MATH231BR HOMEWORK 2 (DUE THURSDAY, 2/12/26)

Exercise 1. If X is a connected, compact, complex manifold, argue that $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, where we recall $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions on X.

Exercise 2.

(1) Let $f\colon S_1 \to S_2$ be a morphism of sheaves over X. Give a reasonable definition of an *image* subsheaf $\mathrm{im}(f) \subseteq S_2$.
(2) Give a reasonable definition of a *cokernel* sheaf $\mathrm{coker}(f)$ over f.
(3) Prove an analogue of the first isomorphism theorem for sheaves of abelian groups over spaces.
(4) How does your first isomorphism theorem look on stalks?

Exercise 3. Let X be a space, F a presheaf, and $\mathcal{U}$ an open cover. For any $q \geq 0$, verify that the composite

$$\check{C}^q(\mathcal{U}, F) \xrightarrow{d^q} \check{C}^{q+1}(\mathcal{U}, F) \xrightarrow{d^{q+1}} \check{C}^{q+2}(\mathcal{U}, F)$$

is zero (that is, the Čech cochains complex is, indeed, a complex).

Exercise 4. Find a cover $\mathcal{U}$ of S^2 for which $\check{H}^*(S^2, \underline{\mathbb{Z}})$ agrees with the singular cohomology $H^*(S^2, \mathbb{Z})$.

Exercise 5 (Important)**.** If X is a space, F is a presheaf on X, and $\mathcal{U}, \mathcal{V}$ are open covers for which $\mathcal{V}$ is a refinement of $\mathcal{U}$, construct a natural map

$$\check{H}^q(\mathcal{U}, F) \to \check{H}^q(\mathcal{V}, F).$$

Show that any choices you made in the process of refinement don't matter, and the resulting math is well-defined.