MIT TALBOT WORKSHOP 2021: AMBIDEXTERITY IN CHROMATIC HOMOTOPY THEORY

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Abstract. Notes from the MIT Talbot Workshop 2021: Ambidexterity in Chromatic Homotopy Theory, mentored by Jacob Lurie and Tomer Schlank. Please fix any typos or errors by submitting a pull request at https://github.com/tbrazel/talbot2021.

Contents

1. Overview (Jacob Lurie)

Let G be a finite group, and consider a class in the cohomology of G, denoted $\eta \in H^n(BG;\mathbb{C}^*)$. Cohomology classes are things that you can integrate against cycles. So if we had a compact oriented manifold M of dimension n, together with a map $f : M \to BG$, then we could take the fundamental class of M, and pair it to get an invertible complex number

$$\langle [M], f^*\eta \rangle \in \mathbb{C}^*.$$

If we want a quantity that only depends on M, we could form a combination which considers *all* maps from M into BG.

There are lots of maps $M \to BG$, and if we wanted to classify them up to homotopy, we obtain a bijection with the set of principal G-bundles on M, considered up to isomorphism. To be really concrete, assume further that M is connected and we have chosen a basepoint. Then we have a bijection

$$[M, BG] \leftrightarrow \{\text{principal } G\text{-bundles on } M\} \leftrightarrow \{\text{group homomorphisms } \pi_i \to M\}/\text{conjugation}$$

Since group homomorphisms are determined by where the generators go, we only have finitely many such homomorphisms. We could sum over all group homomorphisms $\pi_1(M) \to G$ to get a complex number of the form

$$\frac{1}{|G|} \sum_{\pi_1 M \to G} \langle [M], f^*\eta \rangle .$$

This is almost the same as

$$\sum_{\substack{\text{homomotopy classes} \\ f: M \to BG}} \langle [M], f^*\eta \rangle$$

This would be true if G were acting by group conjugation here, but it isn't. What we are really doing here is summing with multiplicity

$$\sum_{G\text{-bundles } P \text{ on } M} \frac{\langle [M], f^*\eta \rangle}{|\mathrm{Aut}(P)|}.$$

We will define this quantity the *integral* over the space of all maps $M \to BG$:

$$Z(M) := \int_{f: M \to BG} \langle [M], f^*\eta \rangle := \frac{1}{|G|} \sum_{\pi_1 M \to G} \langle [M], f^*\eta \rangle .$$

If we fix G and M, then this complex number $Z(M)$ is an invariant of manifolds. This is what is assigned to M by Dijkgraaf–Witten theory. It is not a very interesting invariant, because it only depended on the fundamental group of M and the homotopy class of M.

Goal: Find more constructions "like this."

That is, take this construction and vary the ingredients you used to make it. We could try to vary the field $\mathbb{C}$, but we couldn't use a field of characteristic p if $p \mid |G|$. This is because we are normalizing by $|G|$, and we can't throw out the normalization without losing some structure.

Given a manifold M, we can look at the space of maps $\mathrm{Map}(M, BG)$. We have an evaluation and projection

$$\begin{tikzcd} M \times \mathrm{Map}(M, BG) \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & BG \\ \mathrm{Map}(M, BG) & \end{tikzcd}$$

So we are beginning with η, pulling it back to $\mathrm{ev}^*\eta$, and then pushing it forward along π to obtain $\pi_*\mathrm{ev}^*\eta$, which is a cohomology class of degree zero, i.e. a function of the form $\mathrm{Map}(M, BG) \to \mathbb{C}^*$. This is what happens when M is an n-manifold.

Suppose M is now an $(n-1)$-manifold. Then the pullback and pushforward will be a class

$$\mathcal{L} := \pi_*\mathrm{ev}^*\eta \in H^1\left(\mathrm{Map}(M, BG), \mathbb{C}^*\right).$$

That is, it corresponds to a local system of 1-dimensional complex vector spaces. This cohomology class determines the local system $\mathcal{L}$ up to isomorphism.

Given a 1-dimensional local system, we can try to integrate it, to obtain a single vector space. We could look at the cohomology $H^0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$, or we could look at the homology $H_0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$. We want to assign a complex vector space $Z(M)$ which corresponds to either of these. These two vector spaces *turn out to be the same.*

We have that $\pi_0\mathrm{Map}(M, BG)$ corresponds to G-bundles on M, so

$$\mathrm{Map}(M, BG) = \coprod_{\text{iso classes of G-bundles } P} B\mathrm{Aut}(P).$$

So let's start by thinking about local systems on things like $B\mathrm{Aut}(P)$.

Suppose that H is a finite group, and let's consider complex local systems $\mathcal{L}$ on BH. This is the same thing as a complex representation V of H. We have that

$$H^0\left(BG, \mathcal{L}\right) = V^H = \left\{v \in V \colon hv = v \forall h \in H\right\}.$$

The homology is

$$H_0\left(BG, \mathcal{L}\right) = V_H = V \big/ \mathbb{C} \cdot \left\{hv - v\right\}.$$

This is the minimal quotient of V you can form on which H acts trivially. When H is a finite group, there is an obvious relation between these. We have an averaging function

$$\begin{aligned} V &\to V \\ v &\mapsto \sum_{h \in H} hv. \end{aligned}$$

This map factors through the subspace V^H, but it also factors through the quotient given by the coinvariants, since it annihilates vectors of the form $hv - v$. So we get a norm

$$\begin{tikzcd} V \arrow[rr] \arrow[dr] & & V \\ & V_H \arrow[r, "\mathrm{Nm}"'] & V^H \arrow[u] \end{tikzcd}$$

Basic fact: This map is an isomorphism (assuming characteristic zero).

Proof. We should write down the inverse map. There is an obvious map in the other direction:

$$V^H \subseteq V \twoheadrightarrow V_H.$$

Let's call this map λ. We see that $\lambda \circ \mathrm{Nm} = \mathrm{Nm} \circ \lambda$, which is multiplication by the order of H. Over characteristic zero, this multiplication is an isomorphism. □

Remark 1.0.1. Recall that if M is an n-manifold, we thought about this function $\mathrm{Map}(M, BG) \to \mathbb{C}^*$, given by $f \mapsto \langle [M], f^*\eta \rangle$. This map gives us something in $H^0(\mathrm{Map}(M, BG), \mathbb{C}^*)$. This integration procedure was

$$Z(M) = \int_{\mathrm{Map}(M,BG)} \langle [M], f^*\eta \rangle,$$

took the class in the degree zero cohomology, but we identified cohomology with homology by doing this norm map on every component.

Thinking as an algebraic topologist, we can turn fields K into cohomology theories HK. Thinking about fields from a very large distance, there are fields of characteristic zero, and those of characteristic p. Morava realized that in the world of cohomology theories, there are a hierarchy of examples which interpolate between things like $H\mathbb{Q}$ and things like $H\mathbb{F}_p$. Fixing a prime number p, we have that Morava K-theories are an infinite sequence of cohomology theories, with

$$H\mathbb{Q} = K(0) \subseteq K(1) \subseteq \cdots \subseteq K(\infty) = H\mathbb{F}_p.$$

Question: Do these constructions make sense "over $K(n)$?"

Morava K-theories are characteristic p objects, since multiplication by p is the zero map $K(n) \to K(n)$ for $n > 0$. If we think characteristic p is bad, we might think Morava K-theories are bad. However the answer to this question is yes!

Theorem 1.0.2. (Hovey–Sadofsky) Let V be a $K(n)$-module with an action of a finite group H. Then the norm map

$$\mathrm{Nm}_H : V_{hH} \to V^{hH}$$

is an isomorphism for $n < \infty$.

Suppose that V and W are $K(n)$-modules and suppose we have a family of maps $f_x : V \to W$ parametrized by $x \in BH$. That is, a continuous map $f : BH \to \mathrm{Map}(V, W)$. Yet another way to think about this data is considering f as an element of $H^0\left(BH, \underline{\mathrm{Map}(V, W)}\right)$. Since V and W were $K(n)$-modules, we have that $\underline{\mathrm{Map}(V, W)}$ is a $K(n)$-module (with two $K(n)$-module structures). This theorem earlier tells us that

$$H^0\left(BH, \underline{\mathrm{Map}(V, W)}\right) \cong H_0\left(BH, \underline{\mathrm{Map}(V, W)}\right) \to H_0\left(*, \underline{\mathrm{Map}(V, W)}\right) = \pi_0\mathrm{Map}(V, W),$$

by mapping along $BH \to *$. Thus using this theorem from earlier, we can go from a *family* of maps, to a *single* map $V \to W$. We denote this procedure by

$$\mathrm{Map}(BH, \mathrm{Map}(V, W)) \to \pi_0\mathrm{Map}(V, W)$$
$$f \mapsto \int f.$$

We saw this earlier when $n = 0$ and when $V = W = \mathbb{C}$.

Now let's assume that H is abelian. Then BH is an abelian group object in spaces. What if we want to study representations of BH? That is, local systems on $B(BH) = K(H, 2)$. This is a simply connected space, so there should be no local systems on it, that is, this doesn't make sense classically.

So instead we want to study representations of BH on $K(n)$-modules, that is, local systems $\mathcal{L}$ of BH-modules. We could study the analogue of the coinvariants and invariants, which are the homotopy (co)limits over $\mathcal{L}_x$, where $x \in BH$. The Hovey–Sadofsky theorem gives

$$\lim_{BH} : \mathrm{hocolim}_{y \in K(H,2)}\mathcal{L}_y \to \mathrm{holim}_{x \in K(H,2)}\mathcal{L}_x.$$

To give such a map is to give a family of maps $f_{x,y} : \mathcal{L}_x \to \mathcal{L}_y$, and these should vary continuously in x and y. Any path $p : [0, 1] \to K(H, 2)$ satisfying $p(0) = x$ and $p(1) = y$ determines an isomorphism $p_! : \mathcal{L}_y \to \mathcal{L}_x$. This depends not only on x and y but also on the path that we chose. The collection of such paths is parametrized by a space $\{x\} \times^h_{K(H,2)} \{y\} =: P_{x,y}$. So we have a collection of isomorphisms $\mathcal{L}_y \xrightarrow{\sim} \mathcal{L}_x$ parametrized by the space $P_{x,y} \simeq K(H, 1) = BH$.

We can then use that integration procedure to get

$$f_{x,y} = \int_{p \in P_{x,y}} p_!,$$

which is a single morphism $\mathcal{L}_y \to \mathcal{L}_x$ (not necessarily an isomorphism anymore). So allowing x and y to vary, we get a single map

$$\mathrm{Nm}_{K(H,2)} : \mathrm{hocolim}\mathcal{L} \to \mathrm{holim}\mathcal{L}.$$

Theorem 1.0.3. This map is also a homotopy equivalence.

We can now do this again — suppose we are interested in representations of $K(H,2)$, then $K(H,3)$ and so on. This yields the following.

Theorem 1.0.4. Let X be a space with finitely many homotopy groups, and all homotopy groups are assumed to be finite[1] and let $\mathcal{L}$ be a local system of $K(n)$-modules on X. Then there is a canonical isomorphism

$$\mathrm{Nm}_X : \mathrm{hocolim}\mathcal{L}_x \to \mathrm{holim}\mathcal{L}_x.$$

That is, there is some natural map which induces isomorphisms $H_*(X,\mathcal{L}) \xrightarrow{\sim} H^*(X,\mathcal{L})$.

This is an interesting statement even when $\mathcal{L}$ is a trivial local system. In particular if X has finitely many homotopy groups, there is a canonical isomorphism

$$K(n)_*(X) \xrightarrow{\sim} K(n)^*(X).$$

We can think about this as a statement about X: if X is a nice space it satisfis a Poincaré duality with respect to Morava K-theory. We could also think about it as a statement about the category of $K(n)$-local spectra — it is not just an *additive* category, but it has some kind of fancier additivity where we can take a collection of morphisms indexed over a space and "add" or integrate the maps together. This theorem is also addressing the question that we started with — are there other constructions of Dijkgraaf–Witten theory? Yes, we can replace the height zero complex numbers by things of higher height, like Lubin–Tate spectra.

Question: Why is this true (in an easy example)?

If $X = K(H,2)$, the Hovey–Sadofsky theorem gives us a map

$$K(n)_*(X) \to K(n)^*(X).$$

There is an element $1 \in K(n)^0(X)$, and suppose we could find something, call it $y \in K(n)_0(X)$, mapping to it under the norm. Then if we had such a y, we would have that multiplication by y will induce a map from

$$\Theta : \mathrm{holim}\mathcal{L} \to \mathrm{hocolim}\mathcal{L}.$$

In classical ordinary homology this is called the cap product. The condition $\mathrm{Nm}(y) = 1$ is equivalent to the statement that Θ is inverse to the norm map.

Example 1.0.5. Let $X = BH$ for a finite p-group H, and $n = 1$. Then we have a map

$$K(n)_*(BH) \to K(n)^*(BH).$$

In height one, we know what these mean — these lift to characteristic zero, since $K(1) = \widehat{\mathrm{KU}}/p$. Complex K-theory of BH is described by the Atiyah–Segal completion theorem,

[1] For example $BH, B^2H, \ldots$ where H is finite abelian.

so we have that

$$\widehat{\mathrm{KU}}^0(BH) = \mathtt{Rep}(H)^\wedge$$
$$K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

So our map would be

$$\mathtt{Rep}(H)^\vee/p = K(1)_0(BH) \to K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

By character theory, $\mathtt{Rep}(H) \otimes \mathbb{C}$ is the conjugation-invariant functions $H \to \mathbb{C}$, by sending V to its character χ_V. If we p-adically complete, we are really getting a map

$$\mathtt{Rep}(H)^\vee \to \mathtt{Rep}(H).$$

Rationally, everything is computable, and we can compute that it is an isomorphism. We can study the inverse isomorphism then

$$\mathbb{Q} \otimes \mathtt{Rep}(H)^\vee \leftarrow \mathbb{Q} \otimes \mathtt{Rep}(H).$$

Over $\mathbb{C}$, this bilinear form is given by $V, W \mapsto \frac{1}{|H|}\sum_{h\in H} \chi_V(h)\chi_W(h)$. To know that this isomorphism exists integrally and not rationally, we need to check this value is an integer. But we can rewrite this as

$$\begin{aligned} \frac{1}{|H|}\sum_{h\in H} \chi_V(h)\chi_W(h) &= \frac{1}{|H|}\sum_{h\in H} \chi_{V\otimes W}(h) \\ &= \dim_{\mathbb{C}} \left(V \otimes W\right)^H. \end{aligned}$$

So this is a sketch of the proof of the Hovey–Sadofsky theorem in height one.

2. Review of chromatic homotopy theory (Elizabeth Tatum)

2.1. **Formal group laws.** Let R be a commutative ring, then we can consider formal group laws over that ring $F \in R[[x, y]]$. We say that F is a *formal group law* if

(1) $F(x, 0) = x$
(2) $F(x, y) = F(y, x)$
(3) $F(x, F(y, z)) = F(F(x, y), z)$.

Let $f \in R[[x]]$. We say that it is a *homomorphism* from a formal group law F to a formal group law G if $f(F(x, y)) = G(f(x), f(y))$.

We let the *n-series* on a formal group law F be given by

$$[n]_F(x) := F(x, F(x, \dots F(x, x))).$$

That is, F applied n times. In particular when $n = p$ is a prime, we get that

$$[p]_F(x) \equiv ax^{p^h} + \text{higher order terms}.$$

We say that F has height $\geq h$ if there are no higher order terms, and that F has height exactly h if a is a unit.

Example 2.1.1. We have the *additive formal group law* $F_a(x, y) = x + y$, with height ∞.

Example 2.1.2. We have the *multiplicative formal group law* $F_m(x, y) = x+y+xy$, which has height 1.

Theorem 2.1.3. (Lazard) Over an algebraically closed field, F and G should have the same height if and only if F is isomorphic to G.

2.2. **Complex oriented cohomology theories.** We say that a ring spectrum E is *complex orientable* if the map $E^2(\mathbb{C}\mathrm{P}^\infty) \to E^2(S^2)$ is surjective. In particular in the reduced cohomology, an orientation is a choice of generator x mapping to $1 \in \pi_0(E)$ under the composite

$$\widetilde{E}(\mathbb{C}\mathrm{P}^\infty) \to \widetilde{E}(S^2) \simeq \pi_0 E.$$

We have that $\mathbb{C}\mathrm{P}^\infty$ has a natural multiplication, so by applying E^*, we get

$$E^*\mathbb{C}\mathrm{P}^\infty \to E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$$
$$x \mapsto F(x^L, x^R.$$

So any choice of complex orientation yields a formal group law.

- $H(\mathbb{Z}/p)$ carries F_a
- KU carries F_m
- MU carries the universal formal group law F_{MU}.

This universal fgl is characterized by the property that for any formal group law F over R, there is a $\theta : \mathrm{MU}_* \to R$ so that $\theta(F_{\mathrm{MU}}) = F$. Here $\mathrm{MU}_* \cong \mathbb{Z}[x_1, x_2, \dots]$, where $|x_i| = 2i$.

2.3. **Morava K-theories and related spectra.** The *Brown–Peterson spectrum*, for a fixed prime p, is a wedge summand in complex cobordism

$$\mathrm{MU}_{(p)} \simeq \wedge \mathrm{BP}.$$

We have that BP is a ring spectrum such that $\mathrm{BP}_* \cong \mathrm{MU}_{(p)}/(x_i,\ i \neq p^k - 1)$. Thus

$$\mathrm{BP}_* \cong \mathbb{Z}_{(p)}\left[v_1, v_2, \ldots\right].$$

The x_i's are *not* living in powers of the form $2(p^k - 1)$, and we are quotienting them out. So the v_i's *are* living in those powers — $|v_i| = 2(p^i - 1)$.

Applying $[p]_{F_{\mathrm{MU}}} \to [p]_F$, we are getting that the height of F was the coefficient a appearing in $[p]_F = ax^{p^h}$. So where the v_i's land after this map, v_h is landing on a.

Morava E-theory: The Johnson–Wilson spectrum has homology $v_n^{-1}\mathrm{BP}_*/\left(v_{n+1}, v_{n+2}, \ldots\right) \cong \mathbb{Z}_{(p)}\left[v_1, \ldots, v_{n-1}, v_n^{\pm}\right]$. Morava E-theory is the completion of this – we delinate this from the Johnson–Wilson spectrum $E(n)$ by writing a subscript E_n:

$$(E_n)_* \cong \mathbb{W}(k)\left[\left[u_1, \ldots, u_{n-1}\right]\right]\left[u_n^{\pm}\right].$$

Morava E-theory tells you about the deformations of a formal group law of height n.

Deformations: If $\phi : R \twoheadrightarrow k$ is a nice ring homomorphism, then a formal group law F over R is a deformation of some formal group law G over k if $\phi(F) = G$. We think e.g. about R being some infinitesimal thickening of the field k.

Morava K-theory: We have a $K(n)$ so that $K(n)_* \cong \mathbb{F}_p\left[v_n^{\pm}\right]$. This is a formal group law of height exactly n. At each prime we have Morava K-theories $K(1), K(2), \ldots$. The Morava E-theories $E(n)$ are telling you about the open sets containing the $K(i)$'s for $i < n$. Morava K-theories are like residue fields, and Morava E-theories are like complete local rings at these points.

The $K(n)$'s are like fields in ring spectra. We would say that E is a *field* if $E_*(X)$ is a sum of free E_*-modules.

Theorem 2.3.1. E is a field if and only if E is a $K(n)$.

Furthermore, we have that $K(n) \wedge X \simeq \wedge \Sigma^? K(n)$ is a wedge sum of suspensions of $K(n)$.

2.4. **Bousfield localization.** Fix a ring spectrum E. We say that X is *E-acyclic* if $E_*X = 0$. We say that X is *E-local* if for every E-acyclic Y, we have that $[Y, X] \simeq *$. Finally we say that $f : X \to Y$ is an *E-equivalence* if $E_*(f)$ is an isomorphism.

A *localization functor* is a functor of the form

$$L : Sp \to Sp,$$

together with a natural transformation $\eta : \mathrm{id} \to L$ so that

(1) $L\eta : L \to L^2$ is an equivalence (localizing twice doesn't do anything)
(2) $L\eta \simeq \eta L$.

Theorem 2.4.1. (Bousfield) For every spectrum E, there exists a localization functor $L_E : Sp \to Sp$ with a natural transformation η_E such that for every X, we have that $\eta_E : X \to L_E X$ is the initial E-equivalence.

That is,

(1) $E_*(\eta_X) : E_*X \to E_*L_E X$ is an isomorphism
(2) If $f : X \to Y$ is an E-equivalence and Y is E-local then there is a unique map making the diagram commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ L_E X \arrow[ur, dashed] & \end{tikzcd}$$

Let $\langle E \rangle$ denote the Bousfield class of E — that is, the "collection of E-local spectra." We say that $\langle E \rangle \subseteq \langle F \rangle$ if X is E-local implies that X is F-local.

Proposition 2.4.2. We have that

(1) If $\langle E \rangle = \langle F \rangle$ then there is a natural isomorphism $L_E \simeq L_F$.
(2) If $\langle E \rangle \subseteq \langle F \rangle$, then we have that $L_E L_F \simeq L_E$, and there is a natural transformation $\eta : L_F \to L_E$.

Fact 2.4.3. We have that L_{E_n} is smashing — this means that $L_{E_n}(X) \simeq \left(L_{E_n}(S^0)\right) \wedge X$, and the localization $L_E : Sp \to Sp$ preserves direct sums.[2]

So we get an algebraic chromatic tower

$$L_{E_0}X \leftarrow L_{E_1}X \leftarrow L_{E_2}X \leftarrow \cdots$$

These have monochromatic layers $M_i(X) = \ker\left(L_{E_i}X \to L_{E_{i-1}}X\right)$, which come with maps $M_i(X) \to L_{K(i)}X$. The monochromatic layers and the localization at K-theory are not the same as spectra, but they contain exactly the same information.

Chromatic convergence theorem 2.4.4. If X is a p-local finite spectrum, then

$$X \simeq \operatorname{holim} L_n(X)$$

Fact 2.4.5. Let $\mathscr{C}_0$ denote the full subcategory of p-local finite spectra. Then denote by $\mathscr{C}_n$ the full subcategory of $K(n)$-acyclics, so we have a chain of inclusions

$$\mathscr{C}_0 \supseteq \mathscr{C}_1 \supseteq \cdots \supseteq \mathscr{C}_\infty = \{*\}\,.$$

[2] Localization always sends direct sums in *Sp* to direct sums in E-local spectra. What this condition means is that it preserves direct sums *in spectra*. This is really telling you that the *inclusion* of E-local spectra into spectra preserves direct sums (and actually arbitrary colimits).

Generally a sum of local things won't be local (think about p-completion). However smashing localizations will have this property.

Localization generally feels like the analogue of localization and then completion for rings. Smashing localizations are just localizations.

A full subcategory is called *thick* if it is closed under

(1) retracts
(2) weak equivalences
(3) cofiber sequences.

Example 2.4.6. We have that E-acyclics, E-local objects are thick subcategories.

Thick subcategory theorem 2.4.7. If $\mathscr{C}$ is a thick subcategory p-local finite spectra, then $\mathscr{C}$ is one of the $\mathscr{C}_n$'s from the filtration above.

We say that a finite spectrum F is *type* n if $K(i)_*(F) = 0$ for all $i < n$, and $K(n)_*(F) \neq 0$.

Let F be any type n spectrum. Then a v_n*-self map* is a map $f : \Sigma^i F \to F$ so that

$$K(m)_*(f) = \begin{cases} \text{multiplication by a rational number} & m = n = 0 \\ \text{an isomorphism} & m = n \neq 0 \\ \text{nilpotent} & m \neq n. \end{cases}$$

Periodicity theorem 2.4.8. Any finite type n spectrum admits a v_n-self map. The telescope of this map is

$$\mathrm{Tel}(F) = \operatorname{hocolim}\left(F \xrightarrow{v_n} F \xrightarrow{v_n} F \to \cdots\right),$$

and this is independent of the choice of v_n-self map and the choice of finite type n spectrum. So we can call this $T(n)$.

Fact 2.4.9. $T(n)$ is $K(m)$-acyclic for all $m \neq n$. Applying $K(m)_*$ to the map above, we are taking a homotopy colimit along nilpotents, so this vanishes.

There is a natural transformation $\lambda : L_{T(n)} \to L_{K(n)}$. For finite spectra, we know that $T(n)$-acyclics and $K(n)$-acyclics are the same. Knowing this for all spectra would imply the localizations are the same, which is the telescope conjecture.[3]

[3]The category of $T(n)$-local things contain all $K(n)$-local things. It might be larger. This implies that every $T(n)$-acyclic is always $K(n)$-acyclic. There are certain spectra for which $T(n)$ and $K(n)$-acyclic coincide — we know this for finite spectra and for ring spectra (it follows from the nilpotence theorem).

3. The Tate construction (Andres Mejia)

[todo — get tex from Lucy]

4. Ambidexterity (Thomas Brazelton)

4.1. **Local systems.** A local system is, very roughly speaking, anything you might want to take cohomology in. Classically speaking, a *local system* of abelian groups on a space X is a locally constant sheaf $\mathcal{L}$ on X.

Example 4.1.1. Local systems subsume singular cohomology — this is because for any abelian group A, we can take the constant sheaf $\underline{A}$ considered as a local system.

If X is path-connected, and $\mathcal{L}$ is a local system on X, then we can take any two points x and y, and a path $\gamma : [0,1] \to X$ between them (that is, $\gamma(0) = x$ and $\gamma(1) = y$). We see that $\gamma^*\mathcal{L}$ is constant, giving an isomorphism between the fibers $\mathcal{L}_x$ and $\mathcal{L}_y$. We can check that homotoping γ will not affect the isomorphism $\mathcal{L}_x \xrightarrow{\sim} \mathcal{L}_y$. That is, we can restate $\mathcal{L}$ as the assignment of the data:

- an abelian group $\mathcal{L}_x$ for every $x \in X$
- an isomorphism $\mathcal{L}_x \xrightarrow{\sim} \mathcal{L}_y$ for every homotopy class of paths $x \to y$,

subject to some extra coherence data. From this we can get a new definition of a local system.

Definition 4.1.2. A *local system* on X valued in a 1-category $\mathscr{C}$ is a functor

$$\mathcal{L} : \Pi_1(X) \to \mathscr{C}.$$

Suppose now we want something a little stronger. If γ, γ' are homotopic maps from x to y in X, they provide isomorphisms $\mathscr{C}_x \xrightarrow{\sim} \mathscr{C}_y$ in $\mathscr{C}$. If $\mathscr{C}$ is a 2-category, we might ask for a witness of the homotopy $\gamma \Rightarrow \gamma'$ to be witnessed by a 2-cell in $\mathscr{C}$, and for a different witness to be witnessed by a different 2-cell. Similarly if we have a 3-cell between these, we might ask for a 3-cell witnessing a higher homotopy in $\mathscr{C}$, provided $\mathscr{C}$ has this higher categorical structure.

This leads us to a higher-categorical definition of local systems.

Definition 4.1.3. A *local system* on X valued in an ∞-category $\mathscr{C}$ is an ∞-functor

$$\mathcal{L} : \Pi_\infty(X) \to \mathscr{C},$$

where $\Pi_\infty(X)$ is the fundamental ∞-groupoid of X.

Viewing X as a Kan complex, we might just say a local system is an ∞-functor

$$\mathcal{L} : X \to \mathscr{C}.$$

4.2. **Pullback and adjoints.** Let $f : X \to Y$ be any map of spaces. Then given a local system $\mathcal{L} : Y \to \mathscr{C}$ on Y, we can pull it back to a local system $f^*\mathcal{L}$ on X, by pre-composing with f. For any fixed ∞-category $\mathscr{C}$, this defines a functor

$$f^* : \mathrm{Fun}(Y, \mathscr{C}) \to \mathrm{Fun}(X, \mathscr{C}).$$

If $\mathscr{C}$ admits small colimits, then we may left Kan extend to define a left adjoint to f^* (Higher Topos Theory, 4.3.3). We denote this by $f_!$:

$$f_! : \mathrm{Fun}(X, \mathscr{C}) \leftrightarrows \mathrm{Fun}(Y, \mathscr{C}) : f^*.$$

Dually when $\mathscr{C}$ admits small limits, we may right Kan extend to define a right adjoint to f^*, which we denote by f_*. This gives

$$f_! \dashv f^* \dashv f_*.$$

Example 4.2.1. Let S be a set, viewed as a discrete space, and consider the map $f : S \to *$. Pullback is then the diagonal map $f^* : \mathscr{C} \to \mathrm{Fun}(S, \mathscr{C})$. We see that any functor $S \to \mathscr{C}$ picks out a collection $\{C_s\}$ of objects in $\mathscr{C}$ for each $s \in S$. Assume that $\mathscr{C}$ has all products and coproducts. Then we can see that

$$\begin{aligned} f_! : \mathrm{Fun}(S, \mathscr{C}) &\to \mathscr{C} \\ \{C_s\} &\mapsto \coprod_{s \in S} C_s, \end{aligned}$$

and that

$$\begin{aligned} f_* : \mathrm{Fun}(S, \mathscr{C}) &\to \mathscr{C} \\ \{C_s\} &\mapsto \prod_{s \in S} C_s. \end{aligned}$$

There is always a natural transformation from products to coproducts here, given by $f_! \to f_*$. In particular when products and coproducts agree, e.g. in $\mathtt{Ab}$, we will have that this is a natural isomorphism $f_! \simeq f_*$.

Example 4.2.2. Consider $f : BG \to *$. In this case, since $\mathrm{Fun}(*, \mathscr{C}) \simeq \mathscr{C}$, we have that pullback is of the form

$$f^* : \mathscr{C} \to \mathrm{Fun}(BG, \mathscr{C}),$$

assigning to every object in $\mathscr{C}$ the trivial G-action.

In this case, the adjoints yield, for every G-equivariant object $C \in \mathscr{C}$, the coinvariants $f_! C = C_G$ and the invariants $f_* C = C^G$. Denoting by $C^{tG} = \mathrm{cofib}\left(C_G \to C^G\right)$, we have that a canonical equivalence $f_! \simeq f_*$ would imply that the Tate construction vanishes for every G-equivariant object of $\mathscr{C}$.

Associated to these types of adjunction we have the so-called "calculus of mates," which allows us to take commutative squares of spaces and discuss how the induced functors relate to one another.

Another example of where the calculus of mates appears is in the types of natural isomorphisms of restriction and extension of scalars for modules that come out of commutative diagrams of rings.

Proposition 4.2.3. If f and g are composable, then there is a canonical equivalence $(gf)^* \simeq f^*g^*$. This induces a canonical equivalence $(gf)! \simeq g_! f_!$ by the formalism of adjunctions.

Definition 4.2.4. Consider a commutative diagram of spaces

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "g"'] & Y. \end{tikzcd}$$

Then there is a *Beck–Chevalley exchange transformation* (think about this as top-left to bottom-right), denoted by

$$\mathrm{Ex}_!^* : j_! i^* \to f^* g_!.$$

This is defined by first starting with $j_! i^*$, and tacking on the counit $\mathrm{id}_B \to g^* g_!$ on the end of it. We then get $j_! i^* g^* g_!$. Since the diagram commutes, there is a canonical equivalence $i^* g^* \simeq j^* f^*$, getting us to $j_! j^* f^* g_!$. Finally, we may apply the counit $j_! j^* \to \mathrm{id}$ to conclude. The entire composite gives us:

$$j_! i^* \mathrm{id}_B \to j_! i^* g^* g_! \simeq j_! j^* f^* g_! \to f^* g_!.$$

Proposition 4.2.5. If we have a pullback square, the Beck–Chevalley exchange transformation is an equivalence.

Q: Let $f : X \to Y$, and consider the adjunction $f_! \dashv f^*$. When will $f_!$ also be a *right adjoint* to f^*?

Given a fixed category $\mathscr{C}$ admitting finite limits and colimits, we will define a class of *$\mathscr{C}$-ambidextrous* maps $f : X \to Y$. These will have the property that if $f : X \to Y$ is $\mathscr{C}$-ambidextrous, then there is a canonical equivalence $f_! \simeq f_*$.

4.3. **Ambidextrous morphisms.**

Example 4.3.1. Suppose that $f : X \xrightarrow{\sim} Y$ is a homotopy equivalence. Then $f^* : \mathrm{Fun}(Y, \mathscr{C}) \to \mathrm{Fun}(X, \mathscr{C})$ is an equivalence of categories, and it can be easily promoted to an adjoint equivalence, so that $f_! \simeq f_*$ canonically. In particular, there is a unit map $\mu_f : \mathrm{id} \to f_! f^*$, exhibiting $f_!$ as a right adjoint to f^*.

Homotopy equivalences provide our first class of morphisms which we call *ambidextrous*. Somehow these are the "most" ambidextrous, in the sense that they have the strongest structure. However as we might expect, there exist morphisms which are $\mathscr{C}$-ambidextrous without being homotopy equivalences.

We will define ambidexterity inductively, with homotopy equivalences being the base case. For indexing reasons that will become clear later, we would like to start at $n = -2$. So we will define, for each $n \geq -2$:

- A collection of *n-ambidextrous morphisms* in `Top`
- For each n-ambidextrous morphism $f : X \to Y$, a natural transformation $\mu_f^{(n)} : \mathrm{id} \to f_! f^*$, well-defined up to homotopy, exhibiting $f_!$ as a right adjoint to f^*.

Base case $n = -2$: We say f is (-2)-ambidextrous if and only if f is an equivalence. In this case, we define $\mu_f^{(-2)}$ to be any homotopy inverse to the counit $f_! f^* \to \mathrm{id}$.

Inductive step: Suppose that we have defined n-ambidextrous morphisms for some n. We will define $(n+1)$-ambidextrous maps in two steps: first we define *weakly* $(n+1)$-ambidextrous maps, and then $(n+1)$-*ambidextrous* maps.

Let $f : X \to Y$ be arbitrary, and consider the diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

By Beck–Chevalley, there is an exchange isomorphism $(\pi_1)_! \pi_2^* \simeq f^* f_!$. We say that f is *weakly* $(n+1)$-*ambidextrous* if δ is n-ambidextrous. In this context, we define a counit $\nu_f^{(n+1)}$ to be the composite

$$f^* f_! \xrightarrow{\left(\mathrm{Ex}_!^*\right)^{-1}} (\pi_1)_! \pi_2^* \xrightarrow{\mu_\delta^{(n)}} (\pi_1)_! \delta_! \delta^* \pi_2^* = (\mathrm{id}_X)_! \mathrm{id}_X^* = \mathrm{id}_{\mathrm{Fun}(X,\mathscr{C})}$$

We say f is $(n+1)$-*ambidextrous* if the following conditions hold:

(1) The transformation $\nu_f^{(n+1)} : f^* f_! \to \mathrm{id}$ is the counit for an adjunction $f^* \dashv f_!$, with some unit $\mu_f^{(n+1)}$

(2) Weak $(n+1)$-ambidexterity is closed under pullback along f. That is, for every pullback square

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

we have that g is weakly ambidextrous, with counit $\nu_g^{(n+1)} : g^* g_! \to \mathrm{id}$ defined in the Beck–Chevalley process above

(3) Property (1) is closed under pullback along f. That is, for any pullback square as above, we have that $\nu_g^{(n+1)}$ is the counit of an adjunction $g^* \dashv g_!$.

From this definition, the following are immediate.

Proposition 4.3.2. (Weak) n-ambidexterity is closed under pullback.

Moreover from our inductive definitions, we have the following:

Proposition 4.3.3. Let $-2 \le m \le n$.

(1) If f is weakly m-ambidextrous,[4] then f is weakly n-ambidextrous, and $\nu_f^{(m)}$ and $\nu_f^{(n)}$ agree up to homotopy.
(2) If f is m-ambidextrous, then f is n-ambidextrous, and $\mu_f^{(m)}$ and $\mu_f^{(n)}$ agree up to homotopy.

Proof idea. It suffices to let $n = m+1$, and induct. The inductive step is basically immediate from definitions, and the base case is very direct. $\square$

Definition 4.3.4. We say that f is *weakly ambidextrous* if it is weakly ambidextrous for some $n \ge -1$, and we say that f is *ambidextrous* if it is ambidextrous for some n. We let $\nu_f : f^* f_! \to \mathrm{id}$ denote the counit and $\mu_f : \mathrm{id} \to f_! f^*$ denote the unit. This notation is well-defined up to homotopy by the previous proposition.

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \left\{\substack{n\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] \arrow[d, hook] & \left\{\substack{(n+1)\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \{\text{ambidextrous maps}\} \arrow[d, hook] \\
\cdots \arrow[r, hook] & \left\{\substack{\text{weakly} \\ n\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] & \left\{\substack{\text{weakly} \\ (n+1)\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] & \cdots \arrow[r, hook] & \left\{\substack{\text{weakly} \\ \text{ambidextrous maps}}\right\}
\end{tikzcd}$$

4.4. **Norms.** Suppose $\mathscr{C}$ is an ∞-category with small limits and colimits. Let $f : X \to Y$ be a continuous map of spaces, and let $f_! \dashv f^* \dashv f_*$ be the associated left and right adjoints to pullback provided by Kan extensions. Suppose that f is *weakly ambidextrous* but **not necessarily ambidextrous** (recall this means inductively that the diagonal is weakly ambidextrous of one degree lower, and crucially that there is a natural transformation $\nu_f : f^* f_! \to \mathrm{id}$). Then by adjunction we have a natural homotopy equivalence of mapping spaces

$$\mathrm{Map}\left(f^* f_!, \mathrm{id}\right) \simeq \mathrm{Map}\left(f_!, f_*\right).$$

In particular ν_f maps to a natural transformation, which by definition is the composite

$$f_! \xrightarrow{\eta \cdot f_!} f_* f^* f_! \xrightarrow{f_* \cdot \nu_f} f_*.$$

We call this the *norm* of f and denote it by $\mathrm{Nm}_f : f_! \to f_*$.

Proposition 4.4.1. Let f be weakly ambidextrous as above. Then it is ambidextrous if and only if

[4] Weak ambidexterity isn't defined for $m = -2$ but that's ok

(1) Weak ambidexterity is preserved under pullback along f
(2) The norm map $\mathrm{Nm} : f_! \to f_*$ is an equivalence
(3) The norm map for any map obtained by pullback along f is an equivalence.

Example 4.4.2. We can rephrase our example from earlier to say that the following are equivalent for $f : BG \to *$:

(1) BG is $\mathscr{C}$-ambidextrous
(2) The norm Nm_f is an equivalence
(3) For every G-equivariant object of $\mathscr{C}$, the Tate construction vanishes.

Proposition 4.4.3. Weak ambidexterity is closed under composition — that is, if f and g are composable and weakly ambidextrous, we can take μ_{gf} to be the composite

$$(gf)^*(gf)_! \simeq f^* g^* g_! f_! \xrightarrow{\mu_g} f^* f_! \xrightarrow{\mu_f} \mathrm{id}.$$

5. Higher semiadditivity as span-modules (Kaif Hilman)

5.1. **Introduction.** In [Seg74] Segal showed that the structure of commutative monoids in an arbitrary category $\mathbb{C}$ with finite products can be cleanly encoded as product-preserving functors $\mathrm{Fin}_* \to \mathbb{C}$ where Fin_* is the 1-category of finite pointed sets. Moreover, if we write $\mathrm{CMon}(\mathbb{C}) := \mathrm{Fun}^{\times}(\mathrm{Fin}_*, \mathbb{C})$ for the category of commutative monoids, it turns out that if $\mathbb{C}$ were presentable, then $\mathrm{CMon}(\mathbb{C})$ is the free semiadditive category generated by $\mathbb{C}$. It would be desirable to show that something similar holds for *higher semiadditivity.*

To this end, observe that Fin_* can also be thought of as a category of *spans* whose objects are finite sets and a morphism from X to Y is a span

$$X \hookleftarrow Z \to Y$$

where $X \hookleftarrow Z$ is injective. Hence, it is natural to expect that the span construction might be fruitful in encoding the notion of higher semiadditivity. And indeed, this was what was worked out in [Har19] and we will try to explain some of the highlights from the paper in this note. As a guide to the reader, in §2 we will introduce the basic notion of spans of finite spaces; §3 will define the notion of higher semiadditivity and formulate the universal property of these spans; finally, the punchlines of this note will appear in §4, where we will see that formal consequences of the results in §3 include: (1) the fact that we can view the property of being higher semiadditive equivalently as being a module over the spans introduced in §2, and (2) that for presentable categories, we have a Segal-style method of producing higher semiadditive categories.

5.2. Spans of π-finite spaces.

5.2.1. *Basic definitions.*

Definition 5.2.1 (Truncatedness and π-finiteness, [Har19] 2.5-2.6)**.** Let X, Y be spaces and $n \geq -2$. Then we say that:

- If $n \geq 0$, X is *n-truncated* if $\pi_i(X, x) = 0$ for every $i > n$ and every $x \in X$.
- If $n = -1$, then X is *(-1)-truncated* if it is either empty or contractible.
- If $n = -2$, then X is *(-2)-truncated* if it is contractible.
- A map $f : X \to Y$ is *n-truncated* if $\mathrm{fib}(f, y)$ is n-truncated for all $y \in Y$.

We say that X is *π-finite* if it is n-truncated for some n and all its homotopy groups/sets are finite. If we want to specify the n-truncatedness, we will also say that a space is *π-n-finite.*

Observation 5.2.2. A map is (-1)-truncated if it is an inclusion of path components, and it is (-2)-truncated if it is an equivalence.

Notation 5.2.3. Let $\mathcal{K}_n = \mathrm{Ho}(\mathcal{S}_n{}^{\simeq})$ be the set of representatives of all π-finite n-truncated spaces. We will be thinking of this as the set of indexing diagrams whose colimits we will be interested in.

Construction 5.2.4 (Spans, [Har19] §2.1, [Bar17])**.** Let $\mathbb{C}^\dagger \subset \mathbb{C}$ be a wide subcategory whose morphisms are closed under pullbacks. Then we can construct a new $(\infty, 1)$-category $\mathrm{Span}(\mathbb{C}, \mathbb{C}^\dagger)$ called *the category of spans* whose objects are objects of $\mathbb{C}$ and for $X, Y \in \mathbb{C}$, morphisms $X \to Y$ in $\mathrm{Span}(\mathbb{C}, \mathbb{C}^\dagger)$ are spans $X \leftarrow Z \to Y$ where $X \leftarrow Z$ is in $\mathbb{C}^\dagger$ and compositions of morphisms are given by taking pullbacks.

Fact 5.2.5 (Mapping spaces of spans, [Har19] 2.4)**.** For $X, Y \in \mathbb{C}$, we have that $\mathrm{Map}_{\mathrm{Span}(\mathbb{C},\mathbb{C}^\dagger)}(X, Y)$ is given by the subspace of $(\mathbb{C}_{/X\times Y})^{\simeq}$ on those spans $X \leftarrow Z \to Y$ such that $X \leftarrow Z$ is in $\mathbb{C}^\dagger$.

The following will be the main object of study in this notes.

Definition 5.2.6. Let $n \geq -2$ and $m \leq n$. Then we write:

- $\mathcal{S}_n \subseteq \mathcal{S}$ be the full subcategory of π-finite n-truncated spaces.
- $\mathcal{S}_{n,m} \subseteq \mathcal{S}_n$ be the non-full wide subcategory whose mapping spaces are spanned by m-truncated maps.

Given these notations, we define $\mathcal{S}_n^m := \mathrm{Span}(\mathcal{S}_n, \mathcal{S}_{n,m})$.

Observation 5.2.7. Since (-2)-truncatedness of a map is the same as being an equivalence, we see that $\mathcal{S}_{n,-2} \simeq \mathcal{S}_n{}^{\simeq}$ so that $\mathcal{S}_n^{-2} \simeq \mathcal{S}_n$.

Observation 5.2.8. The inclusion $\mathcal{S}_{n-1}^m \hookrightarrow \mathcal{S}_n^m$ is fully faithful. This is because $m \leq n$, and so if $f : Z \to X$ is m-truncated and X was $(n-1)$-truncated, then Z is $(n-1)$-truncated as well.

Observation 5.2.9. $(\mathcal{S}_n^m)^{\simeq} \subseteq (\mathcal{S}_n)^{\simeq} \subseteq \mathcal{S}_n$.

5.2.2. *Colimits in spans.* Here is an important lemma to check preservation of $\mathcal{K}_n$-colimits out of $\mathcal{S}_n$: the upshot is that in this special case it can be checked just on the constant diagrams.

Lemma 5.2.10 ([Har19] 2.11)**.** Let $\mathcal{D}$ admit $\mathcal{K}_n$-colimits and $F : \mathcal{S}_n \to \mathcal{D}$ be a functor. Then F preserves $\mathcal{K}_n$-colimits iff it preserves those which are constant at $* \in \mathcal{S}_n$.

Proof. The only if direction is immediate. To see the reverse, we use the satisfying classical trick of using the Grothendieck construction to compute colimits in spaces. Let $Y \in \mathcal{K}_n$ and $\mathcal{G} : Y \to \mathcal{S}_n$ be a Y-indexed diagram. Unstraightening we obtain a left fibration $p_{\mathcal{G}} : Z \to Y$ which in particular implies that Z is also a space so that we obtain a fibre sequence of spaces $W \to Z \to Y$ where by construction W was π-n-finite. The upshot of this paragraph is that since Y was π-n-finite also by hypothesis, we see that Z must be too so we can consider Z as living in $\mathcal{S}_n$.

Here's the fun part: for each $y \in Y$, the space $\mathcal{G}(y) \in \mathcal{S}_n$ is the colimit of the $\mathcal{G}(y)$-indexed constant diagram with value $*$ so that by the pointwise left Kan extension formula we see that $\mathcal{G} \simeq p_! \operatorname{const}_*$. In particular, this means that

$$\operatorname{colim}(Z \xrightarrow{\operatorname{const}_*} \mathcal{S}_n) \simeq \operatorname{colim}(Y \xrightarrow{\mathcal{G}} \mathcal{S}_n) \tag{1}$$

To summarise, we now have the diagram

$$\begin{tikzcd} Z \arrow[d, "p"'] \arrow[dr, "\operatorname{const}_*"] & & \\ Y \arrow[r, "\mathcal{G}\simeq p_! \operatorname{const}_*"'] & \mathcal{S}_n \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

Again, by the pointwise left Kan extension formula, we see that $\mathcal{G} \simeq p_! *$ was computed pointwise as $\mathcal{K}_n$-space-indexed diagrams with constant value $*$. Hence, since F preserved these by hypothesis, we see that $F \circ \mathcal{G} \simeq p_!(F \operatorname{const}_*)$. Therefore we obtain

$$\begin{aligned} \operatorname*{colim}_Y F \circ \mathcal{G} &:= \operatorname{colim}(Y \xrightarrow{F\circ\mathcal{G}} \mathcal{D}) \\ &\simeq \operatorname{colim}(Z \xrightarrow{F \operatorname{const}_*} \mathcal{D}) \\ &\simeq F \operatorname{colim}(Z \xrightarrow{\operatorname{const}_*} \mathcal{D}) \\ &\simeq F \operatorname*{colim}_Y \mathcal{G} \end{aligned}$$

where the penultimate line is by our assumption on F and the last line is by (1). $\square$

Lemma 5.2.11 ([Har19] 2.12)**.** For every $-2 \leq m \leq n$ the inclusion $j : \mathcal{S}_n \hookrightarrow \mathcal{S}_n^m$ preserves $\mathcal{K}_n$-colimits.

Proof. By the criterion (5.2.10) we need to show that for each $X \in \mathcal{S}_n$,

$$X \simeq \operatorname{colim}(X \xrightarrow{\operatorname{const}_*} \mathcal{S}_n^m) \in \mathcal{S}_n^m$$

In other words, by Yoneda we need to show that for all $Y \in \mathcal{S}_n^m$, the map

$$\operatorname{Map}_{\mathcal{S}_n^m}(X, Y) \longrightarrow \lim_X \operatorname{Map}_{\mathcal{S}_n^m}(\operatorname{const}_*, Y) \simeq \operatorname{Map}_{\mathcal{S}}(X, \operatorname{Map}_{\mathcal{S}_n^m}(*, Y))$$

is an equivalence. Here the second equivalence is by the usual formula for limits of constant diagrams in spaces (in our case, with value $\operatorname{Map}_{\mathcal{S}_n^m}(*, Y)$). Now by (5.2.5) we know that

$$\operatorname{Map}_{\mathcal{S}_n^m}(X, Y) \simeq (\mathcal{S}_{n/X_m \times Y})^{\simeq} \quad \text{and} \quad \operatorname{Map}_{\mathcal{S}_n^m}(*, Y) \simeq (\mathcal{S}_{n/*_m \times Y})^{\simeq}$$

where the subscript m in $\mathcal{S}_{n/X_m \times Y}$ for example denotes the full subcategory of $\mathcal{S}_{n/X \times Y}$ spanned by those maps $Z \to X \times Y$ such that $Z \to X \times Y \xrightarrow{\pi_X} X$ is m-truncated. But then since X was already n-truncated and $m \leq n$, any space with an m-truncated map to X must itself have been n-truncated, and so in fact

$$\mathcal{S}_{n/X_m \times Y} \simeq \mathcal{S}_{/X_m \times Y}$$

By a similar reasoning, we see that

$$\mathcal{S}_{n/*_m \times Y} \simeq \mathcal{S}_{m/Y}$$

Now the straightening-unstraightening equivalence gives

$$\mathcal{S}_{/X\times Y} \xrightarrow{\simeq} \mathrm{Fun}(X\times Y,\mathcal{S}) \xrightarrow{\simeq} \mathrm{Fun}(X,\mathrm{Fun}(Y,\mathcal{S}))$$

which on objects is given by $\big(q : Z \to X\times Y\big) \mapsto \big(x \mapsto (y \mapsto q^{-1}(x,y))\big)$. Applying core groupoid everywhere we obtain an equivalence

$$(\mathcal{S}_{/X\times Y})^{\simeq} \xrightarrow{\simeq} \mathrm{Map}(X,\mathrm{Map}(Y,\mathcal{S}^{\simeq}))$$

Writing $\mathrm{Map}_m(Y,\mathcal{S}^{\simeq})$ for the components of $\mathrm{Map}(Y,\mathcal{S}^{\simeq})$ such that taking colimits produce m-π-finite spaces, we see clearly that the preceding equivalence restricts to an equivalence

$$(\mathcal{S}_{/X_m\times Y})^{\simeq} \xrightarrow{\simeq} \mathrm{Map}(X,\mathrm{Map}_m(Y,\mathcal{S}^{\simeq}))$$

On the other hand, $\mathrm{Map}_m(Y,\mathcal{S}^{\simeq}) \simeq (\mathcal{S}_{m/Y})^{\simeq}$, and so we're done. □

Corollary 5.2.12 ([Har19] 2.16)**.** A functor $F : \mathcal{S}_n^m \to \mathcal{D}$ preserves $\mathcal{K}_n$-colimits iff the restriction $F : \mathcal{S}_n \hookrightarrow \mathcal{S}_n^m \to \mathcal{D}$ preserves $\mathcal{K}_n$-colimits.

Proof. By (5.2.11) the only if direction is clear. To obtain the reverse direction, note that since objects of $\mathcal{K}_n$ are groupoids, by the observation (5.2.7)(3) we see that $\mathcal{K}_n$-diagrams in $\mathcal{S}_n^m$ in fact land in $\mathcal{S}_n$, and the hypothesis implies the desired statement. □

5.2.3. *Spans as commutative algebras.*

Construction 5.2.13 (Symmetric monoidality of $\mathcal{S}_n^m$)**.** It is standard that span categories inherit the symmetric monoidal structure on the original category, and so the cartesian symmetric monoidal structure on $\mathcal{S}_n$ induces a symmetric monoidal structure on $\mathcal{S}_n^m$ given by taking products of spaces. Note however that this is *no longer* a cartesian symmetric monoidal structure on $\mathcal{S}_n^m$.

Proposition 5.2.14 ([Har19], 2.17)**.** The symmetric monoidal product $\mathcal{S}_n^m \times \mathcal{S}_n^m \to \mathcal{S}_n^m$ preserves $\mathcal{K}_n$-colimits in each variable.

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{S}_n \times \mathcal{S}_n \arrow[r, hook] \arrow[d, "\times"'] & \mathcal{S}_n^m \times \mathcal{S}_n^m \arrow[d, "\times"] \\ \mathcal{S}_n \arrow[r, hook] & \mathcal{S}_n^m \end{tikzcd}$$

where we know that the left vertical multiplication preserves colimits in each variable separately and the horizonal maps preserve $\mathcal{K}_n$-colimits by (5.2.11). The point is that since if $X \in \mathcal{K}_n$, then it's a groupoid, and so any diagram $d : X \to \mathcal{S}_n^m$ factors through $\mathcal{S}_n \subseteq \mathcal{S}_n^m$. Together with (5.2.11) this says that X-colimits in $\mathcal{S}_n^m$ are computed in $\mathcal{S}_n \subseteq \mathcal{S}_n^m$ and so the desired conclusion, which is true for the left vertical, transfers to that on the right vertical. □

Construction 5.2.15 (Spans as a commutative algebra object)**.** By [Lur17] §4.8.1 we know that $\mathrm{Cat}_{\mathcal{K}_n}$ has a symmetric monoidal structure $\otimes_{\mathcal{K}_n}$ where for $\mathbb{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Cat}_{\mathcal{K}_n}$, the tensor product $\mathbb{C} \otimes_{\mathcal{K}_n} \mathcal{D}$ has the universal property

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathbb{C} \otimes_{\mathcal{K}_n} \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}_{\mathcal{K}_n, \mathcal{K}_n}(\mathbb{C} \times \mathcal{D}, \mathcal{E})$$

where the right hand side consists of functors which preserve $\mathcal{K}_n$-colimits in each variable. Hence we can get from (5.2.14) that $\mathcal{S}_n^m$ is a commutative algebra object in $\mathrm{Cat}_{\mathcal{K}_n}$.

5.2.4. *Duality in spans.*

Construction 5.2.16 (Trace and diagonals)**.** Let $\mathbb{C}$ be a category with final object $*$ and admitting finite limits. Then for $X \in \mathrm{Span}(\mathbb{C})$, we define the *trace map* in $\mathrm{Span}(\mathbb{C})$ to be the span

$$\left(X \times X \xrightarrow{\mathrm{tr}_X} *\right) := \left(X \times X \xleftarrow{\Delta} X \rightarrow *\right)$$

and the *diagonal* in $\mathrm{Span}(\mathbb{C})$ to be the span

$$\left(* \xrightarrow{\Delta_X} X \times X\right) := \left(* \leftarrow X \xrightarrow{\Delta} X \times X\right)$$

Proposition 5.2.17 (Self-duality in spans)**.** Let $\mathbb{C}$ be a category with final object $*$ and admitting finite limits. Then the trace map and diagonal constructed above exhibits every object as self-dual in $\mathrm{Span}(\mathbb{C})$.

Proof. Let $X \in \mathrm{Span}(\mathbb{C})$. Note that being dualisable can be checked at the level of homotopy categories, and so it *really is* enough to check that the composites

$$X \xrightarrow{1 \times \Delta_X} X \times X \times X \xrightarrow{\mathrm{tr}_X \times 1} X \quad \text{and} \quad X \xrightarrow{\Delta_X \times 1} X \times X \times X \xrightarrow{1 \times \mathrm{tr}_X} X$$

are homotopic to the identity. We will only show the first. Since composition in span categories are given by pullbacks, we get that the first composite is given by the span

$$\begin{tikzcd}
 & & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[ddll, equal, bend right] \arrow[ddrr, equal, bend left] \arrow[dd, phantom, "\vee"] & & \\
 & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "1 \times \Delta"'] & & X \times X \arrow[dl, "\Delta \times 1"] \arrow[dr, "\pi_2"] & \\
X & & X \times X \times X & & X
\end{tikzcd}$$

which is the identity span, as required. $\square$

5.3. **Higher semiadditivity.**

5.3.1. *Basic notions.*

Definition 5.3.1 ([Har19] 3.1)**.** Let $m \geq -2$ and $\mathcal{D}$ a category. We say that $\mathcal{D}$ is *m-semiadditive* if $\mathcal{D}$ admits $\mathcal{K}_m$-colimits and every m-truncated π-finite space is $\mathcal{D}$-ambidextrous.

Remark 5.3.2. Two consequences which we will not prove here but which are intuitively clear, namely:

- That an m-semiadditive $\mathcal{D}$ automatically admits $\mathcal{K}_m$-limits, essentially because the $\mathcal{D}$-ambidextrousness of any $X \in \mathcal{K}_m$ already gives that the colimit also computes the limit. Given this, the intuition of m-semiadditivity is just that the canonically constructed norm map
$$\operatorname*{colim}_X \Longrightarrow \lim_X$$
is an equivalence in $\mathrm{Fun}(\mathcal{D}^X, \mathcal{D})$ for all $X \in \mathcal{K}_m$.
- The opposite of an m-semiadditive category is again m-semiadditive.

Observation 5.3.3. For $m \leq n$, n-semiadditivity implies m-semiadditivity since $\mathcal{K}_m \subseteq \mathcal{K}_n$.

Example 5.3.4. Here are some important first examples, the second of which justifies the terminology of semiadditivity.

(1) $\mathcal{D}$ is (-1)-semiadditive iff it is pointed. This is because the only nontrivial π-finite space that is (-1)-$\mathcal{D}$-ambidextrous is given by the map $\varnothing \to *$, and $\mathrm{colim}_{\varnothing}$ is the initial object and $\lim_{\varnothing}$ is the final object.

(2) $\mathcal{D}$ is 0-semiadditive iff it is semiadditive in the usual sense. To see this, recall that 0-semiadditivity implies (-1)-semiadditivity and so by the point above, $\mathcal{D}$ is pointed. Now observe that 0-truncated maps to the point $*$ in $\mathcal{S}_0$ consist precisely of maps of form $\coprod^k * \to *$ for $k < \infty$. Then pointedness allows us to construct the canonical norm map
$$\coprod^k \simeq \operatorname*{colim}_{\coprod^k *} \Longrightarrow \lim_{\coprod^k *} \simeq \prod^k$$
and being 0-semiadditive exactly requires these to be equivalences.

(3) An important class of examples for 1-semiadditivity was furnished by Lecture 3 by the Tate-vanishing of $\mathrm{Sp}_{T(n)}$. To see this, note that a map $X \to *$ where X is a π-finite space is 1-truncated iff $X = \coprod_{i=1}^k BG_i$ is a finite coproduct of Eilenberg-MacLane spaces of finite groups, and so the norm map will become the usual one
$$\bigoplus_{i=1}^k (-)_{hG_i} \Longrightarrow \bigoplus_{i=1}^k (-)^{hG_i}$$
whose cofibre $\bigoplus_i^k (-)^{tG_i}$ vanishes as we saw in Lecture 3.

5.3.2. *Modules over spans are semiadditive.* The goal of this subsubsection is to obtain an obstruction for $\mathcal{D}$ satisfying the following assumptions moreover to be m-semiadditive.

Assumption 5.3.5. $\mathcal{D}$ is $(m-1)$-semiadditive which furthermore:

(1) admits $\mathcal{K}_m$-colimits.
(2) admits a structure of an $\mathcal{S}_m^{m-1}$-module in $\mathrm{Cat}_{\mathcal{K}_m}$. This in particular means that there is an action map $\mathcal{S}_m^{m-1} \times \mathcal{D} \to \mathcal{D}$ which preserves $\mathcal{K}_m$-colimits in each variable.

Notation 5.3.6. For $\mathcal{D}$ satisfying the assumptions (5.3.5) and $X \in \mathcal{S}_m^{m-1}$, we write

$$[X] : \mathcal{D} \longrightarrow \mathcal{D}$$

for $X \otimes (-)$ afforded by the action map.

Proposition 5.3.7 (Trace obstruction, [Har19] 3.17, compare with [HL13] 5.1.8)**.** Let $\mathcal{D}$ be as in assumptions (5.3.5). Then $\mathcal{D}$ is m-semiadditive iff for all $X \in \mathcal{S}_m^{m-1}$ the transformation

$$[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{id}$$

exhibits the functor $[X] : \mathcal{D} \to \mathcal{D}$ as self-adjoint.

Theorem 5.3.8 (Modules imply m-semiadditivity, [Har19] 3.19)**.** Let $\mathcal{D}$ be tensored over $\mathcal{S}_m^m$ such that the action functor $\mathcal{S}_m^m \times \mathcal{D} \to \mathcal{D}$ preserves $\mathcal{K}_m$-colimits in each variable. Then $\mathcal{D}$ is m-semiadditive.

Proof. We will prove that $\mathcal{D}$ is m'-semiadditive for every $-2 \leq m' \leq m$ by induction on m'. Since every category is (-2)-semiadditive, the base case $m' = -2$ is done. Now suppose that $\mathcal{D}$ is m'-semiadditive for some $-2 \leq m' < m$. We want to use the trace criterion (5.3.7) to see that $\mathcal{D}$ is $(m'+1)$-semiadditive, and so let $X \in \mathcal{S}_{m'+1}^{m'}$. We want to show that

$$[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{id}$$

exhibits as $[X] : \mathcal{D} \to \mathcal{D}$ as self-adjoint. In other words, by the triangle identity characterisation of adjunctions, we need to see that the triangles

$$\begin{tikzcd} {[X]} \arrow[r, "{[X]([\Delta])}"] \arrow[dr, equal] & {[X] \circ [X] \circ [X]} \arrow[d, "{[\mathrm{tr}_X]_{[X]}}"] \\ & {[X]} \end{tikzcd} \qquad \begin{tikzcd} {[X]} \arrow[r, "{[\Delta]_{[X]}}"] \arrow[dr, equal] & {[X] \circ [X] \circ [X]} \arrow[d, "{[X]([\mathrm{tr}_X])}"] \\ & {[X]} \end{tikzcd}$$

commute. But then these are given precisely by the triangles witnessing self-duality of X in a span category (5.2.17), and so we're done. $\square$

5.3.3. *Universality of spans.* The key result for everything else in the paper is the identification of the universal property of m-spans. Once we have this, the rest follow more or less formally as in the case of ordinary commutative monoids.

Theorem 5.3.9 (Universal property of m-spans, [Har19] 4.1)**.** Let $-2 \leq m \leq n$ and $\mathcal{D}$ be m-semiadditive which admits $\mathcal{K}_n$-colimits. Then evaluation at $* \in \mathcal{S}_n^m$ induces an equivalence of categories

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathcal{S}_n^m, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

5.4. **Formal consequences.**

5.4.1. *Semiadditivity as modules.* We want now to formulate and prove the equivalence between m-semiadditivity and being modules over spans. To this end, we will analyse the forgetful functor

$$\mathcal{U} : \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \longrightarrow \mathrm{Cat}_{\mathcal{K}_m}$$

Notation 5.4.1. Let $\mathrm{SAdd}_m \subseteq \mathrm{Cat}_{\mathcal{K}_m}$ be the full subcategory spanned by m-semiadditive categories.

Lemma 5.4.2 (Idempotence of m-spans, [Har19] 5.1)**.** Let $\mathbb{C}$ be an $\mathcal{S}_m^m$-module. Then the counit map

$$\nu_C : \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C}) \longrightarrow \mathbb{C}$$

from the adjunction $\mathcal{S}_m^m \otimes_{\mathcal{K}_m} (-) : \mathrm{Cat}_{\mathcal{K}_m} \rightleftarrows \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) : \mathcal{U}$ is an equivalence of $\mathcal{S}_m^m$-modules. In particular, this means that the adjunction is a smashing localisation and $\mathcal{S}_m^m$ is an idempotent commutative algebra object.

Proof. Since the forgetful functor $\mathcal{U}$ is conservative it will suffice to show that $\mathcal{U}(\nu_C)$ is an equivalence. Now by the triangle identity of adjunctions we have that the composite

$$\mathcal{U}(\mathbb{C}) \xrightarrow{u_{\mathcal{U}(\mathbb{C})}} \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C}) \xrightarrow{\mathcal{U}(\nu_C)} \mathcal{U}(\mathbb{C})$$

is the identity. Hence it will be enough to show that the first map

$$u_{\mathcal{U}(\mathbb{C})} : \mathcal{U}(\mathbb{C}) \to \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C})$$

is an equivalence. Since both sides admit canonical structures of $\mathcal{S}_m^m$-module (where for the right hand term we use the $\mathcal{S}_m^m \otimes_{\mathcal{K}_m} -$ part for the module structure), by Yoneda it will suffice to show that

$$u^*_{\mathcal{U}(\mathbb{C})} : \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D}) \tag{2}$$

is an equivalence for all $\mathcal{D} \in \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m)$. Now since $\mathcal{D}$ was an $\mathcal{S}_m^m$-module, we get that $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D})$ is too (since $\mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \subseteq \mathrm{Cat}_{\mathcal{K}_m}$ is closed under cotensors). Hence by the universal property of m-spans (5.3.9) we see that

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D})$$

is an equivalence, and so by currying, the map (2) is an equivalence, as required. $\square$

Theorem 5.4.3 (Semiadditivity as modules, [Har19] 5.2)**.** The forgetful functor induces an equivalence

$$\mathcal{U} : \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \xrightarrow{\simeq} \mathrm{SAdd}_m$$

Hence we have the adjunctions

$$\begin{tikzcd} \mathrm{SAdd}_m \arrow[r, hook] & \mathrm{Cat}_{\mathcal{K}_m} \arrow[l, bend right, "\mathcal{S}_m^m \otimes_{\mathcal{K}_m}(-)"'] \arrow[l, bend left, "\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, -)"] \end{tikzcd}$$

where the top adjunction is a smashing localisation. In particular means that for any $\mathcal{D} \in \mathrm{Cat}_{\mathcal{K}_m}$, $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D})$ is the universal m-semiadditive category equipped with a $\mathcal{K}_m$-colimit preserving functor to $\mathcal{D}$.

Proof. We have a few things to show, namely:

(1) That $\mathcal{S}_m^m$-modules are m-semiadditive.
(2) That the forgetful map is essentially surjective on SAdd_m.
(3) That the forgetful map is fully faithful.

Point (1) is by (5.3.8), and point (2) is by the universal property of m-spans (5.3.9) since we can write $\mathcal{D} \simeq \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D})$ which then attains a canonical structure of an $\mathcal{S}_m^m$-module by evaluation. Finally, point (3) is just because (5.4.2) says that $\mathcal{S}_m^m \otimes_{\mathcal{K}_m}(-)$ is a smashing localisation, and so in particular the whole forgetful functor

$$\mathcal{U} : \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \to \mathrm{SAdd}_m \hookrightarrow \mathrm{Cat}_{\mathcal{K}_m}$$

is fully faithful. Since the second map in this factorisation is fully faithful, so is the first map, as required. □

Via this equivalence we can then obtain a symmetric monoidal structure $\mathrm{SAdd}_m^{\otimes}$ on the m-semiadditives, and the following statements are standard consequences of the equivalence.

Corollary 5.4.4 ([Har19] 5.6-5.8)**.** The fully faithful inclusion $\mathrm{SAdd}_m \hookrightarrow \mathrm{Cat}_{\mathcal{K}_m}$ can be canonically refined to a lax symmetric monoidal functor and $\mathcal{S}_m^m$ is the initial object in $\mathrm{CAlg}(\mathrm{SAdd}_m)$.

5.4.2. *Higher commutative monoids.*

Notation 5.4.5. Let $X \in \mathcal{S}_m$. Note that the inclusion of a point $x \in X$, $i_x : * \to X$, is an $(m-1)$-truncated map by the π_*-long exact sequence. We then write $\widehat{i_x}$ to denote the span $X \xleftarrow{i_x} * \to *$ which is in $\mathcal{S}_m^{m-1}$.

Definition 5.4.6. Let $\mathcal{D}$ be a category admitting $\mathcal{K}_m$-limits. Then an *m-commutative monoid* is a functor $F : \mathcal{S}_m^{m-1} \to \mathcal{D}$ such that for every $X \in \mathcal{K}_m$, the set of maps $\{\widehat{i_x} : X \leftarrow$

$*\}_{x\in X}$ induce an equivalence $F(X) \xrightarrow{\simeq} \lim_X^{\mathcal{D}} F(*)$. We write $\mathrm{CMon}_m(\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ for the full subcategory of the m-commutative monoids.

Remark 5.4.7. In the case where $m = 0$, we see that $\mathcal{S}_0^{-1} = \mathrm{Span}(\mathrm{Fin}, \mathrm{Fin}^{\mathrm{inj}}) = \mathrm{Fin}_*$. Moreover, the 0-commutative monoid condition is precisely demanding that $F : \mathrm{Fin}_* \to \mathcal{D}$ preserves products (recall that the categorical products in Fin_* are given by disjoint unions). Hence 0-commutative monoids agree with Segal's notion of commutative monoids mentioned in the introduction.

Lemma 5.4.8 ([Har19] 5.13, 5.14)**.** Let $m \geq -1$. For $\mathcal{D}$ admitting $\mathcal{K}_m$-limits, then the restriction $\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}) \to \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^{m-1}, \mathcal{D})$ factors through an equivalence $\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}) \xrightarrow{\simeq} \mathrm{CMon}_m(\mathcal{D})$, and so we can just as well think of m-commutative monoids in these terms.

Proof. We only argue essential surjectivity, which is [Har19] 5.13. For this just consider the sequence of equivalences:

$$\begin{array}{ll}
 & \mathcal{S}_m^m \xrightarrow{F} \mathcal{D} \text{ preserves } \mathcal{K}_m\text{-limits} \\
\text{iff} & (\mathcal{S}_m^m)\mathrm{op} \xrightarrow{F\mathrm{op}} \mathcal{D}\mathrm{op} \text{ preserves } \mathcal{K}_m\text{-colimits} \\
\text{iff} & \mathcal{S}_m \hookrightarrow (\mathcal{S}_m^m)\mathrm{op} \xrightarrow{F\mathrm{op}} \mathcal{D}\mathrm{op} \text{ preserves } \mathcal{K}_m\text{-colimits} \\
\text{iff} & \text{the set of maps } \{i_x : * \to X\}_{x\in X} \text{ induce an equivalence } \mathrm{colim}_X^{\mathcal{D}\mathrm{op}} F\mathrm{op}(*) \xrightarrow{\simeq} F\mathrm{op}(X) \text{ for all } X \in \mathcal{K}_m \\
\text{iff} & \text{the set of maps } \{\widehat{i}_x : X \leftarrow *\}_{x\in X} \text{ in } \mathcal{S}_m^{m-1} \text{ induce an equivalence } F(X) \xrightarrow{\simeq} \lim_X^{\mathcal{D}} F(*) \text{ for all } X \in \mathcal{K}_m \\
\text{iff} & F|_{\mathcal{S}_m^{m-1}} \text{ is } m\text{-commutative monoid.}
\end{array}$$

where the third line is by (5.2.12), the fourth by (5.2.10), and the fifth just by taking opposites everywhere of the fourth line: here we are using that the span $i_x : * \leftarrow * \xrightarrow{i_x} X$ gets sent to $\widehat{i}_x : X \xleftarrow{i_x} * \to *$. □

Observation 5.4.9 (An alternate life of m-commutative monoids)**.** We have the identification $\mathrm{CMon}_m(\mathcal{S}) \simeq \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m^m)$ since by construction $\mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m^m) := \mathrm{Fun}^{\mathcal{K}_m}((\mathcal{S}_m^m)\mathrm{op}, \mathcal{S})$, and $(\mathcal{S}_m^m)\mathrm{op} \simeq \mathcal{S}_m^m$ since spans are always self-dual.

Lemma 5.4.10 ([Har19] 5.15)**.** Let $\mathcal{D}$ admit $\mathcal{K}_m$-limits. Then $\mathrm{CMon}_m(\mathcal{D})$ is m-semiadditive and the restriction along $\{*\} \hookrightarrow \mathcal{S}_m^m$ induces a functor

$$\mathrm{CMon}_m(\mathcal{D}) \to \mathcal{D}$$

which is the universal $\mathcal{K}_m$-limit preserving functor to $\mathcal{D}$ from an m-semiadditive category. In particular, $\mathcal{D}$ is m-semiadditive iff this functor is an equivalence.

Proof. By hypothesis $\mathcal{D}\mathrm{op}$ admits $\mathcal{K}_m$-colimits. Hence by (5.4.3) we get that

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}\mathrm{op}) \to \mathcal{D}\mathrm{op}$$

is the universal $\mathcal{K}_m$-colimit preserving functor from an m-semiadditive category to $\mathcal{D}\mathrm{op}$, so by taking opposites everywhere and using the result that says that opposites of m-semiadditives are m-semiadditive, we obtain the desired statement. □

Corollary 5.4.11. If $\mathbb{C}$ is an m-semiadditive presentable category, then $\mathbb{C} \simeq \mathrm{CMon}_m(\mathcal{S}) \otimes \mathbb{C}$. In particular, $\mathbb{C}$ attains a canonical $\mathrm{CMon}_m(\mathcal{S})$-module structure.

Proof. The equivalence is essentially due to the formula for the Lurie tensor product of presentables [Lur17] 4.8.1.17: for $\mathcal{D}, \mathcal{E}$ presentables, we have $\mathcal{D} \otimes \mathcal{E} \simeq \mathrm{RFun}(\mathcal{D}\mathrm{op}, \mathcal{E})$ where RFun is the full subcategory spanned by functors which are right adjoints. To wit,

$$\begin{aligned}
\mathbb{C} &\simeq \mathrm{CMon}_m(\mathbb{C}) \\
&:= \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathbb{C}) \\
&\simeq \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathbb{C} \otimes \mathcal{S}) \\
&\simeq \mathrm{Fun}^{\mathcal{K}_m}\big(\mathcal{S}_m^m, \mathrm{RFun}(\mathbb{C}\mathrm{op}, \mathcal{S})\big) \\
&\simeq \mathrm{RFun}\left(\mathbb{C}\mathrm{op}, \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{S})\right) \\
&\simeq \mathbb{C} \otimes \mathrm{CMon}_m(\mathcal{S})
\end{aligned}$$

where the first equivalence is by (5.4.10). This completes the proof. □

Construction 5.4.12. Let $\widehat{\mathrm{Cat}}_{\mathrm{small}}$ be the category of not necessarily small categories admitting small colimits and functors preserving these.

Lemma 5.4.13. $\mathrm{CMon}_m(\mathcal{S}) \in \mathrm{Pr}^L$ is an idempotent commutative algebra object.

Proof. By [Lur17] 4.8.1.16 and 4.8.1.17 we know that the inclusion $\mathrm{Pr}^L \subseteq \widehat{\mathrm{Cat}}_{\mathrm{small}}$ is symmetric monoidal. Moreover, [Lur17] 4.8.1.10 gives that the functor $\mathcal{P}_{\mathcal{K}_m} : \mathrm{Cat}_{\mathcal{K}_m} \to \widehat{\mathrm{Cat}}_{\mathrm{small}}$ is symmetric monoidal and so in particular preserves idempotent commutative algebra objects. Now by (5.4.9) we know that $\mathrm{CMon}_m(\mathcal{S}) \simeq \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m^m)$ and by (5.4.2) we know that $\mathcal{S}_m^m$ is an idempotent commutative algebra object, and so we're done. □

Theorem 5.4.14 ([Har19] 5.21)**.** There is a smashing localisation

$$\mathrm{Pr}^L \underset{i}{\overset{\mathrm{CMon}_m(\mathcal{S})\otimes(-)}{\rightleftarrows}} \mathrm{Mod}_{\mathrm{Pr}^L}(\mathrm{CMon}_m(\mathcal{S}))$$

where the essential image of the fully faithful inclusion i consists precisely of the m-semiadditive presentable categories.

Proof. We need to show two things:

(1) That we have the smashing localisation (easy and formal, given by idempotence of $\mathcal{S}_m^m$)
(2) To identify the essential image as the m-semiadditives.

Point (1) is by idempotence of $\mathrm{CMon}_m(\mathcal{S})$ (5.4.13) and point (2) is just because (5.4.11) implies that the inclusion i is essentially surjective onto the m-semiadditive presentables. □

6. The main theorem of ambidexterity (Songqi Han)

Tomer: m-good means A is connected, m-truncated, p-finite, and $\pi_m(A) \neq 0$.

Theorem 6.0.1. The $T(n)$-local stable homotopy theory is ∞-semiadditive.

We will prove by induction that $Sp_{T(n)}$ is m-semiadditive.

The base case: $Sp_{T(n)}$ is 1-semiadditive, proven by Kuhn.

Remark 6.0.2. We cannot start from $m = -2$ because some constructions are valid only for $m \geq -1$.

For the inductive step, assume that $Sp_{T(n)}$ is m-semiadditive. The goal is to show that for every $(m+1)$-finite space B, and diagram $F : B \to Sp_{T(n)}$, we have that the norm map

$$\mathrm{Nm}_B : \mathrm{colim}(F) \to \lim(F)$$

exists and is invertible.

Step 1: Reduction.

For a fibration $Z \to Y \to X$ of truncated spaces, if both the fiber Z and the base X are $\mathscr{C}$-ambidextrous then Y is $\mathscr{C}$-ambidextrous. This implies that we can apply the Postnikov decomposition of a space, and reduce to the case where B is an Eilenberg–Maclane space $B = K(A, m+1)$.

For every such $A \in \mathtt{Ab}$, we can fit it into a short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

so we can induct on the size of the group A. In particular we can decompose A into cyclic groups, and reduce to the case $K(\mathbb{Z}/q, m+1)$. Since $Sp_{T(n)}$ is p-local, we have that q is invertible for $q \neq p$, so we can reduce to $K(\mathbb{Z}/p, m+1)$.

Step 2: Resolve $B^{m+1}C_p$ with a fibration. The prototype is the natural one:

$$B^m C_p \to * \to B^{m+1} C_p.$$

This doesn't work completely, so some modification is needed.

Lemma 6.0.3. Let $A \to E \to B$ with A and E both m-finite spaces, and B is $(m+1)$-finite. Suppose that multiplication by $|A|$ is a unit in $\pi_0 \mathbb{S}^{\times}_{T(n)}$. Then B is $Sp_{T(n)}$-ambidextrous.

$|A| = \int_A 1$, the A-fold sum of 1.

It suffices to find a fibration $A \to E \to B^{m+1}C_p$ where A and E are m-finite and $|A| \in \pi_0 \mathbb{S}^{\times}_{T(n)}$. We want to take $E = BG_p$ to be the classifying space of the p-Sylow subgroup, and $B = BG$, so that A is the classifying space of a group of order prime to p.

Step 3: Drop E and the fibration and only focus on A.

Observation: Every m-good A fits into a fibration $A \to E \to B$ with $E \in \mathbb{S}^{m-fin}$ an m-finite space.

Reduces to finding an m-good A.

Step 4: Linearization: transfer from $Sp_{T(n)}$ to Morava E-theory. Let E_n be the ring spectrum of the Morava E-theory of height n, and let $\widehat{\mathtt{Mod}}_{E_n}$ be the ∞-category of $K(n)$-local E_n-modules. Then the functor

$$L_{K(n)}\left(E_n \otimes -\right) : Sp_{T(n)} \to \widehat{\mathtt{Mod}}_{E_n}$$

is symmetric monoidal, so it induces a $\mathtt{CRing}$ morphism of the unit

$$f : \pi_0 \mathbb{S}_{T(n)} \to \pi_0 E_n \cong \mathbb{Z}_p[[u_1, \ldots, u_n]].$$

With the nilpotence theorem and some chromatic techniques, we have that an element is invertible on the left hand side if and only if its image is invertible on the right hand side.

So multiplication by $|A|$ lands in the units $\pi_0 \mathbb{S}_{T(n)}^{\times}$ if and only if $f(|A|)$ lands in the units $\pi_0 E_n^{\times}$.

Since $\widehat{\mathtt{Mod}}_{E_n}$ is m-semiadditive, we have that $f(|A|) = f \int_A 1 = \int_A 1 = |A|$. So the problem reduces to finding m-good A with $|A|$ invertible in $\pi_0 E_n$. We will see later why you can interchange f with the integral.

We also have that $\mathrm{im}(f)$ lands inside $\mathbb{Z}_p$, so for all $g \in \mathbb{Z}_p$ its invertibility can be detected b ythe evaluation map:

$$y \in \mathbb{Z}_p^{\times} \iff v_p(y) = 0.$$

So the condition that multiplication by $|A|$ is a unit can be rephrased to $v_p(|A|) = 0$.

Step 5: Recall the Fermat quotient on $\mathbb{Z}_p$, of the form

$$\widetilde{\delta} : x \mapsto \frac{x - x^p}{p}.$$

A lifting δ of $\widetilde{\delta}$ is constructed, so that $\delta|_{\mathbb{Z}_p} = \widetilde{\delta}$, so that δ is a map $\delta : \pi_0 E_n \to \pi_0 E_n$ satisfying the following property: for m-good A, we have that $\delta\left(|A|\right) = |A'| - |A''|$, where both A' and A'' are m-good. Moreover, δ satisfies the property that $v_p(\delta(|A|)) < v_p(|A|)$ unless $|A| = 0 \in \mathbb{Z}_p$ or $|A| \in \mathbb{Z}_p^{\times}$. We also know that

$$v_p\left(|A'| - |A''|\right) \geq \min\left\{v_p|A'|, v_p|A''|\right\}.$$

As long as there exists an A which is m-good with $|A| \neq 0$, then $\min\left\{v_p(|A|) \colon A\ m-\mathrm{good}\right\} = 0$.

It suffices to find any A which is m-good with $|A| \neq 0$ in $\pi_0 \mathbb{E}_n$.

Step 6: $A = B^m C_p$, but we don't yet know that $|B^m C_p| \neq 0$ in $\pi_0 E_n$. We know that $B^{m-1}C_p$ is a loop space, so we have that

$$\begin{aligned} |B^m C_p| \left|B^{m-1}C_p\right| &= \left|\mathrm{Map}(S^1, B^m C_p)\right| \\ &= \dim\left(E_n \otimes B^m C_p\right), \end{aligned}$$

where $E_n \otimes B^m C_p$ is the module defined by the constant map $\mathrm{colim}_{B^m C_p} \underline{E_n}$. We know that this dimensino is equal to the mod-2 Euler characteristic $\chi_n\left(B^m C_p\right)$. This can be computed as

$$\dim_{\mathbb{F}_p} K(n)_0\left(B^m C_p\right) - \dim_{\mathbb{F}_p} K(n)_1\left(B^m C_p\right).$$

We know this latter term is zero, and the first term is nonzero, so we have that $|B^m C_p| \neq 0$.

6.1. **Properties of integrations.**

Definition 6.1.1. Let $q^* : \mathscr{C} \to \mathscr{D}$ be a functor. We say it is a *normed functor* if both $q_!$, and q_* exist, and we have $\mathrm{Nm}_q : q_! \to q_*$. We say it is *iso-normed* if Nm_q is an isomorphism

Definition 6.1.2. Let q be a normed functor. Then we define *integration* as follows: for every $X, Y \in \mathscr{C}$, we have

$$\int_q : \mathrm{Map}\left(q^* X, q^* Y\right) \to \mathrm{Map}(X, Y),$$

defined as the composition

$$\mathrm{Map}\left(q^* X, q^* Y\right) \xrightarrow{q_*} \mathrm{Map}\left(q_* q^* X, q_* q^* Y\right) \overset{\mathrm{Nm}_q}{\longleftarrow} \mathrm{Map}\left(q_* q^* X, q_! q^* Y\right) \xrightarrow{c \circ - \circ u} \mathrm{Map}(X, Y).$$

We will study a pair of normed functors q^* and $\widetilde{q}^*$ that behave well with respect to integration.

Question:Given a commutative diagram up to homotopy

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "q^*"'] & \widetilde{\mathscr{C}} \arrow[d, "\widetilde{q}^*"] \\ \mathscr{D} \arrow[r, "G"] & \widetilde{\mathscr{D}}, \end{tikzcd}$$

when are $\int_q$ and $\int_{\widetilde{q}}$ related with respect to F, G?

We can use Beck–Chevalley to get

$$\begin{tikzcd} \widetilde{q}_! G \arrow[r] \arrow[d, "\mathrm{Nm}_{\widetilde{q}}"] & F q_! \arrow[d, "\mathrm{Nm}_q"] \\ \widetilde{q}_* G & F q_* \arrow[l] \end{tikzcd}$$

The diagram commutes if and only if AmbSq ??

Theorem 6.1.3. AmbSq and BC and BC_* implies that

$$F\left(\int_q f\right) = \int_{\widetilde{q}} Gf$$

for $X, Y \in \mathscr{C}$ and $f : q^*X \to q^*Y$.

Example 6.1.4. A pullback diagram of spaces

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ \widetilde{B} \arrow[r] & B \end{tikzcd}$$

induces a base change square

$$\begin{tikzcd} \mathscr{C}^B \arrow[r] \arrow[d, "q^*"'] & \mathscr{C}^{\widetilde{B}} \arrow[d, "\widetilde{q}^*"] \\ \mathscr{C}^A \arrow[r] & \mathscr{C}^{\widetilde{A}}. \end{tikzcd}$$

(1) If $q, \widetilde{q}$ are $\mathscr{C}$-ambidextrous, then q^* and $\widetilde{q}^*$ are isonormed
(2) ???

Corollary 6.1.5. (Distributivity) Assume that we have two maps $q_1 : A_1 \to B$ and $q_2 : A_2 \to B$ which ar eboth $\mathscr{C}$-ambidextrous. Then $q : A_1 \times_B A_2 \to B$ is $\mathscr{C}$-ambidextrous, and

$$\int_q \pi_2^* f_2 \circ \pi_1^* f_1 = \int_{q_2} f_2 \circ \int_{q_1} f_1,$$

for any $f_i : q_i^* X \to q_i^* Y$.

Corollary 6.1.6. (Additivity) Let $\mathscr{C}$ be 0-semiadditive, and we have finitely many $q_i : A_i \to B$. Then if all the q_i's are $\mathscr{C}$-ambidextrous, then the induced map

$$q = \amalg q_i : \amalg_i A_i \to B$$

is also $\mathscr{C}$-ambidextrous, and

$$\int_q \amalg_i f_i = \sum_i \int_{q_i} f_i.$$

Proof. We can reduce to $k = 2$, and 0-semiadditivity is used to imply that the fold map $\nabla : B \amalg B \to B$ is $\mathscr{C}$-ambidextrous. $\square$

Let $F : \mathscr{C} \to \mathscr{D}$, and $q : A \to B$. Then

$$\begin{tikzcd} \mathscr{C}^A \arrow[r, "F_*"] \arrow[d, "q^*"] & \mathscr{D}^B \arrow[d, "q^*"] \\ \mathscr{C}^A \arrow[r, "F_*"] & \mathscr{D}^A. \end{tikzcd}$$

(1) If q is $\mathscr{C}$- and $\mathscr{D}$-ambidextrous then q^* is iso-normed
(2) If $\mathscr{C}, \mathscr{D}, F$ compatible with q-(co)limits, then $BC_!$ and BC_*
(3) If q is both $\mathscr{C}$-amb and $\mathscr{D}-$amb and F preserves $(m+1)$-colimits, then AmbSq
(4) If both 2 and 3 then $F \int_q f = \int_q Ff$.
(5) If $\mathscr{C}, \mathscr{D}$ are m-semiadditive, and q is m-finite, then F preserves m-finite colimits. Then for such an f we will call it m-semiadditive.

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