MATH231BR HOMEWORK 4 (DUE THURSDAY, 2/26/26)

Exercise 1. For this exercise, we fill in the last bit of homological algebra that lets us conclude that the Čech cohomology of a Leray cover computes sheaf cohomology. Suppose we have a commutative diagram of abelian groups and abelian group homomorphisms laid out in an infinite grid like so:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B_0 \arrow[r] \arrow[d] & C_{00} \arrow[r] \arrow[d] & C_{01} \arrow[r] \arrow[d] & C_{02} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & C_{10} \arrow[r] \arrow[d] & C_{11} \arrow[r] \arrow[d] & C_{12} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B_2 \arrow[r] \arrow[d] & C_{20} \arrow[r] \arrow[d] & C_{21} \arrow[r] \arrow[d] & C_{22} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Suppose that:

- ▷ Every row (except the one with all the A_i's) is exact
- ▷ Every column (except the one with all the B_j's) is exact

Prove that the cohomology of the row consisting of the A_i's is isomorphic to the cohomology of the column consisting with the B_j's.

Exercise 2. This exercise is to convince yourself that null-homotopies on chain complexes work the way we do.

(1) Let $f^\bullet \colon C^\bullet \to C^\bullet$ be an endomorphism of a cochain complex. Suppose we have maps

$$h^n \colon C^n \to C^{n-1}$$

for all n, so that the following equality holds for all n:

$$h^{n+1} \circ d^n + d^{n+1} \circ h^n = \mathrm{id}$$

Use this to prove that $C^\bullet$ has no cohomology (it is exact).

(2) More generally, suppose we have two chain maps $f^\bullet, g^\bullet \colon C^\bullet \to C^\bullet$ and h^n's satisfying the following relation for all n:

$$h^{n+1} \circ d^n + d^{n+1} \circ h^n = f^n - g^n.$$

Then prove that $f^\bullet$ and $g^\bullet$ induce the same map on cohomology of $C^\bullet$.

Exercise 3. Give an example illustrating why C^q_{sing} is not a sheaf.