THE TOPOLOGY OF ALGEBRAIC MANIFOLDS

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Abstract. Course notes from MATH231b: algebraic topology II, taught at Harvard in Spring 2026.

https://github.com/tbrazel/math231br

0. About

These notes are from a MATH213BR, a second-semester graduate algebraic topology class with a free-floating curriculum taught at Harvard in spring of 2026. This particular course is modeled off of Hirzebruch's amazing 1962 text *Neue topologische Methoden in der algebraischen Geometrie*. Theorem counters and page references refer to the 1995 reprinting in English [Hir78].

0.1. **Overview. Part 1**: We begin with the theory of sheaves of sets and discrete groups on topological spaces, developing their cohomology and comparing it with singular and de Rham cohomology. We continue with fiber bundles and develop their basic theory. We then discuss vector bundle theory and Chern and Pontryagin classes.

Part 2: We discuss oriented and complex cobordism, genera, the L-genus and Todd genus, and the index of a $4n$-manifold. We prove Hirzebruch's signature theorem as well as Riemann-Roch for algebraic manifolds.

Contents

0.2. **References.** Aside from the main text, which we deviate from at various points, some other references which have informed the presentation here include:

- ▷ *sheaf theory*: Godement's original work [God58], and of course Serre's [Ser55]
- ▷ *fiber bundles*: Steenrod's 1951 printing [Ste51]
- ▷ *Kähler manifolds*: Michael Wong's 2013 course notes
- ▷ *cobordism*: Dan Freed's notes [Fre12] and Haynes Miller's notes [Mil]
- ▷ *genera*: Hirzebruch's book [HBJ94]
- ▷ *complex manifolds*: Griffiths and Harris [GH78]

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1. Sheaves

The definition of a *sheaf* in Hirzebruch is perhaps a bit different from what we're used to. He wants to think of sheaves as *living over* a space X, whereas we might be comfortable thinking about sheaves as local data across a space. We'll see how these perspectives are equivalent.

Definition 1.1. A *sheaf of abelian groups* over a topological space X is a topological space S with the data of a surjective continuous map $\pi\colon S \to X$, for which

- ▷ π is a *local homeomorphism*, meaning for every point $s \in S$ there is an open neighborhood $U \ni s$ for which
$$\pi_{|U}\colon U \to \pi(U)$$
is a homeomorphism
- ▷ for every $x \in X$, the *stalk* $S_x := \pi^{-1}(x)$ has the structure of an *abelian group*
- ▷ the abelian group structure on S_x is *continuous in* x. Precisely, forming inverses in each fiber defines a continuous function:
$$\begin{aligned} S &\to S \\ s &\mapsto -s, \end{aligned}$$
and if we denote by $S \times_X S$ the subspace
$$S \times_X S := \{(s_1, s_2) \in S \times S \colon \pi(s_1) = \pi(s_2)\} \subseteq S \times S,$$
then the operation of addition is a continuous function:
$$\begin{aligned} S \times_X S &\to S \\ ((\alpha, x), (\beta, x)) &\mapsto (\alpha + \beta, x). \end{aligned}$$

We should think about a sheaf of abelian groups as being a space where we can do abelian group operations (addition, subtraction), but where the abelian group in which we're working is *changing* according to the topology of X.

Example 1.2 (Zero sheaf)**.** The identity map $X \to X$ can be considered a sheaf, where the abelian group structure on each stalk is just the trivial group with one element.

Example 1.3 (Constant sheaves)**.** For any abelian group A and any space X, we get a *constant sheaf*
$$X \times A \to X,$$
where π is the projection onto X. When $A = \{0\}$, this recovers the zero sheaf above.

Example 1.4 (Skyscraper sheaves)**.** If we want build a sheaf with some *prescribed* fiber A over a point $x \in X$ (we want a name for the map of the inclusion of a point, so let's call it $i\colon \{x\} \to X$), we have an nice way to do this – namely, we can take the discrete set underlying the abelian group A, and build the space which we call i_*A, defined as
$$i_*A = \frac{X \times A}{(y, a_1) \sim (y, a_2) \text{ for } y \neq x}.$$
That is, we identify all the points of A together over every point in X *except* our specified point x. We call it a *skyscraper sheaf* because it has this big tall stalk at x and is flat everywhere else. This space comes equipped with a projection back to X which is a continuous local homeomorphism,

and the abelian group structure on the stalks over points other than x are just the structure on the one-element group. Note that this resulting space i_*A is very much not Hausdorff.

Example 1.5 (c.f. [Bre97, 1.3])**.** The *line with doubled origin* is the skyscraper sheaf over $\mathbb{R}$ defined by $i_*(\mathbb{Z}/2)$, where $i\colon \{0\} \hookrightarrow \mathbb{R}$ is the inclusion of the origin.

Notation 1.6. For any sheaf of abelian groups $\pi\colon S \to X$, we have the so-called *zero section* $z\colon X \to S$, sending each element $x \in X$ to the zero element in S_x. This map is continuous and has the property that

$$\pi \circ z = \mathrm{id}_X.$$

Exercise 1.7. Let $\pi\colon X \to S$ be any sheaf and $z\colon X \to S$ the zero section. Then $\mathrm{im}(z) \subseteq S$ is open.

Example 1.8. The projection $\mathbb{R} \to S^1$ cannot be given the structure of a sheaf of abelian groups.

Proof. We're tempted to say it looks like a sheaf, since each fiber looks like $\mathbb{Z}$, however it turns out there is no way to endow these fibers simultaneously with the additive structure of the integers. Suppose towards a contradiction there was. Then composing with the zero section, we get a composite which is the identity:

$$S^1 \xrightarrow{z} \mathbb{R} \xrightarrow{\pi} S^1.$$

Applying π_1 or H_1 for instance, we have that the identity on $\mathbb{Z}$ factors through zero, which is a contradiction. $\square$

Remark 1.9. Just as we have a sheaf of abelian groups, we can have a sheaf with other structures as well — we could have a sheaf of R-modules for instance. All that's important is that:

(1) our structure is a set with extra data
(2) we have that data in every stalk
(3) the data varies continuously in the stalks

Definition 1.10. A *morphism of sheaves* between $\pi\colon S \to X$ and $\widetilde{\pi}\colon \widetilde{S} \to X$ is a continuous map $f\colon S \to \widetilde{S}$ satisfying the following properties:

- $\widetilde{\pi} \circ f = \pi$, that is, we have a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \widetilde{S} \arrow[dl, "\widetilde{\pi}"] \\ & X & \end{tikzcd}$$

  Note this implies that f restricts to a map on each stalk, that is, we get maps $f_x\colon S_x \to \widetilde{S}_x$ for each $x \in X$
- for each $x \in X$, the induced map on stalks

$$f_x\colon S_x \to \widetilde{S}_x$$

  is an abelian group homomorphism.

Exercise 1.11. Show that every morphism between sheaves of abelian groups is a local homeomorphism.

Definition 1.12. We say a morphism of sheaves $f\colon S \to \widetilde{S}$ is:

(1) *injective* if each f_x is injective
(2) *surjective* if each f_x is surjective.

We say a sequence of maps of sheaves

$$S \xrightarrow{f} T \xrightarrow{g} U$$

is *short exact* if $S_x \xrightarrow{f_x} T_x \xrightarrow{g_x} U_x$ is a short exact sequence of abelian groups for every $x \in X$.

1.1. **Sheaves and presheaves.** Given any sheaf $S \to X$, we can ask when we have *sections*, which are maps $X \to S$ that are right inverses to the projection map π.

Definition 1.13. We define the set of *sections* by

$$\Gamma(X, S) := \{s \colon X \to S \text{ continuous} \mid \pi \circ s = \mathrm{id}_X\}.$$

Note that $\Gamma(X, S)$ is an abelian group by adding sections pointwise. The zero element of $\Gamma(X, S)$ is the zero section.

Note that we don't need sections to be defined on all of X — we could look at sections defined on subspaces. For instance if $U \subseteq X$ is an open subspace, we can define

$$\Gamma(U, S) := \{s \colon U \to S \mid \pi \circ s = \mathrm{id}_U\}.$$

If $V \subseteq U$ is an open subspace, there is a natural *restriction* map

$$\begin{aligned} \mathrm{res}^U_V \colon \Gamma(U, S) &\to \Gamma(V, S) \\ s &\mapsto s_{|V}. \end{aligned}$$

Notation 1.14. For X a topological space, we denote by $\mathrm{Open}(X)$ the category whose objects are open subspaces of X and whose morphisms are inclusions. Then the constructions above tell us that $\Gamma(-, S)$ defines a functor

$$\Gamma(-, S) \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}.$$

This is called the *presheaf* associated to S.

Terminology 1.15. If $S \to X$ is a sheaf, an element of $\Gamma(X, S)$ will be called a *global section*. This is to indicate it is defined globally, i.e. everywhere on X, in contrast with sections only defined locally on some open subspace U.

Let's think about what structures of S we can recover from its presheaf.

Question 1.16. Can we recover the stalk S_x from the presheaf $\Gamma(-, S)$?

We can't just take Γ and evaluate it on the one-point set $\{x\}$ since this won't be open in most reasonable topological spaces. Instead, we can take elements in $\Gamma(U, S)$ for any $U \ni x$, and glue sections together along restriction of open sets containing x. This is called a *colimit* (in classical terminology, a *direct limit*):

$$\mathrm{colim}_{U \ni s} \Gamma(U, S).$$

Proposition 1.17. For every sheaf of abelian groups $S \to X$ and every $x \in X$, the canonical "evaluation at x" map

$$\mathrm{ev}_x \colon \mathrm{colim}_{U \ni s} \Gamma(U, S) \to S_x$$

is an isomorphism of abelian groups.

Proof. If $s_1(x) = s_2(x) = a$, then since π is a local homeomorphism, there is some small neighborhood $V \ni a$ on which $\pi_{|V} \colon V \to \pi(V)$ is a local homeomorphism. In particular since $s_{1|\pi(V)}$ and $s_{2|\pi(V)}$ are both sections (right inverses) of $\pi_{|V}$ they must agree. Since they agree on a sufficiently small open neighborhood around x, they agree in the colimit. This establishes injectivity.

To see surjectivity, if $a \in S_x$, we want to create a section s defined in a neighborhood U of x for which $s(x) = a$. We again leverage the fact that π is a local homeomorphism to find some neighborhood $V \ni a$ for which $\pi \colon V \xrightarrow{\sim} \pi(V)$ is a homeomorphism. Then the inverse to this map satisfies $\pi^{-1}(x) = a$. ☐

This gives us an alternative way to look at stalks. Now we know we can recover stalks from the underlying presheaf.

Question 1.18. Can we recover the sheaf S from the presheaf $\Gamma(-, S)$?

That is, suppose someone hands you a functor

$$F \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab},$$

and tells you it is of the form $\Gamma(-, S)$ for some sheaf S, but they don't tell you what S is. Can you rebuild S? Let's denote by

$$F_x := \mathrm{colim}_{U \ni x} F(U)$$

the stalk at a point $x \in X$. Then we know how to construct the sheaf as a *set*, namely it is

$$\amalg_{x \in X} F_x \to X.$$

The question then becomes how to topologize the thing on the left. Each F_x has the discrete topology as an abelian group, and the disjoint union has the discrete topology a priori – but if we accept this topology, then in general we won't have the property that the projection map is a local homeomorphism. So we need to hunt for a different topological structure. The trick to finding it is that we know how to write down sections, and we want to *force sections to be continuous.*

We denote by

$$\mathrm{germ}_x \colon F(U) \to F_x$$

the structure map (part of the data of the colimit) to the stalk (c.f. Proposition 1.17; we called this an evaluation map earlier when we knew $F(U)$ was comprised of actual sections. We don't know that a priori now). For any $s \in F(U)$, we then get a composite

$$\begin{aligned} U &\to \amalg_x F_x \\ x &\mapsto \mathrm{germ}_x(s). \end{aligned}$$

This should be like a section of the sheaf, so we want it to be continuous. We want the preimage of opens to be open, and since U is indeed open, an easy way to try to force continuity is to ask for the image of this map above to be open.

Definition 1.19. We define a *basis*[1] for a topology on $\amalg_x F_x$ by

$$[(s, U)] := \{\mathrm{germ}_x(s) \mid x \in U\}$$

for any $s \in F(U)$.

It now suffices to verify that this satisfies the axioms for a basis:

B1 *The basis elements cover our space*: Take some $x \in X$ and $a \in F_x$. We want to argue there is a basis element containing it. By the construction of F_x as a colimit of abelian groups, there is some $V \in \mathrm{Open}(X)$ and some $s \in F(V)$ for which $\mathrm{germ}_x(s) = a$. That is, by definition, $a \in [(s, V)]$.

[1]See [Mun00, II§13] if this is an unfamiliar term.

B2 *Common refinement*: Suppose we have $[(s_1, U_1)]$ and $[(s_2, U_2)]$ and $a \in [(s_1, U_1)] \cap [(s_2, U_2)]$. Then we want to find another basis element contained in the intersection of these two and containing a. Let $x = \pi(a)$, then $x \in U_1 \cap U_2$ by definition. So we can look in $F(U_1 \cap U_2)$. By the construction of the colimit, since we have a commutative diagram

$$\begin{tikzcd}
F(U_1) \arrow[dr] \arrow[drr] & & \\
& F(U_1 \cap U_2) \arrow[r] & F_x \\
F(U_2) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

since $\mathrm{germ}_x(s_1) = \mathrm{germ}_x(s_2)$, their images in $F(U_1 \cap U_2)$ agree. Call that image s. Then we have that

$$a \in [(s, U_1 \cap U_2)] \subseteq [(s_1, U_1)] \cap [(s_2, U_2)] \subseteq \amalg_{x \in X} F_x.$$

So we've checked that we get a basis for a topology.

Remark 1.20. In general, the sheaf $\amalg_{x \in X} F_x$ constructed in this way will be very far from being Hausdorff ([Bre97, p. 3], [Ser55]).

It turns out we have in fact built a sheaf! That is, the projection map will be a local homeomorphism.

Lemma 1.21. Let $F \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$ be a presheaf, and topologize $\amalg_{x \in X} F_x$ by giving it the basis of Definition 1.19. Then the natural projection

$$\pi \colon \amalg_{x \in X} F_x \to X$$

is a local homeomorphism.

Proof. Pick any $a \in F_x$. Then by construction, there exists some open set U and some $s \in F(U)$ so that $\mathrm{germ}_x(s) = a$. In particular, we claim the map

$$\pi_{|[(s,U)]} \colon [(s, U)] \to U$$

is a local homeomorphism. It is clearly continuous, and it admits a continuous inverse given by $s \colon U \to F(U) \to [(s, U)]$ by construction. $\square$

We now have a way to build a sheaf out of a presheaf! So we have a way to go from presheaves to sheaves. I think we've answered Question 1.18, which was if someone hands you a presheaf $F \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$, and tells you it is of the form $\Gamma(-, S)$ for some sheaf S, then we can reconstruct S.

Let's now give a slightly different question pandering to a more cynical worldview. Suppose someone hands you a presheaf $F \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$ and tells you it is the presheaf of sections attached to some sheaf, but we don't believe them. How might we argue that this cannot come from any sheaf?

Here's an idea of how we might find a way to disagree:

- ▷ if F was of the form $\Gamma(-, S)$ for some S, then $F(U)$ would be equal to $\Gamma(U, S)$
- ▷ we now know how we would reconstruct S: it is $\amalg_{x \in X} F_x$ equipped with the topology defined in Definition 1.19
- ▷ so we can check: is the canonical map $F(U) \xrightarrow{\theta_U} \Gamma(U, \amalg_x F_x)$ a bijection?

Let's recall how this map θ_U worked — each $s \in F(U)$ determines a function $U \to \amalg_{x\in U} F_x$ given by

$$\begin{aligned}\theta_U \colon F(U) &\to \Gamma(U, \amalg_x F_x)\\ s &\mapsto [x \mapsto \mathrm{germ}_x(s)]\,.\end{aligned} \tag{1.22}$$

If θ_U is not a bijection for some open U, then F cannot have come from a sheaf. So let's see when θ_U could fail to be a bijection - that is, when is it injective and when is it surjective? We phrase the following as lemmas, although there's no mathematical content, it's just unwinding definitions.

Lemma 1.23 (When is θ_U a monomorphism?)**.** Let U, θ_U as above. Let's first see what it means for θ_U to send two elements in $F(U)$ to the same thing in $\Gamma(U, \amalg_{x\in X} F_x)$.

(1) We have that $\theta_U(s) = \theta_U(t)$ if and only if s and t *agree locally* – that is, for every $x \in U$ there is some open $V \ni x$ for which $s_{|V} = t_{|V}$.

This tells us the content of being a monomorphism is that "being equal locally implies you are equal." In other words:

(2) θ_U is a monomorphism if and only if, for every $x \in X$, if there is a neighborhood $V \ni x$ with $s_{|V} = t_{|V} \in F(V)$, then $s = t$ in $F(U)$.

As the V's chosen this way form an open cover of U, we can reword this as:

(3) θ_U is a monomorphism if and only if, for every open cover $\{U_i\}$ of U, if $s, t \in F(U)$ satisfies $s_{|U_i} = t_{|U_i}$ for all i, then $s = t$.

Proof. For the first point, we have that $\theta_U(s) = \theta_U(t)$ if and only if the functions $u \mapsto \mathrm{germ}_u(s)$ and $u \mapsto \mathrm{germ}_u(t)$ agree, that is, only if $\mathrm{germ}_u(s) = \mathrm{germ}_u(t)$ in F_u for all $u \in U$. By construction of F_u as a colimit, we have that there must exist some neighborhood $V \ni u$ for which $s_{|V} = t_{|V}$ in $F(V)$. The second point is a rephrasing of the first, and (3) is a rephrasing of the second point. □

Lemma 1.24 (When is θ_U an epimorphism?)**.** Let U, θ_U be as above, and suppose Lemma 1.23(3) holds. We want to know when θ_U hits some element $a \in \Gamma(U, \amalg_{x\in X} F_x)$

(1) Let $a \in \Gamma(U, \amalg_{x\in X} F_x)$. Then for each $x \in U$, we have that $a(x) \in F_x$. By the construction of F_x as a colimit, this means there is some open neighborhood $U_x \ni x$, and some element $s_x \in F(U_x)$ so that, under the structure map $F(U_x) \to F_x$, we have that $s_x \mapsto a(x)$. Another way to say that is that the section $\theta_{U_x}(s_x)$ agrees with a at the point x. Since π is a local homeomorphism, we have that a and $\theta_{U_x}(s_x)$ must agree on some open neighborhood $V_x \ni x$ which, without loss of generality, we may assume to be a subspace of U_x. So we have that $\theta_{V_x}(s_x) = a_{|V_x}$ and $\theta_{V_y}(s_y) = a_{|V_y}$, so under restriction we ge

$$\theta_{V_x\cap V_y}(s_x) = a_{V_x\cap V_y} = \theta_{V_x\cap V_y}(s_y).$$

If we assumed that the θ's were injective, Lemma 1.23(3), this says that

$$s_{x|V_x\cap V_y} = s_{y|V_x\cap V_y}.$$

And we have that a is in the image of θ_U if and only if there is some $s \in F(U)$ so that $s_{|V_x} = s_x$ for each $x \in U$. In other words, θ_U is surjective if, any time locally defined sections agree on overlaps, there is a section defined on all of U restricting to them.

(2) With the hypotheses above, we have that θ_U is surjective if and only if, for every cover $\{U_i\}_{i\in I}$ of U, and elements $s_i \in F(U_i)$ satisfying $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$, there exists an element $s \in F(U)$ so that $s_{|U_i} = s_i$ for each i.

So what conditions were essential to showing that θ_U was a bijection for every U? It was Lemma 1.23(3), which tells us if sections agree *locally* then they agree *globally*, and Lemma 1.24(2), which told us that sections defined locally and agreeing on overlaps can be *glued* together to get a global section. These two conditions are called *locality* and *gluing*. Let's summarize our observations:

Theorem 1.25. A presheaf $F\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$ is the presheaf of sections attached to a sheaf if and only if the two conditions hold:

Sh1 *(Locality)* For any open set $U \subseteq X$ and any open cover $\{U_i\}_{i\in I}$ of U, if $s, t \in F(U)$ satisfy $s_{U_i} = t_{U_i}$ for each $i \in I$, then $s = t$. Phrased differently, the map

$$F(U) \to \prod F(U_i)$$

is injective.

Sh2 *(Gluing)* If $\{U_i\}_{i\in I}$ is an open cover of U, and we have elements $s_i \in F(U_i)$ for each i so that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ for all i, j, then there exists some $s \in F(U)$ so that $s_{|U_i} = s_i$ for each i. Phrased differently, $F(U)$ surjects onto the equalizer of the parallel restriction maps

$$\prod F(U_i) \rightrightarrows \prod F(U_i \cap U_j).$$

Remark 1.26. Let E denote the equalizer $E = \mathrm{eq}\left(\prod F(U_i) \rightrightarrows \prod F(U_i \cap U_j)\right)$. By functoriality, the map $F(U) \to \prod F(U_i)$ factors through E, and $E \to \prod F(U_i)$ is always a monomorphism. So for presheaves of abelian groups, locality reduces to the statement that $F(U) \to E$ is injective, while gluing restricts to the statement that $F(U) \to E$ is surjective. Hence altogether the sheaf condition can be compressed to the assertion that, for any $U \subseteq X$ and any open set $\{U_i\}$ of U, the sequence is an equalizer:

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \cap U_j).$$

Remark 1.27. In an abelian category, an equalizer of two maps $f, g\colon A \rightrightarrows B$ is the same as the kernel of their difference $A \xrightarrow{f-g} B$. So the locality and gluing conditions can be condensed into the statement that, for every open subspace $U \subseteq X$ and every open cover $\{U_i\}$ of U, the sequence

$$0 \to F(U) \to \prod F(U_i) \to \prod F(U_{ij})$$

is exact, where that rightmost map is the difference of the two restrictions. If we ever want to talk about sheaves of *sets* we can't do this, because subtracting two functions doesn't make sense unless they are group homomorphisms.

Corollary 1.28. There is an isomorphism of categories between

$$\{\text{sheaves on } X\} \simeq \left\{\begin{matrix}\text{the full subcategory of presheaves on } X \\ \text{satisfying locality and gluing}\end{matrix}\right\}.$$

Proof. We have shown a bijection on objects, so it suffices to verify that there is also a bijection on morphisms of each type.

Let S and S' be sheaves. Then a map of sheaves $f\colon S \to S'$ clearly induces a map of presheaves, since any section $U \to S$ can be postcomposed with f to get a section $U \to S'$, hence we get an induced map $\Gamma(U, S) \to \Gamma(U, S')$ which is easily seen to be compatible with restriction.

Conversely, suppose we have two presheaves $F, G \in \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Ab})$ which satisfy locality and gluing, and a morphism of presheaves $F \to G$. We want to see how this induces a morphism of sheaves $\amalg_x F_x \to \amalg_x G_x$. We first observe that, by taking colimits, there is a natural induced map on stalks $F_x \to G_x$ for every X, so we get a commutative diagram of *sets*

$$\begin{tikzcd} \amalg_{x\in X} F_x \arrow[rr, "f"] \arrow[dr] & & \amalg_{x\in X} G_x \arrow[dl] \\ & X. & \end{tikzcd}$$

We now want to check that this induced map f is actually continuous. Since we can check continuity on a basis of the target space (reference needed), we can take some $[(t, U)] \subseteq \amalg_{x\in X} G_x$ and try to see its preimage is open in $\amalg_{x\in X} F_x$. We can check that

$$f^{-1}([(t,U)]) = \{\mathrm{germ}_x(s) \mid x \in U \text{ and } \mathrm{germ}_x(f\circ s) = \mathrm{germ}_x(t)\}\,.$$

Pick one such point $\mathrm{germ}_x(s) \in f^{-1}([(t,U)])$. Since $\mathrm{germ}_x(f\circ s) = \mathrm{germ}_x(t)$, there is some open neighborhood $V \ni x$ for which $(f\circ s)_{|V} = t_{|V}$. This means every germ of s over V lies in the preimage of f, that is,

$$\mathrm{germ}_x(s) \in [(s,V)] \subseteq f^{-1}([(t,U)]).$$

In particular this implies that $f^{-1}([(t,U)])$ is open and therefore that f is continuous.

Verifying these two assignments are inverse is immediate by construction in one direction, and is the content of θ_U being bijective in the other direction direction. ☐

Remark 1.29 (Important)**.** What we have been calling sheaves (following Hirzebruch, Bredon, Serre) are now more commonly called the *espace étalé* of a sheaf, while the associated presheaf, satisfying locality and gluing, is what is known contemporarily as a sheaf. The equivalence of categories above tells us it doesn't quite matter which one we talk about, but it's important to keep in mind our definition is technically different than the modern one.

Remark 1.30. The big advantage now is that we can describe a sheaf *in terms of its sections*! We don't even need the sections to form a sheaf, we can always construct a sheaf out of a presheaf if need be.

Example 1.31 (Orientation sheaf)**.** Let M be an n-dimensional manifold, and define a sheaf by the presheaf

$$\begin{aligned} o_M\colon \mathrm{Open}(M)^{\mathrm{op}} &\to \mathrm{Ab} \\ U &\mapsto H_n(M, M-U;\mathbb{Z}). \end{aligned}$$

We call this the *orientation sheaf* of M (something to check: relative homology classes satisfy locality and gluing). Then an *orientation* of M is exactly a global section $\omega \in \Gamma(M, o_M)$ with the property that the stalk of ω at each $x \in M$ is a generator of the abelian group $o_{M,x} = H_n(M, M-\{x\};\mathbb{Z}) \cong \mathbb{Z}$. This is just a rephrasing of [Hat01, pp. 233–236].

Remark 1.32. The éspace étalé of the orientation sheaf is *not* the double cover of M, since this will not be a sheaf of abelian groups, only a sheaf of $\mathbb{Z}/2$-torsors. The éspace étalé of o_M will have fiber $\mathbb{Z}$ (if we are defining o_M by homology with integral coefficients) and will be the infinite-sheeted cover corresponding to the orientation character $\pi_1(M) \to \mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}/2$ given by the first Stiefel-Whitney class. We'll learn more about this later on.

1.2. **Digression: big and small sites.** To briefly tie what we've been doing into the language of Grothendieck topologies, we have the slice category $\mathrm{Top}_{/X}$ of spaces over X, which comes equipped with a Grothendieck topology given by open subsets. We could call $\mathrm{Top}_{/X}$ the *big site*, and in this terminology $\mathrm{Open}(X) \subseteq \mathrm{Top}_{/X}$ is the *small site*, which is the full subcategory on open subsets of X.

1.3. **Sheaves of functions.** The following is a space-level analogue of the idea that representable presheaves are sheaves.

Example 1.33 (Representable sheaves)**.** Let X and Y be any spaces. Then the assignment

$$\begin{aligned} C^0(-,Y)\colon \mathrm{Open}(X)^{\mathrm{op}} &\to \mathrm{Set} \\ U &\mapsto \mathrm{Hom}_{\mathrm{Top}}(U,Y) \end{aligned}$$

is a sheaf of sets. We call this the *representable sheaf* attached to Y, although this is maybe a slight abuse of terminology?[2]

Proof. We just have to verify that the presheaf of functions valued in Y satisfies locality and gluing, which is immediate. □

Question 1.34. When would this be a sheaf of groups or of abelian groups? That is, when we can endow $\mathrm{Hom}_{\mathrm{Top}}(U, Y)$ with a group structure compatibly for all U. One way to insist on this is to consider $\mathrm{Hom}_{\mathrm{Top}}(-, Y)$ by extending its domain to all of Top, and then ask for it to be valued in groups rather than sets. This is the same data (by the Yoneda lemma) as asking for Y to come equipped with continuous multiplication and inversion maps which turn Y into a group compatibly with its topology. This is the data of being a *topological group*, which we'll define in **??**.

Example 1.35. Let X be any space, and denote by

$$\mathbb{C}_{\mathfrak{c}} \colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$$

the sheaf sending $U \subseteq X$ to the abelian group of complex-valued continuous functions $U \to \mathbb{C}$. This is the "representable sheaf" attached to $\mathbb{C}$, in the sense of Example 1.33. We call $\mathbb{C}_{\mathfrak{c}}$ the *sheaf of germs of local complex-valued continuous functions*, following [Hir78, p. 23].

In general for spaces, we can take a sheaf of continuous functions valued in $\mathbb{R}$, in $\mathbb{C}$, in some other space, whatever we like. If X has more structure, we can ask for these functions to respect the additional structure!

Example 1.36. Let X be a differentiable manifold. We denote by $\mathbb{C}_{\mathfrak{b}}$ the sheaf of local complex-valued differentiable functions.[3]

Example 1.37. Let X be a complex manifold. We let $\mathbb{C}_{\omega}$ denote the sheaf of local holomorphic functions. We will prefer the notation $\mathcal{O}_X$ in this class.

Proposition 1.38. If X is any complex manifold and $p \in X$ is any point, there is an isomorphism between $\mathcal{O}_{X,p}$ and the ring of convergent power series in n variables.

Proof. An element of $\mathcal{O}_{X,p}$ is an equivalence class of tuple (f, U) where $U \ni p$ is open and f is holomorphic on U. By cofinality we can assume that (U, p) is homeomorphic $(B, 0)$ where B is some open ball around the origin in Euclidean space. In particular f can be identified with its Taylor series expansion on the open ball given by its radius of convergence. We note that $(f_1, U_1) \sim (f_2, U_2)$ if the Taylor series expansions of f_1 and f_2 agree in a neighborhood of p. □

Example 1.39. Similarly the stalks of the sheaf $\mathcal{M}_X$ of germs of meromorphic functions are given by the ring of convergent Laurent series with finitely many negative variables (finite principal parts) [For81, p. 42].

Exercise 1.40. If X is a compact connected complex manifold, show that $\Gamma(X, \mathbb{C}_{\omega}) = \mathbb{C}$.

1.4. **Sheafification.** Let $\mathrm{Ab}(X)$ denote the category of sheaves of abelian groups. It is clearly a full subcategory of $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Ab})$. Our goal is to show the following:

Theorem 1.41. The category $\mathrm{Ab}(X)$ is a localization of $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Ab})$, that is, the inclusion admits a left adjoint called *sheafification*. Moreover sheafification preserves stalks.

Our goal is to define the sheafification of a presheaf F over X. We claim we've already seen this construction, it's simply the space $\amalg_{x \in X} F_x \to X$ we built previously.

[2] Technically speaking, Y should be viewed as a representable sheaf on $\mathrm{Top} = \mathrm{Top}_{/*}$, which is the big site over the one-point space. We can pull this back along the projection $X \to *$ and we get a sheaf $\mathrm{Hom}_{\mathrm{Top}_{/X}}(-, X \times Y)$, which, when restricted to the small site $\mathrm{Open}(X)$, agrees with $\mathrm{Hom}_{\mathrm{Top}}(-, Y)$ by adjunction.

[3] Recall a complex-valued function is differentiable if and only if both its real and imaginary parts are.

Definition 1.42. Let $F\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$ be a presheaf. We will define the *sheafification* of F to be the sheaf $F^\sharp$ defined as $\amalg_{x\in X} F_x$, topologized as in Definition 1.19.

Remark 1.43. We can interpret $F^\sharp(U) = \Gamma(U, \amalg_{x\in X} F_x)$ as the set of those tuples $(f_x)_{x\in U} \in \prod_{x\in U} F_x$ for which, for any $x \in U$ there exists an open neighborhood $V \ni x$ and a section $s \in F(V)$ satisfying $s_v = f_v$ for all $v \in V$ (see e.g. [Aut, 007X]).

There is clearly a natural map of presheaves $F \to F^\sharp$, and we've already seen it! We called it θ_U in Equation (1.22). It induces an isomorphism on stalks by definition. The thing to check is that the sheafification construction is functorial (omitted) and defines a left adjoint to the inclusion of sheaves in presheaves. Adjunction follows from the following universal property:

Proposition 1.44. Let $F \in \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Ab})$ and $G \in \mathrm{Ab}(X)$. Then any map $F \to G$ factors uniquely through $F^\sharp$.

Proof. By construction there is a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F^\sharp \arrow[d] \\ G \arrow[r] & G^\sharp, \end{tikzcd}$$

so it suffices to check that $G \to G^\sharp$ is an equivalence of sheaves, but this is true because it is a stalkwise isomorphism.[4] □

Exercise 1.45 (Constant sheaf)**.** Let $A \in \mathrm{Ab}$. Then there is a silly presheaf on X which sends every open subset $U \subseteq X$ to A, and every restriction map to the identity on A. We call this the *constant presheaf* valued at A. We will denote by $\underline{A}$ the sheafification of this presheaf, and call this the *constant sheaf* valued at A. By construction.

(1) Verify that, under the isomorphism of categories Corollary 1.28, the constant sheaf $\underline{A}$ corresponds to the sheaf defined as $X \times A \to X$, where A has the discrete topology.
(2) Show that $\Gamma(U, \underline{A})$ can be described as locally constant functions valued in A.

1.5. **Subsheaves, kernels and cokernels.**

Notation 1.46. For a topological space X, we denote by $\mathrm{Ab}(X)$ the category of *sheaves of abelian groups* on X, and morphisms between them. Again following the isomorphism of categories of Corollary 1.28 we're allowed to work with local homeomorphisms of abelian groups over X or presheaves on X satisfying the sheaf condition. So every definition and statement we make here will have two equivalent formulations in each model of sheaves. We will provide both at the start but will begin dropping one for convenience as we continue.

Definition 1.47 (Subsheaf)**.** Let $F\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$ be a sheaf. We say that a sheaf G is a *subsheaf* of F if $G(U) \leq F(U)$ is a subgroup for each $U \in \mathrm{Open}(X)$, and for any $V \subseteq U$, the restriction map $\mathrm{res}^V_U\colon F(U) \to F(V)$ is equal to the restriction map $G(U) \to G(V)$ for G.

Definition 1.48 (Subsheaf, éspace étalé model)**.** We say that a sheaf $\pi_1\colon S_1 \to X$ is a *subsheaf* of $\pi_2\colon S_2 \to X$ if

(1) $S_1 \subseteq S_2$ is an open subspace
(2) $\pi_1 = \pi_{2|S_1}$
(3) The stalk $(S_1)_x$ is a subgroup of the stalk $(S_2)_x$ for all $x \in X$.

Example 1.49. Let X be a complex manifold.

[4]This wouldn't work if G was just a presheaf, as although $G \to G^\sharp$ is a stalkwise isomorphism that does not imply it is an isomorphism of presheaves, only of sheaves.

(1) We can think of subsheaves
$$\mathcal{O}_X \subseteq \mathbb{C}_{\mathfrak{d}} \subseteq \mathbb{C}_{\mathfrak{c}}$$
as the subsheaves of complex valued continuous functions of differentiable, and then holomorphic sections.
(2) Let $\mathcal{M}_X$ denote the sheaf of local meromorphic sections. Then $\mathcal{O}_X \subseteq \mathcal{M}_X$ is a subsheaf, since every holomorphic function is meromorphic.
(3) Let $\mathcal{O}_X^*$ denote the sheaf of germs of nowhere-vanishing holomorphic functions:
$$\mathcal{O}_X^*(U) := \{f : U \to \mathbb{C} \text{ holomorphic, and } f(p) \neq 0 \text{ for all } p \in U\}.$$
Note that, while we have a subset inclusion $\mathcal{O}_X^*(U) \subseteq \mathcal{O}_X(U)$ for every U, this is only a subsheaf of *sets*. It is not a subsheaf of groups because the group operation in $\mathcal{O}_X^*(U)$ is multiplicative, while in $\mathcal{O}_X(U)$ it is additive.

Example 1.50. For X a complex manifold and $p \in X$, we have that the stalk $\mathcal{O}_{X,p}^*$ can be identified with the multiplicative group of convergent power series around p whose constant term is nonzero (c.f. Proposition 1.38).

Just as subgroups of abelian groups are kernels of group homomorphisms, we can construct subsheaves as kernels of abelian sheaf homomorphisms.

Proposition 1.51. If $f : A \to B$ is a morphism of sheaves over X, we denote by $\ker(f)$ the *kernel*, defined to be the sheaf of A consisting of those points $a \in A$ so that $f_{\pi(a)}(a) = 0$ in $B_{\pi(a)}$.

Proof. We have to verify this is a subsheaf. Let K denote the kernel of f, and let's write out the diagram so we can picture it:
$$\begin{tikzcd} K \arrow[r] \arrow[dr, "\pi_K"'] & A \arrow[r, "f"] \arrow[d, "\pi_A"] & B \arrow[dl, "\pi_B"] \\ & X. & \end{tikzcd}$$
It is clear that π_K is just π_A restricted to K by construction, and moreover the stalks K_x are subgroups of A_x, again by construction. So the only thing to verify is that $K \subseteq A$ is an open subspace. To see this, we note that the kernel of f is the preimage of zero in B, which is the image of the zero section $z_B : X \to B$. This is open in B by Exercise 1.7, and $K = f^{-1}(\mathrm{im}(z_B))$, which will also be open since f is continuous. ☐

Exercise 1.52.
(1) Let $f : S_1 \to S_2$ be a morphism of sheaves over X. Give a reasonable definition of an *image* subsheaf $\mathrm{im}(f) \subseteq S_2$.
(2) Give a reasonable definition of a *cokernel* sheaf $\mathrm{coker}(f)$ over f.
(3) Prove an analogue of the first isomorphism theorem for sheaves of abelian groups over spaces.
(4) How does your first iso theorem look on stalks? (c.f. Definition 1.12)

Now that we have a notion of *subsheaf* and *cokernel*, we can build short exact sequences of sheaves out of subsheaves (recall the definition of exactness from Definition 1.12).

Example 1.53. Let $\mathbb{C}_{\mathfrak{c}}^*$ be the sheaf of germs of locally never zero complex valued continuous functions. There is a morphism of sheaves exp, defined on sections by
$$\begin{aligned} \mathbb{C}_{\mathfrak{c}}(U) &\to \mathbb{C}_{\mathfrak{c}}^*(U) \\ f &\mapsto \exp(2\pi i \cdot f). \end{aligned}$$
The kernel of this is isomorphic to the constant sheaf $\mathbb{Z}$, because f and $f+n$ map to the same thing for any $n \in \mathbb{Z}$. We claim that we get a short exact sequence of sheaves
$$0 \to \underline{\mathbb{Z}} \to \mathbb{C}_{\mathfrak{c}} \xrightarrow{\exp} \mathbb{C}_{\mathfrak{c}}^* \to 0,$$

where as a reminder $\underline{\mathbb{Z}}$ denotes the constant sheaf at the integers (Exercise 1.45). The fact that exponentiation is an epimorphism of sheaves is a stalkwise condition, saying that for every value $x \in X$, we can find a sufficiently small neighborhood in which we can take a branch and define $\log(z)$.

Example 1.54. Let X be any complex manifold. Let $\mathcal{O}_X \subseteq \mathcal{M}_X$ be the subsheaf of holomorphic sections inside meromorphic sections, as in Example 1.49. Then we can *define* the *sheaf of principal parts* (see e.g. [Gun18, (4.2)]) as

$$0 \to \mathcal{O}_X \to \mathcal{M}_X \to \mathcal{P}_X \to 0.$$

The terminology comes from the *principal part* of a meromorphic function around a pole [SS03, p. 75].

Proposition 1.55. If X is a Riemann surface, and $p \in X$ is any point, we have that $\mathcal{P}_{X,p}$ is given by the ring of finite negative Laurent expansions in a local coordinate around p.

Proof. As taking stalks is exact, we can describe $\mathcal{P}_{X,p}$ as the cokernel of the natural map $\mathcal{O}_{X,p} \to \mathcal{M}_{X,p}$. This inclusion sits a Taylor series inside the world of convergent Laurent series, so the quotient is exactly those terms which do not come from Taylor series, i.e. the principal part of a convergent Laurent expansion. □

Example 1.56. For X a complex manifold we define the *sheaf of divisors* Div_X to be the cokernel

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathrm{Div}_X \to 0.$$

By our understanding of the stalks of the first two sheaves we can understand the stalks of the latter.

Exercise 1.57. If X is a Riemann surface, show that $\mathrm{Div}_{X,p} \cong \mathbb{Z}$ for every point $p \in X$.

2. Sheaf cohomology

A big result here that we won't prove is the following:

Theorem 2.1. We have that $\mathrm{Ab}(X)$ is an abelian category.

Proof. This is a standard example of an abelian category. See [God58, II§2] or [Gro57]. □

An abelian category is a setting in which we can do *homological algebra*. In particular, we will be allowed to resolve sheaves by nicer sheaves, or ones which are easier to deal with in some respect.

Remark 2.2. For technical reasons, we're going to restrict our attention to sheaf cohomology over paracompact spaces. This is due to the presence of partitions of unity subordinate to open covers (Theorem B.14).

Example 2.3 (Representability of sections)**.** Let $U \subseteq X$ be an open set. Then the functor

$$\begin{aligned} \Gamma(U, -)\colon \mathrm{Ab}(X) &\to \mathrm{Ab} \\ S &\mapsto \Gamma(U, S) \end{aligned}$$

is corepresentable, meaning there exists a sheaf we will denote by $\mathbb{Z}_U$ for which $\mathrm{Hom}_{\mathrm{Ab}(X)}(\mathbb{Z}_U, -) \cong \Gamma(U, -)$ are naturally isomorphic as abelian groups.[5]

[5] This is poor notation. Better notation is to let $j\colon U \hookrightarrow X$ denote the inclusion, and write $j_!\mathbb{Z}_U$.

Proof. Denote by $h_U = \mathrm{Hom}_{\mathrm{Open}(X)}(-, U)$ the presheaf of sets[6]

$$h_U(V) = \begin{cases} * & V \subseteq U \\ \varnothing & V \subsetneq U. \end{cases}$$

By the Yoneda lemma, we have a natural bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Set})}(h_U, F) \cong F(U) \in \mathrm{Set}$$

for any sheaf of sets F.

Let's now bootstrap to abelian groups. Let $\mathbb{Z}[h_U]$ be the free presheaf of abelian groups on the sheaf of sets h_U — by this we mean it is defined as

$$\mathbb{Z}[h_U](V) = \begin{cases} \mathbb{Z} & V \subseteq U \\ \{0\} & V \subsetneq U. \end{cases}$$

It is clear then that we have a natural group isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Ab})} (\mathbb{Z}[h_U], F) \cong F(U) \in \mathrm{Ab}$$

for any presheaf of abelian groups F. Finally if F was a sheaf, then any morphism of presheaves $\mathbb{Z}[h_U] \to F$ factors uniquely through the sheafification of $\mathbb{Z}[h_U]$ by Proposition 1.44. We denote this by

$$\mathbb{Z}_U := (\mathbb{Z}[h_U])^\sharp.$$

It is clear now that we have a natural isomorphism $\mathbb{Z}_U \cong \Gamma(U, -)$ (c.f. [Aut, 03CP,03CQ]. ☐

Remark 2.4. We can describe $\mathbb{Z}_U(V)$ explicitly via our understanding of sheafification in Remark 1.43 — it is exactly given as the abelian group of locally constant functions $f \colon U \cap V \to \mathbb{Z}$ which extend to a locally constant function on all of V, but where the function satisfies $f(v) = 0$ if $v \in V \smallsetminus (U \cap V)$. This latter condition is equivalent to the statement that the support of f is closed in V, which is often how you'll see $\mathbb{Z}_U(V)$ described:

$$\mathbb{Z}_U(V) = \left\{ \substack{\text{locally constant functions } U \cap V \to \mathbb{Z} \\ \text{with support closed in } V} \right\}$$

Observe that if $U_1 \subseteq U_2$ then there is a natural morphism of sheaves $\mathbb{Z}_{U_2} \to \mathbb{Z}_{U_1}$. This extends to a functor

$$\begin{aligned} \mathrm{Open}(X)^{\mathrm{op}} &\to \mathrm{Ab}(X) \\ U &\mapsto \mathbb{Z}_U \end{aligned} \tag{2.5}$$

We'll use this later on.

Exercise 2.6. Verify the claim in [Gro57, 1.9.2] that the collection $\{\mathbb{Z}_U\}_{U \in \mathrm{Open}(X)}$ forms a system of *generators* for the category $\mathrm{Ab}(X)$.

2.1. **Čech cohomology of a presheaf.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of X, let F be a presheaf, and define

$$C^q(\mathcal{U}, F) = \prod_{i_0, \dots, i_q \in I} F(U_{i_0} \cap \dots \cap U_{i_q}). \tag{2.7}$$

We call this the *group of cochains*. We can define a *coboundary map*

$$\delta^q \colon C^q(\mathcal{U}, F) \to C^{q+1}(\mathcal{U}, F), \tag{2.8}$$

[6]Note this is *different* than the "representable" presheaf we defined in Example 1.33. There we looked at all continuous maps into U, whereas here we only want the natural inclusion of V in U, if it exists.

defined as

$$(\delta^q f)(i_0, \ldots, i_{q+1}) = \sum_{k=0}^{q+1} (-1)^k f(i_0, \ldots, \widehat{i_k}, \ldots, i_{q+1}).$$

Remark 2.9.

(1) As a remark, $f(i_0, \ldots, \widehat{i_k}, \ldots, i_{q+1})$ is naturally valued in $U_{i_0} \cap \cdots \cap U_{i_{k-1}} \cap U_{i_{k+1}} \cap \cdots \cap U_{i_{q+1}}$, but we want to think about it as being valued in $\bigcap_{j=0}^{q+1} U_{i_j}$ under the restriction map in order to form the sum above. We've chosen to suppress this restriction from the notation, but Hirzebruch includes it (c.f. [Hir78, p. 26].

(2) The coboundary condition in Equation (2.8) is written additively, since we are assuming F is a sheaf of abelian groups. In later contexts we might write it multiplicatively (if F isn't assumed to be abelian),

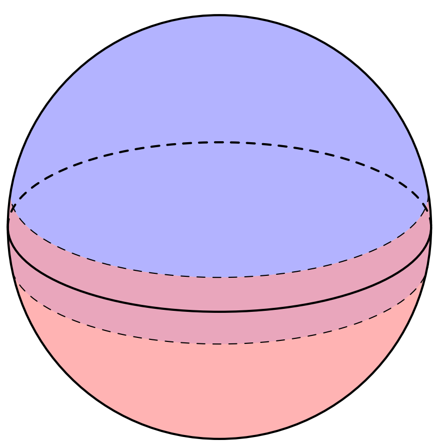
Exercise 2.10. Show that $\delta^{q+1} \circ \delta^q = 0$, so that we get a cochain complex

$$\cdots \to C^{q-1}(\mathcal{U}, F) \xrightarrow{\delta^{q-1}} C^q(\mathcal{U}, F) \xrightarrow{\delta^q} C^{q+1}(\mathcal{U}, F) \xrightarrow{\delta^{q+1}} \cdots \tag{2.11}$$

Definition 2.12. We define the *Cech cohomology* of the presheaf F over the cover $\mathcal{U}$ to be

$$\hat{H}^q(\mathcal{U}, F) = \ker(\delta^q)/\mathrm{im}(\delta^{q-1}).$$

Example 2.13. Consider the sphere S^2, equipped with an open cover $\mathcal{U} = \{U_n, U_s\}$, where U_n and U_s denote the northern and southern hemispheres, respectively, plus a little overlap on the equator. Let's compute the sheaf cohomology in the constant sheaf $\underline{\mathbb{Z}}$ which we defined in Exercise 1.45:



We have that:

- ▷ $\check{C}^0(\mathcal{U}, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_n) \times \underline{\mathbb{Z}}(U_s)$. Since each of these open sets is connected, a locally constant function $U_n \to \mathbb{Z}$ is just constant – so we get that $\underline{\mathbb{Z}}(U_n) \cong \mathbb{Z}$ by evaluation at the north pole, say, and similarly for U_s. Hence $\check{C}^0(\mathcal{U}, \underline{\mathbb{Z}}) \cong \mathbb{Z} \times \mathbb{Z}$.
- ▷ $\check{C}^1(\mathcal{U}, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_n \cap U_s)$, since there is only one double overlap to consider. Again since it is connected, we get that $\check{C}^1(\mathcal{U}, \underline{\mathbb{Z}}) \cong \mathbb{Z}$.
- ▷ $\check{C}^q(\mathcal{U}, \underline{\mathbb{Z}}) = 0$ for $q \geq 2$ since there are no triple overlaps in our cover.

So our chain complex looks like

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

We want to understand the differential δ^0. Given $f \in \check{C}^0(\mathcal{U}, \underline{\mathbb{Z}})$, it consists of two functions which we'll call $f_n \colon U_n \to \mathbb{Z}$ and $f_s \colon U_s \to \mathbb{Z}$. They are both locally constant, hence constant. The differential is then of the form

$$\begin{aligned} \delta^0(f) \colon U_n \cap U_s &\to \mathbb{Z} \\ u &\mapsto f_n(u) - f_s(u). \end{aligned}$$

Altogether we get the chain complex

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathbb{Z} \to 0.$$

So we get

$$\check{H}^q(\mathcal{U}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q = 1 \\ 0 & q \geq 2 \end{cases}$$

Definition 2.14. Let X be a space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ be two open covers of X. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ so that $V \subseteq U$.

Exercise 2.15. Recall the cover $\mathcal{U}$ of the 2-sphere we constructed in Example 2.13. Find a refinement $\mathcal{V}$ of $\mathcal{U}$ for which the Čech cohomology is exactly

$$\check{H}^q(\mathcal{V}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q = 1 \\ \mathbb{Z} & q = 2 \\ 0 & q > 2. \end{cases}$$

Definition 2.16. Let X be a topological space. We denote by $\mathrm{Cov}(X)$ the poset (category) of open covers of X under refinement. That is, an object of $\mathrm{Cov}(X)$ is an open cover $\mathcal{U}$ of X, and for any two covers $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$, we have exactly one morphism $\mathcal{V} \to \mathcal{U}$ if $\mathcal{V}$ refines $\mathcal{U}$, otherwise there are no morphisms from $\mathcal{V}$ to $\mathcal{U}$.

Exercise 2.17. If X is a space, F is a presheaf on X, and $\mathcal{U}, \mathcal{V}$ are open covers of X for which $\mathcal{V}$ is a refinement of $\mathcal{U}$, define a natural restriction morphism

$$\check{H}^q(\mathcal{U}, F) \to \check{H}^q(\mathcal{V}, F).$$

Hint: You should need to make choices in your construction – argue that the map ultimately doesn't depend on the choices you make.

In particular, the exercise above indicates that Čech cohomology is functorial in the cover. This motivates the following definition:

Definition 2.18. We define the *Čech cohomology* of a (pre)sheaf F on X by

$$\check{H}^q(X, F) := \mathrm{colim}_{\mathcal{U} \in \mathrm{Cov}(X)} \check{H}^q(\mathcal{U}, F).$$

That is, it is the colimit of the Čech cohomology along all possible covers, filtered by refinement.

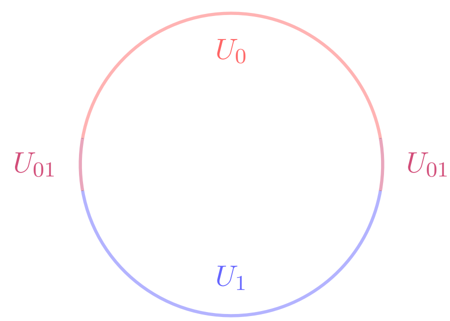
We have very little control over these refinement maps, as the following result illustrates:

Proposition 2.19. Let $\mathcal{V}$ be a refinement of the cover $\mathcal{U}$ on a space X. Then:

- ▷ the refinement map $\check{H}^0(\mathcal{U}, F) \to \check{H}^1(\mathcal{V}, F)$ is an isomorphism
- ▷ the refinement map $\check{H}^1(\mathcal{U}, F) \to \check{H}^1(\mathcal{V}, F)$ is injective
- ▷ higher refinement maps $\check{H}^q(\mathcal{U}, F) \to \check{H}^q(\mathcal{V}, F)$ need not be injective for $q \geq 2$. An explicit example is on Math Overflow here: MO462472.

Proof. The statement $q = 0$ will follow from Proposition 2.21. For $q = 1$ a nice proof is in [For81, 12.4]. □

Example 2.20. On S^1, we can take a simple cover consisting of only all of S^1, letting $F = \underline{\mathbb{Z}}$ be the constant sheaf $\mathbb{Z}$. Then the Čech cochains complex for this is the chain complex $\mathbb{Z}[0]$ concentrated in degree zero. This gives $\check{H}^0(\mathcal{U}, F) = \mathbb{Z}$ and vanishing cohomology elsewhere. On the other hand, we can take a cover $S^1 = U_0 \cup U_1$, where U_0 and U_1 are the upper and lower semicircle, plus a little extra so that they overlap:



Note that $\check{C}^1(\mathcal{U}, F) = \mathbb{Z}^{\times 2}$ in this instance, since $U_0 \cap U_1$ has two connected components. The chain complex here then looks like

$$\check{C}^\bullet(\mathcal{U}, F): \qquad 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \to 0 \to \cdots$$

so we get cohomology in degrees 0 and 1, as expected. This is an example of how *refining* the cover can detect more cohomological features.

Proposition 2.21. For any space X and any *sheaf* $F \in \mathrm{Ab}(X)$, we have a canonical isomorphism

$$\check{H}^0(X, F) \cong \Gamma(X, F).$$

Proof. Given any open cover $\mathcal{U} = \{U_i\}$, an element $f \in \check{H}^0(\mathcal{U}, F)$ is an element $f_i \in F(U_i)$ for each i (this is the condition of lying in $C^0(\mathcal{U}, F)$), which lies in the kernel of $\delta^0 \colon C^0 \to C^1$. This means that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$ for each i, j. Since F is a sheaf, by gluing, the f_i's glue together to give a well-defined global section $f \in \Gamma(X, F)$. This is easily seen to be an isomorphism. □

Exercise 2.22. Verify that Čech cohomology is functorial in the sheaf, that is for any X and any $q \geq 0$ we have a functor

$$\check{H}^q(X, -)\colon \mathrm{Ab}(X) \to \mathrm{Ab}.$$

Before we get to Proposition 2.25 it's worth saying some words about exactness of sheaves versus presheaves.

Remark 2.23 (On exactness)**.** By definition, a sequence of presheaves $F_1 \to F_2 \to F_3$ on X is exact if

$$0 \to F_1(U) \to F_2(U) \to F_3(U) \to 0$$

is an exact sequence of abelian groups for all $U \in \mathrm{Open}(X)$. If these are all sheaves, then we say the sequence is exact if

$$0 \to (F_1)_x \to (F_2)_x \to (F_3)_x \to 0$$

is exact for every point $x \in X$. If a sequence of sheaves is exact (when considered as presheaves) then it is still exact when considered as sheaves, e.g. by Corollary A.7 or by the phrase "sheafification is exact." The converse *does not hold*! Consider the short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to C^0(-, \mathbb{R}) \to C^0(-, S^1) \to 0.$$

This is exact (over any space X) since it is exact in each fiber. However over an arbitrary space it won't be exact as a sequence of presheaves, because not every circle-valued function lifts to a real-valued function (e.g. if it has non-trivial winding number).[7]

Exercise 2.24. If $F \to G$ is a map of presheaves which is a monomorphism of sheaves (it is injective on every stalk) then it is a monomorphism of presheaves (it is injective on every section).

Proposition 2.25. If

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is a short exact sequence *of presheaves*, it induces a long exact sequence

$$\begin{aligned} 0 &\to \check{H}^0(X, F_1) \to \check{H}^0(X, F_2) \to \check{H}^0(X, F_3) \\ &\to \check{H}^1(X, F_1) \to \check{H}^1(X, F_2) \to \check{H}^1(X, F_3) \\ &\to \check{H}^2(X, F_1) \to \cdots \end{aligned}$$

Sketch. The first step is to show that, for any cover $\mathcal{U}$, the functor

$$\check{C}^q(\mathcal{U}, -) \colon \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$$

is exact (subtlety: its restriction to $\mathrm{Ab}(X)$ is not necessarily exact, because $\mathrm{Ab}(X)$ has a different notion of exactness than the presheaf category!). A diagram chase then implies we get a long exact sequence on $\check{H}^*(\mathcal{U}, -)$. Then we take a colimit over all $\mathcal{U}$ in the filtered category of covers $\mathrm{Cov}(X)$, and use that filtered colimits preserve exactness to conclude (Example A.4 and Corollary A.7). The details are spelled out in [Hir78, pp. 28–29]. □

This begs a very interesting question — when do the Čech cohomology of a presheaf F and of its sheafification $F^\sharp$ agree? We need two preliminary results to lead up to a partial answer to this question.

Proposition 2.26. Let F be a presheaf over a space, and suppose $F^\sharp = 0$. For any point $x \in X$, any open $U \ni x$ and any $f \in F(U)$, there exists some smaller open neighborhood $V \subseteq U$ containing x for which $f_{|V} = 0$.

Proof. This is because the stalk $F_x = F^\sharp_x$ must be zero. □

Lemma 2.27. Let X be a paracompact space, and F a presheaf on X. Then if $F^\sharp$ is the zero sheaf, we claim that $\check{H}^n(X, F) = 0$ for all $n \geq 0$.

Proof. We show that no cochains survive refinement. That is, if $f \in \check{C}^q(\mathcal{U}, F)$ for some cover $\mathcal{U} = \{U_i\}_{i \in I}$, we will find a refinement of the cover which kills f. Without loss of generality assume that $\mathcal{U}$ is locally finite. By the Dieudonné shrinking theorem (Theorem B.11), we may find a cover $\mathcal{W} = \{W_i\}_{i \in I}$ for which $\overline{W_i} \subseteq U_i$.

Let $J = X$ as a set— we will produce a refinement of the cover $\mathcal{U}$, now indexed over J. For each $x \in X$, we pick an open neighborhood $V_x \ni x$ for which:

(1) if $x \in U_i$ then $V_x \subseteq U_i$ and if $x \in W_i$ then $V_x \subseteq W_i$
(2) if $V_x \cap W_i \neq \varnothing$ then $V_x \subseteq U_i$
(3) if $x \in U_{i_0} \cap \cdots \cap U_{i_q}$ then $f(i_0, \ldots, i_q)$ restricts to zero in $F(V_x)$.

[7] Indeed we can think of the winding number as measuring the failure of lifting, giving us a class in $H^1(X, \underline{\mathbb{Z}}) = [X, S^1]$.

Since $\mathcal{U}$ and $\mathcal{W}$ are locally finite, there are only finitely many U_i and W_i containing x to worry about, so the first two conditions can be fulfilled. Why are we allowed to guarantee the last condition can hold? By Proposition 2.26 some sufficiently small neighborhood of x in $U_{i_0} \cap \cdots \cap U_{i_q}$ will have the property that $f(i_0, \ldots, i_q)$ restricts to zero in it. We can pick V_x to be that neighborhood, perhaps intersecting it further finitely many times to make it satisfy the first two conditions.

We now claim that f refines to zero in this new cover $\{V_x\}_{x \in X}$. Pick any tuple $(x_0, \ldots, x_q)$, and let i_j be such that $x_j \in W_{i_j} \subseteq U_{i_j}$ for each j. By (1), this tells us that $V_{x_j} \subseteq W_{i_j} \subseteq U_{i_j}$. We want to argue that the image of $f(i_0, \ldots, i_p)$ under the restriction map

$$F(U_{i_0} \cap \cdots \cap U_{i_q}) \to F(V_{x_0} \cap \cdots \cap V_{x_q})$$

is zero. If the intersection of the V_{x_i}'s is empty, there is nothing to show, so suppose it is nonempty. Then $V_{x_0} \cap V_{x_k}$ is nonempty for each k. Since $V_{x_k} \subseteq W_{i_k}$, we have that $V_{x_0} \cap W_{i_k} \neq \varnothing$, and by point (2) this implies that $V_{x_0} \subseteq U_{i_k}$. Since this is true for all k, we have that $V_{x_0} \subseteq U_{i_0} \cap \cdots \cap U_{i_q}$. However by (3), since $x_0 \in V_{x_0} \subseteq U_{i_0} \cap \cdots \cap U_{i_q}$, we have that $f(i_0, \ldots, i_q)$ restricts to zero on V_{x_0} and therefore on the smaller subset $V_{x_0} \cap \cdots \cap V_{x_q}$. □

Theorem 2.28. If X is paracompact, and F is a presheaf with corresponding sheaf $F^\sharp$, then the natural map $\check{H}^q(X, F) \to \check{H}^q(X, F^\sharp)$ is an isomorphism for all $q \geq 0$.

Proof. Consider the sheafification map $F \to F^\sharp$ and take its kernel and cokernel in the category of presheaves

$$0 \to K \to F \to F^\sharp \to C \to 0.$$

As $F \to F^\sharp$ sheafifies to an isomorphism, and since sheafification is an exact functor, we have that K and C sheafify to zero. Hence their higher cohomology vanishes by Lemma 2.27, and using the long exact sequence on cohomology attached to short exact sequences of presheaves, we obtain that the induced maps $\check{H}^q(X, F) \to \check{H}^q(X, F^\sharp)$ are all isomorphisms. □

So over paracompact spaces, there is no difference between the cohomology of a presheaf and of its resulting sheafification. A corollary of this we will use often is the following:

Proposition 2.29. If X is a paracompact space, and $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence of *sheaves* on X, then it induces a long exact sequence

$$0 \to \check{H}^0(X, F_1) \to \check{H}^0(X, F_2) \to \check{H}^0(X, F_3) \to \check{H}^1(X, F_1) \to \cdots$$

Proof. Let $F_1 \to F_2 \to F_3$ be a short exact sequence of *sheaves*. Define a *presheaf* F_2/F_1 to be the cokernel of the map $F_1 \to F_2$ in the category of sheaves. By Proposition 2.25 we get a long exact sequence on Čech cohomology of presheaves

$$0 \to \check{H}^0(X, F_1) \to \check{H}^0(X, F_2) \to \check{H}^0(X, F_2/F_1) \to \check{H}^1(X, F_1) \to \cdots$$

Note that F_1 and F_2 are already sheaves, and we claim that the natural map $(F_2/F_1)^\sharp \to F_3$ is an isomorphism. This is clear since sheafification is exact. Therefore by Theorem 2.28 we have canonical isomorphisms $\check{H}^q(X, F_2/F_1) \xrightarrow{\sim} \check{H}^q(X, F_3)$ for any $q \geq 0$. Plugging these into the sequence above we get our desired result. □

2.2. **Acyclic resolutions.** In classical topology, spaces were called *acyclic* if every cycle on the space was a boundary — in other words the space had the homology type of a point or any contractible space. Similarly a connective cochain complex can be considered acyclic if its cohomology is concentrated in degree zero. For Čech cohomology, we say a sheaf is acyclic if its Čech cochains complex is an acyclic complex. In other words:

Definition 2.30. A sheaf $F \in \mathrm{Ab}(X)$ is *acyclic* if $\check{H}^p(X, F) = 0$ for $p > 0$.

Definition 2.31. Take $F \in \mathrm{Ab}(X)$. An *acyclic resolution* of F is a (possibly infinite) exact sequence

$$0 \to F \to A^0 \to A^1 \to \cdots$$

where each A^i is acyclic for $i \geq 0$.

Exercise 2.32. Argue that the global sections functor

$$\Gamma(X, -)\colon \mathrm{Ab}(X) \to \mathrm{Ab}$$

is left exact, and then argue that it sends an acyclic resolution of F to a chain complex in Ab.

It turns out we can use acyclic resolutions to compute cohomology!

Theorem 2.33 (Čech=sheaf cohomology)**.** Let $F \in \mathrm{Ab}(X)$ be a sheaf over a paracompact space, and consider any acyclic resolution $F \to A^\bullet$. Then $\check{H}^q(X, F)$ is canonically isomorphic to the cohomology of the chain complex obtained by applying $\Gamma(X, -)$ to the acyclic resolution.

Proof. We write out the sequence we're looking at:

$$\Gamma(X, A^0) \xrightarrow{d^0} \Gamma(X, A^1) \xrightarrow{d^1} \cdots$$

We will proceed by induction on q.

Base case $q = 0$: Since $\Gamma(X, -F)$ is a left exact functor (Exercise 2.32), we get that

$$0 \to \Gamma(X, F) \to \Gamma(X, A^0) \xrightarrow{d_*^0} \Gamma(X, A^1)$$

is left exact. In particular, we get that the kernel of d_*^0 is exactly $\Gamma(X, F)$, which agrees with $\check{H}^0(X, F)$ by Proposition 2.21. Hence $\check{H}^0(X, F)$ is the 0th cohomology of $\Gamma(X, -)$ applied to the acyclic resolution, establishing our base case.

Inductive step: For any $p \geq 0$, we let $K^p := \ker(A^p \to A^{p+1})$. Note by definition of our resolution, we have that $K^0 \cong F$. By exactness and the first isomorphism theorem, $A^p/K^p \cong K^{p+1}$, so we get short exact sequences of sheaves for each $p \geq 0$:

$$0 \to K^p \to A^p \to K^{p+1} \to 0.$$

Applying the long exact sequence on cohomology (Proposition 2.29), we get

$$\cdots \to \check{H}^q(X, K^p) \to \check{H}^q(X, A^p) \to \check{H}^q(X, K^{p+1}) \xrightarrow{\partial} \check{H}^{q+1}(X, K^p) \to \check{H}^{q+1}(X, K^p) \to \cdots$$

Since the A^p's are acyclic, the boundary maps ∂ are isomorphisms $\check{H}^q(X, K^{p+1}) \cong \check{H}^{q+1}(X, K^p)$ for $q \geq 1$. Note though that $K^0 = F$, so we get that

$$\check{H}^1(X, K^{q-1}) \cong \check{H}^q(X, F) \qquad \text{for } q \geq 1.$$

We claim $\check{H}^1(K^{q-1})$ is identified canonically with the qth cohomology of the sequence $\left\{\Gamma(X, A^i)\right\}$. Indeed this follows from looking at the long exact sequence attached to $0 \to K^{q-1} \to A^{q-1} \to K^q \to 0$, from which we get

$$H^0(X, A^{q-1}) \xrightarrow{d_*^{q-1}} H^0(X, K^q) \to H^1(X, K^{q-1}) \to 0.$$

Since $H^0(X, K^q) = \ker(d_*^q)$, we have that $H^1(X, K^{q-1})$ is $\ker(d_*^q)$ modulo the image of d_*^{q-1}, hence it is the cohomology of the complex. □

This provides us a new way to compute Čech cohomology over a paracompact space!

Definition 2.34. Recall an *injective sheaf* is an injective object in $\mathrm{Ab}(X)$, meaning that the functor

$$\mathrm{Hom}_{\mathrm{Ab}(X)}(-, I)\colon \mathrm{Ab}(X)^{\mathrm{op}} \to \mathrm{Ab}$$

is exact.

Example 2.35. Let I be an injective abelian group, and $i\colon \{x\} \to X$ the inclusion of a point. Then the skyscraper sheaf $i_*I \in \mathrm{Ab}(X)$ is injective.

Proof. The data of a map into a skyscraper sheaf is completely determined by what happens on the stalk over x (this is a pullback-pushforward adjunction) – this gives us a natural isomorphism of functors $\mathrm{Ab}(X) \to \mathrm{Ab}$:

$$\mathrm{Hom}_{\mathrm{Ab}(X)}(-, i_*I) \cong \mathrm{Hom}_{\mathrm{Ab}}((-)_x, I).$$

the right hand side is exact since taking stalks is an exact functor, hence i_*I is injective. □

Definition 2.36. Let $F \in \mathrm{Ab}(X)$ be a sheaf. An *injective resolution* of F is an exact sequence

$$0 \to F \to I^0 \to I^1 \to I^2 \to \cdots$$

for which each sheaf I^j is injective.

Remark 2.37. In general, one deals with *injective resolutions* rather than acyclic ones. This allows us to reword the above in the language of derived categories, and define cohomology as the right derived functors of the global sections functor. In order to do this, we should show that injective sheaves are in fact acyclic.

First we need a lemma.

Lemma 2.38 ([Aut, 03AS, 03AU])**.** Let X be any space, and $\mathcal{U} = \{U_i\}$ any open cover. Define a chain complex of abelian sheaves $\mathbb{Z}_{\mathcal{U},\bullet}$ by

$$\mathbb{Z}_{\mathcal{U},p} = \bigoplus_{i_0,\dots,i_p} \mathbb{Z}_{U_{i_0}\cap\cdots\cap U_{i_p}},$$

where recall $\mathbb{Z}_U$ is the sheaf corepresenting $\Gamma(U, -)$ as in Example 2.3. The differentials on the chain complex $\mathbb{Z}_{\mathcal{U},\bullet}$ are the sum over $(-1)^k$ times the canonical map

$$\mathbb{Z}_{U_{i_0}\cap\cdots\cap U_{i_p}} \to \mathbb{Z}_{U_{i_0}\cap\cdots\cap\widehat{U_{i_k}}\cap U_{i_p}}.$$

Then we claim that $\mathrm{Hom}_{\mathrm{Ab}(X)}(\mathbb{Z}_{\mathcal{U},p}, F) \cong \check{C}^p(\mathcal{U}, F)$ for any $F \in \mathrm{Ab}(X)$, and that the differentials $\mathbb{Z}_{\mathcal{U},p+1} \to \mathbb{Z}_{\mathcal{U},p}$ induce the Čech cochain differentials $\delta^p\colon \check{C}(\mathcal{U}, F) \to \check{C}^{p+1}(\mathcal{U}, F)$ after applying $\mathrm{Hom}_{\mathrm{Ab}(X)}(-, F)$.

Corollary 2.39. Every injective sheaf is acyclic.

Proof. If $I \in \mathrm{Ab}(X)$ is injective, then $\mathrm{Hom}_{\mathrm{Ab}(X)}(-, I)$ is an exact functor. In particular it sends $\mathbb{Z}_{\mathcal{U},\bullet}$ to a cochain complex which is exact except for in the first term. Since this is exactly the Čech cochains complex, this implies that $\check{H}^q(\mathcal{U}, I) = 0$ for $q \geq 1$. As this is true for every $\mathcal{U}$, we conclude that $\check{H}^q(X, I) = 0$ for all $q \geq 1$. □

The converse does not hold:

Example 2.40. Let A be a non-injective abelian group, and $i\colon \{x\} \to X$ the inclusion of a point. Then we claim the skyscraper sheaf i_*A is acyclic but need not be injective.

Proof. Take any cover of X and refine it so that $U_0 \ni x$ and $U_i \not\ni x$ for $i > 0$. In particular there are no sections of i_*A over overlaps, so $\check{H}^q(\mathcal{U}, i_*A) = 0$ for all $q > 0$. This remains true under refinement, so this vanishing passes to the colimit. Failure of injectivity of the sheaf i_*A reduces to failure of injectivity of A as an abelian group by looking at maps on stalks as in the proof of Example 2.35. □

So Corollary 2.39 lets us conclude what we may have already expected:

Corollary 2.41. Over a paracompact Hausdorff space X, we have that Čech cohomology is canonically isomorphic to the right derived functors of global sections.

Remark 2.42. The contemporary perspective is to *first* define cohomology as these right derived functors, *then* to compare with Čech cohomology. The classical perspective of Serre, Godement and others, and what we presented above, is the exact opposite.

Exercise 2.43. Note that Theorem 2.33 has already told us that cohomology can be computed using an injective resolution, and that this doesn't depend on the choice of injective resolution. The yoga of derived functors lets us conclude this also. If we wanted to try to prove it by hand, we could do it as follows: Prove that if $F \to I^\bullet$ is any injective resolution and $F \to J^\bullet$ is any other resolution, there exists a map of chain complexes $f^\bullet$ making the diagram commute:

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & I^\bullet \arrow[d, "f^\bullet"] \\ & J^\bullet. \end{tikzcd}$$

Show that $f^\bullet$ is a quasi-isomorphism and moreover is well-defined up to chain homotopy.

2.3. **Existence of injectives.** We now know a new way to take cohomology, which is leveraging an acyclic resolution, or if we want to use some category theory, take the derived functors of global sections. This perspective is only valuable if we can actually find acyclic/injective resolutions. How do we know they even exist? Let's argue that they do.

Proposition 2.44. For every $F \in \mathrm{Ab}(X)$, there exists an injective sheaf I and a monomorphism $F \to I$.

Proof. We can embed the abelian group F_x into an injective abelian group I_x. Define a presheaf by

$$\begin{aligned} \mathcal{I} \colon \mathrm{Open}(X)^{\mathrm{op}} &\to \mathrm{Ab} \\ U &\mapsto \prod_{x \in U} I_x. \end{aligned}$$

Note that $\mathcal{I}$ is actually a sheaf, since it is a product of skyscraper sheaves, one for each point of X. Observe that the stalk of the sheaf $\mathcal{I}_x$ need not be equal to I_x, however it will contain I_x as a retract. We claim that

$$\mathrm{Hom}_{\mathrm{Ab}(X)}(F, \mathcal{I}) \cong \prod_{x \in X} \mathrm{Hom}_{\mathrm{Ab}}(F_x, I_x).$$

There is a map from the left to the right by taking $g \colon \mathcal{F}(U) \to \mathcal{I}(U)$ for any open $U \ni x$, then noting that $\mathcal{F}(U) \to \mathcal{I}(U)$ maps to I_x along the projection, then taking a colimit over U, inducing $\mathcal{F}_x \to \mathcal{I}_x$. For the reverse direction, if we have maps $f_x \colon F_x \to I_x$ for each x, we can define $F(U) \to \mathcal{I}(U)$ by the composite

$$F(U) \to \prod_{x \in U} F_x \xrightarrow{\prod f_x} \prod_{x \in U} I_x = \mathcal{I}(U),$$

and we have these maps on each section, which give a map of sheaves. We claim these are inverse operations.

We have to check two things: that $\mathcal{I}$ is injective, and that $F \to \mathcal{I}$ is a monomorphism. That it is a monomorphism can be seen sectionwise, which induces monomorphisms on stalks since filtered colimits preserve monomorphisms.

To see $\mathcal{I}$ is injective, it suffices to show observe each skyscraper sheaf is injective by Example 2.35 and then use that injective objects are closed under arbitrary products. □

Corollary 2.45. For any space X, we have that every $F \in \mathrm{Ab}(X)$ admits an injective resolution.

Proof. By Proposition 2.44, we start with

$$0 \to F \to I^0,$$

and then take the quotient I^0/F and embed it in some injective I^1. The composite $F \to I^0 \to I^1$ is obviously zero since it factors through the quotient by construction, so we get

$$0 \to F \to I^0 \to I^1.$$

We then embed I^1/I^0 into some injective I^2, and so on. □

2.4. **Godement resolutions.** We can expand the results of the previous section by establishing the so-called *Godement resolution*, which is a functorial way of producing an acyclic resolution. We first need some preliminary definitions:

Definition 2.46. Let $f\colon X \to Y$ be a continuous map of spaces.

- ▷ Given a sheaf $F \in \mathrm{Ab}(X)$, we define the *pushforward sheaf* f_*F on Y by the rule

$$\begin{aligned} f_*F\colon \mathrm{Open}(Y) &\to \mathrm{Ab} \\ U &\mapsto F(f^{-1}(U)). \end{aligned}$$

  We claim that f_*F satisfies locality and gluing (exercise) and therefore f_* induces a functor which we call *pushforward*:

$$f_*\colon \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$$

- ▷ Given a sheaf $G \in \mathrm{Ab}(Y)$, we define a presheaf on X by

$$\begin{aligned} f^{-1}_{\mathrm{pre}}G\colon \mathrm{Open}(X) &\to \mathrm{Ab} \\ U &\mapsto \mathrm{colim}_{\substack{V \in \mathrm{Open}(Y) \\ V \supseteq f(U)}} G(U). \end{aligned}$$

  We define the *inverse image sheaf* to be the sheafification of this, that is $f^{-1}G := (f^{-1}_{\mathrm{pre}}G)^\sharp$. Again we claim this is functorial:

$$f^{-1}\colon \mathrm{Ab}(Y) \to \mathrm{Ab}(X).$$

Example 2.47. In the particular case when our map is of the form $i\colon \{x\} \to X$, that is, the inclusion of a point, we have that a sheaf on the one-point space is just a single abelian group A. The pushforward sheaf i_*A is *exactly* the skyscraper sheaf of A at the point $x \in X$. This is the reason for the notation for skyscraper sheaves in Example 1.4.

Proposition 2.48. Let $f\colon X \to Y$ be any map of spaces. Then pullback and pushforward form an adjunction:

$$f^{-1}\colon \mathrm{Ab}(Y) \leftrightarrows \mathrm{Ab}(X)\colon f_*.$$

Proof. For a proof (in slightly more generality), see the arguments in [Aut, 008P] and the previous sections. □

Remark 2.49. Right adjoints preserve injective objects, so the pushforward of an injective sheaf is still injective. In particular pushforward of an injective abelian group (viewed as an injective sheaf over a point) is still injective. This is alternative way to see that skyscraper sheaves of injective abelian groups are injective sheaves (Example 2.35). And indeed, we essentially used the adjunction Proposition 2.48 to prove it.

An interesting case comes from *forgetting* the topology on X, and considering pullback-pushforward along the resulting natural map.

Notation 2.50. For any space X, we denote by X^δ the topological space with the exact same underlying set as X, but with the discrete topology (every subset is open). There is a natural map $\epsilon\colon X^\delta \to X$.[8]

We define a functor

$$\begin{aligned} C^0\colon \mathrm{Ab}(X) &\to \mathrm{Ab}(X) \\ F &\mapsto \epsilon_*\epsilon^{-1}F. \end{aligned}$$

As a presheaf, $\epsilon_*\epsilon^{-1}F$ sends U to $\prod_{x\in U} F_x$, that is, it is the sheaf of germs at all points. It is also clear that it comes equipped with a natural map of sheaves $j\colon F \to C^0(F)$ (which is the unit of the pullback-pushforward adjunction).

Proposition 2.51. Let X be an arbitrary space, and $F \in \mathrm{Ab}(X)$ any sheaf of abelian groups.

(1) The map $j\colon F \to C^0(F)$ is a monomorphism.
(2) The sheaf $C^0(F)$ is acyclic

Proof. The first point is immediate – if $f, g \in F(U)$ have the property that $j(f) = j(g) \in C^0(F)(U) = \prod_{x\in U} F_x$, this means they have the same value on stalks of all points in U, which by the sheaf condition means they must agree. The difficult part is the second point, arguing that $C^0(F)$ is acyclic.

Fix any open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X. We have that the Čech p-cochains of $C^0(F)$ can be rewritten (by swapping products around) as:

$$\begin{aligned} \check{C}^p(\mathcal{U}, C^0(F)) &= \prod_{i_0,\dots,i_p\in I} C^0(F)(U_{i_0}\cap\cdots\cap U_{i_p}) = \prod_{i_0,\dots,i_p\in I}\ \prod_{x\in U_{i_0}\cap\cdots\cap U_{i_p}} F_x \\ &= \prod_{x\in X}\ \prod_{\substack{i_0,\dots,i_p\in I \\ U_{i_0}\cap\cdots\cap U_{i_p}\ni x}} F_x. \end{aligned}$$

So the Čech differential acts pointwise on each x. In other words if we define

$$\check{C}^p(\mathcal{U}, x, F) := \prod_{\substack{i_0,\dots,i_p\in I \\ U_{i_0}\cap\cdots\cap U_{i_p}\ni x}} F_x,$$

then it is clear that $\check{C}^\bullet(\mathcal{U}, C^0(F))$ is a product over the individual complexes $\check{C}^\bullet(\mathcal{U}, x, F)$. It therefore reduces to showing that $\check{C}^\bullet(\mathcal{U}, x, F)$ is an acyclic complex.

Fix an arbitrary point $x \in X$, and find one index $i' \in I$ for which $x \in U_{i'}$. Then (if we use *unnormalized Čech cochains*[9]) we can define a chain homotopy

$$h\colon \check{C}^p(\mathcal{U}, x, F) \to \check{C}^{p-1}(\mathcal{U}, x, F),$$

by defining $(hc)(i_0,\dots,i_{p-1}) = c(i', i_0, \dots, i_{p-1})$. We can check, for $p \geq 1$, this is a null-homotopy for the identity map on the cochain complex. Therefore $\check{C}^p(\mathcal{U}, x, F)$ has no cohomology above H^0, and therefore neither does $\check{C}^p(\mathcal{U}, C^0(F))$. □

Since C^0 is functorial, we can use it to (functorially!) build an acyclic resolution.

[8]There is a "discrete-forgetful" adjunction $(-)^\delta\colon \mathrm{Set} \leftrightarrows \mathrm{Top}\colon U$, where U forgets the topology on a space and returns its underlying set, and $(-)^\delta$ endows a set with the discrete topology. This map $X^\delta \xrightarrow{\epsilon} X$ is simply the counit of this adjunction, hence is natural in X.

[9]There are two notions of Čech cochains, one is *normalized*, where each of $i_0, \dots, i_p$ needs to be distinct, and the other is *unnormalized*, where we're allowed to have repetitions. They end up giving completely the same answer, but some operations like this one are much easier to state in the language of unnormalized cochains.

Notation 2.52. We define $C^1\colon \mathrm{Ab}(X) \to \mathrm{Ab}(X)$ to be the functor sending F to $C^0(\mathrm{coker}(F \xrightarrow{j} C^0(F)))$, that is, we look at $j\colon F \to C^0(F)$, take its cokernel, and then hit it with C^0. We get a composite map which we call d^0:

$$C^0(F) \to \mathrm{coker}(j) \to C^0(\mathrm{coker}(j)) =: C^1(F),$$

and it is clear that the composite $F \to C^0(F) \to C^1(F)$ is zero, since it factors through the cokernel of F.

Iteratively, we define

$$C^{n+1}(F) = C^0(\mathrm{coker}(C^{n-1}(F) \xrightarrow{d^{n+1}} C^n(F)),$$

which receives an analogous map d^n from $C^n(F)$.

Definition 2.53. We define the *Godement resolution* to be the acyclic resolution of F obtained as

$$0 \to F \to C^0(F) \xrightarrow{d^0} C^1(F) \xrightarrow{d^1} C^2(F) \to \cdots$$

as above.

Remark 2.54. A few things about this are worth noting:

- ▷ each of these constructions C^n are *functorial*
- ▷ the functors $C^n(-)\colon \mathrm{Ab}(X) \to \mathrm{Ab}(X)$ are exact for each n.

Exactness of $C^0(-)$ is immediate, since it is a stalkwise construction, and exactness of $C^n(-)$ can be proved inductively.

What is the benefit of all this nonsense? It allows us to give an analogue and a reproof of Proposition 2.29 (on a paracompact space, a short exact sequences of sheaves induces a long exact sequence on cohomology) for sheaf cohomology, and *without paracompactness*.

Proposition 2.55. Let X be any topological space, and let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence of sheaves on X. This induces a long exact sequence on sheaf cohomology

$$0 \to H^0(X, F_1) \to H^0(X, F_2) \to H^0(X, F_3) \to H^1(X, F_1) \to \cdots$$

Proof. We build the Godement resolutions $F_i \to C^\bullet(X, F_i)$ functorially for each $i = 1, 2, 3$, and exactness of the C^n functors implies that $C^\bullet(X, F_1) \to C^\bullet(X, F_2) \to C^\bullet(X, F_3)$ is a short exact sequence *of chain complexes*. A standard diagram chase argument gives the desired long exact sequence on cohomology. □

When X is paracompact, sheaf cohomology agrees with Čech cohomology, so the previous result lets us reprove Proposition 2.29.

Remark 2.56. While $C^n(F)$ is acyclic, there is no reason to expect it to be injective. A modified construction works, however, to produce functorial injective resolutions:

(1) begin with a functorial injective embedding of abelian groups into injective ones
(2) modify the construction of $C^0(F)$ by applying this functor to each stalk
(3) take the inclusion, cokernel, iterate.

Details are in [Aut, 01DG].

2.5. **Fine sheaves.** Another source of acyclic sheaves comes from so-called *fine sheaves* on paracompact spaces.

Definition 2.57. Let F be a sheaf over a paracompact space X. We say that F is *fine* if, for every locally finite open covering $\{U_i\}$ of X, we have a system of homomorphisms

$$h_i\colon F \to F,$$

so that

(1) For every i, the support of h_i is closed in X and contained in U_i
(2) $\sum_i h_i$ is the identity (the sum makes sense because the cover is locally finite).

Example 2.58. The sheaf $C^0(-,\mathbb{C})$ of germs of complex-valued continuous functions over a paracompact space is fine.

Proof. Fix a locally finite open cover $\{U_i\}$, and pick a partition of unity $\{\phi_i\}$. Then we can define

$$\begin{aligned} h_i \colon C^0(-,\mathbb{C}) &\to C^0(-,\mathbb{C}) \\ f &\mapsto f \cdot \phi_i. \end{aligned}$$

It is clear the h_i's sum to the identity, and the closed subsets in the definition of fineness are given by their support. □

Example 2.59. If X is a differentiable manifold, then the sheaf of germs of differentiable functions $\mathbb{C}_{\mathfrak{d}}$ is fine.

Proof. The same proof works, aside from one subtlety – in order for h to be a map of sheaves, we can't just multiply by arbitrary ϕ_i, since this might break differentiability— we need the ϕ_i's themselves to be differentiable. Hence this hinges on the existence of a differentiable partition of unity, which was proven by de Rham (at least in the smooth setting? see [de 84, I.2]). □

Remark 2.60. If X is a complex manifold and $\mathcal{O}_X$ is the sheaf of holomorphic functions, it is not necessarily the case that $\mathcal{O}_X$ is fine. An analogous argument to the above doesn't work because there are no holomorphic partitions of unity (this would violate the maximum modulus principle, for instance). Additionally, if $\mathcal{O}_X$ was fine then $H^1(X,\mathcal{O}_X)$ would vanish by Theorem 2.62, but this isn't true for $X = \mathbb{C}\mathrm{P}^1$, for instance.

Proposition 2.61. Let F be fine over a paracompact Hausdorff space X, and let $Z \subseteq X$ any closed subspace. Define $\Gamma(Z,F)$ to be the colimit of sections defined in open neighborhoods around Z. Then the natural restriction map

$$\Gamma(X,F) \to \Gamma(Z,F)$$

is surjective, that is, every section defined around Z can be extended to a section on X.[10]

Proof. Pick any section $s \in \Gamma(Z,F)$, which by construction of the colimit is defined in some neighborhood $U \supseteq Z$. Pick an open cover $\{U_i\}$ of X, let $V_i = U \cap U_i$, and let s_i be the restriction of s to $U \cap U_i$. Finally let $V = X - Z$, so altogether we have an open cover $\{V\} \cup \{V_i\}_i$ of X. We may assume it is locally finite without loss of generality. Since F is fine, we may pick h_i's which vanish outside $V_i \cap Z$ for all i, and sum to the identity. We also have an h which vanishes outside some closed subset of V, we don't have any control over this. However the sum of all these h's and h_i's is the identity everywhere. Then we can take

$$\sigma = \sum h_i s_i.$$

Away from Z this is zero, and this is equal to s on Z. Hence $\sigma \mapsto s$ under the restriction $\Gamma(X,F) \to \Gamma(Z,F)$ and we are done. □

Theorem 2.62. Let F be a fine sheaf on a paracompact space X. Then F is acyclic.

Proof. It suffices to show that $\check{H}^q(\mathcal{U},F) = 0$ for all $q \geq 1$ and any locally finite open cover $\mathcal{U}$. We will define a *chain homotopy*

$$k^q \colon \check{C}^q(\mathcal{U},F) \to \check{C}^{q-1}(\mathcal{U},F)$$

[10]The terminology for sheaves satisfying this condition is that they are *soft* — in that language, this proposition states "fine sheaves are soft."

using our functions h_i provided by the fineness of the sheaf F. Fix some $f \in \check{C}^q(\mathcal{U}, F)$, then we want to define $(k^q f)(i_0, \ldots, i_{q-1})$. For an arbitrary $i \in I$, we define a section $t_f(i, i_0, \ldots, i_{q-1}) \in F(U_{i_0} \cap \cdots \cap U_{i_{q-1}})$ by

$$t_f(i, i_0, \ldots, i_{q-1})(x) = \begin{cases} h_i \cdot f(i, i_0, \ldots, i_{q-1}) & x \in U_i \cap U_{i_0} \cap \cdots \cap U_{i_{q-1}} \\ 0 & x \notin U_i \cap U_{i_0} \cap \cdots \cap U_{i_{q-1}}. \end{cases}$$

We then define

$$(k^q f)(i_0, \ldots, i_{q-1}) = \sum_{i \in I} t_f(i, i_0, \ldots, i_{q-1}).$$

This sum makes sense since $\mathcal{U}$ is locally finite. If $\delta^q \colon \check{C}^q(\mathcal{U}, F) \to \check{C}^{q+1}(\mathcal{U}, F)$ denotes the boundary operator, then we can verify (exercise) that, for $q \geq 1$ we get:

$$k^{q+1} \circ \delta^q + \delta^{q-1} \circ k^q = \mathrm{id} = \mathrm{id} - 0.$$

This gives us a chain homotopy between the identity and the zero map, implying that $\check{H}^q(\mathcal{U}, F) = 0$ for all $q \geq 1$. □

Example 2.63. On any paracompact space X, the sheaves $C^0(-, \mathbb{R})$ and $C^0(-, \mathbb{C})$ are acyclic.

2.6. **Leray covers.** When is a short exact sequence of sheaves actually a short exact sequence of presheaves? This occurs when it is a *sectionwise* short exact sequence. We claim that a stalkwise monomorphism induces a sectionwise monomorphism, so if $F_1 \to F_2 \to F_3$ is a short exact sequence of sheaves, then by looking at the long exact sequence on *sheaf cohomology*[11], we obtain

$$0 \to F_1(U) \to F_2(U) \to F_3(U) \to H^1(U, F_1) \to \cdots$$

hence failure of exactness as a sequence of presheaves over an open set U is measured by the vanishing of the sheaf cohomology group $H^1(U, F_1)$.

Now what about long exact sequences? Suppose for instance, we had an injective resolution of a sheaf F given by

$$0 \to F \to I^0 \xrightarrow{d^0} I^1 \to \cdots$$

For an open set U, can we see whether

$$0 \to F(U) \to I^0(U) \to I^1(U) \to I^2(U) \to \cdots \tag{2.64}$$

is long exact? Here is where we squint at Equation (2.64) and say that it looks an awful lot like the sequence used to compute the derived functors of $\Gamma(-, F)$, but now over the open subspace U instead of over X. Indeed it is, but only if we know that I^j is still injective (or at least acyclic) when considered as a sheaf on U. This wouldn't necessarily be true if we had used an acyclic resolution (a restriction of an acyclic resolution to an open subspace need not remain an acyclic resolution), but because we used an *injective* resolution, when we restrict to an open subspace it remains an injective resolution (exercise). In summary if $F \to I^\bullet$ is an injective resolution, then Equation (2.64) will be exact exactly if $F_{|U}$ is acyclic.

An interesting case emerges when $U = U_{i_0} \cap \cdots \cap U_{i_q}$ is the overlap of some open spaces coming from an open cover $\mathcal{U}$ of X:

Proposition 2.65. Let X be a space, $\mathcal{U}$ an open cover, and F a sheaf. Suppose that the sheaf cohomology satisfies $H^p(U_{i_0 \cdots i_q}, F_{|U_{i_0 \cdots i_q}}) = 0$ for all $p \geq 1$ and all finite intersections of elements in our cover. Then for any injective resolution of sheaves

$$F \to I^0 \to I^1 \to I^2 \to \cdots$$

[11] We don't know we get a long exact sequence on Čech cohomology for a short exact sequence of sheaves, only of presheaves.

the sequence of Čech cochains is long exact for every q:

$$0 \to \check{C}^q(\mathcal{U}, F) \to \check{C}^q(\mathcal{U}, I^0) \to \check{C}^q(\mathcal{U}, I^1) \to \check{C}^q(\mathcal{U}, I^2) \to \cdots \tag{2.66}$$

Proof. This follows from the formula $\check{C}^q(\mathcal{U}, G) = \prod_{i_0,\dots,i_q} G(U_{i_0\cdots i_q})$ for any sheaf G, together with the above discussion and the observation that an arbitrary product of long exact sequences of abelian groups is still long exact.[12] □

The sheaves F satisfying this nice property deserve a special name for being so very special.

Definition 2.67. Let X be a space and $\mathcal{U}$ an open cover. We say F is *Leray* for $\mathcal{U}$ if

$$H^p(U_{i_0\cdots i_q}, F_{|U_{i_0\cdots i_q}}) = 0$$

for all $p \geq 1$ and all $q \geq 0$.

We'll now see why this key property of Leray covers is so important – it tells us we can compute Čech cohomology along a Leray cover!

Theorem 2.68. Let $\mathcal{U}$ be an open cover of a space X, and suppose F is a Leray cover with respect to $\mathcal{U}$. Then we obtain an isomorphism

$$\check{H}^q(\mathcal{U}, F) \xrightarrow{\sim} H^q(X, F)$$

for all $q \geq 0$.

Proof. Take any injective resolution $F \to A^\bullet$. By functoriality of Čech cochains and naturality of differentials we get a commutative infinite grid (see [Bre97, p. 193]):

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma(X,F) \arrow[r] \arrow[d] & \Gamma(X,I^0) \arrow[r] \arrow[d] & \Gamma(X,I^1) \arrow[r] \arrow[d] & \Gamma(X,I^2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \check{C}^0(\mathcal{U},F) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},I^0) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},I^1) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},I^2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \check{C}^1(\mathcal{U},F) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},I^0) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},I^1) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},I^2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \check{C}^2(\mathcal{U},F) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},I^0) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},I^1) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},I^2) \arrow[r] \arrow[d] & \cdots \\
& \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The first row is not necessarily exact (it computes $H^*(X, F)$) nor is the first column necessarily exact (it computes $\check{H}^*(\mathcal{U}, F)$). However we claim all other rows and columns are exact. The columns are exact because they Čech cohomology in an injective sheaf, and we have seen this vanishes in Corollary 2.39. The rows are exact by the Leray condition – this was the reason for Equation (2.66).

We claim now that the cohomology of the first row is isomorphic to the cohomology of the first column! This can be done by taking an element in a kernel/image and chasing it diagonally through the grid (TODO add details). □

[12]This is axiom AB4* in [Gro57]. Arbitrary products are always left exact, and axiom AB4 says arbitrary coproducts preserve monomorphisms. The dual axioms AB4* which Grothendieck leaves as an exercise to us is that arbitrary products preserve epimorphisms, hence are right exact. We can also just see this by hand in the category of abelian groups.

The procedure of computing Čech cohomology is a bit hairy. For a fixed cover $\mathcal{U}$ it is a fairly easy process to compute $\check{H}^*(\mathcal{U}, F)$, however the process of refining the cover to produce a colimit is a bit unclear. A massive upshot of Theorem 2.68 is that, in the process of computing Čech cohomology over nice spaces, it gives you a way to tell when you've refined your cover *enough* to stop refining and call it a day!

Corollary 2.69. Let X be a paracompact Hausdorff space, and $F \in \mathrm{Ab}(X)$. If $\mathcal{U}$ is any open cover for which F is Leray, then the natural colimiting map

$$\check{H}^q(\mathcal{U}, F) \to \check{H}^q(X, F)$$

is an isomorphism for all q.

Proof. This is a combination of Theorem 2.68 and the comparison between Čech and sheaf cohomology on paracompact Hausdorff spaces we saw in Corollary 2.41. □

3. Comparison with singular cohomology

Theorem 3.1. If X is paracompact, Hausdorff, and locally contractible, and A is any abelian group, there is a natural isomorphism between singular cohomology and sheaf cohomology with constant coefficients

$$H^*_{\mathrm{sing}}(X, A) \xrightarrow{\sim} H^*(X, \underline{A}).$$

In particular as X is paracompact, these are both isomorphic to Čech cohomology $\check{H}^*(X, \underline{A})$ by Theorem 2.33.

Proof (sheafy version). Here we're loosely following [Bre97, III.1] and this note of Mustaţă.

Step 1: Singular cochain sheaves resolve the constant sheaf: For any $q \geq 0$, define a presheaf as

$$\begin{aligned} C^q_{\mathrm{sing}} \colon \mathrm{Open}(X)^{\mathrm{op}} &\to \mathrm{Ab} \\ U &\mapsto C^q_{\mathrm{sing}}(U, A), \end{aligned}$$

where we recall $C_q(U) = \bigoplus_{\Delta^q \xrightarrow{\sigma} U} \mathbb{Z}$ is the free abelian group on the set of singular q-chains, and by definition $C^q_{\mathrm{sing}}(U, A) = \mathrm{Hom}_{\mathrm{Ab}}(C_q(U), A) = \prod_{\sigma \colon \Delta^q \to U} A$ is the set of *singular cochains* on U. Since cochains form a chain complex, it is clear that we get a chain complex of presheaves of abelian groups for any F. Moreover there is a natural map of presheaves

$$\underline{A} \to C^0_{\mathrm{sing}}$$

sending any locally constant function $s \colon U \to A$ to the singular cochain which associates, to the point $u \in U$, the value $s(u) \in A$. So we get a sequence of presheaves

$$\underline{A} \to C^0_{\mathrm{sing}} \to C^1_{\mathrm{sing}} \to \cdots$$

Observe that C^q_{sing} is not a sheaf — this is because we can have simplices landing in U_i's which agree on overlaps, but do not glue to one big simplex (exercise: come up with an explicit example illustrating this). So we now sheafify everything in sight, and we get

$$\underline{A} \to (C^0_{\mathrm{sing}})^\sharp \to (C^1_{\mathrm{sing}})^\sharp \to \cdots \tag{3.2}$$

We first claim this is an exact sequence of sheaves (a resolution of the constant sheaf), so we have to verify exactness on stalks. Since X is locally contractible, there is a cofinal system of open neighborhoods $U \ni x$ which are contractible (Definition B.22). On such a contractible neighborhood, we observe that $\underline{A}(U) = A$ (since U is connected), and we get that all the singular cohomology vanishes, i.e. $H^i_{\mathrm{sing}}(U, A) = 0$ for $i > 0$. In other words, the sequence

$$0 \to \underline{A}(U) \to C^0_{\mathrm{sing}}(U) \to C^1_{\mathrm{sing}}(U) \to \cdots$$

is exact. Hence the stalks are exact and we have that Equation (3.2) is a resolution of sheaves.

Step 2: Singular cochain sheaves are acyclic: The next claim to verify is that these $(C^q_{\text{sing}})^\sharp$ sheaves are acyclic. We'll do this by reference to Theorem 2.28 — since our base space is paracompact it suffices to show the *pre*sheaves C^q_{sing} are acyclic in Čech cohomology. Observe that the Čech cochains for this presheaf on an open cover $\mathcal{U} = \{U_i\}$ look like:

$$\check{C}^q(\mathcal{U}, C^q_{\text{sing}}) = \prod_{i_0,\dots,i_q} C^q_{\text{sing}}(U_{i_0\cdots i_q}) = \prod_{i_0\cdots i_q} \prod_{\Delta^p \xrightarrow{\sigma} U_{i_0\cdots i_q}} A$$

The trick is to rearrange this – if $\sigma\colon \Delta^p \to X$ is any simplex, we'll denote by $I_\sigma = \{i \in I \mid \sigma(\Delta^p) \cap U_i \neq \varnothing\}$, this is the indexing set of the collection of open sets in our cover that a given simplex σ hits. This is because, unless $i_0, \dots, i_q \in I$, then σ does not contribute a copy of A to $C^q_{\text{sing}}(U_{i_0\cdots i_q})$. Letting $\check{C}^q(I_\sigma; A)$ denote the group $\prod_{i_0,\dots,i_q \in I_\sigma} A$, we can rearrange the above to equal

$$\check{C}^q(\mathcal{U}, C^q_{\text{sing}}) = \prod_\sigma \check{C}^q(I_\sigma, A).$$

So acyclicity of C^q_{sing} reduces to checking that the sequence

$$\cdots \to \check{C}^{q-1}(I_\sigma, A) \to \check{C}^q(I_\sigma, A) \to \check{C}^{q+1}(I_\sigma, A) \to \cdots \tag{3.3}$$

is exact (again, because products of long exact sequences of abelian groups are long exact). We do this by constructing an explicit chain homotopy — we've fixed our σ, and let's fix a preferred element which we'll call $j \in I_\sigma$ (it doesn't matter what it is). Then we define a map

$$h\colon \check{C}^q(I_\sigma, A) \to \check{C}^{q-1}(I_\sigma, A)$$

by defining $h(c)(i_0, \dots, i_{q-1}) := c(j, i_0, \dots, i_{q-1})$. We claim that $d^{q-1} \circ h + h \circ d^q = \text{id}$, that is, h exhibits the identity map on Equation (3.3) as null-homotopic. This is because, for any $c \in \check{C}^{q-1}(I_\sigma, A)$, we have

$$(d^{q-1}h)(c)(i_0, \dots, i_q) = \sum_{k=0}^q (-1)^k (hc)(i_0, \dots, \widehat{i_k}, \dots, i_q) = \sum_{k=0}^q (-1)^k c(j, i_0, \dots, \widehat{i_k}),$$

and

$$(hd^q)(c)(i_0, \dots, i_q) = (d^q c)(j, i_0, \dots, i_q) = c(i_0, \dots, i_q) + \sum_{k=1}^q (-1)^{k+1} c(j, i_0, \dots, \widehat{i_k}, \dots, i_q).$$

Summing these two terms, everything cancels except $c(i_0, \dots, i_q)$, so $d^{q-1} \circ h + h \circ d^q = \text{id}$, hence Equation (3.3) is exact, and therefore the C^q_{sing} presheaves are acyclic. No aspect of this used paracompactness of X! However we use paracompactness to conclude that the *sheaves* $(C^q_{\text{sing}})^\sharp$ are acyclic, as desired.

Step 2.5: Pause, what do we have so far: The previous two steps tell us that the constant sheaf cohomology $H^q(X, \underline{A})$ can be computed as derived functors of global sections along the acyclic resolution given by the sheafification of the singular cochain sheaves — explicitly, as the cohomology of the following chain complex:

$$0 \to \Gamma(X, \underline{A}) \to (C^0_{\text{sing}})^\sharp(X) \to (C^1_{\text{sing}})^\sharp(X) \to \cdots$$

Note that we have a natural morphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \underline{A}) \arrow[r] \arrow[d] & (C^0_{\text{sing}})(X) \arrow[r] \arrow[d] & C^1_{\text{sing}}(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma(X, \underline{A}) \arrow[r] & (C^0_{\text{sing}})^\sharp(X) \arrow[r] & (C^1_{\text{sing}})^\sharp(X) \arrow[r] & \cdots
\end{tikzcd}$$

This induces a map between their cohomology, and by what we've seen above, this gives us a map $H^p_{\mathrm{sing}}(X, A) \to H^p(X, \underline{A})$. We're aiming to argue it is an isomorphism by showing that the vertical maps induce a quasi-isomorphism.

Step 3: Cochains to sheafified cochains is levelwise surjective: Suppose we have some $q \geq 0$ and an element $c \in (C^q_{\mathrm{sing}})^\sharp(X)$. We want to lift it to $C^q_{\mathrm{sing}}(X)$. Fix an open cover $\mathcal{U}$. Since we have sheafified, we can repackage the data of c as the data of, for every $U_i \in \mathcal{U}$, a cochain $c_i \in C^q_{\mathrm{sing}}(U_i, A)$, so that the cochains c_i and c_j agree on the overlap. Surjectivity then reduces to checking the existence of a global element $\chi \in C^q_{\mathrm{sing}}(X, A)$ which restricts to c_i on each U_i. Recall what a q-cochain is – it *tells you* a particular value in A for every q-simplex in its domain. The c_i's give us a rule of how to give a value of A for every q-simplex landing in U_i for each i, and these rules agree for simplices on the overlaps, so now we have to assign a value for every simplex landing in X. The trick is to use barycentric subdivision to chop up an arbitrary simplex in X into much smaller simplices, each of which has the property that it either lands completely in a U_i or is disjoint from it. This can't be accomplished on arbitrary spaces, but it can on paracompact Hausdorff ones, since they admit locally finite covers.

Step 4: Cochains to sheafified cochains has null-homologous kernel: We now have a map of chain complexes $C^\bullet_{\mathrm{sing}}(X) \to (C^\bullet_{\mathrm{sing}})^\sharp(X)$ which is levelwise surjective. We want to show that its kernel is null-homologous. We claim that this also follows from barycentric subdivision. In particular we can describe, at each level q, the kernel of $C^q_{\mathrm{sing}}(X) \to (C^q_{\mathrm{sing}})^\sharp(X)$ as those q-cochains c satisfying the following condition: there exists some open cover $\mathcal{U}$ so that, if σ has image inside any one (maybe more) of the U_i's, then $c(\sigma) = 0$. However it is a standard result in barycentric subdivision to see that the complex of such things is null-homologous (see [Hat01, 2.21]). ☐

If we limit our attention to sufficiently nice spaces, e.g. differentiable manifolds, we have an alternative proof of Theorem 3.1 via homotopy theory.

Proof sketch (nerve lemma). Suppose X is paracompact and it admits a *good cover* $\mathcal{U}$ (when X is a Riemannian manifold, we proved this existence in Proposition A.12). The nerve lemma [**?**, §5] (see also [Hat01, 4G.3]) says that $|N(\mathcal{U})| \simeq X$, that is, the geometric realization of the nerve of this cover recovers the homotopy type of the space X. Since singular cohomology is a homotopy invariant, we can replace $H^*_{\mathrm{sing}}(X, A)$ with $H^*_{\mathrm{sing}}(|N(\mathcal{U})|, A)$ without loss of generality. One can prove that the simplicial cochains complex on $N(\mathcal{U})$ is canonically identified with the Čech cochains complex for $\mathcal{U}$. We can compose this isomorphism with the natural map comparing simplicial and singular cochains:

$$\check{C}^*(\mathcal{U}, \underline{A}) \cong C^\bullet_\Delta(N(\mathcal{U}), A) \to C^\bullet_{\mathrm{sing}}(|N(\mathcal{U})|, A).$$

The latter map is a quasi-isomorphism by the standard comparison theorem between simplicial and singular cohomology (see for instance [Hat01, §2.1]).

Thus for any good cover $\mathcal{U}$, we have that Čech cohomology of the cover agrees with singular cohomology. We use cofinality of good covers (Proposition A.10) to conclude. ☐

Remark 3.4. It turns out by a recent theorem of Sella you can weaken this and drop the paracompact+Hausdorff condition, and only assume X is semi-locally contractible [Sel16].

Remark 3.5. When X is a manifold and $A = \mathbb{R}$, we'll provide a different proof of this fact, via the *de Rham theorem* (**??**??).

4. Vector bundles and sheaves of sections

Definition 4.1. A *real vector bundle* $\pi\colon E \to X$ is a map of spaces so that

- ▷ π is a surjection
- ▷ $\pi^{-1}(x)$ has the structure of a real vector space for every $x \in X$

▷ for every $x \in X$ there exists an open neighborhood $U \ni x$, an integer k, and a homeomorphism ϕ fitting into a commutative diagram

$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[rr, "\phi"] \arrow[dr] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U, & \end{tikzcd}$$

so that $\phi(x,-)\colon \mathbb{R}^k \xrightarrow{\sim} \pi^{-1}(x)$ is an isomorphism of real vector spaces for each $x \in U$.

A *complex* vector space is the same definition, but we replace $\mathbb{R}$ with $\mathbb{C}$ above.

Example 4.2. On any manifold, we have a *tangent bundle*

$$TM \to M,$$

which is a vector bundle of rank equal to the dimension of M. A section of the bundle is a vector field on M.

Notation 4.3. If $E \to X$ is a vector bundle and $U \subseteq X$ is some open subset, we denote by $\mathrm{Sec}(U, E)$ the set of continuous sections:

$$\mathrm{Sec}(U,E) := \left\{ s\colon U \to E_{|U} : \pi \circ s = \mathrm{id}_U \right\}.$$

Note that $\mathrm{Sec}(U,E)$ is an $\mathbb{R}$-module. In fact, it is a $C(U,\mathbb{R})$-module.

Proposition 4.4. Let X be a space, and $E \to X$ a vector bundle. Then the functor

$$\mathrm{Sec}(-,E)\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}$$

is a sheaf.

Proof. Sections satisfy locality and gluing. □

Remark 4.5. We can form new vector bundles out of old ones by doing linear algebra operations "fiberwise." That is, we can make sense of direct sums, tensor products, duals, exterior powers, symmetric powers, etc. of existing vector bundles.

Definition 4.6. Let M be a smooth n-manifold. Then we denote by

$$\Omega^k_{\mathrm{dR}} = \mathrm{Sec}(-, \wedge^k T^*M)$$

the sheaf of sections of the kth exterior power of the cotangent bundle.

Example 4.7. We have that Ω^0_{dR} is the sheaf of real-valued differentiable functions on M.

Differential forms are expressible in local coordinates, where they admit a nicer shape. Let's see some examples:

Example 4.8. On $M = \mathbb{R}^3$, an element of $\Gamma(\mathbb{R}^3, \Omega^1_{\mathrm{dR}})$ is expressible as

$$f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz,$$

where the f_i's are differentiable real-valued functions, dx is the section $\mathbb{R}^3 \to T^*\mathbb{R}^3$ which projects a tangent vector to its x-component, and similarly for dy and dz. In multivariable calculus we often rewrite these as *vector fields*

$$F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{\jmath}.$$

Example 4.9. Again on $M = \mathbb{R}^3$, an element of $\Gamma(\mathbb{R}^3, \Omega^2_{\mathrm{dR}})$ is something of the form

$$f_1(x,y,z)dx \wedge dy + f_2(x,y,z)dz \wedge dx + f_3(x,y,z)dy \wedge dx,$$

where again the f_i's are differentiable and real-valued, and $dx \wedge dy \in \Gamma(\mathbb{R}^3, \wedge^2 T^*M)$ is the section which takes an ordered pair of tangent vectors, projects them onto the xy-plane, and spits out the signed volume of the parallelogram they span. In general we make a choice (this is secretly the Hodge star operator which we'll learn later) to identify

$$\begin{aligned} dx \wedge dy &\leftrightarrow \hat{k} \\ dy \wedge dz &\leftrightarrow \hat{i} \\ dz \wedge dx &\leftrightarrow \hat{j}, \end{aligned}$$

and rewrite a differential 2-form as a vector field.

Definition 4.10. In general we define the *exterior derivative* of a differential k-form

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

as

$$d\omega = \sum_{i_1 < \cdots < i_k} df_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Exercise 4.11. Let M be a smooth manifold. Show that the exterior derivative in Definition 4.10 defines a morphism of abelian sheaves.

Terminology 4.12.

- ▷ A form ω is said to be *closed* if $d\omega = 0$.
- ▷ A form ω is said to be *exact* if it is of the form $\omega = d\alpha$ for some α. In this case, α is said to be a *primitive* of ω.

4.1. The Poincaré lemma.

Appendix A. Category theory

If you are unfamiliar with colimits I really recommend reading Chapter 3 of [Rie16].

A.1. Filtered colimits.

Definition A.1. A category is *filtered* if every finite diagram admits a cocone. Dually a category $\mathscr{C}$ is *cofiltered* if every finite diagram admits a cone.

Example A.2. The category $\mathrm{Open}(X)$ is cofiltered.

Proof. We don't have to worry about morphisms commuting, so it suffices to see, for finitely many $U_1, \ldots, U_n \in \mathrm{Open}(X)$, there is a U mapping to them. Clearly we can take $U = U_1 \cap \cdots \cap U_n$. □

Example A.3. For $x \in X$, the category of open subspaces of X containing x is cofiltered.

Example A.4. For a space X, the category $\mathrm{Cov}(X)$ of open covers under refinement (defined in **??**) is cofiltered. This is precisely the assertion that two covers admit a common refinement, and therefore by induction finitely many covers admit a common refinement.

The important thing about filtered categories is that colimits computed over them are very well behaved.

Proposition A.5. Filtered colimits valued in Set or Ab or any other reasonable concrete category (see SE2143601 for details for instance) commute with finite limits.

Corollary A.6. Filtered colimits (valued in Ab, let's say) preserve monomorphisms.

Proof. The property of a morphism $f\colon x \to y$ being a monomorphism is equivalent to the statement that the diagram

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

is a pullback. This is a finite limit, hence commutes with filtered colimits. □

An interesting question in the setting of abelian categories is the interaction between exactness and filtered colimits. Indeed exactness of filtered colimits is a consequence of Grothendieck's axiom AB5 for abelian categories [Gro57, 1.8.1]. We will leverage the following corollary a handful of times:

Corollary A.7. Let $\mathcal{A}$ be an abelian category, and let $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathrm{Ab})$ be the full subcategory of exact functors from $\mathcal{A}$ to abelian groups. Then $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathrm{Ab})$ is preserved under filtered colimits.

Proof. Filtered colimits in this category are computed levelwise, hence they commute with finite limits. Being right exact is a colimit, hence commutes with (filtered) colimits in any context. Being left exact is a pullback condition, hence a finite limit, and therefore commutes with filtered colimits. □

As a remark, this is not true if the colimits are not filtered, as illustrated by the following example:

Example A.8. Let $F = \mathrm{id}_{\mathrm{Ab}}\colon \mathrm{Ab} \to \mathrm{Ab}$ be the (clearly exact) identity functor on the category of abelian groups, and consider the (non-filtered!) equalizer diagram

$$0, 2\colon F \rightrightarrows F,$$

where 0 is the natural transformation given levelwise by the zero homomorphism, and 2 is multiplication by 2. Then we have that the colimit of this diagram is computed by the pointwise coequalizer of 0 and 2. In particular if $C\colon \mathrm{Ab} \to \mathrm{Ab}$ is the colimit of the above diagram, then

$$C(A) = A/2A$$

for every abelian group A. But this is not an exact functor $\mathrm{Ab} \to \mathrm{Ab}$.

A.2. **Cofinality.**

Definition A.9. Let J be a category, and $I \subseteq J$ a subcategory. We say $I \subset J$ is *cofinal* if, for any functor $F\colon J \to \mathscr{C}$, the induced map on colimits

$$\mathrm{colim}_I F \to \mathrm{colim}_J F$$

is an isomorphism in $\mathscr{C}$. We say $I \subseteq J$ is *final* if $I^{\mathrm{op}} \subseteq J^{\mathrm{op}}$ is cofinal.

The key reductive step of the Poincaré lemma relies on a recognition of cofinality (really finality, but the category gets "opped" when we take a contravariant functor out of it):

Proposition A.10. *Let M be an n-dimensional manifold and let $x \in M$ be an arbitrary point. Then the category of open sets of M containing x admits a final system given by those open sets $U \subseteq M$ so that $U \ni x$ and U is homeomorphic to $\mathbb{R}^n$.*

Proof. Fix a chart (U_0, ϕ) around x. For any $V \ni x$ open, we can intersect it with the chart to get $V \cap U_0$. In the image $\phi(V \cap U_0)$ we can find some open ball containing $\phi(x)$, and the preimage of this in M is both homeomorphic to $\mathbb{R}^n$ and contained in V. □

The following result is a categorical rephrasing of an important result in differential topology. Before we can phrase it we need a definition.

Definition A.11. Let $\mathcal{U}$ be an open cover of X. We say that it is a *good cover* if all the $U \in \mathcal{U}$ and all the nonempty intersections of finitely many elements in $\mathcal{U}$ are contractible spaces.

Proposition A.12. *Let M be a Riemannian manifold, and denote by $\mathrm{Cov}^{\mathrm{good}}(M)$ the category of good open covers of M. Then $\mathrm{Cov}^{\mathrm{good}}(M) \subseteq \mathrm{Cov}(M)$ is a final subcategory.*

Proof. The content of this is showing every cover refines to a good one. Fix a cover $\mathcal{U}$ of M, and now pick any point $x \in M$, and any $U \ni x$. Since every point on a Riemannian manifold admits a positive convexity radius, we can obtain a geodesically convex neighborhood around each point, and by potentially shrinking the radius we may assume it is contained in U. This is clearly convex, and it yields a good cover since the intersection of geodesically convex spaces is geodesically convex, hence contractible. □

In particular, any colimiting construction we want to pursue over covers of a Riemannian manifold can be reduced to good open covers.

A wild fun fact is that the poset of open sets and refinement for a good cover can be used to recover the homotopy type of the manifold (or space) you started out with! This is the content of the *nerve lemma*.

Appendix B. Point-set topology

Definition B.1. Let $\{U_i\}_{i \in I}$ be an open cover of a space X. We say it is

- ▷ *point finite* if each $x \in X$ is contained in only finitely many U_i
- ▷ *locally finite* if each $x \in X$ has an open neighborhood $V \ni x$ so that V meets only finitely many U_i.

Clearly a locally finite open cover is point finite, but the converse need not hold (see [Mun00, p. 244] for an explicit example).

We want to rephrase compactness in terms of *refining* the cover, not just throwing out elements of the cover. To that end, it's nicer to work with a different definition. We prove a quick proposition:

Proposition B.2. *Let X be a topological space. Then it is compact if, for every $\mathcal{U} \in \mathrm{Cov}(X)$, there exists a refinement $\mathcal{V} \leq \mathcal{U}$ for which $\mathcal{V}$ is finite (has finitely many elements).*

Proof. If X is compact (in the ordinary sense), then we can just pick $\mathcal{V}$ to be the finite subcollection of $\mathcal{U}$ guaranteed by compactness. For the reverse direction, suppose we have an open cover $\mathcal{U} = \{U_i\}_{i\in I}$, and a finite refinement $\mathcal{V} = \{V_1, \ldots, V_n\}$. Then by the definition of refinement, there is some function $f\colon \{1, 2, \ldots, n\} \to I$ so that $V_i \subseteq U_{f(i)}$ for each $1 \leq i \leq n$. Then it is clear that $\left\{U_{f(1)}, \ldots, U_{f(n)}\right\}$ is a finite open cover of X, which is a subcover of our original cover $\mathcal{U}$. □

As Munkres says, this definition is "awkward... but it suggests a way to generalize."

Definition B.3. Let X be a space. We say it is...

$$\begin{cases} \textit{compact} \\ \textit{paracompact} \\ \textit{point finite/metacompact} \end{cases} \quad \text{if every open cover } \mathcal{U} \text{ admits a refinement } \mathcal{V} \text{ which is } \begin{cases} \text{finite} \\ \text{locally finite} \\ \text{point finite} \end{cases}$$

It is clear that we have the following implications:

$$\text{compact} \Rightarrow \text{paracompact} \Rightarrow \text{point finite}.$$

The converse implications don't hold.

Example B.4. Euclidean space $\mathbb{R}^n$ is paracompact but not compact.

Example B.5 (Point-finite spaces need not be paracompact)**.** The first example was, I believe, [Bin51, Example B]. Another good example is [SS78, Example 89].

Remark B.6. Hirzebruch, Bourbaki, and many other authors assume the spaces receiving the definitions in Definition B.3 are Hausdorff. We will try to be clear when the Hausdorff condition is needed.

Theorem B.7 (Dieudonné, [Hir78, 2.8.2])**.** Every locally compact[13] Hausdorff space with a countable basis is paracompact.

B.1. **Normal spaces.** The idea of a *normal* space can be thought of (loosely) as a strengthening of the Hausdorff condition. On T_1-spaces (spaces in which singletons are closed) this is a literal strengthening of the Hausdorff condition – it says that not only can *points* be separated by open neighborhoods, actually any disjoint *closed subspaces* can be separated by open neighborhoods.

Definition B.8. A space is *normal* if any two disjoint closed subsets can be separated by disjoint open neighborhoods. Explicitly if $Z_1, Z_2 \subseteq X$ are each closed and $Z_1 \cap Z_2 = \varnothing$, then there exist open subspaces $U_1 \supseteq Z_1$ and $U_2 \supseteq Z_2$ so that $U_1 \cap U_2 = \varnothing$.

Theorem B.9 (Dieudonné, [Hir78, 2.8.1])**.** Every paracompact space Hausdorff is *normal*.

In particular since our manifolds are Hausdorff with countable bases, they are in particular normal.

Lemma B.10 (Urysohn's lemma)**.** A space X is normal if and only if, for every two disjoint closed subsets $A, B \subseteq X$, there exists a continuous function

$$f\colon X \to [0, 1]$$

so that $f(A) = 0$ and $f(B) = 1$.

The following crucial result implies that, over a normal space, locally finite covers can always be "shrunk." It is a crucial ingredient for the existence of partitions of unity.

Theorem B.11 (Shrinking theorem, [Hir78, 2.8.3])**.** Let X be normal, and $\{U_i\}_{i\in I}$ a point finite open cover. Then X admits an open cover $\{V_i\}_{i\in I}$, indexed over the same set, so that $\overline{V_i} \subseteq U_i$.

[13] *Locally compact* means for every $x \in X$ there exists some compact subspace $K \subseteq X$ and an open neighborhood $U \ni x$ so that $U \subseteq K$.

Definition B.12. Let $\{U_i\}_{i\in I}$ be an open cover of a space X. We say that a system of continuous functions $\{\phi_i\colon X\to\mathbb{R}\}_{i\in I}$ is a *partition of unity subordinate to* the cover if:

(1) $\phi_i(x)\geq 0$ for all $x\in X$
(2) $\operatorname{supp}(\phi_i)\subseteq U_i$
(3) Each point has an open neighborhood meeting $\operatorname{supp}(\phi_i)$ for only finitely many $i\in I$
(4) We have that

$$\sum_{i\in I}\phi_i(x)=x$$

for all $x\in X$. Note that sum is defined because of the previous point.

Definition B.13. We say an open cover of a space is *numerable* if it admits a subordinate partition of unity. We say a fiber bundle is *numerable* if the base admits a numerable cover over which the total space is trivialized [Dol63, 7.1].

Theorem B.14. A space is paracompact if and only if it is Hausdorff and every open cover admits a partition of unity.

Proof. For a proof see [Hir78, pp. 30–31]. □

The following result is originally due to Stone? An elegant reproof is due to Mary Ellen Rudin in the 60's. We can find the following in [Mun00, 41.4], for instance.

Theorem B.15. Every metric space is paracompact.

Another crucial example is that CW complexes are paracompact. Miyazaki, Dugundji, Bourgin all separately proved in 1952 that simplicial complexes were paracompact. Miyazaki shortly thereafter extended this result to all CW complexes

Theorem B.16 ([Miy52])**.** Every CW complex is paracompact.

Sketch. There's a nice outline on the nLab – we can induct on the skeleton, leveraging that disks and spheres are paracompact, and that paracompact Hausdorff spaces are closed under coproducts and pushouts along closed embeddings. □

B.2. **Properties of open covers and neighborhoods.** We say a *property* $\mathbf{P}$ is any boolean-valued assignment on objects of a category.

Notation B.17. We let

- ▷ $\mathrm{Open}(X)$ be the category of open sets in X, and $\mathrm{Open}^{\mathbf{P}}(X)$ the subcategory of those opens which satisfy $\mathbf{P}$
- ▷ $\mathrm{Open}_x(X)$ the category of open sets in X containing a fixed point $x \in X$, and $\mathrm{Open}_x^{\mathbf{P}}(X)$ the subcategory of those opens containing x and satisfying $\mathbf{P}$
- ▷ $\mathrm{Cov}(X)$ the category of open covers of X and $\mathrm{Cov}^{\mathbf{P}}(X)$ the category of covers comprised of elements satisfying $\mathbf{P}$. Explicitly:

$$\mathrm{Cov}^{\mathbf{P}}(X) = \left\{ \mathcal{U} \in \mathrm{Cov}(X) : U \in \mathcal{U} \Rightarrow U \in \mathrm{Open}^{\mathbf{P}}(X) \right\}.$$

Terminology B.18. A *local basis* of a space X at a point x is any final subcategory of $\mathrm{Open}_x(X)$.

A core idea in the theory of open sets is the following comparison between local bases of opens satisfying a condition and covers built out of opens satisfying that same condition.

Proposition B.19. Let $\mathbf{P}$ be a property of the open subspaces in $\mathrm{Open}(X)$, and assume that X is a T1 space. Then the following are equivalent:

(1) for every $x \in X$, the inclusion $\mathrm{Open}_x^{\mathbf{P}}(X) \to \mathrm{Open}_x(X)$ is final
(2) the inclusion $\mathrm{Cov}^{\mathbf{P}}(X) \to \mathrm{Cov}(X)$ is final

Proof. For the forward direction, pick an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X. For each $x \in X$, it is contained in $U_i \in \mathcal{U}$ for some i. By property (1) we can find some V_x containing x for which $V_x \subseteq U_i$ and V_x satisfies $\mathbf{P}$. Then $\mathcal{V} = \{V_x\}_{x \in X}$ is a cover in $\mathrm{Cov}^{\mathbf{P}}(X)$ which refines $\mathcal{U}$.

Conversely if $x \in X$ and we have an open neighborhood $U \ni x$. We can complete this to a cover $\mathcal{U} = \{U, X - \{x\}\}$ since X is a T1 space.[14] Then by hypothesis there is a refinement $\mathcal{V} \leq \mathcal{U}$ which is in $\mathrm{Cov}^{\mathbf{P}}(X)$. In particular $x \in V \in \mathcal{V}$ for some V, and since $V \notin X \smallsetminus \{x\}$, we must have that $V \subseteq U$. □

Definition B.20. Let X be a T1 space and let $\mathbf{P}$ be the property that a space is connected. Then we say X is *locally connected* if either of the conditions in Proposition B.19 hold.

Remark B.21. The condition of being an n-manifold can be rephrased in this language. If X is second countable and Hausdorff, and $\mathbf{P}$ is the property of "being homeomorphic to $\mathbb{R}^n$," then we say X is an n-manifold if it satisfies the conditions of Proposition B.19. This is essentially our discussion in Proposition A.10.

Definition B.22. Let X be a T1 space and $\mathbf{P}$ the property of being contractible (homotopy equivalent to a one-point space). We say X is *locally contractible* if either of the conditions in Proposition B.19 hold.

What Spanier calls *locally contractible* is something different. We, following Sella, will call that condition *semi-locally contractible.*

Definition B.23. Consider the property $\mathbf{P}$ which is true on an open subset $U \subseteq X$ if and only if the inclusion map $U \hookrightarrow X$ is homotopic to a constant map. We say a space X is *semi-locally contractible* if, for every $U \subseteq X$, the poset $\mathrm{Cov}^{\mathbf{P}}(U)$ is nonempty.

Remark B.24. On nLab (at the time of writing), their definition of semi-locally contractible is that X admits a basis of open sets whose inclusion maps are homotopic to a constant map. This is different than the definition above.

Example B.25. Let X be the *infinite earring.* Then the cone on X, defined as $CX = X \times [0,1]/X \times \{1\}$ is contractible but not semi-locally contractible.

[14] More generally, we start with $U \ni x$ and complete it to a cover for which U is the only open set in the cover containing our point. I think this can only be done for T1 spaces.

Appendix C. Normalized versus unnormalized cochains

In this section of the appendix we cop to a lie by omission (or by obfuscation?). Our definition of the Čech cochains complex for a cover $\mathcal{U} = \{U_i\}_{i\in I}$ was

$$\check{C}^q(\mathcal{U}, F) = \prod_{i_0,\dots,i_q\in I} F(U_{i_0}\cap\cdots\cap U_{i_q})$$

However this notation is a bit ambiguous — are we allowing the $i_0, i_1, \dots, i_q$ to be the same? If so we're allowing the $(q+1)$-fold intersection of the same open set - is this what we want? In examples we did, it seems like we weren't assuming this. We said "ah there are no 3fold overlaps so there are no Čech 2-cochains" but should we really have been allowing for these degenerate intersections where the same open set in our cover is repeated?

The answer is that *it doesn't matter*, they both compute the same cohomology. We're going to sketch how one proves this. We call the one where we don't allow for repeated indices the *normalized* Čech cochains, and the one where we allow for repeated indices the *unnormalized* Čech cochains (definitions to follow).

With that being said, some computations are *much easier* in one setting or the other. When we did explicit computations like Example 2.13 and Example 2.20 it was beneficial to use normalized cochains, since we could conclude there are no higher cochains since there are no higher overlaps. However in the proofs of Proposition 2.51 and Theorem 3.1 it was more beneficial to work with unnormalized cochains since we could construct explicit chain homotopies without worrying about repeated indices.

In this section we'll make the comparison between normalized and unnormalized cochains explicit. We need to define some local (not used outside this section) notation in order to streamline everything:

Definition C.1. Let X be a space, $\mathcal{U} = \{U_i\}_{i\in I}$ some open cover, and F any presheaf on X. Dust off your nearby copy of the axiom of choice and use it to fix a total ordering $\leq$ on the set I. We define

- ▷ the *normalized Čech cochains* to be
$$\check{C}^q_{\mathrm{nor}}(\mathcal{U}, F) := \prod_{i_0<i_1<\cdots<i_q} F(U_{i_0}\cap\cdots\cap U_{i_q}).$$
- ▷ the *unnormalized Čech cochains* to be
$$\check{C}^q_{\mathrm{unnor}}(\mathcal{U}, F) := \prod_{i_0\leq i_1\leq\cdots\leq i_q} F(U_{i_0}\cap\cdots\cap U_{i_q}).$$

In each cases, the differentials are defined the same way, however in the unnormalized cochains, differentials can be built out of identities (when we're omitting an index that was duplicated) while in the normalized cochains this won't occur.

The core result we want to prove is the following:

Theorem C.2. There is a natural map of chain complexes

$$\check{C}^\bullet_{\mathrm{nor}}(\mathcal{U}, F) \to \check{C}^\bullet_{\mathrm{unnor}}(\mathcal{U}, F)$$

and it is a quasi-isomorphism (meaning it induces an isomorphism on cohomology).

Proof sketch. It is clear that normalized cochains sit inside unnormalized cochains as a subcomplex, since at each level q we have a subgroup inclusion $\check{C}^q_{\mathrm{nor}}(\mathcal{U}, F) \subseteq \check{C}^q_{\mathrm{unnor}}(\mathcal{U}, F)$. In fact we will show that this inclusion splits. We say that a cochain $c \in \check{C}^q_{\mathrm{unnor}}(\mathcal{U}, F)$ is *normalized* if it lies in the image of this natural map, or equivalently, it vanishes on any tuple with repeated indices (i.e. one which isn't strict).

For $0 \leq k \leq q-1$, we define an endomorphism of unnormalized cochains, essentially detecting how sensitive a cochain is to doubling up an index in the kth slot. Explicitly:

$$\begin{aligned} P_k \colon \check{C}^q_{\mathrm{unnor}}(\mathcal{U}, F) &\to \check{C}^q_{\mathrm{unnor}}(\mathcal{U}, F) \\ c(i_0 \leq \cdots \leq i_q) &\mapsto c(i_0 \leq \cdots \leq i_q) - c(i_0 \leq \cdots \leq i_k \leq i_k \leq i_{k+2} \leq \cdots \leq i_q). \end{aligned}$$

That is, we output the difference of c and c on the tuple where we have a repeated overlap at the kth spot. Note that if c was normalized, then we have that P_k acts as the identity for each k.

We now define

$$\mathcal{P}_q := P_{q-1} \circ P_{q-2} \circ \cdots \circ P_0 \colon \check{C}^q_{\mathrm{unnor}}(\mathcal{U}, F) \to \check{C}^q_{\mathrm{unnor}}(\mathcal{U}, F).$$

We have a few claims about these maps $\mathcal{P}_q$:

(1) $\mathcal{P}_q$ takes values in *normalized* cochains
(2) $\mathcal{P}_q \circ \mathcal{P}_q = \mathcal{P}_q$, that is, it is idempotent.
(3) $\{\mathcal{P}_q\}_{q \geq 0}$ is a chain map (it commutes with the differentials)

To sketch the first point – we can see that a cochain c is normalized if and only $P_k(c) = c$ for all k. Hence it suffices to verify that $P_k \circ \mathcal{P}_q = \mathcal{P}_q$ for any $0 \leq k \leq q-1$, which is immediate by definition. Once we show this, idempotent-ness is immediate. Checking it is a chain map is just some painful index-bashing, but trust me that it's true.

Hence with all our claims in hand, we get a splitting of chain complexes

$$\begin{tikzcd} \check{C}^\bullet_{\mathrm{nor}}(\mathcal{U}, F) \arrow[r, hook] \arrow[dr, equal] & \check{C}^\bullet_{\mathrm{unnor}}(\mathcal{U}, F) \arrow[d, "\mathcal{P}_\bullet"] \\ & \check{C}^\bullet_{\mathrm{nor}}(\mathcal{U}, F). \end{tikzcd}$$

In particular the inclusion splits (it splits levelwise in fact). The remaining stuff is called the *degenerate* cochains complex. That is, our inclusion and projection exhibit a splitting of abelian groups

$$\check{C}^q_{\mathrm{unnor}} = \check{C}^q_{\mathrm{nor}} \oplus \check{D}^q.$$

Here $\check{D}^q$ is our notation for what is left over – it consists of those cochains which are supported only on $(q+1)$-tuples which are *not* strictly increasing (they have a repeated entry).

The last remaining thing to check is that $\check{D}^\bullet$ has no cohomology. To do this we exhibit an explicit chain homotopy. For $0 \leq k \leq q-1$, we define

$$s^k \colon \check{D}^q \to \check{D}^{q-1}$$

by the formula $(s^k c)(i_0, \ldots, i_{q-1}) = c(i_0, \ldots, i_k, i_k, i_{k+1}, \ldots, i_q)$. We then take h to be the map

$$h = \sum_{k=0}^{q-1} (-1)^k s^k \colon \check{D}^q \to \check{D}^{q-1}.$$

We can check that $dh + hd = \mathrm{id}$ on $\check{D}^\bullet$, hence h exhibits a null-homotopy of the identity map on the degenerate cochains. □

And obviously there's one last sanity check we should do, which is that the definitions above don't depend on the ordering we fixed on I. We skip this argument, and state the big corollary:

Corollary C.3. We can compute Čech cohomology with either normalized or unnormalized cochains and the answers we get in either case will be canonically isomorphic.

Remark C.4. This entire story and proof is *easier and generalizable* if we are willing to do a bit more category theory and work with cosimplicial objects. Indeed this is how it's normally proved. Let Δ denote the category with countably many objects $[0], [1], [2]$, etc., where the object $[n] \in \Delta$ is intended to denote the ordered set with $n+1$ elements $[n] = \{0, 1, \dots, n\}$. The morphisms $[m] \to [n]$ in Δ are weakly order-preserving maps between these sets. We say that a *cosimplicial object* in a category $\mathscr{C}$ is a functor

$$\Delta \to \mathscr{C}.$$

There are special morphisms in Δ which generate all the other morphisms. These are called *degeneracy* and *face* maps:

- ▷ *face maps*: for any $0 \le j \le n$ we have maps $\delta^j \colon [n-1] \to [n]$ uniquely defined as the injective order-preserving map which doesn't contain $j \in [n]$ in its image
- ▷ *degeneracy maps*: for any $0 \le j \le n$ we have maps $\sigma^j \colon [n+1] \to [n]$ uniquely defined as the surjective order-preserving map so that $\sigma^j(j) = \sigma^j(j+1) = j \in [n]$.

These generate all other morphisms in the sense that every morphism can be written as a composite of them [Aut, 0166]. They satisfy some natural identities, called the *simplicial identities*, see for instance [Aut, 0167].

Our Čech cochains are very naturally a cosimplicial object! Explicitly, they form a functor

$$\begin{aligned} \check{C}^{(-)}_{\text{unnor}}(\mathcal{U}, F) \colon \Delta &\to \text{Ab} \\ [q] &\mapsto \prod_{i_{(-)} \in \text{Hom}([q], (I, \le))} F(U_{i_0} \cap \cdots \cap U_{i_q}). \end{aligned}$$

Here the product is over weakly order-preserving functions $[q] \to I$. We could define the normalized cochains by only taking a product over strictly order-preserving maps, if we like.

The Čech cochains have a cosimplicial structure since we have all this extra available stuff we can do with them – we can repeat or skip indices. Moreover the differentials on the cochain complex arise very naturally from the cosimplicial structure. This is a standard thing: any time we have a cosimplicial abelian group we can change it into a cochain complex by taking the alternating sums of the degeneracy maps. If we unwind our definition of the Čech cochains differential in Equation (2.8) we can see that it is exactly built that way.

Once we have all this in mind, the proof above can be rewritten in cosimplicial language:

- ▷ given any cosimplicial abelian group $A^\bullet$ (not just the unnormalized cochains, but *any* one) there is a natural "normalization" of it, defined by taking things that lie in the kernels of all the degeneracy maps:

$$NA^q := \bigcap_{k=0}^{q-1} \ker(A^q \xrightarrow{s^k} A^{q-1}).$$

- ▷ in this language, there is again a natural inclusion $NA^q \subseteq A^q$, and it splits levelwise by a similar projection map:

$$\begin{aligned} P_k &:= (\text{id} - d^k s^{k-1}) \\ \mathcal{P}_q &:= (\text{id} - d^q s^{q-1}) \circ \cdots \circ (\text{id} - d^1 s^0). \end{aligned}$$

- ▷ from this perspective, our three claims (that $\mathcal{P}_q$ is normalized, i.e. that it satisfies $P_k \circ \mathcal{P}_q = \mathcal{P}_q$ for any k, that $\mathcal{P}_q$ is idempotent, and that $\mathcal{P}_q$ is a chain map) follow immediately from the identities of cosimplicial maps
- ▷ the splitting off of a degenerate subcomplex and exhibiting that it has no cohomology is also a standard result in cosimplicial algebra.

The story for simplicial abelian groups can be found in [Mac67, VIII.6] or [GJ99, III.2], and the cosimplicial story is dual.

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