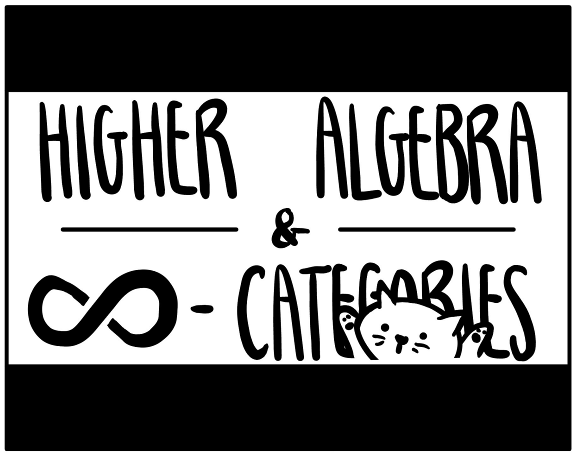
HIGHER ALGEBRA

MAXIMILIEN PÉROUX



Contents

Date: April 18, 2023.

1. Lecture 1: Thursday, January 12th

Today: the **homotopy hypothesis**

Classical algebra: sets, monoids, groups, abelian groups, rings. Each of these are built up on the other. In higher courses, we may see groupoids, which are types of categories. A category is a generalization of a monoid, in some sense. We also have monoidal categories, which in some sense are a generalization of rings.

For higher algebra: spaces, $\mathbb{E}_1$-spaces, spectra, $\mathbf{E}_1$-ring spectra. Underlying this we have ∞-groupoids, ∞-categories, and monoidal ∞-categories.

We study spaces, not up to homeomorphism, but up to *weak homotopy equivalence*. We will study this in a minute. "Spaces" in this class will always mean the study of topological spaces up to weak homotopy equivalence.

We'll give a synthetic definition of what an infinity category is, and circle back to a technical definition in about a month.

What is an ∞-category?

An ∞-category (or $(\infty, 1)$-category) $\mathscr{C}$ should consist of:

(1) a class of objects
(2) a class of morphisms so that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a space
(3) n-morphisms for $n \geq 2$, where for instance 2-morphisms are between 1-morphisms, 3-morphisms between 2-morphisms, etc.
(4) morphisms can be composed in a suitable way
(5) n-morphisms for $n \geq 2$ are invertible in some sense.

An ∞-groupoid (or $(\infty, 0)$-category) should be an ∞-category where all the 1-morphisms are also invertible in some sense.

Why study spaces up to weak homotopy equivalence?

Recall by the Yoneda lemma, we have that

$$X \cong Y \Leftrightarrow \mathrm{Hom}_{\texttt{Top}}(A, X) \cong \mathrm{Hom}_{\texttt{Top}}(A, Y)$$

for all $A \in \texttt{Top}$. Figuring out $\mathrm{Hom}(A, X)$ up to bijection for all A is very difficult, so we prefer to study continuous maps up to homotopy. For X and Y nice enough, we say that $f \simeq g$ in $\mathrm{Hom}(X, Y)$ if there exists some path $I \to \mathrm{Map}(X, Y)$ so that $0 \mapsto f$ and $1 \mapsto g$. We define $[X, Y] = \mathrm{Hom}_{\texttt{Top}}(X, Y)/\simeq$.

We see then that $X \simeq Y$ if and only if $[A, X] \cong [A, Y]$ for all $A \in \texttt{Top}$.

We may ask when $[A, -] : \texttt{Top}_* \to \texttt{Set}$ factors through $\texttt{Grp}$ or $\texttt{Ab}$. We have that $[A, -]$ factors through $\texttt{Grp}$ if and only if A is a co-H-group in $\texttt{Top}$. That is, we have maps

$$\begin{aligned} &A \to A \vee A \\ &A \to *, \end{aligned}$$

which is coassociative, counital, coinvertible.

Example 1.1. S^n, when $n \geq 1$, is a co-H-space. The map $S^n \to S^n \vee S^n$ is the pinch map.

We say that X is *weakly homotopy equivalent* to Y, we write $X \sim Y$, if and only if there is a map $X \to Y$ inducing an isomorphism

$$\pi_n(X) = [S^n, X]_* \cong [S^n, Y]_* = \pi_n(Y),$$

for all $n \geq 0$ (for $n \geq 1$ this is a group isomorphism).

If $X \sim Y$, then $H_n(X) \cong H_n(Y)$ for any n.

Theorem 1.2. (Cellular approximation) For any X in $\texttt{Top}$, there exists $\widetilde{X}$ a CW complex with a canonical map $\widetilde{X} \xrightarrow{\sim} X$ that is a weak equivalence.

Theorem 1.3. (Whitehead) If X, Y are CW complexes, then $X \xrightarrow{\simeq} Y$ is a homotopy equivalence if and only if $X \xrightarrow{\sim} Y$ is a weak homotopy equivalence.

Exercise 1.4. Find spaces X and Y which are weakly homotopy equivalent but not homotopy equivalent.

We denote by Δ the simplex category. Its objects are ordered sets of the form $[n] = \{0, 1, \dots, n\}$, and its morphisms are order-preserving maps. We have that Δ is generated by *cofaces* and *codegeneracies*. The cofaces are of the form

$$d^0, d^1 : [0] \to [1],$$

skipping 0 or 1 in [1], etc. The codegeneracies look like $s^0 : [1] \to [0]$ which "repeat" an element.

The cofaces and codegeneracies satisfy certain *cosimplicial identities*.

If $\mathscr{C}$ is a category, we denote by $s\mathscr{C} = \mathscr{C}^{\Delta^{\mathrm{op}}}$ the simplicial objects in $\mathscr{C}$. If $\mathscr{C} = \mathtt{Set}$, we write $\mathtt{sSet}$ as the category of simplicial sets. A simplicial set $X_\bullet \in \mathtt{sSet}$ consists of sets $X_0, X_1, \dots$ together with face and degeneracy maps satisfying the simplicial identities.

Example 1.5. The *nerve of a small category*. Let $\mathscr{C} \in \mathtt{Cat}$ a small category. We denote by $N_\bullet\mathscr{C}$ the simplicial set with $N_0\mathscr{C} = \mathrm{ob}\mathscr{C}$, $N_1\mathscr{C} = \mathrm{mor}\mathscr{C}$, and $N_n\mathscr{C}$ the set of n composable morphisms in $\mathscr{C}$. That is,

$$N_n\mathscr{C} = N_1\mathscr{C} \times_{N_0\mathscr{C}} \cdots \times_{N_0\mathscr{C}} N_1\mathscr{C}.$$

The face maps are source/target/composition. The degeneracies insert an identity morphism.

Example 1.6. Via Yoneda, we get a functor

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathtt{Set}.$$

If $X_\bullet$ is a simplicial set, we get that the set of n-simplices X_n is in bijection with $\mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, X_\bullet)$.

Example 1.7. (Dold–Kan) We have $\mathtt{Ch}_R^{\geq 0} \xrightarrow{\Gamma} s\mathtt{Mod}_R$ is an isomorphism, where $\Gamma_m C_\bullet = \oplus_{[n] \twoheadrightarrow [k]} C_k$, with faces and degeneracies left as an exercise.

Example 1.8. Let $\Delta^n_{\mathtt{Top}} \subseteq \mathbb{R}^{n+1}$ be defined by

$$\left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \colon 0 \leq t_i \leq 1, \ \sum t_i = 1 \right\}.$$

We can view $[n] = \{v_0, \dots, v_n\}$, and $v_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the ith place. Then if $\alpha : [m] \to [n]$ in Δ, we can define $\alpha(v_i) = v_{\alpha(i)}$. Extend linearly to get $\alpha_* : \Delta^m_{\mathtt{Top}} \to \Delta^n_{\mathtt{Top}}$. We get then that $\Delta^\bullet_{\mathtt{Top}}$ is a cosimplicial topological space.

Example 1.9. If $X \in \mathtt{Top}$, we have $\mathrm{Sing}_\bullet(X) \in \mathtt{sSet}$ defined by $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathtt{Top}}\left(\Delta^n_{\mathtt{Top}}, X\right)$.

Definition 1.10. If $X_\bullet \in \mathtt{sSet}$, we define its *geometric realization* to be

$$|X_\bullet| = \amalg_{n\geq 0} X_n \times \Delta^n_{\mathtt{Top}}/\sim,$$

where $(x,s) \sim (y,t)$ if and only if there is some $\alpha : [m] \to [n]$ so that $\alpha^* y = x$ and $\alpha_* s = t$.

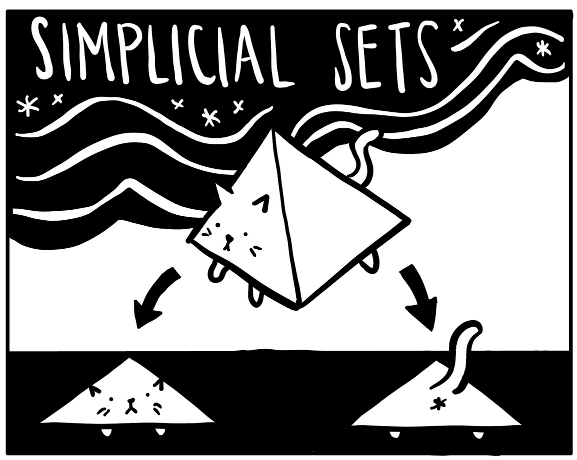
Example 1.11. $|\Delta^n_\bullet| \cong \Delta^n_{\mathtt{Top}}$.

Exercise 1.12. $|X_\bullet|$ is always a CW complex for any $X_\bullet \in \mathtt{sSet}$.

Exercise 1.13. We have an adjunction $|-| : \mathtt{sSet} \rightleftarrows \mathtt{Top} : \mathrm{Sing}(-)$

Definition 1.14. $X_\bullet \to Y_\bullet$ is a *weak homotopy equivalence* in $\mathtt{sSet}$ if $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ is a weak homotopy equivalence of spaces.

Theorem 1.15. (Quillen) Simplicial sets up to weak equivalence is equivalent to topological spaces up to weak homotopy equivalence. Moreover, for any $X \in \mathtt{Top}$, we have that $|\mathrm{Sing}(X)|$ is weakly equivalent to X.



2. Lecture 2: Tuesday, January 17th

Today: the homotopy hypothesis (continued).

Recall we are interested in studying $\mathtt{Top}$ up to weak homotopy equivalences. Equivalently, we are interested in studying $\mathtt{sSet}$ up to weak equivalence, and the relationship between the two was given by the geometric realization / singular complex adjunction.

Recall we've defined $\Delta^n = \mathrm{Hom}_\Delta(-,[n])$. We will define the *kth horn* $\Lambda^n_k \subseteq \Delta^n$ as a coequalizer in `sSet`

$$\left(\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{i\neq k} \Delta^{n-1}\right) \to \Lambda^n_k,$$

where the two maps are δ^{j-1} and δ^i. The geometric realization of Λ^n_k is the topological n-simplex, with the middle and the face opposite the kth edge removed.

Definition 2.1. We say that $Y \in$ `sSet` is a *Kan complex* if for all $k \le n$, and for every $\Lambda^n_k \to Y$, there exists a (not necessarily unique) lift:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Y \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Exercise 2.2. Y is a Kan complex if and only if for any $(n-1)$-simplices $y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$ such that $d_i y_j = d_{j-1} y_i$ for $i < j$, $i, j \neq k$, there exists an n-simplex y such that $d_i y = y_i$ for all $i \neq k$.

Exercise 2.3. We have that $\mathrm{Sing}(X)$ is always a Kan complex for any $X \in$ `Top`.

Exercise 2.4. We have that Δ^n is not a Kan complex for $n \geq 1$.

Exercise 2.5. If $X \in s$`Grp`, then the underlying simplicial set of X is always a Kan complex.

Up to weak homotopy equivalence, every simplicial set is a Kan complex (will see this later).

Recall the Dold-Kan correspondence

$$s\mathtt{Mod}_{\mathbb{Z}} \cong \mathtt{Ch}^{\geq 0}_{\mathbb{Z}},$$

which sends weak homotopy equivalences to quasi-isomorphisms. Given a simplicial set X_*, we can take an associated simplicial abelian group $\mathbb{Z}[X_*]$ by taking the free group on n-simplices at level n. We can ask what $\mathbb{Z}[X_*]$ corresponds to as a chain complex. One answer is that

$$\mathbb{Z}[\mathrm{Sing}(X_*)] \leftrightarrow C_*(X;\mathbb{Z}).$$

This tells us that

$$\pi_*\left(\mathbb{Z}\left[\mathrm{Sing}(X)\right]\right) \cong H_*(X;\mathbb{Z}).$$

In some sense we can view $\mathbb{Z}[\mathrm{Sing}(X)]$ as being (equivalent to) the *free commutative monoid* on X. This is what is known as the *Dold-Thom theorem.*

Homotopy hypothesis: Spaces (up to weak equivalence) are ∞-groupoids. For us, spaces up to weak equivalences correspond to Kan complexes.

Given $X \in \texttt{Kan}$, we can call X_0 the objects, and X_1 the morphisms. The horn filling conditions on horns tell you that you can *compose* and *invert* morphisms in X_1, witnessed by simplices in X_2.

Definition 2.6. A *quasi-category* (i.e. ∞-category) is a simplicial set with inner horn lifting property. That is, we can lift against horns Λ^n_k for $0 < k < n$.

Exercise 2.7. A quasi-category has unique horn filling if and only if it is isomorphic to the nerve of a 1-category.

Model categories

Vista: Every nice infinity category is equivalent in some sense to a model category. This will pretty much be the goal of this class.

Notation 2.8. Let $\mathcal{M}$ be a category, and $\chi \subseteq \mathcal{M}$ a class of morphisms. We define $\mathrm{LLP}(\chi)$ to be the class of morphisms in $\mathcal{M}$ so that f has left lifting property with respect to all morphisms in χ:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "\in \chi"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Similarly we can define $f \in \mathrm{RLP}(\chi)$ by

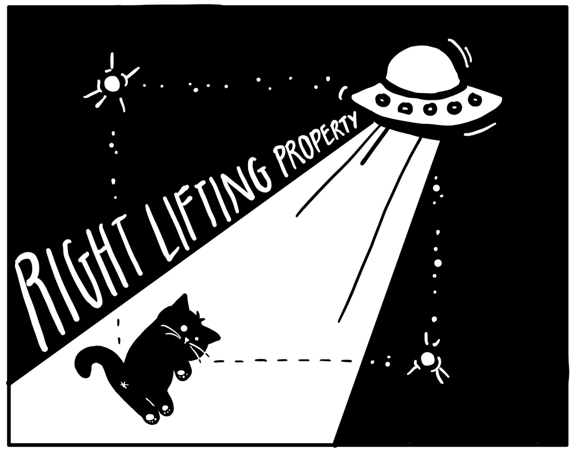
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\chi \ni"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Definition 2.9. A *weak factorization system* on a category $\mathcal{M}$ consists of a pair $(\mathscr{C}, \mathscr{F})$ of classes of morphisms such that

(1) Given any $f : X \to Y$ in $\mathcal{M}$, it factors (not necessarily uniquely) as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathscr{C}\ni"'] & & Y \\ & W \arrow[ur, "\in\mathscr{F}"'] & \end{tikzcd}$$

(2) $\mathscr{C} = \mathrm{LLP}(\mathscr{F})$ and $\mathscr{F} = \mathrm{RLP}(\mathscr{C})$.



Example 2.10. In `Set`, we have that mono and epimorphisms give a weak factorization system. A factorization is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{id}_X \times f"'] & & Y \\ & X \times Y \arrow[ur, "\pi_Y"'] & \end{tikzcd}$$

Definition 2.11. A *model structure* on $\mathcal{M}$ consists of three classes of morphisms:

W	weak equivalences
Cof	cofibrations
Fib	fibrations

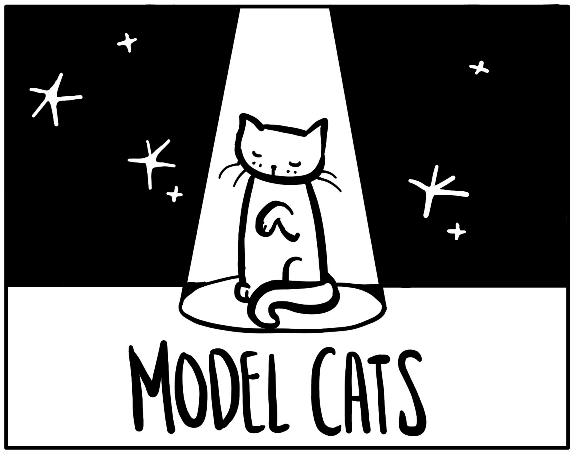
We denote by $\widetilde{\mathrm{Cof}} := \mathrm{Cof} \cap W$ and $\widetilde{\mathrm{Fib}} = \mathrm{Fib} \cap W$, and call these *trivial cofibrations* (resp. *trivial fibrations*). These are subject to the constraint that

(1) $\mathcal{M}$ is bicomplete (all limits and colimits)[1]
(2) W satisfies 2-out-of-3 property[2]
(3) $\left(\mathrm{Cof}, \widetilde{\mathrm{Fib}}\right)$ and $\left(\widetilde{\mathrm{Cof}}, \mathrm{Fib}\right)$ are weak factorization systems.

Terminology 2.12. A category with a model structure is referred to as a *model category.*

Notation 2.13. We will decorate each class of morphisms as

[1]We might also require *finitely* bicomplete.
[2]If f and g are composable, and any two of f, g, gf are in W then so is the third.



$$\begin{array}{c|c} W & \xrightarrow{\sim} \\ \mathrm{Cof} & \hookrightarrow \\ \mathrm{Fib} & \twoheadrightarrow \end{array}$$

Exercise 2.14. W, Cof, and Fib are closed under retracts: that is,

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[rr, bend right, equal] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

then if $g \in W$ (resp. Cof or Fib) then $f \in W$ (resp. Cof or Fib).

Definition 2.15. Let $\mathcal{M}$ be a model category, and let $\varnothing \in \mathcal{M}$ the initial object and $* \in \mathcal{M}$ the terminal object.

- We say that $X \in \mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to X$ is a cofibration.
- We say that $X \in \mathcal{M}$ is *fibrant* if the unique map $X \to *$ is a fibration.
- We say that $\widetilde{X}$ is a *cofibrant replacement* of X if

$$\begin{tikzcd} \varnothing \arrow[rr] \arrow[dr, hook] & & X \\ & \widetilde{X} \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

- We say that $\widetilde{X}$ is a *fibrant replacement* of X if

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim"'] & & * \\ & \widetilde{X} \arrow[ur, two heads] & \end{tikzcd}$$

Example 2.16. $\mathcal{M} = \texttt{Top}$, W = weak homotopy equivalences, Cof = relative CW complexes[3] The fibrations are determined by Fib = RLP($\widetilde{\text{Cof}}$). The fibrations are equivalently RLP($D^n \to D^n \times I$). Every object here is fibrant, and the cofibrant objects are precisely the CW complexes. Cofibrant replacement is cellular approximation.

3. Lecture 3: Thursday, January 19th

Proposition 3.1. Identities and isomorphisms are weak equivalences in a model category.

Proof. For any $X \in \mathcal{M}$, we can fibrantly replace it to get $X \overset{\sim}{\hookrightarrow} \widetilde{X}$. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}"] \arrow[dr, hook, "\sim"'] & & X \arrow[dl, "\sim"] \\ & \widetilde{X}. & \end{tikzcd}$$

By 2-out-of-3, we have that $\mathrm{id} : X \to X$ is also a weak equivalence.

More generally if $f : X \to Y$ is an isomorphism in $\mathcal{M}$, then by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, equal] \arrow[d, "f"'] & Y \arrow[r, "f^{-1}"] \arrow[d, equal] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y, \end{tikzcd}$$

we see that f is contained in W. $\square$

If $(\mathscr{C}, \mathscr{F})$ is a weak factorization system, then both $\mathscr{C}$ and $\mathscr{F}$ are closed under retracts. Hence $\text{Cof}, \widetilde{\text{Cof}}, \text{Fib}, \widetilde{\text{Fib}}$ are closed under retracts. W is also closed under retracts (exercise).

Exercise 3.2. We have that $\mathcal{M}$ is a model category if and only if $\mathcal{M}^{\text{op}}$ is a model category.

[3] $A \hookrightarrow X$ is a *relative CW complex* if X is built out of A by attaching cells.

Theorem 3.3. Cofibrations are closed under pushouts and coproducts.

Proof. Given any test square, we can try to lift:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Z \arrow[r] \arrow[urr, dashed] & P \arrow[ul, phantom, "\ulcorner"] \arrow[r] & B. \end{tikzcd}$$

This map is constructed by universal property of the pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, dashed, bend right] & P \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "\exists!"] & \\ & & A. \end{tikzcd}$$

For coproducts, we can take $X_i \hookrightarrow Y_i$ for $i \in J$. Let's try to lift:

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, hook] & \amalg_i X_i \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Y_i \arrow[r] \arrow[urr, dashed] & \amalg_i Y_i \arrow[r] & B. \end{tikzcd}$$

We know that each $X_i \hookrightarrow Y_i$ is a cofibration hence it lifts against the big square. By universal property a map $\amalg_i Y_i \to A$ exists. $\square$

Example 3.4. If $\mathscr{C}$ is a bicomplete category, then $\mathscr{C}$ has a model structure where W is the isomorphisms, and $\mathrm{Cof} = \mathrm{Fib} = \mathrm{mor}\mathscr{C}$.

Example 3.5. If $\mathcal{M} = \mathtt{Top}$, we have the *Quillen model structure*, with

- W = weak homotopy equivalences
- Cof = retracts of relative CW complexes
- Fib = Serre fibrations (RLP($D^n \hookrightarrow D^n \times I$)).

Example 3.6. The Strøm (or Hurewicz) model structure on `Top`:

- W = homotopy equivalences
- Fib = Hurewicz fibrations (RLP($A \to A \times I$) for all $A \in \mathtt{Top}$)
- Cof = closed cofibrations in `Top`.

Fibrant replacement in the Strøm model structure looks like

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook] & & Y \\ & M_f \arrow[ur, two heads, "\simeq"'] & \end{tikzcd}$$

Where $M_f = (X \times I) \cup_X Y$ is the mapping cylinder.

Example 3.7. The *Kan model structure* on `sSet` with

- W = weak homotopy equivalences
- Cof = monomorphisms (levelwise injections)
- Fib = Kan fibrations (RLP($\Lambda^n_k \to \Delta^n$) for all $0 \le k \le n$).

Everything is cofibrant here (since the empty simplicial set injects into everything). Fibrant things are Kan complexes. This tells us that every simplicial set is weakly equivalent to a Kan complex!

Theorem 3.8. (Milnor) The natural map $X \to \mathrm{Sing}(|X|)$ is a weak homotopy equivalence for any simplicial set X. [Kerodon, 3.5.4.1]

Definition 3.9. Let $\mathscr{C}$ be a cat, and $W \subseteq \mathscr{C}$ a subcategory. A functor $F : \mathscr{C} \to \mathscr{D}$ is called the *localization of* $\mathscr{C}$ *with respect to* W if:

(1) $F(f) \in \mathrm{iso}\mathscr{D}$ if $f \in \mathrm{mor}W$
(2) For any other F' satisfying (1), we have

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F'"] \arrow[d, "F"'] & \mathscr{D}' \\ \mathscr{C} \arrow[ur, "\exists!"'] & \end{tikzcd}$$

We denote by $\mathscr{C} \to \mathscr{C}[W^{-1}]$ the localization.

Here is a naive way to construct $\mathscr{C}[W^{-1}]$: we take the free category on $\mathscr{C}$ and "W^{-1}." That is, we take the same objects, but allow morphisms to be "zigzags" of morphisms forward in $\mathscr{C}$ and morphisms backwards in W, and we mod out by the relation that things in W become isomorphisms. There are size issues here.

Theorem 3.10. If $\mathcal{M}$ is a model category, then localization $\mathcal{M} \to \mathcal{M}[W^{-1}]$ exists. We denote by $\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[W^{-1}]$ the homotopy category of $\mathcal{M}$.

Recall in `Top` that $f \simeq g : X \to Y$ if there is a map $H : X \times I \to Y$ so that $H(-,0) = f$ and $H(-,1) = g$.

Definition 3.11. Le t$\mathcal{M}$ be a model category. A *cylinder object* on $X \in \mathcal{M}$ is defined to be

$$\begin{tikzcd} X \amalg X \arrow[rr, "\nabla"] \arrow[dr, hook] & & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

The construction of cylinder objects is *not functorial*.

A *(left) homotopy* from f to g is a map $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. We denote this by $f \simeq g$.

Proposition 3.12. We have that $i_0 : X \to \mathrm{Cyl}(X)$ is a weak equivalence (and same for i_1).

Proof. We have

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, dashed, "i_0"'] & X \amalg X \arrow[r, "\nabla"] \arrow[d] & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

By 2-out-of-3 on the outside maps, the result follows. $\square$

Proposition 3.13. If X is cofibrant, then $i_0, i_1 : X \to \mathrm{Cyl}(X)$ are cofibrations.

Proof. Since cofibrations are preserved under pushouts, we have that i_0 and i_1 are cofibrations:

$$\begin{tikzcd} \varnothing \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "i_0"] \\ X \arrow[r, "i_1"'] & X \amalg X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\square$

Theorem 3.14. (Exercise) If X is cofibrant, then homotopy $\simeq$ gives an equivalence relation on $\mathrm{Hom}(X, Y)$ for any Y.

We can think of a map

$$\mathrm{Hom}_{\mathcal{M}}(X,Y)/\simeq \times \mathrm{Hom}_{\mathcal{M}}(Y,Z)/\simeq \to \mathrm{Hom}_{\mathcal{M}}(X,Z)/\simeq$$
$$(f,g) \mapsto g \circ f.$$

In order for this to be well-defined, we need Z to be fibrant.

Lemma 3.15. If Z is fibrant, and $f \simeq g : X \to Z$, then if $h : X' \to X$, we have that $fh \simeq gh$.

Proof. We have $H : \mathrm{Cyl}(X) \to Y$ with $H_0 = f$ and $H_1 = g$. By lifting, we get

$$\begin{tikzcd}
X' \amalg X' \arrow[r] \arrow[d, hook] & X \amalg X \arrow[r] & \mathrm{Cyl}(X) \arrow[d, two heads, "\sim"] \\
\mathrm{Cyl}(X') \arrow[r] \arrow[urr, dashed] & X' \arrow[r] & X.
\end{tikzcd}$$

This gives the desired map. We used fibrancy of Z to ensure that the map $\mathrm{Cyl}(X) \to X$ was a trivial fibration (or could be replaced with a better cylinder object using a map to Z). ☐

Theorem 3.16. In $\mathcal{M}$, given $f : X \to Y$ with X cofibrant and Y fibrant, then $f \in W$ if and only if f is a homotopy equivalence.[4]

Notation 3.17. $\mathcal{M}_c$ = cofibrant objects in $\mathcal{M}$, and $\mathcal{M}_f$ = fibrant objects in $\mathcal{M}$. We denote by $\mathcal{M}_{cf}$ = objects which are *both* cofibrant and fibrant.

Concretely, we can define $\mathrm{Ho}(\mathcal{M})$ as the objects in $\mathcal{M}$, but where

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = \mathrm{Hom}_{\mathcal{M}_{cf}/\simeq}(RQX, RQY),$$

where R is a fibrant replacement and Q is a cofibrant replacement.

Exercise 3.18. Given $X \to Y$ in $\mathcal{M}$, there exists $QX \xrightarrow{\widetilde{f}} QY$ such that

$$\begin{tikzcd}
QX \arrow[r, "\widetilde{f}"] \arrow[d, two heads, "\sim"] & QY \arrow[d, two heads, "\sim"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Here $\widetilde{f}$ is well-defined up to left homotopy.

Given some $\mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, we just need to check that $W \mapsto$ isos, and it is universal in that way.

[4]Meaning that there is some $g : Y \to X$ with $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$.

4. Lecture 4: Tuesday, January 24th

Definition 4.1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are model categories, and take a functor $F : \mathcal{M} \to \mathcal{N}$. A *left derived functor* of F is an (absolute) right Kan extension of F along $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\ \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed] \arrow[ur, phantom, shift left=3, "\Leftarrow \ell"] & \end{tikzcd}$$

if $G : \mathrm{Ho}(\mathcal{M}) \to \mathcal{N}$ and $s : G \circ \gamma_{\mathcal{M}} \Rightarrow F$, then there exists a unique $s' : G \Rightarrow LF$ so that $\ell \circ (s' \circ \gamma_{\mathcal{M}}) = s$.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\ \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed] \arrow[ur, bend right] \arrow[ur, phantom, shift left=3, "\Leftarrow \ell"] \arrow[ur, phantom, shift right=2, "\Uparrow s'"] & \end{tikzcd}$$

Definition 4.2. Let $F : \mathcal{M} \to \mathcal{N}$. A *total left derived functor* $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is the left derived functor of $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\gamma_{\mathcal{N}}} \mathrm{Ho}(\mathcal{N})$.

Example 4.3. If $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ where if $f \in W$ between cofibrant objects then Ff is a weak equivalence in $\mathcal{N}$, then $\mathbb{L}F$ exists:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d] & \mathcal{N} \arrow[r] & \mathrm{Ho}(\mathcal{N}) \\ \mathrm{Ho}(\mathcal{M}) \arrow[urr, dashed] & & \end{tikzcd}$$

We will have that $\mathbb{L}F(X) \xrightarrow{\sim} F(X)$ whenever X is cofibrant. In general, $\mathbb{L}F(X) = F(Q(X))$.

Definition 4.4. Let $F : \mathcal{M} \to \mathcal{N}$. We say that F is a *left Quillen functor* if

(i) F is a left adjoint
(ii) F preserves cofibrations and trivial cofibrations.

In this case if G is a right adjoint, then we say the adjunction is a *Quillen adjunction / Quillen pair.*[5]

Exercise 4.5. Show that L is left Quillen if and only if G is right Quillen.

[5] There is a dual notion of right Quillen functor, meaning it is a right adjoint which preserves fibrations and trivial fibrations.

Lemma 4.6. (Ken Brown's Lemma) If $F : \mathcal{M} \to \mathcal{N}$ is any functor between model categories which sends trivial cofibrations between cofibrant objects to weak equivalences in $\mathcal{N}$, then F sends any weak equivalence between cofibrant objects to weak equivalences.

Proof. Let $f : A \xrightarrow{\sim} B$, where $A, B \in \mathcal{M}_c$. We need $F(f)$ to be a weak equivalence. Consider the factorization of the coproduct of f and the identity on B:

$$\begin{tikzcd}
A \amalg B \arrow[rr, "f \amalg \mathrm{id}_B"] \arrow[dr, hook, "q"'] & & B \\
& C \arrow[ur, two heads, "\sim", "p"'] &
\end{tikzcd}$$

Then consider the pushout:

$$\begin{tikzcd}
\varnothing \arrow[r, hook] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook, "i_A"] \arrow[ddr, "\sim"] & B \\
B \arrow[r, hook] \arrow[drr, "q"] \arrow[ddrr, equal] & A \amalg B \arrow[dr, "q"] & \\
& & C \arrow[uu, "p"'] \arrow[d, "p"] \\
& & B
\end{tikzcd}$$

We have that

$$B \overset{i_B}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$

$$A \overset{i_A}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$

are both trivial cofibrations, hence their images under F are weak equivalences. We see that

$$F(p) \circ F(q \circ \mathrm{id}_B) = F(p \circ q \circ \mathrm{id}_B) = F(\mathrm{id}_B).$$

Therefore $F(p)$ is a weak equivalence by 2-out-of-3. $\square$

Theorem 4.7. Suppose that $F : \mathcal{M} \to \mathcal{M}$ is left Quillen. Then $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ exists and can be defined as

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{Q} \mathrm{Ho}(\mathcal{M}_c) \xrightarrow{F} \mathrm{Ho}(\mathcal{N}).$$

Moreover, we obtain an adjunction on the homotopy categories:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) : \mathbb{R}G.$$

Proof idea. We have a natural iso

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y),$$

compatible with homotopy equivalence:

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y))/\simeq \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y)/\simeq$$

□

Theorem/Definition: Take a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. Suppose that $f : X \xrightarrow{\sim} G(Y)$, with $X \in \mathcal{M}_c$ and $Y \in \mathcal{N}_f$ is a weak equivalence if and only if $f^\flat : F(X) \to Y$ is. Then $\mathbb{L}F$ and $\mathbb{R}G$ are equivalences of categories, we call this a *Quillen equivalence.*

Example 4.8. We have that

$$|-| : \mathtt{sSet}_{\mathrm{Kan}} \rightleftarrows \mathtt{Top}_{\mathrm{Quillen}} : \mathrm{Sing}(-)$$

is a Quillen equivalence.

Example 4.9. We have that

$$\mathrm{id} : \mathtt{Top}_{\mathrm{Quillen}} \rightleftarrows \mathtt{Top}_{\mathrm{Strøm}} : \mathrm{id}$$

is a Quillen adjunction but not a Quillen equivalence.

Q: If $\mathcal{M}$ and $\mathcal{N}$ are model categories such that there is an equivalence of categories $\mathrm{Ho}(\mathcal{M}) \cong \mathrm{Ho}(\mathcal{N})$, is this always coming from a Quillen equivalence?

A: No! Dugger–Shipley, 2009.

This indicates that Quillen equivalence is a good notion but it is not a *perfect* notion.

Guided example: chain complexes

Let's take $\mathtt{Ch}_{\mathbb{Z}}$ to be homologically graded unbounded chain complexes. There are three model structures of interest. We first start with the projective one:

$(\mathtt{Ch}_{\mathbb{Z}})_{\mathrm{projective}}$:

- weak equivalences are quasi-isomorphisms
- fibrations are levelwise epimorphisms
- cofibrations are levelwise monomorphisms such that the cokernel of each $f_n : X_n \to Y_n$ is free.

If $M \in \texttt{Ab}$, we define $S^n(M)$ to be the chain complex $M[n]$ which is concentrated in M at degree n. If $M = \mathbb{Z}$, we call it S^n. We define $D^n(M)$ to be a chain complex

$$\cdots \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to \cdots$$

with two M's concentrated in degrees n and $n-1$. We call $D^n(\mathbb{Z}) =: D^n$.

Exercise 4.10. Show that fibrations are $\mathrm{RLP}(0 \to D^n)$ for all n. That is,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ D^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

We claim this lifts iff $X \to Y$ is a levelwise epimorphism. We have that $\mathrm{Hom}_{\texttt{Ch}}(D^n, Y) \cong Y_n$, so we are just asking if every element in Y_n lifts to an element in X_n.

Exercise 4.11. Show that $\widetilde{\mathrm{Fib}} = \mathrm{RLP}\left(S^n \hookrightarrow D^{n+1}\right)$ for all n. Consider $\mathrm{Hom}_{\texttt{Ch}}(S^n, Y)$. A map looks like

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & Y_n \arrow[r] & Y_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

That is, it picks out a class in Y_n which maps to zero under the differential. The data of a square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ D^n \arrow[r] & Y \end{tikzcd}$$

is the data of $(y, x) \in Y_n \oplus Z_{n-1}X$ so that $p(x) = dy$. Show that a lift exists if and only if p is a trivial fibration.

Other model structures.

$(\texttt{Ch}_R)_{\text{injective}}$:

- W = quasi-isomorphisms
- Cof = fiberwise monomorphisms[6]
- Fib = fiberwise epimorphisms with fibrant kernel

[6] Here we roughly have that $\mathrm{Cof} = \mathrm{LLP}(D^n \to 0)$ and $\widetilde{\mathrm{Fib}} = \mathrm{LLP}(D^{n+1} \to S^n)$.

We get a Quillen equivalence

$$\mathrm{id} : (\mathtt{Ch}_R)_{\text{projective}} \rightleftarrows (\mathtt{Ch}_R)_{\text{injective}} : \mathrm{id}.$$

We also have have a third one which is *not* Quillen equivalent.

$(\mathtt{Ch}_R)_{\text{Hurewicz}}$:

- W = homotopy equivalences of chain complexes
- Cof = split levelwise monomorphisms
- Fib = split levelwise epimorphisms

We denote by $\mathscr{D}(R) = \mathrm{Ho}\left((\mathtt{Ch}_R)_{\text{proj}}\right)$ the *derived category* of a ring R.

We can also think about *connective* chain complexes (which are zero in negative degrees). We have an adjunction

$$\mathtt{Ch}_R \rightleftarrows \mathtt{Ch}_R^{\geq 0}.$$

This induces a model structure on $\mathtt{Ch}_R^{\geq 0}$ making it into a Quillen adjunction but not a Quillen equivalence. We denote by $\mathrm{Ho}(\mathtt{Ch}_R^{\geq 0}) = \mathscr{D}^{\geq 0}(R)$.

We get a model structure: $\left(\mathtt{Ch}_R^{\geq 0}\right)_{\text{proj}}$

- W = quasi-isomorphisms
- Fib = positive epimorphisms (may not be epi in degree 0)
- Cof = monomorphisms with projective cokernel. The cofibrant objects here are levelwise projective R-modules.

If we take $M \in \mathtt{Mod}_R$, we can view $S^0(M) \in \mathtt{Ch}_R^{\geq 0}$, and take a cofibrant replacement of it $P \overset{\sim}{\twoheadrightarrow} S^0(M)$. This is *exactly* a projective resolution of M!

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Example 4.12. Let $M \in \mathtt{Mod}_R$. Then we can take

$$S^0(M) \otimes_R - : \mathtt{Ch}_R^{\geq 0} \to \mathtt{Ch}_R^{\geq 0}.$$

We can check that this is left Quillen. We can look at its total left derived functor $S^0(M) \otimes_R^{\mathbb{L}} -$. We can see that

$$M \otimes_R^{\mathbb{L}} N := S^0(M) \otimes_R^{\mathbb{L}} S^0(N) \simeq S^0(M) \otimes_R P_\bullet,$$

where $P_\bullet$ is a projective resolution of N. We have that

$$H_i(M \otimes^{\mathbb{L}}_R N) = \mathrm{Tor}_i^R(M, N).$$

Exercise 4.13. In the same way, if we want to derive hom, we can check that

$$\mathrm{Hom}_{\mathscr{D}^{\geq 0}(R)}(S^m(M), S^n(N)) \cong \mathrm{Ext}_R^{n-m}(M, N).$$

Via Dold-Kan, we have a Quillen adjunction

$$R[-] : \mathtt{sSet}_{\mathrm{Kan}} \rightleftarrows \mathtt{sMod}_R : U,$$

with the model structure on $\mathtt{sMod}_R$ given by weak homotopy equivalences as underlying simplicial sets, and fibrations as underlying Kan fibrations.

Then Dold-Kan takes the form of a Quillen equivalence

$$N : (\mathtt{sMod}_R)_{\mathrm{Kan}} \rightleftarrows (\mathtt{Ch}_R^{\geq 0})_{\mathrm{proj}} : \Gamma.$$

In general $N(X \otimes_R Y) \not\cong N(X) \otimes_R N(Y)$, however $N(X \otimes Y) \cong N(X) \otimes_R N(Y)$. They both describe $\mathscr{D}^{\geq 0}(R)$ in a monoidal way.

5. Lecture 5: Thursday, January 26th

For Dold–Kan $\mathtt{Ch}_{\geq 0} \cong \mathtt{sMod}_R$, we have

$$M \otimes N \rightleftarrows M \otimes R \otimes N \rightleftarrows M \otimes R^{\otimes 2} N \cdots$$

we denote this by $B_\bullet(M, R, N)$ and call it the *bar construction.*

Homotopy colimits

Motivation: Limits and colimits are not invariant under (weak) homotopy equivalence.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

However $\Sigma X \not\simeq *$.

Let $\mathcal{M}$ be a model category, and $\mathscr{C}$ a small category. Then we denote by $\mathrm{Fun}(\mathscr{C}, \mathcal{M}) = \mathcal{M}^{\mathscr{C}}$. Let $\mathscr{C}_0 \subseteq \mathscr{C}$ be the discrete subcategory spanned by $\mathrm{ob}(\mathscr{C})$. Let $\mathcal{M}^{\mathscr{C}_0} = \prod_{\mathscr{C}_0} \mathcal{M}$. This has a model structure where W, Fib, and Cof are determined objectwise.

Consider $\iota : \mathscr{C}_0 \hookrightarrow \mathscr{C}$. This induces a map

$$\begin{aligned} \iota^* : \mathcal{M}^{\mathscr{C}} &\to \mathcal{M}^{\mathscr{C}_0} \\ F &\mapsto F|_{\mathscr{C}_0}. \end{aligned}$$

This admits adjoints:

$$\iota_! \dashv i^* \dashv i_*.$$

We have that ι^* creates W and Fib.

We have $\left(\mathcal{M}^{\mathscr{C}}\right)_{\text{proj}}$:

- $W =$ objectwise weak equivalence
- $\text{Fib} =$ objectwise fib
- $\text{Cof} =$? induced by $\iota_! \text{Cof}$

We have that $\mathcal{M}$ is cocomplete, so we get a tensoring

$$\begin{aligned} \mathcal{M} \times \mathtt{Set}^{\mathscr{C}} &\to \mathcal{M}^{\mathscr{C}} \\ (X, F) &\mapsto X \otimes F = \amalg_{F(-)} X. \end{aligned}$$

We have $(X \times F)(c) = \amalg_{F(c)} X$.

There are representable functors

$$\begin{aligned} \mathscr{C}(c, -) : \mathscr{C} &\to \mathtt{Set} \\ d &\mapsto \mathscr{C}(c, d). \end{aligned}$$

By Yoneda, there is a natural iso

$$\mathtt{Set}^{\mathscr{C}}(\mathscr{C}(c, -), F) \cong F(c).$$

Tensoring with a representable functor gives

$$X \otimes \mathscr{C}(c, -) = \amalg_{\mathscr{C}(c,-)} X.$$

This is the *free diagram of X generated at c*.

This gives an adjunction

$$- \otimes \mathscr{C}(c, -) : \mathcal{M} \rightleftarrows \mathcal{M}^{\mathscr{C}} : \text{ev}_c.$$

In this case

$$\iota_!(F) = \amalg_c \amalg_{\mathscr{C}(c,-)} F(c),$$

which is the free diagram in $\mathcal{M}$ generated by F. Evaluating at d gives

$$\iota_!(F)(d) = \amalg_{c \in \mathscr{C}} \amalg_{\mathscr{C}(c,d)} F(c).$$

This is the functor $\iota_! : \mathcal{M}^{\mathscr{C}_0} \to \mathcal{M}^{\mathscr{C}}$. We see that $\iota_! X$ is a left Kan extension

$$\begin{tikzcd} \mathscr{C}_0 \arrow[r, "X"] \arrow[d, hook, "\iota"'] & \mathcal{M} \\ \mathscr{C} \arrow[ur, dashed] & \end{tikzcd}$$

There is a diagonal functor

$$\begin{aligned} \mathcal{M} &\xrightarrow{\Delta} \mathcal{M}^{\mathscr{C}} \\ C &\mapsto \text{constant functor at } X. \end{aligned}$$

This admits adjoints

$$\operatorname{colim} \dashv \Delta \dashv \lim .$$

Proposition 5.1. The adjunction

$$\operatorname{colim} : \left(\mathcal{M}^{\mathscr{C}}\right)_{\text{proj}} \rightleftarrows \mathcal{M} : \Delta$$

is Quillen.

We denote hocolim $:= \mathbb{L}$colim. There is a map hocolim$(-) \to$ colim$(-)$, and

$$\operatorname{hocolim}(F) \simeq \operatorname{colim}(QF).$$

Here QF denotes a cofibrant replacement in $\left(\mathcal{M}^{\mathscr{C}}\right)_{\text{proj}}$. For a general $\mathscr{C}$, QF is very difficult to determine.

Consider $\mathscr{C} = a \leftarrow b \to c$, and let $X \in \mathcal{M}^{\mathscr{C}_0}$. Then $\iota_! X$ is equal to

$$\begin{tikzcd} X(b) \arrow[r] \arrow[d] & X(b) \amalg X(c) \\ X(a) \amalg X(b) & \end{tikzcd}$$

Cofibrant objects in $\mathcal{M}^{\mathscr{C}}$ are of the form

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Z \\ Y & \end{tikzcd}$$

with X cofibrant. Here cofibrant replacement is easy. We start with $Y \xleftarrow{f} X \xrightarrow{g} Z$, and we replace X with $\widetilde{X} \xrightarrow{\sim} X$ to get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

If we cofibrantly replace $\widetilde{X} \to Z$, and similarly for Y, we get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Z} \\ \widetilde{Y} & \end{tikzcd}$$

The maps we used to fibrantly replace induces a fiberwise weak equivalence between this diagram and the one we started out with.

In $(\mathtt{Top})_{\text{Quillen}}$, we can take $\operatorname{hocolim}(* \leftarrow X \rightarrow *)$. We cofibrantly replace X if necessary, and replace $X \to *$ by $X \hookrightarrow CX$, which is a cofibration. In this case we see that

$$\operatorname{hocolim}\,(* \leftarrow X \to *) \simeq \operatorname{colim}(C\widetilde{X} \leftarrow \widetilde{X} \to C\widetilde{X}) = \Sigma\widetilde{X}.$$

More generally, $\operatorname{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z)$ is the double mapping cylinder $M(f,g)$.

Theorem 5.2. If $\mathcal{M}$ is a *left proper model category* then

$$\operatorname{hocolim}(Y \hookleftarrow X \to Z) \cong \operatorname{colim}(Y \hookleftarrow X \to Z).$$

Proof. In the easy case, X is cofibrant, so we can factor the map to Z to get

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & \widetilde{Z} \arrow[r, two heads, "\sim"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & H \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner"] & P. \end{tikzcd}$$

The entire rectangle is a pushout, so $Z \to P$ is a cofibration, and the right square is a pushout by the pasting law, so $H \to P$ is a weak equivalence. ☐

Example 5.3. Let $\mathscr{C} = * \to * \to \cdots$. Show that $X_0 \to X_1 \to \cdots$ is cofibrant in $\mathcal{M}^{\mathscr{C}}$ if and only if X_0 is cofibrant and $X_i \hookrightarrow X_{i+1}$ is a cofibration for each i.

There is a third model structure on $\mathcal{M}^{\mathscr{C}}$ called the *Reedy model structure* (need $\mathscr{C}$ to be a Reedy cat). In this case, $\operatorname{hocolim}_{\Delta^{\text{op}}}(X_\bullet) \cong \left|Q^{\text{Reedy}}X_\bullet\right|$, for X a simplicial object in $\mathcal{M}$.

Bar construction: Let $\mathcal{M}$ a model cat, $\mathscr{C}$ a small cat, $F : \mathscr{C}^{\mathrm{op}} \to \mathcal{M}$, and $G : \mathscr{C} \to \mathcal{M}$. Then we define

$$B_\bullet(F, \mathscr{C}, G) := \amalg_{c_0 \in \mathscr{C}} F(c_0) \times G(c_0) \leftleftarrows \amalg_{c_0 \leftarrow c_1} F(c_0) \times G(c_1) \leftleftarrows \cdots$$

Example 5.4. If $F = * = G$, then

$$B_\bullet(*, \mathscr{C}, *) \cong N_\bullet(\mathscr{C}^{\mathrm{op}}).$$

Pièce de résistance:

Theorem 5.5. (Bousfield–Kan) If $F : \mathscr{C} \to \mathcal{M}$ is a functor, then

$$\mathrm{hocolim}_{\mathscr{C}}(F) \simeq |B_\bullet(*, \mathscr{C}, F)|.$$

6. Lecture 6: Tuesday, January 31st

Combinatorial model categories

Definition 6.1. A model category is *combinatorial* if it is *presentable*[7] and *cofibrantly generated.*

To motivate presentability, let X be a set. Then X is determined by its elements, meaning that

$$\mathrm{Hom}_{\mathtt{Set}}(*, X) \cong X.$$

Then we can present X as $X = \cup_{x \in X} \{*\}$.

Definition 6.2. A colimit is *filtered* if the diagram is filtered, meaning it is nonempty and every subdiagram has a cocone.

Theorem 6.3. (Exercise) In $\mathtt{Set}$, filtered colimits commute with finite limits. That is, if $F : I \times J \to \mathtt{Set}$ with I finite and J filtered, then

$$\mathrm{colim}_J \left(\lim_I F_I \right) \xrightarrow{\sim} \lim_I \left(\mathrm{colim}_J F_J \right)$$

is an isomorphism.

Proposition 6.4. A set X is finite if and only if

$$\mathrm{Hom}_{\mathtt{Set}}(X, -) : \mathtt{Set} \to \mathtt{Set}$$

preserves filtered colimits.

[7] By this we mean "locally presentable."

Proof. For the backwards direction, let $I = \{X_i\}$ be the collection of finite subsets of X. Then $X = \text{colim}_I X_i$. In particular, we have that

$$\text{colim}_I \text{Hom}(X, X_i) \cong \text{Hom}(X, X)$$
$$(X \xrightarrow{f_i} X_i) \xrightarrow{\sim} \text{id}_X?$$

For the forwards direction, $\text{Hom}_{\mathtt{Set}}(*, -) \cong \text{id}_{\mathtt{Set}}$ so it preserves colimits. Since X is finite, we have that $X = \{x_1, \ldots, x_n\}$, hence

$$\text{Hom}(X, -) \cong \text{Hom}(\cup_i \{x_i\}, -) \cong \lim_i \text{Hom}(\{x_i\}, -).$$

Then we use finite limits commuting with filtered colimits. □

Definition 6.5. An object $X \in \mathscr{C}$ is *compact* if $\text{Hom}_{\mathscr{C}}(X, -) : \mathscr{C} \to \mathtt{Set}$ preserves filtered colimits.

Hence if $F : I \to \mathscr{C}$, with I filtered, then a map $X \to \text{colim}_I F$ factors through an $F(i)$.

Examples 6.6. Compact objects:

- $\mathtt{Set}$, compact = finite set
- $\mathtt{Vect}_F$, compact = finite dimensional
- $\mathtt{Mod}_R$, compact = finitely presented
- $\mathtt{Grp}$, compact = finitely presented
- $\mathtt{Top}$, compact = finite sets with discrete topology
- $\mathtt{Ch}$, compact = perfect chain complexes (bounded, levelwise finitely generated and projective)
- $\mathtt{sSet}$, compact = finite simplicial sets (X_n finite for each n, and there exists an m so that all non-degenerate simplices have dimension $\leq m$).

A topological space is (topologically) compact if and only if $X \in \mathcal{O}(X)$ is (categorically) compact.

Lemma 6.7. Finite colimits of compact objects are compact.

Definition 6.8. A category $\mathscr{C}$ is *presentable* if

(1) $\mathscr{C}$ is cocomplete
(2) There exists a set S of compact objects in $\mathscr{C}$ such that every object in $\mathscr{C}$ is a filtered colimit of objects in S.

We also say the "ind-completion" of S is $\mathscr{C}$, denoted $\text{Ind}(S) = \mathscr{C}$.

Theorem 6.9. $\mathscr{C}$ is presentable if and only if there is an adjunction of the form

$$\mathrm{Fun}(K^{\mathrm{op}}, \mathtt{Set}) \rightleftarrows \mathscr{C},$$

where K is some small category, and the right adjoint is fully faithful and preserves filtered colimits.

We might take K for example to to be isomorphism classes of compact objects in $\mathscr{C}$, then we have

$$\begin{aligned} \mathscr{C} &\to \mathrm{Fun}(K^{\mathrm{op}}, \mathtt{Set}) \\ X &\mapsto \left(K^{\mathrm{op}} \to \mathscr{C}\mathrm{op} \xrightarrow{\mathrm{Hom}(-,X)} \mathtt{Set}\right). \end{aligned}$$

Theorem 6.10. Suppose $\mathscr{C}$ and $\mathscr{D}$ presentable. Then $L : \mathscr{C} \to \mathscr{D}$ preserves colimits if and only if L is a left adjoint.

Cofibrantly generated model categories

Definition 6.11. Let I be a set of maps in a cocomplete category, fix λ to be an ordinal, and let $X : \lambda \to \mathscr{C}$ a functor, and suppose that $X(\alpha) \to X(\alpha+1)$ fits into

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & X(\alpha) \arrow[d] \\ B_\alpha \arrow[r] & X(\alpha+1), \end{tikzcd}$$

where $A_\alpha \to B_\alpha$ is in I. Then we say that $X(0) \to \mathrm{colim}_\lambda X$ is a *relative I-cell complex*. We say an object $Y \in \mathscr{C}$ is an *I-cell complex* if $\varnothing \to Y$ is a relative I-cell complex.

If $I = \{S^n \hookrightarrow D^{n+1}\}_{n\geq 0}$, then we are recovering the idea of CW complexes in spaces.

We denote by $\mathrm{Cell}_I(\mathscr{C})$ the class of relative I-cell complexes.

Exercise 6.12. We have that $\mathrm{Cell}_I(\mathscr{C})$ is the smallest class in $\mathscr{C}$ closed under composition, pushouts, and filtered colimits.

Theorem 6.13. *(Small object argument)* Let $\mathscr{C}$ be cocomplete, let I a set of maps in $\mathscr{C}$, and suppose that for all $A \to B$ in I, we have that A is compact with respect to the full subcategory of of I-cells in $\mathscr{C}$. Then there exists a functorial factorization of maps in $\mathscr{C}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & Y \\ & C \arrow[ur, "\delta"'] & \end{tikzcd}$$

with $\gamma \in \mathrm{Cell}_I(\mathscr{C})$ and $\delta \in \mathrm{RLP}(I)$.

Proof idea. Start with $X(0) = X$, and take a map $X(0) \to Y$. Suppose $X(\beta) = \mathrm{colim}_{\alpha<\beta} X(\alpha)$ is constructed with $X(\beta) \to Y$. Look at the set[8]

$$S = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X(\beta) \arrow[d] \\ B \arrow[r] & Y \end{tikzcd} : g \in I \right\}.$$

Denote by g_s the map $A \to B$ appearing in $s \in S$. Then we build

$$\begin{tikzcd} \amalg_{s\in S} A_s \arrow[r] \arrow[d, "\amalg_s g_s"'] & X(\beta) \arrow[d, "\in \mathrm{Cell}_I(\mathscr{C})"] \\ \amalg_{s\in S} B_s \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X(\beta+1) \end{tikzcd}$$

By UP of the pushout, there is an induced map $X(\beta+1) \to Y$. Then we claim that

$$X(0) \to \mathrm{colim}_\beta X(\beta) =: C$$

is in $\mathrm{Cell}_I(\mathscr{C})$. The only thing left to show is that $C \to Y$ is in $\mathrm{RLP}(I)$. Take

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C = \mathrm{colim}_\beta X(\beta) \arrow[d] \\ B \arrow[r] & Y. \end{tikzcd}$$

Since A is compact with respect to I-cells, the map $A \to C$ factors through some $X(\beta)$. Since $B \to Y$ factors through $X(\beta+1)$, we see that it lifts to $B \to C$. $\square$

Definition 6.14. A model category $\mathcal{M}$ is *cofibrantly generated* if there exist sets of maps I, J in $\mathcal{M}$ so that

- Cof = retracts of I-cell complexes, denoted $\widehat{\mathrm{Cell}_I(\mathscr{C})}$[9]
- Cof = $\widehat{\mathrm{Cell}_J(\mathscr{C})}$

[8]Note this set is nonempty because we can take g to be $\mathrm{id} : X(\beta) \to X(\beta)$.
[9]The hat $\widehat{}$ means "retracts of -"

and "I and J permit the small object argument."

Example 6.15. For $\texttt{Top}_{\text{Quillen}}$, we can take

$$\begin{aligned} I &= \left\{ S^n \hookrightarrow D^{n+1} \right\} \\ J &= \left\{ D^n \to D^n \times [0,1] \right\}. \end{aligned}$$

Example 6.16. For $\texttt{sSet}_{\text{Kan}}$, we can take

$$\begin{aligned} I &= \left\{ \partial\Delta^n \to \Delta^n \right\} \\ J &= \left\{ \Lambda^k_n \to \Delta^n \right\}. \end{aligned}$$

Example 6.17. For $(\texttt{Ch}_R)_{\text{proj}}$,

$$\begin{aligned} I &= \left\{ S^n \to D^{n+1} \right\} \\ J &= \left\{ 0 \to D^n \right\}. \end{aligned}$$

Example 6.18. The Strøm model structure is not cofibrantly generated in the definition above.

Theorem 6.19. (Kan — Right transfer) Let $\mathcal{M}$ be a cofibrantly generated model category and $\mathscr{C}$ is any category where there is an adjunction

$$F : \mathcal{M} \rightleftarrows \mathscr{C} : G.$$

Then $\mathscr{C}$ has a model structure where W and Fib are created by G. The model structure is cofibrantly generated by $F(I)$ and $F(J)$ if:

(1) $F(I)$ and $F(J)$ permit the small object argument
(2) $G\left(\text{Cell}_{F(J)}\right)$ are weak equivalences in $\mathcal{M}$.

For combinatorial model categories, we get an inductive argument for building cofibrant replacements.

[Rezk-Schwede-Shipley] Combinatorial model categories are always simplicially enriched.

[Dugger] Any combinatorial model category $\mathcal{M}$ is Quillen equivalent to a localization of a projective Kan one:

$$L_\tau \text{Fun}(K^{\text{op}}, \texttt{sSet}) \rightleftarrows \mathcal{M}.$$

7. Lecture 7: Thursday, February 2nd

[missed]

8. Lecture 8: Tuesday, February 7th

Last time: We had $\mathcal{M}$ a model category, and $\otimes$ a monoidal structure. We used this to give a monoidal structure on $\mathrm{Ho}(\mathcal{M})$, given by $\otimes^{\mathbb{L}}$, the *left derived tensor product*. We used this to give a homotopy theory on $\mathtt{Alg}(\mathcal{M})$, and $\mathtt{Mod}_R(\mathcal{M})$, etc.

Q: What are algebras in the homotopy category of a model structure $\mathcal{M}$? An example of interest is $\mathcal{M} = \mathtt{Top}$.

What are commutative algebras in $\mathtt{Top}$?

Theorem 8.1. (Moore) If $X \in \mathtt{CAlg}(\mathtt{Top})$, then there is a weak equivalence

$$\prod_{i=1}^{\infty} K(\pi_i(X), i) \to X.$$

Proof. Let $G_n = \pi_n(X)$. Then we take

$$0 \to F \to \mathbb{Z}[G_n] \to G_n \to 0.$$

Then we get that $\widetilde{H}_n(\vee_{g\in G_n} S^n) \cong \oplus_{g\in G_n}\widetilde{H}_n(S^n) = \mathbb{Z}[G_n]$. Using the Hurewicz theorem, there is an isomorphism

$$\pi_n(\vee S^n) \xrightarrow{\sim} \widetilde{H}_n(\vee S^n),$$

so we can pick $f_j \in \pi_n(S^n)$ for each e_j in a basis of F. This gives us a pushout

$$\begin{tikzcd} \vee_{j\in J} S^n \arrow[r] \arrow[d] & \vee_{g\in G_n} S^n \arrow[d] \\ * \arrow[r] & M(G_n, n) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This gives a map $\vee_{n\geq 1} M(G_n, n) \to X$. By universal property, we get an algebra homomorphism[10][11]

$$\mathrm{SP}(\vee_{n\geq 1} M(G_n, n)) \to X$$

The Dold–Thom theorem states that $\pi_* \mathrm{SP}(Y) \cong \widetilde{H}_*(Y)$, given some connectedness hypothesis (path-connected?). We get that

$$\mathrm{SP}(\vee_{n\geq 1} M(G_n, n)) \cong \prod_n \mathrm{SP}(M(G_n, n)) = \prod_n K(G_n, n).$$

□

[10] Here $\mathrm{SP}(-)$ denotes the infinite symmetric product, i.e. the free commutative algebra in $\mathtt{Top}$.

[11] The infinite symmetric product is left adjoint to the forgetful functor, i.e. $\mathrm{SP} : \mathtt{Top} \rightleftarrows \mathtt{CAlg}(\mathtt{Top}) : U$.

Definition 8.2. We say that $X \in \texttt{Alg}(\mathrm{Ho}(\texttt{Top}))$ if and only if X is a CW complex, with multiplication and unit

$$\begin{aligned} X \times X &\to X \\ * &\to X \end{aligned}$$

which are associative and unital *up to homotopy*.

These are also called *H-spaces*. The most prototypical example is a loop space.

Example 8.3. If X is a based space, we can build ΩX as the homotopy pullback of the two maps from a point. Concatenation gives a map $\Omega X \times \Omega X \to \Omega X$.

Example 8.4. Eilenberg-MacLane spaces $K(G, n)$ are uniquely determined up to homotopy. We have that

$$\pi_k\left(\Omega K(G, n)\right) \cong \pi_{k+1}(K(G, n))$$

therefore $\Omega K(G, n) = K(G, n-1)$.

Q: Given X an H-space, such that $\pi_0 X$ is a group, is X a loop space?

A: No, there are many grouplike H-spaces that are not equivalent to ΩX. For example $S^7 \subseteq \mathbb{O}$ the unit octonians.

Loop spaces have an extra condition. Given $w, x, y, z \in \Omega X$, there is an association $(xy)z \simeq x(yz)$. There is a pentagon witnessing the different ways to associate four elements.

We can keep going with 5 loops, 6 loops... and we get the Stasheff associahedra $K(n)$, which tell us how to concatenate n loops. These give maps

$$K(n) \times (\Omega X)^n \to \Omega X,$$

witnessing the higher associativities of concatenation. We call this an A_∞-algebra structure.

Theorem 8.5. (Stasheff) Given X connected, we have that $X \simeq \Omega Y$ for some Y if and only if X is an A_∞-algebra in spaces that is grouplike.

Rigidification: We have that $\mathrm{Ho}(\texttt{Alg}(\texttt{sSet}, \times)) \simeq \texttt{Alg}_{A_\infty}(\mathrm{Ho}(\texttt{Top}))$.

Operads

Let $\mathscr{C} = (\mathscr{C}, \otimes, I, [-,-])$ be a closed monoidal category.

Definition 8.6. An *operad* in $\mathscr{C}$ is a collection of objects $\{\mathcal{O}(j)\}_{j \geq 0}$ in $\mathscr{C}$ such that

(1) there is a right action of Σ_j on $\mathcal{O}(j)$

(2) $\mathcal{O}(0) = I$
(3) $I \to \mathcal{O}(1)$ exists in $\mathscr{C}$
(4) composition

$$\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \xrightarrow{\gamma} \mathcal{O}(j_1 + \ldots + j_k)$$

for all $k \geq 0$ and $j_1, \ldots, j_k \geq 0$ such that they are equivariant, unital, and associative.

We think about $\mathcal{O}(j)$ as an abstract way to compose j-ary operations.

Example 8.7. We let Assoc be the operad defined by

$$\mathrm{Assoc}(j) = \amalg_{\sigma \in \Sigma_j} I.$$

We can define $\mathrm{Comm}(j) = I$.

Example 8.8. If $X \in \mathscr{C}$, the *endomorphism operad* is given by

$$\mathrm{End}_X(j) = [X^{\otimes j}, X].$$

Definition 8.9. A *morphism of operads* $\mathcal{O} \to \mathcal{O}'$ is a sequence of maps $\psi_j : \mathcal{O}(j) \to \mathcal{O}'(j)$ for $g \geq 0$ that are equivariant, associative, and unital.

Definition 8.10. Given $\mathcal{O}$ an operad in $\mathscr{C}$, an $\mathcal{O}$-algebra (X, θ) in $\mathscr{C}$ is $X \in \mathscr{C}$ together with a morphism of operads $\theta : \mathcal{O} \to \mathrm{End}_X$, sending $\mathcal{O}(j) \to \mathrm{End}_X(j)$. By adjointness, we think about this as $\mathcal{O}(j) \otimes X^{\otimes j} \to X$ which are associative and unital.

This gives us a category of $\mathcal{O}$-algebras, denoted $\mathtt{Alg}_{\mathcal{O}}(\mathscr{C})$.

Example 8.11. We have that

$$\mathtt{Alg}_{\mathrm{Assoc}}(\mathscr{C}) \cong \mathtt{Alg}(\mathscr{C})$$
$$\mathtt{Alg}_{\mathrm{Comm}}(\mathscr{C}) \cong \mathtt{CAlg}(\mathscr{C}).$$

We have that $\mathcal{M}$ is a monoidal model category if θ is nice enough, i.e. we get an adjunction

$$\mathcal{M} \rightleftarrows \mathtt{Alg}_{\mathcal{O}}(\mathcal{M}).$$

Definition 8.12. A *monad* in $\mathscr{C}$ is an algebra in $(\mathrm{Fun}(\mathscr{C}, \mathscr{C}), \circ, \mathrm{id}_{\mathscr{C}})$. That is, $M \in \mathtt{Alg}(\mathrm{Fun}(\mathscr{C}, \mathscr{C}))$ if we have $M : \mathscr{C} \to \mathscr{C}$ together with $\mu : M \circ M \Rightarrow M$, and $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathscr{C}$ that are associative and unital.

Example 8.13. Every adjunction $L : \mathscr{C} \rightleftarrows \mathscr{D} : R$ defines a monad RL.

Definition 8.14. An *algebra* (X, θ) over a monad (M, μ, η) in $\mathscr{C}$ is $X \in \mathscr{C}$ together with maps $\theta : M(X) \to X$ such that they are associative and unital, meaning that the diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & M(X) \arrow[d, "\theta"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} M(M(X)) \arrow[r, "\mu_{MX}"] \arrow[d, "M(\theta)"'] & M(X) \arrow[d, "\theta"] \\ M(X) \arrow[r, "\theta"'] & X. \end{tikzcd}$$

Definition 8.15. If M is a monad, a *morphism of M-algebras* $(X, \theta) \to (X', \theta')$ is a map $f : X \to X'$ in $\mathscr{C}$ so that the diagram commutes

$$\begin{tikzcd} MX \arrow[r, "\theta"] \arrow[d, "Mf"'] & X \arrow[d, "f"] \\ MX' \arrow[r, "\theta'"'] & X'. \end{tikzcd}$$

Example 8.16. Consider R a commutative ring, and the adjunction

$$- \otimes_{\mathbb{Z}} R : \mathtt{Ab} \rightleftarrows \mathtt{Mod}_R : U.$$

This forms a monad $M := - \otimes_{\mathbb{Z}} R : \mathtt{Ab} \to \mathtt{Ab}$. Then $\mathtt{Alg}_M(\mathtt{Ab})$ is equivalent to $\mathtt{Mod}_R$.

This is not always true! When this happens we say the adjunction is *monadic*.

Given a monadic adjunction

$$\mathscr{C} \rightleftarrows \mathscr{D} = \mathtt{Alg}_{RL}(\mathscr{C}),$$

we get a ton of things for free:

- R will preserve colimits if RL does
- get things like free monadic resolutions, bar constructions, etc.

9. Lecture 9: Thursday, February 9th

[missed]

10. Lecture 10: Thursday, February 16th

Definition 10.1. A simplicial set $\mathscr{C}$ is an ∞-*category* (or *quasi-category*) if it has inner horn filling — for all $0 < k < n$, we have

$$\begin{tikzcd} \Lambda^k_n \arrow[r] \arrow[d] & \mathscr{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

We shall see that ∞-categories are fibrant objects in `sSet` with the Joyal model structure.

Example 10.2.

(1) If $\mathscr{C}$ is a Kan complex, then it is an ∞-category
(2) If $\mathscr{C}$ is a category, then $N\mathscr{C}$ is an ∞-category.

Definition 10.3. Given an ∞-category $\mathscr{C}$, the *objects* of $\mathscr{C}$ are the vertices,[12] the *morphisms* are 1-simplices. We have *source* and *target* maps $d^1, d^0 : \mathscr{C}_1 \to \mathscr{C}_0$.[13] We define the *set of morphisms* from X to Y as the pullback

$$\begin{tikzcd} \hom_{\mathscr{C}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{C}_1 \arrow[d, "{(s,t)}"] \\ \mathscr{C}_1 \arrow[r, "{(X,Y)}"'] & \mathscr{C}_0 \times \mathscr{C}_0. \end{tikzcd}$$

We have that $\hom_{\mathscr{C}}(X,Y)$ is the set of vertices of a simplicial set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$, which forms a Kan complex.

Definition 10.4. Given $X \in \mathscr{C}$ we define $\mathrm{id}_X \in \mathscr{C}_1$ by $s^0(X)$.

How do we compose? Composition won't be unique, but it will be unique *up to homotopy*.

Given $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, this determines a map of simplicial sets $\Lambda^2_1 \to \mathscr{C}$. By inner horn lifting, we have

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & \mathscr{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

[12] $X \in \mathscr{C}$ means $X \in \mathscr{C}_0$

[13] We write $f : X \to Y$ in $\mathscr{C}$ to mean $f \in \mathscr{C}_1$ with $s(f) = X$ and $t(f) = Y$.

We refer to the filling as a *composition*:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z. \end{tikzcd}$$

Exercise 10.5. Given an ∞-category $\mathscr{C}$, how can we define $\mathscr{C}^{\mathrm{op}}$? Would want that $N(\mathscr{C}^{\mathrm{op}}) \cong (N\mathscr{C})^{\mathrm{op}}$.[14]

Detour: Let $A \in \mathtt{Cat}$, and let $\mathscr{C}$ be a cocomplete category. Recall that $\mathrm{Fun}(A^{\mathrm{op}}, \mathtt{Set})$ is the free cocompletion. Given a functor $A \to \mathscr{C}$, by universal property there is a map

$$\begin{tikzcd} A \arrow[r, "Q"] \arrow[d, hook] & \mathscr{C} \\ \mathrm{Fun}(A^{\mathrm{op}}, \mathtt{Set}) \arrow[ur, dashed, "|-|_Q"'] & \end{tikzcd}$$

This gives us an adjunction

$$|-|_Q : \mathrm{Fun}(A^{\mathrm{op}}, \mathtt{Set}) \rightleftarrows \mathscr{C} : \mathrm{Sing}_Q(-).$$

Here $\mathrm{Sing}_Q(-) = \mathrm{Hom}_{\mathscr{C}}(Q(-), X)$.

Example 10.6. If $\mathscr{C} = \mathtt{Top}$, then we can take $\Delta_{\mathtt{Top}} : \Delta \to \mathtt{Top}$, sending $[n]$ to $\Delta^n_{\mathtt{Top}}$. In this case, we recover the usual $|-|$ and $\mathrm{Sing}(-)$ adjunction.

Example 10.7. If $\mathscr{C} = \mathtt{Cat}$, there is a functor $\Delta \to \mathtt{Cat}$ sending $[n]$ to the associated poset category. We get an associated adjunction:

$$\tau : \mathtt{sSet} \rightleftarrows \mathtt{Cat} : N,$$

since $N = \mathrm{Hom}_{\mathtt{Cat}}([-], \mathscr{C})$.

Exercise 10.8. Describe $\tau : \mathtt{sSet} \to \mathtt{Cat}$ explicitly.

We call τ the fundamental category functor, essentially it will produce the homotopy category of an ∞-category.

Definition 10.9. Given an ∞-category $\mathscr{C}$, two morphisms $f : X \to Y$ and $g : Y \to Z$ are *homotopic*, written $f \simeq g$, if there exists a 2-simplex $\sigma : \Delta^2 \to \mathscr{C}$ with

[14] Every Kan complex has that $\mathscr{C}^{\mathrm{op}} \cong \mathscr{C}$.

boundary (g, f, id_X):

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

Example 10.10. If $\mathscr{C}$ is an ordinary category, then in $N\mathscr{C}$, we have that $f \simeq g$ if and only if $f = g$.

Proposition 10.11. Given $\mathscr{C}$ an ∞-category, and $X, Y \in \mathscr{C}$, the homotopy relation provides an equivalence relation on $\hom_{\mathscr{C}}(X, Y)$.

Definition 10.12. We denote by $[f]$ the homotopy class of f.

Sketch. We first need to show reflexivity, so we want to find a 2-cell witnessing

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, equal] \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

We check that this is $s_0(f)$, where $f \in \mathscr{C}_1$, and $s_0 : \mathscr{C}_1 \to \mathscr{C}_2$.

For symmetry, suppose we have $f \simeq g$. We want to show $g \simeq f$. We can fill a Λ^3_2 witnessing this.

Transitivity is left as an exercise. □

Definition 10.13. Given $\mathscr{C}$ an ∞-category, define the 1-category $\mathrm{Ho}(\mathscr{C})$ to be the *homotopy category*, given by

$$\begin{aligned} \mathrm{ob}\mathrm{Ho}(\mathscr{C}) &= \mathscr{C}_0 \\ \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(X, Y) &= \hom_{\mathscr{C}}(X, Y)/\simeq . \end{aligned}$$

In order to show this, we need to argue that composition is well-defined up to homotopy.

Suppose we have two compositions

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h_1"'] & & Z. \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h_2"'] & & Z. \end{tikzcd}$$

We want to argue that $h_1 \simeq h_2$. This can be done by filling the horn of a 3-simplex.

Proposition 10.14. When we restrict the adjunction $\tau \dashv N$ to ∞-categories, we get an adjunction

$$\mathrm{Ho}(-) : \mathtt{Cat}_\infty \rightleftarrows \mathtt{Cat} : N.$$

The way to compose arrows is contractible.

Theorem 10.15. The inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a map

$$\mathrm{Hom}_*(\Delta^2, \mathscr{C}) \to \mathrm{Hom}_*(\Lambda^2_1, \mathscr{C})$$

which is a trivial Kan fibration for any $\mathscr{C} \in \mathtt{Cat}_\infty$.

Here Hom_* is the *internal hom*, where $\mathrm{Hom}_*(X, Y) := \mathrm{Hom}_{\mathtt{sSet}}(\Delta^* \times X, Y)$.

Proof. Exercise $\square$

As a consequence, we can take a pullback diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_*(\Delta^2, \mathscr{C}) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Hom}_*(\Lambda^2_1, \mathscr{C}). \end{tikzcd}$$

Then the pullback $P \to \Delta^0$ should be a trivial fibration, meaning that P is a contractible Kan complex.

Definition 10.16. Given $\mathscr{C}$ an ∞-category and $X, Y \in \mathscr{C}$, recall that a map $f : X \to Y$ corresponds to $\Delta^1 \to \mathscr{C}$ whose faces are X and Y. An *n-morphism* from X to Y is simply a map $\Delta^n \to \mathscr{C}$ such that $\Delta^{\{0,\dots,n-1\}} = X$ and $\Delta^{\{n\}} = Y$.

For $n \geq 2$, all n-morphisms are invertible in some sense.

Definition 10.17. Two objects X and Y in $\mathscr{C}$ are *equivalent*, written $X \simeq Y$, if there exists a 1-morphism $f : X \to Y$ in $\mathscr{C}$ such that $[f]$ in $\mathrm{Ho}(\mathscr{C})$ is an *isomorphism.*

Definition 10.18. An ∞-*groupoid* is an ∞-category for which $\mathrm{Ho}(\mathscr{C})$ is a groupoid, meaning all the 1-morphisms are equivalences.

Theorem 10.19. (Homotopy hypothesis) We get that $\mathscr{C}$ is an ∞-groupoid if and only if $\mathscr{C}$ is a Kan complex.

11. Lecture 11: Tuesday, February 21st

[missed]

12. Lecture 12: Thursday, February 23rd

Adjoint functors and colimits

Last time: Recall that a 1-morphism in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$[15] is precisely a natural transformation $\eta : F \to G$, where $F, G : \mathscr{C} \to \mathscr{D}$. In other words, it is $\eta : \Delta^1 \times \mathscr{C} \to \mathscr{D}$.

We have $\texttt{hQCat} = \mathrm{Ho}(\texttt{Cat}_\infty)$, where objects are infinity categories, and the morphisms are

$$\mathrm{Hom}_{\texttt{hQCat}}(\mathscr{C}, \mathscr{D}) = \Pi_0 \left(\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\simeq}\right).$$

That is, it is the set of equivalence classes of functors $\mathscr{C} \to \mathscr{D}$.

If $\mathscr{C}$ is an ∞-category, and $X, Y \in \mathscr{C}$, we defined $\mathrm{Hom}_{\mathscr{C}}(X, Y)_\bullet$ to be the simplicial set given by the pullback

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X,Y)_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathscr{C}) \arrow[d] \\
\Delta^0 \arrow[r] & \mathrm{Fun}(\{0\}, \mathscr{C})_\bullet \times \mathrm{Fun}(\{1\}, \mathscr{C}).
\end{tikzcd}$$

Proposition 12.1. We have that $\mathrm{Hom}_{\mathscr{C}}(X, Y) \in \texttt{Kan}$.

Sketch. This follows from a more general fact that for $A \hookrightarrow B$ a subsimplicial set with $A_0 = B_0$, and $\mathscr{C}$ an ∞-category, then P is always a Kan complex

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(B, \mathscr{C}) \arrow[d] \\
\Delta^0 \arrow[r, "f"'] & \mathrm{Fun}(A, \mathscr{C}).
\end{tikzcd}$$

Need to show that every u in $\mathrm{Fun}(B, \mathscr{C})_1$ *in the pullback* is a weak equivalence. We have an evaluation map for every $b \in B_0 = A_0$, given by $\mathrm{ev}_b : \mathrm{Fun}(B, \mathscr{C}) \to \mathrm{Fun}(\{b\}, \mathscr{C})$, mapping u to $u_{f(b)}$. We claim that $u_{f(b)} = \mathrm{id}_{f(b)}$, since the diagram commutes

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathscr{C}) \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\{b\}, \mathscr{C}) \\
& \mathrm{Fun}(A, \mathscr{C}) \arrow[ur] &
\end{tikzcd}$$

□

Adjoint functors

[15] The simplicial set $\mathrm{Fun}(\Delta^\bullet \times \mathscr{C}, \mathscr{D})$

Definition 12.2. Let $F : \mathscr{C} \to \mathscr{D}$, and $G : \mathscr{D} \to \mathscr{C}$ be functors of ∞-categories. We say that $F \dashv G$ if there exist natural transformations $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ ad $\varepsilon : FG \to \mathrm{id}_{\mathscr{D}}$ so that:

(1) there exists $\Delta^2 \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ witnessing

$$\begin{tikzcd} & FGF \arrow[dr, "\varepsilon \mathrm{id}_F"] & \\ F\mathrm{id}_{\mathscr{C}} \arrow[ur, "\mathrm{id}_{\mathscr{C}}\eta"] \arrow[rr, "\mathrm{id}"'] & & \mathrm{id}_{\mathscr{C}} F. \end{tikzcd}$$

(2) there exists $\Delta^2 \to \mathrm{Fun}(\mathscr{D}, \mathscr{C})$ witnessing

$$\begin{tikzcd} & GFG \arrow[dr, "\mathrm{id}\varepsilon"] & \\ \mathrm{id}_{\mathscr{C}} G \arrow[ur, "\eta \mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & & G\mathrm{id}_{\mathscr{C}}. \end{tikzcd}$$

Remark 12.3. We have that $\eta : \mathrm{id} \to GF$ depends only on $[\eta]$ in $\mathrm{Ho}(\mathrm{Fun}(\mathscr{C}, \mathscr{D}))$. If η is given, then ε is unique up to homotopy.

Example 12.4. If $\mathscr{C}$ and $\mathscr{D}$ are ordinary categories, then we have a 1-categorical adjunction

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : G$$

if and only if we have an ∞-categorical adjunction

$$NF : N\mathscr{C} \rightleftarrows N\mathscr{D} : NG.$$

Example 12.5. If $X, Y \in \mathtt{Kan}$, then $F : X \to Y$ is an adjoint if and only if F is a homotopy equivalence of simplicial sets. The unit and counit become the witnesses of homotopy equivalence.

Remark 12.6. If we have an adjunction $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ of ∞-categories, then F and G are homotopy equivalences of simplicial sets. The converse is not true in general.

Exercise 12.7. If $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories, then it is both a left and right adjoint functor.

Proposition 12.8. Given $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ of ∞-categories, then

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathscr{C}) \rightleftarrows \mathrm{Ho}(\mathscr{D}) : \mathrm{Ho}(G)$$

is an adjunction of 1-categories. That is, **if** we know $F \dashv G$ in ∞-categories, then to check if $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ is a unit, it is enough to check that $\mathrm{Ho}(\eta)$ is the unit.

However the converse is not true!

Warning: Suppose we take $F : \Delta^0 \to X$ with $X \in \mathtt{Kan}$ simply connected, and F picks $x \in X_0$. Then $\mathrm{Ho}(F) \dashv \mathrm{Ho}(G)$ because $\mathrm{Ho}(X)$ will be simply connected. But it does not imply that $F \dashv G$ unless X is contractible.

There $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{D})}(FC, D) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(C, GD)$ for any $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

Theorem 12.9. Take $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ functors of ∞-categories. Then $F \dashv G$ with unit η if and only if the composite

$$\mathrm{Hom}_{\mathscr{D}}(FC, D) \xrightarrow{G} \mathrm{Hom}_{\mathscr{C}}(GFC, GD) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathscr{C}}(C, GD)$$

is a weak homotopy equivalence between Kan complexes (aka a homotopy equivalence) for all C, D.

The forward direction is straightforward, but the backwards direction uses (co)cartesian fibration stuff.

Limits and colimits

Recall that if $\mathscr{C}$ is an ordinary category, then $i \in \mathscr{C}$ is *initial* if for all $X \in \mathscr{C}$, there is a unique $i \xrightarrow{!} X$. That is, $\mathrm{Hom}_{\mathscr{C}}(i, X) = *$.

Definition 12.10. In an ∞-category $\mathscr{C}$, we have that $i \in \mathscr{C}$ is *initial* if $\mathrm{Hom}_{\mathscr{C}}(i, X) \simeq *$ is contractible for all $X \in \mathscr{C}$.

Definition 12.11. Let $\mathscr{C}$ be an ∞-category, and $K_\bullet \in \mathtt{sSet}$. Then for any $X \in \mathscr{C}$, denote by $\underline{X} \in \mathrm{Fun}(K, \mathscr{C})$ the constant functor valued at X. The assignment $X \mapsto \underline{X}$ defines a diagonal map

$$\Delta : \mathscr{C} \to \mathrm{Fun}(K, \mathscr{C}).$$

This is defined by precomposing with $K \to \Delta^0$, and looking at $\mathscr{C} \simeq \mathrm{Fun}(\Delta^0, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$.

Definition 12.12. Let $u : K \to \mathscr{C}$ be a diagram. We say a natural transformation $\alpha : \underline{L} \to u$ exhibits $L \in \mathscr{C}$ as *a limit of* u if for all $X \in \mathscr{C}$, we have that the composite

$$\mathrm{Hom}_{\mathscr{C}}(X, L) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{X}, \underline{L}) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{X}, u)$$

is a (weak) homotopy equivalence of Kan complexes.

Definition 12.13. We say that $\beta : u \to \underline{C}$ exhibits C as a *colimit of* u if, for all $Y \in \mathscr{C}$, the composite

$$\mathrm{Hom}_{\mathscr{C}}(C, Y) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{C}, \underline{Y}) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(u, \underline{C})$$

is a (weak) homotopy equivalence.

Note that if α or β exist, they are unique up to equivalence.

Example 12.14. If $\mathscr{C}$ is an ordinary category, then $u : K \to N\mathscr{C}$ is equivalent to a map $\tau(u) : \tau K \to \mathscr{C}$. We can check that $L \in \mathscr{C}$ is $\lim(\tau u)$ in a 1-categorical sense if and only if $L \in \mathscr{C}$ is a limit of u in an ∞-categorical sense.

Example 12.15. Let $f : X \to Y$ in an ∞-cat $\mathscr{C}$. Then f is an equivalence if and only if f exhibits Y as a colimit $\{X\} \to \mathscr{C}$, if and only if f exhibits X as a limit $\{Y\} \to \mathscr{C}$.

Example 12.16. Taking the identity diagram $\varnothing \to \mathscr{C}$, the notion of limit/colimit matches the notion of terminal/initial object.

Proposition 12.17. A limit $L \in \mathscr{C}$ is unique up to homotopy. Therefore we usually define it as $\lim_K(u)$.

Proposition 12.18. We have that $\mathscr{C}$ admits all K-indexed limits if and only if

$$\Delta : \mathscr{C} \to \mathrm{Fun}(K, \mathscr{C})$$

is a left adjoint. The right adjoint is given by $\lim_K(-)$.

Equalizers are limits along $\Delta^1 \amalg_{\partial\Delta^1} \Delta^1$, pullbacks are limits along $\Delta^1 \times \Delta^1 - (0,0)$, etc.

13. Lecture 13: February 28th

[missed]

14. Lecture 14: March 21st

Straightening/unstraightening[16]

Motivation: Let X be a space, and let $\mathrm{Cov}(X)$ denote the 1-category of covering spaces of X, so that in particular the fibers f^{-1} of $f : E \to X$ are discrete sets. This defines a map in `Top` from

$$X \to \mathtt{Set}^{\cong},$$

to sets with the discrete topology. Another way to think about this is as a functor

$$\mathrm{St} : \mathrm{Cov}(X) \to \mathrm{Fun}(\Pi_1(X), \mathtt{Set})$$
$$(E \xrightarrow{p} X) \mapsto \left[x \mapsto f^{-1}(x)\right].$$

[16] Also called the *Grothendieck construction* or the ∞-*category of elements*.

A path from x to y (a morphism in $\Pi_1(X)$) induces a set map $f^{-1}(x) \to f^{-1}(y)$.

This is an equivalence of categories! This is called the *fundamental theorem of covering spaces.*

This is a first instance of *straightening*.

If we view X as an ∞-groupoid, then $\Pi_1(X) = \mathrm{Ho}(X)$ is its homotopy category, and we have that

$$\mathrm{Fun}(\Pi_1(X), \mathtt{Set}) \cong \mathrm{Fun}(X, N(\mathtt{Set})),$$

since nerve is right adjoint to the homotopy category.

We can denote by $\mathrm{Cov}_X \subseteq \mathcal{S}/X$ to be the full subcategory of the infinity category of spaces over X spanned by covering spaces. Then we want to show that

$$\mathrm{Cov}_X \simeq \mathrm{Fun}(X, N(\mathtt{Set})).$$

We have an unstraightening functor

$$\mathrm{Unst} : \mathrm{Fun}(X, N(\mathtt{Set})) \to \mathrm{Cov}_X,$$

given by sending some $F : X \to N(\mathtt{Set})$ to the pullback[17]

$$\begin{tikzcd} E \arrow[r] \arrow[d] & N(\mathtt{Set}_*)^{\simeq} \arrow[d] \\ X \arrow[r, "F"'] & N(\mathtt{Set})^{\simeq} \end{tikzcd}$$

More generally, if we don't require the fibers to be discrete, then we can take $f : E \to X$ to be any continuous map. Then we get a functor[18]

$$\begin{aligned} \mathrm{St} : \mathcal{S}/X &\to \mathrm{Fun}(X, \mathcal{S}) \\ (E \xrightarrow{f} X) &\mapsto \left[x \mapsto f^{-1}(x)\right]. \end{aligned}$$

Unstraightening is of the form

$$\begin{aligned} \mathrm{Unst} : \mathrm{Fun}(X, \mathcal{S}) &\to \mathcal{S}/X \\ F &\mapsto \mathrm{hocolim}_X F = \cup_{x \in X} F^{-1}(x)/\sim. \end{aligned}$$

Let X be connected and suppose $X \simeq BG$. Then we define *G-modules in spaces* to be

$$\mathtt{Mod}_G(\mathcal{S}) := \mathrm{Fun}(BG, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}/BG.$$

[17]Note that $N(\mathtt{Set}^{\simeq}) = N(\mathtt{Set})^{\simeq}$.

[18]By $\mathrm{Fun}(X, \mathcal{S})$ we might mean $\mathrm{Fun}(\mathrm{Sing}(X), N_\Delta(\mathtt{Kan}))$.

If we take some $M : BG \to \mathcal{S}$, and we post-compose with sections $\mathcal{S}/BG \to \mathcal{S}$, then M maps to M^{hG}.

More generally, given $F : X \to \mathcal{S}$, the limit $\lim_X \mathcal{S}$ is given by

$$\mathrm{Fun}(X, \mathcal{S}) \xrightarrow{\text{Unst}} \mathcal{S}/X \xrightarrow{\text{sections}} \mathcal{S}.$$

Goal: Generalize this approach where X is replaced by an ∞-category $\mathscr{C}$ and $\mathcal{S}$ is replaced by $\mathtt{Cat}_\infty$. That is, we want to relate $\mathrm{Fun}(\mathscr{C}, \mathtt{Cat}_\infty)$ with some subcategory of $\mathtt{Cat}_\infty/\mathscr{C}$.

If $f : \mathcal{E} \to \mathscr{C}$, what requirement do we need to make sense of an associated functor

$$\begin{aligned} F : \mathscr{C} &\to \mathtt{Cat}_\infty \\ X &\mapsto f^{-1}(X). \end{aligned}$$

That is, how can we coherently choose our fibers.

Given $X \in \mathscr{C}$, we could take a pullback in $\mathtt{Cat}_\infty$:

$$\begin{tikzcd} f^{-1}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d] \\ \Delta^0 \arrow[r, "X"'] & \mathscr{C}. \end{tikzcd}$$

If we choose $\mathtt{sSet}_{\text{Joyal}}$ as our model, we would need $\mathcal{E} \to \mathscr{C}$ to be an inner fibration (RLP wrto inner horns) to get the pullback $f^{-1}(X)$ to be a quasi-category. If we instead say "pullback in quasi-categories," this requirement goes away.

Given $f : \mathcal{E} \to \mathscr{C}$ and $X \to Y$ in $\mathscr{C}$, how can we define $f^{-1}(X) \to f^{-1}(Y)$ in $\mathtt{Cat}_\infty$?

Need: If $\phi : X \to Y$ in $\mathscr{C}$ and $E_X \in \mathcal{E}$ such that $f(E_X) = X$, then there exists some $E_Y \in \mathcal{E}$ and $\phi_! : E_X \to E_Y$ in $\mathcal{E}$ so that $f(\phi_!) = \phi$, and that is universal in the following sense: for all $Z \in \mathscr{C}$ and for all $\psi : X \to Z$ in $\mathscr{C}$ for all $\overline{\psi} : E_X \to E_Z$ in $\mathcal{E}$ where $f(\overline{\psi}) = \psi$, if there exists $\gamma : Y \to Z$ then there exists a unique map $\overline{\gamma} : E_Y \to E_Z$ in $\mathcal{E}$ so that $f(\overline{\gamma}) = \gamma$ and $\overline{\gamma} \circ \phi_! = \overline{\psi}$.

We say that $\phi_! : E_X \to E_Y$ is a *cocartesian lift* of ϕ.

Definition 14.1. We say that $f : \mathcal{E} \to \mathscr{C}$ is a *cocartesian fibration* if for all $E_X \in \mathcal{E}$, for all $\phi : X \to Y$ with $f(E_X) = X$, there exists a cocartesian lift of ϕ.

Two cocartesian lifts over the same map are equivalent.

Given $f : \mathcal{E} \to \mathscr{C}$, $X \in \mathscr{C}$, $\phi : X \to Y$ in $\mathscr{C}$, we say $\phi_! : E_X \to E_Y$ is a cocartesian lift if the following is a pullback diagram in spaces:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{E}}(E_Y, E_Z) \arrow[r, "(\phi_!)^*"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{\mathscr{C}}(E_X, E_Z) \arrow[d, "f"] \\ \mathrm{Hom}_{\mathscr{C}}(Y, Z) \arrow[r, "\phi^*"'] & \mathrm{Hom}_{\mathscr{C}}(X, Z), \end{tikzcd}$$

for any $Z \in \mathscr{C}$. In particular, taking maps from Δ^0 to the top right and bottom left picks out $\overline{\psi}$ and γ, respectively, so that $\gamma \circ \phi = \psi$, and the universal property of the pullback says that there exists $\overline{\gamma} : E_Y \to E_Z$ so that $\overline{\gamma}\phi_! = \overline{\psi}$ and $f(\overline{\gamma}) = \gamma$.

Definition 14.2. We define $\mathrm{coCart}(\mathscr{C}) \subseteq \mathtt{Cat}_\infty/\mathscr{C}$ to be the subcategory of cocartesian fibrations $\mathcal{E} \to \mathscr{C}$, with morphisms

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "G"] \arrow[dr, "f"'] & & \mathcal{E}' \arrow[dl, "f'"] \\ & \mathscr{C}, & \end{tikzcd}$$

so that G sends f-cocartesian lifts to f'-cocartesian lifts.

In this case, straightening defines a functor

$$\mathrm{St} : \mathrm{coCart}(\mathscr{C}) \to \mathrm{Fun}(\mathscr{C}, \mathtt{Cat}_\infty),$$

sending $f : \mathcal{E} \to \mathscr{C}$ to the functor

$$\begin{aligned} \mathscr{C} &\to \mathtt{Cat}_\infty \\ X &\mapsto f^{-1}(X) \\ (X \xrightarrow{\phi} Y) &\mapsto \left[f^{-1}(X) \xrightarrow{\phi_!} f^{-1}(Y) \right]. \end{aligned}$$

Example 14.3. Let $f : X \to Y$ in $\mathcal{S}$. All lifts are cocartesian lifts. We say that a *left fibration* is a cocartesian fibration where every lift is cocartesian.

Example 14.4. Suppose $\mathscr{C}$ is an ordinary category. Then we can define a new category whose objects are $f : X \to Y$ in $\mathscr{C}$, and whose morphisms are

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "v"] \\ X' \arrow[u, "u"] \arrow[r, "f'"'] & Y'. \end{tikzcd}$$

This defines what we call the *twisted arrow category* $\mathrm{Tw}(\mathscr{C})$. There is a natural functor

$$\begin{aligned} \mathrm{Tw}(\mathscr{C}) &\xrightarrow{\mathrm{Ev}} \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \\ (X \xrightarrow{f} Y) &\mapsto (X, Y). \end{aligned}$$

This is a left fibration, by composition. Straightening this, we get

$$\begin{aligned} \mathrm{St}(\mathrm{Ev}) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} &\to \mathtt{Set} \\ (X, Y) &\mapsto \mathrm{Ev}^{-1}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y). \end{aligned}$$

Example 14.5. If $\mathscr{C}$ is an ∞-category, we can define a twisted arrow category in a similar way

$$\begin{aligned} \mathrm{Tw}(\mathscr{C}) : \Delta^{\mathrm{op}} &\to \mathtt{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathtt{sSet}}(\Delta^{2n+1}, \mathscr{C}), \end{aligned}$$

where the n-simplices of $\mathrm{Tw}(\mathscr{C})$ should be thought of as

$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & X_n \arrow[l] \arrow[d] \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_n. \end{tikzcd}$$

We can define

$$\begin{aligned} \ell : \mathrm{Tw}(\mathscr{C}) &\to \mathscr{C}^{\mathrm{op}} \\ r : \mathrm{Tw}(\mathscr{C}) &\to \mathscr{C}, \end{aligned}$$

by precomposition with $\Delta^n \hookrightarrow \Delta^{2n+1}$. These assemble to give

$$\mathrm{Tw}(\mathscr{C}) \xrightarrow{\mathrm{Ev}} \mathscr{C}^{\mathrm{op}} \times \mathscr{C},$$

and we have $\mathrm{Hom}_{\mathscr{C}}(X, Y) = \mathrm{Ev}^{-1}(X, Y) \in \mathcal{S}$. This evaluation map is a left fibration, left fibrations are preserved under pullback, and left fibrations over Δ^0 are Kan complexes. Therefore $\mathrm{Ev}^{-1}(X)$ is a space.

Example 14.6. Let $X \in \mathscr{C}$. Then we can take

$$\begin{tikzcd} \ell^{-1}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tw}(\mathscr{C}) \arrow[d, "\ell"] \\ \Delta^0 \arrow[r, "X"'] & \mathscr{C}^{\mathrm{op}}. \end{tikzcd}$$

We define $\mathscr{C}_{X/} := \ell^{-1}(X)$, and $r^{-1}(Y) := \mathscr{C}_{/Y}$.

Theorem 14.7. *(Straightening/unstraightening)* If $\mathscr{C}$ is an ∞-category, we can define its *unstraightening* as

$$\begin{aligned} \mathrm{Unst} : \mathrm{Fun}(\mathscr{C}, \mathtt{Cat}_\infty) &\to \mathrm{coCart}(\mathscr{C}) \\ F &\mapsto \mathrm{colim}\left(\mathrm{Tw}(\mathscr{C}) \xrightarrow{\mathrm{Ev}} \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \xrightarrow{\mathscr{C}_{/\cdot} \times F} \mathtt{Cat}_\infty\right). \end{aligned}$$

That composite sends

$$\begin{aligned} \mathrm{Tw}(\mathscr{C}) \xrightarrow{\mathrm{Ev}} \mathscr{C}^{\mathrm{op}} \times \mathscr{C} &\xrightarrow{\mathscr{C}_{/\cdot} \times F} \mathtt{Cat}_\infty \\ (X \xrightarrow{f} Y) &\mapsto \mathscr{C}_{X/} \times F(Y). \end{aligned}$$

This forms an equivalence with St.

There is an equivalence

$$\mathrm{St} : \mathrm{LFib}(\mathscr{C}) \leftrightarrows \mathrm{Fun}(\mathscr{C}, \mathcal{S}) : \mathrm{Unst}.$$

If $\mathscr{C} = X \in \mathcal{S}$, then $\mathrm{coCart}(X) = \mathtt{Cat}_\infty/X$.

If $\mathscr{C} = N(\mathscr{D})$, this recovers the usual Grothendieck construction.

If $F : \mathscr{C} \to \mathtt{Cat}_\infty$, then

$$\mathrm{colim} F = \mathrm{Unst}(\mathscr{C})[\text{cocart. edges}^{-1}]$$

15. Lecture 15: March 23rd

Unstraightening monoidal structures

Recall $\mathcal{S} \simeq N(\mathtt{sSet})[W_{\mathrm{Kan}}^{-1}]$ the ∞-category of spaces. If $X \to Y$ is a map in $\mathcal{S}$ we are meaning that $X \to Y$ is a map in $\mathrm{Ho}(\mathtt{sSet})$ *not* that $X \to Y$ is any map in $\mathtt{sSet}$.

Example 15.1. If we have $X \to Y$ in $\mathcal{S}$, then $X \to Y$ is a left fibration. If X and Y are in $\mathtt{Kan}$ and $X \to Y$ this *does not imply* that $X \to Y$ must be a left fibration. What is true is that if $X \to Y$ is a Kan fibration, then $X \to Y$ is a left fibration.

We have $\mathtt{Cat}_\infty \simeq N(\mathtt{sSet})[W_{\mathrm{Joyal}}^{-1}]$, so $f : \mathscr{C} \to \mathscr{D}$ in $\mathtt{Cat}_\infty$ means

$$\begin{tikzcd} f^{-1}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{C} \arrow[d] \\ \Delta^0 \arrow[r, "X"'] & \mathscr{D}. \end{tikzcd}$$

So we always want it to be a fibration.

That is, a map $f : \mathscr{C} \to \mathscr{D}$ in $\mathtt{Cat}_\infty$ is not the same as $\mathscr{C} \to \mathscr{D}$ of quasi-categories in $\mathtt{sSet}$.

In $\mathtt{Cat}_\infty$, $\mathscr{C} \to \mathscr{D}$ is a cocartesian fibration if there exists a cocartesian lift on any fiber.

If $\mathscr{C}, \mathscr{D}$ are quasi-categories in $\mathtt{sSet}_{\mathrm{Joyal}}$, then $f : \mathscr{C} \to \mathscr{D}$ is a cocartesian fibration if f is an *inner fibration* (RLP inner horns) AND there is a cocartesian lift of any fiber. The inner fibration condition guarantees that the fibers are also infinity categories.

Straightening definition last time was wrong. Last time, we had

$$\begin{aligned} \mathrm{Unst} : \mathrm{Fun}(\mathscr{C}, \mathtt{Cat}_\infty) &\xrightarrow{\sim} \mathrm{coCart}(\mathscr{C}) \\ F &\mapsto \left(\mathcal{E} \xrightarrow{\mathrm{Unst}(F)} \mathscr{C}\right). \end{aligned}$$

is an equivalence of categories, where

$$\mathcal{E} = \mathrm{colim}\left(\mathrm{Tw}(\mathscr{C})^{\mathrm{op}} \to \mathscr{C} \times \mathscr{C}^{\mathrm{op}} \xrightarrow{F \times \mathscr{C}_{\bullet /}} \mathtt{Cat}_\infty\right).$$

Example 15.2. Take $\mathscr{C} = *$. Then $\mathrm{Fun}(*, \mathtt{Cat}_\infty) = \mathtt{Cat}_\infty$. We have that $\mathrm{coCart}(*) = \mathtt{Cat}_\infty$, and that $\mathrm{Tw}(*) = *^{\mathrm{op}} = *$. The composite sends

$$\begin{aligned} \mathrm{Tw}(*)^{\mathrm{op}} \to * \times *^{\mathrm{op}} \to \mathtt{Cat}_\infty& \\ * \mapsto (*, *) \mapsto *A \times * &= A. \end{aligned}$$

Example 15.3. Take $\mathscr{C} = 1 = 0 \to 1$. A functor $F : 1 \to \mathtt{Cat}_\infty$ is exactly a functor $F : \mathscr{A} \to \mathscr{D}$ in $\mathtt{Cat}_\infty$. We see that $\mathrm{Tw}(1)$ has three objects, being $0 = 0$, $0 \to 1$ and $1 = 1$. The identity ones both map to $0 \to 1$ so it is a span-op category. When we op $\mathrm{Tw}(1)^{\mathrm{op}}$ we get the span category, so a colimit becomes a pushout. We see that

$1_{0/} = 1$ and $1_{1/} = *$. Then

$$\mathcal{E} = \operatorname{colim}\left(\begin{array}{l} \mathcal{A} \times 1_{1/} \xrightarrow{\mathrm{id}\times(0\to 1)} \mathcal{A} \times 1_{0/} \\ {\scriptstyle F\times\mathrm{id}}\big\downarrow \\ \mathcal{B} \times 1_{1/} \\ \\ \end{array}\right)$$

$$= \operatorname{colim}\left(\begin{array}{l} \mathcal{A} \xrightarrow{\mathrm{id}\times 1} \mathcal{A} \times 1 \\ \big\downarrow \\ \mathcal{B} \\ \\ \end{array}\right)$$

Then $\mathcal{E}$ is a cocartesian fibration over 1, whose fiber over 0 is $\mathcal{A}$, whose fiber over 1 is $\mathcal{B}$, and with maps $F(A) \to B$ over $0 \to 1$.

Goal: Redefine a symmetric monoidal category $(\mathscr{C}, \otimes, I)$ as a cocartesian fibration $\mathscr{C}^{\otimes} \to \mathtt{Fin}_*$ as certain "pseudo"functors $\mathtt{Fin}_* \to \mathtt{Cat}$. We could take $\mathtt{Fin}_* \to \mathtt{Cat}$ sending $\langle n \rangle$ to $\mathscr{C}^{\times n}$.

Q: Given a psuedofunctor $F : \mathtt{Fin}_* \to \mathtt{Cat}$, when is it defining a symmetric monoidal category?

We would need $F(\langle n \rangle) \cong F(\langle 1 \rangle)^{\times n}$ with Segal's condition $F(\langle 0 \rangle) = 0$.

Theorem 15.4. Symmetric monoidal categories are pseudofunctors $\mathtt{Fin}_* \to \mathtt{Cat}$ with the Segal condition.

16. Lecture 16: March 28th

Monoidal functors and algebra objects

Last time we defined a symmetric monoidal infinity category to be a cocartesian fibration over $\mathtt{Fin}_*$ with a Segal condition. Here $\mathscr{C} = f^{-1}(\langle 1 \rangle)$. We got this by straightening $N(\mathtt{Fin}_*) \to \mathtt{Cat}_\infty$, with $\langle n \rangle \mapsto \mathscr{C}^{\otimes n}$.

Suppose we had a natural transformation η between functors

$$\mathscr{C}, \mathscr{D} : N(\mathtt{Fin}_*) \to \mathtt{Cat}_\infty.$$

This corresponds to a map $\mathscr{C}^\otimes \to \mathscr{D}^\otimes$ over $\mathtt{Fin}_*$ sending p-cocartesian lifts to q-cocartesian lifts:

$$\begin{tikzcd} \mathscr{C}^\otimes \arrow[rr] \arrow[dr, "p"'] & & \mathscr{D}^\otimes \arrow[dl, "q"] \\ & N(\mathtt{Fin}_*). & \end{tikzcd}$$

Think about this as $F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$.

Now suppose we have $F^\otimes : \mathscr{C}^\otimes \to \mathscr{D}^\otimes$ between symmetric monoidal ∞-categories. Then we know the fiber over $\langle 1 \rangle$ must be sent to the fiber over $\langle 1 \rangle$. Then we get $F^\otimes_{\langle n \rangle} : \mathscr{C}^\otimes_{\langle n \rangle} \to \mathscr{D}^\otimes_{\langle n \rangle}$ for all n.

Denote $F = F^\otimes_{\langle 1 \rangle}$. Then $F^\otimes_{\langle n \rangle} \simeq F^{\times n}$.

Let $\rho^i_! : \langle n \rangle \to \langle 1 \rangle$ send everything to 0 except i to 1.

$$\begin{tikzcd} \mathscr{C}^\otimes_{\langle 2 \rangle} \arrow[r, "F^\otimes_{\langle 2 \rangle}"] \arrow[d, "(\rho^1_!, \rho^2_!)"'] & \mathscr{D}^\otimes_{\langle 2 \rangle} \arrow[d, "(\rho^1_!, \rho^2_!)"] \\ \mathscr{C} \times \mathscr{C} \arrow[r, "F \times F"'] & \mathscr{D} \times \mathscr{D}. \end{tikzcd}$$

$F(\rho^1_!) \simeq \rho^1_!$ and $F(\rho^2_!) \simeq \rho^2_!$. For all i we need that $F(\rho^i_!)$ is a q-cocartesian lift of ρ^i. This means that for all n, $F^\otimes_{\langle n \rangle}(X_1, \ldots, X_n) \simeq (F(X_1), \ldots, F(X_n))$.

Definition 16.1. A map $\alpha : \langle n \rangle \to \langle k \rangle$ in $\mathtt{Fin}_*$ is *inert* if $\alpha^{-1}(i)$ is precisely a singleton for $1 \leq i \leq n$.

Fact 16.2. Inert morphisms are generated by ρ^i and τ (here τ is the swap of 1 and 2 on $\langle 2 \rangle$).

Let $F^\otimes : \mathscr{C}^\otimes \to \mathscr{D}^\otimes$ that sends p-cocartesian lifts of inert maps to q-cocartesian lifts. We claim this already gives a lax monoidal structure. Consider $m : \langle 2 \rangle \to \langle 1 \rangle$ the multiplication, and consider $(X, Y) \in \mathscr{C}^{\times 2}$. There is a map $m_! : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ sending $(X, Y) \mapsto X \otimes Y$.

$$\begin{tikzcd} & F(X) \otimes F(Y) \arrow[dr, dashed] & \\ (F(X), F(Y)) \arrow[ur, "m_!"] \arrow[rr, "F(m_!)"'] & & F(X \otimes Y). \end{tikzcd}$$

Note we're not saying that $F(m_!)$ is a cocartesian lift, we're saying that $m_!$ is. If $F(m_!)$ was a cocartesian lift, then this would give $F(X) \otimes F(Y) \to F(X \otimes Y)$ is an equivalence.

Exercise 16.3. Show that $\iota : \langle 0 \rangle \to \langle 1 \rangle$ induces $I_{\mathscr{D}} \to F(I_{\mathscr{C}})$.

Definition 16.4. For $\mathscr{C}^\otimes$ and $\mathscr{D}^\otimes$ symmetric monoidal ∞-categories, a *lax symmetric monoidal functor* $F^\otimes : \mathscr{C}^\otimes \to \mathscr{D}^\otimes$ is a functor that sends lifts of p-cocartesian inert maps in $\mathtt{Fin}_*$ to q-cocartesian lifts.

Definition 16.5. We say $F^\otimes$ is strong symmetric monoidal if it sends *all* p-cocartesian lifts to q-cocartesian lifts.

We can define

$$\begin{tikzcd} \mathrm{Fun}_{N(\mathtt{Fin}_*)}(\mathscr{C}^\otimes, \mathscr{D}^\otimes) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\mathscr{C}^\otimes, \mathscr{D}^\otimes) \arrow[d, "q^*"] \\ \Delta^0 \arrow[r, "p"'] & \mathrm{Fun}(\mathscr{C}^\otimes, \mathtt{Fin}_*). \end{tikzcd}$$

Define $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathscr{C}^\otimes, \mathscr{D}^\otimes)$ to be the full subcategory of lax monoidal functors, and just $\mathrm{Fun}^\otimes(\mathscr{C}^\otimes, \mathscr{D}^\otimes)$ the full subcategory of strong monoidal functors.

Example 16.6. Commutative algebras. We have that Δ^0 is a symmetric monoidal ∞-category with trivial structure, then we have

$$N(\mathtt{Fin}_*) \to \mathtt{Cat}_\infty$$

sending everything to Δ^0. The associated cocartesian fibration is $N(\mathtt{Fin}_*) \to N(\mathtt{Fin}_*)$.

We define $\mathtt{Alg}_\infty(\mathscr{C})$ to be $\mathrm{Fun}^{\otimes,\mathrm{lax}}(N(\mathtt{Fin}_*), \mathscr{C})$. That is,

$$\begin{tikzcd} N(\mathtt{Fin}_*) \arrow[rr, "A^\otimes"] \arrow[dr, equal] & & \mathscr{C}^\otimes \arrow[dl, "p"] \\ & N(\mathtt{Fin}_*) & \end{tikzcd}$$

That is, $A^\otimes$ is a section of p that sends inert maps in $\mathtt{Fin}_*$ to p-cocartesian lifts. We have that $A^\otimes(\langle 1 \rangle) \in \mathscr{C}^\otimes_{\langle 1 \rangle} = \mathscr{C}$, and $A \otimes A \to A$. We have that $A^\otimes(\langle 0 \rangle) = I$.

Q: Can we localize a symmetric monoidal category in such a way that it preserves the symmetric monoidal structure?

Definition 16.7. (HA 4.1.7.4) Given $\mathscr{C}^\otimes$ a symmetric monoidal ∞-category, let $W \subseteq \mathscr{C}$ a collection of edges. Assume W is closed under $\otimes$ (meaning that if $Y \to Y'$ is in W, and X is arbitrary, then $X \otimes Y \to X \otimes Y'$ and $Y \otimes X \to Y' \otimes X$ are in W as well). The *symmetric monoidal localization* of $\mathscr{C}^{\otimes}$ with W is a symmetric monoidal ∞-category $\mathscr{C}[W^{-1}]^{\otimes}$ together with a strong symmetric monoidal functor

$$\ell : \mathscr{C}^{\otimes} \to \mathscr{C}[W^{-1}]^{\otimes}$$

with the following universal property: for any symmetric monoidal ∞-category $\mathscr{D}^{\otimes}$, we get an equivalence of ∞-categories:

$$\mathrm{Fun}^{\otimes}(\mathscr{C}[W^{-1}]^{\otimes}, \mathscr{D}^{\otimes}) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}_{W}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes}),$$

where $\mathrm{Fun}_W(-)$ means sending W to equivalences.

This always exists. We have that $\mathscr{C}[W^{-1}]_{\langle 1\rangle} \simeq \mathscr{C}[W^{-1}]$. In terms of cocartesian fibrations it is maybe(?) some kind of Kan extension

$$\begin{tikzcd}
\mathscr{C} \arrow[r] \arrow[d] & \mathtt{Fin}_* \\
\mathscr{C}[W^{-1}] \arrow[ur, dashed, "?"] &
\end{tikzcd}$$

Definition 16.8. Let $(M, \otimes, I)$ be a symm mon model category (with functorial cofibrant replacement). Suppose I is cofibrant. Then the Dwyer-Kan localization $N(M)[W^{-1}]$ can be given a symmetric monoidal ∞-structure as follows:

- Take the cofibrant objects M_c
- Take the category of operators $M_c^{\otimes}$ as an ordinary category (objects are pairs $\langle n\rangle, c_1, \dots, c_n)$ and morphisms are $\otimes_i c_i \to c'_j$ over $\mathtt{Fin}_*$
- $N(M_c^{\otimes})$ is a symmetric monoidal ∞-category, with class W of edges in $N(M_c)$
- Recall that $X \otimes - : M_c \to M_c$ preserves weak equivalences between cofibrant objects, under the hypothesis that X is cofibrant.
- Thus $N(M_c^{\otimes}) \to N(M_c)[W^{-1}]^{\otimes}$ is called the *symmetric monoidal Dwyer-Kan localization*.

This gives a sym mon structure on the ∞-category $N(M)[W^{-1}] \simeq N(M_c)[W^{-1}]$.

This shows that the derived tensor product $\otimes$ of a monoidal model category M endows $N(M)[W^{-1}]$ with a monoidal structure.

Example 16.9. Spaces $\mathcal{S}$ have a symmetric monoidal ∞-category structure, since we can view them as $N(\mathtt{sSet})[W^{-1}_{\mathrm{Kan}}]$ with the cartesian product. Here $\mathtt{Alg}_{E\infty}(\mathcal{S})$ are equivalent to E_∞-algebras in spaces.

Example 16.10. We have that $\mathtt{Cat}_\infty \simeq N(\mathtt{sSet})[W^{-1}_{\mathrm{Joyal}}]$ with the cartesian product. Then $\mathtt{Alg}_{E_\infty}(\mathtt{Cat}_\infty)$ are symmetric monoidal ∞-categories. This is exactly because $\mathtt{Alg}_{E_\infty}(\mathtt{Cat}_\infty) = \mathrm{Fun}^{\otimes,\mathrm{lax}}(N(\mathtt{Fin}_*), \mathtt{Cat}_\infty)$ which guarantees the Segal condition.

Example 16.11. If R is a commutative ring, then $D(R) \simeq \mathtt{Ch}_R[W_{\mathrm{proj}}^{-1}]$ is a symmetric monoidal ∞-category. The injective model structure does not give you a monoidal model category.

We also have the connective case with two models

$$D^{\geq 0}(R) \simeq N(s\mathtt{Mod}_R)[W^{-1}] \simeq N(\mathtt{Ch}_R^{\geq 0})[W^{-1}].$$

Every symmetric monoidal ∞-category $\mathscr{C}^{\otimes}$ which is presentable and for which $\otimes$ preserves colimits is the symmetric monoidal DK localization of a combinatorial monoidal model category (Lurie-Sagave).

17. Lecture 17: March 30th

Stable ∞-categories

Universal property for $\mathcal{S}$ (spaces). Given $K \in \mathtt{sSet}$, there is a Yoneda embedding

$$K \hookrightarrow \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{S}) =: \mathcal{P}(K),$$

which is the adjoint of "internal hom"[19]

$$K^{\mathrm{op}} \times K \to \mathcal{S}.$$

Given $\mathscr{C}$ an ∞-category, we can call $\mathcal{P}(\mathscr{C})$ the *universal cocompletion* of $\mathscr{C}$. That is, for all $\mathscr{D}$ cocomplete, there is an equivalence

$$\mathrm{Fun}^L(\mathcal{P}(\mathscr{C}), \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathscr{D}),$$

where L denotes colimit-preserving functors.[20]

If we choose $\mathscr{C} = \Delta^0$, we get

$$\mathrm{Fun}^L(\mathcal{S}, \mathscr{D}) = \mathrm{Fun}(\Delta^0, \mathscr{D}) = \mathscr{D}.$$

Hence we can think of $\mathcal{S}$ as the "free cocompletion of Δ^0." Just as a set can be viewed as a union of its points, we can think of any cocomplete ∞-category as gluing its paths together.

Definition 17.1. An ∞-category is *pointed* if it has an object with is both initial and terminal. That is, some $0 \in \mathscr{C}$ so that

$$\mathrm{Hom}_{\mathscr{C}}(0, X) \simeq * \simeq \mathrm{Hom}_{\mathscr{C}}(X, 0)$$

[19] K isn't necessarily an ∞-category, so it doesn't make sense to have internal hom, but this is the straightening of $\mathrm{Tw}(K) \to K^{\mathrm{op}} \times K$ which is always well-defined.

[20] For presentable ∞-categories, being a left adjoint is equivalent to preserving colimits, hence the superscript "L"

for any $X \in \mathscr{C}$.

Example 17.2. If $\mathscr{C}$ is an ∞-category and $* \in \mathscr{C}$ is a terminal object, we can define

$$\mathscr{C}_* := \mathscr{C}_{*/}.$$

This will be pointed and we will have an adjunction

$$(-)_+ : \mathscr{C} \leftrightarrows \mathscr{C}_*.$$

For example, we have

$$\mathcal{S} \leftrightarrows \mathcal{S}_* = N(\mathtt{sSet}_*)[W_{\mathrm{Kan}}^{-1}].$$

If $\mathscr{C}$ is a pointed presentable stable ∞-category, then

$$\mathrm{Fun}^L(\mathcal{S}_*, \mathscr{C}) \simeq \mathscr{C}.$$

Here $\mathcal{S}_*$ is the free presentable pointed ∞-category generated by $*_+ = S^0$.

Now we introduce stable ∞-categories, which behave like $D(R) \simeq N(\mathtt{Ch}_R)[W_{\mathrm{qiso}}^{-1}]$.

Definition 17.3. Let $\mathscr{C}$ be a pointed ∞-category. A *triangle* in $\mathscr{C}$ is a square of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z. \end{tikzcd}$$

This is specified by a functor $N(\Delta^1 \times \Delta^1) \to \mathscr{C}$ sending the bottom corner to 0.

We say a triangle is *exact* if it is a pullback, and *coexact* if it is a pushout.

Example 17.4. If $f : E \to X$ in $\mathcal{S}_*$, then an exact triangle looks like

$$\begin{tikzcd} f^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "f"] \\ * \arrow[r] & X. \end{tikzcd}$$

Example 17.5. We have loops and suspension in $\mathcal{S}_*$ given by the (homotopy) pullback and pushout squares

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma X \end{tikzcd}$$

Our goal is to define $\Sigma : \mathscr{C} \to \mathscr{C}$ and $\Omega : \mathscr{C} \to \mathscr{C}$ for a general pointed ∞-category.

Definition 17.6. For $\mathscr{C}$ finitely bicomplete, we define $\mathscr{C}^\Sigma \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ to be the full subcategory spanned by diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Note that maps between such diagrams are the same as maps $X \to Y$. Thus there is an equivalence

$$\mathscr{C}^\Sigma \xrightarrow{\sim} \mathscr{C},$$

and similarly $\mathscr{C}^\Omega \xrightarrow{\sim} \mathscr{C}$.

We have

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathscr{C}, \mathscr{C}^\Sigma) \arrow[d, "\simeq"] \\ * \arrow[r, "\mathrm{id}"'] & \mathrm{Fun}(\mathscr{C}, \mathscr{C}). \end{tikzcd}$$

Thus there is a unique section $s_\Sigma : \mathscr{C} \to \mathscr{C}^\Sigma$. So now we can define $\Sigma : \mathscr{C} \to \mathscr{C}$ to be

$$\Sigma : \mathscr{C} \xrightarrow{s_\Sigma} \mathscr{C}^\Sigma \xrightarrow{\sim} \mathscr{C}.$$

Analogously we can define Ω.

Theorem 17.7. If $\mathscr{C}$ is a pointed and finitely bicomplete category, we have an adjunction

$$\Sigma : \mathscr{C} \leftrightarrows \mathscr{C} : \Omega.$$

In particular, for $X, Y \in \mathscr{C}$ we have

$$\mathrm{Hom}_{\mathscr{C}}(\Sigma X, Y) \simeq \Omega \mathrm{Hom}_{\mathscr{C}}(X, Y).$$

This is because maps from $\Sigma X \to Y$ are in bijection with

$$\begin{tikzcd} \Omega\mathrm{Hom}(X, Y) \arrow[r] \arrow[d] & \mathrm{Hom}(0, Y) \arrow[d] \\ \mathrm{Hom}(0, Y) \arrow[r] & \mathrm{Hom}(\Sigma X, Y). \end{tikzcd}$$

This tells us that

$$\pi_0 \mathrm{Hom}_{\mathscr{C}}(\Sigma X, Y) = \pi_1 \mathrm{Hom}_{\mathscr{C}}(X, Y),$$

which is a group. Similarly we get that $\pi_0 \mathrm{Hom}(\Sigma^2 X, Y)$ is an abelian group.

Definition 17.8. Given $f : X \to Y$ in $\mathscr{C}$, we can define the *fiber* and *cofiber* as

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ * \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ * \arrow[r] & \mathrm{cof}(f) \end{tikzcd}$$

Definition 17.9. An ∞-category is *stable* if it is

- pointed
- finitely bicomplete
- triangles are exact if and only if they are coexact.

This last condition is equivalent to any of the following

- a square is a pullback iff it is a pushout
- $\Sigma : \mathscr{C} \leftrightarrows \mathscr{C} : \Omega$ is an equivalence
- $\mathrm{cof} : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathrm{Fun}(\Delta^1, \mathscr{C}) : \mathrm{cof}$ is an equivalence.

Let $\mathscr{C}$ be a stable ∞-category. Then

$$\pi_0 \mathrm{Hom}(X, Y) \cong \pi_0(\mathrm{Hom}(\Sigma X', Y)) \cong \pi_0 \mathrm{Hom}(\Sigma^2 X'', Y)$$

for some X, X''. Thus $\mathrm{Ho}(\mathscr{C})$ is an additive category.

We furthermore have that $\mathrm{Ho}(\mathscr{C})$ is triangulated. Given $f : X \to Y$ in $\mathscr{C}$,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{cof}(f) \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

Example 17.10. $\mathscr{C} = D(R)$. Show this has all the properties mentioned above.

Stabilization/linearization

Given $\mathscr{C}$ pointed, we want it to be stable. We can force $\Omega : \mathscr{C} \to \mathscr{C}$ to be an equivalence by considering

$$Sp(\mathscr{C}) := \lim \left(\cdots \xrightarrow{\Omega} \mathscr{C} \xrightarrow{\Omega} \mathscr{C} \right).$$

Historically, we tried to invert Σ (Freudenthal theorem).

We could take $\mathit{Sp}^{\text{naive}}$, whose objects are finite pointed spaces, and morphisms are stable maps $[X, Y]$. The problem is that Σ is not an equivalence on this category.

We could instead take Sp^{Wh}, where objects are pairs (X, n) with X a pointed finite CW complex, and

$$\operatorname{Hom}((X, n), (Y, m)) := \operatorname{colim}_k \left[\Sigma^{n+k} X, \Sigma^{m+k} Y\right].$$

Then we have

$$\begin{aligned} \mathit{Sp}^{\text{naive}} &\hookrightarrow \mathit{Sp}^{\text{Wh}} \\ X &\mapsto (X, 0). \end{aligned}$$

The suspension takes the form

$$\begin{aligned} \Sigma : \mathit{Sp}^{\text{Wh}} &\to \mathit{Sp}^{\text{Wh}} \\ (X, n) &\mapsto (X, n+1). \end{aligned}$$

Thus

$$\mathit{Sp}^{\text{Wh}} = \operatorname{colim}\left(\mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \cdots\right).$$

and we have that

$$\begin{aligned} \mathit{Sp}(\mathcal{S}_*) &= \operatorname{Ind}(\mathit{Sp}^{\text{Wh}}) \\ &= \operatorname{Ind}\operatorname{colim}\left(\mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \cdots\right) \\ &= \lim\left(\operatorname{Ind}(\mathcal{S}_*^{\text{fin}}) \xleftarrow{\Omega} \operatorname{Ind}(\mathcal{S}_*^{\text{fin}}) \xleftarrow{\Omega} \cdots\right) \\ &= \lim\left(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \cdots\right). \end{aligned}$$

Note that $\operatorname{colim}\left(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \cdots\right)$ won't work.

Definition 17.11. If $\mathscr{C}$ is a pointed finitely bicomplete ∞-category, a *prespectrum* in $\mathscr{C}$ is defined to be a functor

$$N(\mathbb{Z} \times \mathbb{Z}) \to \mathscr{C},$$

where $X_{i,j} = 0$ for $i \neq j$. Note that we get induced structure maps $\alpha_n : \Sigma X_n \to X_{n+1}$ and $\beta_n : X_n \to \Omega X_{n+1}$.

A prespectrum is called a *spectrum* in $\mathscr{C}$ if β_n's are equivalences for all n. We define $\mathit{Sp}(\mathscr{C})$ to be the full subcategory of spectra.

Let

$$\mathit{Sp}(\mathscr{C}) \simeq \lim\left(\mathscr{C} \xleftarrow{\Omega} \mathscr{C} \xleftarrow{\Omega} \cdots\right).$$

If $\mathscr{C} = \mathcal{S}_*$, we will write $Sp = Sp(\mathcal{S}_*)$ as the ∞-category of spectra. We define $\mathrm{Ho}(Sp)$ to be the stable homotopy category.

There is a functor

$$\widetilde{\Sigma}^\infty : \mathscr{C} \to \mathrm{PSp}(\mathscr{C}),$$

given by sending X to the prespectrum whose (i,i)th entry is $\Sigma^i X$.

Then there is a functor for $\mathscr{C}$ presentable

$$\mathrm{PSp}(\mathscr{C}) \to Sp(\mathscr{C})$$

sending a prespectrum X to $\widetilde{X}$, defined by

$$\widetilde{X}_n := \mathrm{colim}(X_n \xrightarrow{\beta_n} \Omega X_{n+1} \to \cdots).$$

Then $\widetilde{X}_n \simeq \mathrm{colim}_k \Omega^k X_{n+k} \simeq \mathrm{colim}_k \Omega^{k+1} X_{n+k+1}$. As Ω is a right adjoint it commutes with filtered colimits (using presentable here), so this can be rewritten as

$$\Omega \mathrm{colim}_k \Omega^k X_{n+k+1} \simeq \Omega \widetilde{X}_{n+1}.$$

18. Lecture 18: Tuesday, April 4th

Symmetric monoidal structure on spectra

Last time we had a universal property for $\mathscr{C} \xrightarrow{\Sigma^\infty} Sp(\mathscr{C})$, where $\mathscr{C}$ was a pointed presentable ∞-category. We had that

$$\mathrm{Fun}^L(Sp(\mathscr{C}), \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

for any stable presentable ∞-category $\mathscr{D}$.

We denote by $\Sigma^\infty S^0 =: \mathbb{S} \in Sp = Sp(\mathcal{S}_*)$, and recall that

$$\mathrm{Fun}^L(Sp, \mathscr{D}) \simeq \mathrm{Fun}^L(\mathcal{S}_*, \mathscr{D}) \simeq \mathscr{D}.$$

So we call Sp the free stable ∞-category generated by ∞.

Q: Can we give a symmetric monoidal structure on Sp analogous to $\otimes_{\mathbb{Z}}$ in `Ab`?

Spanier-Whitehead category: Recall Freudenthal says that if X and Y are finite CW complexes, then the sequence $[\Sigma^k X, \Sigma^k Y]$ stabilizes in k. So Sp^{naive} has objects given by finite CW complexes, and homs given by stable maps.

To invert Σ, we introduced Sp^{Wh}, where objects are (X, n) and homs $(X,n) \to (Y,m)$ are

$$\mathrm{colim}_k \left[\Sigma^{n+k} X, \Sigma^{m+k} Y\right].$$

Formally in ∞-categories, we have that

$$Sp^{\mathrm{Wh}} = \operatorname{colim}\left(Sp_* \xrightarrow{\Sigma} Sp_* \xrightarrow{\Sigma} \cdots \right).$$

Then $Sp \simeq \mathrm{Ind}(Sp^{\mathrm{Wh}})$.

Why finiteness? By adjunction we can see

$$\begin{aligned} \mathrm{Hom}((X,0),(Y,0)) &= \operatorname{colim}_k \left[\Sigma^k X, \Sigma^k Y\right] \\ &= \operatorname{colim}_k \left[X, \Omega^k \Sigma^k Y\right] \\ &= [X, \operatorname{colim}_k \Omega^k \Sigma^k Y], \end{aligned}$$

which holds if X is compact (e.g. finite CW). Thus if $\{-,-\}$ is a hom for spectra, we would have

$$\{X, Y\} = \{X, \Omega^n \Sigma^n Y\}.$$

What is the monoidal structure on Sp^{Wh}? Recall in $\mathcal{S}_*$ we have a smash product, so we could define

$$(X, n) \wedge (Y, m) := (X \wedge Y, n + m).$$

The unit is $(S^0, 0)$. This smash product is difficult to translate to spectra however.

Definition 18.1. For all $X \in Sp$, we define

$$\pi_n(X) = \mathrm{Hom}_{\mathrm{Ho}(Sp)}(\Sigma^n \mathbb{S}, X) =: [\Sigma^n \mathbb{S}, X] \in \mathtt{Ab}.$$

In particular if X is a suspension spectrum, we get

$$\begin{aligned} \pi_k(\Sigma^\infty X) &= [\Sigma^n \mathbb{S}, \Sigma^\infty X] \\ &= \operatorname{colim}_k \left[\Sigma^{n+k} S^0, \Sigma^k X\right] \\ &= \operatorname{colim}_k \pi_{n+k}(X) \\ &= \pi_n^s(X). \end{aligned}$$

This is the *stable homotopy group* of X. It gives us a functor

$$\begin{aligned} Sp &\to N(\mathtt{Ab}) \\ X &\mapsto \pi_n(X). \end{aligned}$$

This factors through

$$Sp \xrightarrow{\Omega^\infty} \mathcal{S}_* \xrightarrow{\pi_n} N(\mathtt{Ab})$$

for $n \geq 2$.

(HA 1.4.3.8) The collection of these functors reflect equivalences. That is, if $\pi_n(X) \xrightarrow{\sim} \pi_n(Y)$ for all n, then $X \xrightarrow{\sim} Y$ in *Sp*.

Definition 18.2. We define $Sp^{\geq 0}$ to be the ∞-category of *connective spectra*, the full subcategory of *Sp* on those X for which $\pi_n(X) = 0$ for $n < 0$.

Example 18.3. For all $X \in \mathcal{S}_*$, we have that $\Sigma^\infty X \in Sp^{\geq 0}$.

We get an adjunction

$$Sp^{\geq 0} \leftrightarrows Sp : \tau_{\geq 0},$$

where the right adjoint to the inclusion is the *connective cover*.

If $X \in \mathcal{S}_*$ and $Y \in Sp^{\geq 0}$, we have that

$$[\Sigma^\infty X, Y] \simeq [X, Y_0].$$

That is, $\Omega^\infty Y \simeq Y_0$.

If $Y \in Sp^{\geq 0}$ then $\Omega^\infty Y = Y_0$ is an infinite loop space. That is, for all $k \geq 0$, we have that $Y_0 \simeq \Omega^k Y_k$. May recognition tells us that

$$\mathtt{Alg}^{\text{gplike}}_{E_\infty}(\mathcal{S}_*) \simeq Sp^{\geq 0}.$$

If $\mathscr{C}$ is a symmetric monoidal category, then $\mathtt{CAlg}(\mathscr{C})$ is also a sym mon cat with *some* underlying tensor product.

For example if X, Y are E_∞-algebras which are grouplike in spaces, then $X \wedge Y$ is an E_∞-algebra in $\mathcal{S}_*$. It is *not true* that if X and Y are infinite loop spaces then $X \wedge Y$ is an infinite loop space.

Example 18.4. Let G be an abelian group, then $K(G, 0)$ is an ∞-loop space, with $K(G, 0) \simeq \Omega^n K(G, n)$. Let $HG \in Sp^{\geq 0}$ be its corresponding spectrum, called the *Eilenberg-Maclane spectrum* of G. This gives a functor

$$\begin{aligned} N(\mathtt{Ab}) &\to Sp^{\geq 0} \\ G &\mapsto HG. \end{aligned}$$

We want a monoidal structure on *Sp* and $Sp^{\geq 0}$ for this functor to be compatible with $\otimes_{\mathbb{Z}}$ in $\mathtt{Ab}$.

Ideas for monoidal structure on *Sp*:

- On Sp^{Wh} we had $(X, n) \wedge (Y, m) = (X \wedge Y, n + m)$
- $\mathtt{Alg}_{E_\infty}(\mathcal{S}_*)$
- $\mathtt{Ab}, \otimes_{\mathbb{Z}}$

Boardman: We could define $(X \wedge Y)_n = X_{a(n)} \wedge Y_{b(n)}$ where $a(n) + b(n) = n$, and then we could "Ω-spectrify." There are lots of choices for $a(n)$ and $b(n)$.

Adams: We could define

$$(X \wedge Y)_n \simeq \bigvee_{e_{ij}} \Sigma^{n-i-j-d} X_i \wedge Y_j \wedge M(\tau) / \sim$$

where e_{ij} is the square on the $\mathbb{Z} \times \mathbb{Z}$ grid with bottom left corner based at (i, j), open on the top and right sides, and $M(\tau)$ is the Thom complex of a bundle over e_{ij}.

Indexing on $\mathbb{Z} \times \mathbb{Z}$ is hard because we need to understand choices. Model categories allow us to switch $\mathbb{Z} \times \mathbb{Z}$ to something that records the choices.

Symmetric spectra: we get a model category Sp^{Σ} indexed on finite sets and injective morphisms (Hovey-Shipley-Smith).

Orthogonal spectra: (or EKMM spectra) $Sp^{\mathcal{O}}$, indexed on real inner product spaces. This is by Mandell-May-Schwede-Shipley.

Theorem 18.5. (Lewis, '91) There is no good 1-category Sp^1 that describes Sp with a monoidal structure so that:

(1) Sp^1 is symmetric monoidal
(2) There is an adjunction $\Sigma^\infty : \mathtt{Top}_* \leftrightarrows Sp^1 : \Omega^\infty$
(3) We have that $\Sigma^\infty S^0$ is the unit
(4) Ω^∞ is lax symmetric monoidal
(5) For any pointed space, $\Omega^\infty \Sigma^\infty X \simeq \mathrm{colim}_k \Omega^k \Sigma^k X$. (that is, these functors are really doing stabilization of spaces)

For symmetric and orthogonal spectra, it is (3) that messes up — you really need a fibrant replacement. In EKMM they force (3) to be true, but fail (5).

How to think of $X \wedge Y$ in Sp? We use the universal properties, and try to understand its homotopy groups. There is a Künneth spectra sequence to compute $\pi_n(X \wedge Y)$.

Recall that $\mathrm{Fun}^L(\mathcal{S}, \mathscr{C}) \simeq \mathscr{C}$ for $\mathscr{C}$ any presentable ∞-category. This should remind us of the statement that $\mathrm{Hom}_R(R, M) = M$ for M an R-module. So we want to think of $\mathrm{Fun}^L(-,-)$ as an internal hom somewhere.

Definition 18.6. Let Pr^L denote the (very large) ∞-category of presentable ∞-categories, where

$$\mathrm{Hom}_{\mathrm{Pr}^L}(\mathscr{C}, \mathscr{D}) := \mathrm{Fun}^L(\mathscr{C}, \mathscr{D}).$$

Fact 18.7. This is an internal hom — i.e. if $\mathscr{C}$ and $\mathscr{D}$ are presentable, then $\mathrm{Fun}^L(\mathscr{C}, \mathscr{D})$ is presentable.[21]

We have that

$$\mathrm{Fun}^L(\mathscr{C}_1, \mathrm{Fun}^L(\mathscr{C}_2, \mathscr{D})) \simeq \mathrm{Fun}^{BL}(\mathscr{C}_1 \times \mathscr{C}_2, \mathscr{D}),$$

that is, functors $\mathscr{C}_1 \times \mathscr{C}_2 \to \mathscr{D}$ which are colimit preserving in each variable.

So we want some tensor product so that the above is equivalent to $\mathrm{Fun}^L(\mathscr{C}_1 \otimes \mathscr{C}_2, \mathscr{D})$.

Fact 18.8. If $\mathscr{C}$ is closed monoidal, then $\mathscr{C}^{\mathrm{op}}$ becomes closed monoidal, but where the tensor product and hom switch roles.

The op of Pr^L is Pr^R, where we take limit-preserving functors! So we can check that

$$\mathscr{C}_1 \otimes \mathscr{C}_2 \simeq \mathrm{Fun}^R(\mathscr{C}_1^{\mathrm{op}}, \mathscr{C}_2).$$

By construction $\mathcal{S}$ is the monoidal unit, since

$$\begin{aligned} \mathcal{S} \otimes \mathscr{C} &= \mathrm{Fun}^R(\mathcal{S}^{\mathrm{op}}, \mathscr{C}) \\ &= \left(\mathrm{Fun}^L(\mathcal{S}, \mathscr{C}^{\mathrm{op}})\right)^{\mathrm{op}} \\ &= (\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} \\ &= \mathscr{C}. \end{aligned}$$

Here we are using that

$$\mathrm{Fun}^R(-,-) = \mathrm{Fun}^L(-^{\mathrm{op}}, -^{\mathrm{op}})^{\mathrm{op}}.$$

So we need to create our operator category $\left(\mathrm{Pr}^L\right)^{\otimes} \subseteq \mathtt{Cat}_\infty^{\otimes} \simeq N(\mathtt{sSet}^{\otimes})[W_{\mathrm{Joyal}}^{-1}]$. We had a cocartesian fibration $\mathtt{Cat}_\infty^{\otimes} \to \mathtt{Fin}_*$, and we're going to restrict fibers to get the correct thing. The fibers will look like $(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ with $\mathscr{C}_i$ presentable, and appropriate morphisms.

So the construction we just did argues that $\mathrm{Pr}^L \hookrightarrow \mathtt{Cat}_\infty$ is a lax symmetric monoidal functor. Then

$$\mathtt{Alg}_{E_\infty}(\overset{L}{\mathrm{Pr}}) = \{\text{presentably symmetric monoidal } \infty\text{-cats}\},$$

and $\mathcal{S}$ is the initial object. This provides the universal property of spaces with its monoidal structure $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$, colimit-preserving in each variable, with the point as the unit.

[21] If we took Fun instead of Fun^L, the size might increase, but in fact $\mathrm{Fun}^L(\mathscr{C}, \mathscr{D})$ is presentable in the same size sense that m$\mathscr{C}$ and $\mathscr{D}$ are.

19. Lecture 19: Thursday, April 6th

Brown representability

We've seen that the monoidal product on spectra has two intuitions:

(1) $Sp^{\geq 0} \simeq \mathtt{Alg}^{\text{gplike}}_{E_\infty}(\mathcal{S}_*)$
(2) $Sp^{\text{Wh}} = \operatorname{colim}\left(\mathcal{S}_* \xrightarrow{\Sigma} \cdots\right)$.[22] This had a smash product.

Recall $(\mathcal{S}, \times, *)$ was the initial object in $\mathtt{Alg}_{E_\infty}\left(\mathrm{Pr}^L\right)$. We saw we had

$$\left(\overset{L}{\mathrm{Pr}}, \otimes \mathcal{S}\right) \to \left(\mathtt{Cat}_\infty, \times, \Delta^0\right),$$

with tensor $\mathscr{C} \otimes \mathscr{D} = \mathrm{Fun}^R(\mathscr{C}^{\text{op}}, \mathscr{D})$ and internal hom $\mathrm{Fun}^L(\mathscr{C}, \mathscr{D})$.

We have $\mathtt{Cat}^{\text{st}}_\infty \subseteq \mathtt{Cat}_\infty$ on stable ∞-categories and exact functors, and a corresponding $\mathrm{Pr}^L_{\text{st}} \subseteq \mathrm{Pr}^L$ spanned by stable ∞-categories.

The stabilization functor $\mathscr{C} \mapsto Sp(\mathscr{C})$ can be viewed as left adjoint to the inclusion

$$Sp : \overset{L}{\mathrm{Pr}} \leftrightarrows \underset{\text{st}}{\overset{L}{\mathrm{Pr}}}$$

Tensoring with spectra, we get

$$\begin{aligned} \mathscr{C} \otimes Sp &= \mathrm{Fun}^R(\mathscr{C}^{\text{op}}, Sp) \\ &= \mathrm{Fun}^R(\mathscr{C}^{\text{op}}, \lim\left(\mathcal{S}_* \leftarrow \cdots\right)) \\ &= \lim\left(\mathrm{Fun}^R(\mathscr{C}^{\text{op}}, \mathcal{S}_*) \leftarrow \cdots\right) \\ &= \lim\left(\mathscr{C}_* \leftarrow \cdots\right) \\ &= Sp(\mathscr{C}). \end{aligned}$$

Fact: If $\mathscr{C}, \mathscr{D}$ stable then $\mathrm{Fun}^L(\mathscr{C}, \mathscr{D}) \in \mathrm{Pr}^L_{\text{st}}$.

We can think of stabilization as "extension of scalars" along $\mathcal{S}_* \xrightarrow{\Sigma^\infty} Sp$. We have a monoidal adjunction

$$\left(\overset{L}{\mathrm{Pr}}, \otimes, \mathcal{S}\right) \leftrightarrows \left(\underset{\text{st}}{\overset{L}{\mathrm{Pr}}}, \otimes, Sp\right).$$

Recall Sp is the initial object in $\mathrm{Pr}^L_{\text{st}}$. This characterizes spectra together with

$$Sp \times Sp \xrightarrow{\wedge} Sp$$

[22]We have that $\mathcal{S}_*$ is finite CW complexes, *not* the compact objects in $\mathcal{S}_*$.

monoidal and bicolimit preserving so that $\mathbb{S}$ is the unit.

We have that $\Sigma^\infty_+ : \mathcal{S} \to Sp$ is strong monoidal, and $\Omega^\infty : Sp \to \mathcal{S}$ is lax monoidal, implying that

$$\Sigma^\infty_+ X \wedge \Sigma^\infty_+ Y \simeq \Sigma^\infty_+(X \times Y).$$

We can also shift

$$\Sigma^{\infty-k} : \mathcal{S}_* \leftrightarrows Sp : \Omega^{\infty-k}.$$

We call $E_k = \Omega^{\infty-k}E$.

Formula: For any $E \in Sp$, we have that

$$\begin{aligned} E &\simeq \mathrm{colim}_k \Sigma^{\infty-k}\Omega^{\infty-k}E \\ &\simeq \mathrm{colim}_k \Sigma^{\infty-k}E_k. \end{aligned}$$

For $E, F \in Sp$

$$\begin{aligned} E \wedge F &= (\mathrm{colim}_a \Sigma^{\infty-a}E_a) \wedge \left(\mathrm{colim}_b \Sigma^{\infty-b}F_b\right) \\ &= \mathrm{colim}_{a,b} \Sigma^{\infty-a-b}E_a \wedge F_b. \end{aligned}$$

Example 19.1. Recall Mayer-Vietoris: for $U, V \subseteq X$ open, we have an LES

$$\cdots \to H_*(U \cap V) \to H_*(U) \oplus H_*(V) \to H_*(U \cup V) \to H_{*-1}(U \cap V) \to \cdots$$

Recall that $H_*(X) = H_*(C_*(X))$, and by Dold-Kan, we have that $C_*(X) = \pi_*\mathbb{Z}[\mathrm{Sing}(X)]$. Let's reinterpret Mayer-Vietoris in this setting. It is saying that there is a homotopy pullback in `sSet` of the form

$$\begin{tikzcd} \mathbb{Z}\mathrm{Sing}_*(U \cap V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z}\mathrm{Sing}_* U \arrow[d] \\ \mathbb{Z}\mathrm{Sing}_* V \arrow[r] & \mathbb{Z}\mathrm{Sing}_* U \cup V. \end{tikzcd}$$

We can view homology as

$$\begin{aligned} \mathtt{CW}_* &\to \mathtt{Kan} \\ X &\mapsto \mathbb{Z}\mathrm{Sing}_* X. \end{aligned}$$

Mayer-Vietoris is the statement that this sends homotopy pushouts to homotopy pullbacks. We can view this functor as $\mathcal{S}_* \!\xrightarrow{}\! \mathcal{S}$.

Q: Can we do this for all homology theories?

Definition 19.2. (Eilenberg-Steenrod) A (reduced) homology theory is $\left\{\widetilde{E}_n : \mathtt{CW}_* \!\xrightarrow{}\! \mathtt{Ab}\right\}$ such that

(1) $\widetilde{E}_n$ invariant under homotopy
(2) Excision: $\widetilde{E}^{i+1}(\Sigma X) \cong \widetilde{E}_i(X)$
(3) Additivity: $\widetilde{E}_i(X \vee Y) \cong \widetilde{E}_i(X) \oplus \widetilde{E}_i(Y)$
(4) Exactness: if $f : X \to Y$ then

$$\widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Cf).$$

Goal: We can view $\widetilde{E}_* : \mathtt{CW}_* \to \mathtt{Ab}$ as a certain $\widetilde{E} : \mathcal{S}_*^{\to}\mathcal{S}$.

Axiom (1) allows us to extend $\widetilde{E}_*$ to $\mathrm{Ho}(\mathcal{S}_*^{)}$. Axiom (2) comes from

$$C_*(\Sigma X)[-1] \simeq_{\text{qiso}} C_*(X).$$

If and only if $\Omega\mathbb{Z}\mathrm{Sing}\Sigma X \simeq \mathbb{Z}\mathrm{Sing}X$. So we're rephrasing that

$$\Omega\widetilde{E}(\Sigma X) \simeq \widetilde{E}(X).$$

Axiom (3) comes from $C_*(X \vee Y) \simeq C_*(X) \oplus C_*(Y)$. Translating this over to simplicial sets via Dold-Kan, we get

$$\mathbb{Z}\mathrm{Sing}X \vee Y \simeq \mathbb{Z}\mathrm{Sing}X \times \mathbb{Z}\mathrm{Sing}Y.$$

This gives $\widetilde{E}(X \vee Y) \cong \widetilde{E}(X) \oplus \widetilde{E}(Y)$ and hence $\pi_*(X \times Y) \cong \pi_*(X) \oplus \pi_*(Y)$.

(4) Says $\pi_i(\mathrm{fib}(f)) = \ker(\pi_i(f))$. We have that $C_*(X) \simeq \ker\left(C_*(Y) \to C_*(f)\right)$. Then

$$\mathbb{Z}\mathrm{Sing}_*(X) \xrightarrow{\sim} \mathrm{fib}\left(\mathbb{Z}\mathrm{Sing}Y \to \mathbb{Z}\mathrm{Sing}Cf\right).$$

Hence

$$\widetilde{E}(X) \simeq \mathrm{fib}\left(\widetilde{E}(Y) \to \widetilde{E}(Cf)\right).$$

That is,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & Y \arrow[d] \\ * \arrow[r] & Cf \end{tikzcd}$$

is sent to

$$\begin{tikzcd} \widetilde{E}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \widetilde{E}(Y) \arrow[d] \\ * \arrow[r] & \widetilde{E}(Cf). \end{tikzcd}$$

Definition 19.3. Let $\mathscr{C}$ be an ∞-category. We say a functor $F : \mathcal{S}_*^{\to}\mathscr{C}$ is

(1) *excisive* if F sends pushouts to pullbacks
(2) *reduced/pointed* if $F(*) = *$.

We write $\mathrm{Exc}_*(\mathcal{S}_*;\mathscr{C}) \subseteq \mathrm{Fun}(\mathcal{S}_*;\mathscr{C})$ for excisive and reduced functors.

Given any $\mathcal{S}_* \xrightarrow{\widetilde{E}} \mathcal{S}$ excisive, we obtain a reduced homology theory

$$\mathcal{S}_* \xrightarrow{\widetilde{E}} \mathcal{S} \xrightarrow{\pi_*^s} \mathtt{Ab}.$$

Theorem 19.4. There is an equivalence

$$Sp(\mathscr{C}) \simeq \mathrm{Exc}_*(\mathcal{S}_*;\mathscr{C}).$$

Proof. For some $\widetilde{E} \in \mathrm{Exc}_*(\mathcal{S}_*;\mathcal{S})$, we want to define $E \in Sp$. We define $E_0 = \widetilde{E}(S^0)$, and $E_1 = \widetilde{E}(S^1)$, etc. We can define $E_{-n} = \Omega^n E_0$. This works because

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & S^{n+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is sent to

$$\begin{tikzcd}
\widetilde{E}(S^n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \widetilde{E}(S^{n+1}).
\end{tikzcd}$$

This gives maps $\widetilde{E}(S^n) \xrightarrow{\sim} \Omega\widetilde{E}(S^{n+1})$.

For the other direction, given $E \in Sp$, we can get an excisive functor sending

$$X \mapsto X \wedge E_0.$$

We can reinterpet

$$\begin{aligned}\Omega^\infty : \mathrm{Exc}_*(\mathcal{S}_*;\mathcal{S}) &\to \mathcal{S} \\ \widetilde{E} &\mapsto \widetilde{E}(S^0).\end{aligned}$$

We can show this is universal. $\square$

Given E a spectrum, we have an associated reduced homology theory where $\widetilde{E}_*(X) := \pi_*^s X \wedge E_0$.

We have that

$$\Sigma_+^\infty X \wedge E = \mathrm{colim}_k \Sigma^{\infty-k} X \wedge E_k.$$

We have that

$$\begin{aligned}\Pi_*(\Sigma^\infty X \wedge E) &= \pi_* \left(\mathrm{colim}_k \Sigma^{\infty-k} X \wedge E_k\right) \\ &= \mathrm{colim}_k \pi_* \left(\Sigma^{\infty-k} X \wedge E_k\right) \\ &= \mathrm{colim}_k \pi_{*+k}(X \wedge E_k).\end{aligned}$$

This is exactly the definition of $\pi_*^s(X \wedge E_0)$.

Thus

$$\widetilde{E}_*(X) = \pi_* \Sigma^\infty X \wedge E.$$

Example 19.5. Sphere spectrum $\mathbb{S} \in Sp$ gives the functor

$$\begin{aligned}&\mathcal{S}_* \to \mathcal{S} \\ &X \mapsto X \wedge QS^0,\end{aligned}$$

where $Q(-) = \Omega^\infty \Sigma^\infty(-) = \mathrm{colim}_k \Omega^k \Sigma^k(-)$. Model categorically they think of this as just the natural inclusion $\mathcal{S}_* \overset{\to}{} \mathcal{S}$ because they derive after including. The homology theory is $\widetilde{\mathcal{S}}_* = \pi_*^s(-)$.

Example 19.6. We have that $\widetilde{H\mathbb{Z}}_*(X) = H_*(X; \mathbb{Z})$. Dold-Thom lets us relate $\Sigma^\infty X \wedge H\mathbb{Z}$ with $\mathbb{Z}\mathrm{Sing}(X)$ somehow.

Definition 19.7. For F a spectrum, we can define

$$\widetilde{E}_*(F) = \pi_*(E \wedge F).$$

Theorem 19.8. (Brown representability) If $\widetilde{E}_*(-) : \mathtt{CW}_*^{\to}\mathtt{Ab}$ is a reduced homology theory, then there exists $E \in Sp$ such that $\widetilde{E}_n(X) = \pi_n^{s*}(X \wedge E_0)$.

We looked at π_* of $E \wedge -$. Taking the same thing for its adjoint, we call $F(E, -)$ the right adjoint to $E \wedge -$ (this exists because colimit-preserving + presentable). Can take the internal hom to be

$$F(E, E')_n = \mathrm{Hom}_{Sp}(E, \Sigma^n E').$$

Can define $\widetilde{E}^n(X) = [X, E_n] = [\Sigma^{\infty-n} X, E]$.

20. Lecture 20: Tuesday April 11th

From last time: $F(E, E')_n = \mathrm{Hom}_{Sp}(E, \Sigma^n E')$. This is because

$$\mathrm{Hom}_{Sp}(E \wedge \mathbb{S}, F) \simeq \mathrm{Hom}_{Sp}(\mathbb{S}, F(E, F)).$$

Think $\mathrm{Hom}_R(R, M) = M$ and $\mathrm{Hom}_{\mathtt{Ch}_R}(R, M_*) = M_0$. Then $\mathrm{Hom}_{Sp}(\mathbb{S}, E) \simeq E_0$. This follows from the loops suspension adjunction:

$$\mathrm{Hom}(\Sigma^\infty_+ *, E) = \mathrm{Hom}(\mathbb{S}, E) = \mathrm{Hom}(*, \Omega^\infty E) = E_0.$$

For $E \in Sp$ can define reduced associated cohomology theory for $X \in \mathcal{S}_*$

$$\widetilde{E}^n(X) = [X, E_n] = \pi_n F(\Sigma^\infty X, E).$$

Monoidal categories which are not symmetric:

- Let $\mathscr{C}$ be any category, and look at $\mathrm{End}(\mathscr{C})$ with composition and the identity
- G any non-abelian monoid, defines a discrete monoidal category.
- Bimodules over any non-commutative ring

Recall a sm ∞-cat was $\mathscr{C}^\otimes \to N(\mathtt{Fin}_*)$ a cocartesian fibration + Segal condition. This gave $N(\mathtt{Fin}_*) \to \mathtt{Cat}_\infty$.

We had $\tau : \langle 2\rangle \to \langle 2\rangle$, sending $0 \mapsto 0$ and swapping 1,2. This alone gave a symmetric structure on $\otimes$. We want to restrict from $\mathtt{Fin}_*$ to throw out τ and its friends.

There are multiple ways to do this: can view $\Delta^{\mathrm{op}} \subseteq \mathtt{Fin}_*$ sending $[n] \mapsto \langle n\rangle$. Given $\alpha : [k] \to [n]$ we send it to a map

$$\begin{aligned} \langle n\rangle &\to \langle k\rangle \\ j &\mapsto \begin{cases} i & \exists i\colon j \in [\alpha(i-1)+1, \alpha(i)] \\ * & \text{else} \end{cases} \end{aligned}$$

The composite

$$\Delta^{\mathrm{op}} \to \mathtt{Fin}_* \subseteq \mathtt{Set}_*$$

defines the pointed simplicial set $S^1 = \Delta^1/\partial\Delta^1 \in \mathtt{sSet}_*$.

Definition 20.1. A monoidal ∞-cat is a cocart fibration $\mathscr{C}^\otimes \to N(\Delta^{\mathrm{op}})$ with the Segal condition $\mathscr{C}^\otimes_{[n]} \to \left(\mathscr{C}^\otimes_{[1]}\right)^{\times n}$ given by cocartesian lifts of $p^i : [1] \to [n]$, $0 \mapsto i-1$, $1 \mapsto i$.

By straightening we get $N(\Delta^{\mathrm{op}}) \to \mathtt{Cat}_\infty$ sending $[n] \to \mathscr{C}^{\times n}$. This is some kind of bar construction.

Definition 20.2. $\alpha \in \Delta$ is inert if $\alpha : [n] \to [k]$ is injective, and $\mathrm{im}(\alpha) \subseteq [k]$ is convex. Inert things in Δ^{op} map to inert things in $\mathtt{Fin}_*$ under the map defined above.

Definition 20.3. A *lax monoidal functor* $F^\otimes : \mathscr{C}^\otimes \to \mathscr{D}^\otimes$ is a functor

$$\begin{tikzcd} \mathscr{C}^\otimes \arrow[rr, "F^\otimes"] \arrow[dr] & & \mathscr{D}^\otimes \arrow[dl] \\ & N(\Delta^{\mathrm{op}}), & \end{tikzcd}$$

so that $F^\otimes$ sends cocart lifts of inert to cocart lifts.

A lax monoidal functor $F^\otimes$ is one that sends *all* cocartesian lifts to cocartesian lifts.

Given $\mathscr{C}$ a monoidal ∞-cat, we have that

$$\mathtt{Alg}_{E_1}(\mathscr{C}) = \mathrm{Fun}^{\mathrm{lax}}_{E_1}(N(\Delta^{\mathrm{op}}), \mathscr{C}^\otimes).$$

Every symmetric monoidal ∞-cat can be viewed as a monoidal ∞-cat via

$$\begin{tikzcd} \widetilde{\mathscr{C}}^\otimes \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C}^\otimes \arrow[d] \\ N(\Delta^{\mathrm{op}}) \arrow[r] & N(\mathtt{Fin}_*). \end{tikzcd}$$

We could also straighten then precompose with $N(\Delta^{\mathrm{op}}) \to N(\mathtt{Fin}_*)$.

To define modules over a ring, we will use the bar construction $[n] \mapsto N \otimes R^{\otimes n} \otimes M$.

Definition 20.4. Let $p : \mathscr{C}^\otimes \to N(\Delta^{\mathrm{op}})$ be a monoidal ∞-cat. An ∞-cat $\mathcal{M}$ is said to be *left tensored over* $\mathscr{C}$ if there is a cocart $q : \mathcal{E} \to N(\Delta^{\mathrm{op}})$ so that

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathscr{C}^\otimes \arrow[dl, "p"] \\ & N(\Delta^{\mathrm{op}}). & \end{tikzcd}$$

that sends cocart lifts to cocart lifts, such that

$$\mathcal{E}_{[n]} \xrightarrow{\sim} \mathscr{C}^\otimes_{[n]} \times \mathcal{E}^\otimes_{\{n\}}$$

for $\{n\} \subseteq [n]$, with $\mathcal{M} = \mathcal{E}_{[0]}$, and $\mathcal{E}_{[1]} \simeq \mathscr{C} \times \mathcal{M}$. This is formalizing a functor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ compatible with monoidal structure on $\mathscr{C}$.

Example 20.5. $\mathscr{C}$ is left tensored over itself. Then $\mathcal{E} = \widetilde{\mathscr{C}}^\otimes$ with $\mathcal{E}_{[n]} = \mathscr{C}^{\times(n+1)}$.

Definition 20.6. Given $\mathcal{M}$ left tensored over $\mathscr{C}$, a left module of $\mathcal{M}$ is a map $s : N(\Delta^{\mathrm{op}}) \to \mathcal{M}^\otimes$ such that

$$N(\Delta^{\mathrm{op}}) \xrightarrow{s} \mathcal{M}^\otimes \xrightarrow{f} \mathscr{C}^\otimes$$

is a lax monoidal functor (if $\alpha : [k] \to [n]$ is inert in Δ then $f(\alpha)$ is a cocart fib of $\mathscr{C}^{\otimes}$).

We write $\mathcal{L}\mathtt{Mod}(\mathcal{M}) \subseteq \mathrm{Fun}_{N(\Delta^{\mathrm{op}})}(N(\Delta^{\mathrm{op}}), \mathcal{M}^{\otimes})$ spanned by left modules. We are interested in when $\mathcal{M} = \mathscr{C}$. In that case

$$\begin{aligned} \mathcal{L}\mathtt{Mod}(\mathscr{C}) &\to \mathtt{Alg}_{E_1}(\mathscr{C}) \\ (\mathcal{M}, \mathcal{A}) &\mapsto \mathcal{A}. \end{aligned}$$

Given $A \in \mathtt{Alg}_{E_1}(\mathscr{C})$, we can define A-modules as

$$\begin{tikzcd} {}_A\mathtt{Mod}(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{L}\mathtt{Mod}(\mathscr{C}) \arrow[d] \\ * \arrow[r, "A"'] & \mathtt{Alg}_{E_1}(\mathscr{C}). \end{tikzcd}$$

Can define left A-modules and (A, A)-bimodules in a similar way.

Can defgine left modules in a sym mon ∞-cat

$$\begin{tikzcd} \mathcal{L}\mathtt{Mod}_{E_\infty}(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{L}\mathtt{Mod}(\mathscr{C}) = \mathcal{L}\mathtt{Mod}_{E_1}(\mathscr{C}) \arrow[d] \\ \mathtt{Alg}_{E_\infty}(\mathscr{C}) \arrow[r] & \mathtt{Alg}_{E_1}(\mathscr{C}). \end{tikzcd}$$

Can check that if $A \in \mathtt{Alg}_{E_\infty}(\mathscr{C})$ then ${}_A\mathtt{Mod}(\mathscr{C}) \cong \mathtt{Mod}_A(\mathscr{C})$.

21. Lecture 21: Thursday, April 13th

Schwede-Shipley theorem

Goal: generalize the Freyd-Mitchell and Gabriel theeorems.

Given a lax monoidal functor $F : \mathscr{C} \to \mathscr{D}$ between (symmetric) monoidal ∞-cats, it induces a functor

$$\begin{aligned} \mathtt{Alg}_{E_1}(\mathscr{C}) &\to \mathtt{Alg}_{E_1}(\mathscr{D}) \\ A &\mapsto F(A). \end{aligned}$$

1-categorically if $A \in \mathscr{C}$ is an associative algebra, then $F(A)$ is an associative algebra.

So lax monoidal is the correct notion needed to lift algebras.

∞-categorically, $\mathtt{Alg}_{E_1}(\mathscr{C})$ are lax monoidal functors $* \to \mathscr{C}$. The statement follows from the fact that composition of lax monoidal functors is lax monoidal.

If F is lax *symmetric* monoidal, then we can lift

$$F : \mathtt{Alg}_{E_\infty}(\mathscr{C}) \to \mathtt{Alg}_{E_\infty}(\mathscr{D}).$$

We also have, for all $A \in \mathtt{Alg}_{E_1}(\mathscr{C})$,

$$F : \mathtt{Mod}_A(\mathscr{C}) \to \mathtt{Mod}_{F(A)}(\mathscr{D}).$$

Recall

$$\begin{aligned} N(\mathtt{Ab}) &\to Sp \\ A &\mapsto HA. \end{aligned}$$

Here

(1) $(HA)_n = K(A, n)$ for $n > 0$ and $*$ for $n < 0$
(2) $HA : \mathcal{S}_*^{\to}\mathcal{S}$ sends $X \mapsto X \wedge K(A, 0)$
(3) By Brown representability, $\widetilde{H}^n(X, A) \cong [X, K(A, n)]$.
(4) $\pi_n(HA) = A$ if $n = 0$ and 0 otherwise

$HA \in Sp^{\geq 0}$ then the associated element in $\mathtt{Alg}^{\mathrm{gplike}}_{E_\infty}(\mathcal{S}_*)$ is A as a discrete pointed space.

Given $A, B \in \mathtt{Ab}$ we can compare $HA \wedge HB$ with $A \otimes_{\mathbb{Z}} B$. These are not the same.

$$\begin{aligned} \pi_0 HA \wedge HB &\cong A \otimes_{\mathbb{Z}} B \\ \pi_n HA \wedge HB &\neq 0 \qquad n > 0. \end{aligned}$$

If $A = B = \mathbb{F}_2$, then

$$\pi_* (H\mathbb{F}_2 \wedge H\mathbb{F}_2) = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

with $|\xi_i| = 2^i - 1$. This is the *dual Steenrod algebra*.

We can get a map $HA \wedge HB \to H(A \otimes_{\mathbb{Z}} B)$ by adjunction

$$\pi_0 : Sp^{\geq 0} \leftrightarrows N(\mathtt{Ab}) : H(-).$$

Then

$$\pi_0(E \wedge F) \cong \pi_0(E) \otimes_{\mathbb{Z}} \pi_0(F).$$

Thus π_0 is strong symmetric monoidal.

Exercise 21.1. If $L : \mathscr{C} \leftrightarrows \mathscr{D} : R$ is an adjunction between symmetric monoidal categories, if L is strong monoidal then R is lax monoidal.

Warning: $\pi_0 : Sp \to N(\mathtt{Ab})$ is not strong monoidal on the entire category of spectra.

Since the inclusion $Sp^{\geq 0} \hookrightarrow Sp$ is lax symmetric monoidal, we have the composite $N(\mathtt{Ab}) \xrightarrow{H} Sp^{\geq 0} \to Sp$ is, hence we get

$$\begin{aligned} N(\mathtt{Ring}) = \mathtt{Alg}_{E_1}(N(\mathtt{Ab})) &\to \mathtt{Alg}_{E_1}(Sp) \\ R &\mapsto HR. \end{aligned}$$

We call $\mathtt{Alg}_{E_1}(Sp)$ *ring spectra*.

Can we compare with $\mathtt{Ab} = \mathtt{Mod}_{\mathbb{Z}} \to D(\mathbb{Z})$? Yes we can view $D(\mathbb{Z})$ as $\mathtt{Mod}_{H\widetilde{\mathbb{Z}}}(Sp)$ in a monoidal way.

Recall that for $R \in \mathtt{CRing}$, we get $D(R) = N(\mathtt{Ch}_R)[W_{\text{proj}}^{-1}]$, which is symmetric monoidal ∞-cat with $\otimes_R^{\mathbb{L}}$. We want a monoidal structure on $\mathtt{Mod}_{HR}(Sp)$ that mimics the derived tensor product.

Recall 1-categorically that $R \in \mathtt{Alg}(\mathscr{C}, \otimes, I)$ and $M \in \mathtt{Mod}_R(\mathscr{C})$ and $N \in {}_R\mathtt{Mod}(\mathscr{C})$, we define $\otimes_R$ by the coequalizer

$$M \otimes R \otimes N \rightrightarrows M \otimes N \to M \otimes_R N.$$

So on spectra we want a *relative smash product.*

We have to kill off much higher terms

$$M \wedge_{HR} N := \operatorname{colim}\left(\cdots \rightrightarrows M \wedge HR^{\wedge 2} \wedge N \rightrightarrows M \wedge HR \wedge N \rightrightarrows M \wedge N\right)$$

More generally, given $R \in \mathtt{Alg}_{E_1}(\mathscr{C})$, we can define $M \otimes_R N$ as the colimit of a bar construction. In a 1-category the higher maps don't matter and we just recover the coequalizer definition.

We have

$$\begin{aligned} N(\Delta^{\text{op}}) \to N(\mathtt{Fin}_*) \to \mathscr{C} &\hookrightarrow \mathtt{Cat}_\infty, \\ [n] &\mapsto M \otimes R^{\otimes n} \otimes N. \end{aligned}$$

For $\mathscr{C} = Sp$, this defines $(\mathtt{Mod}_R(Sp), \wedge_R, R)$. We can also define $F_R(M, -) : \mathtt{Mod}_R(Sp) \to \mathtt{Mod}_R(Sp)$ to be the right adjoint of

$$M \wedge_R - : \mathtt{Mod}_R \to \mathtt{Mod}_R.$$

Notation 21.2. If $R \in \mathtt{Alg}_{E_\infty}(Sp)$, and $M, N \in \mathtt{Mod}_R(Sp)$, we can define

$$\begin{aligned} \operatorname{Tor}_*^R(M, N) &:= \pi_* \left(M \wedge_R N\right) \\ \operatorname{Ext}_R^*(M, N) &:= \pi_{-*} F_R(M, N). \end{aligned}$$

We shall see that

$$\pi_* (HM \wedge_{HR} HN) \cong \mathrm{Tor}_*^R(M, N),$$

where $R \in \mathtt{CAlg}(\mathtt{Ab})$ and $M, N \in \mathtt{Mod}_R(\mathtt{Ab})$.

We have *change of algebras*: if $f : A \to B$ in $\mathtt{Alg}_{E_\infty}(\mathscr{C})$, we get a monoidal adjunction

$$- \otimes_A B : \mathtt{Mod}_A \leftrightarrows \mathtt{Mod}_B : f^*,$$

where extension is strong monoidal and restriction f^* is lax.

In spectra this becomes

$$- \wedge R : \mathtt{Mod}_{\mathbb{S}} = Sp \leftrightarrows \mathtt{Mod}_R : U.$$

Theorem 21.3. *(Schwede-Shipley)* Let $\mathscr{C}$ be a stable ∞-category. Then $\mathscr{C} \simeq \mathtt{Mod}_R Sp$ if and only if $\mathscr{C}$ is presentable, and there exists $C \in \mathscr{C}$ compact generator such that if $D \in \mathscr{C}$ and $\mathrm{Ext}^n_{\mathscr{C}}(C, D) \cong 0$ then $D \simeq 0$.

Lemma 21.4. If $\mathscr{C}$ is a stable ∞-category, and $X, Y \in \mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(X, Y) \in Sp$.

Proof. We have that $\mathrm{Hom}_{\mathscr{C}}(X, Y) \in \mathcal{S}_*$, so

$$\Omega \mathrm{Hom}_{\mathscr{C}}(X, Y) \simeq \mathrm{Hom}_{\mathscr{C}}(\Sigma X, Y) \simeq \mathrm{Hom}_{\mathscr{C}}(X, Y).$$

So these are infinite loop spaces. $\square$

Proof of theorem: if $\mathscr{C} \simeq \mathtt{Mod}_R(Sp)$, then $\mathscr{C}$ is presentable. Take $C = R$, then $\mathrm{Ext}^n_{\mathscr{C}}(R, D) \cong \pi_n D$. Then $D \simeq 0$ if and only if $\pi_{-n} D = 0$ for all n.

For the other direction, if $\mathscr{C} \in \mathrm{Pr}^L$, then as $\mathscr{C}$ is stable, there is a map

$$\begin{aligned} Sp \otimes \mathscr{C} &\to \mathscr{C} \\ (E, C) &\mapsto E \otimes C, \end{aligned}$$

adjoint to $\mathrm{Hom}_{\mathscr{C}}(C, -)$ valued in Sp. That is, $\mathscr{C}$ is tensored and cotensored over spectra.

We have

$$- \otimes C : Sp \leftrightarrows \mathscr{C} : \mathrm{Hom}_{\mathscr{C}}(C, -).$$

Let $G = \mathrm{Hom}_{\mathscr{C}}(C, -)$, then the idea is that this is monadic and the monad is equivalent to $- \wedge_{\mathbb{S}} R$ for some R.

Let $\alpha : D \to D'$ in $\mathscr{C}$ such that $G(\alpha)$ is an equivalence in Sp. Then $G(C\alpha) \simeq 0$.

$$\pi_n C\alpha \simeq \mathrm{Ext}^{-n}_{\mathscr{C}}(C, G\alpha) = 0,$$

so $C\alpha \simeq 0$, so α equivalence in $\mathscr{C}$.

Then $R := G(C) = \mathrm{Hom}_{\mathscr{C}}(C, C) = \mathrm{End}_{\mathscr{C}}(C) \in \mathtt{Alg}_{E_1}(Sp)$.

With $E \in Sp$ and $D \in \mathscr{C}$, get $E \wedge G(D) \simeq G(E \otimes D)$. This is true as G preserves all colimits, suffices to check for $E = \mathbb{S}$ then obvious. $R = G(C)$, $E \wedge R = G(X \otimes C)$, Barr Beck Lurie monadicity.

If $R = \mathrm{End}_{\mathscr{C}}(C)$ get an monoidal variant

$$\mathtt{Alg}_{E_\infty}(Sp) \to \mathtt{Alg}_{E_\infty}(\overset{L}{\mathrm{Pr}})$$
$$R \mapsto \mathtt{Mod}_R.$$

So we can say that $\mathscr{C} \in \mathtt{Alg}_{E_\infty}(\mathrm{Pr}^L)$ belongs to the image above if and only if there is some $I \in \mathscr{C}$ a compact generator.

Theorem 21.5. *(Stable Dold Kan)* Let R be a commutative ring. Then

$$(\mathtt{Mod}_{HR}(Sp), \wedge_R, HR) \simeq \left(D(R), \otimes_R^{\mathbb{L}}, R\right).$$

Proof sketch. Take $D_* \in \mathtt{Ch}_R$, then $H_n(D_*) = \mathrm{Ext}_R^{-n}(R, D_*) \cong \mathrm{Ext}_{D(R)}^{-n}(R, D_*)$. Thus R is a compact generator.

Thus $D(R) \simeq \mathtt{Mod}_A(Sp)$, where $A = \mathrm{End}_{D(R)}(R)$, but we check

$$\pi_n(A) \cong \mathrm{Ext}_{D(R)}^{-n}(R, R)$$
$$= \begin{cases} R & n = 0 \\ 0 & \text{else}, \end{cases}$$

so $A \simeq HR$. ∎

Shipley proved this in model categories in early 2000's.

22. Lecture 22: Tuesday, April 18th

Universal trace methods for algebraic K-theory

Recall: for $R \in \mathtt{Ring}$, we can define $K_0(R) = K_0(\mathcal{P}(R))$. The latter K_0 is Grothendieck group completion of commutative monoids, and here $\mathcal{P}(R)$ is iso classes of finitely generated projective (right) R-modules.

If $M \oplus N \cong R^n$, then $[M] + [N] = [R^n]$ in $K_0(R)$. That is, exact sequences split in $K_0(R)$.

Eilenberg swindle: If we just did projective, not also finitely generated, we would get 0 because any projective M has $M \oplus N \cong R^n$ for some N, n, hence we could take

$$R^\infty = M \oplus N \oplus M \oplus N \oplus \cdots$$

Since $M \oplus R^\infty \cong R^\infty$, this would imply $[M] = 0$.

Definition 22.1. $K_n(R) = \pi_n(\mathrm{BGL}(R)^+ \times K_0(R))$.

Here $\mathrm{GL}(R) = \mathrm{colim}_n \mathrm{GL}_n(R)$, and the plus construction is the universal H-space receiving a map from $\mathrm{BGL}(R)$, abelianizing π_1, ...

Note that $\mathrm{BGL}(R)^+ \times K_0(R)$ is an infinite loop space. It admits a Gersten-Wagoner delooping.

K-theory can be generalized to a much wider context, e.g. exact categories, and stable ∞-categories.

For example R corresponds to the stable ∞-category $\mathtt{Mod}_{HR}^{\mathrm{cpct}}(Sp)$. Taking compact objects is again to avoid size issues.

Blumberg-Gepner-Tabuada: Define connective K-theory as a functor

$$\mathtt{Cat}_\infty^{\mathrm{st}} \to Sp^{\geq 0},$$

where $\mathtt{Cat}_\infty^{\mathrm{st}}$ is the category of stable ∞-categories and exact functors (preserves finite limits and colimits).

Definition 22.2. Let $\mathtt{Cat}_\infty^{\mathrm{perf}} \subseteq \mathtt{Cat}_\infty^{\mathrm{st}}$ be the full subcategory spanned by idempotent-complete categories.

We have that $\mathscr{C}$ is idempotent complete if for all $X \in \mathscr{C}$, and any $e : X \to X$ in $\mathscr{C}$ such that $e^2 \simeq e$, we have a splitting onto its image.

F.g. projective modules are idempotent complete, free modules are not.

Idempotent completion is a left adjoint to the inclusion:

$$\mathrm{Idem} : \mathtt{Cat}_\infty^{\mathrm{st}} \leftrightarrows \mathtt{Cat}_\infty^{\mathrm{perf}} : i.$$

We have that $\mathrm{Idem}(\mathscr{C}) = \mathrm{Ind}(\mathscr{C})^\omega$ (BGT 2.20).

Think of an idempotent complete stable ∞-category as the compact objects of a presentable stable ∞-category.

To define $K(\mathscr{C})$ for $\mathscr{C} \in \mathtt{Cat}_\infty^{\mathrm{perf}}$, we can take

$$K(\mathscr{C}) = |S_\bullet \mathscr{C}^\simeq| .$$

K-theory is comprised of two concepts:

- abelian group completion
- splitting exact sequences

Definition 22.3. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be exact functors btw stable ∞-categories.

(1) Say F is a *Morita equivalence* if $\mathrm{Idem}(F)$ is an equivalence of ∞-categories
(2) The sequence is *exact* if F is fully faithful, $G \circ F \simeq 0$, and $\mathcal{C} \simeq \mathcal{B}/\mathcal{A}$ in $\mathtt{Cat}_\infty^{\mathrm{perf}}$.
(3) The sequence is *split exact* if there exist right adjoint functors F', G' to F, G, respectively, so that $F'F = \mathrm{id}$ and $GG' = \mathrm{id}$.

Definition 22.4. Let $E : \mathtt{Cat}_\infty^{\mathrm{st}} \to \mathscr{D}$ where $\mathscr{D} \in \mathrm{Pr}^L_{\mathrm{st}}$. We say E is *additive* if it:

(1) inverts Morita equivalences
(2) preserves filtered colimits
(3) sends split exact sequences to split (co)fiber sequences in $\mathscr{D}$, i.e. $E(\mathcal{B}) \simeq E(\mathcal{A}) \vee E(\mathcal{B})$.

Take

$$\mathtt{Cat}_\infty^{\mathrm{st}} \xrightarrow{\mathrm{Idem}} \mathtt{Cat}_\infty^{\mathrm{perf}} \hookrightarrow \mathrm{Fun}\left(\left(\mathtt{Cat}_\infty^{\mathrm{perf}}\right)^{\mathrm{op}}, \mathcal{S}\right) \xrightarrow{Sp} \mathrm{Fun}(\left(\mathtt{Cat}_\infty^{\mathrm{perf}}\right)^{\mathrm{op}}, Sp) \to \mathrm{Fun}(\left(\mathtt{Cat}_\infty^{\mathrm{perf}}\right)^{\mathrm{op}}, Sp)/\sim$$

where we mod out by split exact sequences.

We call the resulting object $\mathcal{M}_{\mathrm{add}}$, and the composition

$$\mathcal{U}_{\mathrm{add}} : \mathtt{Cat}_\infty^{\mathrm{st}} \to \mathcal{M}_{\mathrm{add}}.$$

This functor is the universal additive invariant, in the sends that

$$\mathrm{Fun}^L(\mathcal{M}_{\mathrm{add}}, \mathscr{D}) \xrightarrow{\mathcal{U}^*_{\mathrm{add}}} \mathrm{Fun}_{\mathrm{add}}(\mathtt{Cat}_\infty^{\mathrm{st}}, \mathscr{D})$$

for all $\mathscr{D} \in \mathrm{Pr}^L_{\mathrm{st}}$.

Definition 22.5. For $\mathscr{C} \in \mathtt{Cat}_\infty^{\mathrm{perf}}$, we define

$$K(\mathscr{C}) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{add}}}(\mathcal{U}_{\mathrm{add}}(Sp^{\mathrm{Wh}}), \mathcal{U}_{\mathrm{add}}(\mathscr{C})) \in Sp^{\geq 0}.$$

We can make this universal property monoidal: if $\mathscr{C}$ is a symmetric monoidal ∞-category, then $K(\mathscr{C})$ is an E_∞ ring spectrum.

Can construct $\otimes$ in $\mathtt{Cat}_\infty^{\mathrm{perf}}$ similar to Pr^L. This induces a monoidal structure on $\mathrm{Fun}_{\mathrm{add}}(\mathtt{Cat}_\infty^{\mathrm{perf}}, Sp)$ by Day convolution.

Here $\mathcal{U}_{\mathrm{add}}$ is strong monoidal. Then for all $\mathscr{D} \in \mathtt{Alg}_{E_\infty}(\mathrm{Pr}^L)$, we get

$$\mathrm{Fun}^{L,lax}(\mathcal{M}_{\mathrm{add}}, \mathscr{D}) \simeq \mathrm{Fun}^{lax}_{\mathrm{add}}(\mathtt{Cat}_\infty^{\mathrm{perf}}, \mathscr{D}).$$

Application: Dennis trace $K(R) \to \mathrm{THH}(R)$. Here $\mathrm{THH}(R) = R \wedge_{R \wedge R^{\mathrm{op}}} R$. If R is a k-algebra, get

$$\mathrm{HH}_*(R) = H_*(R \otimes^{\mathbb{L}}_{R \otimes R^{\mathrm{op}}} R) = \mathrm{Tor}_*^{R \otimes R^{\mathrm{op}}}(R, R).$$

Can replace R by any stable ∞-cat $\mathscr{C}$. Here

$$\mathrm{THH}(\mathscr{C}) = \mathrm{colim}\left(\cdots \amalg_{(c_0, \dots, c_n)} \mathscr{C}(c_{n-1}, c_n) \wedge \cdots \wedge \mathscr{C}(c_n, c_0)\right).$$

Here $\mathrm{THH}(\mathtt{Mod}_R^{\mathrm{perf}}) = \mathrm{THH}(R)$.

We have $\mathrm{THH} : \mathtt{Cat}_\infty^{\mathrm{st}} \to Sp^{\geq 0}$. It is an additive invariant (clearly preserves Morita equivalence and filtered colimits). Can use Dennis-Waldhausen-Morita argument to show it sends split exact sequences to cofiber sequences.

Theorem 22.6. Let E be any additive invariant, i.e. $E \in \mathrm{Fun}_{\mathrm{add}}(\mathtt{Cat}_\infty^{\mathrm{st}}, Sp)$. Then $\mathrm{Nat}(K, E) \simeq E(Sp^{\mathrm{Wh}})$.

We see that

$$\mathrm{Nat}(K(-), \mathrm{THH}(-)) \cong \mathrm{THH}(Sp^{\mathrm{Wh}}) \simeq \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}.$$

Applying π_0, we get that

$$[K(\mathscr{C}), \mathrm{THH}(\mathscr{C})] \cong \pi_0 \mathbb{S} \cong \mathbb{Z}.$$

Given $F : K(\mathscr{C}) \to \mathrm{THH}(\mathscr{C})$, we get

$$\mathbb{S} \to \mathrm{Map}(\mathcal{U}_{\mathrm{add}}(Sp^{\mathrm{Wh}}), \mathcal{U}_{\mathrm{add}}(Sp^{\mathrm{Wh}})) \simeq K(\mathbb{S}) \xrightarrow{F} \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}.$$

The Dennis trace picks up $1 \in \mathbb{Z}$.

We can view $K(R) \to \mathrm{THH}(R)$ via

$$\mathrm{BGL}_n(R) \to B^{\mathrm{cyc}}\mathrm{GL}_n(R) \to B^{\mathrm{cyc}} M_n(R) \to B^{\mathrm{cyc}} R.$$

On π_0, we get

$$K_0(R) \to HH_0(R).$$

For $R \in \mathtt{Ring}$, we send $[P] \mapsto \mathrm{tr}(\mathrm{id}_P \oplus 0)$.

References