AN INTRO TO THE NORM RESIDUE ISOMORPHISM THEOREM

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Abstract. Talk notes for the *Juvitop* seminar, Fall 2025.

0. About

Intro talk for Juvitop Fall 2025, setting up the background of the *norm residue isomorphism theorem*.

1. History

By the end of the 1960's, the first few algebraic K-groups K_0, K_1, and K_2 had been defined and extensively explored, but no full constructions of the higher K-groups had yet appeared. In a 1967 course at Princeton, Milnor wrote down the first definition of K_2 of a ring in terms of Steinberg modules. A few results followed quickly thereafter, including Matsumoto's 1968 PhD thesis, in which he gave a presentation for K_2 of a field.

Theorem 1.1 (Matsumoto)**.** If F is a field, we have that $K_2(F)$ is the abelian group generated by *symbols* $\{x, y\}$ with $x, y \in F^\times$, modulo the relations:

- ▷ $\{x_1x_2, y\} = \{x_1, y\}\{x_2, y\}$ and $\{x, y_1y_2\} = \{x, y_1\}\{x, y_2\}$
- ▷ $\{x, 1-x\} = 0$.

Exercise 1.2. Using this presentation, show that $K_2(\mathbb{F}_q) = 0$.

It turns out there is a close connection between algebraic K_2 and the Brauer group of a field – we recall that the Brauer group classifies central simple algebras over a field F.

Notation 1.3 ([Mil71, §15] [Wei13, III.6.9])**.** If ζ is a primitive nth root of unity in F, and $\alpha, \beta \in F^\times$, we can define the *cyclic algebra*, which is central and simple, defined to be the free unital associative F-algebra with the following generators and relations:

$$A_\zeta(\alpha, \beta) = F\langle x, y \mid x^n = \alpha = \beta,\ xy = \zeta yx\rangle.$$

For $n = 2$ these are quaternion algebras.

Proposition 1.4 ([Mil71, 15.4])**.** For F as above, the function

$$\begin{aligned} F^\times \times F^\times &\to \mathrm{Br}(F) \\ (\alpha, \beta) &\mapsto A_\zeta(\alpha, \beta) \end{aligned}$$

satisfies the relations in Matsumoto's work, hence extends to what's called a *Galois symbol* (or *Steinberg symbol* sometimes)[1]

$$K_2(F) \to \mathrm{Br}(F).$$

Exercise 1.5.

[1] Note that this definition *depends on a choice* of primitive nth root of unity. See [Mil71, 15.5] for more on this.

- ▷ We have that $A_\zeta(\alpha,\beta)^{\otimes_F n}$ is a matrix algebra (hence trivial in the Brauer group), so the Galois symbol lands in the n-torsion elements of the Brauer group $\mathrm{Br}(F)[n]$, and therefore
- ▷ the Galois symbol factors through $K_2(F)/n$
- ▷ $\mathrm{Br}(F)[n] = H^2_{\mathrm{et}}(F,\mu_n)$, and if $\mu_n \subseteq F^\times$ we have that this is also equivalent to $H^2_{\mathrm{et}}(F,\mu_n^{\otimes 2})$.

So altogether we are getting a map we also call the Galois symbol, or maybe the *norm residue map*:

$$K_2(F)/n \to H^2_{\mathrm{et}}(F,\mu_n^{\otimes 2}).$$

and we can still define this even if F doesn't contain primitive nth roots of unity, just by using the cup product structure on étale cohomology.

Theorem 1.6 (Merkurjev-Suslin, 1980's)**.** This is an isomorphism for every field.

This was proven for global fields by Tate in 1976, and by Merkurjev and Suslin in 1980-1981.

1.1. **A connection to Iwasawa theory.**

Theorem 1.7 (Garland)**.** If F is a number field, $K_2(\mathcal{O}_F)$ is finite.

So what is its size? What does it mean?

Notation 1.8. For a number field F, we define

$$w_2(F) := \max\{m \mid \mathrm{Gal}(F(\mu_m)/F) \text{ is 2-torsion}\}.$$

Conjecture 1.9 (Birch-Tate)**.** For a totally real number field F, we have that

$$\#K_2(\mathcal{O}_F) = w_2(F)\cdot\zeta_F(-1).$$

The odd parts of these numbers agree by Mazur-Wiles. The 2-adic part is equivalent to part of a huge conjecture in Iwasawa theory, apparently.

Contrast this with the observation that $K_2(\mathbb{R})$ is uncountably infinite [Wei13, III.5.9.1].

1.2. **Enter quadratic forms.** In a total change of pace, we can consider a field F (assumed to be of characteristic $\neq 2$ here) and study quadratic forms over it. Recall that quadratic forms are O_n-torsors on the étale site, we'll use this later. We define the *Grothendieck–Witt ring* $\mathrm{GW}(F)$ to be the group completion of the monoid of isomorphism classes of quadratic forms over F. Each quadratic form has a well-defined *rank*, just being the number of different variables used.

Definition 1.10. The *fundamental ideal* of F is defined to be the kernel of the rank map:

$$I(F) := \ker\left(\mathrm{GW}(F) \xrightarrow{\text{rank}} \mathbb{Z}\right).$$

As an abelian group, $I(F)$ is generated by the *Pfister forms*

$$\langle\langle a\rangle\rangle := \langle 1\rangle - \langle a\rangle.$$

Theorem 1.11 (Arason-Pfister's Hauptidealsatz)**.** Given two elements $\alpha,\beta \in \mathrm{GW}(F)$, they are equal if and only if they are equal modulo $I^{n+1}(F)$ for every $n \geq 0$. There is an isomorphism

$$\mathrm{GW}(F) \cong \bigoplus_n I^n(F)/I^{n+1}(F)$$

Slogan 1.12. To form invariants of quadratic forms, we should better understand the associated graded parts of the filtration on the Grothendieck-Witt ring by powers of the fundamental ideal.

Example 1.13. We can compute that there are isomorphisms:

$$\text{rank}\colon I^0(F)/I(F) \xrightarrow{\sim} \mathbb{Z}$$
$$\text{determinant}\colon I(F)/I^2(F) \xrightarrow{\sim} F^\times/\left(F^\times\right)^2$$
$$w_2\colon I^2(F)/I^3(F) \xrightarrow{\sim} \mathrm{Br}(F)[2].$$

By precomposing with the projection $\mathrm{GW}(F) \to I^n(F)/I^{n+1}(F)$, these isomorphisms give *invariants* of quadratic forms. The first two are rank and determinant,[2] let's define this latter one:

Definition 1.14. If $q = \sum a_i x_i^2$ is a diagonalized quadratic form, we define $w_2(q) \in \mathrm{Br}(F)$ to be the product of all the quaternion algebras $\prod_{i<j}\left(\frac{a_i,a_j}{F}\right)$.[3] This w_2 is unrelated to the w_2 in the Birch-Tate conjecture as far as I know, it's just unfortunate overloaded notation.

Observation 1.15 (Milnor)**.** The Hasse invariant w_2 factors through the Galois symbol,

$$\begin{tikzcd} I^2(F) \arrow[rr, "w_2"] \arrow[dr, two heads] & & \mathrm{Br}(F)[2] \\ & K_2(M)/2 \arrow[ur] & \end{tikzcd}$$

and moreover the kernel of the surjective map $I^2(F) \to K_2(F)/2$ is exactly $I^3(F)$. Hence we have an isomorphism

$$K_2(F)/2 \cong I^2(F)/I^3(F).$$

Milnor's idea was the following: whatever higher algebraic K-theory is (we don't know yet), perhaps it should be something where $K_n(F)/2$ comes with a natural symbol map to $I^n(F)/I^{n+1}(F)$. This led Milnor to define what is now known as *Milnor K-theory*, which we'll see more about in the talk next week. We write it as $K_n^M(F)$, with a superscript M for Milnor. It comes with a natural map to honest algebraic K-theory, but this starts failing to be an isomorphism at $n = 3$. Nevertheless Milnor K-theory is important in its own right.

As requested, it supports some symbol maps to the associated graded for $\mathrm{GW}(F)$:

Proposition 1.16 (Milnor)**.** *There is a symbol map*

$$K_n^M(F)/2 \to I^n(F)/I^{n+1}(F)$$
$$\{a_1, \ldots, a_n\} \mapsto \prod_{i=1}^n \langle\langle a_i \rangle\rangle\,,$$

which is surjective.

Conjecture 1.17 (Milnor Conjecture 2)**.** *This map is a bijection.*

Proven by Kato in characteristic 2 and by Orlov-Vishik-Voevodsky in characteristic $\neq 2$.

This leads us to a natural question:

Question 1.18. Can we construct symbol maps out of mod two Milnor K-theory valued in étale cohomology?

[2] Here $I^0(F)$ means $\mathrm{GW}(F)$ by convention, so the rank isomorphism is just from the definition of $I(F)$.

[3] This notation $\left(\frac{a,b}{F}\right)$ is shorthand for the free unital associative F-algebra given by the generators and relations $\langle x, y \mid x^2 = a,\ y^2 = b, xy = -yx\rangle$. This might look more familiar to some after replacing x by $\hat{\imath}$, y by $\hat{\jmath}$ and xy by $\hat{k}$, in which case it is clearly a quaternion algebra.

Milnor also did this for us! Moreover, these make sense not just for μ_2 coefficients, but for all μ_n coefficients.

Theorem 1.19 (Bass-Tate, Milnor)**.** Let F be a field containing nth roots of unity, where n is prime to the characteristic of F. Then the cup product map

$$(F^\times)^{\otimes r} \cong H^1_{\text{et}}(F, \mu_n)^{\otimes r} \xrightarrow{\cup} H^r_{\text{et}}(F, \mu_n^{\otimes r})$$

factors through the Steinberg identity, inducing a graded ring homomorphism

$$K^M_*(F)/n \to H^*_{\text{et}}(F, \mu_n^{\otimes *}).$$

Proof. It suffices to check the Steinberg identity holds on $H^2_{\text{et}}(F, \mu_n^{\otimes 2})$ □

Theorem 1.20. This map is an isomorphism.

Proven by Merkurjev in $n = 2$, by Merkurjev-Suslin-Rost for $n = 3$, and by Voevodsky in general. It is a corollary of the more general norm residue isomorphism theorem.

Application: Milnor K-theory is defined very naturally in terms of generators and relations. This gives us a *presentation* for the étale cohomology ring.

Sub-application: This can make either side easier to compute, since we can compute étale cohomology via Milnor K-theory or Milnor K-theory via étale cohomology.

1.3. **Aside: higher Hasse-Witt invariants.** Let's pretend that we're over a field of characteristic $\neq 2$, so that we can write $\mathbb{Z}/2$ instead of $\mu_2^{\otimes n}$ everywhere. Let's also assume that we know the Milnor conjectures are true. Then we have a string of isomorphisms for every n of the form:

$$I^n(F)/I^{n+1}(F) \xleftarrow{\sim} K^M_n(F)/2 \xrightarrow{\sim} H^n_{\text{et}}(F, \mathbb{Z}/2).$$

Altogether we are getting maps

$$\text{GW}(F) \xrightarrow{\sim} \oplus_n H^n_{\text{et}}(F, \mathbb{Z}/2)$$

which jointly classify all quadratic forms. We know the second one w_2 for instance is the Hasse-Witt invariant. We can ask what the others are – are they universal in some sense?

If we take a page from algebraic topology, we are asking for some universal étale cohomology classes which classify quadratic forms. Since quadratic forms of rank n are étale O_n torsors, they are represented by the stack BO_n. Hence in looking for cohomological invariants, we might ask – what is the étale cohomology of the stack BO_n? This was computed by Jardine:

Theorem 1.21 (Jardine)**.** We have that $H^*(\text{BO}_n; \mathbb{Z}/2)$ is the free $H^*(F, \mathbb{Z}/2)$-algebra on generators $w_1, \dots, w_n$, with $|w_i| = i$.

In other words, there are some *universal invariants* w_i for quadratic forms, and these are valued in $H^i_{\text{et}}(F, \mathbb{Z}/2)$. These are the *Hasse-Witt invariants.*

2. Another appearance of Milnor K-theory

(The following story comes from §2.4 of Gillet's paper in the Handbook, but is essentially the construction of the Rost complex.)

Let X be an integral Noetherian scheme, $U \subseteq X$ an open subscheme and $Z \subseteq X$ its closed complement. Then there is an exact sequence

$$? \to \text{CH}(Y) \to \text{CH}(X) \to \text{CH}(U) \to 0. \tag{2.1}$$

Can we extend this to the left? Recall that the Chow group is the cokernel

$$\mathrm{CH}(X) = \mathrm{coker}(R(X) \xrightarrow{\mathrm{div}} Z(X)),$$

where

$$R(X) = \oplus_{\xi \in X} k(\xi)^\times,$$
$$Z(X) = \oplus_{\xi \in X} \mathbb{Z},$$

are the group of K_1-chains and the group of cycles, respectively. By basic homological algebra, the thing extending Equation (2.1) to the left would have to be

$$\ker(R(U) \xrightarrow{\mathrm{div}} Z(U)).$$

What things have divisor zero on U? This is now just a question about U, it doesn't depend on Z or X, so we can forget about them.

Suppose we have an element in $k(\xi)$ mapping to zero under the divisor map, then it must have valuation zero at each discrete valuation. The only such elements in the field $k(\xi)$ are of the form ± 1 in general, so we know we need a formal combination of two or more rational functions.

Exercise 2.1. If the divisors associated to $f, g \in k(U)^\times$ have no components in common, then

$$g_{|\mathrm{div}(f)} - f_{\mathrm{div}(g)}$$

is zero in $Z(U)$.

Proposition 2.2. If U is integral and ϕ, ψ are Cartier divisors with $\mathrm{div}(\phi) = \sum n_i [Y_i]$ and $\mathrm{div}(\psi) = \sum m_j [Z_j]$ then

$$\sum n_i \mathrm{div}(\psi_{|Y_i}) = \sum_j m_j \mathrm{div}(\phi_{|Z_j}).$$

Proof. Can be proven with intersection theory (in Fulton), purely algebraically, or with the coniveau spectral sequence in algebraic K-theory (original proof). □

So we should extend

$$\oplus_x k(x)^\times \otimes k(x)^\times \to R(U) \to Z(U) \to 0.$$

What is this "divisor" map? It should send $f \otimes g$ to $f_{\mathrm{div}(g)} - g_{|\mathrm{div}(f)}$. Hence the kernel of this map is generated by elements $f \otimes g + g \otimes f$.

Exercise 2.3. Check another valid presentation of $K_2(F)$ is

$$F^\times \otimes F^\times / (x \otimes y + y \otimes x).$$

Slogan 2.4. Milnor K-theory appears in nature when attempting to develop a long exact localization sequence for Chow groups.

3. Generalizing everything

In the 80's, Beilinson and Lichtenbaum had conjectured the existence of certain chain complexes of Nisnevich sheaves, denoted $\mathbb{Z}(q)$ or $\mathbb{Z}/\ell(q)$ for a prime ℓ, with a laundry list of desirable properties. We can take hypercohomology of these and we obtain a bigraded ring

$$\oplus_{p,q} H^p(X, \mathbb{Z}(q)).$$

Roughly speaking these should form some graded parts of the algebraic K-theory of X. These complexes were later constructed explicitly by Voevodsky.

If X is a smooth variety over k and $\ell^{-1} \in k$, then we will soon see there is a natural map

$$H^p(X, \mathbb{Z}/\ell(q)) \to H^p_{\text{et}}(X, \mu_\ell^{\otimes q}).$$

A more general version of the étale cohomology version of the Milnor conjectures is the following:

Conjecture 3.1. In the situation above,

$$H^p(X, \mathbb{Z}/\ell(q)) \to H^p_{\text{et}}(X, \mu_\ell^{\otimes q})$$

is an isomorphism for $p \leq q$.

Example 3.2. If $X = \operatorname{Spec}(F)$, then

$$K_p^M(F)/\ell \cong H^{p,p}(F, \mathbb{Z}/\ell),$$

so the conjecture would imply the Bloch-Kato conjecture as phrased for Milnor K-theory modulo ℓ.

This can again be generalized, since it doesn't really depend upon the scheme X, but rather the complex of sheaves $\mathbb{Z}/\ell(q)$. If we let π denote the change of site functor from étale to Nisnevich sheaves, we have that

$$H^n_{\text{et}}(X, \mu_\ell^{\otimes q}) \cong H^n_{\text{Nis}}(X, R\pi_*\mu_\ell^{\otimes q}).$$

The conjecture above then arises from a map in the derived category of Nisnevich sheaves:

$$\mathbb{Z}/\ell(q) \to R\pi_*\mu_\ell^{\otimes q}.$$

The general conjecture is then the following (which we could call the norm residue isomorphism theorem):

Conjecture 3.3. The map

$$\mathbb{Z}/\ell(q) \to \tau^{\leq q} R\pi_*\mu_\ell^{\otimes q}$$

is an isomorphism in the derived category of Nisnevich sheaves.

This is the form of the conjecture we'll work towards proving this semester.

References

[Mil71] John Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies, no. 72, Princeton University Press, 1971.

[Wei13] Charles A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, R.I., 2013.