MATH231BR HOMEWORK 3 (DUE THURSDAY, 2/19/26)

Exercise 1. Let $\phi\colon F \to G$ be a morphism of sheaves. Show that if $\phi_x\colon F_x \to G_x$ is injective for all x, then $\phi_U\colon F(U) \to G(U)$ is injective for all U. (That is, a monomorphism of sheaves is also a monomorphism of presheaves).

Exercise 2. Write a quick argument that Čech cohomology is functorial in the sheaf, that is,

$$\check{H}^q(X,-)\colon \mathrm{Ab}(X) \to \mathrm{Ab}$$

is a functor for any $q \geq 0$.

Exercise 3. Verify some of the claims from class about this chain complex of sheaves $\mathbb{Z}_{\mathcal{U},\bullet}$ we defined:

(1) $\mathbb{Z}_{\mathcal{U},p}$ corepresents the functor $\check{C}^p(\mathcal{U},-)$
(2) the differentials on $\mathbb{Z}_{\mathcal{U},\bullet}$ induce the Čech differentials under the natural isomorphism above
(3) the differentials on $\mathbb{Z}_{\mathcal{U},\bullet}$ make it an exact sequence of sheaves for $\bullet \geq 1$.

Exercise 4. Let X be a Hausdorff space. Prove that it is paracompact if and only if every open cover of X admits a locally finite refinement which has an associated partition of unity. (you may use the shrinking lemma and Urysohn's lemma)