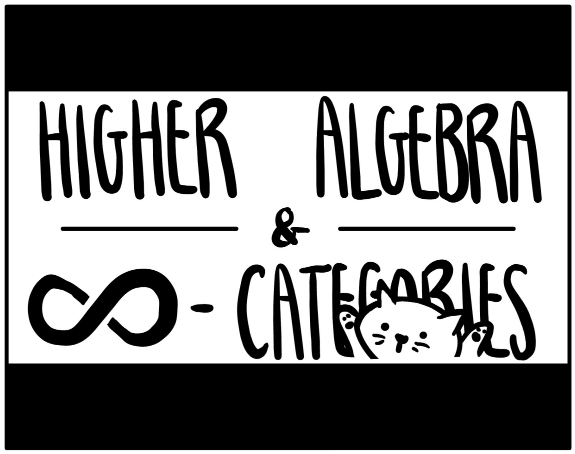
HIGHER ALGEBRA

MAXIMILIEN PÉROUX



1. Lecture 1: Thursday, January 12th

Today: the **homotopy hypothesis**

Classical algebra: sets, monoids, groups, abelian groups, rings. Each of these are built up on the other. In higher courses, we may see groupoids, which are types of categories. A category is a generalization of a monoid, in some sense. We also have monoidal categories, which in some sense are a generalization of rings.

For higher algebra: spaces, $\mathbb{E}_1$-spaces, spectra, $\mathbf{E}_1$-ring spectra. Underlying this we have ∞-groupoids, ∞-categories, and monoidal ∞-categories.

We study spaces, not up to homeomorphism, but up to *weak homotopy equivalence*. We will study this in a minute. "Spaces" in this class will always mean the study of topological spaces up to weak homotopy equivalence.

We'll give a synthetic definition of what an infinity category is, and circle back to a technical definition in about a month.

What is an ∞-category?

Date: January 31, 2023.

An ∞-category (or $(\infty, 1)$-category) $\mathscr{C}$ should consist of:

(1) a class of objects
(2) a class of morphisms so that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a space
(3) n-morphisms for $n \geq 2$, where for instance 2-morphisms are between 1-morphisms, 3-morphisms between 2-morphisms, etc.
(4) morphisms can be composed in a suitable way
(5) n-morphisms for $n \geq 2$ are invertible in some sense.

An ∞-groupoid (or $(\infty, 0)$-category) should be an ∞-category where all the 1-morphisms are also invertible in some sense.

Why study spaces up to weak homotopy equivalence?

Recall by the Yoneda lemma, we have that

$$X \cong Y \Leftrightarrow \mathrm{Hom}_{\mathtt{Top}}(A, X) \cong \mathrm{Hom}_{\mathtt{Top}}(A, Y)$$

for all $A \in \mathtt{Top}$. Figuring out $\mathrm{Hom}(A, X)$ up to bijection for all A is very difficult, so we prefer to study continuous maps up to homotopy. For X and Y nice enough, we say that $f \simeq g$ in $\mathrm{Hom}(X, Y)$ if there exists some path $I \to \mathrm{Map}(X, Y)$ so that $0 \mapsto f$ and $1 \mapsto g$. We define $[X, Y] = \mathrm{Hom}_{\mathtt{Top}}(X, Y)/\simeq$.

We see then that $X \simeq Y$ if and only if $[A, X] \cong [A, Y]$ for all $A \in \mathtt{Top}$.

We may ask when $[A, -] : \mathtt{Top}_* \to \mathtt{Set}$ factors through $\mathtt{Grp}$ or $\mathtt{Ab}$. We have that $[A, -]$ factors through $\mathtt{Grp}$ if and only if A is a co-H-group in $\mathtt{Top}$. That is, we have maps

$$A \to A \vee A$$
$$A \to *,$$

which is coassociative, counital, coinvertible.

Example 1.1. S^n, when $n \geq 1$, is a co-H-space. The map $S^n \to S^n \vee S^n$ is the pinch map.

We say that X is *weakly homotopy equivalent* to Y, we write $X \sim Y$, if and only if there is a map $X \to Y$ inducing an isomorphism

$$\pi_n(X) = [S^n, X]_* \cong [S^n, Y]_* = \pi_n(Y),$$

for all $n \geq 0$ (for $n \geq 1$ this is a group isomorphism).

If $X \sim Y$, then $H_n(X) \cong H_n(Y)$ for any n.

Theorem 1.2. (Cellular approximation) For any X in $\mathtt{Top}$, there exists $\widetilde{X}$ a CW complex with a canonical map $\widetilde{X} \xrightarrow{\sim} X$ that is a weak equivalence.

Theorem 1.3. (Whitehead) If X, Y are CW complexes, then $X \xrightarrow{\simeq} Y$ is a homotopy equivalence if and only if $X \xrightarrow{\sim} Y$ is a weak homotopy equivalence.

Exercise 1.4. Find spaces X and Y which are weakly homotopy equivalent but not homotopy equivalent.

We denote by Δ the simplex category. Its objects are ordered sets of the form $[n] = \{0, 1, \ldots, n\}$, and its morphisms are order-preserving maps. We have that Δ is generated by *cofaces* and *codegeneracies*. The cofaces are of the form

$$d^0, d^1 : [0] \to [1],$$

skipping 0 or 1 in [1], etc. The codegeneracies look like $s^0 : [1] \to [0]$ which "repeat" an element.

The cofaces and codegeneracies satisfy certain *cosimplicial identities*.

If $\mathscr{C}$ is a category, we denote by $s\mathscr{C} = \mathscr{C}^{\Delta^{\mathrm{op}}}$ the simplicial objects in $\mathscr{C}$. If $\mathscr{C} = \mathtt{Set}$, we write $\mathtt{sSet}$ as the category of simplicial sets. A simplicial set $X_\bullet \in \mathtt{sSet}$ consists of sets $X_0, X_1, \ldots$ together with face and degeneracy maps satisfying the simplicial identities.

Example 1.5. The *nerve of a small category*. Let $\mathscr{C} \in \mathtt{Cat}$ a small category. We denote by $N_\bullet\mathscr{C}$ the simplicial set with $N_0\mathscr{C} = \mathrm{ob}\mathscr{C}$, $N_1\mathscr{C} = \mathrm{mor}\mathscr{C}$, and $N_n\mathscr{C}$ the set of n composable morphisms in $\mathscr{C}$. That is,

$$N_n\mathscr{C} = N_1\mathscr{C} \times_{N_0\mathscr{C}} \cdots \times_{N_0\mathscr{C}} N_1\mathscr{C}.$$

The face maps are source/target/composition. The degeneracies insert an identity morphism.

Example 1.6. Via Yoneda, we get a functor

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathtt{Set}.$$

If $X_\bullet$ is a simplicial set, we get that the set of n-simplices X_n is in bijection with $\mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, X_\bullet)$.

Example 1.7. (Dold–Kan) We have $\mathtt{Ch}_R^{\geq 0} \xrightarrow{\Gamma} s\mathtt{Mod}_R$ is an isomorphism, where $\Gamma_m C_\bullet = \oplus_{[n]\twoheadrightarrow[k]} C_k$, with faces and degeneracies left as an exercise.

Example 1.8. Let $\Delta^n_{\mathtt{Top}} \subseteq \mathbb{R}^{n+1}$ be defined by

$$\left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \colon 0 \leq t_i \leq 1,\ \sum t_i = 1\right\}.$$

We can view $[n] = \{v_0, \ldots, v_n\}$, and $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the ith place. Then if $\alpha : [m] \to [n]$ in Δ, we can define $\alpha(v_i) = v_{\alpha(i)}$. Extend linearly to get $\alpha_* : \Delta^m_{\mathtt{Top}} \to \Delta^n_{\mathtt{Top}}$. We get then that $\Delta^\bullet_{\mathtt{Top}}$ is a cosimplicial topological space.

Example 1.9. If $X \in \texttt{Top}$, we have $\mathrm{Sing}_\bullet(X) \in \texttt{sSet}$ defined by $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\texttt{Top}}\left(\Delta^n_{\texttt{Top}}, X\right)$.

Definition 1.10. If $X_\bullet \in \texttt{sSet}$, we define its *geometric realization* to be

$$|X_\bullet| = \amalg_{n \geq 0} X_n \times \Delta^n_{\texttt{Top}} / \sim,$$

where $(x, s) \sim (y, t)$ if and only if there is some $\alpha : [m] \to [n]$ so that $\alpha^* y = x$ and $\alpha_* s = t$.

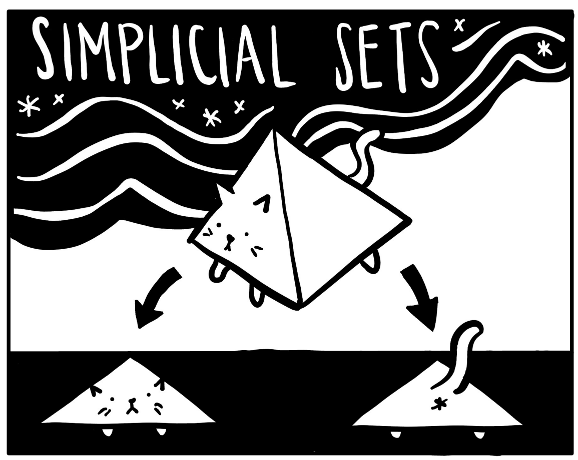
Example 1.11. $|\Delta^n_\bullet| \cong \Delta^n_{\texttt{Top}}$.

Exercise 1.12. $|X_\bullet|$ is always a CW complex for any $X_\bullet \in \texttt{sSet}$.

Exercise 1.13. We have an adjunction $|-| : \texttt{sSet} \leftrightarrows \texttt{Top} : \mathrm{Sing}(-)$

Definition 1.14. $X_\bullet \to Y_\bullet$ is a *weak homotopy equivalence* in $\texttt{sSet}$ if $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ is a weak homotopy equivalence of spaces.

Theorem 1.15. (Quillen) Simplicial sets up to weak equivalence is equivalent to topological spaces up to weak homotopy equivalence. Moreover, for any $X \in \texttt{Top}$, we have that $|\mathrm{Sing}(X)|$ is weakly equivalent to X.



2. Lecture 2: Tuesday, January 17th

Today: the homotopy hypothesis (continued).

Recall we are interested in studying $\texttt{Top}$ up to weak homotopy equivalences. Equivalently, we are interested in studying $\texttt{sSet}$ up to weak equivalence, and the relationship between the two was given by the geometric realization / singular complex adjunction.

Recall we've defined $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$. We will define the *kth horn* $\Lambda^n_k \subseteq \Delta^n$ as a coequalizer in `sSet`

$$\left(\coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{i \neq k} \Delta^{n-1} \right) \to \Lambda^n_k,$$

where the two maps are δ^{j-1} and δ^i. The geometric realization of Λ^n_k is the topological n-simplex, with the middle and the face opposite the kth edge removed.

Definition 2.1. We say that $Y \in$ `sSet` is a *Kan complex* if for all $k \le n$, and for every $\Lambda^n_k \to Y$, there exists a (not necessarily unique) lift:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Y \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Exercise 2.2. Y is a Kan complex if and only if for any $(n-1)$-simplices $y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$ such that $d_i y_j = d_{j-1} y_i$ for $i < j$, $i, j \neq k$, there exists an n-simplex y such that $d_i y = y_i$ for all $i \neq k$.

Exercise 2.3. We have that $\mathrm{Sing}(X)$ is always a Kan complex for any $X \in$ `Top`.

Exercise 2.4. We have that Δ^n is not a Kan complex for $n \geq 1$.

Exercise 2.5. If $X \in s$`Grp`, then the underlying simplicial set of X is always a Kan complex.

Up to weak homotopy equivalence, every simplicial set is a Kan complex (will see this later).

Recall the Dold-Kan correspondence

$$s\mathtt{Mod}_{\mathbb{Z}} \cong \mathtt{Ch}^{\geq 0}_{\mathbb{Z}},$$

which sends weak homotopy equivalences to quasi-isomorphisms. Given a simplicial set X_*, we can take an associated simplicial abelian group $\mathbb{Z}[X_*]$ by taking the free group on n-simplices at level n. We can ask what $\mathbb{Z}[X_*]$ corresponds to as a chain complex. One answer is that

$$\mathbb{Z}[\mathrm{Sing}(X_*)] \leftrightarrow C_*(X; \mathbb{Z}).$$

This tells us that

$$\pi_* \left(\mathbb{Z} \left[\mathrm{Sing}(X) \right] \right) \cong H_*(X; \mathbb{Z}).$$

In some sense we can view $\mathbb{Z}[\mathrm{Sing}(X)]$ as being (equivalent to) the *free commutative monoid* on X. This is what is known as the *Dold-Thom theorem.*

Homotopy hypothesis: Spaces (up to weak equivalence) are ∞-groupoids. For us, spaces up to weak equivalences correspond to Kan complexes.

Given $X \in \mathtt{Kan}$, we can call X_0 the objects, and X_1 the morphisms. The horn filling conditions on horns tell you that you can *compose* and *invert* morphisms in X_1, witnessed by simplices in X_2.

Definition 2.6. A *quasi-category* (i.e. ∞-category) is a simplicial set with inner horn lifting property. That is, we can lift against horns Λ^n_k for $0 < k < n$.

Exercise 2.7. A quasi-category has unique horn filling if and only if it is isomorphic to the nerve of a 1-category.

Model categories

Vista: Every nice infinity category is equivalent in some sense to a model category. This will pretty much be the goal of this class.

Notation 2.8. Let $\mathcal{M}$ be a category, and $\chi \subseteq \mathcal{M}$ a class of morphisms. We define $\mathrm{LLP}(\chi)$ to be the class of morphisms in $\mathcal{M}$ so that f has left lifting property with respect to all morphisms in χ:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "\in \chi"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Similarly we can define $f \in \mathrm{RLP}(\chi)$ by

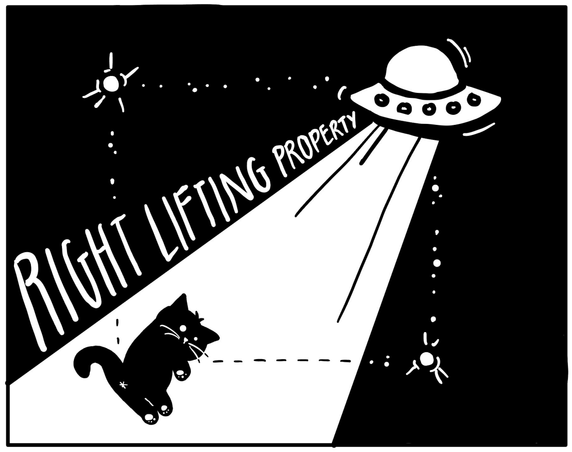
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\chi \ni"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Definition 2.9. A *weak factorization system* on a category $\mathcal{M}$ consists of a pair $(\mathscr{C}, \mathscr{F})$ of classes of morphisms such that

(1) Given any $f : X \to Y$ in $\mathcal{M}$, it factors (not necessarily uniquely) as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathscr{C}\ni"'] & & Y \\ & W \arrow[ur, "\in\mathscr{F}"'] & \end{tikzcd}$$

(2) $\mathscr{C} = \mathrm{LLP}(\mathscr{F})$ and $\mathscr{F} = \mathrm{RLP}(\mathscr{C})$.



Example 2.10. In `Set`, we have that mono and epimorphisms give a weak factorization system. A factorization is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{id}_X \times f"'] & & Y \\ & X \times Y \arrow[ur, "\pi_Y"'] & \end{tikzcd}$$

Definition 2.11. A *model structure* on $\mathcal{M}$ consists of three classes of morphisms:

W	weak equivalences
Cof	cofibrations
Fib	fibrations

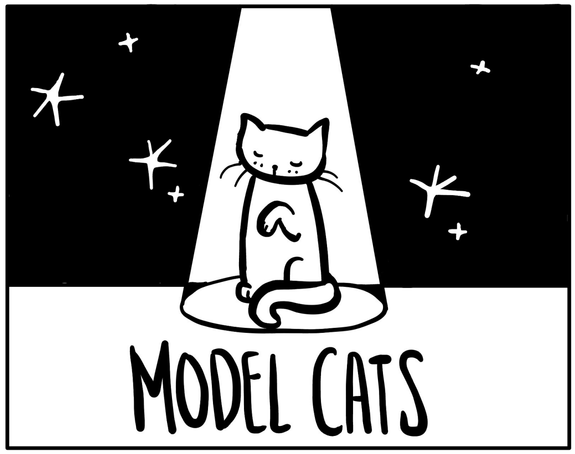
We denote by $\widetilde{\mathrm{Cof}} := \mathrm{Cof} \cap W$ and $\widetilde{\mathrm{Fib}} = \mathrm{Fib} \cap W$, and call these *trivial cofibrations* (resp. *trivial fibrations*). These are subject to the constraint that

(1) $\mathcal{M}$ is bicomplete (all limits and colimits)[1]
(2) W satisfies 2-out-of-3 property[2]
(3) $\left(\mathrm{Cof}, \widetilde{\mathrm{Fib}}\right)$ and $\left(\widetilde{\mathrm{Cof}}, \mathrm{Fib}\right)$ are weak factorization systems.

Terminology 2.12. A category with a model structure is referred to as a *model category.*

Notation 2.13. We will decorate each class of morphisms as

[1]We might also require *finitely* bicomplete.
[2]If f and g are composable, and any two of f, g, gf are in W then so is the third.



$$
\begin{array}{l|l}
W & \xrightarrow{\sim} \\
\mathrm{Cof} & \hookrightarrow \\
\mathrm{Fib} & \twoheadrightarrow
\end{array}
$$

Exercise 2.14. W, Cof, and Fib are closed under retracts: that is,

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[rr, bend right, equal] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

then if $g \in W$ (resp. Cof or Fib) then $f \in W$ (resp. Cof or Fib).

Definition 2.15. Let $\mathcal{M}$ be a model category, and let $\varnothing \in \mathcal{M}$ the initial object and $* \in \mathcal{M}$ the terminal object.

- We say that $X \in \mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to X$ is a cofibration.
- We say that $X \in \mathcal{M}$ is *fibrant* if the unique map $X \to *$ is a fibration.
- We say that $\widetilde{X}$ is a *cofibrant replacement* of X if

$$
\begin{tikzcd}
\varnothing \arrow[rr] \arrow[dr, hook] & & X \\
& \widetilde{X} \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}
$$

- We say that $\widetilde{X}$ is a *fibrant replacement* of X if

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim"'] & & * \\ & \widetilde{X} \arrow[ur, two heads] & \end{tikzcd}$$

Example 2.16. $\mathcal{M} = \texttt{Top}$, $W =$ weak homotopy equivalences, Cof $=$ relative CW complexes[3] The fibrations are determined by Fib $= \mathrm{RLP}(\widetilde{\mathrm{Cof}})$. The fibrations are equivalently $\mathrm{RLP}(D^n \to D^n \times I)$. Every object here is fibrant, and the cofibrant objects are precisely the CW complexes. Cofibrant replacement is cellular approximation.

3. Lecture 3: Thursday, January 19th

Proposition 3.1. Identities and isomorphisms are weak equivalences in a model category.

Proof. For any $X \in \mathcal{M}$, we can fibrantly replace it to get $X \overset{\sim}{\hookrightarrow} \widetilde{X}$. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}"] \arrow[dr, hook, "\sim"'] & & X \arrow[dl, "\sim"] \\ & \widetilde{X}. & \end{tikzcd}$$

By 2-out-of-3, we have that $\mathrm{id} : X \to X$ is also a weak equivalence.

More generally if $f : X \to Y$ is an isomorphism in $\mathcal{M}$, then by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] \arrow[rr, bend left, equal] & Y \arrow[r, "f^{-1}"] \arrow[d, equal] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y, \end{tikzcd}$$

we see that f is contained in W. $\square$

If $(\mathscr{C}, \mathscr{F})$ is a weak factorization system, then both $\mathscr{C}$ and $\mathscr{F}$ are closed under retracts. Hence $\mathrm{Cof}, \widetilde{\mathrm{Cof}}, \mathrm{Fib}, \widetilde{\mathrm{Fib}}$ are closed under retracts. W is also closed under retracts (exercise).

Exercise 3.2. We have that $\mathcal{M}$ is a model category if and only if $\mathcal{M}^{\mathrm{op}}$ is a model category.

[3] $A \hookrightarrow X$ is a *relative CW complex* if X is built out of A by attaching cells.

Theorem 3.3. Cofibrations are closed under pushouts and coproducts.

Proof. Given any test square, we can try to lift:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Z \arrow[r] \arrow[urr, dashed] & P \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & B. \end{tikzcd}$$

This map is constructed by universal property of the pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, dashed, bend right] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists!"] & \\ & & A. \end{tikzcd}$$

For coproducts, we can take $X_i \hookrightarrow Y_i$ for $i \in J$. Let's try to lift:

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, hook] & \amalg_i X_i \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Y_i \arrow[r] \arrow[urr, dashed] & \amalg_i Y_i \arrow[r] & B. \end{tikzcd}$$

We know that each $X_i \hookrightarrow Y_i$ is a cofibration hence it lifts against the big square. By universal property a map $\amalg_i Y_i \to A$ exists. $\square$

Example 3.4. If $\mathscr{C}$ is a bicomplete category, then $\mathscr{C}$ has a model structure where W is the isomorphisms, and $\mathrm{Cof} = \mathrm{Fib} = \mathrm{mor}\mathscr{C}$.

Example 3.5. If $\mathcal{M} = \mathtt{Top}$, we have the *Quillen model structure*, with

- W = weak homotopy equivalences
- Cof = retracts of relative CW complexes
- Fib = Serre fibrations ($\mathrm{RLP}(D^n \hookrightarrow D^n \times I)$).

Example 3.6. The Strøm (or Hurewicz) model structure on `Top`:

- W = homotopy equivalences
- Fib = Hurewicz fibrations ($\mathrm{RLP}(A \to A \times I)$ for all $A \in \mathtt{Top}$)
- Cof = closed cofibrations in `Top`.

Fibrant replacement in the Strøm model structure looks like

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook] & & Y \\ & M_f \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Where $M_f = (X \times I) \cup_X Y$ is the mapping cylinder.

Example 3.7. The *Kan model structure* on `sSet` with

- W = weak homotopy equivalences
- Cof = monomorphisms (levelwise injections)
- Fib = Kan fibrations (RLP($\Lambda^n_k \to \Delta^n$) for all $0 \leq k \leq n$).

Everything is cofibrant here (since the empty simplicial set injects into everything). Fibrant things are Kan complexes. This tells us that every simplicial set is weakly equivalent to a Kan complex!

Theorem 3.8. (Milnor) The natural map $X \to \mathrm{Sing}(|X|)$ is a weak homotopy equivalence for any simplicial set X. [Kerodon, 3.5.4.1]

Definition 3.9. Let $\mathscr{C}$ be a cat, and $W \subseteq \mathscr{C}$ a subcategory. A functor $F : \mathscr{C} \to \mathscr{D}$ is called the *localization of* $\mathscr{C}$ *with respect to* W if:

(1) $F(f) \in \mathrm{iso}\mathscr{D}$ if $f \in \mathrm{mor}W$
(2) For any other F' satisfying (1), we have

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F'"] \arrow[d, "F"'] & \mathscr{D}' \\ \mathscr{C} \arrow[ur, "\exists!"'] & \end{tikzcd}$$

We denote by $\mathscr{C} \to \mathscr{C}[W^{-1}]$ the localization.

Here is a naive way to construct $\mathscr{C}[W^{-1}]$: we take the free category on $\mathscr{C}$ and "W^{-1}." That is, we take the same objects, but allow morphisms to be "zigzags" of morphisms forward in $\mathscr{C}$ and morphisms backwards in W, and we mod out by the relation that things in W become isomorphisms. There are size issues here.

Theorem 3.10. If $\mathcal{M}$ is a model category, then localization $\mathcal{M} \to \mathcal{M}[W^{-1}]$ exists. We denote by $\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[W^{-1}]$ the homotopy category of $\mathcal{M}$.

Recall in `Top` that $f \simeq g : X \to Y$ if there is a map $H : X \times I \to Y$ so that $H(-, 0) = f$ and $H(-, 1) = g$.

Definition 3.11. Le t$\mathcal{M}$ be a model category. A *cylinder object* on $X \in \mathcal{M}$ is defined to be

$$\begin{tikzcd} X \amalg X \arrow[rr, "\nabla"] \arrow[dr, hook] & & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

The construction of cylinder objects is *not functorial*.

A *(left) homotopy* from f to g is a map $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. We denote this by $f \simeq g$.

Proposition 3.12. We have that $i_0 : X \to \mathrm{Cyl}(X)$ is a weak equivalence (and same for i_1).

Proof. We have

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, dashed, "i_0"'] & X \amalg X \arrow[r, "\nabla"] \arrow[d] & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

By 2-out-of-3 on the outside maps, the result follows. □

Proposition 3.13. If X is cofibrant, then $i_0, i_1 : X \to \mathrm{Cyl}(X)$ are cofibrations.

Proof. Since cofibrations are preserved under pushouts, we have that i_0 and i_1 are cofibrations:

$$\begin{tikzcd} \varnothing \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "i_0"] \\ X \arrow[r, "i_1"'] & X \amalg X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

□

Theorem 3.14. (Exercise) If X is cofibrant, then homotopy $\simeq$ gives an equivalence relation on $\mathrm{Hom}(X, Y)$ for any Y.

We can think of a map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(X, Y)/\simeq \times \mathrm{Hom}_{\mathcal{M}}(Y, Z)/\simeq &\to \mathrm{Hom}_{\mathcal{M}}(X, Z)/\simeq \\ (f, g) &\mapsto g \circ f. \end{aligned}$$

In order for this to be well-defined, we need Z to be fibrant.

Lemma 3.15. If Z is fibrant, and $f \simeq g : X \to Z$, then if $h : X' \to X$, we have that $fh \simeq gh$.

Proof. We have $H : \mathrm{Cyl}(X) \to Y$ with $H_0 = f$ and $H_1 = g$. By lifting, we get

$$\begin{tikzcd} X' \amalg X' \arrow[r] \arrow[d, hook] & X \amalg X \arrow[r] & \mathrm{Cyl}(X) \arrow[d, two heads, "\sim"] \\ \mathrm{Cyl}(X') \arrow[r] \arrow[urr, dashed] & X' \arrow[r] & X. \end{tikzcd}$$

This gives the desired map. We used fibrancy of Z to ensure that the map $\mathrm{Cyl}(X) \to X$ was a trivial fibration (or could be replaced with a better cylinder object using a map to Z). ☐

Theorem 3.16. In $\mathcal{M}$, given $f : X \to Y$ with X cofibrant and Y fibrant, then $f \in W$ if and only if f is a homotopy equivalence.[4]

Notation 3.17. $\mathcal{M}_c$ = cofibrant objects in $\mathcal{M}$, and $\mathcal{M}_f$ = fibrant objects in $\mathcal{M}$. We denote by $\mathcal{M}_{cf}$ = objects which are *both* cofibrant and fibrant.

Concretely, we can define $\mathrm{Ho}(\mathcal{M})$ as the objects in $\mathcal{M}$, but where

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = \mathrm{Hom}_{\mathcal{M}_{cf}/\simeq}(RQX, RQY),$$

where R is a fibrant replacement and Q is a cofibrant replacement.

Exercise 3.18. Given $X \to Y$ in $\mathcal{M}$, there exists $QX \xrightarrow{\widetilde{f}} QY$ such that

$$\begin{tikzcd} QX \arrow[r, "\widetilde{f}"] \arrow[d, two heads, "\sim"] & QY \arrow[d, two heads, "\sim"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Here $\widetilde{f}$ is well-defined up to left homotopy.

Given some $\mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, we just need to check that $W \mapsto$ isos, and it is universal in that way.

[4]Meaning that there is some $g : Y \to X$ with $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$.

4. Lecture 4: Tuesday, January 24th

Definition 4.1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are model categories, and take a functor $F : \mathcal{M} \to \mathcal{N}$. A *left derived functor* of F is an (absolute) right Kan extension of F along $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\
\mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed, bend right] \arrow[u, phantom, shift left=4ex, "\overset{\ell}{\Leftarrow}"]
\end{tikzcd}$$

if $G : \mathrm{Ho}(\mathcal{M}) \to \mathcal{N}$ and $s : G \circ \gamma_{\mathcal{M}} \Rightarrow F$, then there exists a unique $s' : G \Rightarrow LF$ so that $\ell \circ (s' \circ \gamma_{\mathcal{M}}) = s$.

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\
\mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed] \arrow[ur, bend right] \arrow[u, phantom, shift left=4ex, "\overset{\ell}{\Leftarrow}"] \arrow[ur, phantom, shift right=2ex, "\Uparrow s'"]
\end{tikzcd}$$

Definition 4.2. Let $F : \mathcal{M} \to \mathcal{N}$. A *total left derived functor* $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is the left derived functor of $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\gamma_{\mathcal{N}}} \mathrm{Ho}(\mathcal{N})$.

Example 4.3. If $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ where if $f \in W$ between cofibrant objects then Ff is a weak equivalence in $\mathcal{N}$, then $\mathbb{L}F$ exists:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d] & \mathcal{N} \arrow[r] & \mathrm{Ho}(\mathcal{N}) \\
\mathrm{Ho}(\mathcal{M}) \arrow[urr, dashed, bend right]
\end{tikzcd}$$

We will have that $\mathbb{L}F(X) \xrightarrow{\sim} F(X)$ whenever X is cofibrant. In general, $\mathbb{L}F(X) = F(Q(X))$.

Definition 4.4. Let $F : \mathcal{M} \to \mathcal{N}$. We say that F is a *left Quillen functor* if

(i) F is a left adjoint
(ii) F preserves cofibrations and trivial cofibrations.

In this case if G is a right adjoint, then we say the adjunction is a *Quillen adjunction / Quillen pair*.[5]

Exercise 4.5. Show that L is left Quillen if and only if G is right Quillen.

[5]There is a dual notion of right Quillen functor, meaning it is a right adjoint which preserves fibrations and trivial fibrations.

Lemma 4.6. (Ken Brown's Lemma) If $F : \mathcal{M} \to \mathcal{N}$ is any functor between model categories which sends trivial cofibrations between cofibrant objects to weak equivalences in $\mathcal{N}$, then F sends any weak equivalence between cofibrant objects to weak equivalences.

Proof. Let $f : A \xrightarrow{\sim} B$, where $A, B \in \mathcal{M}_c$. We need $F(f)$ to be a weak equivalence. Consider the factorization of the coproduct of f and the identity on B:

$$\begin{tikzcd}
A \amalg B \arrow[rr, "f \amalg \mathrm{id}_B"] \arrow[dr, hook, "q"'] & & B \\
& C \arrow[ur, two heads, "\sim", "p"'] &
\end{tikzcd}$$

Then consider the pushout:

$$\begin{tikzcd}
\varnothing \arrow[r, hook] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook, "i_A"] \arrow[ddr, "\sim"] & B \\
B \arrow[r, hook] \arrow[ddrr, equal] \arrow[drr, "q"] & A \amalg B \arrow[dr, "q"] & \\
& & C \arrow[uu, "p"] \arrow[d, "p"] \\
& & B
\end{tikzcd}$$

We have that

$$B \overset{i_B}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$

$$A \overset{i_A}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$

are both trivial cofibrations, hence their images under F are weak equivalences. We see that

$$F(p) \circ F(q \circ \mathrm{id}_B) = F(p \circ q \circ \mathrm{id}_B) = F(\mathrm{id}_B).$$

Therefore $F(p)$ is a weak equivalence by 2-out-of-3. $\square$

Theorem 4.7. Suppose that $F : \mathcal{M} \to \mathcal{M}$ is left Quillen. Then $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ exists and can be defined as

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{Q} \mathrm{Ho}(\mathcal{M}_c) \xrightarrow{F} \mathrm{Ho}(\mathcal{N}).$$

Moreover, we obtain an adjunction on the homotopy categories:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \leftrightarrows \mathrm{Ho}(\mathcal{N}) : \mathbb{R}G.$$

Proof idea. We have a natural iso

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y),$$

compatible with homotopy equivalence:

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y))/\simeq \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y)/\simeq$$

□

Theorem/Definition: Take a Quillen adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$. Suppose that $f : X \xrightarrow{\sim} G(Y)$, with $X \in \mathcal{M}_c$ and $Y \in \mathcal{N}_f$ is a weak equivalence if and only if $f^\flat : F(X) \to Y$ is. Then $\mathbb{L}F$ and $\mathbb{R}G$ are equivalences of categories, we call this a *Quillen equivalence.*

Example 4.8. We have that

$$|-| : \mathtt{sSet}_{\mathrm{Kan}} \leftrightarrows \mathtt{Top}_{\mathrm{Quillen}} : \mathrm{Sing}(-)$$

is a Quillen equivalence.

Example 4.9. We have that

$$\mathrm{id} : \mathtt{Top}_{\mathrm{Quillen}} \leftrightarrows \mathtt{Top}_{\mathrm{Strøm}} : \mathrm{id}$$

is a Quillen adjunction but not a Quillen equivalence.

Q: If $\mathcal{M}$ and $\mathcal{N}$ are model categories such that there is an equivalence of categories $\mathrm{Ho}(\mathcal{M}) \cong \mathrm{Ho}(\mathcal{N})$, is this always coming from a Quillen equivalence?

A: No! Dugger–Shipley, 2009.

This indicates that Quillen equivalence is a good notion but it is not a *perfect* notion.

Guided example: chain complexes

Let's take $\mathtt{Ch}_{\mathbb{Z}}$ to be homologically graded unbounded chain complexes. There are three model structures of interest. We first start with the projective one:

$(\mathtt{Ch}_{\mathbb{Z}})_{\mathrm{projective}}$:

- weak equivalences are quasi-isomorphisms
- fibrations are levelwise epimorphisms
- cofibrations are levelwise monomorphisms such that the cokernel of each $f_n : X_n \to Y_n$ is free.

If $M \in \mathtt{Ab}$, we define $S^n(M)$ to be the chain complex $M[n]$ which is concentrated in M at degree n. If $M = \mathbb{Z}$, we call it S^n. We define $D^n(M)$ to be a chain complex

$$\cdots \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to \cdots$$

with two M's concentrated in degrees n and $n-1$. We call $D^n(\mathbb{Z}) =: D^n$.

Exercise 4.10. Show that fibrations are $\mathrm{RLP}(0 \to D^n)$ for all n. That is,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ D^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

We claim this lifts iff $X \to Y$ is a levelwise epimorphism. We have that $\mathrm{Hom}_{\mathtt{Ch}}(D^n, Y) \cong Y_n$, so we are just asking if every element in Y_n lifts to an element in X_n.

Exercise 4.11. Show that $\widetilde{\mathrm{Fib}} = \mathrm{RLP}\left(S^n \hookrightarrow D^{n+1}\right)$ for all n. Consider $\mathrm{Hom}_{\mathtt{Ch}}(S^n, Y)$. A map looks like

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & Y_n \arrow[r] & Y_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

That is, it picks out a class in Y_n which maps to zero under the differential. The data of a square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ D^n \arrow[r] & Y \end{tikzcd}$$

is the data of $(y, x) \in Y_n \oplus Z_{n-1}X$ so that $p(x) = dy$. Show that a lift exists if and only if p is a trivial fibration.

Other model structures.

$(\mathtt{Ch}_R)_{\text{injective}}$:

- W = quasi-isomorphisms
- Cof = fiberwise monomorphisms[6]
- Fib = fiberwise epimorphisms with fibrant kernel

[6]Here we roughly have that $\mathrm{Cof} = \mathrm{LLP}(D^n \to 0)$ and $\widetilde{\mathrm{Fib}} = \mathrm{LLP}(D^{n+1} \to S^n)$.

We get a Quillen equivalence

$$\mathrm{id} : (\mathtt{Ch}_R)_{\text{projective}} \leftrightarrows (\mathtt{Ch}_R)_{\text{injective}} : \mathrm{id}.$$

We also have have a third one which is *not* Quillen equivalent.

$(\mathtt{Ch}_R)_{\text{Hurewicz}}$:

- W = homotopy equivalences of chain complexes
- Cof = split levelwise monomorphisms
- Fib = split levelwise epimorphisms

We denote by $\mathscr{D}(R) = \mathrm{Ho}\left((\mathtt{Ch}_R)_{\text{proj}}\right)$ the *derived category* of a ring R.

We can also think about *connective* chain complexes (which are zero in negative degrees). We have an adjunction

$$\mathtt{Ch}_R \leftrightarrows \mathtt{Ch}_R^{\geq 0}.$$

This induces a model structure on $\mathtt{Ch}_R^{\geq 0}$ making it into a Quillen adjunction but not a Quillen equivalence. We denote by $\mathrm{Ho}(\mathtt{Ch}_R^{\geq 0}) = \mathscr{D}^{\geq 0}(R)$.

We get a model structure: $\left(\mathtt{Ch}_R^{\geq 0}\right)_{\text{proj}}$

- W = quasi-isomorphisms
- Fib = positive epimorphisms (may not be epi in degree 0)
- Cof = monomorphisms with projective cokernel. The cofibrant objects here are levelwise projective R-modules.

If we take $M \in \mathtt{Mod}_R$, we can view $S^0(M) \in \mathtt{Ch}_R^{\geq 0}$, and take a cofibrant replacement of it $P \overset{\sim}{\twoheadrightarrow} S^0(M)$. This is *exactly* a projective resolution of M!

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Example 4.12. Let $M \in \mathtt{Mod}_R$. Then we can take

$$S^0(M) \otimes_R - : \mathtt{Ch}_R^{\geq 0} \to \mathtt{Ch}_R^{\geq 0}.$$

We can check that this is left Quillen. We can look at its total left derived functor $S^0(M) \otimes_R^{\mathbb{L}} -$. We can see that

$$M \otimes_R^{\mathbb{L}} N := S^0(M) \otimes_R^{\mathbb{L}} S^0(N) \simeq S^0(M) \otimes_R P_\bullet,$$

where $P_\bullet$ is a projective resolution of N. We have that

$$H_i(M \otimes^{\mathbb{L}}_R N) = \mathrm{Tor}_i^R(M, N).$$

Exercise 4.13. In the same way, if we want to derive hom, we can check that

$$\mathrm{Hom}_{\mathscr{D}^{\geq 0}(R)}(S^m(M), S^n(N)) \cong \mathrm{Ext}_R^{n-m}(M, N).$$

Via Dold-Kan, we have a Quillen adjunction

$$R[-] : \mathtt{sSet}_{\mathrm{Kan}} \leftrightarrows \mathtt{sMod}_R : U,$$

with the model structure on $\mathtt{sMod}_R$ given by weak homotopy equivalences as underlying simplicial sets, and fibrations as underlying Kan fibrations.

Then Dold-Kan takes the form of a Quillen equivalence

$$N : (\mathtt{sMod}_R)_{\mathrm{Kan}} \leftrightarrows (\mathtt{Ch}_R^{\geq 0})_{\mathrm{proj}} : \Gamma.$$

In general $N(X \otimes_R Y) \ncong N(X) \otimes_R N(Y)$, however $N(X \otimes Y) \cong N(X) \otimes_R N(Y)$. They both describe $\mathscr{D}^{\geq 0}(R)$ in a monoidal way.

5. Lecture 5: Thursday, January 26th

For Dold–Kan $\mathtt{Ch}_{\geq 0} \cong \mathtt{sMod}_R$, we have

$$M \otimes N \leftrightarrows M \otimes R \otimes N \leftrightarrows M \otimes R^{\otimes 2} N \cdots$$

we denote this by $B_\bullet(M, R, N)$ and call it the *bar construction*.

Homotopy colimits

Motivation: Limits and colimits are not invariant under (weak) homotopy equivalence.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & CX \arrow[d] \\ CX \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & * \end{tikzcd}$$

However $\Sigma X \not\simeq *$.

Let $\mathcal{M}$ be a model category, and $\mathscr{C}$ a small category. Then we denote by $\mathrm{Fun}(\mathscr{C}, \mathcal{M}) = \mathcal{M}^{\mathscr{C}}$. Let $\mathscr{C}_0 \subseteq \mathscr{C}$ be the discrete subcategory spanned by $\mathrm{ob}(\mathscr{C})$. Let $\mathcal{M}^{\mathscr{C}_0} = \prod_{\mathscr{C}_0} \mathcal{M}$. This has a model structure where W, Fib, and Cof are determined objectwise.

Consider $\iota : \mathscr{C}_0 \hookrightarrow \mathscr{C}$. This induces a map

$$\begin{aligned} \iota^* : \mathcal{M}^{\mathscr{C}} &\to \mathcal{M}^{\mathscr{C}_0} \\ F &\mapsto F|_{\mathscr{C}_0}. \end{aligned}$$

This admits adjoints:

$$\iota_! \dashv i^* \dashv i_*.$$

We have that ι^* creates W and Fib.

We have $\left(\mathcal{M}^{\mathscr{C}}\right)_{\text{proj}}$:

- W = objectwise weak equivalence
- Fib = objectwise fib
- Cof = ? induced by $\iota_!$Cof

We have that $\mathcal{M}$ is cocomplete, so we get a tensoring

$$\begin{aligned} \mathcal{M} \times \mathtt{Set}^{\mathscr{C}} &\to \mathcal{M}^{\mathscr{C}} \\ (X, F) &\mapsto X \otimes F = \amalg_{F(-)} X. \end{aligned}$$

We have $(X \times F)(c) = \amalg_{F(c)} X$.

There are representable functors

$$\begin{aligned} \mathscr{C}(c, -) : \mathscr{C} &\to \mathtt{Set} \\ d &\mapsto \mathscr{C}(c, d). \end{aligned}$$

By Yoneda, there is a natural iso

$$\mathtt{Set}^{\mathscr{C}}(\mathscr{C}(c, -), F) \cong F(c).$$

Tensoring with a representable functor gives

$$X \otimes \mathscr{C}(c, -) = \amalg_{\mathscr{C}(c,-)} X.$$

This is the *free diagram of X generated at c.*

This gives an adjunction

$$- \otimes \mathscr{C}(c, -) : \mathcal{M} \leftrightarrows \mathcal{M}^{\mathscr{C}} : \mathrm{ev}_c.$$

In this case

$$\iota_!(F) = \amalg_c \amalg_{\mathscr{C}(c,-)} F(c),$$

which is the free diagram in $\mathcal{M}$ generated by F. Evaluating at d gives

$$\iota_!(F)(d) = \amalg_{c \in \mathscr{C}} \amalg_{\mathscr{C}(c,d)} F(c).$$

This is the functor $\iota_! : \mathcal{M}^{\mathscr{C}_0} \to \mathcal{M}^{\mathscr{C}}$. We see that $\iota_! X$ is a left Kan extension

$$\begin{tikzcd}
\mathscr{C}_0 \arrow[r, "X"] \arrow[d, hook, "\iota"'] & \mathcal{M} \\
\mathscr{C} \arrow[ur, dashed] &
\end{tikzcd}$$

There is a diagonal functor

$$\begin{aligned}
\mathcal{M} &\xrightarrow{\Delta} \mathcal{M}^{\mathscr{C}} \\
C &\mapsto \text{constant functor at } X.
\end{aligned}$$

This admits adjoints

$$\operatorname{colim} \dashv \Delta \dashv \lim .$$

Proposition 5.1. The adjunction

$$\operatorname{colim} : \left(\mathcal{M}^{\mathscr{C}}\right)_{\text{proj}} \leftrightarrows \mathcal{M} : \Delta$$

is Quillen.

We denote hocolim $:= \mathbb{L}$colim. There is a map $\operatorname{hocolim}(-) \to \operatorname{colim}(-)$, and

$$\operatorname{hocolim}(F) \simeq \operatorname{colim}(QF).$$

Here QF denotes a cofibrant replacement in $\left(\mathcal{M}^{\mathscr{C}}\right)_{\text{proj}}$. For a general $\mathscr{C}$, QF is very difficult to determine.

Consider $\mathscr{C} = a \leftarrow b \to c$, and let $X \in \mathcal{M}^{\mathscr{C}_0}$. Then $\iota_! X$ is equal to

$$\begin{tikzcd}
X(b) \arrow[r] \arrow[d] & X(b) \amalg X(c) \\
X(a) \amalg X(b) &
\end{tikzcd}$$

Cofibrant objects in $\mathcal{M}^{\mathscr{C}}$ are of the form

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook] & Z \\
Y &
\end{tikzcd}$$

with X cofibrant. Here cofibrant replacement is easy. We start with $Y \xleftarrow{f} X \xrightarrow{g} Z$, and we replace X with $\widetilde{X} \xrightarrow{\sim} X$ to get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

If we cofibrantly replace $\widetilde{X} \to Z$, and similarly for Y, we get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Z} \\ \widetilde{Y} & \end{tikzcd}$$

The maps we used to fibrantly replace induces a fiberwise weak equivalence between this diagram and the one we started out with.

In $(\texttt{Top})_{\text{Quillen}}$, we can take $\operatorname{hocolim}(* \leftarrow X \to *)$. We cofibrantly replace X if necessary, and replace $X \to *$ by $X \hookrightarrow CX$, which is a cofibration. In this case we see that

$$\operatorname{hocolim}(* \leftarrow X \to *) \simeq \operatorname{colim}(C\widetilde{X} \leftarrow \widetilde{X} \to C\widetilde{X}) = \Sigma\widetilde{X}.$$

More generally, $\operatorname{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z)$ is the double mapping cylinder $M(f,g)$.

Theorem 5.2. If $\mathcal{M}$ is a *left proper model category* then

$$\operatorname{hocolim}(Y \hookleftarrow X \to Z) \cong \operatorname{colim}(Y \hookleftarrow X \to Z).$$

Proof. In the easy case, X is cofibrant, so we can factor the map to Z to get

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & \widetilde{Z} \arrow[r, two heads, "\sim"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & H \arrow[r, dashed] & P. \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

The entire rectangle is a pushout, so $Z \to P$ is a cofibration, and the right square is a pushout by the pasting law, so $H \to P$ is a weak equivalence. $\square$

Example 5.3. Let $\mathscr{C} = * \to * \to \cdots$. Show that $X_0 \to X_1 \to \cdots$ is cofibrant in $\mathcal{M}^{\mathscr{C}}$ if and only if X_0 is cofibrant and $X_i \hookrightarrow X_{i+1}$ is a cofibration for each i.

There is a third model structure on $\mathcal{M}^{\mathscr{C}}$ called the *Reedy model structure* (need $\mathscr{C}$ to be a Reedy cat). In this case, $\operatorname{hocolim}_{\Delta^{\text{op}}}(X_\bullet) \cong \left|Q^{\text{Reedy}} X_\bullet\right|$, for X a simplicial object in $\mathcal{M}$.

Bar construction: Let $\mathcal{M}$ a model cat, $\mathscr{C}$ a small cat, $F : \mathscr{C}^{\mathrm{op}} \to \mathcal{M}$, and $G : \mathscr{C} \to \mathcal{M}$. Then we define

$$B_\bullet(F, \mathscr{C}, G) := \amalg_{c_0 \in \mathscr{C}} F(c_0) \times G(c_0) \leftleftarrows \amalg_{c_0 \leftarrow c_1} F(c_0) \times G(c_1) \leftleftarrows \cdots$$

Example 5.4. If $F = * = G$, then

$$B_\bullet(*, \mathscr{C}, *) \cong N_\bullet(\mathscr{C}^{\mathrm{op}}).$$

Pièce de résistance:

Theorem 5.5. (Bousfield–Kan) If $F : \mathscr{C} \to \mathcal{M}$ is a functor, then

$$\mathrm{hocolim}_{\mathscr{C}}(F) \simeq |B_\bullet(*, \mathscr{C}, F)|.$$

6. Lecture 6: Tuesday, January 31st

Combinatorial model categories

Definition 6.1. A model category is *combinatorial* if it is *presentable*[7] and *cofibrantly generated*.

To motivate presentability, let X be a set. Then X is determined by its elements, meaning that

$$\mathrm{Hom}_{\mathtt{Set}}(*, X) \cong X.$$

Then we can present X as $X = \cup_{x \in X} \{*\}$.

Definition 6.2. A colimit is *filtered* if the diagram is filtered, meaning it is nonempty and every subdiagram has a cocone.

Theorem 6.3. (Exercise) In $\mathtt{Set}$, filtered colimits commute with finite limits. That is, if $F : I \times J \to \mathtt{Set}$ with I finite and J filtered, then

$$\mathrm{colim}_J \left(\lim_I F_I \right) \xrightarrow{\sim} \lim_I \left(\mathrm{colim}_J F_J \right)$$

is an isomorphism.

Proposition 6.4. A set X is finite if and only if

$$\mathrm{Hom}_{\mathtt{Set}}(X, -) : \mathtt{Set} \to \mathtt{Set}$$

preserves filtered colimits.

[7]By this we mean "locally presentable."

Proof. For the backwards direction, let $I = \{X_i\}$ be the collection of finite subsets of X. Then $X = \mathrm{colim}_I X_i$. In particular, we have that

$$\mathrm{colim}_I \mathrm{Hom}(X, X_i) \cong \mathrm{Hom}(X, X)$$
$$(X \xrightarrow{f_i} X_i) \xrightarrow{\sim} \mathrm{id}_X?$$

For the forwards direction, $\mathrm{Hom}_{\mathtt{Set}}(*, -) \cong \mathrm{id}_{\mathtt{Set}}$ so it preserves colimits. Since X is finite, we have that $X = \{x_1, \ldots, x_n\}$, hence

$$\mathrm{Hom}(X, -) \cong \mathrm{Hom}(\cup_i \{x_i\}, -) \cong \lim_i \mathrm{Hom}(\{x_i\}, -).$$

Then we use finite limits commuting with filtered colimits. □

Definition 6.5. An object $X \in \mathscr{C}$ is *compact* if $\mathrm{Hom}_{\mathscr{C}}(X, -) : \mathscr{C} \to \mathtt{Set}$ preserves filtered colimits.

Hence if $F : I \to \mathscr{C}$, with I filtered, then a map $X \to \mathrm{colim}_I F$ factors through an $F(i)$.

Examples 6.6. Compact objects:

- $\mathtt{Set}$, compact = finite set
- $\mathtt{Vect}_F$, compact = finite dimensional
- $\mathtt{Mod}_R$, compact = finitely presented
- $\mathtt{Grp}$, compact = finitely presented
- $\mathtt{Top}$, compact = finite sets with discrete topology
- $\mathtt{Ch}$, compact = perfect chain complexes (bounded, levelwise finitely generated and projective)
- $\mathtt{sSet}$, compact = finite simplicial sets (X_n finite for each n, and there exists an m so that all non-degenerate simplices have dimension $\leq m$).

A topological space is (topologically) compact if and only if $X \in \mathcal{O}(X)$ is (categorically) compact.

Lemma 6.7. Finite colimits of compact objects are compact.

Definition 6.8. A category $\mathscr{C}$ is *presentable* if

(1) $\mathscr{C}$ is cocomplete
(2) There exists a set S of compact objects in $\mathscr{C}$ such that every object in $\mathscr{C}$ is a filtered colimit of objects in S.

We also say the "ind-completion" of S is $\mathscr{C}$, denoted $\mathrm{Ind}(S) = \mathscr{C}$.

Theorem 6.9. $\mathscr{C}$ is presentable if and only if there is an adjunction of the form

$$\mathrm{Fun}(K^{\mathrm{op}}, \mathtt{Set}) \leftrightarrows \mathscr{C},$$

where K is some small category, and the right adjoint is fully faithful and preserves filtered colimits.

We might take K for example to to be isomorphism classes of compact objects in $\mathscr{C}$, then we have

$$\begin{aligned} &\mathscr{C} \to \mathrm{Fun}(K^{\mathrm{op}}, \mathtt{Set}) \\ &X \mapsto \left(K^{\mathrm{op}} \to \mathscr{C}\mathrm{op} \xrightarrow{\mathrm{Hom}(-,X)} \mathtt{Set}\right). \end{aligned}$$

Theorem 6.10. Suppose $\mathscr{C}$ and $\mathscr{D}$ presentable. Then $L : \mathscr{C} \to \mathscr{D}$ preserves colimits if and only if L is a left adjoint.

Cofibrantly generated model categories

Definition 6.11. Let I be a set of maps in a cocomplete category, fix λ to be an ordinal, and let $X : \lambda \to \mathscr{C}$ a functor, and suppose that $X(\alpha) \to X(\alpha+1)$ fits into

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & X(\alpha) \arrow[d] \\ B_\alpha \arrow[r] & X(\alpha+1), \end{tikzcd}$$

where $A_\alpha \to B_\alpha$ is in I. Then we say that $X(0) \to \mathrm{colim}_\lambda X$ is a *relative I-cell complex*. We say an object $Y \in \mathscr{C}$ is an *I-cell complex* if $\varnothing \to Y$ is a relative I-cell complex.

If $I = \{S^n \hookrightarrow D^{n+1}\}_{n\geq 0}$, then we are recovering the idea of CW complexes in spaces.

We denote by $\mathrm{Cell}_I(\mathscr{C})$ the class of relative I-cell complexes.

Exercise 6.12. We have that $\mathrm{Cell}_I(\mathscr{C})$ is the smallest class in $\mathscr{C}$ closed under composition, pushouts, and filtered colimits.

Theorem 6.13. *(Small object argument)* Let $\mathscr{C}$ be cocomplete, let I a set of maps in $\mathscr{C}$, and suppose that for all $A \to B$ in I, we have that A is compact with respect to the full subcategory of of I-cells in $\mathscr{C}$. Then there exists a functorial factorization of maps in $\mathscr{C}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & Y \\ & C \arrow[ur, "\delta"'] & \end{tikzcd}$$

with $\gamma \in \mathrm{Cell}_I(\mathscr{C})$ and $\delta \in \mathrm{RLP}(I)$.

Proof idea. Start with $X(0) = X$, and take a map $X(0) \to Y$. Suppose $X(\beta) = \mathrm{colim}_{\alpha<\beta} X(\alpha)$ is constructed with $X(\beta) \to Y$. Look at the set[8]

$$S = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X(\beta) \arrow[d] \\ B \arrow[r] & Y \end{tikzcd} : g \in I \right\}.$$

Denote by g_s the map $A \to B$ appearing in $s \in S$. Then we build

$$\begin{tikzcd} \amalg_{s\in S} A_s \arrow[r] \arrow[d, "\amalg_s g_s"'] & X(\beta) \arrow[d, "\in \mathrm{Cell}_I(\mathscr{C})"] \\ \amalg_{s\in S} B_s \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X(\beta+1) \end{tikzcd}$$

By UP of the pushout, there is an induced map $X(\beta+1) \to Y$. Then we claim that

$$X(0) \to \mathrm{colim}_\beta X(\beta) =: C$$

is in $\mathrm{Cell}_I(\mathscr{C})$. The only thing left to show is that $C \to Y$ is in $\mathrm{RLP}(I)$. Take

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C = \mathrm{colim}_\beta X(\beta) \arrow[d] \\ B \arrow[r] & Y. \end{tikzcd}$$

Since A is compact with respect to I-cells, the map $A \to C$ factors through some $X(\beta)$. Since $B \to Y$ factors through $X(\beta+1)$, we see that it lifts to $B \to C$. $\square$

Definition 6.14. A model category $\mathcal{M}$ is *cofibrantly generated* if there exist sets of maps I, J in $\mathcal{M}$ so that

- Cof = retracts of I-cell complexes, denoted $\widehat{\mathrm{Cell}_I(\mathscr{C})}$[9]
- Cof = $\widehat{\mathrm{Cell}_J(\mathscr{C})}$

[8]Note this set is nonempty because we can take g to be $\mathrm{id} : X(\beta) \to X(\beta)$.

[9]The hat $\widehat{-}$ means "retracts of -"

and "I and J permit the small object argument."

Example 6.15. For $\texttt{Top}_{\text{Quillen}}$, we can take

$$I = \left\{ S^n \hookrightarrow D^{n+1} \right\}$$
$$J = \left\{ D^n \to D^n \times [0,1] \right\}.$$

Example 6.16. For $\texttt{sSet}_{\text{Kan}}$, we can take

$$I = \left\{ \partial\Delta^n \to \Delta^n \right\}$$
$$J = \left\{ \Lambda^k_n \to \Delta^n \right\}.$$

Example 6.17. For $(\texttt{Ch}_R)_{\text{proj}}$,

$$I = \left\{ S^n \to D^{n+1} \right\}$$
$$J = \left\{ 0 \to D^n \right\}.$$

Example 6.18. The Strøm model structure is not cofibrantly generated in the definition above.

Theorem 6.19. (Kan — Right transfer) Let $\mathcal{M}$ be a cofibrantly generated model category and $\mathscr{C}$ is any category where there is an adjunction

$$F : \mathcal{M} \leftrightarrows \mathscr{C} : G.$$

Then $\mathscr{C}$ has a model structure where W and Fib are created by G. The model structure is cofibrantly generated by $F(I)$ and $F(J)$ if:

(1) $F(I)$ and $F(J)$ permit the small object argument
(2) $G\left(\mathrm{Cell}_{F(J)}\right)$ are weak equivalences in $\mathcal{M}$.

For combinatorial model categories, we get an inductive argument for building cofibrant replacements.

[Rezk-Schwede-Shipley] Combinatorial model categories are always simplicially enriched.

[Dugger] Any combinatorial model category $\mathcal{M}$ is Quillen equivalent to a localization of a projective Kan one:

$$L_\tau \mathrm{Fun}(K^{\mathrm{op}}, \texttt{sSet}) \leftrightarrows \mathcal{M}.$$

References