Math 619 Notes

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Contents

1 A preview of unstable vs stable homotopy theory

Unstable homotopy theory is the study of spaces up to homotopy, whereas stable homotopy is the study of spectra, which can be viewed as stable version of spaces, or from a different perspective, as a generalization of abelian groups. We will start with constructing the category of spaces and its homotopy category and work our way up to the construction the *the stable homotopy category.*

1.1 From spaces to spectra

The basic construction that we stabilize with respect to is the suspension functor Σ, which for a based space X is defined as $\Sigma X = X \wedge S^1$. Recall that for based spaces (X, x_0) and (Y, y_0), the smash product is defined as

$$X \wedge Y := \frac{X \times Y}{(\{x_0\} \times Y) \cup (X \times \{y_0\})}.$$

Example 1.1. There is a homeomorphism $S^1 \wedge S^1 \cong S^2$.

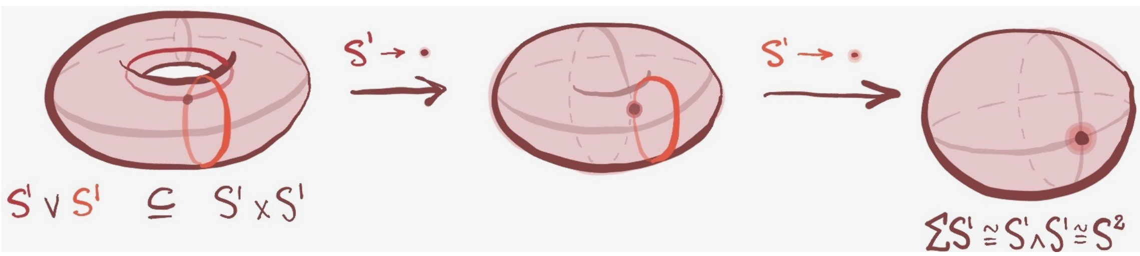

Figure 1: The homeomorphism $S^1 \wedge S^1 \cong S^2$.

In general, we have a similar homeomorphism

$$S^n \wedge S^m \cong S^{n+m}.$$

Let Ω be the loop functor, which for a based space X is defined as $\Omega X := \mathrm{Map}_*(S^1, X)$, i.e., the space of loops in X based at the basepoint of X. There is an adjunction $\Sigma \dashv \Omega$, which means that we have a natural homeomorphism

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y),$$

where this homeomorphism holds in our "convenient" category of spaces. Alternatively this is the data of a unit $X \to \Omega\Sigma X$ and a counit $\Sigma\Omega X \to X$ satisfying certain triangle identities. From this adjunction, we will end up seeing that the set of homotopy classes of based maps are isomorphic:

$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*.$$

Higher homotopy groups of a space based space X are defined as homotopy classes of maps from higher dimensional spheres $\pi_n(X) = [S^n, X]_*$. The theorem that forms the bridge from unstable to stable homotopy theory is the *Freudenthal suspension theorem*. Before stating it, we need a preliminary definition.

Definition 1.2. X is *n-connected* if $\pi_i(X) = 0$ for all $i \leq n$. For example, 0-connected means path connected and 1-connected means simply connected.

We may also define a suspension map on homotopy groups

$$\begin{aligned}\Sigma : \pi_q(X) &\to \pi_{q+1}(\Sigma X)\\ f &\mapsto \left[f \wedge \mathrm{id} : S^{q+1} = S^q \wedge S^1 \to X \wedge S^1 = \Sigma X\right].\end{aligned}$$

Theorem 1.3. *(Freudenthal suspension theorem)* Assume that X is a based space which is n-connected. Then the morphism $\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ is:

- a bijection for $q \leq 2n$
- a surjection for $q = 2n + 1$.

By Freudenthal, $\pi_{q+n}(\Sigma^n X)$ stabilizes as n increases. This stable value is an interesting invariant on the space X.

Definition 1.4. The *qth stable homotopy group* of a space X is defined to be

$$\pi_q^s(X) := \operatorname*{colim}_n \pi_{q+n}(\Sigma^n X).$$

We observe that if X is $(n-1)$-connected, then for $q < n-1$, the maps in the colimit system are isomorphisms. Thus we have that $\pi_q^s(X) = \pi_{q+n}(\Sigma^n X)$ for $q < n - 1$.

Remark 1.5. The computations of stable homotopy groups of spheres $\pi_q^s(S^0)$ is one of the deepest problems in algebraic topology. We will observe that these groups are deeply related to many questions throughout geometry and algebra as well.

We can take the idea of stabilizing a space that we have seen, and make the following definition of a *spectrum*, a stable analogue of a space.

Definition 1.6. A *(pre)spectrum* X is a sequence of based spaces $X_0, X_1, \ldots$ equipped with structure maps

$$\sigma_i : \Sigma X_i \to X_{i+1}$$

for each i.

Example 1.7. There is a *sphere spectrum*, denoted $\mathbb{S}$, whose nth space is the n-sphere S^n. The structure maps are the homeomorphisms $S^1 \wedge S^n \xrightarrow{\sim} S^{n+1}$ given by Example 1.1.

Example 1.8. Given any based space X, we denote by $\Sigma^\infty X$ its *suspension spectrum*, whose nth space is $\Sigma^n X$, and whose structure maps are the obvious homeomorphisms $\Sigma\Sigma^n X \xrightarrow{\sim} \Sigma^{n+1} X$. As an example, we have that $\mathbb{S} = \Sigma^\infty S^0$.

Definition 1.9. Given a spectrum X, we may define the *nth stable homotopy group of* X as

$$\pi_n(X) = \operatorname*{colim}_k \left(\cdots \to \pi_{n+k}(X_k) \to \pi_{n+k+1}(\Sigma X_k) \xrightarrow{\sigma_{k,*}} \to \cdots \right)$$

Exercise 1.10. If X is a based space, then its qth stable homotopy group is the qth homotopy group of its suspension spectrum.

The collection of spectra form a category.

Definition 1.11. A *morphism of spectra* $X \to Y$ is a collection of maps $f_i : X_i \to Y_i$ at each level, which commute with the structure maps:

$$\begin{tikzcd} \Sigma X_i \arrow[r, "\Sigma f_i"] \arrow[d, "\sigma_i"'] & \Sigma Y_i \arrow[d, "\sigma_i"] \\ X_{i+1} \arrow[r, "f_{i+1}"'] & Y_{i+1}. \end{tikzcd}$$

Knowing what a weak homotopy equivalence of topological spaces is, we might want to define a notion of having a weak homotopy equivalence of spectra. One naive notion would be to define an equivalence to be a levelwise equivalence (i.e. $f : X \to Y$ is an equivalence if $f_i : X_i \to Y_i$ is a weak homotopy equivalence for each i). A better notion is to ask for a *stable equivalence*.

Definition 1.12. A *stable equivalence* of spectra is a morphism $f : X \to Y$ so that the induced map on homotopy groups

$$(f_q)_* : \pi_q(X) \xrightarrow{\sim} \pi_q(Y)$$

is an isomorphism for each q.

We may also define a category Ho$\mathtt{Sp}$ which is the localization of the category of spectra at the stable equivalences. That is, a functor $\mathtt{Sp} \to \mathrm{Ho}\mathtt{Sp}$ which turns stable equivalences into isomorphisms, and is initial among functors with this property — that is, if there is another functor $\mathtt{Sp} \to \mathscr{C}$ which sends stable equivalences to isomorphisms, we get a unique factorization

$$\begin{tikzcd} \mathtt{Sp} \arrow[r] \arrow[dr] & \mathrm{Ho}\mathtt{Sp} \arrow[d, dashed] \\ & \mathscr{C}. \end{tikzcd}$$

However Ho$\mathtt{Sp}$ is a difficult category to work in, since it does not have many limits or colimits.

$$\begin{tikzcd} \mathtt{CW} \arrow[r, "\text{disjoint}", "\text{basept.}"'] \arrow[d] & \mathtt{CW}_* \arrow[r, "\Sigma^\infty"] \arrow[d] & \mathtt{Sp} \arrow[d] & \mathtt{Ab} \arrow[l, "H-"'] \arrow[d, "\text{0th component}"] \\ \mathrm{Ho}(\mathtt{CW}) \arrow[r] & \mathrm{Ho}(\mathtt{CW}_*) \arrow[r] & \mathrm{Ho}(\mathtt{Sp}) \arrow[r, "\pi_*"] & \mathtt{Ab}^{\mathbb{Z}_{\geq 0}} \end{tikzcd}$$

For every abelian group G, there is an *Eilenberg-Maclane spectrum HG* which is described up to homotopy by the property that $\pi_0(HG) = G$ and $\pi_i(HG) = 0$ for $i \neq 0$, where $(HG)_n = K(G, n)$ for each n. We will construct this spectrum later on in the class.

1.2 Why should we care about spectra?

There is an amazing theorem called *Brown representability* that states that every cohomology theory $E^* : \mathrm{Ho}\mathtt{CW} \to \mathtt{Ab}^{\mathbb{Z}_{\geq 0}}$, satisfying certain axioms, is represented by a spectrum E, that is, $E^n(X) \cong [X, E_n]$ and E is unique up to isomorphism in $\mathrm{Ho}\mathtt{Sp}$. Equivalently, any spectrum yields a cohomology theory. Thus we can study generalized cohomology theories by directly studying spectra.

Here are some questions and applications where spectra are key.

Example 1.13. The enumeration of differentiable structures on spheres S^n for $n \geq 5$ is reduced to the computation of the stable homotopy groups of spheres.

Example 1.14. One can impose an equivalence relation on closed manifolds called *cobordism.* (Draw picture) The classification question of closed manifolds up to cobordism is a central problem in differential topology. Amazingly, it turns out that this problem can be translated into stable homotopy theory. The groups Ω_n of closed n-manifolds up to cobordism was shown by Thom to be isomorphic to the homotopy groups homotopy groups of a spectrum MO.

Example 1.15. Consider the question: *For which n is $\mathbb{R}^n$ is a division algebra?* Of course, we know four examples, $\mathbb{R}, \mathbb{C} = \mathbb{R}^2, \mathbb{H} = \mathbb{R}^4, \mathbb{O} = \mathbb{R}^8$), are there any others? This is also equivalent to asking when S^{n-1} is an H-space, i.e., it has a continuous multiplication map with a two-sided identity element. This question is also known as *the Hopf invariant one problem* because it is also equivalent to the existence of a Hopf invariant one map $S^{2n-1} \to S^n$. This question was solved by Adams, initially in a 100 page paper, where he introduced the Adams spectral sequence for $\pi_q^s(S^0)$. He later gave a much shorter solution using K-theory. Stable homotopy theory is key in both solutions.

Example 1.16. Consider the question: *How many linearly independent everywhere nonzero vector fields can there be on S^{n-1}?* This question was answered by Adams using K-theory, an extraordinary cohomology theory. The answer is that S^{n-1} admits exactly $\rho(n) - 1$ linearly independent everywhere nonzero vector fields, where $\rho(n)$ is the Radon-Hurwitz number equal to $2^c + 8d$, where $n = (2a + 1)2^b$ and $b = c + 4d$ with $0 \leq c < 4$.

Example 1.17. A very old question in differential topology dating back to the 1960s was: *For which n does there exist a stably framed n-manifold with the Kervaire invariant one?* This is known as the *Kervaire invariant one problem.* The Kervaire invariant of a smooth stably framed manifold is the so-called Arf-invariant of an associated quadratic form $H^{\frac{n}{2}}(M, \mathbb{Z}/2) \to \mathbb{Z}/2$. The answer to the Kervaire invariant 1 problem is yes in dimensions $n = 2, 6, 14, 62$ and possibly 126, but it was ruled out in all other dimensions in 2009 by a recent breakthrough result of Hill, Hopkins and Ravenel [HHR16]. This question can again be translated into the survival of certain elements in the Adams spectral sequence for the stable homotopy groups of spheres, and it was further translated by the authors into a question in equivariant stable homotopy theory and solved in that setting.

2 The category of spaces

We start with some recollections about basic category theory and we construct the "convenient category of topological spaces" which modern algebraic topologists work in.

2.1 Category theory review

We refer the reader to the following excellent sources for the basics of category theory. The sets of notes [GS],[Tor] and [Joh15] are great introductions and so are the textbooks [Lei14] and [Rie16]. The standard reference for category, which was the first book on the subject is still [Mac71]. We skip the definition of category but recall a list of standard examples of categories that you have certainly encountered before.

Table 1: Examples of categories

Category	**Objects**	**Morphisms**
$\mathtt{Set}$	sets	set functions
$\mathtt{Grp}$	groups	group homomorphisms
$\mathtt{Ab}$	abelian groups	group homomorphisms
$\mathtt{Ring}$	rings	ring homomorphisms
$\mathtt{Top}$	topological spaces	continuous maps
$\mathtt{Top}_*$	based spaces	based continuous maps
$\mathtt{Vect}(k)$	vector spaces over a field k	linear maps
$\mathbb{1}$	single object	only identity arrow
$\mathbb{2}$	two objects, denoted 0 and 1	single non-identity morphism $0 \to 1$.

Also, here are some examples of functors between some of these categories, which we have probably encountered before:

- the forgetful functor $U : \mathtt{Grp} \to \mathtt{Set}$ which forgets the group structure
- the free functor $F : \mathtt{Set} \to \mathtt{Grp}$ which takes the free group on a set
- homology $H_i \colon \mathtt{Top}_* \to \mathtt{Grp}$
- homotopy groups $\pi_i \colon \mathtt{Top}_* \to \mathtt{Grp}$
- cohomology $H^i : \mathtt{Top}^* \to \mathtt{Grp}$, which is a *contravariant* functor.

Example 2.1. Suppose that G is a group. Then we may form a category G with one object, and whose morphism set is all of G. Composition is given by the group law. A functor $G \to \mathtt{Set}$ is exactly the data of a G-set, that is, a set equipped with an action of G. Similarly, a functor $G \to \mathscr{C}$ is referred to as a G-object in $\mathscr{C}$.

Definition 2.2. For any category $\mathscr{C}$, we can define its *opposite category*, denoted $\mathscr{C}^{\mathrm{op}}$, whose objects are the same as $\mathscr{C}$, but whose morphisms have the domain and codomain swapped, and whose composition is defined to mirror the composition in $\mathscr{C}$. That is, a morphism $x \xrightarrow{f} y$ in $\mathscr{C}$ corresponds to a morphism $y \xrightarrow{f^{\mathrm{op}}} x$, and we define $g^{\mathrm{op}} \circ f^{\mathrm{op}} := (f \circ g)^{\mathrm{op}}$.

Note that for any category $\mathscr{C}$, we have that $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$ is just $\mathscr{C}$. The construction $\mathscr{C}^{\mathrm{op}}$ occurs frequently in mathematics. The opposite category $\mathscr{C}^{\mathrm{op}}$ can be very different than the category $\mathscr{C}$. Some interesting examples follow:

- the category of affine schemes is equivalent to $\mathtt{Ring}^{\mathrm{op}}$
- the opposite category of finite sets is equivalent to the category of finite Boolean algebras
- the category $\mathtt{Top}^{\mathrm{op}}$ is, in some sense, equivalent to the category of frames (for a discussion of this, see [Wof])

Note on terminology: The classical notion of a contravariant functor from $\mathscr{C} \to \mathscr{D}$ can be thought of as a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$, thus so long as we specify the domain category, we may often refer to a functor with the implicit assumption that it is covariant.

Definition 2.3. Let $F, G : \mathscr{C} \to \mathscr{D}$ be covariant functors. A *natural transformation* $\eta : F \Rightarrow G$ is a collection of morphisms $\eta_x : Fx \to Gx$ for each $x \in \mathscr{C}$, such that, for every $x \xrightarrow{f} y$ in $\mathscr{C}$, we have that the diagram commutes

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\eta_x"'] & Fy \arrow[d, "\eta_y"] \\ Gx \arrow[r, "Gf"'] & Gy. \end{tikzcd}$$

Definition 2.4. A *natural isomorphism* is a natural transformation $\eta : F \Rightarrow G$ such that the components $\eta_c : Fc \xrightarrow{\cong} Gc$ is an isomorphism for each $c \in \mathscr{C}$.

For any two categories $\mathscr{C}$ and $\mathscr{D}$, we can define the product category $\mathscr{C} \times \mathscr{D}$, with objects and morphisms given by pairs of objects and morphisms, and identities and composition defined componentwise.

Exercise 2.5. Show that the data of a natural transformation is the same as a functor $H : \mathscr{C} \times \mathbb{2} \to \mathscr{D}$, such that $H(-,0) = F$ and $H(-,1) = G$.

Example 2.6. Let $(-)^{**} : \mathtt{Vect}(\mathbb{R}) \to \mathtt{Vect}(\mathbb{R})$ denote the double dual functor. We have a natural transformation $\eta : \mathrm{id}_{\mathtt{Vect}(\mathbb{R})} \Rightarrow (-)^{**}$ whose components are the isomorphisms $\eta_V \to V^{**}$ given by $x \mapsto [\mathrm{ev}_x : V^* \to \mathbb{R}]$. One may check that the corresponding diagram commutes, and gives a natural isomorphism.

Remark 2.7. For every V, there also exists an isomorphism $V \cong V^*$, however there is no natural isomorphism $\mathrm{id} \Rightarrow (-)^*$.

Definition 2.8. We define $\mathscr{D}^{\mathscr{C}}$ or $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ as the *functor category* whose objects are functors $\mathscr{C} \to \mathscr{D}$ and whose morphisms are natural transformations.

Thomas: is this an okay convention though? I feel like in general, $\mathscr{D}^{\mathscr{C}}$ denotes $\mathscr{D}$-valued presheaves on $\mathscr{C}$, so functors $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$

Let $\mathscr{C}$ be a *locally small* category, meaning that $\mathrm{Hom}_{\mathscr{C}}(a,b)$ is a set for each pair of objects in $\mathscr{C}$. For each $c \in \mathscr{C}$, we obtain functors

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(c,-) &: \mathscr{C} \to \mathtt{Set} \\ \mathrm{Hom}_{\mathscr{C}}(-,c) &: \mathscr{C}^{\mathrm{op}} \to \mathtt{Set} \end{aligned}$$

Functors of this form are called *representable*. The Yoneda lemma says that the set of natural transformations $\mathrm{Hom}_{\mathscr{C}}(-,c) \Rightarrow F$ is in bijection with $F(c)$, naturally in c.

Lemma 2.9 (Yoneda Lemma, covariant version)**.** Let $F\colon \mathscr{C} \to \mathtt{Set}$. There is a bijection

$$\mathrm{Hom}_{\mathtt{Set}^{\mathscr{C}}}(\mathrm{Hom}_{\mathscr{C}}(c,-), F) \cong F(c),$$

naturally in both c and F.

Exercise 2.10. Prove the Yoneda lemma.

Hint: There are two parts to this. First, define the required set isomorphism. Second, draw out the naturality squares, along maps $c \to c'$ and along natural transformations $F \Rightarrow F'$ and check that with the definition of the isomorphism you gave, these squares commute. A reference for the proof is for example [Joh15, Lemma 2.5].

There is an equivalent contravariant version of the Yoneda lemma. Moreover, it turns out we can prove two objects in a category are isomorphic by showing that the representables are naturally isomorphic.

Lemma 2.11. Suppose that there is a natural isomorphism of functors

$$\eta\colon \mathrm{Hom}_{\mathscr{C}}(c,-) \cong \mathrm{Hom}_{\mathscr{C}}(d,-).$$

Then $c \cong d$ in $\mathscr{C}$.

Proof. We can show that $\eta_c(\mathrm{id}_c)\colon d \to c$ is an isomorphism. Since η_d is an isomorphism, there exists an $f\colon c \to d$ such that $\eta_d(f) = \mathrm{id}_d$. Considering the naturality sqaure

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(c,c) \arrow[r, "\eta_c"] \arrow[d, "\circ f"'] & \mathrm{Hom}_{\mathscr{C}}(d,c) \arrow[d, "\circ f"] \\ \mathrm{Hom}_{\mathscr{C}}(c,d) \arrow[r, "\eta_d"'] & \mathrm{Hom}(d,d) \end{tikzcd}$$

we can see that $f \circ \eta_c(\mathrm{id}_c) \circ f = \mathrm{id}_d$. Similarly considering the naturality square for $\eta_c(\mathrm{id}_c)\colon d \to c$ we can show that $\eta_c(\mathrm{id}_c) \circ f = \mathrm{id}_c$. □

2.2 Adjoint pairs of functors

Let `Cat` be the category of categories and functors. Note that in any category, there is a looser way that objects can be "the same" than equality: two objects C and D are defined to be isomorphic if there are morphisms $f\colon C \to D$ and $g\colon D \to C$ such that $fg = \mathrm{id}$ and $gf = \mathrm{id}$. In particular, in the category `Cat`, two categories $\mathscr{C}$ and $\mathscr{D}$ are isomorphic if there are functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that $FG = \mathrm{id}$ and $GF = \mathrm{id}$. However, we have seen that there is a looser notion that we can use to compare functors than equality: we have defined a notion of isomorphic functors, which leads to the definition of *equivalence of categories*.

Definition 2.12. A functor $F\colon \mathscr{C} \to \mathscr{D}$ is an *equivalence of categories* if there exists $G : \mathscr{D} \to \mathscr{C}$ such that $FG \cong \mathrm{id}_{\mathscr{D}}$ and $GF \cong \mathrm{id}_{\mathscr{C}}$ are natural isomorphisms of functors. We say that F is an *isomorphism of categories* if $FG = \mathrm{id}_{\mathscr{D}}$ and $GF = \mathrm{id}_{\mathscr{C}}$.

Recall that for a set function having an inverse is equivalent to being injective and surjective. An analogous result regarding equivalences of categories is the following.

Proposition 2.13. $F\colon \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if and only if F is:

- *faithful*, meaning that $\mathrm{Hom}_{\mathscr{C}}(c, c') \hookrightarrow \mathrm{Hom}_{\mathscr{D}}(Fc, Fc')$ is injective for each c, c'
- *full*, meaning that $\mathrm{Hom}_{\mathscr{C}}(c, c') \twoheadrightarrow \mathrm{Hom}_{\mathscr{D}}(Fc, Fc')$ is surjective for each c, c'
- *essentially surjective on objects*, meaning for each $d \in \mathscr{D}$ there exists $c \in \mathscr{C}$ such that $F(c) \cong d$.

Exercise 2.14. Prove the proposition. *A reference for the proof is [GS, Proposition 1].*

More generally we have a notion of having a map between functors, i.e., a natural transformation. We can ask for two categories to be related in an even weaker sense than the notion of equivalence we have defined by simply requiring that we have natural transformations between the two composites and identities, but without requiring that these are isomorphisms. We do impose a condition that ensures that the two natural transformations do interact nicely with each other.

Definition 2.15. Given a pair of functors $F\colon \mathscr{C} \leftrightarrows \mathscr{D} :G$, we say they are *adjoint* if there exist natural transformations $\eta\colon \mathrm{id}_{\mathscr{C}} \Rightarrow GF$ (called the *unit*) and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathscr{D}}$ (called the *counit*) satisfying the triangle identities:

$$\begin{tikzcd} F(a) \arrow[r, "F\eta_a"] \arrow[dr, equal] & FGF(a) \arrow[d, "\epsilon_{F(a)}"] \\ & F(a) \end{tikzcd} \qquad \begin{tikzcd} G(b) \arrow[r, "\eta_{G(b)}"] \arrow[dr, equal] & GFG(b) \arrow[d, "G(\epsilon_b)"] \\ & G(b), \end{tikzcd}$$

for each $a \in \mathscr{C}$, $b \in \mathscr{D}$.

We will give three more definitions of an adjunction, which we claim are all equivalent to each other. We can define an adjunction in terms of just the unit or the counit together with a universal property.

Definition 2.16. Given a pair of functors $F\colon \mathscr{C} \leftrightarrows \mathscr{D}\colon G$, we say they are *adjoint* if there exists a natural transformations $\eta\colon \mathrm{id}_{\mathscr{C}} \Rightarrow GF$ (called the *unit*) such that for any object C of $\mathscr{C}$ and D of $\mathscr{D}$ and any map $f\colon C \to G(D)$ there exists a unique map $g\colon F(C) \to D$ so that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"] & GF(C) \arrow[dl, dotted, "Ug"] \\ U(D) & \end{tikzcd}$$

Definition 2.17. (Dual definition specifying the counit instead of the unit.)

Lastly, we can give a definition in terms of a natural isomorphism of hom-sets when the categories $\mathscr{C}$ and $\mathscr{D}$ are both locally small so that $\mathrm{Hom}_{\mathscr{D}}(F(a), b)$ and $\mathrm{Hom}_{\mathscr{C}}(a, G(b))$ are sets. Oftentimes, the following is given as the definition of an adjunction.

Definition 2.18. Given a pair of functors $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$, we say they are *adjoint* if, for each $a \in \mathscr{C}$, $b \in \mathscr{D}$, there exists an isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(F(a), b) \cong \mathrm{Hom}_{\mathscr{C}}(a, G(b)),$$

which is natural in both a and b.

Exercise 2.19. Demonstrate the equivalence of these four definitions.

These equivalences of definitions can all be found in the nLab entry on adjunctions. Also, the proof of the equivalence of Definition 2.15 and Definition 2.18 can be found as [Lei14, Theorem 2.2.5] or [Joh15, Theorem 3.9]. The equivalence of Definition 2.16 and Definition 2.18 can be found in [Hen08, Section 3] or [Lei14, Theorem 2.3.6.], where the universal property of the unit is formulated in terms of initial objects. The argument using Definition 2.17 instead of Definition 2.16 should be exactly analogous.

Examples 2.20. Here are some examples of adjunctions:

- Fix an abelian group B. Then for any abelian groups A and C, we have a natural isomorphism of sets

  $$\mathrm{Hom}_{\mathtt{Ab}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathscr{C}}(A, \mathrm{Hom}_{\mathtt{Ab}}(B, C)).$$

  That is, the functors $- \otimes B$ and $\mathrm{Hom}_{\mathtt{Ab}}(B, -)$ are adjoint.
- We have an free-forgetful adjunction $F : \mathtt{Set} \leftrightarrows \mathtt{Grp} : U$
- $(-)_+ : \mathtt{Top} \leftrightarrows \mathtt{Top}_* : U$ is an adjunction. We see this since

  $$\mathtt{Top}(X, UY) \cong \mathtt{Top}_*(X_+, Y).$$

Proposition 2.21. Adjoints compose.

Exercise 2.22. Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ be an adjunction. Then

1. F is fully faithful if and only if the unit $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ is a natural isomorphism.

2. G is fully faithful if and only if the counit $\epsilon : FG \to \mathrm{id}_{\mathscr{D}}$ is a natural isomorphism.

Note that in Definition 2.12 of equivalence of categories we only required that $GF \cong \mathrm{id}$ and $FG \cong \mathrm{id}$ were natural isomorphisms, but not necessarily that F and G are adjoint (namely, the triangle identities were not imposed). It turns out that every equivalence of categories can be modified to an adjoint equivalence by possible changing the unit or counit. For a proof, see [Joh15, Lemma 3.10].

2.3 A convenient category of spaces

> For many years, algebraic topologists have been laboring under the handicap of not knowing in which category of spaces they should work. Our need is to be able to make a variety of constructions and to know that the results have good properties without the tedious spelling out at each step of lengthy hypotheses such as countably paracompact, normal, completely regular, first axiom of countability, metrizable, and so forth. It may be good research technique and an enjoyable exercise to analyse the precise circumstances for which an argument works; but if a developing theory is to be handy for research workers and attractive to students, then the simplicity of the fundamentals must be the goal.
>
> [Ste67]

The main references for this section are [Ste67] and [Str], and we refer the reader to those papers for all of the proofs of point set topology propositions. An excellent set of notes on this topic is also [Fra13].

Recall that in `Set` we have an adjuction between the product and the hom functor given by

$$\mathrm{Hom}_{\mathtt{Set}}(A \times B, C) \cong \mathrm{Hom}_{\mathtt{Set}}(A, \mathrm{Hom}_{\mathtt{Set}}(B, C)).$$

Of course, the bijection follows from comparing cardinalities, but the point is that these bijections are natural, so they are compatible along morphisms in the category. We want a similar adjunction in the category of spaces.

Let Y^X or $\mathrm{Map}(X, Y)$ denote the *mapping space* between two topological spaces, whose underlying set is the collection of continuous maps $X \to Y$, and which is equipped with the *compact-open topology*, generated by the subbasis

$$W_{K,U} = \{f : X \to Y \ : \ f(K) \subseteq U\},$$

indexed over $K \subseteq X$ compact and $U \subseteq Y$ open. We have a map

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z)), \tag{1}$$

which is always a homeomorphism onto its image, but it is not always surjective. Thus $\mathrm{Map}(Y, -)$ and $- \times Y$ are not adjoint. As we will see in the next section, any left adjoint in `Top` would need to preserve quotients, which is a type of colimit. However, by [Mun00, Section 22, Example 7], The product of the quotient map $\mathbb{R} \to \mathbb{R}/\sim$, which identifies all of $\mathbb{N}$ to a point, with $\mathbb{Q}$ is not a quotient. Thus $- \times \mathbb{Q}$ cannot possibly be a left adjoint.

In order to turn Equation 1 into a homeomorphism, we define a class of spaces for which this is a homeomorphism.

Definition 2.23. A subspace $A \subseteq X$ is *k-closed in X* if, for all compact Hausdorff spaces K and continuous maps $K \xrightarrow{f} X$, we have that $f^{-1}(A)$ is closed in K.

Exercise 2.24. The collection of k-closed subsets of X forms a topology which contains the original topology on X (since closed implies k-closed).

Let kX denote the set X equipped with the k-closed topology. So kX has the same underlying set as X but its topology has more closed sets. We note that the identity map $kX \to X$ is continuous.

Definition 2.25. We say that a space X is *compactly generated (CG)* if the identity map $kX \to X$ is a homeomorphism. That is, if every k-closed set is closed in the original topology on X.

Some examples of CG spaces are metric spaces and all locally compact spaces, so in particular, CW complexes. The proofs that these spaces are CG can be found in [Str, Propositions 1.6, 1.7]

Note that immediately from the definitions, if X is a compactly generated space, and Y is any space, then $f : X \to Y$ is continuous if and only if $f : X \to kY$ is continuous.

Upshot 2.26. For the functors $i : \mathtt{CG} \leftrightarrows \mathtt{Top} : k$, we have that

$$\mathrm{Hom}_{\mathtt{Top}}(iX, Y) \cong \mathrm{Hom}_{\mathtt{CG}}(X, kY),$$

so i is left adjoint to k.

Note 2.27. We have that $k^2X = kX$.

Note 2.28. The category $\mathtt{CG}$ is not closed under product and mapping spaces. So we must define

$$X \times_k Y := k(X \times Y)$$
$$\mathrm{Map}_k(X, Y) := k\mathrm{Map}(X, Y).$$

Theorem 2.29. There exists a homeomorphism

$$\mathrm{Map}(X \times_k Y, Z) \cong \mathrm{Map}_k(X, \mathrm{Map}_k(Y, Z)),$$

so $\mathtt{CG}$ is Cartesian closed. For a proof of this theorem, read [Str, Proposition 2.11].

The category $\mathtt{CG}$ is good enough for most applications, but to make things a little better, we impose an additional separation axiom.

Definition 2.30. A topological space is *weak Hausdorff* if for all compact Hausdorff spaces K and every continuous map $f : K \to X$, we have that $f(K)$ is closed in X.

Exercise 2.31. Hausdorff implies weak Hausdorff.

Some examples of weak Hausdorff spaces are again CW complexes, metric spaces, etc. One of the reasons why we work with weak Hausdorff spaces as opposed to Hausdorff spaces is for example, for any closed inclusion $A \hookrightarrow X$ where X is weak Hausdorff, the quotient is also weak Hausdorff, though this would not necessarily be true in Hausdorff spaces. For example, the quotient of the C_2-action on a pair of real lines $\mathbb{R}$ which swaps the points between the lines outside of 0, and fixes 0 on both lines is the "line with two origins," which is not Haurdorff. Another example is the quotient of the action of the additive subgroup $\mathbb{Q}$ on $\mathbb{R}$, which is $\mathbb{R}/\mathbb{Q}$, which has the trivial topology so it is not Hausdorff.

Proposition 2.32. For a weak Hausdorff space X, any larger topology on X (i.e. a topology containing the original topology on X) will also be weak Hausdorff.

Another reason why we like weak Hausdorff spaces is because they interact nicely with compactly generated spaces. For example, by [Str, Lemma 1.4. and Proposition 2.14]

- for X is a k-space, it is weak Hausdorff if and only if the diagonal $X \xrightarrow{\Delta} X \times_k X$ is closed,
- for X weak Hausdorff, it is a k-space if and only if the closed set C are precisely those for which $C \cap K$ is closed for every compact Hausdorff $K \subseteq X$.

Proposition 2.33. For $X \in \mathtt{CG}$, we have that $X/\sim$ is weak Hausdorff for some equivalence class on X if and only if $\sim$ is closed (meaning the equivalence relation is closed as a subspace of $X \times X$).

For the proof of this, read [Str, Corollary 2.21]. Now, by [Str, Proposition 2.22], there is a smallest closed equivalence relation on a CG space X, so we can make the following definition.

Definition 2.34. For $X \in \mathtt{CG}$, let $hX = X/\sim$, where $\sim$ is the smallest closed equivalence relation on X. Let $j : \mathtt{CGWH} \to \mathtt{CG}$ denote the inclusion. Thus we have an adjunction

$$h : \mathtt{CG} \leftrightarrows \mathtt{CGWH} : j.$$

That is, $\mathrm{Hom}_{\mathtt{CG}}(X, jY) \cong \mathrm{Hom}_{\mathtt{CGWH}}(hX, Y)$.

Proposition 2.35. If $X, Y \in \mathtt{CGWH}$ then $\mathrm{Map}_k(X, Y)$ and $X \times_k Y$ are in $\mathtt{CGWH}$. So $\mathtt{CGWH}$ is cartesian closed.

In other words, once in $\mathtt{CG}$, passing to weak Hausdorff spaces, does not change products and mapping spaces. For the proofs, again see [Str, Corollary 2.16 and Proposition 2.24].

2.4 Limits and colimits in categories

Definition 2.36. Let $F : I \to \mathscr{C}$ be a functor. We call this an *I-shaped diagram in $\mathscr{C}$*. The *colimit* of F, denoted $\operatorname{colim} F$, is an object in $\mathscr{C}$ together with a family of morphisms $F(X) \to \operatorname{colim} F$ for each $X \in I$, such that

$$\begin{tikzcd} F(X) \arrow[rr] \arrow[dr] & & F(Y) \arrow[dl] \\ & \operatorname{colim} F & \end{tikzcd}$$

commutes for each $X \to Y$ in I, and moreover it satisfies a universal property so that for any other object $A \in \mathscr{C}$ with the above properties, we have that there exists a unique map $\operatorname{colim} F \to A$ such that the diagram commutes

$$\begin{tikzcd}
F(X) \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & F(Y) \arrow[dl] \arrow[ddl, bend left] \\
& \operatorname{colim} F \arrow[d, dashed] & \\
& A &
\end{tikzcd}$$

for each $X \to Y$.

The *limit* of a diagram $F : I \to \mathscr{C}$ is dual to this definition, meaning that a limit is a colimit in the opposite category $\mathscr{C}^{\mathrm{op}}$, i.e. the colimit of the composite functor $I \xrightarrow{F} \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$.

Proposition 2.37. Limits and colimits, when they exist, are unique up to isomorphism.

Exercise 2.38. Prove Proposition 2.37.

Before moving on to many examples, we state a proposition whose usefulness is hard to overestimate. It says that left adjoints preserve colimits and right adjoints preserve limits.

Proposition 2.39. (*LAPC and RAPL*[1]) Let $F \dashv G$ be adjoint functors. Then F preserves colimits and G preserves limits.

Exercise 2.40. Prove Proposition 2.39.

Hint: Note that if you prove that right adjoint preserve limits, it will dually follow that left adjoints preserve colimits by considering opposite categories. Now, to prove RAPL, the simplest way to approach it is straight from the hom bijection definition of adjoints and the definition of limit. Namely, show that if L is a limit of a diagram in the domain category $\mathscr{C}$ then GL satisfies the universal property of the limit in the target category $\mathscr{D}$. Another way would be to show that the proposition holds for representables and then go from there. Or, we using Remark 2.51, which we will see in the next section, assuming limits exist, you can use that adjunction to prove that a right adjoint preserves limits.

Definition 2.41. Let $I = \bullet \quad \bullet$ be the discrete category with two objects and no nontrivial morphisms. Then the limit over I is called the *product*, and the colimit over I is called the *coproduct*. Explicitly, where 0 and 1 denote the two objects in I, we have that

$$\lim_I F = F(0) \Pi F(1)$$
$$\operatorname*{colim}_I F = F(0) \amalg F(1).$$

Examples 2.42. What follows are some examples of products and coproducts:

[1] Left adjoints preserve colimits and right adjoints preserve limits.

- In `Set`, a functor determines two sets $F(0) = A$ and $F(1) = B$, and their colimit $A \amalg B$ satisfies the universal property that, for any other set X equipped with morphisms $A \to X$ and $B \to X$, there is a unique map $A \amalg B \to X$ such that

$$\begin{tikzcd} A \arrow[dr] \arrow[ddr, bend right] & & B \arrow[dl] \arrow[ddl, bend left] \\ & A \amalg B \arrow[d, dashed] & \\ & X & \end{tikzcd}$$

  commutes. This is exactly the coproduct of sets that we are familiar with. Dually, the limit $A \prod B$ satisfies the dual property that, for any X equipped with maps $X \to A$ and $X \to B$, there is a unique map such that

$$\begin{tikzcd} & X \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & A \prod B \arrow[dl] \arrow[dr] & \\ A & & B. \end{tikzcd}$$

  Thus the product in `Set` is just the cartesian product.
- In `Top`, the coproduct is disjoint union and the product is the product of spaces with the product topology.
- In $\texttt{Top}_*$, the coproduct is the wedge $A \vee B$, and the product is just the product of spaces $(A, a_0) \times (B, b_0)$ based at the point (a_0, b_0).
- In `Poset`, the product $p \wedge q$ is the greatest lower bound, and the coproduct $p \vee q$ is the least upper bound (if they exist).
- In `Grp`, the product is the direct product, and the coproduct is the free product of groups.

Definition 2.43. If $I = \bullet \rightrightarrows \bullet$, then the colimit over I is called the *coequalizer* and the limit over I is called the *equalizer*.

Example 2.44. In `Top`, let $E \subseteq X \times X$ be an equivalence relation on X. Then

$$\operatorname{colim}\left(E \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X \right) = X/\sim .$$

Definition 2.45. If $I = \bullet \leftarrow \bullet \rightarrow \bullet$, then the colimit over I is called the *pushout*. The limit over $I^{\mathrm{op}} = \bullet \rightarrow \bullet \leftarrow \bullet$ is called the *pullback*. We denote these by

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \amalg_Z Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & Z. \end{tikzcd}$$

Examples 2.46. Here are some examples of pushouts:

- In `Set`, the pushout

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"'] & X \amalg_Z Y \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

  is given by $X \amalg_Z Y = (X \amalg Y)\big/(i_1 f(z) \sim i_2 g(z))$.

  If $X, Y \subseteq S$ are both subsets of some larger set with $X \cap Y = Z$, then the pushout is $X \cup Y$.
- In `Top` the pushout is a gluing construction

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] & X \cup_Z Y, \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

  where $Z \to X$ is some nice map like a closed inclusion, then we have that

$$X \cup_f Y = X \cup_f Y = X \amalg Y\big/f(z) \sim z.$$

  For example, given the boundary inclusion $S^1 \to D^2$, we have that

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d, hook] & D^2 \arrow[d] \\ D^2 \arrow[r] & S^2. \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

- In `Set`, the pullback

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  is given by

$$X \times_Z Y := \{(x, y) \in X \times Y \ : \ f(x) = g(y)\}.$$

  For example, given the parity map $\mathbb{Z} \to \mathbb{Z}/2$, the pullback along the trivial map from 0 is:

$$\begin{tikzcd} 2\mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{Z} \arrow[d] \\ 0 \arrow[r] & \mathbb{Z}/2. \end{tikzcd}$$

  More generally, given $f : X \to A$, the pullback of f along the inclusion $* \to A$ which picks out a point $a \in A$ is given by:

$$\begin{tikzcd} f^{-1}(a) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ * \arrow[r, "a"'] & X \end{tikzcd}$$

Example 2.47. Recall that a CW complex X has a discrete set X_0 and has skeleta X_{n+1} which is built from X_n via a pushout diagram

$$\begin{tikzcd} \coprod_\alpha S^n \arrow[r] \arrow[d, "\amalg_\alpha f_\alpha"'] \arrow[dr, phantom, "\ulcorner" very near end] & \coprod_\alpha D^{n+1} \arrow[d] \\ X_n \arrow[r] & X_{n+1}. \end{tikzcd}$$

where $\amalg_\alpha f_\alpha$ are the attaching maps. Then we define

$$X = \bigcup_n X_n = \operatorname{colim}(X_0 \to X_1 \to \cdots)$$

with the *weak topology*, meaning that $A \subseteq X$ is closed if and only if $A \cap X_n$ is closed for each n.

Proposition 2.48. CW complexes are in `CGWH`.

Note 2.49. The product of CW complexes $X \times Y$ gets a cell structure with n-cells of $X \times Y$ corresponding to a p-cell of X and a q-cell of Y where $p + q = n$. The idea behind this is that

$$D^{p+q} \cong I^{p+q} \cong I^p \times I^q \cong D^p \times D^q.$$

The issue is that the product of two CW complexes $X \times Y$ is only a CW complex if the product topology is the same as the weak topology. We also note that the product topology is generally not as fine as the weak topology.

There were some partial results known before the introduction of CGWH spaces.

1. Whitehead '49 showed that if X and Y are CW complexes, and one is locally finite, then $X \times Y$ is CW citation sneeded
2. Milnor '56 showed that if both are locally countable, then $X \times Y$ is CW
3. Tanaka '82 counterexample: if X and Y are CW and neither are locally countable, then $X \times Y$ is *not* CW.

But the good news is that in `CGWH` we have that $X \times_k Y$ is always a CW complex. A proof can be found in [Hat03, Theorem A.6].

2.5 Existence of limits and colimits in `Top`

Definition 2.50. A category with all limits is called *complete*, and a category with all colimits is called *cocomplete*.

We start with a remark about a way to describe limits when they exist.

Remark 2.51. We may check that, in any $\mathscr{C}$ which has J-shaped limits,

$$\operatorname{Hom}_{\mathscr{C}}(Z, \lim_J F) \cong \operatorname{Hom}_{\mathscr{C}^J}(\Delta Z, F)$$

where $\Delta : \mathscr{C} \to \mathscr{C}^J$ is the diagonal functor which sends c to the constant diagram at c.

Proposition 2.52. If $\mathscr{C}$ has arbitrary coproducts and coequalizers, then it is cocomplete. Dually, if $\mathscr{C}$ has arbitrary products and equalizers, then it is complete.

Proposition 2.53. `Top` is complete and cocomplete.

Proof idea. Note that we have coproducts given by disjoint union, and coequalizers are given by quotients

$$\operatorname{coeq}\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) := X/f(x) \sim g(x).$$

Products are given by products, and we may check that equalizers are a subspace given by equality of two functions.

$\square$

Proposition 2.54. The category `CGWH` is complete and cocomplete.

Proof. Recall that we had adjunctions

$$\mathtt{Top} \underset{i}{\overset{k}{\rightleftarrows}} \mathtt{CG} \underset{h}{\overset{j}{\leftrightarrows}} \mathtt{CGWH},$$

where the right adjoints are displayed on top.

1. We have that i preserves colimits, so colimits of CG spaces are computed in `Top` when they exist. So we must check that coequalizers and coproducts of CG spaces are CG (exercise).

   For limits, we claim that $\lim_J D$ for $D\colon J \to \mathtt{CG}$ is $k \lim_J (i \circ G)$ in `CG`. First note that

   $$\operatorname{Hom}_{\mathtt{CG}}(X, k \lim_J (i \circ D)) \cong \operatorname{Hom}_{\mathtt{Top}}(iX, \lim_J (i \circ D)).$$

   Using Remark 2.51, we have that

   $$\begin{aligned} \operatorname{Hom}_{\mathtt{Top}}(iX, \lim_J (i \circ D)) &\cong \operatorname{Hom}_{\mathtt{Top}^J}(\Delta(iX), iD) \\ &\cong \operatorname{Hom}_{\mathtt{Top}^J}(i \circ \Delta X, iD) \\ &\cong \operatorname{Hom}_{\mathtt{CG}^J}(\Delta X, k \circ iD) \\ &\cong \operatorname{Hom}_{\mathtt{CG}^J}(\Delta X, D), \end{aligned}$$

   where this last isomorphism is since D already takes values in `CG`. Finally, we see that

   $$\operatorname{Hom}_{\mathtt{CG}^J}(\Delta X, D) \cong \operatorname{Hom}_{\mathtt{CG}}(X, \lim_J D).$$

   Since this is true for all objects X, we may check that $k \lim_J (i \circ D) = \lim_J D$.

   Upshot: We have that `CG` is complete and cocomplete and we know how to compute limits and colimits:

- colimits: do nothing except compute them in `Top`
- limits: just apply $k(-)$ to it.

2. Since j preserves limits, we have that `CGWH` is complete if all limits of `CG` spaces exist in `CGWH`. We must check this on products and equalizers (exercise). For colimits, we need to apply h to the colimit in `CG`, similarly to as we did above.

So we have that `CGWH` is complete, cocomplete, and cartesian closed. $\square$

Notation 2.55. From here on we redefine the category of topological spaces to be `Top` $:=$ `CGWH`, and redefine $\times := \times_k$, $\mathrm{Map} := k\mathrm{Map}$.

2.6 The category of based topological spaces

Let $\mathtt{Top}_*$ be the category of based (CGWH) spaces and based maps. Let $\mathrm{Map}_*(X, Y)$ be the space of based maps, and note that we may base this mapping space at the constant map sending every $x \in X$ to the basepoint y_0 of Y.

The functor $\mathrm{Map}_*(X, -)$ has an adjoint, but it is not given by the product! Instead, the functor $X \wedge -$ is an adjoint. We can verify that, for any other based spaces Y and Z we have a natural isomorphism

$$\mathrm{Hom}_{\mathtt{Top}_*}(Y \wedge X, Z) \cong \mathrm{Hom}_{\mathtt{Top}_*}(Y, \mathrm{Map}_*(X, Z)).$$

Lemma 2.56. There exist natural isomorphisms

1. $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$
2. $X \wedge Y \cong Y \wedge X$
3. $X \wedge S^0 \cong X \cong S^0 \wedge X$.

Proof idea. For (1), it is important that we are in `CGWH`. By [Str, Proposition 2.20], products preserve quotients, so that

$$X \times Y \times Z \to (X \wedge Y) \times Z \to (X \wedge Y) \wedge Z.$$

is a composite of quotient maps, and may be identified with the quotient

$$X \times Y \times Z \to \frac{X \times Y \times Z}{X \vee Y \vee Z}.$$

Thus it does not depend on the order.

$\square$

We remark that the associativity isomorphism does not hold in the category of all topological spaces. A counterexample is given by $(\mathbb{N} \wedge \mathbb{Q}) \wedge \mathbb{Q})$ which is not homeomorphic to $\mathbb{N} \wedge (\mathbb{Q} \wedge \mathbb{Q})$. For a discussion see this mathoverflow post.

Definition 2.57. A *symmetric monoidal category* is a triple $(\mathscr{C}, \otimes, \mathbb{1})$, where $\mathscr{C}$ is a category, $\mathbb{1} \in \mathscr{C}$ is an object, and $\otimes$ is a bifunctor

$$- \otimes - : \mathscr{C} \times \mathscr{C} \to \mathscr{C},$$

such that there are natural isomorphisms

$$\begin{aligned} (c \otimes d) \otimes e &\cong c \otimes (d \otimes e) \\ c \otimes d &\cong d \otimes c \\ \mathbb{1} \otimes c \cong c &\cong c \otimes \mathbb{1}, \end{aligned}$$

for any objects in $\mathscr{C}$, such that all possible diagrams involving these commute (infinite data, really hard to check). However by MacLane's coherence theorem, we only need to check a few diagrams (in particular, the pentagon), and it then follows that all higher diagrams commute.

Definition 2.58. A symmetric monoidal category is *closed* if $- \otimes X$ has an adjoint for all X.

Proposition 2.59. $(\mathtt{Top}_*, \wedge, S^0)$ is a closed symmetric monoidal category.

A More category theory: properties of morphisms

As we have seen, limits and colimits allow us ways to construct new objects, morphisms, and universal properties, over indexing diagrams. In particular we might be interested in relating certain properties of morphisms in a category to induced morphisms produced out of a colimit. As a motivating example, consider the following question.

Question A.1. Let I be a small indexing diagram, and let $\alpha, \beta : I \to \mathtt{Set}$ denote two functors, and let $A_i := \alpha(i)$ and $B_i = \beta(i)$ denote the sets at each object $i \in I$. Suppose we have a natural transformation between these functors, consisting of functions $f_i : A_i \to B_i$ for each i. *If each f_i is injective, is it true that the induced map $f : \operatorname{colim} \alpha \to \operatorname{colim} \beta$ is injective as well?*

It turns out the answer to this diagram depends on the shape of I. It is true if the colimits are *filtered*, which is a condition on the indexing diagram which tells us that it interacts well with finite limits. This motivates the more broad question of what properties of morphisms are preserved under limits and colimits. In general this is a hard question, but it will be important when we investigate fibrations and cofibrations in the category of topological spaces.

A.1 Stability and closure definitions

Let $\mathscr{C}$ be a category, not necessarily assumed to be locally small. We start with an easy definition.

Definition A.2. Let $\mathbf{P}$ be a property of morphisms in $\mathscr{C}$. We say that $\mathbf{P}$ is *closed under composition* if, anytime we have two composable morphisms $f : x \to y$ and $g : y \to z$, if both f and g have property $\mathbf{P}$, then the composite $g \circ f$ does as well.

Definition A.3. Let $\mathbf{P}$ be a property of morphisms in $\mathscr{C}$. We say that $\mathbf{P}$ *satisfies 2-out-of-3* if for every commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C, \end{tikzcd}$$

if any two of f, g, or $g \circ f$ have property $\mathbf{P}$, then the third does as well.

Exercise A.4.

1. Prove that isomorphisms satisfy 2-out-of-3.
2. In the category `Set`, prove that injections and surjections do not satisfy 2-out-of-3.

Definition A.5. Let $\mathbf{P}$ be a property of morphisms. Then we say that $\mathbf{P}$ is *stable under pullback* (often also called *stable under base change*) if for any pullback diagram of the form

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "f"] \\ A \arrow[r, "g"'] & B, \end{tikzcd}$$

if f has property $\mathbf{P}$, then k has property $\mathbf{P}$ as well. Dually, we say that $\mathbf{P}$ is *stable under pushout* if for any pushout diagram of the form

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "k"'] & D \arrow[d, "f"] \\ A \arrow[r, "g"'] \arrow[ur, phantom, "\ulcorner", very near end] & B, \end{tikzcd}$$

if k has property $\mathbf{P}$, then f has property $\mathbf{P}$ as well.

Exercise A.6.

1. Prove that isomorphisms are always stable under pullback.
2. Prove in the category `Set` that injective functions are stable under pullback, and surjective functions are stable under pushout.

Thus far we have discussed closure properties internal to a category. We might wonder how properties translate across functors.

Definition A.7. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and let $\mathbf{P}$ be a property of morphisms.

1. We say that F *preserves property* $\mathbf{P}$ if any time $f : x \to y$ is a morphisms in $\mathscr{C}$ with property $\mathbf{P}$, we have that $Ff : Fx \to Fy$ has property $\mathbf{P}$ as well.
2. We say that F *reflects property* $\mathbf{P}$ if any tie $f : x \to y$ is a morphism in $\mathscr{C}$, if $Ff : Fx \to Fy$ has property $\mathbf{P}$, then f must necessarily have property $\mathbf{P}$ as well.

Remark A.8. We have that F always preserves isomorphisms (this is part of the property of being a functor). It is *not true* that F needs to reflect isomorphisms. For example if X is a topological space with a non-trivial topology, then the map $\mathrm{id} : X \to X_{\mathrm{disc}}$ from X to itself with the discrete topology is not a homeomorphism, however its underlying set map is a bijection. (That is, the forgetful functor $\texttt{Top} \to \texttt{Set}$ does not reflect isomorphisms).

Exercise A.9. If F is full and faithful, it reflects isomorphisms.

A.2 Monomorphisms and epimorphisms

Exercise A.6 generalizes, as the notions of being injective and surjective in `Set` are equivalent to more abstract categorical conditions. We begin by generalizing injections.

Definition A.10. Let $\mathscr{C}$ be a category. Then a morphism $f : x \to y$ is a *monomorphism* if any of the following equivalent conditions hold:

1. For any pair of morphisms $g, h : z \to x$ so that $f \circ g = f \circ h$, that is:
$$z \rightrightarrows x \to y,$$
we have that $g = h$. (Another way of phrasing this is that f is *left cancellable*).
2. The following diagram is a pullback:
$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & y. \end{tikzcd}$$

   If $\mathscr{C}$ is locally small, there is another equivalent condition:

3. The induced functor $\mathrm{Hom}_{\mathscr{C}}(-, x) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathscr{C}}(-, y)$ is a natural injection, meaning that $\mathrm{Hom}_{\mathscr{C}}(z, x) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathscr{C}}(z, y)$ is an injective function for any $z \in \mathscr{C}$.

Exercise A.11. Prove that the definitions in Definition A.10 are equivalent.

Example A.12. We have that isomorphisms and equalizers are always monomorphisms. Any morphism from a terminal object is a monomorphism.

Exercise A.13. *(Closure properties of monomorphisms)*

1. Monomorphisms are closed under composition
2. Monomorphisms are stable under pullback
3. Any time $f \circ g$ is a monomorphism, then g is a monomorphism.

Exercise A.14. *(Left adjoints preserve monomorphisms)* Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ be an adjunction betweem locally small categories. Then if $f : x \to y$ is a monomorphism in $\mathscr{C}$, we have that $Ff : Fx \to Fy$ is a monomorphism in $\mathscr{D}$. (Hint: use the natural bijection associated to the adjunction, together with Definition A.10, Definition (iii)).

There is a dual notion, called an *epimorphism*.

Definition A.15. Let $\mathscr{C}$ be a category. Then a morphism $f : x \to y$ is a *epimorphism* if any of the following equivalent conditions hold:

1. For any pair of morphisms $g, h : y \to z$ so that $g \circ f = h \circ f$, that is:

$$x \to y \rightrightarrows z,$$

   we have that $g = h$. (Another way of phrasing this is that f is *right cancellable*).

2. The following diagram is a pushout:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d, "\mathrm{id}_y"] \\ y \arrow[r, "\mathrm{id}_y"'] & y. \end{tikzcd}$$

   If $\mathscr{C}$ is locally small, there is another equivalent condition:

3. The induced functor $\mathrm{Hom}_{\mathscr{C}}(y, -) \xrightarrow{- \circ f} \mathrm{Hom}_{\mathscr{C}}(x, -)$ is a natural injection, meaning that $\mathrm{Hom}_{\mathscr{C}}(y, z) \xrightarrow{- \circ f} \mathrm{Hom}_{\mathscr{C}}(x, z)$ is an injective function for any $z \in \mathscr{C}$.

Epimorphisms satisfy all the dual properties to monomorphisms.

Exercise A.16. *(Properties of epimorphisms)*

1. Every isomorphism is an epimorphism, as is every coequalizer.
2. Any morphism to an initial object is an epimorphism.
3. Epimorphisms are closed under composition.
4. Epimorphisms are stable under pushout.

5. If $f \circ g$ is an epimorphism, then f is an epimorphism.
6. Right adjoints preserve epimorphisms.

As we have hinted at, monomorphisms and epimorphisms generalize the notions of injectivity and surjectivity in `Set`. We can ask then whether categories which we think of as "sets with extra data" have the property that their monomorphisms are just those underlain by injections. Phrased differently, does the forgetful functor $U : \mathscr{C} \to \texttt{Set}$ reflect mono and epimorphisms? This turns out to be true in some generality, but we must first make rigorous what it means to be a "set with extra structure."

Definition A.17. A *concrete category* is a locally small category $\mathscr{C}$, together with faithful functor to sets $U : \mathscr{C} \to \texttt{Set}$. We think of this functor as "forgetting" the data.

Examples A.18. The following categories are concrete: `Grp`, `Poset`, `Vect`, `Top`, . add more

Proposition A.19. Any faithful functor reflects monomorphisms and epimorphisms.

Thus we see that all the categories in Examples A.18 have the property that their monomorphisms are underlying injections, and epimorphisms are underlying surjections. We summarize this in the following table.

Table 2: Examples of monomorphisms and epimorphisms

Category	**Monomorphisms**	**Epimorphisms**
`Set`	injections	surjections
`Grp`	underlying injections	underlying surjections

Counterexample A.20. We have that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category `Ring` of unital rings. This tells us that the forgetful functor $U : \texttt{Ring} \to \texttt{Set}$ is *not* faithful.

A.3 Preservation properties under limits and colimits

[[todo]]

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