and suppose that for all $A \rightarrow B$ in I , we have that A is compact with respect to the full subcategory of I -cells in \mathcal{C} . Then there exists a functorial factorization

of maps in \mathcal{C} :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \gamma & \nearrow \delta \\ & C & \end{array}$$

with $\gamma \in \text{Cell}_I(\mathcal{C})$ and $\delta \in \text{RLP}(I)$.

Proof idea. Start with $X(0) = X$, and take a map $X(0) \rightarrow Y$. Suppose $X(\beta) = \text{colim}_{\alpha < \beta} X(\alpha)$ is constructed with $X(\beta) \rightarrow Y$. Look at the set⁸

$$S = \left\{ \begin{array}{ccc} A & \longrightarrow & X(\beta) \\ g \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array} : g \in I \right\}.$$

Denote by g_s the map $A \rightarrow B$ appearing in $s \in S$. Then we build

$$\begin{array}{ccc} \coprod_{s \in S} A_s & \longrightarrow & X(\beta) \\ \coprod_{s \in S} g_s \downarrow & \lrcorner & \downarrow \in \text{Cell}_I(\mathcal{C}) \\ \coprod_{s \in S} B_s & \longrightarrow & X(\beta + 1) \end{array}$$

By UP of the pushout, there is an induced map $X(\beta + 1) \rightarrow Y$. Then we claim that

$$X(0) \rightarrow \text{colim}_{\beta} X(\beta) =: C$$

is in $\text{Cell}_I(\mathcal{C})$. The only thing left to show is that $C \rightarrow Y$ is in $\text{RLP}(I)$. Take

$$\begin{array}{ccc} A & \longrightarrow & C = \text{colim}_{\beta} X(\beta) \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y. \end{array}$$

Since A is compact with respect to I -cells, the map $A \rightarrow C$ factors through some $X(\beta)$. Since $B \rightarrow Y$ factors through $X(\beta + 1)$, we see that it lifts to $B \rightarrow C$. \square

Definition 6.14. A model category \mathcal{M} is *cofibrantly generated* if there exist sets of maps I, J in \mathcal{M} so that

- $\text{Cof} = \text{retracts of } I\text{-cell complexes, denoted } \widehat{\text{Cell}_I(\mathcal{C})}$ ⁹
- $\text{Cof} = \widehat{\text{Cell}_J(\mathcal{C})}$

⁸Note this set is nonempty because we can take g to be $\text{id} : X(\beta) \rightarrow X(\beta)$.

⁹The hat $\widehat{}$ means “retracts of -”

and “ I and J permit the small object argument.”

Example 6.15. For $\mathbf{Top}_{\text{Quillen}}$, we can take

$$\begin{aligned} I &= \{S^n \hookrightarrow D^{n+1}\} \\ J &= \{D^n \rightarrow D^n \times [0, 1]\}. \end{aligned}$$

Example 6.16. For $\mathbf{sSet}_{\text{Kan}}$, we can take

$$\begin{aligned} I &= \{\partial \Delta^n \rightarrow \Delta^n\} \\ J &= \{\Lambda_n^k \rightarrow \Delta^n\}. \end{aligned}$$

Example 6.17. For $(\mathbf{Ch}_R)_{\text{proj}}$,

$$\begin{aligned} I &= \{S^n \rightarrow D^{n+1}\} \\ J &= \{0 \rightarrow D^n\}. \end{aligned}$$

Example 6.18. The Strøm model structure is not cofibrantly generated in the definition above.

Theorem 6.19. (Kan — Right transfer) Let \mathcal{M} be a cofibrantly generated model category and \mathcal{C} is any category where there is an adjunction

$$F : \mathcal{M} \rightleftarrows \mathcal{C} : G.$$

Then \mathcal{C} has a model structure where W and Fib are created by G . The model structure is cofibrantly generated by $F(I)$ and $F(J)$ if:

- (1) $F(I)$ and $F(J)$ permit the small object argument
- (2) $G(\text{Cell}_{F(J)})$ are weak equivalences in \mathcal{M} .

For combinatorial model categories, we get an inductive argument for building cofibrant replacements.

[Rezk-Schwede-Shipley] Combinatorial model categories are always simplicially enriched.

[Dugger] Any combinatorial model category \mathcal{M} is Quillen equivalent to a localization of a projective Kan one:

$$L_\tau \text{Fun}(K^{\text{op}}, \mathbf{sSet}) \rightleftarrows \mathcal{M}.$$

7. LECTURE 7: THURSDAY, FEBRUARY 2ND

[missed]

8. LECTURE 8: TUESDAY, FEBRUARY 7TH

Last time: We had \mathcal{M} a model category, and \otimes a monoidal structure. We used this to give a monoidal structure on $\mathrm{Ho}(\mathcal{M})$, given by $\otimes^{\mathbb{L}}$, the *left derived tensor product*. We used this to give a homotopy theory on $\mathrm{Alg}(\mathcal{M})$, and $\mathrm{Mod}_R(\mathcal{M})$, etc.

Q: What are algebras in the homotopy category of a model structure \mathcal{M} ? An example of interest is $\mathcal{M} = \mathbf{Top}$.

What are commutative algebras in \mathbf{Top} ?

Theorem 8.1. (Moore) If $X \in \mathbf{CAlg}(\mathbf{Top})$, then there is a weak equivalence

$$\prod_{i=1}^{\infty} K(\pi_i(X), i) \rightarrow X.$$

Proof. Let $G_n = \pi_n(X)$. Then we take

$$0 \rightarrow F \rightarrow \mathbb{Z}[G_n] \rightarrow G_n \rightarrow 0.$$

Then we get that $\tilde{H}_n(\vee_{g \in G_n} S^n) \cong \oplus_{g \in G_n} \tilde{H}_n(S^n) = \mathbb{Z}[G_n]$. Using the Hurewicz theorem, there is an isomorphism

$$\pi_n(\vee S^n) \xrightarrow{\sim} \tilde{H}_n(\vee S^n),$$

so we can pick $f_j \in \pi_n(S^n)$ for each e_j in a basis of F . This gives us a pushout

$$\begin{array}{ccc} \vee_{j \in J} S^n & \longrightarrow & \vee_{g \in G_n} S^n \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & M(G_n, n) \end{array}$$

This gives a map $\vee_{n \geq 1} M(G_n, n) \rightarrow X$. By universal property, we get an algebra homomorphism¹⁰¹¹

$$\mathrm{SP}(\vee_{n \geq 1} M(G_n, n)) \rightarrow X$$

The Dold–Thom theorem states that $\pi_* \mathrm{SP}(Y) \cong \tilde{H}_*(Y)$, given some connectedness hypothesis (path-connected?). We get that

$$\mathrm{SP}(\vee_{n \geq 1} M(G_n, n)) \cong \prod_n \mathrm{SP}(M(G_n, n)) = \prod_n K(G_n, n).$$

□

¹⁰Here $\mathrm{SP}(-)$ denotes the infinite symmetric product, i.e. the free commutative algebra in \mathbf{Top} .

¹¹The infinite symmetric product is left adjoint to the forgetful functor, i.e. $\mathrm{SP} : \mathbf{Top} \rightleftarrows \mathbf{CAlg}(\mathbf{Top}) : U$.

Definition 8.2. We say that $X \in \mathbf{Alg}(\mathbf{Ho}(\mathbf{Top}))$ if and only if X is a CW complex, with multiplication and unit

$$\begin{aligned} X \times X &\rightarrow X \\ * &\rightarrow X \end{aligned}$$

which are associative and unital *up to homotopy*.

These are also called *H-spaces*. The most prototypical example is a loop space.

Example 8.3. If X is a based space, we can build ΩX as the homotopy pullback of the two maps from a point. Concatenation gives a map $\Omega X \times \Omega X \rightarrow \Omega X$.

Example 8.4. Eilenberg-MacLane spaces $K(G, n)$ are uniquely determined up to homotopy. We have that

$$\pi_k(\Omega K(G, n)) \cong \pi_{k+1}(K(G, n))$$

therefore $\Omega K(G, n) = K(G, n-1)$.

Q: Given X an *H-space*, such that $\pi_0 X$ is a group, is X a loop space?

A: No, there are many grouplike *H-spaces* that are not equivalent to ΩX . For example $S^7 \subseteq \mathbb{O}$ the unit octonians.

Loop spaces have an extra condition. Given $w, x, y, z \in \Omega X$, there is an association $(xy)z \simeq x(yz)$. There is a pentagon witnessing the different ways to associate four elements.

We can keep going with 5 loops, 6 loops... and we get the Stasheff associahedra $K(n)$, which tell us how to concatenate n loops. These give maps

$$K(n) \times (\Omega X)^n \rightarrow \Omega X,$$

witnessing the higher associativities of concatenation. We call this an A_∞ -algebra structure.

Theorem 8.5. (Stasheff) Given X connected, we have that $X \simeq \Omega Y$ for some Y if and only if X is an A_∞ -algebra in spaces that is grouplike.

Rigidification: We have that $\mathbf{Ho}(\mathbf{Alg}(\mathbf{sSet}, \times)) \simeq \mathbf{Alg}_{A_\infty}(\mathbf{Ho}(\mathbf{Top}))$.

Operads

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, [-, -])$ be a closed monoidal category.

Definition 8.6. An *operad* in \mathcal{C} is a collection of objects $\{\mathcal{O}(j)\}_{j \geq 0}$ in \mathcal{C} such that

- (1) there is a right action of Σ_j on $\mathcal{O}(j)$

- (2) $\mathcal{O}(0) = I$
- (3) $I \rightarrow \mathcal{O}(1)$ exists in \mathcal{C}
- (4) composition

$$\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \xrightarrow{\gamma} \mathcal{O}(j_1 + \cdots + j_k)$$

for all $k \geq 0$ and $j_1, \dots, j_k \geq 0$ such that they are equivariant, unital, and associative.

We think about $\mathcal{O}(j)$ as an abstract way to compose j -ary operations.

Example 8.7. We let Assoc be the operad defined by

$$\text{Assoc}(j) = \coprod_{\sigma \in \Sigma_j} I.$$

We can define $\text{Comm}(j) = I$.

Example 8.8. If $X \in \mathcal{C}$, the *endomorphism operad* is given by

$$\text{End}_X(j) = [X^{\otimes j}, X].$$

Definition 8.9. A *morphism of operads* $\mathcal{O} \rightarrow \mathcal{O}'$ is a sequence of maps $\psi_j : \mathcal{O}(j) \rightarrow \mathcal{O}'(j)$ for $j \geq 0$ that are equivariant, associative, and unital.

Definition 8.10. Given \mathcal{O} an operad in \mathcal{C} , an \mathcal{O} -algebra (X, θ) in \mathcal{C} is $X \in \mathcal{C}$ together with a morphism of operads $\theta : \mathcal{O} \rightarrow \text{End}_X$, sending $\mathcal{O}(j) \rightarrow \text{End}_X(j)$. By adjointness, we think about this as $\mathcal{O}(j) \otimes X^{\otimes j} \rightarrow X$ which are associative and unital.

This gives us a category of \mathcal{O} -algebras, denoted $\text{Alg}_{\mathcal{O}}(\mathcal{C})$.

Example 8.11. We have that

$$\begin{aligned} \text{Alg}_{\text{Assoc}}(\mathcal{C}) &\cong \text{Alg}(\mathcal{C}) \\ \text{Alg}_{\text{Comm}}(\mathcal{C}) &\cong \text{CAlg}(\mathcal{C}). \end{aligned}$$

We have that \mathcal{M} is a monoidal model category if θ is nice enough, i.e. we get an adjunction

$$\mathcal{M} \rightleftarrows \text{Alg}_{\mathcal{O}}(\mathcal{M}).$$

Definition 8.12. A *monad* in \mathcal{C} is an algebra in $(\text{Fun}(\mathcal{C}, \mathcal{C}), \circ, \text{id}_{\mathcal{C}})$. That is, $M \in \text{Alg}(\text{Fun}(\mathcal{C}, \mathcal{C}))$ if we have $M : \mathcal{C} \rightarrow \mathcal{C}$ together with $\mu : M \circ M \Rightarrow M$, and $\eta : \text{id}_{\mathcal{C}} \Rightarrow M$ that are associative and unital.

Example 8.13. Every adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ defines a monad RL .

Definition 8.14. An *algebra* (X, θ) over a monad (M, μ, η) in \mathcal{C} is $X \in \mathcal{C}$ together with maps $\theta : M(X) \rightarrow X$ such that they are associative and unital, meaning that the diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & M(X) \\ & \searrow & \downarrow \theta \\ & & X \end{array} \quad \begin{array}{ccc} M(M(X)) & \xrightarrow{\mu_{MX}} & M(X) \\ M(\theta) \downarrow & & \downarrow \theta \\ M(X) & \xrightarrow{\theta} & X. \end{array}$$

Definition 8.15. If M is a monad, a *morphism of M -algebras* $(X, \theta) \rightarrow (X', \theta')$ is a map $f : X \rightarrow X'$ in \mathcal{C} so that the diagram commutes

$$\begin{array}{ccc} MX & \xrightarrow{\theta} & X \\ Mf \downarrow & & \downarrow f \\ MX' & \xrightarrow{\theta'} & X'. \end{array}$$

Example 8.16. Consider R a commutative ring, and the adjunction

$$- \otimes_{\mathbb{Z}} R : \mathbf{Ab} \rightleftarrows \mathbf{Mod}_R : U.$$

This forms a monad $M := - \otimes_{\mathbb{Z}} R : \mathbf{Ab} \rightarrow \mathbf{Ab}$. Then $\mathbf{Alg}_M(\mathbf{Ab})$ is equivalent to \mathbf{Mod}_R .

This is not always true! When this happens we say the adjunction is *monadic*.

Given a monadic adjunction

$$\mathcal{C} \rightleftarrows \mathcal{D} = \mathbf{Alg}_{RL}(\mathcal{C}),$$

we get a ton of things for free:

- R will preserve colimits if RL does
- get things like free monadic resolutions, bar constructions, etc.

9. LECTURE 9: THURSDAY, FEBRUARY 9TH

[missed]

10. LECTURE 10: THURSDAY, FEBRUARY 16TH

Definition 10.1. A simplicial set \mathcal{C} is an ∞ -category (or *quasi-category*) if it has inner horn filling — for all $0 < k < n$, we have

$$\begin{array}{ccc} \Lambda_n^k & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array}$$

We shall see that ∞ -categories are fibrant objects in \mathbf{sSet} with the Joyal model structure.

Example 10.2.

- (1) If \mathcal{C} is a Kan complex, then it is an ∞ -category
- (2) If \mathcal{C} is a category, then $N\mathcal{C}$ is an ∞ -category.

Definition 10.3. Given an ∞ -category \mathcal{C} , the *objects* of \mathcal{C} are the vertices,¹² the *morphisms* are 1-simplices. We have *source* and *target* maps $d^1, d^0 : \mathcal{C}_1 \rightarrow \mathcal{C}_0$.¹³ We define the *set of morphisms* from X to Y as the pullback

$$\begin{array}{ccc} \mathrm{hom}_{\mathcal{C}}(X, Y) & \longrightarrow & \mathcal{C}_1 \\ \downarrow & \lrcorner & \downarrow (s, t) \\ \mathcal{C}_1 & \xrightarrow{(X, Y)} & \mathcal{C}_0 \times \mathcal{C}_0. \end{array}$$

We have that $\mathrm{hom}_{\mathcal{C}}(X, Y)$ is the set of vertices of a simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, which forms a Kan complex.

Definition 10.4. Given $X \in \mathcal{C}$ we define $\mathrm{id}_X \in \mathcal{C}_1$ by $s^0(X)$.

How do we compose? Composition won't be unique, but it will be unique *up to homotopy*.

Given $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , this determines a map of simplicial sets $\Lambda_1^2 \rightarrow \mathcal{C}$. By inner horn lifting, we have

$$\begin{array}{ccc} \Lambda_1^2 & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^2 & & \end{array}$$

¹² $X \in \mathcal{C}$ means $X \in \mathcal{C}_0$

¹³We write $f : X \rightarrow Y$ in \mathcal{C} to mean $f \in \mathcal{C}_1$ with $s(f) = X$ and $t(f) = Y$.

We refer to the filling as a *composition*:

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z. \end{array}$$

Exercise 10.5. Given an ∞ -category \mathcal{C} , how can we define \mathcal{C}^{op} ? Would want that $N(\mathcal{C}^{\text{op}}) \cong (N\mathcal{C})^{\text{op}}$.¹⁴

Detour: Let $A \in \mathbf{Cat}$, and let \mathcal{C} be a cocomplete category. Recall that $\text{Fun}(A^{\text{op}}, \mathbf{Set})$ is the free cocompletion. Given a functor $A \rightarrow \mathcal{C}$, by universal property there is a map

$$\begin{array}{ccc} A & \xrightarrow{Q} & \mathcal{C} \\ \downarrow & \dashrightarrow & \uparrow \\ \text{Fun}(A^{\text{op}}, \mathbf{Set}) & & \end{array} \quad \begin{array}{c} \\ \\ | - |_Q \end{array}$$

This gives us an adjunction

$$| - |_Q : \text{Fun}(A^{\text{op}}, \mathbf{Set}) \rightleftarrows \mathcal{C} : \text{Sing}_Q(-).$$

Here $\text{Sing}_Q(-) = \text{Hom}_{\mathcal{C}}(Q(-), X)$.

Example 10.6. If $\mathcal{C} = \mathbf{Top}$, then we can take $\Delta_{\mathbf{Top}} : \Delta \rightarrow \mathbf{Top}$, sending $[n]$ to $\Delta_{\mathbf{Top}}^n$. In this case, we recover the usual $| - |$ and $\text{Sing}(-)$ adjunction.

Example 10.7. If $\mathcal{C} = \mathbf{Cat}$, there is a functor $\Delta \rightarrow \mathbf{Cat}$ sending $[n]$ to the associated poset category. We get an associated adjunction:

$$\tau : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : N,$$

since $N = \text{Hom}_{\mathbf{Cat}}([-], \mathcal{C})$.

Exercise 10.8. Describe $\tau : \mathbf{sSet} \rightarrow \mathbf{Cat}$ explicitly.

We call τ the fundamental category functor, essentially it will produce the homotopy category of an ∞ -category.

Definition 10.9. Given an ∞ -category \mathcal{C} , two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are *homotopic*, written $f \simeq g$, if there exists a 2-simplex $\sigma : \Delta^2 \rightarrow \mathcal{C}$ with

¹⁴Every Kan complex has that $\mathcal{C}^{\text{op}} \cong \mathcal{C}$.

boundary (g, f, id_X) :

$$\begin{array}{ccc} & X & \\ \text{id}_X \nearrow & & \searrow g \\ X & \xrightarrow{f} & Y. \end{array}$$

Example 10.10. If \mathcal{C} is an ordinary category, then in $N\mathcal{C}$, we have that $f \simeq g$ if and only if $f = g$.

Proposition 10.11. Given \mathcal{C} an ∞ -category, and $X, Y \in \mathcal{C}$, the homotopy relation provides an equivalence relation on $\text{hom}_{\mathcal{C}}(X, Y)$.

Definition 10.12. We denote by $[f]$ the homotopy class of f .

Sketch. We first need to show reflexivity, so we want to find a 2-cell witnessing

$$\begin{array}{ccc} & X & \\ \parallel \nearrow & & \searrow f \\ X & \xrightarrow{f} & Y. \end{array}$$

We check that this is $s_0(f)$, where $f \in \mathcal{C}_1$, and $s_0 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$.

For symmetry, suppose we have $f \simeq g$. We want to show $g \simeq f$. We can fill a Λ_2^3 witnessing this.

Transitivity is left as an exercise. □

Definition 10.13. Given \mathcal{C} an ∞ -category, define the 1-category $\text{Ho}(\mathcal{C})$ to be the *homotopy category*, given by

$$\begin{aligned} \text{obHo}(\mathcal{C}) &= \mathcal{C}_0 \\ \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y) &= \text{hom}_{\mathcal{C}}(X, Y) / \simeq. \end{aligned}$$

In order to show this, we need to argue that composition is well-defined up to homotopy.

Suppose we have two compositions

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h_1} & Z. \end{array} \quad \begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h_2} & Z. \end{array}$$

We want to argue that $h_1 \simeq h_2$. This can be done by filling the horn of a 3-simplex.

Proposition 10.14. When we restrict the adjunction $\tau \dashv N$ to ∞ -categories, we get an adjunction

$$\mathrm{Ho}(-) : \mathbf{Cat}_\infty \rightleftarrows \mathbf{Cat} : N.$$

The way to compose arrows is contractible.

Theorem 10.15. The inclusion $\Lambda_1^2 \hookrightarrow \Delta^2$ induces a map

$$\mathrm{Hom}_*(\Delta^2, \mathcal{C}) \rightarrow \mathrm{Hom}_*(\Lambda_1^2, \mathcal{C})$$

which is a trivial Kan fibration for any $\mathcal{C} \in \mathbf{Cat}_\infty$.

Here Hom_* is the *internal hom*, where $\mathrm{Hom}_*(X, Y) := \mathrm{Hom}_{\mathbf{sSet}}(\Delta^* \times X, Y)$.

Proof. Exercise □

As a consequence, we can take a pullback diagram:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Hom}_*(\Delta^2, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \longrightarrow & \mathrm{Hom}_*(\Lambda_1^2, \mathcal{C}). \end{array}$$

Then the pullback $P \rightarrow \Delta^0$ should be a trivial fibration, meaning that P is a contractible Kan complex.

Definition 10.16. Given \mathcal{C} an ∞ -category and $X, Y \in \mathcal{C}$, recall that a map $f : X \rightarrow Y$ corresponds to $\Delta^1 \rightarrow \mathcal{C}$ whose faces are X and Y . An n -morphism from X to Y is simply a map $\Delta^n \rightarrow \mathcal{C}$ such that $\Delta^{\{0, \dots, n-1\}} = X$ and $\Delta^{\{n\}} = Y$.

For $n \geq 2$, all n -morphisms are invertible in some sense.

Definition 10.17. Two objects X and Y in \mathcal{C} are *equivalent*, written $X \simeq Y$, if there exists a 1-morphism $f : X \rightarrow Y$ in \mathcal{C} such that $[f]$ in $\mathrm{Ho}(\mathcal{C})$ is an *isomorphism*.

Definition 10.18. An ∞ -groupoid is an ∞ -category for which $\mathrm{Ho}(\mathcal{C})$ is a groupoid, meaning all the 1-morphisms are equivalences.

Theorem 10.19. (Homotopy hypothesis) We get that \mathcal{C} is an ∞ -groupoid if and only if \mathcal{C} is a Kan complex.

11. LECTURE 11: TUESDAY, FEBRUARY 21ST

[missed]

12. LECTURE 12: THURSDAY, FEBRUARY 23RD

Adjoint functors and colimits

Last time: Recall that a 1-morphism in $\text{Fun}(\mathcal{C}, \mathcal{D})$ ¹⁵ is precisely a natural transformation $\eta : F \rightarrow G$, where $F, G : \mathcal{C} \rightarrow \mathcal{D}$. In other words, it is $\eta : \Delta^1 \times \mathcal{C} \rightarrow \mathcal{D}$.

We have $\mathbf{hQC}at = \text{Ho}(\text{Cat}_\infty)$, where objects are infinity categories, and the morphisms are

$$\text{Hom}_{\mathbf{hQC}at}(\mathcal{C}, \mathcal{D}) = \Pi_0(\text{Fun}(\mathcal{C}, \mathcal{D})^\simeq).$$

That is, it is the set of equivalence classes of functors $\mathcal{C} \rightarrow \mathcal{D}$.

If \mathcal{C} is an ∞ -category, and $X, Y \in \mathcal{C}$, we defined $\text{Hom}_{\mathcal{C}}(X, Y)_\bullet$ to be the simplicial set given by the pullback

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, Y)_\bullet & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \longrightarrow & \text{Fun}(\{0\}, \mathcal{C})_\bullet \times \text{Fun}(\{1\}, \mathcal{C}). \end{array}$$

Proposition 12.1. We have that $\text{Hom}_{\mathcal{C}}(X, Y) \in \text{Kan}$.

Sketch. This follows from a more general fact that for $A \hookrightarrow B$ a subsimplicial set with $A_0 = B_0$, and \mathcal{C} an ∞ -category, then P is always a Kan complex

$$\begin{array}{ccc} P & \longrightarrow & \text{Fun}(B, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{f} & \text{Fun}(A, \mathcal{C}). \end{array}$$

Need to show that every u in $\text{Fun}(B, \mathcal{C})_1$ in the pullback is a weak equivalence. We have an evaluation map for every $b \in B_0 = A_0$, given by $\text{ev}_b : \text{Fun}(B, \mathcal{C}) \rightarrow \text{Fun}(\{b\}, \mathcal{C})$, mapping u to $u_{f(b)}$. We claim that $u_{f(b)} = \text{id}_{f(b)}$, since the diagram commutes

$$\begin{array}{ccc} \text{Fun}(B, \mathcal{C}) & \longrightarrow & \text{Fun}(\{b\}, \mathcal{C}) \\ & \searrow & \nearrow \\ & \text{Fun}(A, \mathcal{C}) & \end{array}$$

□

Adjoint functors

¹⁵The simplicial set $\text{Fun}(\Delta^\bullet \times \mathcal{C}, \mathcal{D})$

Definition 12.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$, and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors of ∞ -categories. We say that $F \dashv G$ if there exist natural transformations $\eta : \text{id}_{\mathcal{C}} \rightarrow GF$ and $\varepsilon : FG \rightarrow \text{id}_{\mathcal{D}}$ so that:

- (1) there exists $\Delta^2 \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$ witnessing

$$\begin{array}{ccc} & FGF & \\ \text{id}_{\mathcal{C}}\eta \nearrow & & \searrow \varepsilon \text{id}_F \\ F\text{id}_{\mathcal{C}} & \xrightarrow{\text{id}} & \text{id}_{\mathcal{C}}F. \end{array}$$

- (2) there exists $\Delta^2 \rightarrow \text{Fun}(\mathcal{D}, \mathcal{C})$ witnessing

$$\begin{array}{ccc} & GFG & \\ \eta \text{id} \nearrow & & \searrow \text{id}_{\mathcal{D}}\varepsilon \\ \text{id}_{\mathcal{C}}G & \xrightarrow{\text{id}} & G\text{id}_{\mathcal{C}}. \end{array}$$

Remark 12.3. We have that $\eta : \text{id} \rightarrow GF$ depends only on $[\eta]$ in $\text{Ho}(\text{Fun}(\mathcal{C}, \mathcal{D}))$. If η is given, then ε is unique up to homotopy.

Example 12.4. If \mathcal{C} and \mathcal{D} are ordinary categories, then we have a 1-categorical adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

if and only if we have an ∞ -categorical adjunction

$$NF : N\mathcal{C} \rightleftarrows N\mathcal{D} : NG.$$

Example 12.5. If $X, Y \in \mathbf{Kan}$, then $F : X \rightarrow Y$ is an adjoint if and only if F is a homotopy equivalence of simplicial sets. The unit and counit become the witnesses of homotopy equivalence.

Remark 12.6. If we have an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ of ∞ -categories, then F and G are homotopy equivalences of simplicial sets. The converse is not true in general.

Exercise 12.7. If $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of ∞ -categories, then it is both a left and right adjoint functor.

Proposition 12.8. Given $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ of ∞ -categories, then

$$\text{Ho}(F) : \text{Ho}(\mathcal{C}) \rightleftarrows \text{Ho}(\mathcal{D}) : \text{Ho}(G)$$

is an adjunction of 1-categories. That is, **if** we know $F \dashv G$ in ∞ -categories, then to check if $\eta : \text{id}_{\mathcal{C}} \rightarrow GF$ is a unit, it is enough to check that $\text{Ho}(\eta)$ is the unit.

However the converse is not true!

Warning: Suppose we take $F : \Delta^0 \rightarrow X$ with $X \in \mathbf{Kan}$ simply connected, and F picks $x \in X_0$. Then $\mathrm{Ho}(F) \dashv \mathrm{Ho}(G)$ because $\mathrm{Ho}(X)$ will be simply connected. But it does not imply that $F \dashv G$ unless X is contractible.

There $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FC, D) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(C, GD)$ for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

Theorem 12.9. Take $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ functors of ∞ -categories. Then $F \dashv G$ with unit η if and only if the composite

$$\mathrm{Hom}_{\mathcal{D}}(FC, D) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(GFC, GD) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathcal{C}}(C, GD)$$

is a weak homotopy equivalence between Kan complexes (aka a homotopy equivalence) for all C, D .

The forward direction is straightforward, but the backwards direction uses (co)cartesian fibration stuff.

Limits and colimits

Recall that if \mathcal{C} is an ordinary category, then $i \in \mathcal{C}$ is *initial* if for all $X \in \mathcal{C}$, there is a unique $i \xrightarrow{!} X$. That is, $\mathrm{Hom}_{\mathcal{C}}(i, X) = *$.

Definition 12.10. In an ∞ -category \mathcal{C} , we have that $i \in \mathcal{C}$ is *initial* if $\mathrm{Hom}_{\mathcal{C}}(i, X) \simeq *$ is contractible for all $X \in \mathcal{C}$.

Definition 12.11. Let \mathcal{C} be an ∞ -category, and $K_{\bullet} \in \mathbf{sSet}$. Then for any $X \in \mathcal{C}$, denote by $\underline{X} \in \mathrm{Fun}(K, \mathcal{C})$ the constant functor valued at X . The assignment $X \mapsto \underline{X}$ defines a diagonal map

$$\Delta : \mathcal{C} \rightarrow \mathrm{Fun}(K, \mathcal{C}).$$

This is defined by precomposing with $K \rightarrow \Delta^0$, and looking at $\mathcal{C} \simeq \mathrm{Fun}(\Delta^0, \mathcal{C}) \rightarrow \mathrm{Fun}(K, \mathcal{C})$.

Definition 12.12. Let $u : K \rightarrow \mathcal{C}$ be a diagram. We say a natural transformation $\alpha : \underline{L} \rightarrow u$ exhibits $L \in \mathcal{C}$ as a *limit of u* if for all $X \in \mathcal{C}$, we have that the composite

$$\mathrm{Hom}_{\mathcal{C}}(X, L) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{C})}(\underline{X}, \underline{L}) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{C})}(\underline{X}, u)$$

is a (weak) homotopy equivalence of Kan complexes.

Definition 12.13. We say that $\beta : u \rightarrow \underline{C}$ exhibits C as a *colimit of u* if, for all $Y \in \mathcal{C}$, the composite

$$\mathrm{Hom}_{\mathcal{C}}(C, Y) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{C})}(\underline{C}, \underline{Y}) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{C})}(u, \underline{C})$$

is a (weak) homotopy equivalence.

Note that if α or β exist, they are unique up to equivalence.

Example 12.14. If \mathcal{C} is an ordinary category, then $u : K \rightarrow N\mathcal{C}$ is equivalent to a map $\tau(u) : \tau K \rightarrow \mathcal{C}$. We can check that $L \in \mathcal{C}$ is $\lim(\tau u)$ in a 1-categorical sense if and only if $L \in \mathcal{C}$ is a limit of u in an ∞ -categorical sense.

Example 12.15. Let $f : X \rightarrow Y$ in an ∞ -cat \mathcal{C} . Then f is an equivalence if and only if f exhibits Y as a colimit $\{X\} \rightarrow \mathcal{C}$, if and only if f exhibits X as a limit $\{Y\} \rightarrow \mathcal{C}$.

Example 12.16. Taking the identity diagram $\emptyset \rightarrow \mathcal{C}$, the notion of limit/colimit matches the notion of terminal/initial object.

Proposition 12.17. A limit $L \in \mathcal{C}$ is unique up to homotopy. Therefore we usually define it as $\lim_K(u)$.

Proposition 12.18. We have that \mathcal{C} admits all K -indexed limits if and only if

$$\Delta : \mathcal{C} \rightarrow \text{Fun}(K, \mathcal{C})$$

is a left adjoint. The right adjoint is given by $\lim_K(-)$.

Equalizers are limits along $\Delta^1 \amalg_{\partial \Delta^1} \Delta^1$, pullbacks are limits along $\Delta^1 \times \Delta^1 - (0, 0)$, etc.

13. LECTURE 13: FEBRUARY 28TH

[missed]

14. LECTURE 14: MARCH 21ST

Straightening/unstraightening¹⁶

Motivation: Let X be a space, and let $\text{Cov}(X)$ denote the 1-category of covering spaces of X , so that in particular the fibers f^{-1} of $f : E \rightarrow X$ are discrete sets. This defines a map in **Top** from

$$X \rightarrow \mathbf{Set}^{\cong},$$

to sets with the discrete topology. Another way to think about this is as a functor

$$\text{St} : \text{Cov}(X) \rightarrow \text{Fun}(\Pi_1(X), \mathbf{Set})$$

$$(E \xrightarrow{p} X) \mapsto [x \mapsto f^{-1}(x)].$$

¹⁶Also called the *Grothendieck construction* or the *∞ -category of elements*.

A path from x to y (a morphism in $\Pi_1(X)$) induces a set map $f^{-1}(x) \rightarrow f^{-1}(y)$.

This is an equivalence of categories! This is called the *fundamental theorem of covering spaces*.

This is a first instance of *straightening*.

If we view X as an ∞ -groupoid, then $\Pi_1(X) = \mathrm{Ho}(X)$ is its homotopy category, and we have that

$$\mathrm{Fun}(\Pi_1(X), \mathbf{Set}) \cong \mathrm{Fun}(X, N(\mathbf{Set})),$$

since nerve is right adjoint to the homotopy category.

We can denote by $\mathrm{Cov}_X \subseteq \mathcal{S}/X$ to be the full subcategory of the infinity category of spaces over X spanned by covering spaces. Then we want to show that

$$\mathrm{Cov}_X \simeq \mathrm{Fun}(X, N(\mathbf{Set})).$$

We have an unstraightening functor

$$\mathrm{Unst} : \mathrm{Fun}(X, N(\mathbf{Set})) \rightarrow \mathrm{Cov}_X,$$

given by sending some $F : X \rightarrow N(\mathbf{Set})$ to the pullback¹⁷

$$\begin{array}{ccc} E & \longrightarrow & N(\mathbf{Set}_*)^{\simeq} \\ \downarrow & & \downarrow \\ X & \xrightarrow{F} & N(\mathbf{Set})^{\simeq} \end{array}$$

More generally, if we don't require the fibers to be discrete, then we can take $f : E \rightarrow X$ to be any continuous map. Then we get a functor¹⁸

$$\begin{aligned} \mathrm{St} : \mathcal{S}/X &\rightarrow \mathrm{Fun}(X, \mathcal{S}) \\ (E \xrightarrow{f} X) &\mapsto [x \mapsto f^{-1}(x)]. \end{aligned}$$

Unstraightening is of the form

$$\begin{aligned} \mathrm{Unst} : \mathrm{Fun}(X, \mathcal{S}) &\rightarrow \mathcal{S}/X \\ F &\mapsto \mathrm{hocolim}_X F = \cup_{x \in X} F^{-1}(x) / \sim. \end{aligned}$$

Let X be connected and suppose $X \simeq BG$. Then we define *G-modules in spaces* to be

$$\mathrm{Mod}_G(\mathcal{S}) := \mathrm{Fun}(BG, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}/BG.$$

¹⁷Note that $N(\mathbf{Set}^{\simeq}) = N(\mathbf{Set})^{\simeq}$.

¹⁸By $\mathrm{Fun}(X, \mathcal{S})$ we might mean $\mathrm{Fun}(\mathrm{Sing}(X), N_{\Delta}(\mathbf{Kan}))$.

If we take some $M : BG \rightarrow \mathcal{S}$, and we post-compose with sections $\mathcal{S}/BG \rightarrow \mathcal{S}$, then M maps to M^{hG} .

More generally, given $F : X \rightarrow \mathcal{S}$, the limit $\lim_X \mathcal{S}$ is given by

$$\mathrm{Fun}(X, \mathcal{S}) \xrightarrow{\mathrm{Unst}} \mathcal{S}/X \xrightarrow{\mathrm{sections}} \mathcal{S}.$$

Goal: Generalize this approach where X is replaced by an ∞ -category \mathcal{C} and \mathcal{S} is replaced by \mathbf{Cat}_∞ . That is, we want to relate $\mathrm{Fun}(\mathcal{C}, \mathbf{Cat}_\infty)$ with some subcategory of $\mathbf{Cat}_\infty/\mathcal{C}$.

If $f : \mathcal{E} \rightarrow \mathcal{C}$, what requirement do we need to make sense of an associated functor

$$\begin{aligned} F : \mathcal{C} &\rightarrow \mathbf{Cat}_\infty \\ X &\mapsto f^{-1}(X). \end{aligned}$$

That is, how can we coherently choose our fibers.

Given $X \in \mathcal{C}$, we could take a pullback in \mathbf{Cat}_∞ :

$$\begin{array}{ccc} f^{-1}(X) & \longrightarrow & \mathcal{E} \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{X} & \mathcal{C}. \end{array}$$

If we choose $\mathbf{sSet}_{\mathrm{Joyal}}$ as our model, we would need $\mathcal{E} \rightarrow \mathcal{C}$ to be an inner fibration (RLP wrto inner horns) to get the pullback $f^{-1}(X)$ to be a quasi-category. If we instead say “pullback in quasi-categories,” this requirement goes away.

Given $f : \mathcal{E} \rightarrow \mathcal{C}$ and $X \rightarrow Y$ in \mathcal{C} , how can we define $f^{-1}(X) \rightarrow f^{-1}(Y)$ in \mathbf{Cat}_∞ ?

Need: If $\phi : X \rightarrow Y$ in \mathcal{C} and $E_X \in \mathcal{E}$ such that $f(E_X) = X$, then there exists some $E_Y \in \mathcal{E}$ and $\phi_! : E_X \rightarrow E_Y$ in \mathcal{E} so that $f(\phi_!) = \phi$, and that is universal in the following sense: for all $Z \in \mathcal{C}$ and for all $\psi : X \rightarrow Z$ in \mathcal{C} for all $\bar{\psi} : E_X \rightarrow E_Z$ in \mathcal{E} where $f(\bar{\psi}) = \psi$, if there exists $\gamma : Y \rightarrow Z$ then there exists a unique map $\bar{\gamma} : E_Y \rightarrow E_Z$ in \mathcal{E} so that $f(\bar{\gamma}) = \gamma$ and $\bar{\gamma} \circ \phi_! = \bar{\psi}$.

We say that $\phi_! : E_X \rightarrow E_Y$ is a *cocartesian lift* of ϕ .

Definition 14.1. We say that $f : \mathcal{E} \rightarrow \mathcal{C}$ is a *cocartesian fibration* if for all $E_X \in \mathcal{E}$, for all $\phi : X \rightarrow Y$ with $f(E_X) = X$, there exists a cocartesian lift of ϕ .

Two cocartesian lifts over the same map are equivalent.

Given $f : \mathcal{E} \rightarrow \mathcal{C}$, $X \in \mathcal{C}$, $\phi : X \rightarrow Y$ in \mathcal{C} , we say $\phi_! : E_X \rightarrow E_Y$ is a cocartesian lift if the following is a pullback diagram in spaces:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{E}}(E_Y, E_Z) & \xrightarrow{(\phi_!)^*} & \mathrm{Hom}_{\mathcal{C}}(E_X, E_Z) \\ f \downarrow & \lrcorner & \downarrow f \\ \mathrm{Hom}_{\mathcal{C}}(Y, Z) & \xrightarrow{\phi^*} & \mathrm{Hom}_{\mathcal{C}}(X, Z), \end{array}$$

for any $Z \in \mathcal{C}$. In particular, taking maps from Δ^0 to the top right and bottom left picks out $\bar{\psi}$ and γ , respectively, so that $\gamma \circ \phi = \psi$, and the universal property of the pullback says that there exists $\bar{\gamma} : E_Y \rightarrow E_Z$ so that $\bar{\gamma}\phi_! = \bar{\psi}$ and $f(\bar{\gamma}) = \gamma$.

Definition 14.2. We define $\mathrm{coCart}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty}/\mathcal{C}$ to be the subcategory of cocartesian fibrations $\mathcal{E} \rightarrow \mathcal{C}$, with morphisms

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{G} & \mathcal{E}' \\ & \searrow f & \swarrow f' \\ & \mathcal{C}, & \end{array}$$

so that G sends f -cocartesian lifts to f' -cocartesian lifts.

In this case, straightening defines a functor

$$\mathrm{St} : \mathrm{coCart}(\mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_{\infty}),$$

sending $f : \mathcal{E} \rightarrow \mathcal{C}$ to the functor

$$\begin{aligned} \mathcal{C} &\rightarrow \mathrm{Cat}_{\infty} \\ X &\mapsto f^{-1}(X) \\ (X \xrightarrow{\phi} Y) &\mapsto \left[f^{-1}(X) \xrightarrow{\phi_!} f^{-1}(Y) \right]. \end{aligned}$$

Example 14.3. Let $f : X \rightarrow Y$ in \mathcal{S} . All lifts are cocartesian lifts. We say that a *left fibration* is a cocartesian fibration where every lift is cocartesian.

Example 14.4. Suppose \mathcal{C} is an ordinary category. Then we can define a new category whose objects are $f : X \rightarrow Y$ in \mathcal{C} , and whose morphisms are

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ u \uparrow & & \downarrow v \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

This defines what we call the *twisted arrow category* $\mathrm{Tw}(\mathcal{C})$. There is a natural functor

$$\begin{aligned} \mathrm{Tw}(\mathcal{C}) &\xrightarrow{\mathrm{Ev}} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \\ (X \xrightarrow{f} Y) &\mapsto (X, Y). \end{aligned}$$

This is a left fibration, by composition. Straightening this, we get

$$\begin{aligned} \mathrm{St}(\mathrm{Ev}) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\rightarrow \mathbf{Set} \\ (X, Y) &\mapsto \mathrm{Ev}^{-1}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y). \end{aligned}$$

Example 14.5. If \mathcal{C} is an ∞ -category, we can define a twisted arrow category in a similar way

$$\begin{aligned} \mathrm{Tw}(\mathcal{C}) : \Delta^{\mathrm{op}} &\rightarrow \mathbf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathbf{sSet}}(\Delta^{2n+1}, \mathcal{C}), \end{aligned}$$

where the n -simplices of $\mathrm{Tw}(\mathcal{C})$ should be thought of as

$$\begin{array}{ccccccc} X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots & \longleftarrow & X_n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n. \end{array}$$

We can define

$$\begin{aligned} \ell : \mathrm{Tw}(\mathcal{C}) &\rightarrow \mathcal{C}^{\mathrm{op}} \\ r : \mathrm{Tw}(\mathcal{C}) &\rightarrow \mathcal{C}, \end{aligned}$$

by precomposition with $\Delta^n \hookrightarrow \Delta^{2n+1}$. These assemble to give

$$\mathrm{Tw}(\mathcal{C}) \xrightarrow{\mathrm{Ev}} \mathcal{C}^{\mathrm{op}} \times \mathcal{C},$$

and we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Ev}^{-1}(X, Y) \in \mathcal{S}$. This evaluation map is a left fibration, left fibrations are preserved under pullback, and left fibrations over Δ^0 are Kan complexes. Therefore $\mathrm{Ev}^{-1}(X)$ is a space.

Example 14.6. Let $X \in \mathcal{C}$. Then we can take

$$\begin{array}{ccc} \ell^{-1}(X) & \longrightarrow & \mathrm{Tw}(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \ell \\ \Delta^0 & \xrightarrow{X} & \mathcal{C}^{\mathrm{op}}. \end{array}$$

We define $\mathcal{C}_{X/} := \ell^{-1}(X)$, and $r^{-1}(Y) := \mathcal{C}_{/Y}$.

Theorem 14.7. (*Straightening/unstraightening*) If \mathcal{C} is an ∞ -category, we can define its *unstraightening* as

$$\begin{aligned} \text{Unst} : \text{Fun}(\mathcal{C}, \mathbf{Cat}_\infty) &\rightarrow \text{coCart}(\mathcal{C}) \\ F &\mapsto \text{colim} \left(\text{Tw}(\mathcal{C}) \xrightarrow{\text{Ev}} \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}/ \cdot \times F} \mathbf{Cat}_\infty \right). \end{aligned}$$

That composite sends

$$\begin{aligned} \text{Tw}(\mathcal{C}) &\xrightarrow{\text{Ev}} \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}/ \cdot \times F} \mathbf{Cat}_\infty \\ (X \xrightarrow{f} Y) &\mapsto \mathcal{C}_{X/} \times F(Y). \end{aligned}$$

This forms an equivalence with St .

There is an equivalence

$$\text{St} : \text{LFib}(\mathcal{C}) \rightleftarrows \text{Fun}(\mathcal{C}, \mathcal{S}) : \text{Unst}.$$

If $\mathcal{C} = X \in \mathcal{S}$, then $\text{coCart}(X) = \mathbf{Cat}_\infty/X$.

If $\mathcal{C} = N(\mathcal{D})$, this recovers the usual Grothendieck construction.

If $F : \mathcal{C} \rightarrow \mathbf{Cat}_\infty$, then

$$\text{colim} F = \text{Unst}(\mathcal{C})[\text{cocart. edges}^{-1}]$$

15. LECTURE 15: MARCH 23RD

Unstraightening monoidal structures

Recall $\mathcal{S} \simeq N(\mathbf{sSet})[W_{\text{Kan}}^{-1}]$ the ∞ -category of spaces. If $X \rightarrow Y$ is a map in \mathcal{S} we are meaning that $X \rightarrow Y$ is a map in $\text{Ho}(\mathbf{sSet})$ *not* that $X \rightarrow Y$ is any map in \mathbf{sSet} .

Example 15.1. If we have $X \rightarrow Y$ in \mathcal{S} , then $X \rightarrow Y$ is a left fibration. If X and Y are in \mathbf{Kan} and $X \rightarrow Y$ this *does not imply* that $X \rightarrow Y$ must be a left fibration. What is true is that if $X \rightarrow Y$ is a Kan fibration, then $X \rightarrow Y$ is a left fibration.

We have $\mathbf{Cat}_\infty \simeq N(\mathbf{sSet})[W_{\text{Joyal}}^{-1}]$, so $f : \mathcal{C} \rightarrow \mathcal{D}$ in \mathbf{Cat}_∞ means

$$\begin{array}{ccc} f^{-1}(X) & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{X} & \mathcal{D}. \end{array}$$

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So we always want it to be a fibration.

That is, a map $f : \mathcal{C} \rightarrow \mathcal{D}$ in \mathbf{Cat}_∞ is not the same as $\mathcal{C} \rightarrow \mathcal{D}$ of quasi-categories in \mathbf{sSet} .

In \mathbf{Cat}_∞ , $\mathcal{C} \rightarrow \mathcal{D}$ is a cocartesian fibration if there exists a cocartesian lift on any fiber.

If \mathcal{C}, \mathcal{D} are quasi-categories in $\mathbf{sSet}_{\text{Joyal}}$, then $f : \mathcal{C} \rightarrow \mathcal{D}$ is a cocartesian fibration if f is an *inner fibration* (RLP inner horns) AND there is a cocartesian lift of any fiber. The inner fibration condition guarantees that the fibers are also infinity categories.

Straightening definition last time was wrong. Last time, we had

$$\begin{aligned} \text{Unst} : \text{Fun}(\mathcal{C}, \mathbf{Cat}_\infty) &\xrightarrow{\sim} \text{coCart}(\mathcal{C}) \\ F &\mapsto \left(\mathcal{E} \xrightarrow{\text{Unst}(F)} \mathcal{C} \right). \end{aligned}$$

is an equivalence of categories, where

$$\mathcal{E} = \text{colim} \left(\text{Tw}(\mathcal{C})^{\text{op}} \rightarrow \mathcal{C} \times \mathcal{C}^{\text{op}} \xrightarrow{F \times \mathcal{C}_{\bullet}/} \mathbf{Cat}_\infty \right).$$

Example 15.2. Take $\mathcal{C} = *$. Then $\text{Fun}(*, \mathbf{Cat}_\infty) = \mathbf{Cat}_\infty$. We have that $\text{coCart}(*) = \mathbf{Cat}_\infty$, and that $\text{Tw}(*) = *^{\text{op}} = *$. The composite sends

$$\begin{aligned} \text{Tw}(*)^{\text{op}} &\rightarrow * \times *^{\text{op}} \rightarrow \mathbf{Cat}_\infty \\ * &\mapsto (*, *) \mapsto *A \times * = A. \end{aligned}$$

Example 15.3. Take $\mathcal{C} = 1 = 0 \rightarrow 1$. A functor $F : 1 \rightarrow \mathbf{Cat}_\infty$ is exactly a functor $F : \mathcal{A} \rightarrow \mathcal{D}$ in \mathbf{Cat}_∞ . We see that $\text{Tw}(1)$ has three objects, being $0 = 0$, $0 \rightarrow 1$ and $1 = 1$. The identity ones both map to $0 \rightarrow 1$ so it is a span-op category. When we op $\text{Tw}(1)^{\text{op}}$ we get the span category, so a colimit becomes a pushout. We see that

$1_{0/} = 1$ and $1_{1/} = *$. Then

$$\begin{aligned} \mathcal{E} &= \operatorname{colim} \left(\begin{array}{ccc} \mathcal{A} \times 1_{1/} & \xrightarrow{\operatorname{id} \times (0 \rightarrow 1)} & \mathcal{A} \times 1_{0/} \\ F \times \operatorname{id} \downarrow & & \\ \mathcal{B} \times 1_{1/} & & \end{array} \right) \\ &= \operatorname{colim} \left(\begin{array}{ccc} \mathcal{A} & \xrightarrow{\operatorname{id} \times 1} & \mathcal{A} \times 1 \\ \downarrow & & \\ \mathcal{B} & & \end{array} \right) \end{aligned}$$

Then \mathcal{E} is a cocartesian fibration over 1, whose fiber over 0 is \mathcal{A} , whose fiber over 1 is \mathcal{B} , and with maps $F(A) \rightarrow B$ over $0 \rightarrow 1$.

Goal: Redefine a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ as a cocartesian fibration $\mathcal{C}^{\otimes} \rightarrow \mathbf{Fin}_*$ as certain “pseudo”functors $\mathbf{Fin}_* \rightarrow \mathbf{Cat}$. We could take $\mathbf{Fin}_* \rightarrow \mathbf{Cat}$ sending $\langle n \rangle$ to $\mathcal{C}^{\times n}$.

Q: Given a psuedofunctor $F : \mathbf{Fin}_* \rightarrow \mathbf{Cat}$, when is it defining a symmetric monoidal category?

We would need $F(\langle n \rangle) \cong F(\langle 1 \rangle)^{\times n}$ with Segal’s condition $F(\langle 0 \rangle) = 0$.

Theorem 15.4. Symmetric monoidal categories are pseudofunctors $\mathbf{Fin}_* \rightarrow \mathbf{Cat}$ with the Segal condition.

16. LECTURE 16: MARCH 28TH

Monoidal functors and algebra objects

Last time we defined a symmetric monoidal infinity category to be a cocartesian fibration over \mathbf{Fin}_* with a Segal condition. Here $\mathcal{C} = f^{-1}(\langle 1 \rangle)$. We got this by straightening $N(\mathbf{Fin}_*) \rightarrow \mathbf{Cat}_{\infty}$, with $\langle n \rangle \mapsto \mathcal{C}^{\otimes n}$.

Suppose we had a natural transformation η between functors

$$\mathcal{C}, \mathcal{D} : N(\mathbf{Fin}_*) \rightarrow \mathbf{Cat}_{\infty}.$$

This corresponds to a map $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ over \mathbf{Fin}_* sending p -cocartesian lifts to q -cocartesian lifts:

$$\begin{array}{ccc} \mathcal{C}^\otimes & \xrightarrow{\quad} & \mathcal{D}^\otimes \\ & \searrow p \quad \swarrow q & \\ & N(\mathbf{Fin}_*) & \end{array}$$

Think about this as $F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$.

Now suppose we have $F^\otimes : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ between symmetric monoidal ∞ -categories. Then we know the fiber over $\langle 1 \rangle$ must be sent to the fiber over $\langle 1 \rangle$. Then we get $F_{\langle n \rangle}^\otimes : \mathcal{C}_{\langle n \rangle}^\otimes \rightarrow \mathcal{D}_{\langle n \rangle}^\otimes$ for all n .

Denote $F = F_{\langle 1 \rangle}^\otimes$. Then $F_{\langle n \rangle}^\otimes \simeq F^{\times n}$.

Let $\rho_i^i : \langle n \rangle \rightarrow \langle 1 \rangle$ send everything to 0 except i to 1.

$$\begin{array}{ccc} \mathcal{C}_{\langle 2 \rangle}^\otimes & \xrightarrow{F_{\langle 2 \rangle}^\otimes} & \mathcal{D}_{\langle 2 \rangle}^\otimes \\ (\rho_1^1, \rho_1^2) \downarrow & & \downarrow (\rho_1^1, \rho_1^2) \\ \mathcal{C} \times \mathcal{C} & \xrightarrow{F \times F} & \mathcal{D} \times \mathcal{D}. \end{array}$$

$F(\rho_i^1) \simeq \rho_i^1$ and $F(\rho_i^2) \simeq \rho_i^2$. For all i we need that $F(\rho_i^i)$ is a q -cocartesian lift of ρ_i^i . This means that for all n , $F_{\langle n \rangle}^\otimes(X_1, \dots, X_n) \simeq (F(X_1), \dots, F(X_n))$.

Definition 16.1. A map $\alpha : \langle n \rangle \rightarrow \langle k \rangle$ in \mathbf{Fin}_* is *inert* if $\alpha^{-1}(i)$ is precisely a singleton for $1 \leq i \leq n$.

Fact 16.2. Inert morphisms are generated by ρ^i and τ (here τ is the swap of 1 and 2 on $\langle 2 \rangle$).

Let $F^\otimes : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ that sends p -cocartesian lifts of inert maps to q -cocartesian lifts. We claim this already gives a lax monoidal structure. Consider $m : \langle 2 \rangle \rightarrow \langle 1 \rangle$ the multiplication, and consider $(X, Y) \in \mathcal{C}^{\times 2}$. There is a map $m_! : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ sending $(X, Y) \mapsto X \otimes Y$.

$$\begin{array}{ccc} & F(X) \otimes F(Y) & \\ m_! \nearrow & & \searrow \\ (F(X), F(Y)) & \xrightarrow{F(m_!)} & F(X \otimes Y). \end{array}$$

Note we're not saying that $F(m_l)$ is a cocartesian lift, we're saying that m_l is. If $F(m_l)$ was a cocartesian lift, then this would give $F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ is an equivalence.

Exercise 16.3. Show that $\iota : \langle 0 \rangle \rightarrow \langle 1 \rangle$ induces $I_{\mathcal{D}} \rightarrow F(I_{\mathcal{C}})$.

Definition 16.4. For \mathcal{C}^{\otimes} and \mathcal{D}^{\otimes} symmetric monoidal ∞ -categories, a *lax symmetric monoidal functor* $F^{\otimes} : \mathcal{C}^{\otimes} \rightarrow \mathcal{D}^{\otimes}$ is a functor that sends lifts of p -cocartesian inert maps in \mathbf{Fin}_* to q -cocartesian lifts.

Definition 16.5. We say F^{\otimes} is strong symmetric monoidal if it sends *all* p -cocartesian lifts to q -cocartesian lifts.

We can define

$$\begin{array}{ccc} \mathrm{Fun}_{N(\mathbf{Fin}_*)}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \\ \downarrow & \lrcorner & \downarrow q^* \\ \Delta^0 & \xrightarrow{p} & \mathrm{Fun}(\mathcal{C}^{\otimes}, \mathbf{Fin}_*). \end{array}$$

Define $\mathrm{Fun}^{\otimes, \mathrm{lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ to be the full subcategory of lax monoidal functors, and just $\mathrm{Fun}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ the full subcategory of strong monoidal functors.

Example 16.6. Commutative algebras. We have that Δ^0 is a symmetric monoidal ∞ -category with trivial structure, then we have

$$N(\mathbf{Fin}_*) \rightarrow \mathbf{Cat}_{\infty}$$

sending everything to Δ^0 . The associated cocartesian fibration is $N(\mathbf{Fin}_*) \rightarrow N(\mathbf{Fin}_*)$.

We define $\mathbf{Alg}_{\infty}(\mathcal{C})$ to be $\mathrm{Fun}^{\otimes, \mathrm{lax}}(N(\mathbf{Fin}_*), \mathcal{C})$. That is,

$$\begin{array}{ccc} N(\mathbf{Fin}_*) & \xrightarrow{A^{\otimes}} & \mathcal{C}^{\otimes} \\ & \searrow p & \swarrow \\ & N(\mathbf{Fin}_*) & \end{array}$$

That is, A^{\otimes} is a section of p that sends inert maps in \mathbf{Fin}_* to p -cocartesian lifts. We have that $A^{\otimes}(\langle 1 \rangle) \in \mathcal{C}_{\langle 1 \rangle}^{\otimes} = \mathcal{C}$, and $A \otimes A \rightarrow A$. We have that $A^{\otimes}(\langle 0 \rangle) = I$.

Q: Can we localize a symmetric monoidal category in such a way that it preserves the symmetric monoidal structure?

Definition 16.7. (HA 4.1.7.4) Given \mathcal{C}^{\otimes} a symmetric monoidal ∞ -category, let $W \subseteq \mathcal{C}$ a collection of edges. Assume W is closed under \otimes (meaning that if $Y \rightarrow Y'$ is in W , and X is arbitrary, then $X \otimes Y \rightarrow X \otimes Y'$ and $Y \otimes X \rightarrow Y' \otimes X$ are in W as

well). The *symmetric monoidal localization* of \mathcal{C}^\otimes with W is a symmetric monoidal ∞ -category $\mathcal{C}[W^{-1}]^\otimes$ together with a strong symmetric monoidal functor

$$\ell : \mathcal{C}^\otimes \rightarrow \mathcal{C}[W^{-1}]^\otimes$$

with the following universal property: for any symmetric monoidal ∞ -category \mathcal{D}^\otimes , we get an equivalence of ∞ -categories:

$$\mathrm{Fun}^\otimes(\mathcal{C}[W^{-1}]^\otimes, \mathcal{D}^\otimes) \xrightarrow{\sim} \mathrm{Fun}_W^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes),$$

where $\mathrm{Fun}_W(-)$ means sending W to equivalences.

This always exists. We have that $\mathcal{C}[W^{-1}]_{(1)} \simeq \mathcal{C}[W^{-1}]$. In terms of cocartesian fibrations it is maybe(?) some kind of Kan extension

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad} & \mathbf{Fin}_* \\ \downarrow & \nearrow ? & \\ \mathcal{C}[W^{-1}] & & \end{array}$$

Definition 16.8. Let (M, \otimes, I) be a symm mon model category (with functorial cofibrant replacement). Suppose I is cofibrant. Then the Dwyer-Kan localization $N(M)[W^{-1}]$ can be given a symmetric monoidal ∞ -structure as follows:

- Take the cofibrant objects M_c
- Take the category of operators M_c^\otimes as an ordinary category (objects are pairs $\langle n \rangle, c_1, \dots, c_n$) and morphisms are $\otimes_i c_i \rightarrow c'_j$ over \mathbf{Fin}_*
- $N(M_c^\otimes)$ is a symmetric monoidal ∞ -category, with class W of edges in $N(M_c)$
- Recall that $X \otimes - : M_c \rightarrow M_c$ preserves weak equivalences between cofibrant objects, under the hypothesis that X is cofibrant.
- Thus $N(M_c^\otimes) \rightarrow N(M_c)[W^{-1}]^\otimes$ is called the *symmetric monoidal Dwyer-Kan localization*.

This gives a sym mon structure on the ∞ -category $N(M)[W^{-1}] \simeq N(M_c)[W^{-1}]$.

This shows that the derived tensor product \otimes of a monoidal model category M endows $N(M)[W^{-1}]$ with a monoidal structure.

Example 16.9. Spaces \mathcal{S} have a symmetric monoidal ∞ -category structure, since we can view them as $N(\mathbf{sSet})[W_{\mathrm{Kan}}^{-1}]$ with the cartesian product. Here $\mathrm{Alg}_{E_\infty}(\mathcal{S})$ are equivalent to E_∞ -algebras in spaces.

Example 16.10. We have that $\mathbf{Cat}_\infty \simeq N(\mathbf{sSet})[W_{\mathrm{Joyal}}^{-1}]$ with the cartesian product. Then $\mathrm{Alg}_{E_\infty}(\mathbf{Cat}_\infty)$ are symmetric monoidal ∞ -categories. This is exactly because $\mathrm{Alg}_{E_\infty}(\mathbf{Cat}_\infty) = \mathrm{Fun}^{\otimes, \mathrm{lax}}(N(\mathbf{Fin}_*), \mathbf{Cat}_\infty)$ which guarantees the Segal condition.

Example 16.11. If R is a commutative ring, then $D(R) \simeq \mathbf{Ch}_R[W_{\text{proj}}^{-1}]$ is a symmetric monoidal ∞ -category. The injective model structure does not give you a monoidal model category.

We also have the connective case with two models

$$D^{\geq 0}(R) \simeq N(\mathbf{sMod}_R)[W^{-1}] \simeq N(\mathbf{Ch}_R^{\geq 0})[W^{-1}].$$

Every symmetric monoidal ∞ -category \mathcal{C}^{\otimes} which is presentable and for which \otimes preserves colimits is the symmetric monoidal DK localization of a combinatorial monoidal model category (Lurie-Sagave).

17. LECTURE 17: MARCH 30TH

Stable ∞ -categories

Universal property for \mathcal{S} (spaces). Given $K \in \mathbf{sSet}$, there is a Yoneda embedding

$$K \hookrightarrow \text{Fun}(K^{\text{op}}, \mathcal{S}) =: \mathcal{P}(K),$$

which is the adjoint of “internal hom”¹⁹

$$K^{\text{op}} \times K \rightarrow \mathcal{S}.$$

Given \mathcal{C} an ∞ -category, we can call $\mathcal{P}(\mathcal{C})$ the *universal cocompletion* of \mathcal{C} . That is, for all \mathcal{D} cocomplete, there is an equivalence

$$\text{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}(\mathcal{C}, \mathcal{D}),$$

where L denotes colimit-preserving functors.²⁰

If we choose $\mathcal{C} = \Delta^0$, we get

$$\text{Fun}^L(\mathcal{S}, \mathcal{D}) = \text{Fun}(\Delta^0, \mathcal{D}) = \mathcal{D}.$$

Hence we can think of \mathcal{S} as the “free cocompletion of Δ^0 .” Just as a set can be viewed as a union of its points, we can think of any cocomplete ∞ -category as gluing its paths together.

Definition 17.1. An ∞ -category is *pointed* if it has an object which is both initial and terminal. That is, some $0 \in \mathcal{C}$ so that

$$\text{Hom}_{\mathcal{C}}(0, X) \simeq * \simeq \text{Hom}_{\mathcal{C}}(X, 0)$$

¹⁹ K isn’t necessarily an ∞ -category, so it doesn’t make sense to have internal hom, but this is the straightening of $\text{Tw}(K) \rightarrow K^{\text{op}} \times K$ which is always well-defined.

²⁰For presentable ∞ -categories, being a left adjoint is equivalent to preserving colimits, hence the superscript “ L ”

for any $X \in \mathcal{C}$.

Example 17.2. If \mathcal{C} is an ∞ -category and $*$ $\in \mathcal{C}$ is a terminal object, we can define

$$\mathcal{C}_* := \mathcal{C}_{*/}.$$

This will be pointed and we will have an adjunction

$$(-)_+ : \mathcal{C} \rightleftarrows \mathcal{C}_*.$$

For example, we have

$$\mathcal{S} \rightleftarrows \mathcal{S}_* = N(\mathbf{sSet}_*)[W_{\text{Kan}}^{-1}].$$

If \mathcal{C} is a pointed presentable stable ∞ -category, then

$$\text{Fun}^L(\mathcal{S}_*, \mathcal{C}) \simeq \mathcal{C}.$$

Here \mathcal{S}_* is the free presentable pointed ∞ -category generated by $*_+ = S^0$.

Now we introduce stable ∞ -categories, which behave like $D(R) \simeq N(\mathbf{Ch}_R)[W_{\text{qiso}}^{-1}]$.

Definition 17.3. Let \mathcal{C} be a pointed ∞ -category. A *triangle* in \mathcal{C} is a square of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z. \end{array}$$

This is specified by a functor $N(\Delta^1 \times \Delta^1) \rightarrow \mathcal{C}$ sending the bottom corner to 0.

We say a triangle is *exact* if it is a pullback, and *coexact* if it is a pushout.

Example 17.4. If $f : E \rightarrow X$ in \mathcal{S}_* , then an exact triangle looks like

$$\begin{array}{ccc} f^{-1}(x) & \longrightarrow & E \\ \downarrow \lrcorner & & \downarrow f \\ * & \longrightarrow & X. \end{array}$$

Example 17.5. We have loops and suspension in \mathcal{S}_* given by the (homotopy) pull-back and pushout squares

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow \lrcorner & & \downarrow \\ * & \longrightarrow & X \end{array} \quad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

Our goal is to define $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ and $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ for a general pointed ∞ -category.

Definition 17.6. For \mathcal{C} finitely bicomplete, we define $\mathcal{C}^\Sigma \subseteq \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ to be the full subcategory spanned by diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

Note that maps between such diagrams are the same as maps $X \rightarrow Y$. Thus there is an equivalence

$$\mathcal{C}^\Sigma \xrightarrow{\sim} \mathcal{C},$$

and similarly $\mathcal{C}^\Omega \xrightarrow{\sim} \mathcal{C}$.

We have

$$\begin{array}{ccc} \Gamma & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{C}^\Sigma) \\ \sim \downarrow & \lrcorner & \downarrow \simeq \\ * & \xrightarrow{\text{id}} & \text{Fun}(\mathcal{C}, \mathcal{C}). \end{array}$$

Thus there is a unique section $s_\Sigma : \mathcal{C} \rightarrow \mathcal{C}^\Sigma$. So now we can define $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ to be

$$\Sigma : \mathcal{C} \xrightarrow{s_\Sigma} \mathcal{C}^\Sigma \xrightarrow{\sim} \mathcal{C}.$$

Analogously we can define Ω .

Theorem 17.7. If \mathcal{C} is a pointed and finitely bicomplete category, we have an adjunction

$$\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega.$$

In particular, for $X, Y \in \mathcal{C}$ we have

$$\text{Hom}_{\mathcal{C}}(\Sigma X, Y) \simeq \Omega \text{Hom}_{\mathcal{C}}(X, Y).$$

This is because maps from $\Sigma X \rightarrow Y$ are in bijection with

$$\begin{array}{ccc} \Omega \text{Hom}(X, Y) & \longrightarrow & \text{Hom}(0, Y) \\ \downarrow & & \downarrow \\ \text{Hom}(0, Y) & \longrightarrow & \text{Hom}(\Sigma X, Y). \end{array}$$

This tells us that

$$\pi_0 \text{Hom}_{\mathcal{C}}(\Sigma X, Y) = \pi_1 \text{Hom}_{\mathcal{C}}(X, Y),$$

which is a group. Similarly we get that $\pi_0 \text{Hom}(\Sigma^2 X, Y)$ is an abelian group.

Definition 17.8. Given $f : X \rightarrow Y$ in \mathcal{C} , we can define the *fiber* and *cofiber* as

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & Y \end{array} \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathrm{cof}(f) \end{array}$$

Definition 17.9. An ∞ -category is *stable* if it is

- pointed
- finitely bicomplete
- triangles are exact if and only if they are coexact.

This last condition is equivalent to any of the following

- a square is a pullback iff it is a pushout
- $\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega$ is an equivalence
- $\mathrm{cof} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}) : \mathrm{cof}$ is an equivalence.

Let \mathcal{C} be a stable ∞ -category. Then

$$\pi_0 \mathrm{Hom}(X, Y) \cong \pi_0(\mathrm{Hom}(\Sigma X', Y)) \cong \pi_0 \mathrm{Hom}(\Sigma^2 X'', Y)$$

for some X, X'' . Thus $\mathrm{Ho}(\mathcal{C})$ is an additive category.

We furthermore have that $\mathrm{Ho}(\mathcal{C})$ is triangulated. Given $f : X \rightarrow Y$ in \mathcal{C} ,

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ f \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & \mathrm{cof}(f) \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

Example 17.10. $\mathcal{C} = D(R)$. Show this has all the properties mentioned above.

Stabilization/linearization

Given \mathcal{C} pointed, we want it to be stable. We can force $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ to be an equivalence by considering

$$Sp(\mathcal{C}) := \lim \left(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right).$$

Historically, we tried to invert Σ (Freudenthal theorem).

We could take Sp^{naive} , whose objects are finite pointed spaces, and morphisms are stable maps $[X, Y]$. The problem is that Σ is not an equivalence on this category.

We could instead take Sp^{Wh} , where objects are pairs (X, n) with X a pointed finite CW complex, and

$$\text{Hom}((X, n), (Y, m)) := \text{colim}_k [\Sigma^{n+k} X, \Sigma^{m+k} Y].$$

Then we have

$$\begin{aligned} Sp^{\text{naive}} &\hookrightarrow Sp^{\text{Wh}} \\ X &\mapsto (X, 0). \end{aligned}$$

The suspension takes the form

$$\begin{aligned} \Sigma : Sp^{\text{Wh}} &\rightarrow Sp^{\text{Wh}} \\ (X, n) &\mapsto (X, n+1). \end{aligned}$$

Thus

$$Sp^{\text{Wh}} = \text{colim} \left(\mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \dots \right).$$

and we have that

$$\begin{aligned} Sp(\mathcal{S}_*) &= \text{Ind}(Sp^{\text{Wh}}) \\ &= \text{Indcolim} \left(\mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \dots \right) \\ &= \lim \left(\text{Ind}(\mathcal{S}_*^{\text{fin}}) \xleftarrow{\Omega} \text{Ind}(\mathcal{S}_*^{\text{fin}}) \xleftarrow{\Omega} \dots \right) \\ &= \lim \left(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \dots \right). \end{aligned}$$

Note that $\text{colim} \left(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \dots \right)$ won't work.

Definition 17.11. If \mathcal{C} is a pointed finitely bicomplete ∞ -category, a *prespectrum* in \mathcal{C} is defined to be a functor

$$N(\mathbb{Z} \times \mathbb{Z}) \rightarrow \mathcal{C},$$

where $X_{i,j} = 0$ for $i \neq j$. Note that we get induced structure maps $\alpha_n : \Sigma X_n \rightarrow X_{n+1}$ and $\beta_n : X_n \rightarrow \Omega X_{n+1}$.

A prespectrum is called a *spectrum* in \mathcal{C} if β_n 's are equivalences for all n . We define $Sp(\mathcal{C})$ to be the full subcategory of spectra.

Let

$$Sp(\mathcal{C}) \simeq \lim \left(\mathcal{C} \xleftarrow{\Omega} \mathcal{C} \xleftarrow{\Omega} \dots \right).$$

If $\mathcal{C} = \mathcal{S}_*$, we will write $Sp = Sp(\mathcal{S}_*)$ as the ∞ -category of spectra. We define $\mathrm{Ho}(Sp)$ to be the stable homotopy category.

There is a functor

$$\tilde{\Sigma}^\infty : \mathcal{C} \rightarrow \mathrm{PSp}(\mathcal{C}),$$

given by sending X to the prespectrum whose (i, i) th entry is $\Sigma^i X$.

Then there is a functor for \mathcal{C} presentable

$$\mathrm{PSp}(\mathcal{C}) \rightarrow Sp(\mathcal{C})$$

sending a prespectrum X to \tilde{X} , defined by

$$\tilde{X}_n := \mathrm{colim}(X_n \xrightarrow{\beta_n} \Omega X_{n+1} \rightarrow \cdots).$$

Then $\tilde{X}_n \simeq \mathrm{colim}_k \Omega^k X_{n+k} \simeq \mathrm{colim}_k \Omega^{k+1} X_{n+k+1}$. As Ω is a right adjoint it commutes with filtered colimits (using presentable here), so this can be rewritten as

$$\Omega \mathrm{colim}_k \Omega^k X_{n+k+1} \simeq \Omega \tilde{X}_{n+1}.$$

18. LECTURE 18: TUESDAY, APRIL 4TH

Symmetric monoidal structure on spectra

Last time we had a universal property for $\mathcal{C} \xrightarrow{\Sigma^\infty} Sp(\mathcal{C})$, where \mathcal{C} was a pointed presentable ∞ -category. We had that

$$\mathrm{Fun}^L(Sp(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

for any stable presentable ∞ -category \mathcal{D} .

We denote by $\Sigma^\infty S^0 =: \mathbb{S} \in Sp = Sp(\mathcal{S}_*)$, and recall that

$$\mathrm{Fun}^L(Sp, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{S}_*, \mathcal{D}) \simeq \mathcal{D}.$$

So we call Sp the free stable ∞ -category generated by ∞ .

Q: Can we give a symmetric monoidal structure on Sp analogous to $\otimes_{\mathbb{Z}}$ in \mathbf{Ab} ?

Spanier-Whitehead category: Recall Freudenthal says that if X and Y are finite CW complexes, then the sequence $[\Sigma^k X, \Sigma^k Y]$ stabilizes in k . So Sp^{naive} has objects given by finite CW complexes, and homs given by stable maps.

To invert Σ , we introduced Sp^{Wh} , where objects are (X, n) and homs $(X, n) \rightarrow (Y, m)$ are

$$\mathrm{colim}_k [\Sigma^{n+k} X, \Sigma^{m+k} Y].$$

Formally in ∞ -categories, we have that

$$Sp^{\text{Wh}} = \text{colim} \left(Sp_* \xrightarrow{\Sigma} Sp_* \xrightarrow{\Sigma} \dots \right).$$

Then $Sp \simeq \text{Ind}(Sp^{\text{Wh}})$.

Why finiteness? By adjunction we can see

$$\begin{aligned} \text{Hom}((X, 0), (Y, 0)) &= \text{colim}_k [\Sigma^k X, \Sigma^k Y] \\ &= \text{colim}_k [X, \Omega^k \Sigma^k Y] \\ &= [X, \text{colim}_k \Omega^k \Sigma^k Y], \end{aligned}$$

which holds if X is compact (e.g. finite CW). Thus if $\{-, -\}$ is a hom for spectra, we would have

$$\{X, Y\} = \{X, \Omega^n \Sigma^n Y\}.$$

What is the monoidal structure on Sp^{Wh} ? Recall in \mathcal{S}_* we have a smash product, so we could define

$$(X, n) \wedge (Y, m) := (X \wedge Y, n + m).$$

The unit is $(S^0, 0)$. This smash product is difficult to translate to spectra however.

Definition 18.1. For all $X \in Sp$, we define

$$\pi_n(X) = \text{Hom}_{\text{Ho}(Sp)}(\Sigma^n \mathbb{S}, X) =: [\Sigma^n \mathbb{S}, X] \in \mathbf{Ab}.$$

In particular if X is a suspension spectrum, we get

$$\begin{aligned} \pi_k(\Sigma^\infty X) &= [\Sigma^n \mathbb{S}, \Sigma^\infty X] \\ &= \text{colim}_k [\Sigma^{n+k} S^0, \Sigma^k X] \\ &= \text{colim}_k \pi_{n+k}(X) \\ &= \pi_n^s(X). \end{aligned}$$

This is the *stable homotopy group* of X . It gives us a functor

$$\begin{aligned} Sp &\rightarrow N(\mathbf{Ab}) \\ X &\mapsto \pi_n(X). \end{aligned}$$

This factors through

$$Sp \xrightarrow{\Omega^\infty} \mathcal{S}_* \xrightarrow{\pi_n} N(\mathbf{Ab})$$

for $n \geq 2$.

(HA 1.4.3.8) The collection of these functors reflect equivalences. That is, if $\pi_n(X) \xrightarrow{\sim} \pi_n(Y)$ for all n , then $X \xrightarrow{\sim} Y$ in Sp .

Definition 18.2. We define $Sp^{\geq 0}$ to be the ∞ -category of *connective spectra*, the full subcategory of Sp on those X for which $\pi_n(X) = 0$ for $n < 0$.

Example 18.3. For all $X \in \mathcal{S}_*$, we have that $\Sigma^\infty X \in Sp^{\geq 0}$.

We get an adjunction

$$Sp^{\geq 0} \rightleftarrows Sp : \tau_{\geq 0},$$

where the right adjoint to the inclusion is the *connective cover*.

If $X \in \mathcal{S}_*$ and $Y \in Sp^{\geq 0}$, we have that

$$[\Sigma^\infty X, Y] \simeq [X, Y_0].$$

That is, $\Omega^\infty Y \simeq Y_0$.

If $Y \in Sp^{\geq 0}$ then $\Omega^\infty Y = Y_0$ is an infinite loop space. That is, for all $k \geq 0$, we have that $Y_0 \simeq \Omega^k Y_k$. May recognition tells us that

$$\mathbf{Alg}_{E_\infty}^{\text{gp-like}}(\mathcal{S}_*) \simeq Sp^{\geq 0}.$$

If \mathcal{C} is a symmetric monoidal category, then $\mathbf{CAlg}(\mathcal{C})$ is also a sym mon cat with *some* underlying tensor product.

For example if X, Y are E_∞ -algebras which are grouplike in spaces, then $X \wedge Y$ is an E_∞ -algebra in \mathcal{S}_* . It is *not true* that if X and Y are infinite loop spaces then $X \wedge Y$ is an infinite loop space.

Example 18.4. Let G be an abelian group, then $K(G, 0)$ is an ∞ -loop space, with $K(G, 0) \simeq \Omega^n K(G, n)$. Let $HG \in Sp^{\geq 0}$ be its corresponding spectrum, called the *Eilenberg-MacLane spectrum* of G . This gives a functor

$$\begin{aligned} N(\mathbf{Ab}) &\rightarrow Sp^{\geq 0} \\ G &\mapsto HG. \end{aligned}$$

We want a monoidal structure on Sp and $Sp^{\geq 0}$ for this functor to be compatible with $\otimes_{\mathbb{Z}}$ in \mathbf{Ab} .

Ideas for monoidal structure on Sp :

- On Sp^{Wh} we had $(X, n) \wedge (Y, m) = (X \wedge Y, n + m)$
- $\mathbf{Alg}_{E_\infty}(\mathcal{S}_*)$
- $\mathbf{Ab}, \otimes_{\mathbb{Z}}$

Boardman: We could define $(X \wedge Y)_n = X_{a(n)} \wedge Y_{b(n)}$ where $a(n) + b(n) = n$, and then we could “ Ω -spectrify.” There are lots of choices for $a(n)$ and $b(n)$.

Adams: We could define

$$(X \wedge Y)_n \simeq \bigvee_{e_{ij}} \Sigma^{n-i-j-d} X_i \wedge Y_j \wedge M(\tau) / \sim$$

where e_{ij} is the square on the $\mathbb{Z} \times \mathbb{Z}$ grid with bottom left corner based at (i, j) , open on the top and right sides, and $M(\tau)$ is the Thom complex of a bundle over e_{ij} .

Indexing on $\mathbb{Z} \times \mathbb{Z}$ is hard because we need to understand choices. Model categories allow us to switch $\mathbb{Z} \times \mathbb{Z}$ to something that records the choices.

Symmetric spectra: we get a model category Sp^Σ indexed on finite sets and injective morphisms (Hovey-Shipley-Smith).

Orthogonal spectra: (or EKMM spectra) $Sp^\mathcal{O}$, indexed on real inner product spaces. This is by Mandell-May-Schwede-Shipley.

Theorem 18.5. (Lewis, '91) There is no good 1-category Sp^1 that describes Sp with a monoidal structure so that:

- (1) Sp^1 is symmetric monoidal
- (2) There is an adjunction $\Sigma^\infty : \mathbf{Top}_* \rightleftarrows Sp^1 : \Omega^\infty$
- (3) We have that $\Sigma^\infty S^0$ is the unit
- (4) Ω^∞ is lax symmetric monoidal
- (5) For any pointed space, $\Omega^\infty \Sigma^\infty X \simeq \text{colim}_k \Omega^k \Sigma^k X$. (that is, these functors are really doing stabilization of spaces)

For symmetric and orthogonal spectra, it is (3) that messes up — you really need a fibrant replacement. In EKMM they force (3) to be true, but fail (5).

How to think of $X \wedge Y$ in Sp ? We use the universal properties, and try to understand its homotopy groups. There is a Künneth spectra sequence to compute $\pi_n(X \wedge Y)$.

Recall that $\text{Fun}^L(\mathcal{S}, \mathcal{C}) \simeq \mathcal{C}$ for \mathcal{C} any presentable ∞ -category. This should remind us of the statement that $\text{Hom}_R(R, M) = M$ for M an R -module. So we want to think of $\text{Fun}^L(-, -)$ as an internal hom somewhere.

Definition 18.6. Let Pr^L denote the (very large) ∞ -category of presentable ∞ -categories, where

$$\text{Hom}_{\text{Pr}^L}(\mathcal{C}, \mathcal{D}) := \text{Fun}^L(\mathcal{C}, \mathcal{D}).$$

Fact 18.7. This is an internal hom — i.e. if \mathcal{C} and \mathcal{D} are presentable, then $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is presentable.²¹

We have that

$$\mathrm{Fun}^L(\mathcal{C}_1, \mathrm{Fun}^L(\mathcal{C}_2, \mathcal{D})) \simeq \mathrm{Fun}^{BL}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}),$$

that is, functors $\mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{D}$ which are colimit preserving in each variable.

So we want some tensor product so that the above is equivalent to $\mathrm{Fun}^L(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D})$.

Fact 18.8. If \mathcal{C} is closed monoidal, then $\mathcal{C}^{\mathrm{op}}$ becomes closed monoidal, but where the tensor product and hom switch roles.

The op of Pr^L is Pr^R , where we take limit-preserving functors! So we can check that

$$\mathcal{C}_1 \otimes \mathcal{C}_2 \simeq \mathrm{Fun}^R(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2).$$

By construction \mathcal{S} is the monoidal unit, since

$$\begin{aligned} \mathcal{S} \otimes \mathcal{C} &= \mathrm{Fun}^R(\mathcal{S}^{\mathrm{op}}, \mathcal{C}) \\ &= (\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}^{\mathrm{op}}))^{\mathrm{op}} \\ &= (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \\ &= \mathcal{C}. \end{aligned}$$

Here we are using that

$$\mathrm{Fun}^R(-, -) = \mathrm{Fun}^L(-^{\mathrm{op}}, -^{\mathrm{op}})^{\mathrm{op}}.$$

So we need to create our operator category $(\mathrm{Pr}^L)^{\otimes} \subseteq \mathbf{Cat}_{\infty}^{\otimes} \simeq N(\mathbf{sSet}^{\otimes})[W_{\mathrm{Joyal}}^{-1}]$. We had a cocartesian fibration $\mathbf{Cat}_{\infty}^{\otimes} \rightarrow \mathbf{Fin}_{*}$, and we're going to restrict fibers to get the correct thing. The fibers will look like $(\mathcal{C}_1, \dots, \mathcal{C}_n)$ with \mathcal{C}_i presentable, and appropriate morphisms.

So the construction we just did argues that $\mathrm{Pr}^L \hookrightarrow \mathbf{Cat}_{\infty}$ is a lax symmetric monoidal functor. Then

$$\mathbf{Alg}_{E_{\infty}}^L(\mathrm{Pr}) = \{\text{presentably symmetric monoidal } \infty\text{-cats}\},$$

and \mathcal{S} is the initial object. This provides the universal property of spaces with its monoidal structure $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$, colimit-preserving in each variable, with the point as the unit.

²¹If we took Fun instead of Fun^L , the size might increase, but in fact $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is presentable in the same size sense that \mathcal{C} and \mathcal{D} are.

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REFERENCES