THE EVOLUTION OF ENUMERATIVE GEOMETRY: A NARRATIVE FROM CLASSICAL PROBLEMS TO ENRICHED INVARIANTS

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Abstract. Enumerative geometry, which is the art and science of counting geometric objects satisfying specific conditions, has seen a resurgence of activity in recent years due to an influx of new techniques that allows for *enriched* computations. This paper offers a historical survey of enumerative geometry, starting with its classical origins and real counterparts, to the new frontier of quadratic enrichment. We include a brief survey of the paradigm shift initiated by Gromov-Witten theory, whose impact can be seen in recent results in quadratically enriched enumerative geometry. Finally, we conclude with an overview of emerging directions including random and equivariant enumerative geometry. By weaving these strands into a single narrative, we provide a panoramic overview of the field's evolution and its ongoing transformation.

1. Introduction

Enumerative geometry is, very roughly speaking, the study of counting geometric things. When a mathematician asks "how many X satisfy property Y" they are often met with the answers zero or infinitely many. Enumerative geometry deals with questions in the liminal space between these, where we have a finite amount of data to latch onto and study. Infamous examples in this area include the Apollonian problem of finding the eight circles tangent to three on the plane, Salmon and Cayley's computation that there are 27 lines on a smooth cubic surface, and Kontsevich's recursive formula for the finite numbers N_d of degree d rational curves interpolating $3d-1$ points on the projective plane.

The power of enumerative geometry comes from the statement of *conservation of number*, which says that the answer to the problem is well-defined and independent of initial parameters, provided the parameters are generically chosen. For example any two smooth cubic surfaces have 27 lines, or very pedantically, the set of lines on any smooth cubic surface has cardinality 27. When we ask a specific enumerative geometry problem, e.g. finding the flexes on a smooth planar cubic, we are naturally handed a *finite set* of solutions, and conservation of number says that given two instances of the problem, the finite sets of solutions are in bijection. To that end we posit the following:

Pseudo-definition: We say a theory of enumerative geometry is *enriched* if its solutions are sets equipped with additional structure (called a *weight* or *mass*), and it has a conservation of number statement that the formal sums of these structures agree for any two generic instantiations of an enumerative problem.

In a sense classical enumerative geometry is already enriched, in that the definition of conservation of number floated above is a mild lie – solutions should be counted *with*

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multiplicity, and conservation of number states that the sums of the solutions, weighted by their multiplicity, is always conserved. In this note, we explore a few instances of enriched enumerative geometry, including:

- ▷ *Signed real enumerative geometry*: Here we work over the real numbers, and each solution is weighted by a sign $+$ or $-$. The sum of these signed contributions is conserved.
- ▷ *Gromov–Witten theory*: Here we are generally over the complex numbers, but often over other fields. Each solution is weighted by a finite group of automorphisms, and in general the sum of the reciprocals of the orders of these groups is conserved.
- ▷ *Quadratically enriched enumerative geometry*: We work over an arbitrary field k, and each solution is weighted by a (virtual) symmetric bilinear form over k. The solution is conserved in the group completion of the monoid of isomorphism classes of symmetric bilinear forms, which is the *Grothendieck–Witt ring* $\mathrm{GW}(k)$.
- ▷ *Random enumerative geometry*: We work over the real numbers, and instantiate a problem according to some notion of randomness. The finite set of solutions is instead replaced by an *expected value* of the number of solutions.[1]
- ▷ *Equivariant enumerative geometry*: We are again over the complex numbers, working over an object with some finite symmetry group G acting on the moduli space of potential solutions. Our solution sets are naturally G-sets, and conservation says that two sets of solutions are in G-equivariant bijection.

The purpose of these conference proceedings is intended to expose readers to the development of some of these ideas and how they flow together. A tremendous amount of detail, nearly all, is missing from these notes, so it is *not intended to be an technical introduction or a thorough historical introduction to any one of these collections of work.* As J.F. Adams once wrote, this work should be viewed as “an essay in machine appreciation; it is not intended to qualify the reader for a mechanic’s certificate” [Ada78]. In particular, this article should be supplemented by the many cited references for anyone interested in legitimately learning the mathematics discussed here. To that end, we have omitted certain definitions and results and skipped certain parts of historical narrative, instead focusing on specific things according to our own interests and within the context of this conference proceedings. Nevertheless we hope this might serve as interesting reading to anyone who has ever heard the phrase “enriched enumerative geometry” and wondered what it was all about.

Acknowledgements: to do

2. Classical enumerative geometry

While enumerative geometry is often said to have begun in antiquity with the work of the great Greek geometers, perhaps its two most important components (algebraic closure and conservation of number) did not crystallize until the 19th century. Throughout the Renaissance, original Greek texts saw direct Latin translations, rather than translation by way of the Arabic translations from the Islamic Golden Age centuries prior. This brought renewed focus to the study of classical Euclidean geometry, and led to the further development

[1]This doesn’t quite fit the pseudo-definition of enrichment above, but it fits nicely into the story we’re telling here.

of fields of mathematics including descriptive and projective geometry. Throughout the 19th century, the development of enumerative geometry is then inextricably intertwined with the so-called "gold era" of projective geometry. We will avoid discussing the development of projective geometry here, referring the reader instead to [DCG25], and focus on the emergence of enumerative methods out of it.

The resurrection of projective geometry in France at the beginning of the 19th century (demarcated by Coolidge as beginning with Monge's 1799 text *Géométrie descriptive*) brought with it a newfound appreciation for the geometry of antiquity. At École Polytechnique, Monge spearheaded the research group on descriptive geometry, graduating such infamous students as Brianchon, Lacroix, and, perhaps most infamously, Poncelet. Monge held this post until he was ousted by Laplace, who detested Monge (and devalued descriptive geometry by extension) in 1815. Jean-Victor Poncelet, who studied at École Polytechnique from 1807—1810, first posited what we view as the fundamental ingredient to many enriched perspectives on enumerative geometry, which was his *principle of continuity of number*. This principle held that enumerative counts should be invariant under small parametrized changes. We will explain by example.

Problem 2.1 (The Circles of Apollonius)**.** Given three circles on the plane, how many are tangent to all three?[2]

The answer to this problem (provided we interpret circles and planes in a projective way over the complex numbers, which the reader may correctly object is ahistorical) is eight. In this context, we may interpret Poncelet's continuity of number as saying the following: given three initial circles and the eight tangent circles, by slightly perturbing the equation of any of our initial circles, the tangent circles grow or shrink or translate in order to accomodate tangency, but no new solutions are created and no existing solutions destroyed. We might be invited to think about this as homotopy invariance, and indeed this can be reinterpreted as the statement that a general algebraic section of $\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}$ is transverse to the zero section at eight points, and this fact is invariant under a small perturbation of the section.

Poncelet's phrasing of his principle was vague, and the machinery needed to make it precise was not yet available. Against the backdrop of the military and civil obligations of mathematicians in Napoleonic France, algebra was decidedly concrete and focused on computations, pitting Poncelet against mathematicians like Cauchy. This dichotomy, between symbolic manipulation and concrete computation, "echoed decades of infighting amongst the protagonists of the early development of enumerative geometry" [Mic23]. Consequently the history of classical enumerative geometry is plagued by vague explanations, governed by manipulations of symbols which often took a century to explain. This is also apparent in the work of Michel Chasles, whose *theory of characteristics* was similarly profound, resolving deep errors and ideas in enumerative geometry, but lacked rigorous justification, at least by modern standards.

Problem 2.2. Given i points and $5-i$ lines on the plane, how many conics pass through these points and are tangent to these lines?

[2]This problem is attributed to Apollonius of Perga (who Geminus of Rhodes nicknamed the *excellent geometer*), although no original writing on this problem exists. The attribution comes from nearly half a millennium later, in work of Pappus of Alexandria, which was resurrected and popularized by Viète [Viè00].

Working with $\mathbb{P}^5$ as a space of conics, Bézout's theorem tells us to compute the degree of the hypersurfaces corresponding to passing through a point or being tangent to a line, which are one and two, respectively. The answer then appears to be 2^{5-i}, a statement implied by de Jonquière's prior work [dJ66]. Appealing to pole-polar duality, we see this cannot be correct, as it implies in the dual projective plane that there are 32 conics passing through five points. The problem with this is of course a defect in $\mathbb{P}^5$ as a moduli space for conics – it contains a locus of doubled lines and line pairs which contributes to any results that leverage Bézout's theorem on this space.

Chasles' insight (see [Kle80]) was that, given a family of conics, the condition μ of passing through a point and ν of being tangent to a line, should both be considered, and any condition on conics is a linear combination of these. The number of conics passing through i points and tangent to $5-i$ lines can then be computed as $2^{\min\{i,5-i\}}$. A very similar argument counts the smooth conics tangent to five given conics as 3264, correcting an erroneous computation of Steiner which fell victim to the same issue creating the number 32 above. In contemporary language, Chasles computations can be thought of as occurring in the Chow ring of the moduli space of complete conics, where μ and ν are the pullback of the hyperplane class to the projective plane and its dual, respectively [EH16, §8.2].

Chasles' theory of characteristics was a historical precursor to Schubert's "calculus of conditions," which were justified by his "principle of conservation of number," which held that solutions to enumerative problems were conserved under perturbations of initial parameters. This was almost identical to Poncelet's prior observation but admitted subtle differences in both scope and application. Poncelet's principle of conservation of number was an attempt to begin an axiomatization of pure geometry which incorporated this idea of continuity, which had been missing from the geometry of the ancients, in the hope that this would return geometry to the forefront on mathematics anongside algebra and analysis. His principle could be thought of, both spiritually and in fact historically, as analogous to the introduction of imaginary numbers in algebra [Che24]. Schubert's principle, on the other hand, was a purely computational tool — as the desired solution is independent of a choice of initial parameters, once may *specialize* to a choice of parameters and then carry out a single computation. This distillation of concept to computation was hugely influential but not universally loved. While Chasles and his student Zeuthen relied on pictures and explanations to justify their computations (the context for which was still ambiguous as mentioned), Schubert focused purely on computation, neglecting intuition, much to the chagrin of Zeuthen, Halphen, Study, Kohn, and others. The shaky bedrock upon which this theory was built led to Hilbert including *Schubert calculus* as one of his fifteen eponymous problems at the turn of the century.

2.1. **Resolving Hilbert's 15th problem.** Schubert's calculus of conditions, Chasles's theory of characteristics, and much of 19th century enumerative geometry, lacked a rigorous foundation. This framework emerged from work of van der Waerden in 1929 [vdW30], leveraging simplicial cohomology, a tool coming from algebraic topology, and work of Lefschetz [Kle76]. The idea was the following: rather than study some algebraic locus (say, a hypersurface representing some geometric condition) we should instead investigate its associated cohomology class. The condition of a circle being tangent to a fixed circle, for instance, is the class $2h \in H^2(\mathbb{CP}^3;\mathbb{Z})$, and changing the fixed circle doesn't affect the cohomology class.

To carry out a computation which combines conditions, we want to compute the number of points in the intersection, which can be done leveraging the graded ring structure on singular cohomology. To compute the number eight in the Circles of Apollonius problem for instance, we compute that the number of circles tangent to three of them is $2^3h^3 \in H^6(\mathbb{CP}^3;\mathbb{Z})$, which after integrating is equal to 8. One may say that the Chow ring is the home for Schubert's computations, as it is the home for intersection theory of algebraic varieties, however most of the spaces over which these classical computations are carried out were cellular varieties, hence the cycle class map from the Chow ring to singular cohomology is an isomorphism, and the computation is agnostic as to whether we view it as occurring in Chow groups or cohomology.[3] Viewing these computations as occurring in singular cohomology is not only historically sound, it is pedagogically useful here – the observation that an enumerative computation can be carried out via a characteristic class is crucial in efforts to enrich classical enumerative geometry to other contexts. It also provides an elegant resolution to the principle of continuity as hinted at earlier, that a characteristic class computed by a section depends only on the homotopy class of the section – i.e. it is invariant under *continuous* deformations, not just algebraic ones.

2.2. **Parameterized problems: towards thinking about stacks.** Following the development of theories like singular cohomology, the difference between counting solutions to an enumerative problem and computing a homology class became more or less indistinguishable. Indeed many enumerative problems whose rigorous solutions eluded 19th century geometers can be now thought of as easy computations in the cohomology ring of a certain moduli space. Schubert calculus has a very different meaning today than it did a century ago; instead of referring to the procedure of solving and enumerating linear intersection problems it now means (to many) the study of cohomology of homogeneous spaces under a simple Lie group.

Many moduli spaces one wishes to study, however, cannot be accurately captured as an algebraic variety or manifold. Often this is due to the moduli space attempting to describe objects which are *parametrized* in some sense. A motivating problem is the following classical interpolation problem:

Problem 2.3. How many rational degree d curves pass through $3d-1$ points in general position on $\mathbb{P}^2$?

The development of *stacks* in the 1960's gave a mathematical foundation for the moduli needed in these sorts of questions, leading to a resurgence of enumerative geometry as well as a shift of focus towards Gromov-Witten theory and related theories, which we discuss more in Section 4.

3. Real counts in enumerative geometry

The study of algebraic geometry over the reals is much more complicated than over the complex numbers due to the lack of algebraic closure. In enumerative algebraic geometry

[3] Viewing computations as Chow-valued can mislead us in attempts to enrich enumerative computations to other settings — for instance attempting to build equivariant enumerative geometry using equivariant Chow groups produces answers valued in the complex representation ring of the group, which is not sufficient to recover the G-action on the solutions themselves (see for instance [Bra25b, Example 5.22]).

this is felt in the failure of conservation of number. Consider for instance the classical count of 27 lines on a smooth complex cubic surface [Cay49].

Theorem 3.1 ([Sch58])**.** A real smooth cubic surface can have 3, 7, 15, or 27 real lines.

One perspective to take on this is that the moduli of smooth complex cubic surfaces is connected, and the incidence variety of cubic surfaces equipped with a line ramifies only over the singular cubic surfaces, hence any smooth cubic surface contains the same number of lines over $\mathbb{C}$. Over the reals, the moduli of smooth real cubic surfaces is disconnected, having five connected components corresponding to the number of real lines and real tritangents [Seg42, §23].

Indeed this type of behavior is the primary complicating factor in studying smooth objects in real algebraic geometry — over the complex numbers a codimension one discriminant locus has real codimension two, whereas over the reals it has real codimension one, chopping up a space into many connected components. As an example to keep in mind when contemplating complexity, every smooth planar curve of degree d over $\mathbb{C}$ looks identical to a topologist, whereas over the real numbers the number of connected components in the moduli space of real smooth planar curves of degree d grows exponentially in d^2 [OK03]; the classification of their possible shapes is part of Hilbert's 16th problem, and is completely open for $d \geq 8$.

Perhaps one of the earliest cases of real counts to enumerative problems is one we have already seen – this is the Circles of Apollonius (Problem 2.1). The problem of counting circles tangent to three over the reals was well-studied by Viète at the turn of the 17th century in his infamous book *Apollonius Gallus* [Viè00], after which it remained a consistent topic of study, enjoying solutions by Newton, Gergonne, and many others.[4] For the purposes of this narrative, however, we view this topic as lying more firmly in classical Euclidean and projective geometry than in enumerative algebraic geometry.

In the late 1930's, Beniamino Segre (a former student of Corrado Segre) had been studying the geometry and topology of real algebraic varieties, particularly intersections, via their limiting behavior. After fleeing fascism in Italy, he was interned in England, and it was during this time he worked on perhaps his most well-known work, which was his treatise on non-singular cubic surfaces [Seg42]. This book begins by studying the behavior of lines under a degeneration from a cubic surface to a union of three planes. This topological perspective, in contrast to the more algebraic one, allows for a more careful geometric analysis of cubic surfaces over the real numbers, as highlighted by Zariski in his review of Segre's book. In particular, given a real line on a cubic surface, and a hyperplane containing that line, it cuts the cubic surface at a residual conic, intersecting the line at two points (Segre called these *parabolic points*). Interchanging pairs of parabolic points gives an involution of any real line, and lines are called *hyperbolic* or *elliptic* corresponding to whether the fixed locus of the involution is real or complex, respectively [Seg42, §27]. This leads to the following theorem.

Theorem 3.2. On a real smooth cubic surface, we have that

$$\#\{\text{hyperbolic lines}\} - \#\{\text{elliptic lines}\} = 3. \tag{3.1}$$

This can be proven in a number of ways, for instance Benedetti and Silhol proved that a real cubic surface inherits a Pin^- structure whose modulo four reduction can distinguish

[4] For an overview of solutions to this problem see [Cou61].

the two types of lines at the level of homology [BS95]. This can be also be proven from the perspective of open Gromov-Witten theory, see for instance [Sol06, HS12]. A perspective we discuss here is what one may call *absolute Euler classes*.

Definition 3.3 ([OT14])**.** Let X be a closed real manifold of dimension n, and $V \to X$ a real topological vector bundle which is *relatively oriented*[5] in the sense that the line bundle $\mathrm{Hom}(\det TX, \det V)$ admits a trivialization θ. Then V admits an Euler class $e(V,\theta)$ which depends on θ up to a sign. We call $|e(V,\theta)|$ the *absolute Euler class* of V.

Analogous to how a Chern class of a corank zero bundle is Poincaré dual to the vanishing locus of a generic section, the absolute Euler class provides a *signed count* of the zeros of a generic section. For example the absolute Euler class of $\mathrm{Sym}^3\mathcal{S}^* \to \mathrm{Gr}_{\mathbb{R}}(2,4)$ is equal to 3, and the local index at a line is equal to $+1$ or -1 corresponding to whether the line is hyperbolic or elliptic, respectively. From this perspective Equation (3.1) can be reinterpreted as a formula arising from an absolute Euler class, and it becomes clear how to generalize this to other settings. In [OT14, FK13] the authors compute the absolute Euler class for the analogous symmetric bundles over the Grassmannian of lines in higher-dimensional space, proving for instance a lower bound of $(2n-1)!!$ for the number of real lines on a general smooth real hypersurface of degree $2n-1$ in $\mathbb{P}^{n+1}$. A beautiful analog of this result for the 240 (-1)-curves on a real degree one del Pezzo can be found in [FK21], and a spiritually similar result for bitangents to real algebraic curves can be found in [BBG24].

3.1. **Signed real enumerative geometry.** The lesson to be learned from Segre's work is the following: while conservation of number may break over $\mathbb{R}$, a certain *signed* count of solutions may remain invariant, and this sign can encode beautiful information about the local geometry. This is the jumping off point for quadratically enriched enumerative geometry, which we touch on more in Section 5. Before doing this, we discuss another key appearance of this idea, which is that of *Welschinger invariants*.

Recall that the number N_d of complex rational curves interpolating $3d-1$ points on $\mathbb{P}^2$ in Problem 2.3 is invariant of the position of the points, provided they are in general position. As we might expect, this same statement fails in the context of real curves – for instance through 8 generic points in $\mathbb{R}\mathrm{P}^2$ there can be 8, 10, or 12 real rational cubics interpolating them [DK00]. Groundbreaking work of Welschinger tells us we should count the cubics interpolating these points weighted by a sign. More explicitly:

Definition 3.4. If C is a rational curve of degree d, we define its *Welschinger invariant* to be $\mathrm{Wel}(C) = (-1)^n$, where n is the number of isolated points of C.[6] This is also sometimes called the *mass* of the curve.

The remarkable theorem is the following:

Theorem 3.5 ([Wel06])**.** For $3d-1 = n_1 + 2n_2$ points on $\mathbb{R}\mathrm{P}^2$, n_1 of them real and n_2 pairs of complex conjugate points, we have that the quantity

[5] C.f. Definition 5.2, note that for real line bundles being a square is the same as being trivial.

[6] Recall a real point on a real rational curve is said to be *isolated* if its directions of tangency form a complex conjugate pair.

$$W_{d,n_1} = \sum_{\substack{C \text{ deg } d \\ \text{through these points}}} \mathrm{Wel}(C).$$

is independent of the choice of points (provided they are chosen generically).

4. The Emergence of Gromov–Witten theory

A survey of enumerative geometry would be remiss to omit a discussion Gromov–Witten theory, currently one of the most active areas of enumerative geometry. The goal of this section is a departure from prior sections; this section primarily serves to introduce high-level aspects of the subject that are most relevant to current trends in quadratically enriched enumerative geometry, discussed in Section 5. This section does not seek to serve as a comprehensive historical review or a complete technical introduction to the subject, which would be beyond the scope of this work. Considering this warning to the reader, we include technical references introducing the subject in varying levels of detail [PT14, FP97, KV07, Ros14].

This section's perspective on Gromov–Witten theory is fundamentally motivated by a classical question in algebraic geometry, which has appeared in various forms throughout this article:

Problem 4.1. How many rational curves lie on a smooth complex projective variety?

Given such a variety X, a rational degree d curve on X is a map $f\colon \mathbb{CP}^1 \to X$ given by $f([s:t]) = [f_0(s,t) : f_1(s,t) : f_2(s,t)]$ where the f_i are degree d homogeneous polynomials. This is question is simple to state and notoriously difficult to answer.

The starting point for Problem 4.1 is for $\mathbb{CP}^2$, which is the question of how many degree d rational plane curves exist for any d. With no additional restrictions, there are infinitely many for each d. We may ask the same question after imposing the condition that we seek to count rational curves that pass through a prescribed number of points in general position in $\mathbb{CP}^2$. A dimension argument shows that the space of degree d rational, nodal plane curves is $3d-1$ dimensional, which leads us to computing the finite solutions N_d to Problem 2.3. Many well-known variations on this problem are considered classical, and they have historically captured the attention of algebraic geometers independent of developments Gromov–Witten theory. A celebrated result of Caporaso-Harris leverages the geometry of the Severi varieties parameterizing nodal plane curves of degree d and geometric genus g to give a recursive formula enumerating the number of nodal plane curves of degree d passing through an appropriate number of points [CH98].

One of the early successes of Gromov–Witten theory was the proof of a recursive formula in d for the number of degree d rational plane curves, denoted N_d, given by Kontsevich and Manin [KM94, 5.2.1]. In rapid succession, Gromov–Witten theory was developed symplectically and algebraically [Kon95, RT94, RT95, MS12, BM96, Beh97], leading to a flurry of results in enumerative geometry that had previously been unattainable using existing methods. Though these results are too numerous to name individually, we mention a few [Giv96, Vak00, BL00].

Given a a smooth, projective, complex algebraic variety X and $\beta \in H_*(X,\mathbb{Z})$, Gromov–Witten theory studies stable maps from nodal curves of arbitrary genus to X. The moduli stack of genus g, n-marked stable maps, denoted $\overline{M}_{g,n}(X,\beta)$ is a proper Deligne-Mumford stack of finite type [Kon95, Section 1.3.1 p.3]. Note when $X = \mathbb{CP}^2$, $\beta = d\cdot\ell$ where $d \geq 1$ and ℓ is the class of a line, and $g = 0$, $\overline{M}_{0,3d-1}(\mathbb{CP}^2, d\cdot\ell)$ is the moduli stack parameterizing nodal, rational plane curves of degree d passing through $3d-1$ points.

Defining Gromov-Witten invariants is best done in cases, we define only the case for counting stable maps passing through points. When X is convex in the sense of [KM94], for example $X = \mathbb{CP}^r$ for some r, $\overline{M}_{g,n}(X,\beta)$ is smooth [Kon95, 1.3.2]. For each $1 \leq i \leq n$ there is an evaluation map

$$\mathrm{ev}_i\colon \overline{M}_{g,n}(X,\beta) \to X, \quad (C \xrightarrow{f} X, p_1, \ldots, p_n) \mapsto f(p_i).$$

Given $p_1,\ldots,p_n$ be n points of X and their Poincaré duals by $\gamma_i \in H^{n_i}(X)$, the cohomology class $\mathrm{ev}_i^*\gamma_i \in H^*(\overline{M}_{g,n}(X,\beta))$ represents the class of stable maps whose image passes through V_i for each i. The n-fold cup product $\mathrm{ev}_1^*\gamma_1 \cup \cdots \cup \mathrm{ev}_n^*\gamma_n$ in $H^*(\overline{M}_{g,n}(X,\beta))$ represents the class of stable maps whose image passes through V_i for *all* $1 \leq i \leq n$. Let X be a smooth, complex projective variety which is convex in the sense of [KM94]. The *Gromov–Witten invariant counting genus g, degree β stable maps passing through* $V_1,\ldots,V_n$ is the degree

$$\langle p_1,\ldots,p_n\rangle^X_{g,\beta} := \deg\left([\overline{M}_{g,n}(X,\beta)] \cap (\cup_{i=1}^n \mathrm{ev}_i^*\gamma_i)\right). \tag{4.1}$$

There are more general ways of defining Gromov–Witten invariants, but this formulation of the definition is most relevant for Section 5.

Remark 4.2. Of course, an immediate consideration in the definition above arises when X is not convex, in which case $\overline{M}_{g,n}(X,\beta)$ is not smooth. In these instances, we cannot hope to take the degree of $\mathrm{ev}_1^*\gamma_1 \cup \cdots \cup \mathrm{ev}_n^*\gamma_n$ by capping with the fundamental class $[\overline{M}_{g,n}(X,\beta)]$ and pushing forward to a point, as $\overline{M}_{g,n}(X,\beta)$ is singular and has irreducible components of varying dimensions. Beautiful work of Li-Tian [LT98] and shortly later Behrend-Fantechi [BF97] constructs a virtual fundamental class for finite type Deligne-Mumford stacks. In particular, there is a virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}_{E^\bullet}$ of the expected dimension given a perfect obstruction theory $E^\bullet$ for $\overline{M}_{g,n}(X,\beta)$ [Beh97]. The virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}_{E^\bullet}$ can be used to define Gromov–Witten invariants in general.

Based on the explanations given thus far, which are far from thorough, the definition of Gromov-Witten invariants leaves much to the imagination in terms of computational feasibility. Stunningly, powerful tools make these invariants computable in a number of cases of interest. We give two examples that will be relevant in Section 5.

Gromov–Witten invariants of blow-ups of projective space. A natural course of study following the curve counting results of Caporaso-Harris [CH98] and Kontsevich-Manin [KM94] on enumerative curve counts for $\mathbb{P}^n$ is the study of Gromov–Witten invariants of smooth, projective rational surfaces, i.e., surfaces which are deformation equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ at finitely many points $x_1,\ldots,x_r$. Gromov–Witten invariants of blow-ups of projective space have been studied by Göttsche-Pandharipande and Gathmann [GP98, Gat01] amongst others; Gathmann studies the case of blow-ups of $\mathbb{P}^n$. In particular, such Gromov–Witten invariants can be interpreted in terms of counts of rational curves in $\mathbb{P}^2$ with specified tangent multiplicities at the points $x_1,\ldots,x_r$.

Equivariant localization. Building on the equivariant localization results of Atiyah-Bott in equivariant cohomology [AB84] and Edidin-Graham in equivariant Chow groups

[EG98], Graber-Pandharipande give a $\mathbb{C}^*$-equivariant virtual localization formula for a $\mathbb{C}^*$-equivariant scheme X with a $\mathbb{C}^*$-equivariant perfect obstruction theory $E^\bullet$, which expresses $[X]^{\mathrm{vir}}_{E^\bullet}$ in terms of the virtual fundamental classes and Euler numbers of the $\mathbb{C}^*$-fixed point loci [GP99]. Graber-Pandharipande show a consequence of this is a virtual localization formula for $\overline{M}_{g,n}(X,\beta)$, which can be leveraged to express Gromov–Witten invariants $X = \mathbb{CP}^r$ as a sum over graphs corresponding to the $\mathbb{C}^*$-fixed point loci [GP99].

These example cases motivate the work to-date on counting rational curves in quadratically enriched enumerative geometry, covered in Section 5. To conclude this section, we give a brief overview of the specific historical developments since the advent of Gromov-Witten theory that most influence current work in quadratically enriched enumerative geometry today.

While powerful and quite general, Gromov–Witten invariants are not perfect. They are often not enumerative, for example in $g > 0$ cases for 3-folds, and they are typically rational numbers rather than integers to account for automorphisms of stable maps. One of the most prominent paths toward resolving some of the difficulties of Gromov–Witten invariants is the introduction of Donaldson–Thomas invariants, which seek to count stable sheaves in a given curve class on a Calabi-Yau 3-fold [DT98, Tho00]. The moduli space of such sheaves has a perfect obstruction theory in the sense of Behrend-Fantechi [BF97] that is in fact symmetric [Beh09, BF08]. Donaldson–Thomas invariants can be defined by integrating over the associated virtual fundamental class of the moduli of stable sheaves.

While Donaldson–Thomas invariants have their own drawbacks, see [PT14] for a discussion, they are integral and can be computed motivically using the Behrend constructible function on the moduli space [Beh09]. Of course, the comparison between Gromov–Witten invariants and Donaldson–Thomas invariants is natural to explore, with several groundbreaking results on their connections [MNOP06a, MNOP06b, BP08, OP10, Par23]. See Thomas-Pandaripande [PT14] for a description of curve counting theories more generally, beautifully elucidating how Gromov–Witten and Donaldson–Thomas invariants fit into broader enumerative theories.

As Section 5.3 focuses on quadratic enrichments of Gromov–Witten invariants, we end this section by noting that enriched Gromov–Witten invariants have appeared in existing work. For example, celebrated work of Y-P Lee constructs Gromov–Witten invariants in K-theory via integration over a virtual structure sheaf [Lee04], and Guéré recently recently studied K-theoretic Gromov–Witten invariants using virtual localization for a finite group action [Gué23].

5. Quadratically enriched enumerative geometry

At the beginning of the 21st century, *motivic homotopy theory* emerged as a popular exciting new direction in mathematics, rising to the forefront after Voevodsky's Fields Medal-winning resolution of the Bloch-Kato conjectures. Following work of Morel, it was understood that motivic spaces over a field k admit a *quadratic Euler characteristic* valued in the Grothendieck–Witt ring $\mathrm{GW}(k)$. Following Marc Hoyois' thesis, which explored further applications of this $\mathbb{A}^1$-enhancement of the Euler characteristic, Marc Levine, and simultaneously Jesse Kass and Kirsten Wickelgren, began exploring potential ways to build an enumerative geometry whose answers take values in $\mathrm{GW}(k)$ rather than $\mathbb{Z}$. This is now called $\mathbb{A}^1$-*enumerative geometry* or sometimes *quadratically enriched enumerative geometry.*

Definition 5.1. The *Grothendieck–Witt ring* $\mathrm{GW}(k)$ of a field k is the group completion of the semi-ring of isomorphism classes of non-degenerate symmetric bilinear forms over k. Explicitly it is generated by the rank one forms

$$\begin{aligned}\langle a \rangle : k \times k &\to k \\ (x, y) &\mapsto xay.\end{aligned}$$

In 2000, Barge and Morel developed a theory of *oriented Chow groups* (also called *Chow–Witt groups*) , which are twisted by line bundles over the input scheme, and which have a natural home in the world of motivic homotopy theory. These should be thought of as an enhancement of Chow groups, in that they are generated by cycles equipped with some extra "orientation data" which we neglect to define precisely here. The properties of these groups were established in further detail by Fasel and Srinivas [Fas08, FS09], and it is natural to think of them as providing an enhanced setting for intersection theory, where the degree takes values in the Grothendieck–Witt ring. This is the primary tool leveraged by Levine to explore quadratically enriched enumerative geometry [Lev20].

Analogous to how the top Chern class provides an enumerative count of the zeros of a section of a corank zero vector bundle along a smooth compact manifold (or how the *absolute Euler class* does the same for relatively oriented bundles in the real setting), one has an *Euler class* for relatively oriented vector bundles in the motivic setting. More precisely:

Definition 5.2. Let X be a smooth proper k-scheme of dimension n, and let $V \to X$ be an algebraic vector bundle of rank n, which is *relatively oriented*, in the sense that there is an isomorphism of line bundles

$$\mathrm{Hom}(\det TX, \det V) \cong \mathcal{L}^{\otimes 2} \tag{5.1}$$

for some line bundle $\mathcal{L} \to X$. Then this bundle admits a well-defined *Euler number* $n(V) \in \mathrm{GW}(k)$.

The remarkable result is that this Euler number provides a quadratically enriched count of the zeros of an algebraic section of the bundle, in a way that is independent of the choice of section. This result can be found in [BW23, 1.1], but is the culmination of a lot of work towards establishing a rigorous theory of motivic Euler classes [BM00, Mor12, AF16, Lev20, LR20, LKW21].

A seminal application of these techniques is the following quadratically enriched count of 27 lines on a cubic surface.

Theorem 5.3 ([LKW21])**.** We have that

$$n\left(\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}_k(2,4)\right) = 15\,\langle 1 \rangle + 12\,\langle -1 \rangle \in \mathrm{GW}(k).$$

The rank of this form is 27, recovering the classical count of 27 lines on a smooth cubic surface. The signature is 3, which recovers Segre's theorem discussed in Section 3. Over finite fields, for examples, this reveals new constraints on the possible fields of definition for hyperbolic and elliptic lines on cubic surfaces. This result has been generalized to provide a count of lines on symmetric hypersurfaces in general (see [Pau22], [Lev19, §8], and [BW23, §6.1]).

5.1. **Computing local indices.** One of the core problems in quadratically enriched enumerative geometry is computing local indices. As the Euler class is the sum of the local indices of the zeros of a section, the local index should be read as some local geometric data that a single solution contributes to the overall count (think: a line being hyperbolic versus elliptic).

In the setting of Definition 5.2, the local index is computed as an $\mathbb{A}^1$-*Brouwer degree* of the intrinsic derivative of the section $\sigma\colon X \to V$ around an isolated zero. One can pass to affine charts (see [LKW21, Lemma 19] for precise details), hence the problem reduces to producing an element in $\mathrm{GW}(k)$ from an endomorphism of affine n-space with an isolated zero at the origin. In [KW19], the authors argued that the $\mathbb{A}^1$-Brouwer degree at a rational point generalizes the *Eisenbud–Khimshiashvili–Levine signature formula* over the reals [EL77, Him77]. This form can be interpreted in a number of ways – it is a quadratic duality pairing explored by Scheja and Storch for complete intersections [SS75], or can be thought of more generally as a trace form arising from coherent duality. The so-called *EKL form* is very computable, and was shortly generalized to local degrees at points with residue fields finite separable over the base [BBM$^+$21] and finally points with arbitrary residue field [BMP23]. It can now be computed explicitly in `Macaulay2` [BBE$^+$24].

5.2. **More results in quadratically enriched enumerative geometry.** Since the seminal work of Kass-Wickelgren and Levine, a tremendous amount of progress has been made leveraging these techniques to provide enriched counts of classical questions in enumerative geometry. We highlight a few of these which are not mentioned elsewhere in this paper.

A quadratically enriched Schubert calculus was developed by studying the Chow–Witt groups of Grassmannians and flag varieties [Wen24, HMW24]. Many of these results are obtained as an $\mathbb{A}^1$ degree or Euler number. Bézout's theorem has seen a quadratic enrichment [McK21], as has the Circles of Apollonius problem explored in Problem 2.1 [McK22]. An enriched count of bitangents to a planar curve has been developed [LV21] and was further explored in [KM24]. An enriched count of lines meeting four lines in three-space [SW19] and higher-dimensional analogues of this [Bra25a] have been developed. Arithmetic inflection for linear series along curves has been enriched [CDH23, CDL$^+$24]. Other results include [AK25, DGGM23, Dar22, EW23b, EMP25, KP24, Kum23, Mur25]. An arithmetic Yau-Zaslow formula was established in [PP25], which is an exampple of a quadratically enriched rational curve count in a linear series. While this is different from the quadratically enriched rational curve counts through given point conditions, this naturally leads to the next subsection.

5.3. **Quadratically enriched curve counting.** A problem that has appeared throughout this article culminates in the following question:

Problem 5.4. Let k be a field. Given general points $p_1, \ldots, p_r$ of $\mathbb{P}^2_k$ such that all residue fields $k(p_i)$ are separable over k, how many degree d rational plane curves pass through $p_1, \ldots, p_r$?

A quadratically enricehd answer to this question would give a quadratic form whose rank is equal to the Gromov–Witten invariant N_d, recovering the complex count of rational degree d plane curves when $r = 3d - 1$, and whose signature is equal to the Welschinger invariant W_d, recovering the signed count of real degree d plane curves. Recent work of J. Kass, M. Levine, J. Solomon, and K. Wickelgren introduce precisely these quadratically enriched counts of rational curves in a certain divisor class on a del Pezzo surfaces of degree ≥ 4 over perfect fields of characteristic $\neq 2, 3$ [KLSW23a]. This builds on prior work of M. Levine defining quadratically enriched Welschinger invariants [Lev18]. Beyond recovering real and complex rational curve counts, their work proposes a definition of rational curve counts over other fields.

Let X denote a del Pezzo surface of degree ≥ 4 over a perfect field of characteristic $\neq 2, 3$. Using the notation of Section 4, let $\overline{M}_{0,n}(X, d)$ denote the moduli space of genus 0 stable maps of degree d to X, and let

$$\mathrm{ev}\colon \overline{M}_{0,n}(X, d) \to X^n, \quad (C \xrightarrow{f} X, p_1, \ldots, p_n) \mapsto (f(p_1), \ldots, f(p_n))$$

denote the total evaluation map. Kass-Levine-Solomon-Wickelgren [KLSW23a] define the quadratically enriched rational curve counts to be

$$\mathrm{N}_{X,d,\sigma} := \deg^{\mathbb{A}^1}(\mathrm{ev}),$$

analogously to the definition in Equation (4.1) defining Gromov–Witten invariants as a degree. This definition relies on relative orientability of the total evaluation map, which is shown to be relatively oriented away from a high codimension locus in X in [KLSW23b]. In general, $\deg^{\mathbb{A}^1}(ev)$ is a quadratic form over X^n, not over k, but a unique class in $\mathrm{GW}(k)$ can be obtained when X is $\mathbb{A}^1$-connected, for example when X is a smooth proper rational surface.

Remarkably, the quadratic forms $\mathrm{N}_{X,d,\sigma}$ depend only on the list of separable field extensions over k determined by the chosen points, $\{k(p_1), \ldots, k(p_r)\}$ [KLSW23a, Section 8]. Equally remarkably, $\mathrm{N}_{X,d,\sigma}$ can be computed as a sum of contributions of each curve, which are quadratic enrichments of the Welschinger mass (Definition 3.4), that take into account the fields of definition of branches of the curve at nodal points. In some cases, the numbers $\mathrm{N}_{X,d,\sigma}$ can be computed using the $\mathbb{A}^1$-Euler number of a certain oriented vector bundle over X, see [KLSW23a, 9.1].

This work has generated significant activity since its inception. This includes connections with tropically enriched enumerative geometry, which is discussed in the next subsection. See also [BW25b, CW24].

Importantly, this work occurs alongside the study of quadratically enriched Donaldson–Thomas invariants. Given a smooth, projective 3-fold X, Viergever and Viergever-Levine define quadratically enriched Donaldson–Thomas invariants using a perfect obstruction theory for the Hilbert scheme $\mathrm{Hilb}^n(X)$ of ideal sheaves of length n with support of dimension 0 on X [Vie23, LV25]. There is an associated virtual fundamental class for $\mathrm{Hilb}^n(X)$ using the construction of [Lev21], which is used to define the enriched Donaldson–Thomas invariants. Viergever and Viergever-Levine leverage work of Levine on virtual localization and a relative orientation for $\mathrm{Hilb}^n(X)$ for computations, see [Lev22a, Lev22b, Lev23]. Work of Espreafico-Walcher in [EW23a] leverages the realization of $\mathrm{Hilb}^n(\mathbb{A}^3)$ as the critical locus of a function and the compactly supported $\mathbb{A}^1$-Euler characteristic to define quadratically enriched Donaldson–Thomas invariants for $\mathbb{A}^3$ in $\mathrm{GW}(k)$.

5.4. **Tropical and quadratic enumerative geometry.** A well-established direction in enumerative geometry (which we do not explore in these notes) is the use of methods from *tropical geometry* to carry out computations in classical enumerative geometry. A beautiful example of this is the *Mikhalkin correspondence*, which states that the number N_d of rational degree d curves through $3d-1$ generic points on $\mathbb{P}^2$ can be computed as a count of rational tropical curves with a given Newton polytope through $3d-1$ generically chosen points on the plane (computed with multiplicity) [Mik05]. Mikhalkin further proved an analogue for real rational curves interpolating points, which provides a tropical way to compute the signed count of these curves weighted by their Welschinger invariants.

With this and $\mathbb{A}^1$-enriched Welschinger invariants in mind, it is natural to ask whether these tropical methods developed by Mikhalkin can be given a quadratic enrichment in order to capture both the classical and real counts simultaneously. This leads to a beautiful sub-field of quadratically enriched enumerative geometry which can leverage tropical methods to compute enriched solutions [MPS23, JPMPR24, JPP25]. For an expository introduction to these ideas see [Pau24].

6. A few emergent directions

There are many exciting new directions in enumerative geometry, and far too many to do justice in this paper. Here we highlight two directions, one coming from the theory of *random algebraic geometry* and the other coming from equivariant mathematics.

6.1. **Random enumerative geometry.** Over the reals, conservation of number fails as previously discussed above, however we can still ask the following question: what is the *expected* number of real solutions? Here we discuss two ways to approach such a problem, one coming from probability theory and the other from Hodge theory and hyperbolic geometry.

The easiest place to see the failure of conservation of number (and hence the easiest place to establish a testbed for a random theory of enumerative geometry) is in the fundamental theorem of algebra — that a univariate polynomial $f \in \mathbb{R}[x]$ of degree n may fail to have n real roots, counted with multiplicity. Motivated by this question, Kac investigated the following question in the 1940's: what is the *expected number* of real roots of a randomly chosen real polynomial of some degree n? One first has to define what they mean by "random," and in Kac's work he assumes that the coefficients are distributed according to a normal distribution on $\mathbb{R}$. With this in mind he gives an exact formula for the expected value, together with an asymptotic of $\frac{2}{\pi}\log(n)$ [Kac42, Kac48]. An entirely different, although spiritually similar, approach to this problem came from the work of Stephen Oswald Rice, a researcher at Bell Labs working in signal processing and random noise [Ric44]. With some results, these results can be extended to provide formulas for random maps on manifolds [BL, §4] – these sorts of generalizations are called *Kac-Rice formulas.*

This flavor of question (what is the expected behavior of a randomly chosen algebraic object over the reals) leads to a new program of mathematics called *random algebraic geometry* [Ler]. A key application of these techniques is to leverage tools like the Kac-Rice formula to compute the expected number of real solutions to an enumerative problem – we may call this *random enumerative geometry.* A motivating result in this direction is the following:

Theorem 6.1 ([BLLP19])**.** The average number of real lines on a random real cubic surface is $6\sqrt{2}-3$.

This theorem is proven by choosing random sections of the symmetric bundle $\mathrm{Sym}^3\mathcal{S}^*$ over the Grassmannian of lines in $\mathbb{RP}^3$, then running the aforementioned machinery to compute the expected number of zeros. Implicit in this work is a choice of probability distribution, and the answer to the problem will very naturally depend on this choice. These methods have been used, among other applications, to study random Schubert calculus [BL20] and to extend these randomized questions away from the reals and toward the p-adics [AEML22].

Returning to lines on real cubic surfaces, an entirely different approach to the same problem above comes from analyzing the moduli of real cubic surfaces, and attempting to compare the volumes of the five connected components. A priori this is a poorly phrased problem, as there is no natural metric to compute volume on the moduli space of cubic surfaces (being, as it is, a quotient of a Zariski open subset of projective space by the action of an infinite group). Miraculously, we can leverage techniques from Hodge theory to endow it with a hyperbolic metric! More precisely, we can study the variation of Hodge structure on real cubic threefolds covering our real cubic surfaces, analyze their images under a period map, and finally compute the volume of each connected component of the moduli space via Vinberg's algorithm or similar techniques. For the specific problem of computing lines on real cubic surfaces, this was accomplished in groundbreaking work of Allcock-Carlson-Toledo [ACT10]. This program of mathematics has already seen wide-ranging applications, from the classification of real cubic fourfolds [FK10] to the study of real binary octics [Chu11].

As to the potential of both approaches for gaining intuition and revealing structure in real enumerative geometry, it seems we have only just scratched the surface of what is possible.

6.2. **Equivariant enumerative geometry.** In this last subsection, we abandon any pretense of modesty to discuss an ongoing program of work by the authors and others to leverage tools from equivariant homotopy theory in order to provide equivariantly enriched counts of classical questions in enumerative geometry. Given an enumerative problem and some symmetry, it is natural to ask how the symmetry group interacts with the solutions to the problem.[7]

When G is a finite group, we can build a theory of equivariant Euler classes for G-equivariant complex topological vector bundles along G-manifolds, which allows us to prove a *equivariant conservation of number* result under mild hypotheses [Bra25b]. Roughly speaking this states that the symmetries of an equivariant enumerative problem are always conserved. For instance given a smooth cubic with automorphism group G, the group always acts on the 27 lines in the same way [Bra25b]. Another application of the equivariant Euler number has been given in [BB24], where the authors give an enriched count of orbits to the 28 bitangents of any smooth, non-hyperelliptic, symmetric quartic curve. Equivariant counts of orbits of solutions to enumerative problems have also appeared in the context of counting solutions to enumerative problems in families that are invariant under a finite group action. Equivariant counts have been provided for counting nodes in a G-invariant pencil of conics [Bet25] and rational cubics interpolating a G-invariant set 8 general points in $\mathbb{CP}^2$ [BW25a], both of these results recover a real signed count of nodal conics and rational cubics respectively when C_2 acts on $\mathbb{CP}^2$ by pointwise complex conjugation. The intersection of real symmetric hypersurfaces has also been explored [LLM24]. Equivariantly enriched Gromov-Witten invariants for smooth, projective complex varieties with the action of a finite group will appear in upcoming work of the first named author and Wickelgren.

[7] Related but orthogonal work in this direction includes [Rob85, Dam91, CHT24]

An interesting direction is to ask how symmetry interplays with the Galois group of an enumerative problem (in the sense of [Her51, Jor70, Har79]). Using variation of Hodge structure techniques analogous to those discussed in Section 6.2, the second-named author and Raman showed the Galois group of lines on a S_4-symmetric cubic surfaces is the Klein four-group [BR25]. Using entirely different techniques derived from the world of stacks, Landi investigated the same question for cubic surfaces with involution and computed the Galois group of their lines [Lan25]. These results will be expanded upon in upcoming work of the second named author, Landi and Raman.

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