MIT TALBOT WORKSHOP 2021: AMBIDEXTERITY IN CHROMATIC HOMOTOPY THEORY

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ABSTRACT. Notes from the MIT Talbot Workshop 2021: Ambidexterity in Chromatic Homotopy Theory, mentored by Jacob Lurie and Tomer Schlank. Please fix any typos or errors by submitting a pull request at https://github.com/tbrazel/talbot2021.

Contents

1. Overview (Jacob Lurie)

Let G be a finite group, and consider a class in the cohomology of G, denoted $\eta \in H^n(BG; \mathbb{C}^*)$. Cohomology classes are things that you can integrate against cycles. So if we had a compact oriented manifold M of dimension n, together with a map $f : M \to BG$, then we could take the fundamental class of M, and pair it to get an invertible complex number

$$\langle [M], f^*\eta \rangle \in \mathbb{C}^*.$$

If we want a quantity that only depends on M, we could form a combination which considers *all* maps from M into BG.

There are lots of maps $M \to BG$, and if we wanted to classify them up to homotopy, we obtain a bijection with the set of principal G-bundles on M, considered up to isomorphism. To be really concrete, assume further that M is connected and we have chosen a basepoint. Then we have a bijection

$$[M, BG] \leftrightarrow \{\text{principal } G\text{-bundles on } M\} \leftrightarrow \{\text{group homomorphisms } \pi_i \to M\}/\text{conjugation}$$

Since group homomorphisms are determined by where the generators go, we only have finitely many such homomorphisms. We could sum over all group homomorphisms $\pi_1(M) \to G$ to get a complex number of the form

$$\frac{1}{|G|} \sum_{\pi_1 M \to G} \langle [M], f^*\eta \rangle .$$

This is almost the same as

$$\sum_{\substack{\text{homomotopy classes} \\ f: M \to BG}} \langle [M], f^*\eta \rangle$$

This would be true if G were acting by group conjugation here, but it isn't. What we are really doing here is summing with multiplicity

$$\sum_{G\text{-bundles } P \text{ on } M} \frac{\langle [M], f^*\eta \rangle}{|\mathrm{Aut}(P)|}.$$

We will define this quantity the *integral* over the space of all maps $M \to BG$:

$$Z(M) := \int_{f: M \to BG} \langle [M], f^*\eta \rangle := \frac{1}{|G|} \sum_{\pi_1 M \to G} \langle [M], f^*\eta \rangle .$$

If we fix G and M, then this complex number $Z(M)$ is an invariant of manifolds. This is what is assigned to M by Dijkgraaf–Witten theory. It is not a very interesting invariant, because it only depended on the fundamental group of M and the homotopy class of M.

Goal: Find more constructions "like this."

That is, take this construction and vary the ingredients you used to make it. We could try to vary the field $\mathbb{C}$, but we couldn't use a field of characteristic p if $p \mid |G|$. This is because we are normalizing by $|G|$, and we can't throw out the normalization without losing some structure.

Given a manifold M, we can look at the space of maps $\mathrm{Map}(M, BG)$. We have an evaluation and projection

$$\begin{tikzcd} M \times \mathrm{Map}(M, BG) \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & BG \\ \mathrm{Map}(M, BG) & \end{tikzcd}$$

So we are beginning with η, pulling it back to $\mathrm{ev}^*\eta$, and then pushing it forward along π to obtain $\pi_*\mathrm{ev}^*\eta$, which is a cohomology class of degree zero, i.e. a function of the form $\mathrm{Map}(M, BG) \to \mathbb{C}^*$. This is what happens when M is an n-manifold.

Suppose M is now an $(n-1)$-manifold. Then the pullback and pushforward will be a class

$$\mathcal{L} := \pi_*\mathrm{ev}^*\eta \in H^1\left(\mathrm{Map}(M, BG), \mathbb{C}^*\right).$$

That is, it corresponds to a local system of 1-dimensional complex vector spaces. This cohomology class determines the local system $\mathcal{L}$ up to isomorphism.

Given a 1-dimensional local system, we can try to integrate it, to obtain a single vector space. We could look at the cohomology $H^0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$, or we could look at the homology $H_0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$. We want to assign a complex vector space $Z(M)$ which corresponds to either of these. These two vector spaces *turn out to be the same.*

We have that $\pi_0\mathrm{Map}(M, BG)$ corresponds to G-bundles on M, so

$$\mathrm{Map}(M, BG) = \coprod_{\text{iso classes of G-bundles } P} B\mathrm{Aut}(P).$$

So let's start by thinking about local systems on things like $B\mathrm{Aut}(P)$.

Suppose that H is a finite group, and let's consider complex local systems $\mathcal{L}$ on BH. This is the same thing as a complex representation V of H. We have that

$$H^0\left(BG, \mathcal{L}\right) = V^H = \left\{v \in V \colon hv = v \forall h \in H\right\}.$$

The homology is

$$H_0\left(BG, \mathcal{L}\right) = V_H = V/\mathbb{C} \cdot \left\{hv - v\right\}.$$

This is the minimal quotient of V you can form on which H acts trivially. When H is a finite group, there is an obvious relation between these. We have an averaging function

$$\begin{aligned} V &\to V \\ v &\mapsto \sum_{h \in H} hv. \end{aligned}$$

This map factors through the subspace V^H, but it also factors through the quotient given by the coinvariants, since it annihilates vectors of the form $hv - v$. So we get a norm

$$\begin{tikzcd} V \arrow[rr] \arrow[dr] & & V \\ & V_H \arrow[r, "\mathrm{Nm}"'] & V^H \arrow[ul] \end{tikzcd}$$

Basic fact: This map is an isomorphism (assuming characteristic zero).

Proof. We should write down the inverse map. There is an obvious map in the other direction:

$$V^H \subseteq V \twoheadrightarrow V_H.$$

Let's call this map λ. We see that $\lambda \circ \mathrm{Nm} = \mathrm{Nm} \circ \lambda$, which is multiplication by the order of H. Over characteristic zero, this multiplication is an isomorphism. $\square$

Remark 1.0.1. Recall that if M is an n-manifold, we thought about this function $\mathrm{Map}(M, BG) \to \mathbb{C}^*$, given by $f \mapsto \langle [M], f^*\eta \rangle$. This map gives us something in $H^0(\mathrm{Map}(M, BG), \mathbb{C}^*)$. This integration procedure was

$$Z(M) = \int_{\mathrm{Map}(M,BG)} \langle [M], f^*\eta \rangle\,,$$

took the class in the degree zero cohomology, but we identified cohomology with homology by doing this norm map on every component.

Thinking as an algebraic topologist, we can turn fields K into cohomology theories HK. Thinking about fields from a very large distance, there are fields of characteristic zero, and those of characteristic p. Morava realized that in the world of cohomology theories, there are a hierarchy of examples which interpolate between things like $H\mathbb{Q}$ and things like $H\mathbb{F}_p$. Fixing a prime number p, we have that Morava K-theories are an infinite sequence of cohomology theories, with

$$H\mathbb{Q} = K(0) \subseteq K(1) \subseteq \cdots \subseteq K(\infty) = H\mathbb{F}_p.$$

Question: Do these constructions make sense "over $K(n)$?"

Morava K-theories are characteristic p objects, since multiplication by p is the zero map $K(n) \to K(n)$ for $n > 0$. If we think characteristic p is bad, we might think Morava K-theories are bad. However the answer to this question is yes!

Theorem 1.0.2. (Hovey–Sadofsky) Let V be a $K(n)$-module with an action of a finite group H. Then the norm map

$$\mathrm{Nm}_H : V_{hH} \to V^{hH}$$

is an isomorphism for $n < \infty$.

Suppose that V and W are $K(n)$-modules and suppose we have a family of maps $f_x : V \to W$ parametrized by $x \in BH$. That is, a continuous map $f : BH \to \mathrm{Map}(V, W)$. Yet another way to think about this data is considering f as an element of $H^0\left(BH, \underline{\mathrm{Map}(V,W)}\right)$. Since V and W were $K(n)$-modules, we have that $\underline{\mathrm{Map}(V,W)}$ is a $K(n)$-module (with two $K(n)$-module structures). This theorem earlier tells us that

$$H^0\left(BH, \underline{\mathrm{Map}(V,W)}\right) \cong H_0\left(BH, \underline{\mathrm{Map}(V,W)}\right) \to H_0\left(*, \underline{\mathrm{Map}(V,W)}\right) = \pi_0\mathrm{Map}(V,W),$$

by mapping along $BH \to *$. Thus using this theorem from earlier, we can go from a *family* of maps, to a *single* map $V \to W$. We denote this procedure by

$$\begin{gathered}\mathrm{Map}(BH, \mathrm{Map}(V,W)) \to \pi_0\mathrm{Map}(V,W)\\ f \mapsto \int f.\end{gathered}$$

We saw this earlier when $n = 0$ and when $V = W = \mathbb{C}$.

Now let's assume that H is abelian. Then BH is an abelian group object in spaces. What if we want to study representations of BH? That is, local systems on $B(BH) = K(H, 2)$. This is a simply connected space, so there should be no local systems on it, that is, this doesn't make sense classically.

So instead we want to study representations of BH on $K(n)$-modules, that is, local systems $\mathcal{L}$ of BH-modules. We could study the analogue of the coinvariants and invariants, which are the homotopy (co)limits over $\mathcal{L}_x$, where $x \in BH$. The Hovey–Sadofsky theorem gives

$$\lim_{BH} : \mathrm{hocolim}_{y\in K(H,2)}\mathcal{L}_y \to \mathrm{holim}_{x\in K(H,2)}\mathcal{L}_x.$$

To give such a map is to give a family of maps $f_{x,y} : \mathcal{L}_x \to \mathcal{L}_y$, and these should vary continuously in x and y. Any path $p : [0,1] \to K(H,2)$ satisfying $p(0) = x$ and $p(1) = y$ determines an isomorphism $p_! : \mathcal{L}_y \to \mathcal{L}_x$. This depends not only on x and y but also on the path that we chose. The collection of such paths is parametrized by a space $\{x\} \times^h_{K(H,2)} \{y\} =: P_{x,y}$. So we have a collection of isomorphisms $\mathcal{L}_y \xrightarrow{\sim} \mathcal{L}_x$ parametrized by the space $P_{x,y} \simeq K(H,1) = BH$.

We can then use that integration procedure to get

$$f_{x,y} = \int_{p\in P_{x,y}} p_!,$$

which is a single morphism $\mathcal{L}_y \to \mathcal{L}_x$ (not necessarily an isomorphism anymore). So allowing x and y to vary, we get a single map

$$\mathrm{Nm}_{K(H,2)} : \mathrm{hocolim}\mathcal{L} \to \mathrm{holim}\mathcal{L}.$$

Theorem 1.0.3. This map is also a homotopy equivalence.

We can now do this again — suppose we are interested in representations of $K(H,2)$, then $K(H,3)$ and so on. This yields the following.

Theorem 1.0.4. Let X be a space with finitely many homotopy groups, and all homotopy groups are assumed to be finite[1] and let $\mathcal{L}$ be a local system of $K(n)$-modules on X. Then there is a canonical isomorphism

$$\mathrm{Nm}_X : \mathrm{hocolim}\mathcal{L}_x \to \mathrm{holim}\mathcal{L}_x.$$

That is, there is some natural map which induces isomorphisms $H_*(X, \mathcal{L}) \xrightarrow{\sim} H^*(X, \mathcal{L})$.

This is an interesting statement even when $\mathcal{L}$ is a trivial local system. In particular if X has finitely many homotopy groups, there is a canonical isomorphism

$$K(n)_*(X) \xrightarrow{\sim} K(n)^*(X).$$

We can think about this as a statement about X: if X is a nice space it satisfis a Poincaré duality with respect to Morava K-theory. We could also think about it as a statement about the category of $K(n)$-local spectra — it is not just an *additive* category, but it has some kind of fancier additivity where we can take a collection of morphisms indexed over a space and "add" or integrate the maps together. This theorem is also addressing the question that we started with — are there other constructions of Dijkgraaf–Witten theory? Yes, we can replace the height zero complex numbers by things of higher height, like Lubin–Tate spectra.

Question: Why is this true (in an easy example)?

If $X = K(H,2)$, the Hovey–Sadofsky theorem gives us a map

$$K(n)_*(X) \to K(n)^*(X).$$

There is an element $1 \in K(n)^0(X)$, and suppose we could find something, call it $y \in K(n)_0(X)$, mapping to it under the norm. Then if we had such a y, we would have that multiplication by y will induce a map from

$$\Theta : \mathrm{holim}\mathcal{L} \to \mathrm{hocolim}\mathcal{L}.$$

In classical ordinary homology this is called the cap product. The condition $\mathrm{Nm}(y) = 1$ is equivalent to the statement that Θ is inverse to the norm map.

Example 1.0.5. Let $X = BH$ for a finite p-group H, and $n = 1$. Then we have a map

$$K(n)_*(BH) \to K(n)^*(BH).$$

In height one, we know what these mean — these lift to characteristic zero, since $K(1) = \widehat{\mathrm{KU}}/p$. Complex K-theory of BH is described by the Atiyah–Segal completion theorem,

[1] For example $BH, B^2H, \ldots$ where H is finite abelian.

so we have that

$$\widehat{\mathrm{KU}}^0(BH) = \mathtt{Rep}(H)^\wedge$$
$$K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

So our map would be

$$\mathtt{Rep}(H)^\vee/p = K(1)_0(BH) \to K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

By character theory, $\mathtt{Rep}(H) \otimes \mathbb{C}$ is the conjugation-invariant functions $H \to \mathbb{C}$, by sending V to its character χ_V. If we p-adically complete, we are really getting a map

$$\mathtt{Rep}(H)^\vee \to \mathtt{Rep}(H).$$

Rationally, everything is computable, and we can compute that it is an isomorphism. We can study the inverse isomorphism then

$$\mathbb{Q} \otimes \mathtt{Rep}(H)^\vee \leftarrow \mathbb{Q} \otimes \mathtt{Rep}(H).$$

Over $\mathbb{C}$, this bilinear form is given by $V, W \mapsto \frac{1}{|H|} \sum_{h \in H} \chi_V(h)\chi_W(h)$. To know that this isomorphism exists integrally and not rationally, we need to check this value is an integer. But we can rewrite this as

$$\begin{aligned} \frac{1}{|H|} \sum_{h \in H} \chi_V(h)\chi_W(h) &= \frac{1}{|H|} \sum_{h \in H} \chi_{V \otimes W}(h) \\ &= \dim_{\mathbb{C}} (V \otimes W)^H. \end{aligned}$$

So this is a sketch of the proof of the Hovey–Sadofsky theorem in height one.

2. Review of chromatic homotopy theory (Elizabeth Tatum)

2.1. **Formal group laws.** Let R be a commutative ring, then we can consider formal group laws over that ring $F \in R[[x, y]]$. We say that F is a *formal group law* if

(1) $F(x, 0) = x$
(2) $F(x, y) = F(y, x)$
(3) $F(x, F(y, z)) = F(F(x, y), z)$.

Let $f \in R[[x]]$. We say that it is a *homomorphism* from a formal group law F to a formal group law G if $f(F(x, y)) = G(f(x), f(y))$.

We let the *n-series* on a formal group law F be given by

$$[n]_F(x) := F(x, F(x, \ldots F(x, x))).$$

That is, F applied n times. In particular when $n = p$ is a prime, we get that

$$[p]_F(x) \equiv ax^{p^h} + \text{higher order terms}.$$

We say that F has height $\geq h$ if there are no higher order terms, and that F has height exactly h if a is a unit.

Example 2.1.1. We have the *additive formal group law* $F_a(x, y) = x + y$, with height ∞.

Example 2.1.2. We have the *multiplicative formal group law* $F_m(x, y) = x+y+xy$, which has height 1.

Theorem 2.1.3. (Lazard) Over an algebraically closed field, F and G should have the same height if and only if F is isomorphic to G.

2.2. **Complex oriented cohomology theories.** We say that a ring spectrum E is *complex orientable* if the map $E^2(\mathbb{C}\mathrm{P}^\infty) \to E^2(S^2)$ is surjective. In particular in the reduced cohomology, an orientation is a choice of generator x mapping to $1 \in \pi_0(E)$ under the composite

$$\widetilde{E}(\mathbb{C}\mathrm{P}^\infty) \to \widetilde{E}(S^2) \simeq \pi_0 E.$$

We have that $\mathbb{C}\mathrm{P}^\infty$ has a natural multiplication, so by applying E^*, we get

$$E^*\mathbb{C}\mathrm{P}^\infty \to E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$$
$$x \mapsto F(x^L, x^R.$$

So any choice of complex orientation yields a formal group law.

- $H(\mathbb{Z}/p)$ carries F_a
- KU carries F_m
- MU carries the universal formal group law F_{MU}.

This universal fgl is characterized by the property that for any formal group law F over R, there is a $\theta : \mathrm{MU}_* \to R$ so that $\theta(F_{\mathrm{MU}}) = F$. Here $\mathrm{MU}_* \cong \mathbb{Z}[x_1, x_2, \ldots]$, where $|x_i| = 2i$.

2.3. **Morava K-theories and related spectra.** The *Brown–Peterson spectrum*, for a fixed prime p, is a wedge summand in complex cobordism

$$\mathrm{MU}_{(p)} \simeq \wedge \mathrm{BP}.$$

We have that BP is a ring spectrum such that $\mathrm{BP}_* \cong \mathrm{MU}_{(p)} / (x_i,\ i \neq p^k - 1)$. Thus

$$\mathrm{BP}_* \cong \mathbb{Z}_{(p)} [v_1, v_2, \ldots] .$$

The x_i's are *not* living in powers of the form $2(p^k - 1)$, and we are quotienting them out. So the v_i's *are* living in those powers — $|v_i| = 2(p^i - 1)$.

Applying $[p]_{F_{\mathrm{MU}}} \to [p]_F$, we are getting that the height of F was the coefficient a appearing in $[p]_F = ax^{p^h}$. So where the v_i's land after this map, v_h is landing on a.

Morava E-theory: The Johnson–Wilson spectrum has homology $v_n^{-1}\mathrm{BP}_* / (v_{n+1}, v_{n+2}, \ldots) \cong \mathbb{Z}_{(p)} [v_1, \ldots, v_{n-1}, v_n^{\pm}]$. Morava E-theory is the completion of this – we delinate this from the Johnson–Wilson spectrum $E(n)$ by writing a subscript E_n:

$$(E_n)_* \cong \mathbb{W}(k) [[u_1, \ldots, u_{n-1}]] \left[u_n^{\pm}\right] .$$

Morava E-theory tells you about the deformations of a formal group law of height n.

Deformations: If $\phi : R \twoheadrightarrow k$ is a nice ring homomorphism, then a formal group law F over R is a deformation of some formal group law G over k if $\phi(F) = G$. We think e.g. about R being some infinitesimal thickening of the field k.

Morava K-theory: We have a $K(n)$ so that $K(n)_* \cong \mathbb{F}_p [v_n^{\pm}]$. This is a formal group law of height exactly n. At each prime we have Morava K-theories $K(1), K(2), \ldots$. The Morava E-theories $E(n)$ are telling you about the open sets containing the $K(i)$'s for $i < n$. Morava K-theories are like residue fields, and Morava E-theories are like complete local rings at these points.

The $K(n)$'s are like fields in ring spectra. We would say that E is a *field* if $E_*(X)$ is a sum of free E_*-modules.

Theorem 2.3.1. E is a field if and only if E is a $K(n)$.

Furthermore, we have that $K(n) \wedge X \simeq \wedge \Sigma^? K(n)$ is a wedge sum of suspensions of $K(n)$.

2.4. **Bousfield localization.** Fix a ring spectrum E. We say that X is *E-acyclic* if $E_*X = 0$. We say that X is *E-local* if for every E-acyclic Y, we have that $[Y, X] \simeq *$. Finally we say that $f : X \to Y$ is an *E-equivalence* if $E_*(f)$ is an isomorphism.

A *localization functor* is a functor of the form

$$L : Sp \to Sp,$$

together with a natural transformation $\eta : \mathrm{id} \to L$ so that

(1) $L\eta : L \to L^2$ is an equivalence (localizing twice doesn't do anything)
(2) $L\eta \simeq \eta L$.

Theorem 2.4.1. (Bousfield) For every spectrum E, there exists a localization functor $L_E : Sp \to Sp$ with a natural transformation η_E such that for every X, we have that $\eta_E : X \to L_E X$ is the initial E-equivalence.

That is,

(1) $E_*(\eta_X) : E_* X \to E_* L_E X$ is an isomorphism
(2) If $f : X \to Y$ is an E-equivalence and Y is E-local then there is a unique map making the diagram commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ L_E X \arrow[ur, dashed] & \end{tikzcd}$$

Let $\langle E \rangle$ denote the Bousfield class of E — that is, the "collection of E-local spectra." We say that $\langle E \rangle \subseteq \langle F \rangle$ if X is E-local implies that X is F-local.

Proposition 2.4.2. We have that

(1) If $\langle E \rangle = \langle F \rangle$ then there is a natural isomorphism $L_E \simeq L_F$.
(2) If $\langle E \rangle \subseteq \langle F \rangle$, then we have that $L_E L_F \simeq L_E$, and there is a natural transformation $\eta : L_F \to L_E$.

Fact 2.4.3. We have that L_{E_n} is smashing — this means that $L_{E_n}(X) \simeq \left(L_{E_n}(S^0)\right) \wedge X$, and the localization $L_E : Sp \to Sp$ preserves direct sums.[2]

So we get an algebraic chromatic tower

$$L_{E_0} X \leftarrow L_{E_1} X \leftarrow L_{E_2} X \leftarrow \cdots$$

These have monochromatic layers $M_i(X) = \ker\left(L_{E_i} X \to L_{E_{i-1}} X\right)$, which come with maps $M_i(X) \to L_{K(i)} X$. The monochromatic layers and the localization at K-theory are not the same as spectra, but they contain exactly the same information.

Chromatic convergence theorem 2.4.4. If X is a p-local finite spectrum, then

$$X \simeq \mathrm{holim} L_n(X)$$

Fact 2.4.5. Let $\mathscr{C}_0$ denote the full subcategory of p-local finite spectra. Then denote by $\mathscr{C}_n$ the full subcategory of $K(n)$-acyclics, so we have a chain of inclusions

$$\mathscr{C}_0 \supseteq \mathscr{C}_1 \supseteq \cdots \supseteq \mathscr{C}_\infty = \{*\}.$$

[2]Localization always sends direct sums in *Sp* to direct sums in E-local spectra. What this condition means is that it preserves direct sums *in spectra*. This is really telling you that the *inclusion* of E-local spectra into spectra preserves direct sums (and actually arbitrary colimits).

Generally a sum of local things won't be local (think about p-completion). However smashing localizations will have this property.

Localization generally feels like the analogue of localization and then completion for rings. Smashing localizations are just localizations.

A full subcategory is called *thick* if it is closed under

(1) retracts
(2) weak equivalences
(3) cofiber sequences.

Example 2.4.6. We have that E-acyclics, E-local objects are thick subcategories.

Thick subcategory theorem 2.4.7. If $\mathscr{C}$ is a thick subcategory p-local finite spectra, then $\mathscr{C}$ is one of the $\mathscr{C}_n$'s from the filtration above.

We say that a finite spectrum F is *type* n if $K(i)_*(F) = 0$ for all $i < n$, and $K(n)_*(F) \neq 0$.

Let F be any type n spectrum. Then a v_n*-self map* is a map $f : \Sigma^i F \to F$ so that

$$K(m)_*(f) = \begin{cases} \text{multiplication by a rational number} & m = n = 0 \\ \text{an isomorphism} & m = n \neq 0 \\ \text{nilpotent} & m \neq n. \end{cases}$$

Periodicity theorem 2.4.8. Any finite type n spectrum admits a v_n-self map. The telescope of this map is

$$\mathrm{Tel}(F) = \operatorname{hocolim}\left(F \xrightarrow{v_n} F \xrightarrow{v_n} F \to \cdots\right),$$

and this is independent of the choice of v_n-self map and the choice of finite type n spectrum. So we can call this $T(n)$.

Fact 2.4.9. $T(n)$ is $K(m)$-acyclic for all $m \neq n$. Applying $K(m)_*$ to the map above, we are taking a homotopy colimit along nilpotents, so this vanishes.

There is a natural transformation $\lambda : L_{T(n)} \to L_{K(n)}$. For finite spectra, we know that $T(n)$-acyclics and $K(n)$-acyclics are the same. Knowing this for all spectra would imply the localizations are the same, which is the telescope conjecture.[3]

[3] The category of $T(n)$-local things contain all $K(n)$-local things. It might be larger. This implies that every $T(n)$-acyclic is always $K(n)$-acyclic. There are certain spectra for which $T(n)$ and $K(n)$-acyclic coincide — we know this for finite spectra and for ring spectra (it follows from the nilpotence theorem).

3. (Andres Mejia)

[todo — get tex from Lucy]

4. Ambidexterity (Thomas Brazelton)

4.1. **Local systems.** A local system is, very roughly speaking, anything you might want to take cohomology in. Classically speaking, a *local system* of abelian groups on a space X is a locally constant sheaf $\mathcal{L}$ on X.

Example 4.1.1. Local systems subsume singular cohomology — this is because for any abelian group A, we can take the constant sheaf $\underline{A}$ considered as a local system.

If X is path-connected, and $\mathcal{L}$ is a local system on X, then we can take any two points x and y, and a path $\gamma : [0,1] \to X$ between them (that is, $\gamma(0) = x$ and $\gamma(1) = y$). We see that $\gamma^*\mathcal{L}$ is constant, giving an isomorphism between the fibers $\mathcal{L}_x$ and $\mathcal{L}_y$. We can check that homotoping γ will not affect the isomorphism $\mathcal{L}_x \xrightarrow{\sim} \mathcal{L}_y$. That is, we can restate $\mathcal{L}$ as the assignment of the data:

- an abelian group $\mathcal{L}_x$ for every $x \in X$
- an isomorphism $\mathcal{L}_x \xrightarrow{\sim} \mathcal{L}_y$ for every homotopy class of paths $x \to y$,

subject to some extra coherence data. From this we can get a new definition of a local system.

Definition 4.1.2. A *local system* on X valued in a 1-category $\mathscr{C}$ is a functor

$$\mathcal{L} : \Pi_1(X) \to \mathscr{C}.$$

Suppose now we want something a little stronger. If γ, γ' are homotopic maps from x to y in X, they provide isomorphisms $\mathscr{C}_x \xrightarrow{\sim} \mathscr{C}_y$ in $\mathscr{C}$. If $\mathscr{C}$ is a 2-category, we might ask for a witness of the homotopy $\gamma \Rightarrow \gamma'$ to be witnessed by a 2-cell in $\mathscr{C}$, and for a different witness to be witnessed by a different 2-cell. Similarly if we have a 3-cell between these, we might ask for a 3-cell witnessing a higher homotopy in $\mathscr{C}$, provided $\mathscr{C}$ has this higher categorical structure.

This leads us to a higher-categorical definition of local systems.

Definition 4.1.3. A *local system* on X valued in an ∞-category $\mathscr{C}$ is an ∞-functor

$$\mathcal{L} : \Pi_\infty(X) \to \mathscr{C},$$

where $\Pi_\infty(X)$ is the fundamental ∞-groupoid of X.

Viewing X as a Kan complex, we might just say a local system is an ∞-functor

$$\mathcal{L} : X \to \mathscr{C}.$$

4.2. **Pullback and adjoints.** Let $f : X \to Y$ be any map of spaces. Then given a local system $\mathcal{L} : Y \to \mathscr{C}$ on Y, we can pull it back to a local system $f^*\mathcal{L}$ on X, by pre-composing with f. For any fixed ∞-category $\mathscr{C}$, this defines a functor

$$f^* : \mathrm{Fun}(Y, \mathscr{C}) \to \mathrm{Fun}(X, \mathscr{C}).$$

If $\mathscr{C}$ admits small colimits, then we may left Kan extend to define a left adjoint to f^* (Higher Topos Theory, 4.3.3). We denote this by $f_!$:

$$f_! : \mathrm{Fun}(X, \mathscr{C}) \leftrightarrows \mathrm{Fun}(Y, \mathscr{C}) : f^*.$$

Dually when $\mathscr{C}$ admits small limits, we may right Kan extend to define a right adjoint to f^*, which we denote by f_*. This gives

$$f_! \dashv f^* \dashv f_*.$$

Example 4.2.1. Let S be a set, viewed as a discrete space, and consider the map $f : S \to *$. Pullback is then the diagonal map $f^* : \mathscr{C} \to \mathrm{Fun}(S, \mathscr{C})$. We see that any functor $S \to \mathscr{C}$ picks out a collection $\{C_s\}$ of objects in $\mathscr{C}$ for each $s \in S$. Assume that $\mathscr{C}$ has all products and coproducts. Then we can see that

$$\begin{aligned} f_! : \mathrm{Fun}(S, \mathscr{C}) &\to \mathscr{C} \\ \{C_s\} &\mapsto \coprod_{s \in S} C_s, \end{aligned}$$

and that

$$\begin{aligned} f_* : \mathrm{Fun}(S, \mathscr{C}) &\to \mathscr{C} \\ \{C_s\} &\mapsto \prod_{s \in S} C_s. \end{aligned}$$

There is always a natural transformation from products to coproducts here, given by $f_! \to f_*$. In particular when products and coproducts agree, e.g. in $\mathtt{Ab}$, we will have that this is a natural isomorphism $f_! \simeq f_*$.

Example 4.2.2. Consider $f : BG \to *$. In this case, since $\mathrm{Fun}(*, \mathscr{C}) \simeq \mathscr{C}$, we have that pullback is of the form

$$f^* : \mathscr{C} \to \mathrm{Fun}(BG, \mathscr{C}),$$

assigning to every object in $\mathscr{C}$ the trivial G-action.

In this case, the adjoints yield, for every G-equivariant object $C \in \mathscr{C}$, the coinvariants $f_! C = C_G$ and the invariants $f_* C = C^G$. Denoting by $C^{tG} = \mathrm{cofib}\left(C_G \to C^G\right)$, we have that a canonical equivalence $f_! \simeq f_*$ would imply that the Tate construction vanishes for every G-equivariant object of $\mathscr{C}$.

Associated to these types of adjunction we have the so-called "calculus of mates," which allows us to take commutative squares of spaces and discuss how the induced functors relate to one another.

Another example of where the calculus of mates appears is in the types of natural isomorphisms of restriction and extension of scalars for modules that come out of commutative diagrams of rings.

Proposition 4.2.3. If f and g are composable, then there is a canonical equivalence $(gf)^* \simeq f^*g^*$. This induces a canonical equivalence $(gf)! \simeq g_! f_!$ by the formalism of adjunctions.

Definition 4.2.4. Consider a commutative diagram of spaces

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "g"'] & Y. \end{tikzcd}$$

Then there is a *Beck–Chevalley exchange transformation* (think about this as top-left to bottom-right), denoted by

$$\mathrm{Ex}_!^* : j_! i^* \to f^* g_!.$$

This is defined by first starting with $j_! i^*$, and tacking on the counit $\mathrm{id}_B \to g^* g_!$ on the end of it. We then get $j_! i^* g^* g_!$. Since the diagram commutes, there is a canonical equivalence $i^* g^* \simeq j^* f^*$, getting us to $j_! j^* f^* g_!$. Finally, we may apply the counit $j_! j^* \to \mathrm{id}$ to conclude. The entire composite gives us:

$$j_! i^* \mathrm{id}_B \to j_! i^* g^* g_! \simeq j_! j^* f^* g_! \to f^* g_!.$$

Proposition 4.2.5. If we have a pullback square, the Beck–Chevalley exchange transformation is an equivalence.

Q: Let $f : X \to Y$, and consider the adjunction $f_! \dashv f^*$. When will $f_!$ also be a *right adjoint* to f^*?

Given a fixed category $\mathscr{C}$ admitting finite limits and colimits, we will define a class of *$\mathscr{C}$-ambidextrous* maps $f : X \to Y$. These will have the property that if $f : X \to Y$ is $\mathscr{C}$-ambidextrous, then there is a canonical equivalence $f_! \simeq f_*$.

4.3. **Ambidextrous morphisms.**

Example 4.3.1. Suppose that $f : X \xrightarrow{\sim} Y$ is a homotopy equivalence. Then $f^* : \mathrm{Fun}(Y, \mathscr{C}) \to \mathrm{Fun}(X, \mathscr{C})$ is an equivalence of categories, and it can be easily promoted to an adjoint equivalence, so that $f_! \simeq f_*$ canonically. In particular, there is a unit map $\mu_f : \mathrm{id} \to f_! f^*$, exhibiting $f_!$ as a right adjoint to f^*.

Homotopy equivalences provide our first class of morphisms which we call *ambidextrous*. Somehow these are the "most" ambidextrous, in the sense that they have the strongest structure. However as we might expect, there exist morphisms which are $\mathscr{C}$-ambidextrous without being homotopy equivalences.

We will define ambidexterity inductively, with homotopy equivalences being the base case. For indexing reasons that will become clear later, we would like to start at $n = -2$. So we will define, for each $n \geq -2$:

- A collection of *n-ambidextrous morphisms* in $\mathtt{Top}$
- For each n-ambidextrous morphism $f : X \to Y$, a natural transformation $\mu_f^{(n)} : \mathrm{id} \to f_! f^*$, well-defined up to homotopy, exhibiting $f_!$ as a right adjoint to f^*.

Base case $n = -2$: We say f is (-2)-ambidextrous if and only if f is an equivalence. In this case, we define $\mu_f^{(-2)}$ to be any homotopy inverse to the counit $f_! f^* \to \mathrm{id}$.

Inductive step: Suppose that we have defined n-ambidextrous morphisms for some n. We will define $(n+1)$-ambidextrous maps in two steps: first we define *weakly* $(n+1)$-ambidextrous maps, and then $(n+1)$-*ambidextrous* maps.

Let $f : X \to Y$ be arbitrary, and consider the diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

By Beck–Chevalley, there is an exchange isomorphism $(\pi_1)_! \pi_2^* \simeq f^* f_!$. We say that f is *weakly* $(n+1)$-*ambidextrous* if δ is n-ambidextrous. In this context, we define a counit $\nu_f^{(n+1)}$ to be the composite

$$f^* f_! \xrightarrow{\left(\mathrm{Ex}_!^*\right)^{-1}} (\pi_1)_! \, \pi_2^* \xrightarrow{\mu_\delta^{(n)}} (\pi_1)_! \, \delta_! \delta^* \pi_2^* = (\mathrm{id}_X)_! \, \mathrm{id}_X^* = \mathrm{id}_{\mathrm{Fun}(X, \mathscr{C})}$$

We say f is $(n+1)$-*ambidextrous* if the following conditions hold:

(1) The transformation $\nu_f^{(n+1)} : f^* f_! \to \mathrm{id}$ is the counit for an adjunction $f^* \dashv f_!$, with some unit $\mu_f^{(n+1)}$

(2) Weak $(n+1)$-ambidexterity is closed under pullback along f. That is, for every pullback square

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

we have that g is weakly ambidextrous, with counit $\nu_g^{(n+1)} : g^* g_! \to \mathrm{id}$ defined in the Beck–Chevalley process above

(3) Property (1) is closed under pullback along f. That is, for any pullback square as above, we have that $\nu_g^{(n+1)}$ is the counit of an adjunction $g^* \dashv g_!$.

From this definition, the following are immediate.

Proposition 4.3.2. (Weak) n-ambidexterity is closed under pullback.

Moreover from our inductive definitions, we have the following:

Proposition 4.3.3. Let $-2 \leq m \leq n$.

(1) If f is weakly m-ambidextrous,[4] then f is weakly n-ambidextrous, and $\nu_f^{(m)}$ and $\nu_f^{(n)}$ agree up to homotopy.
(2) If f is m-ambidextrous, then f is n-ambidextrous, and $\mu_f^{(m)}$ and $\mu_f^{(n)}$ agree up to homotopy.

Proof idea. It suffices to let $n = m+1$, and induct. The inductive step is basically immediate from definitions, and the base case is very direct. □

Definition 4.3.4. We say that f is *weakly ambidextrous* if it is weakly ambidextrous for some $n \geq -1$, and we say that f is *ambidextrous* if it is ambidextrous for some n. We let $\nu_f : f^* f_! \to \mathrm{id}$ denote the counit and $\mu_f : \mathrm{id} \to f_! f^*$ denote the unit. This notation is well-defined up to homotopy by the previous proposition.

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \left\{\substack{n\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] \arrow[d, hook] & \left\{\substack{(n+1)\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \{\text{ambidextrous maps}\} \arrow[d, hook] \\
\cdots \arrow[r, hook] & \left\{\substack{\text{weakly} \\ n\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] & \left\{\substack{\text{weakly} \\ (n+1)\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] & \cdots \arrow[r, hook] & \left\{\substack{\text{weakly} \\ \text{ambidextrous maps}}\right\}
\end{tikzcd}$$

4.4. **Norms.** Suppose $\mathscr{C}$ is an ∞-category with small limits and colimits. Let $f : X \to Y$ be a continuous map of spaces, and let $f_! \dashv f^* \dashv f_*$ be the associated left and right adjoints to pullback provided by Kan extensions. Suppose that f is *weakly ambidextrous* but **not necessarily ambidextrous** (recall this means inductively that the diagonal is weakly ambidextrous of one degree lower, and crucially that there is a natural transformation $\nu_f : f^* f_! \to \mathrm{id}$). Then by adjunction we have a natural homotopy equivalence of mapping spaces

$$\mathrm{Map}\left(f^* f_!, \mathrm{id}\right) \simeq \mathrm{Map}\left(f_!, f_*\right).$$

In particular ν_f maps to a natural transformation, which by definition is the composite

$$f_! \xrightarrow{\eta \cdot f_!} f_* f^* f_! \xrightarrow{f_* \cdot \nu_f} f_*.$$

We call this the *norm* of f and denote it by $\mathrm{Nm}_f : f_! \to f_*$.

Proposition 4.4.1. Let f be weakly ambidextrous as above. Then it is ambidextrous if and only if

[4] Weak ambidexterity isn't defined for $m = -2$ but that's ok

(1) Weak ambidexterity is preserved under pullback along f
(2) The norm map $\mathrm{Nm} : f_! \to f_*$ is an equivalence
(3) The norm map for any map obtained by pullback along f is an equivalence.

Example 4.4.2. We can rephrase our example from earlier to say that the following are equivalent for $f : BG \to *$:

(1) BG is $\mathscr{C}$-ambidextrous
(2) The norm Nm_f is an equivalence
(3) For every G-equivariant object of $\mathscr{C}$, the Tate construction vanishes.

Proposition 4.4.3. Weak ambidexterity is closed under composition — that is, if f and g are composable and weakly ambidextrous, we can take μ_{gf} to be the composite

$$(gf)^*(gf)_! \simeq f^*g^*g_!f_! \xrightarrow{\mu_g} f^*f_! \xrightarrow{\mu_f} \mathrm{id}.$$