MIT TALBOT WORKSHOP 2021: AMBIDEXTERITY IN CHROMATIC HOMOTOPY THEORY

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ABSTRACT. Notes from the MIT Talbot Workshop 2021: Ambidexterity in Chromatic Homotopy Theory, mentored by Jacob Lurie and Tomer Schlank. Please fix any typos or errors by submitting a pull request at https://github.com/tbrazel/talbot2021.

CONTENTS

1. Overview (Jacob Lurie)

Let G be a finite group, and consider a class in the cohomology of G, denoted $\eta \in H^n(BG;\mathbb{C}^*)$. Cohomology classes are things that you can integrate against cycles. So if we had a compact oriented manifold M of dimension n, together with a map $f : M \to BG$, then we could take the fundamental class of M, and pair it to get an invertible complex number

$$\langle [M], f^*\eta\rangle \in \mathbb{C}^*.$$

If we want a quantity that only depends on M, we could form a combination which considers *all* maps from M into BG.

There are lots of maps $M \to BG$, and if we wanted to classify them up to homotopy, we obtain a bijection with the set of principal G-bundles on M, considered up to isomorphism. To be really concrete, assume further that M is connected and we have chosen a basepoint. Then we have a bijection

$$[M, BG] \leftrightarrow \{\text{principal } G\text{-bundles on } M\} \leftrightarrow \{\text{group homomorphisms } \pi_i \to M\}/\text{conjugation}$$

Since group homomorphisms are determined by where the generators go, we only have finitely many such homomorphisms. We could sum over all group homomorphisms $\pi_1(M) \to G$ to get a complex number of the form

$$\frac{1}{|G|}\sum_{\pi_1 M \to G} \langle [M], f^*\eta\rangle\,.$$

This is almost the same as

$$\sum_{\substack{\text{homomotopy classes}\\ f:M\to BG}} \langle [M], f^*\eta\rangle$$

This would be true if G were acting by group conjugation here, but it isn't. What we are really doing here is summing with multiplicity

$$\sum_{G\text{-bundles } P \text{ on } M} \frac{\langle [M], f^*\eta\rangle}{|\mathrm{Aut}(P)|}.$$

We will define this quantity the *integral* over the space of all maps $M \to BG$:

$$Z(M) := \int_{f:M\to BG} \langle [M], f^*\eta\rangle := \frac{1}{|G|}\sum_{\pi_1 M\to G} \langle [M], f^*\eta\rangle\,.$$

If we fix G and M, then this complex number $Z(M)$ is an invariant of manifolds. This is what is assigned to M by Dijkgraaf–Witten theory. It is not a very interesting invariant, because it only depended on the fundamental group of M and the homotopy class of M.

Goal: Find more constructions "like this."

That is, take this construction and vary the ingredients you used to make it. We could try to vary the field $\mathbb{C}$, but we couldn't use a field of characteristic p if $p \mid |G|$. This is because we are normalizing by $|G|$, and we can't throw out the normalization without losing some structure.

Given a manifold M, we can look at the space of maps $\mathrm{Map}(M, BG)$. We have an evaluation and projection

$$\begin{tikzcd} M \times \mathrm{Map}(M, BG) \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & BG \\ \mathrm{Map}(M, BG) & \end{tikzcd}$$

So we are beginning with η, pulling it back to $\mathrm{ev}^*\eta$, and then pushing it forward along π to obtain $\pi_*\mathrm{ev}^*\eta$, which is a cohomology class of degree zero, i.e. a function of the form $\mathrm{Map}(M, BG) \to \mathbb{C}^*$. This is what happens when M is an n-manifold.

Suppose M is now an $(n-1)$-manifold. Then the pullback and pushforward will be a class

$$\mathcal{L} := \pi_*\mathrm{ev}^*\eta \in H^1\left(\mathrm{Map}(M, BG), \mathbb{C}^*\right).$$

That is, it corresponds to a local system of 1-dimensional complex vector spaces. This cohomology class determines the local system $\mathcal{L}$ up to isomorphism.

Given a 1-dimensional local system, we can try to integrate it, to obtain a single vector space. We could look at the cohomology $H^0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$, or we could look at the homology $H_0\left(\mathrm{Map}(M, BG), \mathcal{L}\right)$. We want to assign a complex vector space $Z(M)$ which corresponds to either of these. These two vector spaces *turn out to be the same.*

We have that $\pi_0\mathrm{Map}(M, BG)$ corresponds to G-bundles on M, so

$$\mathrm{Map}(M, BG) = \coprod_{\text{iso classes of G-bundles } P} B\mathrm{Aut}(P).$$

So let's start by thinking about local systems on things like $B\mathrm{Aut}(P)$.

Suppose that H is a finite group, and let's consider complex local systems $\mathcal{L}$ on BH. This is the same thing as a complex representation V of H. We have that

$$H^0\left(BG, \mathcal{L}\right) = V^H = \{v \in V : hv = v \forall h \in H\}.$$

The homology is

$$H_0\left(BG, \mathcal{L}\right) = V_H = V \big/ \mathbb{C} \cdot \{hv - v\}.$$

This is the minimal quotient of V you can form on which H acts trivially. When H is a finite group, there is an obvious relation between these. We have an averaging function

$$\begin{aligned} V &\to V \\ v &\mapsto \sum_{h \in H} hv. \end{aligned}$$

This map factors through the subspace V^H, but it also factors through the quotient given by the coinvariants, since it annihilates vectors of the form $hv - v$. So we get a norm

$$\begin{tikzcd} V \arrow[rr] \arrow[dr] & & V \\ & V_H \arrow[r, "\mathrm{Nm}"'] & V^H \arrow[u] \end{tikzcd}$$

Basic fact: This map is an isomorphism (assuming characteristic zero).

Proof. We should write down the inverse map. There is an obvious map in the other direction:

$$V^H \subseteq V \twoheadrightarrow V_H.$$

Let's call this map λ. We see that $\lambda \circ \mathrm{Nm} = \mathrm{Nm} \circ \lambda$, which is multiplication by the order of H. Over characteristic zero, this multiplication is an isomorphism. □

Remark 1.0.1. Recall that if M is an n-manifold, we thought about this function $\mathrm{Map}(M, BG) \to \mathbb{C}^*$, given by $f \mapsto \langle [M], f^*\eta \rangle$. This map gives us something in $H^0(\mathrm{Map}(M, BG), \mathbb{C}^*)$. This integration procedure was

$$Z(M) = \int_{\mathrm{Map}(M,BG)} \langle [M], f^*\eta \rangle,$$

took the class in the degree zero cohomology, but we identified cohomology with homology by doing this norm map on every component.

Thinking as an algebraic topologist, we can turn fields K into cohomology theories HK. Thinking about fields from a very large distance, there are fields of characteristic zero, and those of characteristic p. Morava realized that in the world of cohomology theories, there are a hierarchy of examples which interpolate between things like $H\mathbb{Q}$ and things like $H\mathbb{F}_p$. Fixing a prime number p, we have that Morava K-theories are an infinite sequence of cohomology theories, with

$$H\mathbb{Q} = K(0) \subseteq K(1) \subseteq \cdots \subseteq K(\infty) = H\mathbb{F}_p.$$

Question: Do these constructions make sense "over $K(n)$?"

Morava K-theories are characteristic p objects, since multiplication by p is the zero map $K(n) \to K(n)$ for $n > 0$. If we think characteristic p is bad, we might think Morava K-theories are bad. However the answer to this question is yes!

Theorem 1.0.2. (Hovey–Sadofsky) Let V be a $K(n)$-module with an action of a finite group H. Then the norm map

$$\mathrm{Nm}_H : V_{hH} \to V^{hH}$$

is an isomorphism for $n < \infty$.

Suppose that V and W are $K(n)$-modules and suppose we have a family of maps $f_x : V \to W$ parametrized by $x \in BH$. That is, a continuous map $f : BH \to \mathrm{Map}(V, W)$. Yet another way to think about this data is considering f as an element of $H^0\left(BH, \underline{\mathrm{Map}(V, W)}\right)$. Since V and W were $K(n)$-modules, we have that $\underline{\mathrm{Map}(V, W)}$ is a $K(n)$-module (with two $K(n)$-module structures). This theorem earlier tells us that

$$H^0\left(BH, \underline{\mathrm{Map}(V, W)}\right) \cong H_0\left(BH, \underline{\mathrm{Map}(V, W)}\right) \to H_0\left(*, \underline{\mathrm{Map}(V, W)}\right) = \pi_0\mathrm{Map}(V, W),$$

by mapping along $BH \to *$. Thus using this theorem from earlier, we can go from a *family* of maps, to a *single* map $V \to W$. We denote this procedure by

$$\begin{gathered}\mathrm{Map}(BH, \mathrm{Map}(V, W)) \to \pi_0\mathrm{Map}(V, W)\\ f \mapsto \int f.\end{gathered}$$

We saw this earlier when $n = 0$ and when $V = W = \mathbb{C}$.

Now let's assume that H is abelian. Then BH is an abelian group object in spaces. What if we want to study representations of BH? That is, local systems on $B(BH) = K(H, 2)$. This is a simply connected space, so there should be no local systems on it, that is, this doesn't make sense classically.

So instead we want to study representations of BH on $K(n)$-modules, that is, local systems $\mathcal{L}$ of BH-modules. We could study the analogue of the coinvariants and invariants, which are the homotopy (co)limits over $\mathcal{L}_x$, where $x \in BH$. The Hovey–Sadofsky theorem gives

$$\lim_{BH} : \mathrm{hocolim}_{y \in K(H,2)}\mathcal{L}_y \to \mathrm{holim}_{x \in K(H,2)}\mathcal{L}_x.$$

To give such a map is to give a family of maps $f_{x,y} : \mathcal{L}_x \to \mathcal{L}_y$, and these should vary continuously in x and y. Any path $p : [0, 1] \to K(H, 2)$ satisfying $p(0) = x$ and $p(1) = y$ determines an isomorphism $p_! : \mathcal{L}_y \to \mathcal{L}_x$. This depends not only on x and y but also on the path that we chose. The collection of such paths is parametrized by a space $\{x\} \times^h_{K(H,2)} \{y\} =: P_{x,y}$. So we have a collection of isomorphisms $\mathcal{L}_y \xrightarrow{\sim} \mathcal{L}_x$ parametrized by the space $P_{x,y} \simeq K(H, 1) = BH$.

We can then use that integration procedure to get

$$f_{x,y} = \int_{p \in P_{x,y}} p_!,$$

which is a single morphism $\mathcal{L}_y \to \mathcal{L}_x$ (not necessarily an isomorphism anymore). So allowing x and y to vary, we get a single map

$$\mathrm{Nm}_{K(H,2)} : \mathrm{hocolim}\mathcal{L} \to \mathrm{holim}\mathcal{L}.$$

Theorem 1.0.3. This map is also a homotopy equivalence.

We can now do this again — suppose we are interested in representations of $K(H,2)$, then $K(H,3)$ and so on. This yields the following.

Theorem 1.0.4. Let X be a space with finitely many homotopy groups, and all homotopy groups are assumed to be finite[1] and let $\mathcal{L}$ be a local system of $K(n)$-modules on X. Then there is a canonical isomorphism

$$\mathrm{Nm}_X : \mathrm{hocolim}\mathcal{L}_x \to \mathrm{holim}\mathcal{L}_x.$$

That is, there is some natural map which induces isomorphisms $H_*(X,\mathcal{L}) \xrightarrow{\sim} H^*(X,\mathcal{L})$.

This is an interesting statement even when $\mathcal{L}$ is a trivial local system. In particular if X has finitely many homotopy groups, there is a canonical isomorphism

$$K(n)_*(X) \xrightarrow{\sim} K(n)^*(X).$$

We can think about this as a statement about X: if X is a nice space it satisfis a Poincaré duality with respect to Morava K-theory. We could also think about it as a statement about the category of $K(n)$-local spectra — it is not just an *additive* category, but it has some kind of fancier additivity where we can take a collection of morphisms indexed over a space and "add" or integrate the maps together. This theorem is also addressing the question that we started with — are there other constructions of Dijkgraaf–Witten theory? Yes, we can replace the height zero complex numbers by things of higher height, like Lubin–Tate spectra.

Question: Why is this true (in an easy example)?

If $X = K(H,2)$, the Hovey–Sadofsky theorem gives us a map

$$K(n)_*(X) \to K(n)^*(X).$$

There is an element $1 \in K(n)^0(X)$, and suppose we could find something, call it $y \in K(n)_0(X)$, mapping to it under the norm. Then if we had such a y, we would have that multiplication by y will induce a map from

$$\Theta : \mathrm{holim}\mathcal{L} \to \mathrm{hocolim}\mathcal{L}.$$

In classical ordinary homology this is called the cap product. The condition $\mathrm{Nm}(y) = 1$ is equivalent to the statement that Θ is inverse to the norm map.

Example 1.0.5. Let $X = BH$ for a finite p-group H, and $n = 1$. Then we have a map

$$K(n)_*(BH) \to K(n)^*(BH).$$

In height one, we know what these mean — these lift to characteristic zero, since $K(1) = \widehat{\mathrm{KU}}/p$. Complex K-theory of BH is described by the Atiyah–Segal completion theorem,

[1] For example $BH, B^2H, \ldots$ where H is finite abelian.

so we have that

$$\widehat{\mathrm{KU}}^0(BH) = \mathtt{Rep}(H)^\wedge$$
$$K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

So our map would be

$$\mathtt{Rep}(H)^\vee/p = K(1)_0(BH) \to K(1)^0(BH) = \mathtt{Rep}(H)/p.$$

By character theory, $\mathtt{Rep}(H) \otimes \mathbb{C}$ is the conjugation-invariant functions $H \to \mathbb{C}$, by sending V to its character χ_V. If we p-adically complete, we are really getting a map

$$\mathtt{Rep}(H)^\vee \to \mathtt{Rep}(H).$$

Rationally, everything is computable, and we can compute that it is an isomorphism. We can study the inverse isomorphism then

$$\mathbb{Q} \otimes \mathtt{Rep}(H)^\vee \leftarrow \mathbb{Q} \otimes \mathtt{Rep}(H).$$

Over $\mathbb{C}$, this bilinear form is given by $V, W \mapsto \frac{1}{|H|} \sum_{h \in H} \chi_V(h)\chi_W(h)$. To know that this isomorphism exists integrally and not rationally, we need to check this value is an integer. But we can rewrite this as

$$\begin{aligned} \frac{1}{|H|} \sum_{h \in H} \chi_V(h)\chi_W(h) &= \frac{1}{|H|} \sum_{h \in H} \chi_{V \otimes W}(h) \\ &= \dim_{\mathbb{C}} (V \otimes W)^H . \end{aligned}$$

So this is a sketch of the proof of the Hovey–Sadofsky theorem in height one.