MIT TALBOT WORKSHOP 2021: AMBIDEXTERITY IN CHROMATIC HOMOTOPY THEORY

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Abstract. Notes from the MIT Talbot Workshop 2021: Ambidexterity in Chromatic Homotopy Theory, mentored by Jacob Lurie and Tomer Schlank. Please fix any typos or errors by submitting a pull request at https://github.com/tbrazel/talbot2021.

Contents

1. Overview (Jacob Lurie)

Let G be a finite group, and consider a class in the cohomology of G, denoted $\eta \in H^n(BG; \mathbb{C}^*)$. Cohomology classes are things that you can integrate against cycles. So if we had a compact oriented manifold M of dimension n, together with a map $f : M \to BG$, then we could take the fundamental class of M, and pair it to get an invertible complex number

$$\langle [M], f^*\eta \rangle \in \mathbb{C}^*.$$

If we want a quantity that only depends on M, we could form a combination which considers *all* maps from M into BG.

There are lots of maps $M \to BG$, and if we wanted to classify them up to homotopy, we obtain a bijection with the set of principal G-bundles on M, considered up to isomorphism. To be really concrete, assume further that M is connected and we have chosen a basepoint. Then we have a bijection

$$[M, BG] \leftrightarrow \{\text{principal } G\text{-bundles on } M\} \leftrightarrow \{\text{group homomorphisms } \pi_i \to M\}/\text{conjugation}$$

Since group homomorphisms are determined by where the generators go, we only have finitely many such homomorphisms. We could sum over all group homomorphisms $\pi_1(M) \to G$ to get a complex number of the form

$$\frac{1}{|G|} \sum_{\pi_1 M \to G} \langle [M], f^*\eta \rangle .$$

This is almost the same as

$$\sum_{\substack{\text{homomotopy classes} \\ f: M \to BG}} \langle [M], f^*\eta \rangle$$

This would be true if G were acting by group conjugation here, but it isn't. What we are really doing here is summing with multiplicity

$$\sum_{G\text{-bundles } P \text{ on } M} \frac{\langle [M], f^*\eta \rangle}{|\mathrm{Aut}(P)|}.$$

We will define this quantity the *integral* over the space of all maps $M \to BG$:

$$Z(M) := \int_{f: M \to BG} \langle [M], f^*\eta \rangle := \frac{1}{|G|} \sum_{\pi_1 M \to G} \langle [M], f^*\eta \rangle .$$

If we fix G and M, then this complex number $Z(M)$ is an invariant of manifolds. This is what is assigned to M by Dijkgraaf–Witten theory. It is not a very interesting invariant, because it only depended on the fundamental group of M and the homotopy class of M.

Goal: Find more constructions "like this."

That is, take this construction and vary the ingredients you used to make it. We could try to vary the field $\mathbb{C}$, but we couldn't use a field of characteristic p if $p \mid |G|$. This is because we are normalizing by $|G|$, and we can't throw out the normalization without losing some structure.

Given a manifold M, we can look at the space of maps $\mathrm{Map}(M, BG)$. We have an evaluation and projection

$$\begin{tikzcd} M \times \mathrm{Map}(M, BG) \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & BG \\ \mathrm{Map}(M, BG) & \end{tikzcd}$$

So we are beginning with η, pulling it back to $\mathrm{ev}^*\eta$, and then pushing it forward along π to obtain $\pi_*\mathrm{ev}^*\eta$, which is a cohomology class of degree zero, i.e. a function of the form $\mathrm{Map}(M, BG) \to \mathbb{C}^*$. This is what happens when M is an n-manifold.

Suppose M is now an $(n-1)$-manifold. Then the pullback and pushforward will be a class

$$\mathcal{L} := \pi_*\mathrm{ev}^*\eta \in H^1\left(\mathrm{Map}(M, BG), \mathbb{C}^*\right).$$

That is, it corresponds to a local system of 1-dimensional complex vector spaces. This cohomology class determines the local system $\mathcal{L}$ up to isomorphism.

Given a 1-dimensional local system, we can try to integrate it, to obtain a single vector space. We could look at the cohomology $H^0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$, or we could look at the homology $H_0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$. We want to assign a complex vector space $Z(M)$ which corresponds to either of these. These two vector spaces *turn out to be the same.*

We have that $\pi_0\mathrm{Map}(M, BG)$ corresponds to G-bundles on M, so

$$\mathrm{Map}(M, BG) = \coprod_{\text{iso classes of G-bundles } P} B\mathrm{Aut}(P).$$

So let's start by thinking about local systems on things like $B\mathrm{Aut}(P)$.

Suppose that H is a finite group, and let's consider complex local systems $\mathcal{L}$ on BH. This is the same thing as a complex representation V of H. We have that

$$H^0\left(BG, \mathcal{L}\right) = V^H = \left\{v \in V \colon hv = v \forall h \in H\right\}.$$

The homology is

$$H_0\left(BG, \mathcal{L}\right) = V_H = V \big/ \mathbb{C} \cdot \left\{hv - v\right\}.$$

This is the minimal quotient of V you can form on which H acts trivially. When H is a finite group, there is an obvious relation between these. We have an averaging function

$$\begin{aligned} V &\to V \\ v &\mapsto \sum_{h \in H} hv. \end{aligned}$$

This map factors through the subspace V^H, but it also factors through the quotient given by the coinvariants, since it annihilates vectors of the form $hv - v$. So we get a norm

$$\begin{tikzcd} V \arrow[rrr] \arrow[dr] & & & V \\ & V_H \arrow[r, "\mathrm{Nm}"'] & V^H \arrow[ur] & \end{tikzcd}$$

Basic fact: This map is an isomorphism (assuming characteristic zero).

Proof. We should write down the inverse map. There is an obvious map in the other direction:

$$V^H \subseteq V \twoheadrightarrow V_H.$$

Let's call this map λ. We see that $\lambda \circ \mathrm{Nm} = \mathrm{Nm} \circ \lambda$, which is multiplication by the order of H. Over characteristic zero, this multiplication is an isomorphism. $\square$

Remark 1.0.1. Recall that if M is an n-manifold, we thought about this function $\mathrm{Map}(M, BG) \to \mathbb{C}^*$, given by $f \mapsto \langle [M], f^*\eta \rangle$. This map gives us something in $H^0(\mathrm{Map}(M, BG), \mathbb{C}^*)$. This integration procedure was

$$Z(M) = \int_{\mathrm{Map}(M,BG)} \langle [M], f^*\eta \rangle \, ,$$

took the class in the degree zero cohomology, but we identified cohomology with homology by doing this norm map on every component.

Thinking as an algebraic topologist, we can turn fields K into cohomology theories HK. Thinking about fields from a very large distance, there are fields of characteristic zero, and those of characteristic p. Morava realized that in the world of cohomology theories, there are a hierarchy of examples which interpolate between things like $H\mathbb{Q}$ and things like $H\mathbb{F}_p$. Fixing a prime number p, we have that Morava K-theories are an infinite sequence of cohomology theories, with

$$H\mathbb{Q} = K(0) \subseteq K(1) \subseteq \cdots \subseteq K(\infty) = H\mathbb{F}_p.$$

Question: Do these constructions make sense "over $K(n)$?"

Morava K-theories are characteristic p objects, since multiplication by p is the zero map $K(n) \to K(n)$ for $n > 0$. If we think characteristic p is bad, we might think Morava K-theories are bad. However the answer to this question is yes!

Theorem 1.0.2. (Hovey–Sadofsky) Let V be a $K(n)$-module with an action of a finite group H. Then the norm map

$$\mathrm{Nm}_H : V_{hH} \to V^{hH}$$

is an isomorphism for $n < \infty$.

Suppose that V and W are $K(n)$-modules and suppose we have a family of maps $f_x : V \to W$ parametrized by $x \in BH$. That is, a continuous map $f : BH \to \mathrm{Map}(V, W)$. Yet another way to think about this data is considering f as an element of $H^0\left(BH, \underline{\mathrm{Map}(V, W)}\right)$. Since V and W were $K(n)$-modules, we have that $\underline{\mathrm{Map}(V, W)}$ is a $K(n)$-module (with two $K(n)$-module structures). This theorem earlier tells us that

$$H^0\left(BH, \underline{\mathrm{Map}(V, W)}\right) \cong H_0\left(BH, \underline{\mathrm{Map}(V, W)}\right) \to H_0\left(*, \underline{\mathrm{Map}(V, W)}\right) = \pi_0\mathrm{Map}(V, W),$$

by mapping along $BH \to *$. Thus using this theorem from earlier, we can go from a *family* of maps, to a *single* map $V \to W$. We denote this procedure by

$$\mathrm{Map}(BH, \mathrm{Map}(V, W)) \to \pi_0\mathrm{Map}(V, W)$$
$$f \mapsto \int f.$$

We saw this earlier when $n = 0$ and when $V = W = \mathbb{C}$.

Now let's assume that H is abelian. Then BH is an abelian group object in spaces. What if we want to study representations of BH? That is, local systems on $B(BH) = K(H, 2)$. This is a simply connected space, so there should be no local systems on it, that is, this doesn't make sense classically.

So instead we want to study representations of BH on $K(n)$-modules, that is, local systems $\mathcal{L}$ of BH-modules. We could study the analogue of the coinvariants and invariants, which are the homotopy (co)limits over $\mathcal{L}_x$, where $x \in BH$. The Hovey–Sadofsky theorem gives

$$\lim_{BH} : \mathrm{hocolim}_{y \in K(H,2)}\mathcal{L}_y \to \mathrm{holim}_{x \in K(H,2)}\mathcal{L}_x.$$

To give such a map is to give a family of maps $f_{x,y} : \mathcal{L}_x \to \mathcal{L}_y$, and these should vary continuously in x and y. Any path $p : [0, 1] \to K(H, 2)$ satisfying $p(0) = x$ and $p(1) = y$ determines an isomorphism $p_! : \mathcal{L}_y \to \mathcal{L}_x$. This depends not only on x and y but also on the path that we chose. The collection of such paths is parametrized by a space $\{x\} \times^h_{K(H,2)} \{y\} =: P_{x,y}$. So we have a collection of isomorphisms $\mathcal{L}_y \xrightarrow{\sim} \mathcal{L}_x$ parametrized by the space $P_{x,y} \simeq K(H, 1) = BH$.

We can then use that integration procedure to get

$$f_{x,y} = \int_{p \in P_{x,y}} p_!,$$

which is a single morphism $\mathcal{L}_y \to \mathcal{L}_x$ (not necessarily an isomorphism anymore). So allowing x and y to vary, we get a single map

$$\mathrm{Nm}_{K(H,2)} : \mathrm{hocolim}\mathcal{L} \to \mathrm{holim}\mathcal{L}.$$

Theorem 1.0.3. This map is also a homotopy equivalence.

We can now do this again — suppose we are interested in representations of $K(H,2)$, then $K(H,3)$ and so on. This yields the following.

Theorem 1.0.4. Let X be a space with finitely many homotopy groups, and all homotopy groups are assumed to be finite[1] and let $\mathcal{L}$ be a local system of $K(n)$-modules on X. Then there is a canonical isomorphism

$$\mathrm{Nm}_X : \mathrm{hocolim}\mathcal{L}_x \to \mathrm{holim}\mathcal{L}_x.$$

That is, there is some natural map which induces isomorphisms $H_*(X,\mathcal{L}) \xrightarrow{\sim} H^*(X,\mathcal{L})$.

This is an interesting statement even when $\mathcal{L}$ is a trivial local system. In particular if X has finitely many homotopy groups, there is a canonical isomorphism

$$K(n)_*(X) \xrightarrow{\sim} K(n)^*(X).$$

We can think about this as a statement about X: if X is a nice space it satisfis a Poincaré duality with respect to Morava K-theory. We could also think about it as a statement about the category of $K(n)$-local spectra — it is not just an *additive* category, but it has some kind of fancier additivity where we can take a collection of morphisms indexed over a space and "add" or integrate the maps together. This theorem is also addressing the question that we started with — are there other constructions of Dijkgraaf–Witten theory? Yes, we can replace the height zero complex numbers by things of higher height, like Lubin–Tate spectra.

Question: Why is this true (in an easy example)?

If $X = K(H,2)$, the Hovey–Sadofsky theorem gives us a map

$$K(n)_*(X) \to K(n)^*(X).$$

There is an element $1 \in K(n)^0(X)$, and suppose we could find something, call it $y \in K(n)_0(X)$, mapping to it under the norm. Then if we had such a y, we would have that multiplication by y will induce a map from

$$\Theta : \mathrm{holim}\mathcal{L} \to \mathrm{hocolim}\mathcal{L}.$$

In classical ordinary homology this is called the cap product. The condition $\mathrm{Nm}(y) = 1$ is equivalent to the statement that Θ is inverse to the norm map.

Example 1.0.5. Let $X = BH$ for a finite p-group H, and $n = 1$. Then we have a map

$$K(n)_*(BH) \to K(n)^*(BH).$$

In height one, we know what these mean — these lift to characteristic zero, since $K(1) = \widehat{\mathrm{KU}}/p$. Complex K-theory of BH is described by the Atiyah–Segal completion theorem,

[1] For example $BH, B^2H, \dots$ where H is finite abelian.

so we have that

$$\widehat{\mathrm{KU}}^0(BH) = \mathtt{Rep}(H)^\wedge$$
$$K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

So our map would be

$$\mathtt{Rep}(H)^\vee/p = K(1)_0(BH) \to K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

By character theory, $\mathtt{Rep}(H) \otimes \mathbb{C}$ is the conjugation-invariant functions $H \to \mathbb{C}$, by sending V to its character χ_V. If we p-adically complete, we are really getting a map

$$\mathtt{Rep}(H)^\vee \to \mathtt{Rep}(H).$$

Rationally, everything is computable, and we can compute that it is an isomorphism. We can study the inverse isomorphism then

$$\mathbb{Q} \otimes \mathtt{Rep}(H)^\vee \leftarrow \mathbb{Q} \otimes \mathtt{Rep}(H).$$

Over $\mathbb{C}$, this bilinear form is given by $V, W \mapsto \frac{1}{|H|} \sum_{h \in H} \chi_V(h)\chi_W(h)$. To know that this isomorphism exists integrally and not rationally, we need to check this value is an integer. But we can rewrite this as

$$\begin{aligned}\frac{1}{|H|} \sum_{h \in H} \chi_V(h)\chi_W(h) &= \frac{1}{|H|} \sum_{h \in H} \chi_{V \otimes W}(h) \\ &= \dim_{\mathbb{C}} (V \otimes W)^H .\end{aligned}$$

So this is a sketch of the proof of the Hovey–Sadofsky theorem in height one.

2. Review of chromatic homotopy theory (Elizabeth Tatum)

2.1. **Formal group laws.**
Let R be a commutative ring, then we can consider formal group laws over that ring $F \in R[[x, y]]$. We say that F is a *formal group law* if

(1) $F(x, 0) = x$
(2) $F(x, y) = F(y, x)$
(3) $F(x, F(y, z)) = F(F(x, y), z)$.

Let $f \in R[[x]]$. We say that it is a *homomorphism* from a formal group law F to a formal group law G if $f(F(x, y)) = G(f(x), f(y))$.

We let the *n-series* on a formal group law F be given by

$$[n]_F(x) := F(x, F(x, \ldots F(x, x))).$$

That is, F applied n times. In particular when $n = p$ is a prime, we get that

$$[p]_F(x) \equiv ax^{p^h} + \text{higher order terms.}$$

We say that F has height $\geq h$ if there are no higher order terms, and that F has height exactly h if a is a unit.

Example 2.1.1. We have the *additive formal group law* $F_a(x, y) = x + y$, with height ∞.

Example 2.1.2. We have the *multiplicative formal group law* $F_m(x, y) = x + y + xy$, which has height 1.

Theorem 2.1.3. (Lazard) Over an algebraically closed field, F and G should have the same height if and only if F is isomorphic to G.

2.2. **Complex oriented cohomology theories.**
We say that a ring spectrum E is *complex orientable* if the map $E^2(\mathbb{CP}^\infty) \to E^2(S^2)$ is surjective. In particular in the reduced cohomology, an orientation is a choice of generator x mapping to $1 \in \pi_0(E)$ under the composite

$$\widetilde{E}(\mathbb{CP}^\infty) \to \widetilde{E}(S^2) \simeq \pi_0 E.$$

We have that $\mathbb{CP}^\infty$ has a natural multiplication, so by applying E^*, we get

$$E^*\mathbb{CP}^\infty \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$
$$x \mapsto F(x^L, x^R.$$

So any choice of complex orientation yields a formal group law.

- $H(\mathbb{Z}/p)$ carries F_a
- KU carries F_m
- MU carries the universal formal group law F_{MU}.

This universal fgl is characterized by the property that for any formal group law F over R, there is a $\theta : \mathrm{MU}_* \to R$ so that $\theta(F_{\mathrm{MU}}) = F$. Here $\mathrm{MU}_* \cong \mathbb{Z}[x_1, x_2, \ldots]$, where $|x_i| = 2i$.

2.3. **Morava K-theories and related spectra.** The *Brown–Peterson spectrum*, for a fixed prime p, is a wedge summand in complex cobordism

$$\mathrm{MU}_{(p)} \simeq \wedge \mathrm{BP}.$$

We have that BP is a ring spectrum such that $\mathrm{BP}_* \cong \mathrm{MU}_{(p)}/(x_i,\ i \neq p^k - 1)$. Thus

$$\mathrm{BP}_* \cong \mathbb{Z}_{(p)}\left[v_1, v_2, \ldots\right].$$

The x_i's are *not* living in powers of the form $2(p^k - 1)$, and we are quotienting them out. So the v_i's *are* living in those powers — $|v_i| = 2(p^i - 1)$.

Applying $[p]_{F_{\mathrm{MU}}} \to [p]_F$, we are getting that the height of F was the coefficient a appearing in $[p]_F = ax^{p^h}$. So where the v_i's land after this map, v_h is landing on a.

Morava E-theory: The Johnson–Wilson spectrum has homology $v_n^{-1}\mathrm{BP}_*/\left(v_{n+1}, v_{n+2}, \ldots\right) \cong \mathbb{Z}_{(p)}\left[v_1, \ldots, v_{n-1}, v_n^{\pm}\right]$. Morava E-theory is the completion of this – we delinate this from the Johnson–Wilson spectrum $E(n)$ by writing a subscript E_n:

$$(E_n)_* \cong \mathbb{W}(k)\left[\left[u_1, \ldots, u_{n-1}\right]\right]\left[u_n^{\pm}\right].$$

Morava E-theory tells you about the deformations of a formal group law of height n.

Deformations: If $\phi : R \twoheadrightarrow k$ is a nice ring homomorphism, then a formal group law F over R is a deformation of some formal group law G over k if $\phi(F) = G$. We think e.g. about R being some infinitesimal thickening of the field k.

Morava K-theory: We have a $K(n)$ so that $K(n)_* \cong \mathbb{F}_p\left[v_n^{\pm}\right]$. This is a formal group law of height exactly n. At each prime we have Morava K-theories $K(1), K(2), \ldots$. The Morava E-theories $E(n)$ are telling you about the open sets containing the $K(i)$'s for $i < n$. Morava K-theories are like residue fields, and Morava E-theories are like complete local rings at these points.

The $K(n)$'s are like fields in ring spectra. We would say that E is a *field* if $E_*(X)$ is a sum of free E_*-modules.

Theorem 2.3.1. E is a field if and only if E is a $K(n)$.

Furthermore, we have that $K(n) \wedge X \simeq \wedge \Sigma^{?} K(n)$ is a wedge sum of suspensions of $K(n)$.

2.4. **Bousfield localization.** Fix a ring spectrum E. We say that X is *E-acyclic* if $E_*X = 0$. We say that X is *E-local* if for every E-acyclic Y, we have that $[Y, X] \simeq *$. Finally we say that $f : X \to Y$ is an *E-equivalence* if $E_*(f)$ is an isomorphism.

A *localization functor* is a functor of the form

$$L : Sp \to Sp,$$

together with a natural transformation $\eta : \mathrm{id} \to L$ so that

(1) $L\eta : L \to L^2$ is an equivalence (localizing twice doesn't do anything)
(2) $L\eta \simeq \eta L$.

Theorem 2.4.1. (Bousfield) For every spectrum E, there exists a localization functor $L_E : Sp \to Sp$ with a natural transformation η_E such that for every X, we have that $\eta_E : X \to L_E X$ is the initial E-equivalence.

That is,

(1) $E_*(\eta_X) : E_* X \to E_* L_E X$ is an isomorphism
(2) If $f : X \to Y$ is an E-equivalence and Y is E-local then there is a unique map making the diagram commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ L_E X \arrow[ur, dashed] & \end{tikzcd}$$

Let $\langle E \rangle$ denote the Bousfield class of E — that is, the "collection of E-local spectra." We say that $\langle E \rangle \subseteq \langle F \rangle$ if X is E-local implies that X is F-local.

Proposition 2.4.2. We have that

(1) If $\langle E \rangle = \langle F \rangle$ then there is a natural isomorphism $L_E \simeq L_F$.
(2) If $\langle E \rangle \subseteq \langle F \rangle$, then we have that $L_E L_F \simeq L_E$, and there is a natural transformation $\eta : L_F \to L_E$.

Fact 2.4.3. We have that L_{E_n} is smashing — this means that $L_{E_n}(X) \simeq \big(L_{E_n}(S^0)\big) \wedge X$, and the localization $L_E : Sp \to Sp$ preserves direct sums.[2]

So we get an algebraic chromatic tower

$$L_{E_0} X \leftarrow L_{E_1} X \leftarrow L_{E_2} X \leftarrow \cdots$$

These have monochromatic layers $M_i(X) = \ker\big(L_{E_i} X \to L_{E_{i-1}} X\big)$, which come with maps $M_i(X) \to L_{K(i)} X$. The monochromatic layers and the localization at K-theory are not the same as spectra, but they contain exactly the same information.

Chromatic convergence theorem 2.4.4. If X is a p-local finite spectrum, then

$$X \simeq \operatorname{holim} L_n(X)$$

Fact 2.4.5. Let $\mathscr{C}_0$ denote the full subcategory of p-local finite spectra. Then denote by $\mathscr{C}_n$ the full subcategory of $K(n)$-acyclics, so we have a chain of inclusions

$$\mathscr{C}_0 \supseteq \mathscr{C}_1 \supseteq \cdots \supseteq \mathscr{C}_\infty = \{*\}\,.$$

[2]Localization always sends direct sums in *Sp* to direct sums in E-local spectra. What this condition means is that it preserves direct sums *in spectra*. This is really telling you that the *inclusion* of E-local spectra into spectra preserves direct sums (and actually arbitrary colimits).

Generally a sum of local things won't be local (think about p-completion). However smashing localizations will have this property.

Localization generally feels like the analogue of localization and then completion for rings. Smashing localizations are just localizations.

A full subcategory is called *thick* if it is closed under

(1) retracts
(2) weak equivalences
(3) cofiber sequences.

Example 2.4.6. We have that E-acyclics, E-local objects are thick subcategories.

Thick subcategory theorem 2.4.7. If $\mathscr{C}$ is a thick subcategory p-local finite spectra, then $\mathscr{C}$ is one of the $\mathscr{C}_n$'s from the filtration above.

We say that a finite spectrum F is *type* n if $K(i)_*(F) = 0$ for all $i < n$, and $K(n)_*(F) \neq 0$.

Let F be any type n spectrum. Then a v_n*-self map* is a map $f : \Sigma^i F \to F$ so that

$$K(m)_*(f) = \begin{cases} \text{multiplication by a rational number} & m = n = 0 \\ \text{an isomorphism} & m = n \neq 0 \\ \text{nilpotent} & m \neq n. \end{cases}$$

Periodicity theorem 2.4.8. Any finite type n spectrum admits a v_n-self map. The telescope of this map is

$$\mathrm{Tel}(F) = \operatorname{hocolim}\left(F \xrightarrow{v_n} F \xrightarrow{v_n} F \to \cdots\right),$$

and this is independent of the choice of v_n-self map and the choice of finite type n spectrum. So we can call this $T(n)$.

Fact 2.4.9. $T(n)$ is $K(m)$-acyclic for all $m \neq n$. Applying $K(m)_*$ to the map above, we are taking a homotopy colimit along nilpotents, so this vanishes.

There is a natural transformation $\lambda : L_{T(n)} \to L_{K(n)}$. For finite spectra, we know that $T(n)$-acyclics and $K(n)$-acyclics are the same. Knowing this for all spectra would imply the localizations are the same, which is the telescope conjecture.[3]

[3]The category of $T(n)$-local things contain all $K(n)$-local things. It might be larger. This implies that every $T(n)$-acyclic is always $K(n)$-acyclic. There are certain spectra for which $T(n)$ and $K(n)$-acyclic coincide — we know this for finite spectra and for ring spectra (it follows from the nilpotence theorem).

3. The Tate construction (Andres Mejia)

[todo — get tex from Lucy]

4. Ambidexterity (Thomas Brazelton)

4.1. **Local systems.**

A local system is, very roughly speaking, anything you might want to take cohomology in. Classically speaking, a *local system* of abelian groups on a space X is a locally constant sheaf $\mathcal{L}$ on X.

Example 4.1.1. Local systems subsume singular cohomology — this is because for any abelian group A, we can take the constant sheaf $\underline{A}$ considered as a local system.

If X is path-connected, and $\mathcal{L}$ is a local system on X, then we can take any two points x and y, and a path $\gamma : [0,1] \to X$ between them (that is, $\gamma(0) = x$ and $\gamma(1) = y$). We see that $\gamma^*\mathcal{L}$ is constant, giving an isomorphism between the fibers $\mathcal{L}_x$ and $\mathcal{L}_y$. We can check that homotoping γ will not affect the isomorphism $\mathcal{L}_x \xrightarrow{\sim} \mathcal{L}_y$. That is, we can restate $\mathcal{L}$ as the assignment of the data:

- an abelian group $\mathcal{L}_x$ for every $x \in X$
- an isomorphism $\mathcal{L}_x \xrightarrow{\sim} \mathcal{L}_y$ for every homotopy class of paths $x \to y$,

subject to some extra coherence data. From this we can get a new definition of a local system.

Definition 4.1.2. A *local system* on X valued in a 1-category $\mathscr{C}$ is a functor

$$\mathcal{L} : \Pi_1(X) \to \mathscr{C}.$$

Suppose now we want something a little stronger. If γ, γ' are homotopic maps from x to y in X, they provide isomorphisms $\mathscr{C}_x \xrightarrow{\sim} \mathscr{C}_y$ in $\mathscr{C}$. If $\mathscr{C}$ is a 2-category, we might ask for a witness of the homotopy $\gamma \Rightarrow \gamma'$ to be witnessed by a 2-cell in $\mathscr{C}$, and for a different witness to be witnessed by a different 2-cell. Similarly if we have a 3-cell between these, we might ask for a 3-cell witnessing a higher homotopy in $\mathscr{C}$, provided $\mathscr{C}$ has this higher categorical structure.

This leads us to a higher-categorical definition of local systems.

Definition 4.1.3. A *local system* on X valued in an ∞-category $\mathscr{C}$ is an ∞-functor

$$\mathcal{L} : \Pi_\infty(X) \to \mathscr{C},$$

where $\Pi_\infty(X)$ is the fundamental ∞-groupoid of X.

Viewing X as a Kan complex, we might just say a local system is an ∞-functor

$$\mathcal{L} : X \to \mathscr{C}.$$

4.2. **Pullback and adjoints.**

Let $f : X \to Y$ be any map of spaces. Then given a local system $\mathcal{L} : Y \to \mathscr{C}$ on Y, we can pull it back to a local system $f^*\mathcal{L}$ on X, by pre-composing with f. For any fixed ∞-category $\mathscr{C}$, this defines a functor

$$f^* : \mathrm{Fun}(Y, \mathscr{C}) \to \mathrm{Fun}(X, \mathscr{C}).$$

If $\mathscr{C}$ admits small colimits, then we may left Kan extend to define a left adjoint to f^* (Higher Topos Theory, 4.3.3). We denote this by $f_!$:

$$f_! : \mathrm{Fun}(X, \mathscr{C}) \leftrightarrows \mathrm{Fun}(Y, \mathscr{C}) : f^*.$$

Dually when $\mathscr{C}$ admits small limits, we may right Kan extend to define a right adjoint to f^*, which we denote by f_*. This gives

$$f_! \dashv f^* \dashv f_*.$$

Example 4.2.1. Let S be a set, viewed as a discrete space, and consider the map $f : S \to *$. Pullback is then the diagonal map $f^* : \mathscr{C} \to \mathrm{Fun}(S, \mathscr{C})$. We see that any functor $S \to \mathscr{C}$ picks out a collection $\{C_s\}$ of objects in $\mathscr{C}$ for each $s \in S$. Assume that $\mathscr{C}$ has all products and coproducts. Then we can see that

$$\begin{aligned} f_! : \mathrm{Fun}(S, \mathscr{C}) &\to \mathscr{C} \\ \{C_s\} &\mapsto \coprod_{s \in S} C_s, \end{aligned}$$

and that

$$\begin{aligned} f_* : \mathrm{Fun}(S, \mathscr{C}) &\to \mathscr{C} \\ \{C_s\} &\mapsto \prod_{s \in S} C_s. \end{aligned}$$

There is always a natural transformation from products to coproducts here, given by $f_! \to f_*$. In particular when products and coproducts agree, e.g. in $\mathtt{Ab}$, we will have that this is a natural isomorphism $f_! \simeq f_*$.

Example 4.2.2. Consider $f : BG \to *$. In this case, since $\mathrm{Fun}(*, \mathscr{C}) \simeq \mathscr{C}$, we have that pullback is of the form

$$f^* : \mathscr{C} \to \mathrm{Fun}(BG, \mathscr{C}),$$

assigning to every object in $\mathscr{C}$ the trivial G-action.

In this case, the adjoints yield, for every G-equivariant object $C \in \mathscr{C}$, the coinvariants $f_! C = C_G$ and the invariants $f_* C = C^G$. Denoting by $C^{tG} = \mathrm{cofib}\left(C_G \to C^G\right)$, we have that a canonical equivalence $f_! \simeq f_*$ would imply that the Tate construction vanishes for every G-equivariant object of $\mathscr{C}$.

Associated to these types of adjunction we have the so-called "calculus of mates," which allows us to take commutative squares of spaces and discuss how the induced functors relate to one another.

Another example of where the calculus of mates appears is in the types of natural isomorphisms of restriction and extension of scalars for modules that come out of commutative diagrams of rings.

Proposition 4.2.3. If f and g are composable, then there is a canonical equivalence $(gf)^* \simeq f^*g^*$. This induces a canonical equivalence $(gf)! \simeq g_! f_!$ by the formalism of adjunctions.

Definition 4.2.4. Consider a commutative diagram of spaces

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "g"'] & Y. \end{tikzcd}$$

Then there is a *Beck–Chevalley exchange transformation* (think about this as top-left to bottom-right), denoted by

$$\mathrm{Ex}_!^* : j_! i^* \to f^* g_!.$$

This is defined by first starting with $j_! i^*$, and tacking on the counit $\mathrm{id}_B \to g^* g_!$ on the end of it. We then get $j_! i^* g^* g_!$. Since the diagram commutes, there is a canonical equivalence $i^* g^* \simeq j^* f^*$, getting us to $j_! j^* f^* g_!$. Finally, we may apply the counit $j_! j^* \to \mathrm{id}$ to conclude. The entire composite gives us:

$$j_! i^* \mathrm{id}_B \to j_! i^* g^* g_! \simeq j_! j^* f^* g_! \to f^* g_!.$$

Proposition 4.2.5. If we have a pullback square, the Beck–Chevalley exchange transformation is an equivalence.

Q: Let $f : X \to Y$, and consider the adjunction $f_! \dashv f^*$. When will $f_!$ also be a *right adjoint* to f^*?

Given a fixed category $\mathscr{C}$ admitting finite limits and colimits, we will define a class of *$\mathscr{C}$-ambidextrous* maps $f : X \to Y$. These will have the property that if $f : X \to Y$ is $\mathscr{C}$-ambidextrous, then there is a canonical equivalence $f_! \simeq f_*$.

4.3. **Ambidextrous morphisms.**

Example 4.3.1. Suppose that $f : X \xrightarrow{\sim} Y$ is a homotopy equivalence. Then $f^* : \mathrm{Fun}(Y, \mathscr{C}) \to \mathrm{Fun}(X, \mathscr{C})$ is an equivalence of categories, and it can be easily promoted to an adjoint equivalence, so that $f_! \simeq f_*$ canonically. In particular, there is a unit map $\mu_f : \mathrm{id} \to f_! f^*$, exhibiting $f_!$ as a right adjoint to f^*.

Homotopy equivalences provide our first class of morphisms which we call *ambidextrous.* Somehow these are the "most" ambidextrous, in the sense that they have the strongest structure. However as we might expect, there exist morphisms which are $\mathscr{C}$-ambidextrous without being homotopy equivalences.

We will define ambidexterity inductively, with homotopy equivalences being the base case. For indexing reasons that will become clear later, we would like to start at $n = -2$. So we will define, for each $n \geq -2$:

- A collection of *n-ambidextrous morphisms* in `Top`
- For each n-ambidextrous morphism $f : X \to Y$, a natural transformation $\mu_f^{(n)}$: $\mathrm{id} \to f_! f^*$, well-defined up to homotopy, exhibiting $f_!$ as a right adjoint to f^*.

Base case $n = -2$: We say f is (-2)-ambidextrous if and only if f is an equivalence. In this case, we define $\mu_f^{(-2)}$ to be any homotopy inverse to the counit $f_! f^* \to \mathrm{id}$.

Inductive step: Suppose that we have defined n-ambidextrous morphisms for some n. We will define $(n+1)$-ambidextrous maps in two steps: first we define *weakly* $(n+1)$-ambidextrous maps, and then $(n+1)$-*ambidextrous* maps.

Let $f : X \to Y$ be arbitrary, and consider the diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

By Beck–Chevalley, there is an exchange isomorphism $(\pi_1)_! \pi_2^* \simeq f^* f_!$. We say that f is *weakly* $(n+1)$-*ambidextrous* if δ is n-ambidextrous. In this context, we define a counit $\nu_f^{(n+1)}$ to be the composite

$$f^* f_! \xrightarrow{\left(\mathrm{Ex}_!^*\right)^{-1}} (\pi_1)_! \, \pi_2^* \xrightarrow{\mu_\delta^{(n)}} (\pi_1)_! \, \delta_! \delta^* \pi_2^* = (\mathrm{id}_X)_! \, \mathrm{id}_X^* = \mathrm{id}_{\mathrm{Fun}(X,\mathscr{C})}$$

We say f is $(n+1)$-*ambidextrous* if the following conditions hold:

(1) The transformation $\nu_f^{(n+1)} : f^* f_! \to \mathrm{id}$ is the counit for an adjunction $f^* \dashv f_!$, with some unit $\mu_f^{(n+1)}$
(2) Weak $(n+1)$-ambidexterity is closed under pullback along f. That is, for every pullback square

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

we have that g is weakly ambidextrous, with counit $\nu_g^{(n+1)} : g^* g_! \to \mathrm{id}$ defined in the Beck–Chevalley process above
(3) Property (1) is closed under pullback along f. That is, for any pullback square as above, we have that $\nu_g^{(n+1)}$ is the counit of an adjunction $g^* \dashv g_!$.

From this definition, the following are immediate.

Proposition 4.3.2. (Weak) n-ambidexterity is closed under pullback.

Moreover from our inductive definitions, we have the following:

Proposition 4.3.3. Let $-2 \leq m \leq n$.

(1) If f is weakly m-ambidextrous,[4] then f is weakly n-ambidextrous, and $\nu_f^{(m)}$ and $\nu_f^{(n)}$ agree up to homotopy.
(2) If f is m-ambidextrous, then f is n-ambidextrous, and $\mu_f^{(m)}$ and $\mu_f^{(n)}$ agree up to homotopy.

Proof idea. It suffices to let $n = m + 1$, and induct. The inductive step is basically immediate from definitions, and the base case is very direct. $\square$

Definition 4.3.4. We say that f is *weakly ambidextrous* if it is weakly ambidextrous for some $n \geq -1$, and we say that f is *ambidextrous* if it is ambidextrous for some n. We let $\nu_f : f^* f_! \to \mathrm{id}$ denote the counit and $\mu_f : \mathrm{id} \to f_! f^*$ denote the unit. This notation is well-defined up to homotopy by the previous proposition.

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \left\{\substack{n\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] \arrow[d, hook] & \left\{\substack{(n+1)\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \{\text{ambidextrous maps}\} \arrow[d, hook] \\
\cdots \arrow[r, hook] & \left\{\substack{\text{weakly} \\ n\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] & \left\{\substack{\text{weakly} \\ (n+1)\text{-ambidextrous} \\ \text{maps}}\right\} \arrow[r, hook] & \cdots \arrow[r, hook] & \left\{\substack{\text{weakly} \\ \text{ambidextrous maps}}\right\}
\end{tikzcd}$$

4.4. **Norms.** Suppose $\mathscr{C}$ is an ∞-category with small limits and colimits. Let $f : X \to Y$ be a continuous map of spaces, and let $f_! \dashv f^* \dashv f_*$ be the associated left and right adjoints to pullback provided by Kan extensions. Suppose that f is *weakly ambidextrous* but **not necessarily ambidextrous** (recall this means inductively that the diagonal is weakly ambidextrous of one degree lower, and crucially that there is a natural transformation $\nu_f : f^* f_! \to \mathrm{id}$). Then by adjunction we have a natural homotopy equivalence of mapping spaces

$$\mathrm{Map}\left(f^* f_!, \mathrm{id}\right) \simeq \mathrm{Map}\left(f_!, f_*\right).$$

In particular ν_f maps to a natural transformation, which by definition is the composite

$$f_! \xrightarrow{\eta \cdot f_!} f_* f^* f_! \xrightarrow{f_* \cdot \nu_f} f_*.$$

We call this the *norm* of f and denote it by $\mathrm{Nm}_f : f_! \to f_*$.

Proposition 4.4.1. Let f be weakly ambidextrous as above. Then it is ambidextrous if and only if

[4] Weak ambidexterity isn't defined for $m = -2$ but that's ok

(1) Weak ambidexterity is preserved under pullback along f
(2) The norm map $\mathrm{Nm} : f_! \to f_*$ is an equivalence
(3) The norm map for any map obtained by pullback along f is an equivalence.

Example 4.4.2. We can rephrase our example from earlier to say that the following are equivalent for $f : BG \to *$:

(1) BG is $\mathscr{C}$-ambidextrous
(2) The norm Nm_f is an equivalence
(3) For every G-equivariant object of $\mathscr{C}$, the Tate construction vanishes.

Proposition 4.4.3. Weak ambidexterity is closed under composition — that is, if f and g are composable and weakly ambidextrous, we can take μ_{gf} to be the composite

$$(gf)^*(gf)_! \simeq f^*g^*g_!f_! \xrightarrow{\mu_g} f^*f_! \xrightarrow{\mu_f} \mathrm{id}.$$

5. Higher semiadditivity as span-modules (Kaif Hilman)

5.1. **Introduction.** In [**?**] Segal showed that the structure of commutative monoids in an arbitrary category $\mathbb{C}$ with finite products can be cleanly encoded as product-preserving functors $\mathrm{Fin}_* \to \mathbb{C}$ where Fin_* is the 1-category of finite pointed sets. Moreover, if we write $\mathrm{CMon}(\mathbb{C}) := \mathrm{Fun}^\times(\mathrm{Fin}_*, \mathbb{C})$ for the category of commutative monoids, it turns out that if $\mathbb{C}$ were presentable, then $\mathrm{CMon}(\mathbb{C})$ is the free semiadditive category generated by $\mathbb{C}$. It would be desirable to show that something similar holds for *higher semiadditivity.*

To this end, observe that Fin_* can also be thought of as a category of *spans* whose objects are finite sets and a morphism from X to Y is a span

$$X \hookleftarrow Z \to Y$$

where $X \hookleftarrow Z$ is injective. Hence, it is natural to expect that the span construction might be fruitful in encoding the notion of higher semiadditivity. And indeed, this was what was worked out in [**?**] and we will try to explain some of the highlights from the paper in this note. As a guide to the reader, in §2 we will introduce the basic notion of spans of finite spaces; §3 will define the notion of higher semiadditivity and formulate the universal property of these spans; finally, the punchlines of this note will appear in §4, where we will see that formal consequences of the results in §3 include: (1) the fact that we can view the property of being higher semiadditive equivalently as being a module over the spans introduced in §2, and (2) that for presentable categories, we have a Segal-style method of producing higher semiadditive categories.

5.2. Spans of π-finite spaces.

5.2.1. *Basic definitions.*

Definition 5.2.1 (Truncatedness and π-finiteness, [**?**] 2.5-2.6)**.** Let X, Y be spaces and $n \geq -2$. Then we say that:

- If $n \geq 0$, X is *n-truncated* if $\pi_i(X, x) = 0$ for every $i > n$ and every $x \in X$.
- If $n = -1$, then X is *(-1)-truncated* if it is either empty or contractible.
- If $n = -2$, then X is *(-2)-truncated* if it is contractible.
- A map $f : X \to Y$ is *n-truncated* if $\mathrm{fib}(f, y)$ is n-truncated for all $y \in Y$.

We say that X is *π-finite* if it is n-truncated for some n and all its homotopy groups/sets are finite. If we want to specify the n-truncatedness, we will also say that a space is *π-n-finite.*

Observation 5.2.2. A map is (-1)-truncated if it is an inclusion of path components, and it is (-2)-truncated if it is an equivalence.

Notation 5.2.3. Let $\mathcal{K}_n = \mathrm{Ho}(\mathcal{S}_n{}^{\simeq})$ be the set of representatives of all π-finite n-truncated spaces. We will be thinking of this as the set of indexing diagrams whose colimits we will be interested in.

Construction 5.2.4 (Spans, [?] §2.1, [?])**.** Let $\mathbb{C}^\dagger \subset \mathbb{C}$ be a wide subcategory whose morphisms are closed under pullbacks. Then we can construct a new $(\infty,1)$-category $\mathrm{Span}(\mathbb{C},\mathbb{C}^\dagger)$ called *the category of spans* whose objects are objects of $\mathbb{C}$ and for $X, Y \in \mathbb{C}$, morphisms $X \to Y$ in $\mathrm{Span}(\mathbb{C},\mathbb{C}^\dagger)$ are spans $X \leftarrow Z \to Y$ where $X \leftarrow Z$ is in $\mathbb{C}^\dagger$ and compositions of morphisms are given by taking pullbacks.

Fact 5.2.5 (Mapping spaces of spans, [?] 2.4)**.** For $X, Y \in \mathbb{C}$, we have that $\mathrm{Map}_{\mathrm{Span}(\mathbb{C},\mathbb{C}^\dagger)}(X,Y)$ is given by the subspace of $(\mathbb{C}_{/X\times Y})^{\simeq}$ on those spans $X \leftarrow Z \to Y$ such that $X \leftarrow Z$ is in $\mathbb{C}^\dagger$.

The following will be the main object of study in this notes.

Definition 5.2.6. Let $n \geq -2$ and $m \leq n$. Then we write:

- $\mathcal{S}_n \subseteq \mathcal{S}$ be the full subcategory of π-finite n-truncated spaces.
- $\mathcal{S}_{n,m} \subseteq \mathcal{S}_n$ be the non-full wide subcategory whose mapping spaces are spanned by m-truncated maps.

Given these notations, we define $\mathcal{S}_n^m := \mathrm{Span}(\mathcal{S}_n, \mathcal{S}_{n,m})$.

Observation 5.2.7. Since (-2)-truncatedness of a map is the same as being an equivalence, we see that $\mathcal{S}_{n,-2} \simeq \mathcal{S}_n{}^{\simeq}$ so that $\mathcal{S}_n^{-2} \simeq \mathcal{S}_n$.

Observation 5.2.8. The inclusion $\mathcal{S}_{n-1}^m \hookrightarrow \mathcal{S}_n^m$ is fully faithful. This is because $m \leq n$, and so if $f : Z \to X$ is m-truncated and X was $(n-1)$-truncated, then Z is $(n-1)$-truncated as well.

Observation 5.2.9. $(\mathcal{S}_n^m)^{\simeq} \subseteq (\mathcal{S}_n)^{\simeq} \subseteq \mathcal{S}_n$.

5.2.2. *Colimits in spans.* Here is an important lemma to check preservation of $\mathcal{K}_n$-colimits out of $\mathcal{S}_n$: the upshot is that in this special case it can be checked just on the constant diagrams.

Lemma 5.2.10 ([?] 2.11)**.** Let $\mathcal{D}$ admit $\mathcal{K}_n$-colimits and $F : \mathcal{S}_n \to \mathcal{D}$ be a functor. Then F preserves $\mathcal{K}_n$-colimits iff it preserves those which are constant at $* \in \mathcal{S}_n$.

Proof. The only if direction is immediate. To see the reverse, we use the satisfying classical trick of using the Grothendieck construction to compute colimits in spaces. Let $Y \in \mathcal{K}_n$ and $\mathcal{G} : Y \to \mathcal{S}_n$ be a Y-indexed diagram. Unstraightening we obtain a left fibration $p_{\mathcal{G}} : Z \to Y$ which in particular implies that Z is also a space so that we obtain a fibre sequence of spaces $W \to Z \to Y$ where by construction W was π-n-finite. The upshot of this paragraph is that since Y was π-n-finite also by hypothesis, we see that Z must be too so we can consider Z as living in $\mathcal{S}_n$.

Here's the fun part: for each $y \in Y$, the space $\mathcal{G}(y) \in \mathcal{S}_n$ is the colimit of the $\mathcal{G}(y)$-indexed constant diagram with value $*$ so that by the pointwise left Kan extension formula we see that $\mathcal{G} \simeq p_! \operatorname{const}_*$. In particular, this means that

$$\operatorname{colim}(Z \xrightarrow{\operatorname{const}_*} \mathcal{S}_n) \simeq \operatorname{colim}(Y \xrightarrow{\mathcal{G}} \mathcal{S}_n) \tag{1}$$

To summarise, we now have the diagram

$$\begin{tikzcd} Z \arrow[d, "p"'] \arrow[dr, "\operatorname{const}_*"] & & \\ Y \arrow[r, "\mathcal{G} \simeq p_! \operatorname{const}_*"'] & \mathcal{S}_n \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

Again, by the pointwise left Kan extension formula, we see that $\mathcal{G} \simeq p_! *$ was computed pointwise as $\mathcal{K}_n$-space-indexed diagrams with constant value $*$. Hence, since F preserved these by hypothesis, we see that $F \circ \mathcal{G} \simeq p_!(F \operatorname{const}_*)$. Therefore we obtain

$$\begin{aligned} \operatorname*{colim}_Y F \circ \mathcal{G} &:= \operatorname{colim}(Y \xrightarrow{F \circ \mathcal{G}} \mathcal{D}) \\ &\simeq \operatorname{colim}(Z \xrightarrow{F \operatorname{const}_*} \mathcal{D}) \\ &\simeq F \operatorname{colim}(Z \xrightarrow{\operatorname{const}_*} \mathcal{D}) \\ &\simeq F \operatorname*{colim}_Y \mathcal{G} \end{aligned}$$

where the penultimate line is by our assumption on F and the last line is by (1). $\square$

Lemma 5.2.11 ([?] 2.12)**.** For every $-2 \leq m \leq n$ the inclusion $j : \mathcal{S}_n \hookrightarrow \mathcal{S}_n^m$ preserves $\mathcal{K}_n$-colimits.

Proof. By the criterion (5.2.10) we need to show that for each $X \in \mathcal{S}_n$,

$$X \simeq \operatorname{colim}(X \xrightarrow{\operatorname{const}_*} \mathcal{S}_n^m) \in \mathcal{S}_n^m$$

In other words, by Yoneda we need to show that for all $Y \in \mathcal{S}_n^m$, the map

$$\operatorname{Map}_{\mathcal{S}_n^m}(X, Y) \longrightarrow \lim_X \operatorname{Map}_{\mathcal{S}_n^m}(\operatorname{const}_*, Y) \simeq \operatorname{Map}_{\mathcal{S}}(X, \operatorname{Map}_{\mathcal{S}_n^m}(*, Y))$$

is an equivalence. Here the second equivalence is by the usual formula for limits of constant diagrams in spaces (in our case, with value $\operatorname{Map}_{\mathcal{S}_n^m}(*, Y)$). Now by (5.2.5) we know that

$$\operatorname{Map}_{\mathcal{S}_n^m}(X, Y) \simeq (\mathcal{S}_{n/X_m \times Y})^{\simeq} \quad \text{and} \quad \operatorname{Map}_{\mathcal{S}_n^m}(*, Y) \simeq (\mathcal{S}_{n/*_m \times Y})^{\simeq}$$

where the subscript m in $\mathcal{S}_{n/X_m \times Y}$ for example denotes the full subcategory of $\mathcal{S}_{n/X \times Y}$ spanned by those maps $Z \to X \times Y$ such that $Z \to X \times Y \xrightarrow{\pi_X} X$ is m-truncated. But then since X was already n-truncated and $m \leq n$, any space with an m-truncated map to X must itself have been n-truncated, and so in fact

$$\mathcal{S}_{n/X_m \times Y} \simeq \mathcal{S}_{/X_m \times Y}$$

By a similar reasoning, we see that

$$\mathcal{S}_{n/*_m \times Y} \simeq \mathcal{S}_{m/Y}$$

Now the straightening-unstraightening equivalence gives

$$\mathcal{S}_{/X\times Y} \xrightarrow{\simeq} \mathrm{Fun}(X\times Y,\mathcal{S}) \xrightarrow{\simeq} \mathrm{Fun}(X,\mathrm{Fun}(Y,\mathcal{S}))$$

which on objects is given by $\big(q : Z \to X\times Y\big) \mapsto \big(x \mapsto (y \mapsto q^{-1}(x,y))\big)$. Applying core groupoid everywhere we obtain an equivalence

$$(\mathcal{S}_{/X\times Y})^{\simeq} \xrightarrow{\simeq} \mathrm{Map}(X,\mathrm{Map}(Y,\mathcal{S}^{\simeq}))$$

Writing $\mathrm{Map}_m(Y,\mathcal{S}^{\simeq})$ for the components of $\mathrm{Map}(Y,\mathcal{S}^{\simeq})$ such that taking colimits produce m-π-finite spaces, we see clearly that the preceding equivalence restricts to an equivalence

$$(\mathcal{S}_{/X_m\times Y})^{\simeq} \xrightarrow{\simeq} \mathrm{Map}(X,\mathrm{Map}_m(Y,\mathcal{S}^{\simeq}))$$

On the other hand, $\mathrm{Map}_m(Y,\mathcal{S}^{\simeq}) \simeq (\mathcal{S}_{m/Y})^{\simeq}$, and so we're done. $\square$

Corollary 5.2.12 ([?] 2.16)**.** A functor $F : \mathcal{S}_n^m \to \mathcal{D}$ preserves $\mathcal{K}_n$-colimits iff the restriction $F : \mathcal{S}_n \hookrightarrow \mathcal{S}_n^m \to \mathcal{D}$ preserves $\mathcal{K}_n$-colimits.

Proof. By (5.2.11) the only if direction is clear. To obtain the reverse direction, note that since objects of $\mathcal{K}_n$ are groupoids, by the observation (5.2.7)(3) we see that $\mathcal{K}_n$-diagrams in $\mathcal{S}_n^m$ in fact land in $\mathcal{S}_n$, and the hypothesis implies the desired statement. $\square$

5.2.3. *Spans as commutative algebras.*

Construction 5.2.13 (Symmetric monoidality of $\mathcal{S}_n^m$)**.** It is standard that span categories inherit the symmetric monoidal structure on the original category, and so the cartesian symmetric monoidal structure on $\mathcal{S}_n$ induces a symmetric monoidal structure on $\mathcal{S}_n^m$ given by taking products of spaces. Note however that this is *no longer* a cartesian symmetric monoidal structure on $\mathcal{S}_n^m$.

Proposition 5.2.14 ([?], 2.17)**.** The symmetric monoidal product $\mathcal{S}_n^m \times \mathcal{S}_n^m \to \mathcal{S}_n^m$ preserves $\mathcal{K}_n$-colimits in each variable.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{S}_n\times\mathcal{S}_n \arrow[r, hook] \arrow[d, "\times"'] & \mathcal{S}_n^m\times\mathcal{S}_n^m \arrow[d, "\times"] \\
\mathcal{S}_n \arrow[r, hook] & \mathcal{S}_n^m
\end{tikzcd}$$

where we know that the left vertical multiplication preserves colimits in each variable separately and the horizonal maps preserve $\mathcal{K}_n$-colimits by (5.2.11). The point is that since if $X\in\mathcal{K}_n$, then it's a groupoid, and so any diagram $d : X \to \mathcal{S}_n^m$ factors through $\mathcal{S}_n \subseteq \mathcal{S}_n^m$. Together with (5.2.11) this says that X-colimits in $\mathcal{S}_n^m$ are computed in $\mathcal{S}_n \subseteq \mathcal{S}_n^m$ and so the desired conclusion, which is true for the left vertical, transfers to that on the right vertical. $\square$

Construction 5.2.15 (Spans as a commutative algebra object)**.** By [**?**] §4.8.1 we know that $\mathrm{Cat}_{\mathcal{K}_n}$ has a symmetric monoidal structure $\otimes_{\mathcal{K}_n}$ where for $\mathbb{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Cat}_{\mathcal{K}_n}$, the tensor product $\mathbb{C} \otimes_{\mathcal{K}_n} \mathcal{D}$ has the universal property

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathbb{C} \otimes_{\mathcal{K}_n} \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}_{\mathcal{K}_n,\mathcal{K}_n}(\mathbb{C} \times \mathcal{D}, \mathcal{E})$$

where the right hand side consists of functors which preserve $\mathcal{K}_n$-colimits in each variable. Hence we can get from (5.2.14) that $\mathcal{S}_n^m$ is a commutative algebra object in $\mathrm{Cat}_{\mathcal{K}_n}$.

5.2.4. *Duality in spans.*

Construction 5.2.16 (Trace and diagonals)**.** Let $\mathbb{C}$ be a category with final object $*$ and admitting finite limits. Then for $X \in \mathrm{Span}(\mathbb{C})$, we define the *trace map* in $\mathrm{Span}(\mathbb{C})$ to be the span

$$\Big(X \times X \xrightarrow{\mathrm{tr}_X} *\Big) := \Big(X \times X \xleftarrow{\Delta} X \to *\Big)$$

and the *diagonal* in $\mathrm{Span}(\mathbb{C})$ to be the span

$$\Big(* \xrightarrow{\Delta_X} X \times X\Big) := \Big(* \leftarrow X \xrightarrow{\Delta} X \times X\Big)$$

Proposition 5.2.17 (Self-duality in spans)**.** Let $\mathbb{C}$ be a category with final object $*$ and admitting finite limits. Then the trace map and diagonal constructed above exhibits every object as self-dual in $\mathrm{Span}(\mathbb{C})$.

Proof. Let $X \in \mathrm{Span}(\mathbb{C})$. Note that being dualisable can be checked at the level of homotopy categories, and so it *really is* enough to check that the composites

$$X \xrightarrow{1\times\Delta_X} X \times X \times X \xrightarrow{\mathrm{tr}_X\times 1} X \quad \text{and} \quad X \xrightarrow{\Delta_X\times 1} X \times X \times X \xrightarrow{1\times\mathrm{tr}_X} X$$

are homotopic to the identity. We will only show the first. Since composition in span categories are given by pullbacks, we get that the first composite is given by the span

$$\begin{tikzcd}
& & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[ddll, equal, bend right] \arrow[ddrr, equal, bend left] \arrow[dd, phantom, "\vee"] & & \\
& X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "1\times\Delta"] & & X \times X \arrow[dl, "\Delta\times 1"'] \arrow[dr, "\pi_2"] & \\
X & & X \times X \times X & & X
\end{tikzcd}$$

which is the identity span, as required. $\square$

5.3. **Higher semiadditivity.**

5.3.1. *Basic notions.*

Definition 5.3.1 ([?] 3.1)**.** Let $m \geq -2$ and $\mathcal{D}$ a category. We say that $\mathcal{D}$ is *m-semiadditive* if $\mathcal{D}$ admits $\mathcal{K}_m$-colimits and every m-truncated π-finite space is $\mathcal{D}$-ambidextrous.

Remark 5.3.2. Two consequences which we will not prove here but which are intuitively clear, namely:

- That an m-semiadditive $\mathcal{D}$ automatically admits $\mathcal{K}_m$-limits, essentially because the $\mathcal{D}$-ambidextrousness of any $X \in \mathcal{K}_m$ already gives that the colimit also computes the limit. Given this, the intuition of m-semiadditivity is just that the canonically constructed norm map
$$\operatorname*{colim}_X \Longrightarrow \lim_X$$
is an equivalence in $\mathrm{Fun}(\mathcal{D}^X, \mathcal{D})$ for all $X \in \mathcal{K}_m$.
- The opposite of an m-semiadditive category is again m-semiadditive.

Observation 5.3.3. For $m \leq n$, n-semiadditivity implies m-semiadditivity since $\mathcal{K}_m \subseteq \mathcal{K}_n$.

Example 5.3.4. Here are some important first examples, the second of which justifies the terminology of semiadditivity.

(1) $\mathcal{D}$ is (-1)-semiadditive iff it is pointed. This is because the only nontrivial π-finite space that is (-1)-$\mathcal{D}$-ambidextrous is given by the map $\varnothing \to *$, and $\mathrm{colim}_\varnothing$ is the initial object and $\lim_\varnothing$ is the final object.

(2) $\mathcal{D}$ is 0-semiadditive iff it is semiadditive in the usual sense. To see this, recall that 0-semiadditivity implies (-1)-semiadditivity and so by the point above, $\mathcal{D}$ is pointed. Now observe that 0-truncated maps to the point $*$ in $\mathcal{S}_0$ consist precisely of maps of form $\coprod^k * \to *$ for $k < \infty$. Then pointedness allows us to construct the canonical norm map
$$\coprod^k \simeq \operatorname*{colim}_{\coprod^k *} \Longrightarrow \lim_{\coprod^k *} \simeq \prod^k$$
and being 0-semiadditive exactly requires these to be equivalences.

(3) An important class of examples for 1-semiadditivity was furnished by Lecture 3 by the Tate-vanishing of $\mathrm{Sp}_{T(n)}$. To see this, note that a map $X \to *$ where X is a π-finite space is 1-truncated iff $X = \coprod_{i=1}^k BG_i$ is a finite coproduct of Eilenberg-MacLane spaces of finite groups, and so the norm map will become the usual one
$$\bigoplus_{i=1}^k (-)_{hG_i} \Longrightarrow \bigoplus_{i=1}^k (-)^{hG_i}$$
whose cofibre $\bigoplus_i^k (-)^{tG_i}$ vanishes as we saw in Lecture 3.

5.3.2. *Modules over spans are semiadditive.* The goal of this subsubsection is to obtain an obstruction for $\mathcal{D}$ satisfying the following assumptions moreover to be m-semiadditive.

Assumption 5.3.5. $\mathcal{D}$ is $(m-1)$-semiadditive which furthermore:

(1) admits $\mathcal{K}_m$-colimits.
(2) admits a structure of an $\mathcal{S}_m^{m-1}$-module in $\mathrm{Cat}_{\mathcal{K}_m}$. This in particular means that there is an action map $\mathcal{S}_m^{m-1} \times \mathcal{D} \to \mathcal{D}$ which preserves $\mathcal{K}_m$-colimits in each variable.

Notation 5.3.6. For $\mathcal{D}$ satisfying the assumptions (5.3.5) and $X \in \mathcal{S}_m^{m-1}$, we write

$$[X] : \mathcal{D} \longrightarrow \mathcal{D}$$

for $X \otimes (-)$ afforded by the action map.

Proposition 5.3.7 (Trace obstruction, [?] 3.17, compare with [?] 5.1.8)**.** Let $\mathcal{D}$ be as in assumptions (5.3.5). Then $\mathcal{D}$ is m-semiadditive iff for all $X \in \mathcal{S}_m^{m-1}$ the transformation

$$[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{id}$$

exhibits the functor $[X] : \mathcal{D} \to \mathcal{D}$ as self-adjoint.

Theorem 5.3.8 (Modules imply m-semiadditivity, [?] 3.19)**.** Let $\mathcal{D}$ be tensored over $\mathcal{S}_m^m$ such that the action functor $\mathcal{S}_m^m \times \mathcal{D} \to \mathcal{D}$ preserves $\mathcal{K}_m$-colimits in each variable. Then $\mathcal{D}$ is m-semiadditive.

Proof. We will prove that $\mathcal{D}$ is m'-semiadditive for every $-2 \leq m' \leq m$ by induction on m'. Since every category is (-2)-semiadditive, the base case $m' = -2$ is done. Now suppose that $\mathcal{D}$ is m'-semiadditive for some $-2 \leq m' < m$. We want to use the trace criterion (5.3.7) to see that $\mathcal{D}$ is $(m'+1)$-semiadditive, and so let $X \in \mathcal{S}_{m'+1}^{m'}$. We want to show that

$$[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{id}$$

exhibits as $[X] : \mathcal{D} \to \mathcal{D}$ as self-adjoint. In other words, by the triangle identity characterisation of adjunctions, we need to see that the triangles

$$\begin{tikzcd} {[X]} \arrow[r, "{[X]([\Delta])}"] \arrow[dr, equal] & {[X] \circ [X] \circ [X]} \arrow[d, "{[\mathrm{tr}_X]_{[X]}}"] \\ & {[X]} \end{tikzcd} \qquad \begin{tikzcd} {[X]} \arrow[r, "{[\Delta]_{[X]}}"] \arrow[dr, equal] & {[X] \circ [X] \circ [X]} \arrow[d, "{[X]([\mathrm{tr}_X])}"] \\ & {[X]} \end{tikzcd}$$

commute. But then these are given precisely by the triangles witnessing self-duality of X in a span category (5.2.17), and so we're done. □

5.3.3. *Universality of spans.* The key result for everything else in the paper is the identification of the universal property of m-spans. Once we have this, the rest follow more or less formally as in the case of ordinary commutative monoids.

Theorem 5.3.9 (Universal property of m-spans, [**?**] 4.1)**.** Let $-2 \leq m \leq n$ and $\mathcal{D}$ be m-semiadditive which admits $\mathcal{K}_n$-colimits. Then evaluation at $* \in \mathcal{S}_n^m$ induces an equivalence of categories

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathcal{S}_n^m, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

5.4. **Formal consequences.**

5.4.1. *Semiadditivity as modules.* We want now to formulate and prove the equivalence between m-semiadditivity and being modules over spans. To this end, we will analyse the forgetful functor

$$\mathcal{U} : \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \longrightarrow \mathrm{Cat}_{\mathcal{K}_m}$$

Notation 5.4.1. Let $\mathrm{SAdd}_m \subseteq \mathrm{Cat}_{\mathcal{K}_m}$ be the full subcategory spanned by m-semiadditive categories.

Lemma 5.4.2 (Idempotence of m-spans, [**?**] 5.1)**.** Let $\mathbb{C}$ be an $\mathcal{S}_m^m$-module. Then the counit map

$$\nu_C : \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C}) \longrightarrow \mathbb{C}$$

from the adjunction $\mathcal{S}_m^m \otimes_{\mathcal{K}_m} (-) : \mathrm{Cat}_{\mathcal{K}_m} \rightleftarrows \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) : \mathcal{U}$ is an equivalence of $\mathcal{S}_m^m$-modules. In particular, this means that the adjunction is a smashing localisation and $\mathcal{S}_m^m$ is an idempotent commutative algebra object.

Proof. Since the forgetful functor $\mathcal{U}$ is conservative it will suffice to show that $\mathcal{U}(\nu_C)$ is an equivalence. Now by the triangle identity of adjunctions we have that the composite

$$\mathcal{U}(\mathbb{C}) \xrightarrow{u_{\mathcal{U}(\mathbb{C})}} \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C}) \xrightarrow{\mathcal{U}(\nu_C)} \mathcal{U}(\mathbb{C})$$

is the identity. Hence it will be enough to show that the first map

$$u_{\mathcal{U}(\mathbb{C})} : \mathcal{U}(\mathbb{C}) \to \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C})$$

is an equivalence. Since both sides admit canonical structures of $\mathcal{S}_m^m$-module (where for the right hand term we use the $\mathcal{S}_m^m \otimes_{\mathcal{K}_m} -$ part for the module structure), by Yoneda it will suffice to show that

$$u^*_{\mathcal{U}(\mathbb{C})} : \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathbb{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D}) \tag{2}$$

is an equivalence for all $\mathcal{D} \in \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m)$. Now since $\mathcal{D}$ was an $\mathcal{S}_m^m$-module, we get that $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D})$ is too (since $\mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \subseteq \mathrm{Cat}_{\mathcal{K}_m}$ is closed under cotensors). Hence by the universal property of m-spans (5.3.9) we see that

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathbb{C}), \mathcal{D})$$

is an equivalence, and so by currying, the map (2) is an equivalence, as required. $\square$

Theorem 5.4.3 (Semiadditivity as modules, [?] 5.2)**.** The forgetful functor induces an equivalence

$$\mathcal{U} : \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \xrightarrow{\simeq} \mathrm{SAdd}_m$$

Hence we have the adjunctions

$$\begin{tikzcd} \mathrm{SAdd}_m \arrow[r, hook] & \mathrm{Cat}_{\mathcal{K}_m} \arrow[l, bend right, "\mathcal{S}_m^m \otimes_{\mathcal{K}_m} (-)"'] \arrow[l, bend left, "\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, -)"] \end{tikzcd}$$

where the top adjunction is a smashing localisation. In particular means that for any $\mathcal{D} \in \mathrm{Cat}_{\mathcal{K}_m}$, $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D})$ is the universal m-semiadditive category equipped with a $\mathcal{K}_m$-colimit preserving functor to $\mathcal{D}$.

Proof. We have a few things to show, namely:

(1) That $\mathcal{S}_m^m$-modules are m-semiadditive.
(2) That the forgetful map is essentially surjective on SAdd_m.
(3) That the forgetful map is fully faithful.

Point (1) is by (5.3.8), and point (2) is by the universal property of m-spans (5.3.9) since we can write $\mathcal{D} \simeq \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D})$ which then attains a canonical structure of an $\mathcal{S}_m^m$-module by evaluation. Finally, point (3) is just because (5.4.2) says that $\mathcal{S}_m^m \otimes_{\mathcal{K}_m} (-)$ is a smashing localisation, and so in particular the whole forgetful functor

$$\mathcal{U} : \mathrm{Mod}_{\mathrm{Cat}_{\mathcal{K}_m}}(\mathcal{S}_m^m) \to \mathrm{SAdd}_m \hookrightarrow \mathrm{Cat}_{\mathcal{K}_m}$$

is fully faithful. Since the second map in this factorisation is fully faithful, so is the first map, as required. □

Via this equivalence we can then obtain a symmetric monoidal structure $\mathrm{SAdd}_m^{\otimes}$ on the m-semiadditives, and the following statements are standard consequences of the equivalence.

Corollary 5.4.4 ([?] 5.6-5.8)**.** The fully faithful inclusion $\mathrm{SAdd}_m \hookrightarrow \mathrm{Cat}_{\mathcal{K}_m}$ can be canonically refined to a lax symmetric monoidal functor and $\mathcal{S}_m^m$ is the initial object in $\mathrm{CAlg}(\mathrm{SAdd}_m)$.

5.4.2. *Higher commutative monoids.*

Notation 5.4.5. Let $X \in \mathcal{S}_m$. Note that the inclusion of a point $x \in X$, $i_x : * \to X$, is an $(m-1)$-truncated map by the π_*-long exact sequence. We then write $\widehat{i_x}$ to denote the span $X \xleftarrow{i_x} * \to *$ which is in $\mathcal{S}_m^{m-1}$.

Definition 5.4.6. Let $\mathcal{D}$ be a category admitting $\mathcal{K}_m$-limits. Then an *m-commutative monoid* is a functor $F : \mathcal{S}_m^{m-1} \to \mathcal{D}$ such that for every $X \in \mathcal{K}_m$, the set of maps $\{\widehat{i_x} : X \leftarrow$

$*\}_{x\in X}$ induce an equivalence $F(X) \xrightarrow{\simeq} \lim_X^{\mathcal{D}} F(*)$. We write $\mathrm{CMon}_m(\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ for the full subcategory of the m-commutative monoids.

Remark 5.4.7. In the case where $m = 0$, we see that $\mathcal{S}_0^{-1} = \mathrm{Span}(\mathrm{Fin}, \mathrm{Fin}^{\mathrm{inj}}) = \mathrm{Fin}_*$. Moreover, the 0-commutative monoid condition is precisely demanding that $F : \mathrm{Fin}_* \to \mathcal{D}$ preserves products (recall that the categorical products in Fin_* are given by disjoint unions). Hence 0-commutative monoids agree with Segal's notion of commutative monoids mentioned in the introduction.

Lemma 5.4.8 ([?] 5.13, 5.14)**.** Let $m \geq -1$. For $\mathcal{D}$ admitting $\mathcal{K}_m$-limits, then the restriction $\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}) \to \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^{m-1}, \mathcal{D})$ factors through an equivalence $\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}) \xrightarrow{\simeq} \mathrm{CMon}_m(\mathcal{D})$, and so we can just as well think of m-commutative monoids in these terms.

Proof. We only argue essential surjectivity, which is [?] 5.13. For this just consider the sequence of equivalences:

$$\begin{array}{ll}
 & \mathcal{S}_m^m \xrightarrow{F} \mathcal{D} \text{ preserves } \mathcal{K}_m\text{-limits} \\
\text{iff} & (\mathcal{S}_m^m)\mathrm{op} \xrightarrow{F\mathrm{op}} \mathcal{D}\mathrm{op} \text{ preserves } \mathcal{K}_m\text{-colimits} \\
\text{iff} & \mathcal{S}_m \hookrightarrow (\mathcal{S}_m^m)\mathrm{op} \xrightarrow{F\mathrm{op}} \mathcal{D}\mathrm{op} \text{ preserves } \mathcal{K}_m\text{-colimits} \\
\text{iff} & \text{the set of maps } \{i_x : * \to X\}_{x\in X} \text{ induce an equivalence } \mathrm{colim}_X^{\mathcal{D}\mathrm{op}} F\mathrm{op}(*) \xrightarrow{\simeq} F\mathrm{op}(X) \text{ for all } X \in \mathcal{K}_m \\
\text{iff} & \text{the set of maps } \{\widehat{i}_x : X \leftarrow *\}_{x\in X} \text{ in } \mathcal{S}_m^{m-1} \text{ induce an equivalence } F(X) \xrightarrow{\simeq} \lim_X^{\mathcal{D}} F(*) \text{ for all } X \in \mathcal{K}_m \\
\text{iff} & F|_{\mathcal{S}_m^{m-1}} \text{ is } m\text{-commutative monoid.}
\end{array}$$

where the third line is by (5.2.12), the fourth by (5.2.10), and the fifth just by taking opposites everywhere of the fourth line: here we are using that the span $i_x : * \leftarrow * \xrightarrow{i_x} X$ gets sent to $\widehat{i}_x : X \xleftarrow{i_x} * \to *$. □

Observation 5.4.9 (An alternate life of m-commutative monoids)**.** We have the identification $\mathrm{CMon}_m(\mathcal{S}) \simeq \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m^m)$ since by construction $\mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m^m) := \mathrm{Fun}^{\mathcal{K}_m}((\mathcal{S}_m^m)\mathrm{op}, \mathcal{S})$, and $(\mathcal{S}_m^m)\mathrm{op} \simeq \mathcal{S}_m^m$ since spans are always self-dual.

Lemma 5.4.10 ([?] 5.15)**.** Let $\mathcal{D}$ admit $\mathcal{K}_m$-limits. Then $\mathrm{CMon}_m(\mathcal{D})$ is m-semiadditive and the restriction along $\{*\} \hookrightarrow \mathcal{S}_m^m$ induces a functor

$$\mathrm{CMon}_m(\mathcal{D}) \to \mathcal{D}$$

which is the universal $\mathcal{K}_m$-limit preserving functor to $\mathcal{D}$ from an m-semiadditive category. In particular, $\mathcal{D}$ is m-semiadditive iff this functor is an equivalence.

Proof. By hypothesis $\mathcal{D}\mathrm{op}$ admits $\mathcal{K}_m$-colimits. Hence by (5.4.3) we get that

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}\mathrm{op}) \to \mathcal{D}\mathrm{op}$$

is the universal $\mathcal{K}_m$-colimit preserving functor from an m-semiadditive category to $\mathcal{D}\mathrm{op}$, so by taking opposites everywhere and using the result that says that opposites of m-semiadditives are m-semiadditive, we obtain the desired statement. □

Corollary 5.4.11. If $\mathbb{C}$ is an m-semiadditive presentable category, then $\mathbb{C} \simeq \mathrm{CMon}_m(\mathcal{S}) \otimes \mathbb{C}$. In particular, $\mathbb{C}$ attains a canonical $\mathrm{CMon}_m(\mathcal{S})$-module structure.

Proof. The equivalence is essentially due to the formula for the Lurie tensor product of presentables [**?**] 4.8.1.17: for $\mathcal{D}, \mathcal{E}$ presentables, we have $\mathcal{D} \otimes \mathcal{E} \simeq \mathrm{RFun}(\mathcal{D}\mathrm{op}, \mathcal{E})$ where RFun is the full subcategory spanned by functors which are right adjoints. To wit,

$$\begin{aligned}
\mathbb{C} &\simeq \mathrm{CMon}_m(\mathbb{C}) \\
&:= \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathbb{C}) \\
&\simeq \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathbb{C} \otimes \mathcal{S}) \\
&\simeq \mathrm{Fun}^{\mathcal{K}_m}\left(\mathcal{S}_m^m, \mathrm{RFun}(\mathbb{C}\mathrm{op}, \mathcal{S})\right) \\
&\simeq \mathrm{RFun}\left(\mathbb{C}\mathrm{op}, \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{S})\right) \\
&\simeq \mathbb{C} \otimes \mathrm{CMon}_m(\mathcal{S})
\end{aligned}$$

where the first equivalence is by (5.4.10). This completes the proof. □

Construction 5.4.12. Let $\widehat{\mathrm{Cat}}_{\mathrm{small}}$ be the category of not necessarily small categories admitting small colimits and functors preserving these.

Lemma 5.4.13. $\mathrm{CMon}_m(\mathcal{S}) \in \mathrm{Pr}^L$ is an idempotent commutative algebra object.

Proof. By [**?**] 4.8.1.16 and 4.8.1.17 we know that the inclusion $\mathrm{Pr}^L \subseteq \widehat{\mathrm{Cat}}_{\mathrm{small}}$ is symmetric monoidal. Moreover, [**?**] 4.8.1.10 gives that the functor $\mathcal{P}_{\mathcal{K}_m} : \mathrm{Cat}_{\mathcal{K}_m} \to \widehat{\mathrm{Cat}}_{\mathrm{small}}$ is symmetric monoidal and so in particular preserves idempotent commutative algebra objects. Now by (5.4.9) we know that $\mathrm{CMon}_m(\mathcal{S}) \simeq \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m^m)$ and by (5.4.2) we know that $\mathcal{S}_m^m$ is an idempotent commutative algebra object, and so we're done. □

Theorem 5.4.14 ([**?**] 5.21)**.** There is a smashing localisation

$$\mathrm{Pr}^L \underset{i}{\overset{\mathrm{CMon}_m(\mathcal{S})\otimes(-)}{\rightleftarrows}} \mathrm{Mod}_{\mathrm{Pr}^L}(\mathrm{CMon}_m(\mathcal{S}))$$

where the essential image of the fully faithful inclusion i consists precisely of the m-semiadditive presentable categories.

Proof. We need to show two things:

(1) That we have the smashing localisation (easy and formal, given by idempotence of $\mathcal{S}_m^m$)
(2) To identify the essential image as the m-semiadditives.

Point (1) is by idempotence of $\mathrm{CMon}_m(\mathcal{S})$ (5.4.13) and point (2) is just because (5.4.11) implies that the inclusion i is essentially surjective onto the m-semiadditive presentables. □

6. The main theorem of ambidexterity (Songqi Han)

Tomer: m-good means A is connected, m-truncated, p-finite, and $\pi_m(A) \neq 0$.

Theorem 6.0.1. The $T(n)$-local stable homotopy theory is ∞-semiadditive.

We will prove by induction that $Sp_{T(n)}$ is m-semiadditive.

The base case: $Sp_{T(n)}$ is 1-semiadditive, proven by Kuhn.

Remark 6.0.2. We cannot start from $m = -2$ because some constructions are valid only for $m \geq -1$.

For the inductive step, assume that $Sp_{T(n)}$ is m-semiadditive. The goal is to show that for every $(m+1)$-finite space B, and diagram $F : B \to Sp_{T(n)}$, we have that the norm map

$$\mathrm{Nm}_B : \mathrm{colim}(F) \to \lim(F)$$

exists and is invertible.

Step 1: Reduction.

For a fibration $Z \to Y \to X$ of truncated spaces, if both the fiber Z and the base X are $\mathscr{C}$-ambidextrous then Y is $\mathscr{C}$-ambidextrous. This implies that we can apply the Postnikov decomposition of a space, and reduce to the case where B is an Eilenberg–Maclane space $B = K(A, m+1)$.

For every such $A \in \mathtt{Ab}$, we can fit it into a short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

so we can induct on the size of the group A. In particular we can decompose A into cyclic groups, and reduce to the case $K(\mathbb{Z}/q, m+1)$. Since $Sp_{T(n)}$ is p-local, we have that q is invertible for $q \neq p$, so we can reduce to $K(\mathbb{Z}/p, m+1)$.

Step 2: Resolve $B^{m+1}C_p$ with a fibration. The prototype is the natural one:

$$B^m C_p \to * \to B^{m+1}C_p.$$

This doesn't work completely, so some modification is needed.

Lemma 6.0.3. Let $A \to E \to B$ with A and E both m-finite spaces, and B is $(m+1)$-finite. Suppose that multiplication by $|A|$ is a unit in $\pi_0\mathbb{S}^{\times}_{T(n)}$. Then B is $Sp_{T(n)}$-ambidextrous.

$|A| = \int_A 1$, the A-fold sum of 1.

It suffices to find a fibration $A \to E \to B^{m+1}C_p$ where A and E are m-finite and $|A| \in \pi_0\mathbb{S}^{\times}_{T(n)}$. We want to take $E = BG_p$ to be the classifying space of the p-Sylow subgroup, and $B = BG$, so that A is the classifying space of a group of order prime to p.

Step 3: Drop E and the fibration and only focus on A.

Observation: Every m-good A fits into a fibration $A \to E \to B$ with $E \in \mathbb{S}^{m-fin}$ an m-finite space.

Reduces to finding an m-good A.

Step 4: Linearization: transfer from $Sp_{T(n)}$ to Morava E-theory. Let E_n be the ring spectrum of the Morava E-theory of height n, and let $\widehat{\texttt{Mod}}_{E_n}$ be the ∞-category of $K(n)$-local E_n-modules. Then the functor

$$L_{K(n)}\left(E_n \otimes -\right) : Sp_{T(n)} \to \widehat{\texttt{Mod}}_{E_n}$$

is symmetric monoidal, so it induces a $\texttt{CRing}$ morphism of the unit

$$f : \pi_0 \mathbb{S}_{T(n)} \to \pi_0 E_n \cong \mathbb{Z}_p[[u_1, \dots, u_n]].$$

With the nilpotence theorem and some chromatic techniques, we have that an element is invertible on the left hand side if and only if its image is invertible on the right hand side.

So multiplication by $|A|$ lands in the units $\pi_0 \mathbb{S}_{T(n)}^\times$ if and only if $f(|A|)$ lands in the units $\pi_0 E_n^\times$.

Since $\widehat{\texttt{Mod}}_{E_n}$ is m-semiadditive, we have that $f(|A|) = f \int_A 1 = \int_A 1 = |A|$. So the problem reduces to finding m-good A with $|A|$ invertible in $\pi_0 E_n$. We will see later why you can interchange f with the integral.

We also have that $\mathrm{im}(f)$ lands inside $\mathbb{Z}_p$, so for all $g \in \mathbb{Z}_p$ its invertibility can be detected b ythe evaluation map:

$$y \in \mathbb{Z}_p^\times \iff v_p(y) = 0.$$

So the condition that multiplication by $|A|$ is a unit can be rephrased to $v_p(|A|) = 0$.

Step 5: Recall the Fermat quotient on $\mathbb{Z}_p$, of the form

$$\widetilde{\delta} : x \mapsto \frac{x - x^p}{p}.$$

A lifting δ of $\widetilde{\delta}$ is constructed, so that $\delta|_{\mathbb{Z}_p} = \widetilde{\delta}$, so that δ is a map $\delta : \pi_0 E_n \to \pi_0 E_n$ satisfying the following property: for m-good A, we have that $\delta\left(|A|\right) = |A'| - |A''|$, where both A' and A'' are m-good. Moreover, δ satisfies the property that $v_p(\delta(|A|)) < v_p(|A|)$ unless $|A| = 0 \in \mathbb{Z}_p$ or $|A| \in \mathbb{Z}_p^\times$. We also know that

$$v_p\left(|A'| - |A''|\right) \geq \min\left\{v_p|A'|, v_p|A''|\right\}.$$

As long as there exists an A which is m-good with $|A| \neq 0$, then $\min\left\{v_p(|A|) \colon A \ m-\mathrm{good}\right\} = 0$.

It suffices to find any A which is m-good with $|A| \neq 0$ in $\pi_0 \mathbb{E}_n$.

Step 6: $A = B^m C_p$, but we don't yet know that $|B^m C_p| \neq 0$ in $\pi_0 E_n$. We know that $B^{m-1}C_p$ is a loop space, so we have that

$$\begin{aligned} |B^m C_p| \left|B^{m-1} C_p\right| &= \left|\mathrm{Map}(S^1, B^m C_p)\right| \\ &= \dim\left(E_n \otimes B^m C_p\right), \end{aligned}$$

where $E_n \otimes B^m C_p$ is the module defined by the constant map $\mathrm{colim}_{B^m C_p} \underline{E_n}$. We know that this dimensino is equal to the mod-2 Euler characteristic $\chi_n\left(B^m C_p\right)$. This can be computed as

$$\dim_{\mathbb{F}_p} K(n)_0\left(B^m C_p\right) - \dim_{\mathbb{F}_p} K(n)_1\left(B^m C_p\right).$$

We know this latter term is zero, and the first term is nonzero, so we have that $|B^m C_p| \neq 0$.

6.1. **Properties of integrations.**

Definition 6.1.1. Let $q^* : \mathscr{C} \to \mathscr{D}$ be a functor. We say it is a *normed functor* if both $q_!$, and q_* exist, and we have $\mathrm{Nm}_q : q_! \to q_*$. We say it is *iso-normed* if Nm_q is an isomorphism

Definition 6.1.2. Let q be a normed functor. Then we define *integration* as follows: for every $X, Y \in \mathscr{C}$, we have

$$\int_q : \mathrm{Map}\left(q^* X, q^* Y\right) \to \mathrm{Map}(X, Y),$$

defined as the composition

$$\mathrm{Map}\left(q^* X, q^* Y\right) \xrightarrow{q_*} \mathrm{Map}\left(q_* q^* X, q_* q^* Y\right) \xleftarrow{\mathrm{Nm}_q} \mathrm{Map}\left(q_* q^* X, q_! q^* Y\right) \xrightarrow{c \circ - \circ u} \mathrm{Map}(X, Y).$$

We will study a pair of normed functors q^* and $\widetilde{q}^*$ that behave well with respect to integration.

Question:Given a commutative diagram up to homotopy

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "q^*"'] & \widetilde{\mathscr{C}} \arrow[d, "\widetilde{q}^*"] \\ \mathscr{D} \arrow[r, "G"] & \widetilde{\mathscr{D}}, \end{tikzcd}$$

when are $\int_q$ and $\int_{\widetilde{q}}$ related with respect to F, G?

We can use Beck–Chevalley to get

$$\begin{tikzcd} \widetilde{q}_! G \arrow[r] \arrow[d, "\mathrm{Nm}_{\widetilde{q}}"] & F q_! \arrow[d, "\mathrm{Nm}_q"] \\ \widetilde{q}_* G & F q_* \arrow[l] \end{tikzcd}$$

The diagram commutes if and only if AmbSq ??

Theorem 6.1.3. AmbSq and BC and BC_* implies that

$$F\left(\int_q f\right) = \int_{\widetilde{q}} Gf$$

for $X, Y \in \mathscr{C}$ and $f : q^*X \to q^*Y$.

Example 6.1.4. A pullback diagram of spaces

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ \widetilde{B} \arrow[r] & B \end{tikzcd}$$

induces a base change square

$$\begin{tikzcd} \mathscr{C}^B \arrow[r] \arrow[d, "q^*"'] & \mathscr{C}^{\widetilde{B}} \arrow[d, "\widetilde{q}^*"] \\ \mathscr{C}^A \arrow[r] & \mathscr{C}^{\widetilde{A}}. \end{tikzcd}$$

(1) If $q, \widetilde{q}$ are $\mathscr{C}$-ambidextrous, then q^* and $\widetilde{q}^*$ are isonormed
(2) ???

Corollary 6.1.5. (Distributivity) Assume that we have two maps $q_1 : A_1 \to B$ and $q_2 : A_2 \to B$ which ar eboth $\mathscr{C}$-ambidextrous. Then $q : A_1 \times_B A_2 \to B$ is $\mathscr{C}$-ambidextrous, and

$$\int_q \pi_2^* f_2 \circ \pi_1^* f_1 = \int_{q_2} f_2 \circ \int_{q_1} f_1,$$

for any $f_i : q_i^* X \to q_i^* Y$.

Corollary 6.1.6. (Additivity) Let $\mathscr{C}$ be 0-semiadditive, and we have finitely many $q_i : A_i \to B$. Then if all the q_i's are $\mathscr{C}$-ambidextrous, then the induced map

$$q = \amalg q_i : \amalg_i A_i \to B$$

is also $\mathscr{C}$-ambidextrous, and

$$\int_q \amalg_i f_i = \sum_i \int_{q_i} f_i.$$

Proof. We can reduce to $k = 2$, and 0-semiadditivity is used to imply that the fold map $\nabla : B \amalg B \to B$ is $\mathscr{C}$-ambidextrous. $\square$

Let $F : \mathscr{C} \to \mathscr{D}$, and $q : A \to B$. Then

$$\begin{tikzcd} \mathscr{C}^A \arrow[r, "F_*"] \arrow[d, "q^*"] & \mathscr{D}^B \arrow[d, "q^*"] \\ \mathscr{C}^A \arrow[r, "F_*"] & \mathscr{D}^A. \end{tikzcd}$$

(1) If q is $\mathscr{C}$- and $\mathscr{D}$-ambidextrous then q^* is iso-normed
(2) If $\mathscr{C}, \mathscr{D}, F$ compatible with q-(co)limits, then $BC_!$ and BC_*
(3) If q is both $\mathscr{C}$-amb and $\mathscr{D}-$amb and F preserves $(m+1)$-colimits, then AmbSq
(4) If both 2 and 3 then $F \int_q f = \int_q Ff$.
(5) If $\mathscr{C}, \mathscr{D}$ are m-semiadditive, and q is m-finite, then F preserves m-finite colimits. Then for such an f we will call it m-semiadditive.

7. The δ-power operation (Shay Ben Moshe)

This won't depend on much material previously, except a brief use of 1-semiadditivity.

7.1. Additive p-derivations.

Definition 7.1.1. Let R be an honest commutative ring. An *additive p-derivation* is a function of sets $\delta : R \to R$ such that

(1) (normalization) we have that $\delta(0) = 0$ and $\delta(1) = 0$
(2) (additivity) we have that $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$.[5]

Remark 7.1.2. If we have such a δ, we can define another function

$$\begin{aligned} \phi_\delta : R &\to R \\ x &\mapsto x^p + p\delta(x). \end{aligned}$$

This is an additive unital lift of the Frobenius, in that it takes sums to sums, and modulo p it is literally Frobenius. This function ϕ_δ does not determine δ, but it does if R is p-torsion free.

We may be familiar with a p-derivation, which is an additive p-derivation satisfying an extra multiplicative axiom. That axiom is exactly making ϕ_δ into a ring homomorphism.

Example 7.1.3. The *Fermat quotient* on $\mathbb{Z}$ is the additive derivation $\widetilde{\delta}$, defined by

$$\begin{aligned} \widetilde{\delta} : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto \frac{x - x^p}{p}. \end{aligned}$$

It is pretty easy to show that this is the unique additive p-derivation on $\mathbb{Z}$.[6]

The crucial property that this satisfies is that it decreases the p-adic valuation by 1. Suppose that $v_p(x) > 0$. Then we see

$$v_p(x^p) = pv_p(x) > v_p(x),$$

and hence

$$v_p\left(\widetilde{\delta}(x)\right) = v_p\left(\frac{x}{p} - \frac{x^p}{p}\right) = v_p\left(\frac{x^p}{p}\right) = v_p(x) - 1.$$

Lemma 7.1.4. Let δ be an additive p-derivation on R, and let $x \in R$ and $n \in \mathbb{N}$. Then

$$\delta(nx) = n\delta(x) + \widetilde{\delta}(n)x^p.$$

Proof. Induct on the additivity axiom. □

[5] This last term is a polynomial, we are not actually dividing by p.
[6] In fact $\widetilde{\delta}$ extends to a function on the rationals.

Having even one δ-operation is a very restrictive condition on a ring.

Example 7.1.5. Suppose that R is p-local and admits an additive p-derivation. Then every torsion element in R is also nilpotent. In particular, if R is torsion ($R \otimes \mathbb{Q} = 0$), then $R = 0$.

Proof. Let $x \in R$ be torsion. Since the ring R is p-local, we may assume that $p^m x = 0$ for some m. Since $\delta(p^m x) = \delta(0) = 0$, we see that

$$0 = \delta(p^m x) = p^m \delta(x) + \widetilde{\delta}(p^m) x^p.$$

Multiplying both sides by x we get

$$\begin{aligned} 0 &= p^m x \delta(x) + \widetilde{\delta}(p^m) x^{p+1} \\ &= \widetilde{\delta}(p^m) x^{p+1}. \end{aligned}$$

But $v_p(\widetilde{\delta}(p^m)) = m - 1$, since $\widetilde{\delta}$ decreases p-adic valuation. Therefore $p^{m-1} x^{p+1} = 0$. Repeating m times, we have that $x^{p+1} = 0$.

If the entire ring is torsion, 1 is torsion, so it is nilpotent, hence $1^n = 0$, so $1 = 0$. □

Anti-example 7.1.6. We have that $\mathbb{Z}/p^n$ admits no additive p-derivations (since it is torsion).

7.2. **Construction of the δ-power operation.** We would like to generalize $\widetilde{\delta}$ from the rational case (height 0) to higher heights. Specifically, let $\mathscr{C}$ be a symmetric monoidal 1-semiadditive stable (p-local presentable) category (e.g. $Sp_{K(n)}$ or $Sp_{T(n)}$) and let $X \in \mathtt{CAlg}(\mathscr{C})$. Then

$$R = \pi_0 X = \pi_0 \mathrm{Hom}\,(1_{\mathscr{C}}, R)\,.$$

We will use that the mapping space between objects in a 1-semiadditive category is a 1-commutative monoid. We will endow R with such a δ. The plan is to define a family of other power operations, and in the rational case we can use those to define $\widetilde{\delta}$. We will make this into $\widetilde{\delta}$ in the higher case.

Let A be a finite set with a G-action, where G is also finite. Consider $(x, \dots, x) \in X^{|A|}$. This comes with an action of $\mathrm{Aut}(A)$. That is, thinking of $(x, \dots, x) \in \mathrm{Map}(1, X)$, we can think of it not only as a map $1 \to \mathrm{Map}(1, X)$ but actually a map of the form

$$B\mathrm{Aut}(A) \to \mathrm{Map}(1, X).$$

So we have a family of maps indexed by $B\mathrm{Aut}(A)$. Since G acts on A, we can precompose to get

$$BG \to B\mathrm{Aut}(A) \to \mathrm{Map}(1, X).$$

This gives a G-action on $X^{|A|}$. We define

$$\alpha_{G,A}(x) = \int_{BG} X^{|A|} \in \pi_0 \mathrm{Hom}(1, X) = R.$$

Proposition 7.2.1. Let $\mathscr{C} = Sp_{\mathbb{Q}}$, and let $X \in \mathtt{CAlg}(\mathscr{C})$. Then we have that

$$\alpha_{G,A}(x) = \frac{x^{|A|}}{|G|}.$$

Proof. Recall that the integral is defined using the norm. In the rational case, Nm_{BG} is multiplication by the size of BG, and the integral is defined using the inverse to the norm, i.e. dividing out by $|G|$. □

Example 7.2.2. Take $A = \{*\}$, and $G = C_p$. Then

$$\alpha_{C_p,*}(x) = \frac{x}{p}.$$

Example 7.2.3. Let $A = C_p$ and $G = C_p$ acting via left translation. Then

$$\alpha_{C_p,C_p}(x) = \frac{x^p}{p}.$$

Corollary 7.2.4. We see that

$$\widetilde{\delta}(x) = \frac{x - x^p}{p} = \alpha_{C_p,*}(x) - \alpha_{C_p,C_p}(x).$$

This holds in more generality.

Definition 7.2.5. Let $\mathscr{C}$ be a 1-semiadditive stable category[7] and $X \in \mathtt{CAlg}(\mathscr{C})$. Let $R = \pi_0 X$. Then we define

$$\begin{aligned} &\delta : R \to R \\ &\delta(x) = \alpha_{C_p,*}(x) - \alpha_{C_p,C_p}(x). \end{aligned}$$

Theorem 7.2.6. We have that δ is an additive p-derivation.

We will defer the proof to the end of the talk.

Corollary 7.2.7. Every torsion element of R is also nilpotent. So if R is torsion in that $R \otimes \mathbb{Q} = 0$, then $X = 0$.

This treatment of semiadditivity is able to prove May's conjecture.

7.3. **May's conjecture.**

Definition 7.3.1. We say that a ring spectrum E *detects nilpotents* if for any ring spectrum X, and $x \in \pi_* X$, we have that the image of x under the Hurewicz map $\pi_* X \to E_* X$ is nilpotent, then x is nilpotent.

Remark 7.3.2. This definition is *equivalent* to the condition that $E \otimes X = 0$ implying that $X = 0$. To see this, plug in $x^{-1}X$.

[7] Since we are 1-semiadditive, we are also 0-semiadditive, so homs are commutative monoids. They don't have to have subtraction though. In the stable setting, however, all things are additive since the homs are abelian groups.

Corollary 7.3.3. Let $X \in \texttt{CAlg}(Sp)$ such that $X \otimes \mathbb{Q} = 0$. Then $L_{T(n)}X = 0$ for all $0 \leq n < \infty$. This implies that $L_{K(n)}X = 0$.

Proof. We know by assumption that $1 \in X$ is torsion. Therefore this is also true in $L_{T(n)}X$ (since the unit in $Sp_{T(n)}$ is just the localization of the unit in Sp). Then $L_{T(n)}X$ is a ring in a 1-semiadditive category, so we can apply the previous corollary to see that $L_{T(n)}X = 0$. □

Using this we can prove May's conjecture.

Corollary 7.3.4. (May's conjecture) Let $X \in \texttt{CAlg}(Sp)$. If $X \otimes \mathbb{Z} = 0$, then $X = 0$. This tells us that $\mathbb{Z}$ detects nilpotents for commutative algebras in spectra.

Proof. We have that $X \otimes \mathbb{Q} = (X \otimes \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, and similarly $X \otimes \mathbb{F}_p = 0$, which implies that $L_{K(n)}X = 0$ for all $0 \leq n \leq \infty$. Then the nilpotence theorem says $X = 0$. □

7.4. **Proof of the main theorem.**

Theorem 7.4.1. We have that

$$\begin{aligned}\delta(x) &= \int_{BC_p} x - \int_{BC_p} x^p \\ &= \int_{BC_p} \alpha_{C_p,*}(x) - \int_{BC_p} \alpha_{C_p,C_p}(x)\end{aligned}$$

is an additive p-derivation on $R = \pi_0 X$.

Proof. We have that $\delta(0) = \int 0 - \int 0 = 0$, and that $\delta(1) = \int 1 - \int 1 = 0$.

For additivity, we want to show that

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}.$$

Since

$$\int_{BC_p} x + y = \int_{BC_p} x + \int_{BC_p} y,$$

it is enough to show that

$$\int_{BC_p} (x+y)^p = \int_{BC_p} x^p + \int_{BC_p} y + \frac{(x+y)^p - x^p - y^p}{p}.$$

We can expand

$$(x+y)^p = x^p + y^p + \sum_{w \in S} w(x,y),$$

where we are summing over all words in x and y of length p which are not just constant letters. On any such word, C_p acts by cycling letters around, so we can rewrite this as

$$(x+y)^p = x^p + y^p + \sum_{[w]\in X/C_p} \sum_{g\in C_p} g\cdot w(x,y).$$

The term $\sum_{g\in C_p} g\cdot w(x,y)$ is a map $BC_p \to \mathrm{Hom}(1,X)$ which is induced by the map $* \to BC_p$, which is always ambidextrous. We get a map $BC_p \to \mathrm{Hom}(1,X)$ by summing along the fibers of $* \to BC_p \to \mathrm{Hom}(1,X)$. That is, our map $BC_p \to \mathrm{Hom}(1,X)$ is integrated from a map $* \to \mathrm{Hom}(1,X)$. In particular this tells us that

$$\int_{BC_p} \sum gw(x,y) = w(x,y) = \frac{px^{w_x}y^{w_y}}{p}.$$

□

8. Amenability and the bootstrap machine (David Chan)

Goal: Describe a detection principle for m-semiadditivity in ∞-categories.

Throughout, we will let $\mathscr{C}$ and $\mathscr{D}$ be stable presentably symmetric monoidal p-local ∞-categories. From p-locality we get that $R_{\mathscr{C}} := \mathrm{Hom}_{\mathscr{C}}(1,1)$ is a p-local ring.

8.1. Amenability.

We might have some functor $q^* : \mathscr{C} \to \mathscr{D}$. We say that this is *iso-normed*[8] if the associated norm map $\mathrm{Nm}_q : q_! \to q_*$ is an isomorphism. With this construction, we can define the integral over q:

$$\int_q : \mathrm{Map}_{h\mathscr{D}}(q^*X, q^*Y) \to \mathrm{Map}_{h\mathscr{C}}(X, Y).$$

We can define the *cardinality* of q at $X \in \mathscr{C}$ to be

$$|q|_X := \int_q \mathrm{id}_{q^*X}.$$

When $q : A \to *$, we write $|A|_X := |q|_X$, and we say that A is *amenable* if

$$|A| : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathrm{id}_{\mathscr{C}}$$

is a natural isomorphism.

Example 8.1.1. Let A be a finite set, and let $x \in \mathscr{C}$ be arbitrary. Then $|A|_X = \int_A \mathrm{id}_{q^*X} : X \to X$ is multiplication by $|A|$. So in this context, amenability allows us to average maps. Given $\{f_a : X \to X\}_{a\in A}$, we have that dividing by $|A|$ is well-defined, so we can take

$$\frac{1}{|A|}\sum_{a\in A} f_a.$$

Amenability is a stronger condition than being ambidextrous with respect to $\mathscr{C}$, so we get some strong properties.

8.2. Properties of amenable maps.

Maschke theorem 8.2.1. Suppose that a functor $q^* : \mathscr{C} \to \mathscr{D}$ is amenable. Then the counit $\varepsilon : q_!q^* \to \mathrm{id}_{\mathscr{C}}$ admits a section. In particular everything in $\mathscr{C}$ is a retract of something in the image of $q_!$.

Proof. We decompose the identity on q^*X using the triangle identity as

$$q^*X \xrightarrow{\eta} q^*q_!q^*X \xrightarrow{\varepsilon} q^*X.$$

[8] Normed = weakly ambidextrous, and iso-normed = ambidextrous.

Integrating both sides, we get, using some integration laws

$$|q|_X = \int_q q^*\varepsilon \circ \eta = \varepsilon \circ \int_q \eta.$$

Thus

$$\mathrm{id}_X = \varepsilon \circ \left(\int_q \eta \circ |q|_X^{-1} \right).$$

So this thing on the right is our section. □

To recover the regular Maschke theorem, we think about $q : * \to BG$, and $\mathscr{C} = \mathtt{Vect}_K$ so that $q^* : \mathtt{Rep}_{K[G]} \to \mathtt{Vect}_k$ is the forgetful functor. We have that

$$q_!(V) = K[G] \otimes V.$$

So we have a counit map $K[G] \otimes V \to V$, which admits a section. This section is one way to phrase the classical Maschke theorem.

Cancellation theorem 8.2.2. If we have two functors $\mathscr{C} \xrightarrow{q^*} \mathscr{D} \xrightarrow{p^*} \mathscr{E}$. Suppose we know that

(1) p^* is amenable
(2) q^* is weakly ambidextrous
(3) p^*q^* is iso-normed.

Then q^* is iso-normed.

Proof. The norm Nm_{qp} is the composition

$$q_!p_! \xrightarrow{\mathrm{Nm}_q p_!} q_*p_! \xrightarrow{q_*\mathrm{Nm}_p} q_*p_*.$$

This is an isomorphism since p^*q^* is iso-normed, and this last map is an isomorphism since p^* is amenable. Then $\mathrm{Nm}_q p_!$ is an isomorphism by 2-out-of-3. But everything in $\mathscr{D}$ is a retract of something in $\mathrm{im}(p_!)$, so Nm_q is an isomorphism. □

Proposition 8.2.3. Let $A \to E \xrightarrow{p} B$ be a fibration with B connected and weakly ambidextrous, A is $\mathscr{C}$-amenable, and suppose E is $\mathscr{C}$-ambidextrous. Then B is $\mathscr{C}$-ambidextrous.

Proof. We have a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ * \arrow[r] & B. \end{tikzcd}$$

By assumption, h is amenable. It would suffice to show that p is amenable, since we could take the following:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \arrow[dr, "qp"] & \\ * \arrow[r, "b"'] & B \arrow[r, "q"'] & * \end{tikzcd}$$

Since E is ambidextrous, we can apply the cancellation theorem to conclude that B is $\mathscr{C}$-ambidextrous.

We have that

$$b^*|h|_X = |p|_{f^*X}.$$

We know that $|h|_X$ is an isomorphism. Since $\pi_0(b)$ is surjective, we have that b^* is conservative on local systems, in particular it reflects isomorphisms. Therefore $b^*|h|_X$ is an isomorphism.[9] □

So we want to reduce to situations in which E is $\mathscr{C}$-ambidextrous, and find fibers which we can compute are amenable, in order to conclude that the base B is $\mathscr{C}$-ambidextrous.

8.3. **Amenability in symmetric monoidal ∞-categories.** It is a lot to check that the natural transformation $|A|$ is a natural isomorphism. In the symmetric monoidal setting, we can reduce this to a single thing, namely we only have to check it at the unit.

Lemma 8.3.1. If $\mathscr{C}$ is m-semiadditive and presentably symmetric monoidal, then an m-finite space A[10] is amenable if and only if $|A|_{1_{\mathscr{C}}} : 1_{\mathscr{C}} \to 1_{\mathscr{C}}$ is an isomorphism.

Proof sketch. We just show that $|A|_X \cong \mathrm{id}_X \otimes |A|_{1_{\mathscr{C}}}$, since $X \cong X \otimes 1$. □

What is the least work we can do to show that A is amenable? We have to check that $|A|_{1_{\mathscr{C}}} \in R_{\mathscr{C}} = \mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, 1_{\mathscr{C}})$ is invertible. Since $R_{\mathscr{C}}$ is p-local, then if $|A|_{1_{\mathscr{C}}}$ is rational in $R_{\mathscr{C}}$, it suffices to show that $v_p(|A|) = 0$. By repeatedly using δ, we are going to lower the p-valuation until we reach a point where $v_p = 0$, so we get a unit and hence we have amenability.

Lemma 8.3.2. Suppose $\mathscr{C}$ is m-semiadditive, and it has all the restrictions from before. Suppose further that there exists some A so that

(1) A is $\mathscr{C}$-amenable
(2) $\pi_m(A) \neq 0$
(3) A is m-finite.

Then $\mathscr{C}$ is $(m+1)$-semiadditive.

[9]The intuition for this proposition is that the norm map of p is doing the norm map at each fiber separately. The fibers don't mix at all.

[10]Which is already m-ambidextrous since $\mathscr{C}$ is m-semiadditive.

Proof. Songqi reduced to checking that $B^{m+1}C_p$ is $\mathscr{C}$-ambidextrous. Since $\pi_m(A) \neq 0$ and A is a p-space, there exists a fibration

$$B^m C_p \to A \twoheadrightarrow E.$$

So E is like A but we killed some copy of C_p in $\pi_m(A)$. Since A is a p-space, it is nilpotent, so we can change this to a new fibration of the form

$$A \to E \overset{p}{\twoheadrightarrow} B^{m+1}C_p.$$

Since A was m-finite, E is still m-finite, and in particular it is $\mathscr{C}$-ambidextrous. In this fibration A is amenable, E is ambidextrous, and $B^{m+1}C_p$ is $(m+1)$-finite in an m-semiadditive category, and hence weakly ambidextrous. Therefore we can conclude that $B^{m+1}C_p$ is ambidextrous by the cancellation theorem. □

Proposition 8.3.3. Suppose that $\mathscr{C}$ is m-semiadditive for $m \geq 1$, and $h : R_{\mathscr{C}} \to S$ is a map of semi δ-rings[11] that detects invertibility. If $h(|BC_p|)$ and $h(|B^m C_p|)$ are rational and nonzero, then $\mathscr{C}$ is $(m+1)$-semiadditive.[12]

Proof. Let A be an m-finite space. We say it is h*-good* if

(1) $h(|A|)$ is rational and nonzero
(2) $\pi_m(A) \neq 0$.

So in order to show that $\mathscr{C}$ is $(m+1)$-semiadditive, it suffices by the last lemma to find an h-good A with $v(A) = v_p(h(|A|)) = 0$. Assume that p is not invertible in S.

Claim: If A is h-good and $v(A) > 0$, then $A \wr C_p = (A^p)_{hC_p}$ is h-good and has the property that $h\left(A \wr C_p\right) = v(A) - 1$.

To prove the claim, we compute[13]

$$\begin{aligned}\delta\left(|A|\right) &= \int_{BC_p} |A| - \int_{BC_p} |A|^p \\ &= |A| \cdot |BC_p| - |A \wr C_p|.\end{aligned}$$

Therefore we have

$$|A \wr C_p| = |A| \cdot |BC_p| - \delta\left(|A|\right).$$

Taking valuations everywhere, we have that

$$v\left(|A||BC_p|\right) \geq v(|A|),$$

[11] Rings equipped with additive p-derivations.

[12] To relate this back to Songqi's talk, $\mathscr{C} = Sp_{T(n)}$, and we have a map to $\mathscr{D} = \widehat{\mathtt{Mod}}_{E_n}$, inducing $R_{\mathscr{C}} \to R_{\mathscr{D}}$. This functor factors as $Sp_{T(n)} \to Sp_{K(n)} \xrightarrow{(E_n \otimes -)^\wedge} \widehat{\mathtt{Mod}}_{E_n}$.

[13] $|A|$ has no C_p action, so we pull it out, and we are taking $|A|$ times $\int_{BC_p} 1$. For the second term, there is a C_p-action, and the integration is exactly the wr product.

while $v(\delta(|A|)) = v(|A|) - 1$. This is because $|A|$ is rational and there is a unique δ-valuation on the rationals, which drops the valuation by 1. $\square$

Interlude: Tomer's recap

Ingredient 1

We had that $Sp_{T(n)}$ is 1-semiadditive. So if A is a π-finite 1-truncated space, and $F : A \to Sp_{T(n)}$, we have that $\mathrm{colim}_A F \cong \lim_A F$.

Given a family of maps $f : A \to \mathrm{Map}(X, Y)$, we can integrate it to get

$$\int_A f \in \mathrm{Map}(X, Y).$$

We can think about $f \in \mathrm{Map}(X, Y^A) = \mathrm{Map}(X, \mathrm{colim}_A \underline{Y}) \xrightarrow{\nabla} \mathrm{Map}(X, Y)$. This composite is the integral $\int_A$.

We want to now prove that $Sp_{T(n)}$ is 2-semiadditive. If we have a p-space we can break it into pieces by a Postnikov tower, and we can reduce to Eilenberg-Maclane spaces. We want to show that B^2C_p is ambidextrous.

For example, suppose we want to do it in $Sp_{T(1)}$. We are going to look at the fiber sequence

$$BC_p \to * \xrightarrow{q} B^2C_p.$$

We want to show that for every local system on B^2C_p, the colimit and limit coincide. Given such a local system $F : B^2C_p \to Sp_{K(1)}$, we want $\mathrm{colim} F \cong \lim F$ through the norm map.

If we take $q_! q^* F$ and $q_* q^* F$, these are the same, since the fiber of q is a 1-type, and we have already seen that $Sp_{T(n)}$ is 1-semiadditive. Calling the map to a point $\pi : B^2C_p \to *$, we have that

$$\begin{aligned} \pi_! X &= \pi_! q_! q^* F = q^* F \\ \pi_* X &= \pi_* q_* q^* F = q^* F. \end{aligned}$$

Thus local systems on B^2C_p which lie in the image of $q_!$, their limits and colimits coincide. In order to extend this to *all* local systems, we use amenability.

At height one, we have that $|BC_p|_X$ is an isomorphism. Once we go to height two, this won't be true anymore — in height two, we have that $|BC_p|$ is essentially p.

At height two, we could take the wreath product $C_p \wr C_p$ which is a p-group with C_p in the center, so we get

$$B\left(C_p \wr C_p\right) \to B\left(\frac{C_p \wr C_p}{C_p}\right) \to B^2C_p.$$

We have that $|B\left(C_p \wr C_p\right)|_{\mathbb{S}_{T(2)}} \in \pi_0\left(\mathbb{S}_{T(2)}\right)^\times$, and that the middle term is the classifying space of a group, so its norm behaves well.

So this whole discussion comes down to finding appropriate 1-types as fibers whose cardinalities are invertible.

At height 3, B^2C_p won't be amenable, but $\left(B^2C_p\right)^p_{hC_p}$ is, and it's π_2 is C_p^p. At each stage, we want an appropriate m-type with invertible cardinality.

Ingredient 2

We don't know a lot about $Sp_{T(n)}$. However the map

$$\begin{aligned} Sp_{T(n)} &\to \widehat{\mathtt{Mod}}_{E_n} \\ \pi_0\mathbb{S}_{T(n)} &\mapsto \pi_0 E_n = \mathbb{Z}_p[[u_1, \dots, u_n]] \end{aligned}$$

detects invertibility. Moreover, the map below factors through $\mathbb{Z}_p$, so it doesn't hit the generators. So the only question is whether it is divisible by p.

We could, for every n, write a specific space A, and then compute that $|A|_{E_n} \in \mathbb{Z}_p$, but we want to avoid these complicated computations. This is where δ comes in.

The 1-semiadditive structure on both these rings is a δ-structure. So it allows us to take an element which is not invertible and not zero in $\pi_0\mathbb{S}_{T(n)}$ and reduce its valuation by 1. So we want to find some space which doesn't map to zero in $\mathbb{Z}_p$, and just grind it down with δ.

So now we can just compute $|B^mC_p|$, and show that they are nonzero. At height n, this is exactly $p^{\binom{n+1}{n}} \neq 0$.

The essential things we haven't seen yet are:

(1) Why does $\pi_0\mathbb{S}_{T(n)} \mapsto \pi_0 E_n$ detect invertibility? This will require some chromatic homotopy theory. The bottom line is that this is essentially the nilpotence theorem.
(2) How is the computation $|B^mC_p|_n = p^{\binom{n+1}{n}}$ done? This will have to do with a relationship between cardinality and dimensions of dualizable objects. This reduces the question to a question about dimensions.
(3) The dimension of B^mC_p in $\pi_0(E_m)$ is a computation of Ravenel and Wilson.

9. $T(n)$-local ambidexterity (Ishan Levy)

Theorem 9.0.1. (Bootstrap machine) If $F : \mathscr{C} \to \mathscr{D}$ is a symmetric monoidal functor between stable categories, and

(1) $\mathscr{C}$ and $\mathscr{D}$ are k-semiadditive for $k \geq 1$
(2) F detects units in $\mathrm{End}\,(1_{\mathscr{C}})$
(3) $|BC_p|_D \in \mathbb{Q} - 0$ and there exists A with $\pi_k A \neq 0$, and $|A|_D \in \mathbb{Q} - 0$

then $\mathscr{C}$ and $\mathscr{D}$ are $(k+1)$-semiadditive.

We want to apply this to $Sp_{T(n)} \to \widehat{\mathtt{Mod}}_{E_n}$, mapping to $K(n)$-local E_n-modules.

WE have seen that $Sp_{T(n)}$ is 1-semiadditive, and since this functor is symmetric monoidal, so is $\widehat{\mathtt{Mod}}_{E_n}$. So property (1) is true by induction on k and Tate vanishing.

For condition (2), once we have a unit in the target category, we can get a unit in the source category. We want to give a criterion for when (2) will be satisfied. First we need some definitions.

Definition 9.0.2. A *weak ring* is a right unital ring. That is, it comes with a multiplication which is unital on the right:

$$\begin{tikzcd} R \otimes R \arrow[r, "\mu"] & R \\ R \otimes \mathbb{S} \arrow[u] \arrow[ur, equal] & \end{tikzcd}$$

We don't require any further structure (associativity, etc.)

$T(n)$ can be made an E_1-ring, and hence a weak ring, by taking $\mathrm{End}\,(\text{type } n)\,[v_n^{-1}]$.

Definition 9.0.3. A symmetric monoidal functor $F : \mathscr{C} \to \mathscr{D}$ is *nilconservative* if $F(R) = 0$ if and only if $R = 0$ for all weak rings R.

Lemma 9.0.4. If a symmetric monoidal exact functor $F : \mathscr{C} \to \mathscr{D}$ is nilconservative, then it detects units.[14]

Proof. We can construct a particular ring and use the definition. We have that x is unit if its cofiber is zero, which occurs if and only if $\mathrm{cof}\,(x) \otimes \mathrm{cof}(x)^* = 0$. This occurs if and only if

$$\mathrm{cof}(Fx) \otimes \mathrm{cof}(Fx)^* = 0,$$

which occurs if and only if Fx is a unit. □

[14] If a functor is nilconservative, then it is conservative on the subcategory of dualizable objects.

Theorem 9.0.5. The map

$$Sp \to \prod_{0 \le n \le \infty} Sp_{K(n)}$$

is nilconservative. That is in order to check a ring is zero, it suffices to check it in all the Morava K-theories.

Given an object X, we can define its *support* to be

$$\mathrm{Supp}(X) := \{n \colon K(n) \otimes X \neq 0\}.$$

We have that $\mathrm{Supp}(X \otimes Y) = \mathrm{Supp}(X) \cap \mathrm{Supp}(Y)$.

Example 9.0.6. We have that

$$\begin{aligned} \mathrm{Supp}(T(n)) &= \{n\} \\ \mathrm{Supp}(K(n)) &= \{n\} \\ \mathrm{Supp}(E_n) &= \{0, \dots, n\} \\ \mathrm{Supp}\,(\text{type } n \text{ spectrum}) &= \{n, \dots, \infty\}. \end{aligned}$$

Given a ring, in the R-local category we can get an analogue of the nilpotence theory looking just at the Morava K-theories in its support.

Proposition 9.0.7. If R is a weak ring, then the map

$$Sp_R \to \prod_{0 \le n \le \infty} Sp_{K(n)}$$

is nilconservative.

Proof. If S is sent to zero, then $\mathrm{Supp}(S) \cap \mathrm{Supp}(R) = 0$, implying that $\mathrm{Supp}(R \otimes S) = 0$. By nilpotence, this says that $S \otimes R = 0$, and therefore $S = 0$. $\square$

Nilpotence here is that a weak ring is zero if and only if it has zero support.

Since $T(n)$ is only supported at n, this tells us that

$$Sp_{T(n)} \to Sp_{K(n)}$$

is nilconservative. We can post-compose with the map to $K(n)$-local E_n-modules, and we have a conservative functor

$$Sp_{K(n)} \xrightarrow{E_n \otimes (-)} \widehat{\mathtt{Mod}}_{E_n}.$$

Nilconservative implies conservative, and nilconservativity is closed under composition. Therefore $Sp_{T(n)} \to \widehat{\mathtt{Mod}}_{E_n}$ is nilconservative. So condition (2) is satisfied.

Claim 9.0.8. In a symmetric monoidal category, if A is any π-finite m-finite space, then the cardinality of the free loop space is

$$\left|A^{S^1}\right| = \dim(A) = (1 \to A \otimes A^* \to A^* \otimes A \to 1).$$

Proof. We can check this in the universal case. The universal n-semiadditive category is spans of n-finite spaces. We see that since A is self-dual, the composition of spans gives us

$$\begin{tikzcd} A^{S^1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r] \arrow[d] & * \\ A \arrow[r] \arrow[d] & A \times A & \\ * & & \end{tikzcd}$$

$\square$

Theorem 9.0.9. (Ravenel, Wilson) We have that

$$\dim_{\mathbb{F}_p} K(n)_0 B^k C_p = p^{\binom{n}{k}},$$

and its $K(n)_1$ is equal to 0.

We can upgrade this to being able to understand the Morava E-theory of $B^k C_p$, because we claim that

$$E_n \widehat{\otimes} B^k C_p = E_n^{p^{\binom{n}{k}}}.$$

The proof is that we run the v_i Bockstein spectral sequence.

Since the dimension of a free module is just its rank, this tells us that

$$\dim_{\widehat{\mathtt{Mod}}_{E_n}} B^k C_p = p^{\binom{n}{k}}.$$

Since $(B^k C_p)^{S^1} = B^k C_p \times \Omega B^k C_p$[15], then this dimension will be equal to

$$\left| B^k C_p \right| \cdot \left| B^{k-1} C_p \right|.$$

Since both of these are nonzero, this completes condition (3).

Moreover, we see that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

We can use this to inductively compute the cardinality of $B^k C_p$ as

$$\left| B^k C_p \right| = p^{\binom{n-1}{k}}.$$

We note that it is 1 if $k > n-1$, i.e. when $k \geq n$. So $B^k C_p$ is *amenable* in $\widehat{\mathtt{Mod}}_{E_n}$ in these cases, and since our functor detects units, this will imply that it is amenable in $Sp_{T(n)}$.

Theorem 9.0.10. If $F \to X \xrightarrow{f} Y$ is a fiber sequence of spaces, where F is π-finite and n-connected, then $\Sigma^\infty_+ f$ is a $T(n)$-equivalence.

[15] This splitting is true for any group object in any ∞-category.

Once we have a space that is sufficiently connected, $T(n)$-homology won't see the bottom finitely many homotopy groups of it.

Proof. We observe that $f = \mathrm{colim}_Y \mathrm{fib}(f)$, so we can reduce to the case of $f : F \to *$. We can further reduce via the Postnikov tower to the case of $F = B^k C_p$, and since F is n-connected, we have $k > n$. So now we are claiming that $B^k C_p$ is $T(n)$-acyclic for $k > n$. This is because the map to a point admits a section

$$B^k C_p \to *$$

which is a retract on $T(n)_*$. The other map is $* \to B^k C_p$, whose fiber is $B^{k-1}C_p$. Since $B^{k-1}C_p$ is amenable, this map is also a retract on $T(n)$-homology by Maschke's theorem.[16] Both of the maps are retracts, so they are both equivalences. □

Thus $B^k C_p$ dies in $Sp_{T(n)}$ if $k \geq n$.

Proposition 9.0.11. The following are equivalent for a weak ring R

(1) $R \otimes \Sigma^\infty B^n C_p = 0$
(2) $R \otimes \{\text{type } n\} = 0$
(3) $\mathrm{Supp}(R) \subseteq \{0, \ldots, n-1\}$.

Proof. Given (1), we can look at the support $\mathrm{Supp}\left(\Sigma^\infty B^n C_p\right) = \{n, \ldots, \infty\}$. So

$$\mathrm{Supp}(R) \cap \mathrm{Supp}\left(\Sigma^\infty B^n C_p\right) = \varnothing,$$

so $\mathrm{Supp}(R) \subseteq \{0, \ldots, n-1\}$.

Given (3), we can see that $R \otimes \mathrm{End}\left(\text{type } n\right) = 0$ by the nilpotence theorem, implying (2).

Finally given (2), we have that L_R factors through L^f_{n-1}. We have seen that $T(i)$-homology vanishes on all the Eilenberg Maclane spaces for $i \leq n-1$. So the result is true for L^f_n, which gives (1). □

We can characterize higher semiadditivity among weak rings.

Proposition 9.0.12. If Sp_R is 1-semiadditive, then $\mathrm{Supp}(R) = \{n\}$ for some $n < \infty$.

Proof. We note that $L_R \mathbb{F}_p = 0$ because π_0 is p-torsion semi-δ ring, and we can't have a torsion thing which is nonzero. Therefore ∞ is not in the support.

If we had $n < m < \infty$ which were both in the support of R, we can look at $L_R\left(E_n \otimes E_m\right)$. This will be nonzero, and it admits maps from E_n and E_m which are R-local maps. We can look at $|BC_p|$ in all of these rings. In E_n we have $|BC_p| = p^{n-1}$, while in E_m we have $|BC_p| = p^{m-1}$. These have to coincide in $E_n \otimes E_m$. This tells us that the map

$$\mathbb{Z} \to \pi_0 \mathrm{End}\left(E_n \otimes E_m\right)$$

[16] If the fiber is amenable, then the map will admit a retract.

is not injective, which can't happen in these semi-δ rings. $\square$

Theorem 9.0.13. The following are equivalent for a nonzero weak ring R

(1) Sp_R sits between $Sp_{K(n)} \subseteq Sp_R \subseteq Sp_{T(n)}$
(2) Either $Sp_R = Sp_{\mathbb{Q}}$ or the functor
$$\Omega^\infty : Sp_R \to \mathcal{S}_*$$
admits a retraction
(3) Sp_R is 1-semiadditive
(4) Sp_R is ∞-semiadditive
(5) $\mathrm{Supp}(R) = \{n\}$ for some $n < \infty$.

Proof. (1) implies (2): This is clear for $n > 0$, since we can take $\Omega^\infty : Sp_R \to \mathcal{S}_*$, then take a Bousfield–Kuhn functor $\Phi : \mathcal{S}_* \to Sp_{T(n)}$, then localize $L_R : Sp_{T(n)} \to Sp_R$.

(2) implies (3): Tate vanishing argument from Clausen–Mathew.

(4) implies (3) is trivial

(1) implies (4): We could use that L_R is a symmetric monoidal functor from $Sp_{T(n)}$, and we know ∞-semiadditivity for $Sp_{T(n)}$.

(3) implies (5): We just proved this

(5) implies (1): $Sp_{K(n)} \subseteq Sp_R$ easily. For the second inclusion, observe that
$$\begin{aligned}\langle R\rangle &= \langle \oplus_{i=0}^n T(i) \otimes R \oplus \text{type } n+1\rangle \\ &= \langle T(n) \otimes R\rangle .\end{aligned}$$
Whenever you are $T(n)$-acyclic, you are $T(n) \otimes R$-acyclic, and hence you are R-acyclic. $\square$

10. (Lucy Yang)

[Lucy has typed notes]

11. (Jan Steinebrunner)

[Still need to TeX]

12. Morava K-theory homology of $K\left(\mathbb{Z}/p^j\mathbb{Z}\right), m$ (Yuqing Shi)

We are going to show Ravenel and Wilson's calculation of the $K(n)$-homology of $K\left(\mathbb{Z}/\pi^j\mathbb{Z}, m\right)$. Recall the result from Jan's talk:

Notation 12.0.1. Let p be a fixed odd prime, and $K(n)$ the nth Morava K-theory with

$$K(n)_* = \mathbb{F}\left[v_n^{\pm 1}\right],$$

with $|v_n| = 2p^n - 2$. Let $K_m := K\left(\mathbb{Z}/p^j\mathbb{Z}, m\right)$ for j fixed.

Structure of $K(n)_*K_m$:

- We have that $K(n)_*K_m$ is a cocommutative $K(n)_*$-coalgebra, with comultiplication

$$\psi : K(n)_*K_m \to K(n)_*K_m \otimes_{K(n)_*} K(n)_*K_m$$

- $K(n)_*K_m$ is an abelian group object in $\texttt{coAlg}_{K(n)_*}$ with multiplication

$$* : K(n)_*K_m \otimes_{K(n)_*} K(n)_*K_m \to K(n)_*K_m$$

  induced by the H-space structure since these are infinite loop spaces
- $K(n)_*K_m$ is a bicommutative $K(n)_*$ Hopf algebra
- The cup product pairing $K_i \times K_n \to K_{i+n}$ induces another multiplication

$$\circ : K(n)_*K_i \otimes_{K(n)_*} K(n)_*K_m \to K(n)_*K_{i+m}.$$

  This multiplication is graded commutative, unital, and distributes over the algebra structure $*$.

So letting m vary, we have that

$$\oplus_{m\geq 0} K(n)_*K_m$$

is a graded commutative monoid in the category of Hopf algebras over $K(n)_*$. These are the graded commutative Hopf rings over $K(n)_*$. In the original paper, they call this a graded commutative ring object in coalgebras.

In particular, $\oplus_{m\geq 0} K(n)_*K_m$ belongs to the subcategory $\texttt{HopfAlg}_{K(n)_*,p^j} \subseteq \texttt{HopfAlg}_{K(n)_*}$, the Hopf algebras which are annihilated by multiplication by p^j. This is the set of $H[p^j]$ in the notation from the last talk. Recall again that p^j is fixed.

The category $\texttt{HopfAlg}_{K(n)_*}$ is symmetric monoidal under $\boxtimes$ with unit $K(n)_*[\mathbb{Z}]$. This tensor product descends to $\texttt{HopfAlg}_{K(n)_*,p^j}$, with unit $K(n)_*\left[\mathbb{Z}/p^j\mathbb{Z}\right] \simeq K(n)_*K_0$.

Main theorem 12.0.2. (Ravenel–Wilson) The Hopf ring $\oplus_{m\geq 0} K(n)_*K_m$ is the free $K(n)_*K_0$-Hopf ring on the Hopf algebra $K(n)_*K_1$. That is,

$$\oplus_{m\geq 0} K(n)_*K_m = K(n)_*K_0 \oplus \left(K(n)_*K_1 \oplus K(n)_*K_1\right)/\Sigma_2 \oplus \cdots$$

Remark 12.0.3. In this situation, given $a_{(i)} \in K(n)_{2p^i}(K_1)$ and $a_{(j)} \in K(n)_{2p^j}(K_1)$, then

$$a_{(i)} \circ a_{(j)} = -a_{(j)}a_{(i)}.$$

This is why we have an exterior algebra structure.

Before explaining this theorem, we should discuss the relation to Dieudonné modules.

Notation 12.0.4. We can define the cyclic graded $K(n)$-homology

$$\overline{K(n)}_{\bar{t}}(X) = K(n)_t(X),$$

with $\bar{t} \in \mathbb{Z}/(2p^n-2)$ the reduction of $t \in \mathbb{Z}$. There is always a map $K(n)_* \to \overline{K(n)}_* \cong \mathbb{F}_p$ sending $v_n \mapsto 1$.

We will also define the Hopf algebra $H_j := \overline{K(n)}_* K_1$ with associated Diedonné module $D_j := \mathrm{DM}(H_j)$.

Recall the Dieudonné ring $D_{\mathbb{F}_p} \cong \mathbb{Z}_p[F,V]/FV = p$.

Applying the Dieudonné module functor DM to the objects in the theorem, we get

$$\mathrm{DM}\left(\oplus_{m\geq 0}\overline{K(n)}_*(K_m)\right) = \mathrm{DM}\left(\overline{K(n)}_*(K_1)\right) \oplus D_j \oplus (D_j \oplus D_j)/\Sigma_2 \oplus \cdots$$

where here $\mathrm{DM}\left(\overline{K(n)}_*(K_1)\right) \cong \mathbb{Z}/p^j\mathbb{Z}$. This whole thing is $\Lambda_{\boxtimes} D_j$.

We first want to know what D_j is. First consider $\mathbb{Z}/p\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \cdots$ whose colimit is $\mathbb{Q}_p/\mathbb{Z}_p$. This gives an isomorphism

$$H^\vee := \lim_j \overline{K(n)}^*\left(K\left(\mathbb{Z}/p^j\mathbb{Z},1\right)\right) \cong \overline{K(n)}^*\left(K\left(\mathbb{Q}_p/\mathbb{Z}_p,1\right)\right) \cong K(n)^*\left(K(\mathbb{Z},2)\right).$$

We have that $H^\vee = \overline{K(n)}_*[[t]]$.

Claim 12.0.5. $\mathrm{DM}(H) = \mathrm{DM}\left(\mathtt{Spf}\, H^\vee\right) \cong \mathbb{Z}_p[V,F]/VF = p, V^{n-1} = F$.

Sketch. There is a bijection

$$\begin{aligned} \{\text{formal groups of finite height}\} &\leftrightarrows \{\text{DM of finite rank}\} \\ \text{height} &\mapsto \text{rank} \\ \dim &\mapsto \text{length of } M/VM \\ \text{Frobenius} &\mapsto \text{Verschiebung}. \end{aligned}$$

□

Corollary 12.0.6. The Dieudonné module D_j corresponding to K_1 is just a mod p reduction:

$$D_j = \mathrm{DM}(H_j) \cong \mathbb{Z}/p^t\mathbb{Z}\left[F,V\right]/VF = p, pv^{n-1} = F.$$

We still want to understand the Frobenius and Verschiebung action on $\Lambda_{\boxtimes} D_j$, and from what we have seen before, it is sufficient to understand it on $\Lambda_{\boxtimes}\mathrm{DM}(H) \bmod p^j$.

We have that $\mathrm{DM}(H)$ is a free $\mathbb{Z}_p$-module generated by $\alpha_{n-1}, \ldots, \alpha_0$. The V and F action on $\Lambda_{\boxtimes}\mathrm{DM}(H) \cong \Lambda\underline{\mathrm{DM}(H)}$, the underlying $\mathbb{Z}_p$-module of $\mathrm{DM}(H)$. We see inductively that

$$V\left(\alpha_{i_1} \circ \cdots \circ \alpha_{i_q}\right) = V\left(\alpha_{i_1} \circ \cdots \circ \alpha_{i_{q-1}}\right) \wedge \alpha_{i_q},$$

and similarly for Frobenius.

We want to prove the main theorem. We first have to show that $\oplus_{m\geq 0} K(n)_* K_m$ is generated by $K(n)_* K_1$.

For simplicity, assume $j = 1$ here, since the case $j > 1$ works exactly the same.

Step 1: Understand $K(n)_*K_1$. The Eilenberg-Maclane spaces fit into

$$K_1 \xrightarrow{\delta} K(\mathbb{Z}, 2) \xrightarrow{\cdot p} K(\mathbb{Z}, 2).$$

Recall that

- As an algebra $K(n)^*\left(\mathbb{CP}^\infty\right) \cong K(n)_*[[c]]$ with $|c| = 2$
- As a $K(n)_*$-module

  $$K(n)_*\left(\mathbb{CP}^\infty\right) \cong K(n)_*\left[\beta_0, \beta_1, \ldots,\right]$$

  with $|\beta_i| = 2i$ and $\langle c_i, \beta_j\rangle = \delta_{ij}$.
- Denote by $\beta_{(i)} = \beta_{p^i}$ and $\beta_{(i)} := 0$ for $i < 0$. Thus as an algebra

  $$K(n)_*\left(\mathbb{CP}^\infty\right) \cong K(n)_*\left[\beta_{(0)}, \beta_{(1)}, \ldots\right]/\beta^{*p}_{(n+i-1} = v_n^{p^i}\beta_{(i)}.$$

  The coproduct here is

  $$\psi(\beta_m) = \sum_{i=0}^{m} \beta_i \otimes \beta_{m-i}.$$

Theorem 12.0.7. We have that

(1) $\delta_* : K(n)_*K_1 \to K(n)_*\mathbb{CP}^\infty$ is a monomorphism of $K(n)_*$-Hopf algebras
(2) As a $K(n)_*$-module, $K(n)_*K_1 \cong K(n)_*[a_0, \ldots, a_{p^n-1}]$ with $|a_m| = 2m$, and $\delta^{-1}(a_m) = \beta_m$.

Corollary 12.0.8. We define $a_{(i)} := a_{p^i}$, so as an algebra

$$K(n)_*K_1 \cong K(n)_*\left[a_{(0)}, \ldots, a_{(n-1)}\right]/a^{*p}_{(n+i-1)} = v_n^{p^i}a_{(i)},$$

with coproduct

$$\psi(a_m) = \sum_{i=0}^{m} a_i \otimes a_{m-i}.$$

Sketch of theorem. We use the Gysin sequence associated to the sphere bundle

$$S^1 = \Omega K(\mathbb{Z},2) \to K_1 \to K(\mathbb{Z},2) \xrightarrow{p} K(\mathbb{Z},2).$$

So we get

$$\cdots \to K(n)_* K_1 \xrightarrow{\delta_*} K(n)_*\left(\mathbb{CP}^\infty\right) \xrightarrow{\cap e_\delta} K(n)_{*-2}\left(\mathbb{CP}^\infty\right).$$

Here e_δ is the Euler class. We claim that capping with it is $y \mapsto y \cap [p]_{K(n)}(c)$. Because of the pairing, we have that $\beta_{n+i} \mapsto \beta_i$. Thus this map is surjective! Thus δ_* is a monomorphism. $\square$

Remark 12.0.9. $K(n)_*K_1$ is a truncated polynomial algebra

$$K(n)_*K_1 \cong K(n)_*\left[a_{(1)},\dots,a_{(n-1)}\right]/a_{(i)}^{*p} = 0,\ 1 \le i \le n-2,\ a_{(n-1)}^{*p^2} = 0.$$

Notation 12.0.10. For $I = (i_1,\dots,i_q)$ with $0 \le i_q < n$, we define $a_I \in K(n)_*K_q$ via the iterated cup product pairing

$$\begin{aligned} \circ^q : K(n)_*K_1 \boxtimes \cdots \boxtimes K(n)_*K_1 &\to K(n)_*K_q \\ a_{(i_1)},\dots,a_{(i_q)} &\mapsto a_I. \end{aligned}$$

Theorem 12.0.11. We have

(1) $a_{(i)} \circ a_{(j)} = -a_{(j)}a_{(i)}$ and $a_{(i)} \circ a_{(i)} = 0$, which follows from the axiom of being a (graded) Hopf ring.
(2) As an algebra,

$$\begin{aligned} K(n)_*K_0 &\cong K(n)_*\left[\mathbb{Z}/p\mathbb{Z}\right] \\ K(n)_*K_\ell &\cong K(n)_* \text{ for } \ell > n \\ K(n)_*K_n &\cong K(n)_*\left[a_I\right]/\left(a_I^{*p} + (-1)^n v_n a_I\right) \text{ with } I = (0,1,\dots,n) \\ K(n)_*K_m &\cong \bigotimes_{I=0<i_1<\cdots<i_m<n} K(n)_*\left[a_I\right]/\text{relations} \end{aligned}$$

(3) The coalgebra structure follows from the coalgebra structure of $K(n)_*K_1$ and the ring map $\circ$, which is a coalgebra map by definition.

Passage from $K(n)_*K_m$ to $K(n)_*K_{m+1}$: We use here that $K_{m+1} = BK_m$, and that K_{m+1} is the geometric realization of $\cdots \rightrightarrows K_m \times K_m \rightrightarrows K_m \rightrightarrows *$.

From the filtration

$$* = B_0K_m \subseteq B_1K_m \subseteq \cdots \subseteq K_{m+1}$$

we get a spectral sequence.

Theorem 12.0.12. There exists a spectral sequence $E_{*,*}(K_m)$ of $K(n)_*$-Hopf algebras converging to $K(n)_*K_{m+1}$ with first page

$$E^1_{s,t}(K_m) = \widetilde{K(n)}_*\left(B_sK_m/B_{s-1}K_m\right) \cong \otimes_s \widetilde{K(n)}_* K_m.$$

This gives the induced bar filtration $E^2_{s,t}(K_m) \cong \mathrm{Tor}^{K(n)_*K_m}_{s,t}\left(K(n)_*, K(n)_*\right) =: H_{s,t}\left(K(n)_*K_m\right)$.

Example 12.0.13. Going from K_1 to K_2, let's take $n = 2$ and $p = 3$, so our periodicity is $2p^n - 2 = 16$. In this case, as an algebra we have

$$K(n)_*K_1 \cong K(n)_*\left[a_{(1)}\right]/a_{(1)}^q,$$

so $\deg a_{(1)} = 6$. The E^2 page here is

$$E^2_{s,t}(K_1) = \Lambda\left(\sigma a_{(1)}\right) \otimes_{K(n)_*} \Gamma\left(\phi\left(a^{*3}_{(1)}\right)\right),$$

with elements $\sigma a_{(1)} \in E^{st}_{1,6}$ and $\gamma_1 = \phi\left(a^{*3}_{(1)}\right) \in E^{st}_{2,54}$. This spectral sequence gives us an algebra isomorphism

$$K(2)_*K_2 \cong K(2)_*\left[a_{(0,1)}\right]/a^{*3}_{(0,1)} = v_2 a_{(0,1)}.$$

13. (Paul VanKoughnett)

[tex'd by Lucy]

14. (Adela Zhang)

[tex'd by Lucy]

15. Tempered cohomology (elliptic III) (Piotr Pstragowski)

Suppose that X is a finite CW complex. We can take complex vector bundles $[V \to X]$ and make it into a Grothendieck group $\mathbb{Z}[V]/[V \oplus W] = [V] + [W]$. This gives $\mathrm{KU}^0(X)$, which extends to a 2-periodic cohomology theory KU.

Given two bundles $V, W \to X$, we can associated to it their tensor product $V \otimes W \to X$, which is a new element in $\mathrm{KU}^0(X)$, making KU^0 into a ring. The associativity and commutativity endows KU with an E_∞-structure.

If G is a finite group acting on X, we can take a Grothendieck group of bundles with actions of G. This is the complex G-equivariant K-theory $\mathrm{KU}^0_G(X)$.

Example 15.0.1. $\mathrm{KU}^0_G(*)$ is the Grothendieck group of representations of G, which is $\mathtt{Rep}(G)$.

We have a comparison map

$$\begin{aligned} \mathrm{KU}^0_G(X) &\to \mathrm{KU}^0(X_{hG}) \\ [V \to X] &\mapsto [V/G \to X/G] \end{aligned}$$

given by quotienting out by the action.

This remembers quite a bit about the equivariant picture.

Theorem 15.0.2. (Atiyah–Segal) Let $I \subseteq \mathtt{Rep}(G)$ denote the augmentation ideal (of virtual representations of rank zero). Then the Atiyah–Segal map

$$\mathrm{KU}^0_G(X) \to \mathrm{KU}^0(X_{hG})$$

is an isomorphism after I-adic completion.

As an example if $X = *$, then

$$\mathtt{Rep}(G)^\wedge_I \simeq \mathrm{KU}^0(BG)^\wedge_I.$$

Assume that G is a p-group. Then the augmentation ideal lands in $I \subseteq p\mathtt{Rep}(G)$. So p-completion is a stronger form of completion than completion at the augmentation ideal, so a priori it remembers less information:

$$\mathtt{Rep}(G)^\wedge_p \simeq \mathrm{KU}^{0,\wedge}_p(BG).$$

At the right we have p-complete K-theory, which is the same as the Lubin–Tate spectrum to the formal group $\mathbb{G}_m$ associated to KU:

$$\mathrm{KU}^\wedge_p = E_1.$$

So chromatic objects are closely related to representation theory if they are evaluated on classifying spaces of finite groups. We are going to try to generalize this relation between chromatic cohomology theories and cohomology of classifying spaces of finite groups.

Plan:

(1) Show that the equivariant generalization of KU (i.e. $\mathrm{KU}_G(-)$) is determined by its chromatic structure
(2) For any E_∞-ring A with a choice of (an orientation on) a p-divisible group G over A, there is an equivariant extension of the cohomology theory determined by A.

Recall If k is a classical ring, then a p-divisible group G is a formal group scheme which looks like "$(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus n}$" which is like $(\mathbb{Z}/p^\infty)^{\oplus n}$.

(1) If $B \in \mathtt{CAlg}_k$, it can be mapped to $\mathrm{Hom}_{\mathbb{Z}}(M, G(B))$. We want to ask that this is representable by a finite flat group scheme
(2) Multiplication by p as a map $G \to G$ is an epimorphism in a suitable category

We want to generalize this to an E_∞-ring. There are two generalizations we will make

(1) Allow A to be an E_∞-ring
(2) We will remove dependence on p. More precisely we can do p-divisible groups for all primes at once.

Notation 15.0.3. Write $\mathcal{P}$ for the set of primes.

Definition 15.0.4. Let A be an E_∞-ring. Then we say a *$\mathcal{P}$-divisible functor* $\mathcal{O}_G : \mathtt{Ab}^{\mathrm{fin}} \to \mathtt{CAlg}_A$ such that

(1) $\mathcal{O}_G$ preserves direct sums – it maps sums to tensor products. This implies $\mathcal{O}_G(G) = A$ and $\mathcal{O}_G(M)$ is a Hopf algebra for any M (it is a cogroup object in $\mathtt{CAlg}_A$).
(2) Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian groups, the corresponding diagram is a pushout:

$$\begin{tikzcd} \mathcal{O}_G(M') \arrow[r] \arrow[d] & A \arrow[d] \\ \mathcal{O}_G(M) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{O}_G(M''), \end{tikzcd}$$

and the vertical maps are finite flat.

Informally, if M is an abelian p-group, then

$$\mathrm{Map}_{\mathtt{Alg}_A}(\mathcal{O}_G(M), B) \cong \mathrm{Map}_{\mathbb{Z}}\left(\check{M}, G_p(B)\right),$$

where G_p is a p-divisible group. Here $\check{M} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual.

So we can show that there is a 1-1 correspondence

$$\{\mathcal{P}\text{-divisible functors } \mathcal{O}_G\} \leftrightarrows \{(G_p)_{p\in\mathcal{P}} : G_p \text{ is } p\text{-divisible}\}.$$

Example 15.0.5. (Constant) We have the constant $\mathcal{P}$-divisible group $\mathbb{Q}/\mathbb{Z}$. This is $M \mapsto A^{\check{M}}$.

Example 15.0.6. (Multiplicative) We can take

$$M \mapsto A \otimes \Sigma^\infty_+ M.$$

In this case,

$$\mathrm{Map}_{\mathtt{Alg}_A}\left(A \otimes \Sigma^\infty_+ M, B\right) \cong \mathrm{Map}_{E_\infty}\left(M, \mathrm{GL}_1(B)\right).$$

This is the multiplicative group.

Example 15.0.7. Suppose A is $K(n)$-local and even periodic (think about $A = E_n$). Then we have the *Quillen $\mathcal{P}$-divisible group* (since we are $K(n)$-local this is just p-divisible since we have already localized at a prime):

$$M \mapsto A^{B\check{M}}.$$

The underlying p-divisible group over π_0 is:

$$\mathrm{Spec}\left(A^0\left(B\mathbb{Z}/p^n\right)\right) \cong A^0\left(\mathbb{CP}^\infty\right)[p^n].$$

Notation 15.0.8. Let $\mathcal{T} \subseteq$ be the full subcategory spanned by classifying spaces of the form BM where M is finite and abelian.

Proposition 15.0.9. For A an E_∞-ring, and $\mathcal{O}_G : \mathtt{Ab}^{\mathrm{fin}} \to \mathtt{CAlg}_A$ some $\mathcal{P}$-divisible functor, the following two sets of data are equivalent:

(1) a natural transformation $\mathcal{O}_G(M) \to A^{B\check{M}}$.[17]
(2) a factorization

$$\begin{tikzcd} \mathtt{Ab}^{\mathrm{fin}} \arrow[r, "\mathcal{O}_G"] \arrow[d, "B(\check{-})"'] & \mathtt{CAlg}_A \\ \mathcal{T}^{\mathrm{op}} \arrow[ur, dashed] & \end{tikzcd}$$

We call this a *preorientation*.

Example 15.0.10. If $A = k$ is a classical ring, then any flat algebra $B \in \mathtt{CAlg}_A$ is discrete. Thus $\mathcal{O}_G$ factors through the category of discrete A-algebras. Since $\mathtt{Ab}^{\mathrm{fin}} \to \mathcal{T}^{\mathrm{op}}$ is an equivalence of homotopy categories, there is a unique preorientation.

$$\begin{tikzcd} \mathtt{Ab}^{\mathrm{fin}} \arrow[r, "\mathcal{O}_G"] \arrow[d, "\sim"'] & \mathtt{CAlg}_A \arrow[r, hook] & \mathtt{CAlg}_A \\ \mathcal{T}^{\mathrm{op}} \arrow[ur, dashed] & & \end{tikzcd}$$

Example 15.0.11. If A is $K(n)$-local even periodic, we defined $\mathcal{O}_G(M) = A^{B\check{M}}$. Here the comparison transformation can be taken to be a natural equivalence.

[17] In the Quillen case this is just an equivalence.

Example 15.0.12. (Oriented $\mathcal{P}$-divisible group over KU) We can associate

$$B\check{M} \mapsto \operatorname{Fun}\left(B\check{M}, \mathtt{Vect}_{\mathbb{C}}^{\simeq}\right).$$

As soon as we fix a basepoint, we could identify this functor category with complex representations of M, but we don't have to fix a basepoint. We could define a spectrum as the group completion at the level of E_∞-spaces

$$(BM) = \operatorname{Gr}\left(\operatorname{Fun}\left(B\check{M}, \mathtt{Vect}_{\mathbb{C}}^{\simeq}\right)\right).$$

Finally, we can define from this connective one

$$\mathrm{KU}(B\check{M}) = (B\check{M}) \otimes_{\mathrm{KU}} \in \mathtt{CAlg}_{\mathrm{KU}}.$$

We can verify this is always finite free since $\pi_0\operatorname{Fun}(\cdots)$ is always the representation ring. This has a preorientation.

Question: What's the corresponding $\mathcal{P}$-divisible group?

The functor sends

$$M \mapsto \mathrm{KU}(B\check{M}) \xrightarrow{\pi_0} \mathtt{Rep}(\check{M}) \cong \mathbb{Z}[M].$$

This is because an element on the right is like a function $\lambda : \check{M} \to \mathbb{Q}/\mathbb{Z} \subseteq \mathbb{C}^\times$, which is a complex representation of $\check{M}$. From this we see that $\pi_0\mathrm{KU}(B\check{M})$ is the multiplicative group. This is also true over KU itself.

This construction is a *preorientation* on μ_∞ over KU.

Definition 15.0.13. An *orbispace* $\mathbb{X} : \mathcal{T}^{\mathrm{op}} \to$ is a presheaf of spaces on $\mathcal{T}$.

Example 15.0.14. If $X \in$, we can take $\mathbb{X} = X \in \mathtt{OSpc}$, so that $\mathbb{X}(S) = \operatorname{Map}(S, X)$. This is the Yoneda embedding, and it admits an inverse, given by geometric realization

$$|\cdot| : \mathtt{OSpc} \to,$$

given by left Kan extension of the inclusion of $\mathcal{T}$ into and $\mathtt{OSpc}$. This is the same as evaluation of $\mathbb{X}$ on a point.

Given $X \in G-$, we can associate to X the *orbispace quotient* $X \to X//G$. This is defined by

$$X//G = \operatorname{colim}_{G/H \to X} BH,$$

indexed over all equivariant maps $G/H \to X$ with H abelian. This is again a left Kan extension.

Definition 15.0.15. Let A be an E_∞-ring, and $\mathcal{O}_G : \mathcal{T}^{\mathrm{op}} \to \mathtt{CAlg}_A$ a preoriented $\mathcal{P}$-divisible group. The *tempered cohomology* cochain spectrum $A_G^{(-)} : \mathtt{OSpc}^{\mathrm{op}} \to \mathtt{CAlg}$ is the unique extension

$$\begin{tikzcd} \mathcal{T}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathtt{CAlg}_A \\ \mathtt{OSpc}^{\mathrm{op}} \arrow[ur, dashed, "A_G^{(-)}"'] & \end{tikzcd}$$

Notation 15.0.16. We denote by $A_G^*(\mathbb{X})) = \pi_{-*}A_G^{\mathbb{X}}$.

Example 15.0.17. (Complex equivariant K-theory) The preorientation is given by the Segal construction as we have seen. Say for now that G is abelian. Then

$$\mathrm{KU}^*_{\mu_\infty}((H\backslash G)//G) = \pi_{-*}\mathcal{O}_{\mu_\infty}(H) = \mathrm{KU}^*_H.$$

Since both sides take colimits to limits, we deduce that in general, given any finite CW complex acted on by G, we have that

$$\mathrm{KU}^*_{\mu_\infty}(X//G) \simeq \mathrm{KU}^*_G(X).$$

What is surprising is that this is still true when G is not abelian.

Theorem 15.0.18. Assume that A is $K(n)$-local and even periodic. Then the comparison map

$$A^*_{G_Q}(\mathbb{X}) \cong A^*(|\mathbb{X}|),$$

where G_Q is the Quillen $\mathcal{P}$-divisible formal group.

Proof. Both sides take colimits to limits, so we just have to check it on BM for M finite. But here $A^*_{B_Q}(BM) \cong A^*(BM)$ by construction. □

Say $A \to B$ is a flat extension of E_∞ rings. Then if $\mathcal{O}_G$ is a $\mathcal{P}$-divisible functor, we get an extension $\mathcal{O}_{G_B}\left(B\check{M}\right) := B \otimes_A \mathcal{O}_G(B\check{M})$, by extending scalars.

Theorem 15.0.19. If X is a finite CW complex with an H-action, and G is an oriented $\mathcal{P}$-divisible group, then

$$B^*_{G_B}(X//H) \simeq B^* \otimes_{A^*} A^*_G(X//H).$$

16. Character theory via tempered cohomology (Peter Haine)

Plan:

(1) Motivation from character theory
(2) How the statements in (1) relate to tempered cohomology
(3) Orientations

Key examples:

(1) $\mathrm{KU}_{\mu_{\mathcal{P}}^{\infty}}$
(2) $A = E_n$ and $G = \mathbb{G}^{\mathrm{Quillen}}$

16.1. **Character theory of finite groups.** Given a finite group G, there is an isomorphism

$$\mathtt{Rep}(G) \xrightarrow{\chi} \{\text{class function} G \to \mathbb{C}\},$$

which is an isomorphism after applying $- \otimes_{\mathbb{Z}} \mathbb{C}$. We know that $\mathtt{Rep}(G) = \mathrm{KU}_G^0(*)$. Since the class functions are conjugation-invariant, they are functions on an adjoint quotient of G by itself to the complex numbers. We think about this as $H^0(-, \mathbb{C})$ of a quotient of G by its conjugation. That is, we think about the set of class functions as

$$H^0(G_{hG}, \mathbb{C}).$$

Theorem 16.1.1. If X is a finite G-space, write $\amalg_{g \in G} X^g$ for the space with G-action given by conjugation on the indexing set G and residual actions on the fixed point spaces $X^g := X^{\langle g \rangle}$ for each g. Then the *equivariant Chern character* defines an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{KU}_G^0(X) \xrightarrow{\sim} H^{\mathrm{even}}\left((\amalg_g X^g)_{hG}, \mathbb{C}\right).$$

Question: What about the p-complete K-theory analogue of this statement?

We have that $G \cong \mathrm{Hom}(\mathbb{Z}, G)$, so we are going to replace this by $\mathrm{Hom}(\mathbb{Z}_p, G)$. This latter group is finite, so they are factoring through some subgroup of finite order. This is

$$\mathrm{Hom}(\mathbb{Z}_p, G) = \left\{ g \in G \colon g^{p^k} = 1 \text{ for } k \gg 0 \right\}.$$

These elements are called *p-singular elements*.

Theorem 16.1.2. Fix a prime p and an embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}$. Then there is a canonical isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}_p} \mathrm{KU}_p^{\wedge,0}(X_{hG}) \xrightarrow{\sim} H^{\mathrm{even}}\left(\left(\amalg_{\alpha : \mathbb{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right)_{hG}, \mathbb{C}\right).$$

On the left we have height 1, while on the right we have height zero.

Question: What about relating height n and height 0, or more generally height m for $m \leq n$. Relating height n with height zero is Hopkins–Ravenel character theory. Height n and m is due to Stapleton.

Setup:

- Let k be a perfect field of characteristic p, let $\widehat{\mathbb{G}}_0$ be a height n formal group, with $\widehat{\mathbb{G}}$, the identity component of a p-divisible group, its universal deformation
- R a Lubin–Tate ring
- $C_0 = R$-algebra classifying isos $\mathbb{G} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^\wedge$
- E the Lubin–Tate ring of $\widehat{\mathbb{G}}_0$

HKR theorem 16.1.3. Let G be a finite group and X a finite G-space:

$$C_0 \otimes_R E^* (X_{hG}) \xrightarrow{\sim} C_0 \otimes_{R_\mathbb{Q}} E^*_\mathbb{Q} \left(\left(\amalg_{\alpha:\mathbb{Z}_p \to \mathbb{C}} X^{\mathrm{im}(\alpha)}\right)_{hG}\right)$$

Key steps:

(1) Understand what happens to tempered cohomology if we use something like $A_{\mathbb{G}_0 \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^n}$
(2) $B \otimes_A A^{\mathbb{X}}_G$ versus $B^{\mathbb{X}}_{\mathbb{G}}$

16.2. **Formal loop spaces.** Recall $\mathcal{T} \subseteq$ denotes the full subcategory on objects BG with $G \in \mathtt{Ab}^{\mathrm{fin}}$, $\mathcal{O} = (\mathcal{T})$

Definition 16.2.1. For $\mathbb{X} \in$, and Λ a torsion abelian subgroup, then the formal loop space $\mathcal{L}^\wedge(\mathbb{X})$ is defined as

$$\mathcal{T}^{\mathrm{op}} \to$$
$$T \mapsto \operatorname{colim}_{\substack{\Lambda_0 \subseteq \Lambda \\ \text{finite}}} \left(\mathbb{X}^{T \times B\Lambda_0^\vee}\right)$$

Computation 16.2.2. If X is a G-space, we can take the formal loop space on its orbispace quotient

$$\mathcal{L}^\wedge (X//G) \simeq \left(\coprod_{\alpha:\Lambda^\vee \to G} X^{\mathrm{im}(\alpha)}\right) //G.$$

If X is π-finite, then $\mathcal{L}^{\mathbb{Q}/\mathbb{Z}}(X) \simeq \mathcal{L}(X)$. Similarly if X is p-finite, then $\mathcal{L}^{\mathbb{Q}_p/\mathbb{Z}_p}(X) \simeq \mathcal{L}(X)$.

Recall That $(\mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \mathbb{Z}_p$. Similarly $(\mathbb{Q}/\mathbb{Z})^\vee \cong \widehat{\mathbb{Z}}$.

Note 16.2.3. If $X \in$, we have $X^{(-)} \in$, There's a map

$$\left|\mathcal{L}^\wedge \left(X^{(-)}\right)\right| \simeq \operatorname{colim}_{\Lambda_0 \subseteq \Lambda} X^{B\Lambda_0^\vee} \to X^{B\Lambda^\vee},$$

where in this last step we are ignoring the profinite topology. That gives you a natural map

$$\mathcal{L}^\wedge X^{-)} \to \left(X^{B\Lambda^\vee}\right)$$

Definition 16.2.4. We say an abelian group Λ is a *collatice* if

(1) Λ is torsion

(2) For all n, the multiplication by n map $\Lambda \xrightarrow{n\cdot -} \Lambda$ is a surjection with finite kernel.

Equivalent, if for all p, we have that $\Lambda_{(p)} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{h_p}$ for various heights.

Proposition 16.2.5. Let X be a π-finite space, and Λ is a colattice, then

$$\mathcal{L}^{\wedge}\left(X^{(-)}\right) \xrightarrow{\sim} \left(X^{B\Lambda^\vee}\right)^{(-)}$$

Character isomorphism 16.2.6. Let $\mathbb{G}_0$ be a preoriented $\mathcal{P}$-divisible group, Λ a colattice, and $\mathbb{G} := \mathbb{G}_0 \oplus \underline{\Lambda}$. Then for any $\mathbb{X} \in$, we have an isomorphism

$$\chi : A^*_{\mathbb{G}} \simeq A^{\mathcal{L}\cdot}_{G_0}$$

Corollary 16.2.7. In the same setup, for G a finite group and X a G-spce, we have

$$\chi : A^*_G(X//G) \simeq \mathbb{A}^*_{\mathbb{G}_0}\left(\left(\amalg_{\alpha:\Lambda^\vee\to\mathbb{C}} X^{\mathrm{im}(\alpha)}\right)//G\right).$$

16.3. **Chern character via tempered cohomology. Setup**: Write $\mathrm{KU}_{\mathbb{C}} := \mathbb{C} \otimes \mathrm{KU}$. Over $\mathbb{C}$ there is an exponential of p-divisible groups

$$\begin{aligned} \exp : \underline{\mathbb{Q}/\mathbb{Z}} &\xrightarrow{\sim} \mu_{\mathcal{P}^\infty} \\ \lambda &\mapsto \exp(2\pi i\lambda). \end{aligned}$$

We have that

$$(\mathrm{KU}_{\mathbb{C}})^{\mathbb{X}}_{\mu_{\mathcal{P}^\infty}} \simeq (\mathrm{KU}_{\mathbb{C}})^{\mathbb{X}}_{\underline{\mathbb{Q}/\mathbb{Z}}} \simeq \mathrm{KU}_{\mathbb{C}}^{\left|\mathcal{L}^{\mathbb{Q}/\mathbb{Z}(\mathbb{X})}\right|}$$

Precomposing with a map from ordinary KU we get a composite we call the Chern character

$$\mathtt{Ch} : \mathrm{KU}^*_{\mu_{\mathcal{P}^\infty}}(\mathbb{X}) \to H^*\left(\left|\mathcal{L}^{\mathbb{Q}/\mathbb{Z}}(\mathbb{X})\right|, \mathbb{C}\right)[\beta^{-1}].$$

We haven't yet used Atiyah–Segal completion in tempered cohomology. We will introduce this, and this is when the base change theorem will hold.

16.4. **Orientations. Goal**: Construct a map

$$\zeta : A^{\mathbb{X}}_{\mathbb{G}} \to A^{|\mathbb{X}|}.$$

Fact 16.4.1. $A^{\underline{X}}_{\mathbb{G}} \simeq A^X$.

So it suffices, for constructing the map above, to construct a map $\underline{|\mathbb{X}|} \to \mathbb{X}$. But we already have one, this is just the counit. So the Atiyah–Segal completion map is just ζ above using the counit.

Note 16.4.2. The point $* \to BG$ determines a surjection $A^0_{\mathbb{G}}(BG) \to A^0_{\mathbb{G}}(*) = \pi_0 A$. We will write I_G for the kernel of this map.

Definition 16.4.3. Let A be an E_∞-ring, $\mathbb{G}$ a preoriented $\mathcal{P}$-divisible group over A. Then we say that $\mathbb{G}$ is *oriented* if:

(1) For all p, the Atiyah–Segal comparison map

$$\zeta : A_{\mathbb{G}}^{BC_p} \to A^{BC_p}$$

exhibits A^{BC_p} as the I_{C_p}-completion of the source.

(2) The Tate construction A^{tC_p} is I_{C_p}-local as an $A_{\mathbb{G}}^{BC_p}$-module.

Theorem 16.4.4. (Base change) If $\mathbb{G}$ is an oriented $\mathcal{P}$-divisible group over A, then for every π-finite space X, given a map $f : A \to B$, we have that

$$B \otimes_A A_{\mathbb{G}}^X \xrightarrow{\sim} B_{\mathbb{G}}^X$$

is an equivalence.

17. (Arpon Raksit)

18. What else? (Tomer Schlank)

With Lior Yanovski and Shachar Carmeli.

Just like in `Ab` we can decompose at primes and reassemble, in homotopy theory we have more primes — these are the chromatic heights. Looking locally at a prime, there are two candidates for a local category — $Sp_{T(n)}$ or $Sp_{K(n)}$. Usually we deal with $Sp_{K(n)}$ because there are more tools available. It is hard to see ambidexterity looking at all heights at once, but zooming in at one prime we can see it.

The $K(n)$-local sphere $\mathbb{S}^0_{K(n)}$ has a Galois closure, given by $E_n\left(\overline{\mathbb{F}}_p\right)$. This has a Galois group, the Morava stabilizer group $\mathbb{G}_n$. In chromatic homotopy theory, our tool to study this Galois theory is the Hopkins–Devinatz spectral sequence.

What about $\mathbb{S}^0_{T(n)}$? Do we know anything about Galois extensions?

Definition 18.0.1. (Rognes) Given $R \in \mathtt{CAlg}(\mathscr{C})$ in a stable symmetric monoidal ∞-category, with an action of a finite group G, we say that it is a *Galois extension* if

$$1 \xrightarrow{\sim} R^{hG}$$

is an equivalence, and the following map is an equivalence:

$$\begin{aligned} R \otimes R &\xrightarrow{\sim} \prod_{g \in G} R \\ (r_1, r_2) &\mapsto r_1 \cdot g r_2. \end{aligned}$$

We don't know what the algebraic closure of $\mathbb{S}^0_{T(n)}$ is, but we can construct some Galois extensions, and ambidexterity is a useful tool. How do we construct Galois extensions?

Given our field $\mathbb{Q}$, we have to add an element and divide by a polynomial to get $\mathbb{Q}[x]/f(x)$. This is too element-dependent to work in the ∞-world.

There are however special Galois extensions that we can describe differently, for example a cyclotomic extension like $\mathbb{Q}\left[\zeta_{p^n}\right]$. Analogous to $\mathrm{Gal}\left(\mathbb{Q}\left[\zeta_p\right]/\mathbb{Q}\right) = C_{p-1}$, we want to construct $\mathrm{Gal}\left(\mathbb{S}^0_{T(n)}\left[\zeta_p\right]/\mathbb{S}^0_{T(n)}\right) = C_{p-1}$.

We can take the group algebra $\mathbb{Q}[C_p] \cong \mathbb{Q}[x]/(x^p - 1)$. This splits as $\mathbb{Q}[\zeta_p] \times \mathbb{Q}$. We define $\varepsilon \in \mathbb{Q}[C_p]$ to be the idempotent element

$$\varepsilon = \frac{\sum_{y \in C_p} y}{|C_p|} \in \mathbb{Q}[C_p].$$

This is an idempotent since

$$\varepsilon^2 = \frac{\sum_{g,h} gh}{|C_p|^2} = \frac{|C_p| \sum_g g}{|C_p|^2} = \varepsilon.$$

This is exactly the idempotent which splits the algebra.

We want to:

(1) take a group algebra
(2) sum things
(3) divide by the order of C_p.

We normally can't do this last thing in an additive category. However we learned this week that

$$|B^n C_p| \in \pi_0 \left(\mathbb{S}^0_{T(n)}\right)^\times.$$

So let's first take

$$\mathbb{S}^0_{T(n)}\left[B^n C_p\right].$$

We have a canonical map

$$B^n C_p \xrightarrow{\text{can}} \operatorname{Map}_{Sp_{T(n)}}\left(\mathbb{S}^0, \mathbb{S}^0\left[B^n C_p\right]\right),$$

sending every point to its corresponding component. Ambidexterity allows us to sum these things, to obtain

$$\int_{g \in B^n C_p} \operatorname{can}(g) \in \operatorname{Map}(\mathbb{S}^0, \mathbb{S}^0\left[B^n C_p\right]),$$

and since $B^n C_p$ is amenable, we can divide by its order to get

$$\varepsilon := \frac{1}{|B^n C_p|} \int_{g \in B^n C_p} \operatorname{can}(g).$$

Then ε will be an idempotent in $\pi_0 \mathbb{S}^0_{T(n)}\left[B^n C_p\right]$. If we invert this idempotent, we get the $T(n)$-local sphere. If we invert $1 - \varepsilon$, we will get

$$\mathbb{S}^0_{T(n)}\left[B^n C_p\right]\left[(1-\varepsilon)^{-1}\right] =: \mathbb{S}^0_{T(n)}\left[\mu_p\right].$$

We get an action here and we can ask if this is a Galois extension in the sense of Rognes, and the answer is yes.

More generally, we can get a cyclotomic extension for the p^n roots of unity.

Given a map of rings $T \to K$, we get a map of Galois groups $\operatorname{Gal}(\overline{K}/K) \to \operatorname{Gal}(\overline{T}/T)$. Similarly we get a map

$$\operatorname{Gal}\left(Sp_{K(n)}\right) \to \operatorname{Gal}\left(Sp_{T(n)}\right).$$

On the left we get $\mathbb{G}$, which maps to its own abelianization $\mathbb{G}^{\text{ab}} = \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$. We know that the kernel lies in the commutator, and that the kernel is a virtually p-Sylow subgroup in the commutator.

So ambidexterity gives us higher analogues of the notions of pth roots of unity. What can we do with pth roots of unity?

Kummer theory tries to understand Galois extensions of fields — given a field K with $\mu_p \subseteq K$ and $\frac{1}{p} \in K$, then $\mathrm{Gal}(K, \mathbb{Z}/p) = H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \cong K^\times/(K^\times)^p$. If we are homologically-minded, we might describe this group as

$$K^\times/(K^\times)^p = \mathrm{Ext}^1\left(\mathbb{Z}/p, K^\times\right).$$

In our culture, we write this as

$$\left[\Sigma^{-1}(\mathbb{Z}/p), K^\times\right] = \left[\Sigma^{-1}(\mathbb{Z}/p), \mathrm{GL}_1(K)\right].$$

Suppose that $R \in \mathtt{CAlg}\left(Sp_{T(n)}\right)$, and $\mu_p \subseteq R$, meaning that there exists a map $\mathbb{S}^0_{T(n)}[\mu_p] \to R$. Then the collection of $\mathbb{Z}/p$-Galois extensions of R are given by[18]

$$\mathrm{Gal}(R, \mathbb{Z}/p) = \left[\Sigma^{h-1}\mathbb{Z}/p, \mathrm{GL}_1(R)\right].$$

Given roots of unity, we can also do a discrete Fourier transform. Given R with $\frac{1}{p} \in R$ and $\mu_{p^\infty} \subseteq R$, and A is a finite abelian p-group, then we can construct a category of representations of A in R-modules, which we might denote by $\mathtt{Rep}_{\mathtt{Mod}_R}(A)$. The discrete Fourier transform tells you that every such representation decomposes into a sum indexed over the Pontryagin dual of A:

$$\mathtt{Rep}_{\mathtt{Mod}_R}(A) = \oplus_{A^*}\mathtt{Mod}_R.$$

We can rewrite these categories as

$$\begin{aligned} \mathtt{Rep}_{\mathtt{Mod}_R}(A) &= \mathrm{Fun}\left(BA, \mathtt{Mod}_R\right) \\ \oplus_{A^*}\mathtt{Mod}_R &= \mathrm{Fun}\left(A^*, \mathtt{Mod}_R\right), \end{aligned}$$

where we are considering A^* as a finite set here. If R is a commutative ring, then the category of representations gets a tensor product computed pointwise, implying that $\mathrm{Fun}\left(A^*, \mathtt{Mod}_R\right)$ has a compatible symmetric monoidal structure. This is Day convolution, by remembering that A^* is a group. We can then rewrite this equivalence as

$$\mathrm{Fun}^{\otimes\text{-ptwise}}\left(B^1A, \mathtt{Mod}_R\right) \cong \mathrm{Fun}^{\otimes\text{-Day}}\left(B^0A^*, \mathtt{Mod}_R\right)$$

Now assume $R \in \mathtt{CAlg}(Sp_{T(n)})$ and $\mu_{p^\infty} \subseteq R$, and let $a + b = n + 1$. Then there exists an equivalence

$$\mathrm{Fun}^{\otimes\text{-ptwise}}\left(B^aA, \mathtt{Mod}_R\right) \cong \mathrm{Fun}^{\otimes\text{-Day}}\left(B^bA^*, \mathtt{Mod}_R\right)$$

sending local systems tensored pointwise to the Day convolution. We could envision an even fancier version of this by noting that $B^aA = \Omega^\infty\Sigma^aA$ and that $B^bA^* = \Omega^\infty\Sigma^bA^* = \Omega^\infty\underline{\mathrm{Hom}}\left(\Sigma^aA, \Sigma^{n+1}I\right)$, where I is the Brown–Comenetz dual. When we do this, we can replace Σ^aA with any spectra with suitable homotopy groups.

[18] Here h is the height, we were secretly working with $h = 0$ before.

Bhatt–Clausen–Mathew has the following result — suppose that R is a ring in which p is invertible. Then we have that

$$L_{K(1)}\left(K\left(R\left[\mu_{p^\infty}\right]\right)\right) \cong K_{K(1)}K(R)\widehat{\otimes}\mathrm{KU}.$$

There is a $\mathbb{Z}/p$-action on $R\left[\mu_{p^\infty}\right]$, and the Adams operations on KU on the right, and this equivalence is equivariant. Since the telescope conjecture is true at height 1, we can replace this by $T(1)$.

We have that KU is a Galois extension, given by

$$\mathrm{KU} = \mathbb{S}^0_{K(1)}\left[\mu_{p^\infty}\right].$$

Tensoring with adding pth roots of unity is just adding pth roots of unity, so we can rewrite their result as

$$L_{K(1)}\left(K\left(R\left[\mu_{p^\infty}\right]\right)\right) \cong K_{K(1)}K(R)\left[\mu_{p^\infty}\right].$$

That is, taking pth roots of unity passes outside taking algebraic K-theory in an equivariant way.

For $R \in \mathtt{CAlg}\left(Sp_{T(n)}\right)$:

$$L_{T(n+1)}K\left(R\left[\mu_{p^\infty}\right]\right) \cong L_{T(n+1)}K(R)\left[\mu_{p^\infty}\right].$$

We call this *cyclotomic redshift* because we moved from height n to height $n+1$ here.

Q: To what extent are results in chromatic homotopy theory given by the presence of ambidexterity and stability?

Think about height — given a semiadditive structure, can we talk about height in a category? Think about a prime as $p = |C_p|$, sitting in an infinite sequence $\left\{|B^iC_p|\right\}_{i=0}^{\infty}$. Given an ∞-semiadditive ∞-category, given any object $X \in \mathscr{C}$ we have a bunch of transformations

$$X \xrightarrow{|B^kC_p|} X.$$

If $|B^kC_p|_X$ is an isomorphism, then so is $|B^{k+1}C_p|_X$. So we can say that the *height* of X is $\leq k$ if $|B^kC_p|_X \in \mathrm{End}(X)^\times$. We say that the height of X is $> k$ if $|B^kC_p|$-complete: that is,

$$X \to \lim_k X/|B^kC_p|_X$$

is an isomorphism. So we can use these to describe when the height of X is *exactly* height k.

All objects in $Sp_{T(n)}$ and $Sp_{K(n)}$ have height exactly n. This is our discussion about amenability.

We can take Sp, the universal stable category, and force it to be ∞-semiadditive by taking $_{\infty}(Sp)$. This is the universal stable ∞-semiadditive. Working p-locally, let's write this as

$$=_{\infty}\left(Sp_{(p)}\right).$$

We have that this splits as

$$=_{0}\times_{\geq 1}.$$

Here $_{0} = Sp_{\mathbb{Q}}$. We see also that $_{\geq 1} =_{1} \times_{\geq 2}$, and so on.

Conjecture 18.0.2. $_{\infty} = 0$.

There is always a nice map $_{n} \to Sp_{T(n)}$. Yuan gave a counterexample to this always being an equivalence, for $n = 1$.