$\mathbb{A}^1$-BROUWER DEGREES IN MACAULAY2

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Abstract. We describe the Macaulay2 package `A1BrouwerDegrees` for computing local and global $\mathbb{A}^1$-Brouwer degrees and studying symmetric bilinear forms over a field.

1. Introduction

In $\mathbb{A}^1$-homotopy theory, the $\mathbb{A}^1$-*Brouwer degree* provides an algebro-geometric analogue of the classical Brouwer degree from differential topology. Morel's $\mathbb{A}^1$-degree homomorphism identifies the zeroth stable stem of the motivic sphere spectrum with the *Grothendieck–Witt ring* of symmetric bilinear forms over a field [13]. Given an endomorphism of affine space with an isolated rational zero, work of Kass and Wickelgren [6] identifies its local $\mathbb{A}^1$-Brouwer degree with the Eisenbud–Khimshiashvili–Levine signature form [4, 5], which was used to compute local Brouwer degrees in real differential topology. Work of Bachmann and Wickelgren [1] identifies the local $\mathbb{A}^1$-Brouwer degree with the *Scheja–Storch form* [16], used to study complete intersections, as well as with a quadratic Grothendieck–Serre duality form.

The $\mathbb{A}^1$-Brouwer degree is now understood as a quadratically enriched analogue of intersection multiplicity, often encoding deeper geometric information than was available over the complex numbers [12]. This perspective has seen a wealth of applications in $\mathbb{A}^1$-*enumerative geometry* [7, 11] (see [2, 15] for an overview). The $\mathbb{A}^1$-Brouwer degree is an important and powerful tool in computing enriched intersection numbers and solving enumerative geometry problems over other fields.

Recent work of the second named author, McKean, and Pauli [3] provides tractable formulas for computing $\mathbb{A}^1$-Brouwer degrees as *Bézoutian bilinear forms*. In this package, we implement these methods and provide a suite of tools whose capabilities include:

(1) computing $\mathbb{A}^1$-Brouwer degrees (both local and global) for endomorphisms of affine space;
(2) decomposing symmetric bilinear forms into their isotropic and anisotropic parts; and
(3) extracting invariants of symmetric bilinear forms (rank, signature, discriminant, Hasse–Witt invariants).

In Section 2, we provide a rapid introduction to the theory of symmetric bilinear forms, highlighting the capacity of our package to build forms, check isomorphisms, and decompose forms. In Section 3, we discuss local and global $\mathbb{A}^1$-Brouwer degrees and provide some computational examples, including quadratically enriched intersection multiplicity of real curves, the $\mathbb{A}^1$-Euler characteristic of the Grassmannian $\mathrm{Gr}(2,4)$ (following [3, Section 8.2]), and local computations for 27 lines on a cubic surface (following [7, 14]).

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1.1. **Software availability.** The software documented here is currently in the `development` branch of the Macaulay2 repository. The package may be installed via the `A1BrouwerDegrees` file and the directory by the same name, which may be found at the following:

https://github.com/Macaulay2/M2/tree/development/M2/Macaulay2/packages

2. The Grothendieck–Witt ring

For this entire section, we assume k is a field not of characteristic 2. We say that a bilinear form $\beta\colon V \times V \to k$ is *symmetric* if $\beta(v,w) = \beta(w,v)$ for all $v, w \in V$. We say β is *non-degenerate* if $\beta(v,-)\colon V \to k$ is identically zero if and only if $v = 0$.

Definition 1. Given a symmetric bilinear form β, by picking a vector space basis $e_1, \ldots, e_n$ for V, we can define a *Gram matrix* of β to be the symmetric matrix with entries $\beta(e_i, e_j)$. Non-degeneracy of β is equivalent to the statement that the determinant of any Gram matrix is nonzero. A change of basis for V corresponds to a congruence of the associated Gram matrices.

Given two symmetric bilinear forms $\beta_i\colon V_i \times V_i \to k$ for $i = 1, 2$, we can define their sum and product:

$$\begin{aligned}(\beta_1 \oplus \beta_2)&\colon V_1 \oplus V_2 \times V_1 \oplus V_2 \to k\\ (\beta_1 \otimes \beta_2)&\colon (V_1 \otimes V_2) \times (V_1 \otimes V_2) \to k.\end{aligned} \tag{1}$$

On Gram matrices, these operations are given by block sum and tensor product, respectively.

Definition 2. We define the *Grothendieck–Witt ring* $\mathrm{GW}(k)$ to be the group completion of the semiring of isomorphism classes of non-degenerate symmetric bilinear forms over k.

When k is the complex numbers, the real numbers, the rational numbers, or a finite field, we define a type called `GrothendieckWittClass`, intended to encode the class $[\beta] \in \mathrm{GW}(k)$ of a symmetric bilinear form β. Grothendieck–Witt classes can be constructed from Gram matrices via the `gwClass` method.

```
i1 : M = matrix(QQ, {{1,3},{3,7}});

                2        2
o1 : Matrix QQ  <--- QQ

i2 : beta = gwClass(M)

o2 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 1 3 |
                                     | 3 7 |

o2 : GrothendieckWittClass
```

The underlying field and associated Gram matrix can be recovered by running `baseField beta` and `beta.matrix`, respectively. Objects of type `GrothendieckWittClass` can be added and multiplied via the `gwAdd` and `gwMultiply` commands, respectively, which correspond to Equation (1).

Example 3. For any unit $a \in k^\times$, there is a symmetric bilinear form of rank one

$$\begin{aligned}\langle a\rangle \colon k \times k &\to k\\ (x,y) &\mapsto axy.\end{aligned}$$

Note that via the change of basis $(x,y) \mapsto (bx,by)$ we observe that $\langle a\rangle = \langle ab^2\rangle$. Hence the representative for a rank one form is determined only by its square class.

The following classical result (see [10, Corollary I.2.4]) implies that the forms $\langle a\rangle$ generate $\mathrm{GW}(k)$.

Theorem 4. *Every class in* $\mathrm{GW}(k)$ *is represented by a diagonal Gram matrix.*

A diagonal representative of a class in $\mathrm{GW}(k)$ can be extracted using the `diagonalClass` method.

```
i3 : diagonalClass(beta)

o3 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 1 0  |
                                     | 0 -2 |

o3 : GrothendieckWittClass
```

We also provide methods for constructing various forms. We can construct a class corresponding to a list of diagonal entries via the `diagonalForm` command.

```
i4 : diagonalForm(GF(13),(2,6))

o4 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 2 0 |
                                     | 0 6 |

o4 : GrothendieckWittClass
```

We can produce hyperbolic forms of various even ranks via the `hyperbolicForm` method, and we can produce Pfister forms, which are important objects of study in the world of quadratic forms [10, Chapter X], via the `PfisterForm` method.

2.1. **Verifying isomorphisms of forms.** Given two symmetric bilinear forms, a natural question to ask is whether they represent the same element of $\mathrm{GW}(k)$. An easy invariant we may check is whether they are defined on vector spaces of the same dimension, i.e. whether the *rank* of the forms (the rank of their Gram matrices) agrees. Since every complex number is a square, rank completely classifies symmetric bilinear forms over the complex numbers.

Since there are two square classes over the reals, namely $+1$ and -1, we can find a Gram matrix representative of any form which is diagonal, with only ± 1 appearing along the diagonal. The trace of such a Gram matrix is an invariant of the form, called the *signature*. Rank and signature jointly classify symmetric bilinear forms over the reals.

```
i1 : gamma = gwClass(matrix(RR, {{3,0,0},{0,-4,0},{0,0,7}}));

i1 : signature(gamma)
```

```
o1 = 1
```

Over finite fields, the *discriminant*, namely the determinant of any Gram matrix representative (valued in square classes), and the rank jointly classify symmetric bilinear forms.

Over the rationals things are a bit more complicated. For each completion of $\mathbb{Q}$, we have a different invariant, and these, together with rank, jointly classify symmetric bilinear forms. At the Archimedean place, we recover the signature, but over the p-adics we obtain the so-called *Hasse–Witt invariants*.

Definition 5. Given any form $\beta \cong \langle a_1, \ldots, a_n \rangle \in \mathrm{GW}(\mathbb{Q})$, its *Hasse–Witt invariant* is defined to be the product

$$\prod_{i<j} (a_i, a_j)_p,$$

where $(-,-)_p$ denotes the *Hilbert symbol*

$$(a,b)_p := \begin{cases} 1 & \text{if } z^2 = ax^2 + by^2 \text{ has a nonzero solution in } \mathbb{Q}_p, \\ -1 & \text{otherwise.} \end{cases}$$

We can compute the Hilbert symbol $(a,b)_p$ via `HilbertSymbol`(a,b,p) and the Hasse–Witt invariant of a form β at p by `HasseWittInvariant`(β, p).

These methods together form one of our core methods `gwIsomorphic`, which is a Boolean-valued method determining whether two symmetric bilinear forms are isomorphic. This is done by reference to the relevant invariants over $\mathbb{C}$, $\mathbb{R}$, $\mathbb{F}_q$, or $\mathbb{Q}$.

2.2. **Decomposing forms.** *Witt's Decomposition Theorem* (see [10, I.4.1]) implies that any non-degenerate symmetric bilinear form decomposes into an anisotropic part and an isotropic part that is a sum of hyperbolic forms. This decomposition is crucial in simplifying an element of $\mathrm{GW}(k)$. While this decomposition is fairly routine over $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{F}_q$, to decompose forms over $\mathbb{Q}$ we must implement existing algorithms from the literature. An important mathematical stepping stone is the following local-to-global principle for isotropy, a reference for which is [10, VI.3.1].

Theorem 6 (Hasse–Minkowski Principle)**.** *A form $\beta \in \mathrm{GW}(\mathbb{Q})$ is isotropic if and only if it is isotropic over $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes p.*

Our method `anisotropicDimensionQp`, an implementation of [8, Algorithm 8], determines the dimension of the anisotropic part of a form over $\mathbb{Q}_p$. The method `anisotropicDimension` returns the anisotropic dimension of a form defined over the real numbers, the complex numbers, a finite field, or the rational numbers.

Given a form, we can therefore decompose it as

$$\beta \cong \beta_a \oplus n\mathbb{H},$$

where β_a is anisotropic, where $\mathbb{H}$ denotes the hyperbolic form $\langle 1, -1 \rangle$, and where n is the *Witt index* (implemented as `WittIndex`). The dimension of β_a can be extracted via `anisotropicDimension`(β).

The Boolean-valued method `isAnisotropic` returns whether a form is anisotropic; the method `isIsotropic` is its negation.

```
i1 : alpha = diagonalForm(QQ,(1,2,-3));

i2 : isAnisotropic(alpha)

o2 = false

i3 : isIsotropic(alpha)

o3 = true
```

Over $\mathbb{Q}$, the computation of the anisotropic part of β is carried out inductively by reduction of the anisotropic dimension of β, following recently published algorithms for quadratic forms over number fields by Koprowski and Rothkegel [9]. The anisotropic part of a form can be computed via `anisotropicPart`.

```
i4 : beta = diagonalForm(QQ,(3,-3,2,5,1,-9));

i5 : anisotropicPart(beta)

o6 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 2 0  |
                                     | 0 20 |
```

A quick string reading off the decomposition of a form can be obtained by running the `sumDecompositionString` method.

```
i7 : sumDecompositionString(beta)

o8 = 2H+ <2>+ <20>
```

3. $\mathbb{A}^1$-Brouwer degrees

In [3], the authors show that local and global $\mathbb{A}^1$-Brouwer degrees for endomorphisms of affine space with isolated zeros can be expressed in terms of a bilinear form associated to the Bézoutian of the endomorphism.

More explicitly, let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$, $f_i \in k[x_1, \ldots, x_n]$. Introducing new variables $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, we can construct the matrix Δ with entries

$$\Delta_{i,j} = \frac{f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n) - f_i(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n)}{X_j - Y_j}.$$

One can think of this matrix Δ as a Jacobian of formal derivatives. Define $Q(f) = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ and $Q_p(f) = k[x_1, \ldots, x_n]_{\mathfrak{m}}/(f_1, \ldots, f_n)$ for $\mathfrak{m}$ the maximal ideal of a closed point p in the preimage of 0. The *Bézoutian* of f is defined to be the image of $\det(\Delta)$ in the algebra $Q(f) \otimes Q(f)$ (respectively, in the local algebra $Q_p(f) \otimes Q_p(f)$). Given $a_1, \ldots, a_m$ a k-linear basis of $Q(f)$ (resp. $Q_p(f)$), there are $b_{i,j} \in k$ such that

$$\det(\Delta) = \sum_{1 \le i \le j \le m} b_{i,j}(a_i \otimes a_j)$$

in $Q(f) \otimes Q(f)$ (resp. $Q_p(f) \otimes Q_p(f)$). The *Bézoutian bilinear form*, the symmetric bilinear form with Gram matrix given by the $b_{i,j}$, gives the global (resp. local) $\mathbb{A}^1$-degree [3, Theorem 1.2].

In the case of the $\mathbb{A}^1$-global degree, a theorem of Macaulay [17, Proposition 2.1] tells us that the k-basis of the algebra $Q(f)$ is given by the standard monomials. In the case of the local $\mathbb{A}^1$-degree, a result of Sturmfels states that the k-basis for the local ring can be calculated as the quotient of $k[x_1, \ldots, x_n]$ by the saturation of $(f_1, \ldots, f_n)$ at $\mathfrak{m}$.

Proposition 7 ([17, Proposition 2.5])**.** *There is an isomorphism of rings*

$$k[x_1, \ldots, x_n]_{\mathfrak{m}}/I \cong k[x_1, \ldots, x_n]/(I : (I : \mathfrak{m}^\infty)),$$

where I is an ideal of $k[x_1, \ldots, x_n]$ and $(I : (I : \mathfrak{m}^\infty))$ is the quotient of I by the saturation of I at $\mathfrak{m}$.

We implement Proposition 7 as the method `localAlgebraBasis`.

These methods for computing k-bases for $Q(f)$ and $Q_p(f)$ allow us to algorithmically implement techniques to compute the global and local $\mathbb{A}^1$-degrees (see also [3, Section 5A]).

3.1. **A univariate polynomial.** A univariate polynomial over a field k defines an endomorphism of affine space $\mathbb{A}^1_k \to \mathbb{A}^1_k$. Consider the endomorphism $f\colon \mathbb{A}^1_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$ by

$$f(x) = (x^2 + x + 1)(x - 3)(x + 2).$$

We can compute the global degree.

```
i1 : R = QQ[x];

i2 : f = {x^4 - 6*x^2 - 7*x - 6};

i3 : alpha = globalA1Degree(f)

o3 = GrothendieckWittClass{cache => CacheTable{}  }
                           matrix => | -7 -6 0 1 |
                                     | -6 0  1 0 |
                                     | 0  1  0 0 |
                                     | 1  0  0 0 |

o3 : GrothendieckWittClass
```

Given the decomposition of the polynomial into its factors, we can also compute the local degrees at the ideals $(x^2 + x + 1)$, $(x - 3)$, and $(x + 2)$, respectively.

```
i4 : I1 = ideal(x^2 + x + 1);

o4 : Ideal of R

i5 : alpha1 = localA1Degree(f, I1);

i6 : I2 = ideal(x - 3);

o6 : Ideal of R
```

```
i7 : alpha2 = localA1Degree(f, I2)

o7 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 65 |

o7 : GrothendieckWittClass

i8 : I3 = ideal(x + 2);

o8 : Ideal of R

i9 : alpha3 = localA1Degree(f, I3)

o9 = GrothendieckWittClass{cache => CacheTable{}}
                            matrix => | -15 |

o9 : GrothendieckWittClass
```

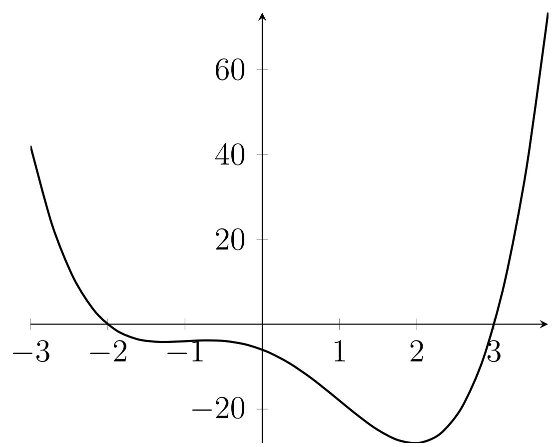
We can then use the `gwIsomorphic` method (see also Section 2.1) to verify that the local $\mathbb{A}^1$-degrees sum to the global $\mathbb{A}^1$-degree.

```
i10 : alpha' = gwAdd(alpha1, gwAdd(alpha2, alpha3));

i11 : gwIsomorphic(alpha,alpha')

o11 = true
```

We consider the graph of $f(x)$.



Following [12], we can interpret $\mathbb{A}^1$-degrees as enriched intersection numbers, allowing us to interpret $\alpha_2 = \langle 65 \rangle$, the local $\mathbb{A}^1$-degree at $(x-3)$, and $\alpha_3 = \langle -15 \rangle$, the local $\mathbb{A}^1$-degree at $(x+2)$, as signs of the derivative at these points.

3.2. **The Euler characteristic of the Grassmannian of lines in $\mathbb{P}^3$.** For k a field of characteristic not 2, let $\mathrm{Gr}_k(2,4)$ be the Grassmannian of lines in $\mathbb{P}^3_k$. Following [3, Example 8.2], we can compute the $\mathbb{A}^1$-Euler characteristic of the Grassmannian over $k = \mathbb{F}_9$

as the $\mathbb{A}^1$-degree of the section $\sigma\colon \mathbb{A}^4_{\mathbb{F}_9} \to \mathbb{A}^4_{\mathbb{F}_9}$ defined by

$$(x_1, x_2, x_3, x_4) \mapsto (x_2 - x_1x_3, x_4 - x_1 - x_3^2, 1 - x_1x_4, -x_2 - x_3x_4).$$

We compute the $\mathbb{A}^1$-Euler characteristic as follows.

```
i1 : k = GF(9);

i2 : R = k[x_1, x_2, x_3, x_4];

i3 : f = {x_2 - x_1*x_3, x_4 - x_1 - x_3^2, 1 - x_1*x_4, -x_2 - x_3*x_4};

i4 : beta = globalA1Degree(f)

o4 = GrothendieckWittClass{cache => CacheTable{}        }
                           matrix => | 0  0  0 0 0  -1 |
                                     | 0  -1 0 0 0  0  |
                                     | 0  0  0 1 0  0  |
                                     | 0  0  1 0 0  0  |
                                     | 0  0  0 0 -1 0  |
                                     | -1 0  0 0 0  0  |

o4 : GrothendieckWittClass
```

We can subsequently use the SumDecompositionString method to decompose the symmetric bilinear form β.

```
i5 : sumDecompositionString(beta)

o5 = 3H
```

Over $\mathbb{F}_9$, one has that -1 is in the same square class as 1, so our computation agrees with the result given in [3, Example 8.2], thus showing $\chi(\mathrm{Gr}_k(2,4)) = 2\mathbb{H} + \langle 1\rangle + \langle 1\rangle$.

3.3. **Local geometry of the 27 lines on a cubic surface.** In their pioneering paper [7], Kass and Wickelgren give a Grothendieck–Witt class-valued count of the number of lines on a smooth cubic surface, providing an interpretation of the local $\mathbb{A}^1$-degree as the topological type of the line.

Let k be a field, and let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for k^4. By [7, Lemma 45], we can define local coordinates on $\mathrm{Spec}(k[y_1, y_2, y_3, y_4]) \cong \mathbb{A}^4_k$ around the point of $\mathrm{Gr}_k(2,4)$ defined by the span of $\{e_3, e_4\}$ such that y_1, y_2, y_3, y_4 corresponds to the span of $\{\widetilde{e}_3, \widetilde{e}_4\}$, where

$$\widetilde{e}_i = \begin{cases} e_i & \text{for } i \in \{1,2\}, \\ e_1y_1 + e_2y_2 + e_3 & \text{for } i = 3, \\ e_1y_3 + e_2y_4 + e_4 & \text{for } i = 4. \end{cases}$$

These coordinates give a trivialization of the vector bundle $\mathrm{Sym}^3 \mathcal{S}^\vee$ over

$$U \cong \mathrm{Spec}(k[y_1, y_2, y_3, y_4]) \subseteq \mathrm{Gr}_k(2,4).$$

A cubic surface X defines a section $\sigma_X|_U\colon U \to \mathrm{Sym}^3 \mathcal{S}^\vee|_U$ that vanishes on the lines on X that, when treated as affine two-dimensional subspaces of k^4, contain e_3 and e_4 in their span.

Let us consider the Fermat cubic surface defined by the homogeneous cubic equation $x_0^3 + x_1^3 + x_2^3 + x_3^3$. That is,

$$X = \left\{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3_k : x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\right\} \subseteq \mathbb{P}^3_k.$$

Working over $\mathbb{Q}$, these lines are defined over the cyclotomic extension $\mathbb{Q}(\zeta)$ for ζ the third root of unity. We can explicitly compute the 27 lines as

$$[s : t : -\zeta^i t : -\zeta^j s], [s : t : -\zeta^i s : -\zeta^j t], [s : -\zeta^i s : t : -\zeta^j t]$$

for $0 \leq i, j \leq 2$ and $[s : t] \in \mathbb{P}^1_{\mathbb{Q}}$. We note that there are only 18 lines containing e_3 and e_4 in their span. We thus expect the section to vanish at 18 points. Our section is of the form $\sigma_X|_U = (f_1, f_2, f_3, f_4)$ (see [14, Section 2.2]), where

$$\begin{aligned} f_1(y_1, y_2, y_3, y_4) &= y_1^3 + y_3^3 + 1 \\ f_2(y_1, y_2, y_3, y_4) &= 3y_1^2 y_2 + 3y_3^2 y_4 \\ f_3(y_1, y_2, y_3, y_4) &= 3y_1 y_2^2 + 3y_3 y_4^2 \\ f_4(y_1, y_2, y_3, y_4) &= y_2^3 + y_4^3 + 1. \end{aligned}$$

We compute the global $\mathbb{A}^1$-degree, which is rank 18, as expected.

```
i1 : R = QQ[y_1, y_2, y_3, y_4];

i2 : f = {y_1^3 + y_3^3 + 1,
    3*y_1^2*y_2 + 3*y_3^2*y_4,
    3*y_1*y_2^2 + 3*y_3*y_4^2,
    y_2^3 + y_4^3 + 1};

i3 : alpha = globalA1Degree(f);

i4 : sumDecompositionString(alpha)

o4 = 8H+ <1>+ <1>
```

To compute the local degree, we find an isolated zero using the `minimalPrimes` method of Macaulay2.

```
i5 : I = (minimalPrimes ideal f)_0

o5 = ideal (y , y  + 1, y  + 1, y )
             4   3       2       1

o5 : Ideal of R
```

We then compute the local $\mathbb{A}^1$-degree at this point.

```
i6 : beta = localA1Degree(f,I)

o6 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 81 |

o6 : GrothendieckWittClass
```

```
i7 : sumDecomposition(beta)

o7 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 1 |

o7 : GrothendieckWittClass
```

This indicates that the line spanned by $\{-e_2+e_3, -e_1+e_4\}$ on the Fermat cubic surface is a hyperbolic line. We show this agrees with the type as defined in [7, Definition 9].

We compute the intersection of the Fermat cubic surface with the tangent plane at a point along the line defined by $(y_1, y_2, y_3, y_4) = (0, -1, -1, 0) \in U \subseteq \mathrm{Gr}_{\mathbb{Q}}(2, 4)$ spanned by $\{-e_2+e_3, -e_1+e_4\}$. One easily verifies the point $p = [-1 : -1 : 1 : 1]$ lies on X and on the line. We compute the tangent space at this point as

$$T_pX = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3_{\mathbb{Q}} : 3x_0 + 3x_1 + 3x_2 + 3x_3 = 0\} \subseteq \mathbb{P}^3_{\mathbb{Q}},$$

and its intersection with X in Macaulay2.

```
i8 : R = QQ[x_1, x_2, x_3,x_4];

i9 : J = ideal(x_1^3 + x_2^3 + x_3^3 + x_4^3,
     3*x_1 + 3*x_2 + 3*x_3 + 3*x_4);

o9 : Ideal of R

i10 : minimalPrimes J

o10 = {ideal (x  + x , x  + x ), ideal (x  + x , x  + x ), ideal (x  + x ,
               3    4   1    2             2    4   1    3             2    3
      ----------------------------------------------------------------------
       x  + x )}
        1    4

o10 : List
```

This shows the intersection of the tangent plane with X at a point along the line is the union of three lines; our line of interest L is defined by the third ideal in the list. Setting `I1` to be the sum of the first and third ideals and `I2` to be the sum of the latter two ideals in the list, we compute the intersection points of L with the other two lines.

```
i11 : I1 = (minimalPrimes J)_0 + (minimalPrimes J)_2;

o11 : Ideal of R

i12 : I2 = (minimalPrimes J)_1 + (minimalPrimes J)_2;

o12 : Ideal of R

i13 : minimalPrimes I1
```

```
o13 = {ideal (x  + x , x  - x , x  + x )}
               3    4   2    4   1    4

o14 : List

i14 : minimalPrimes I2

o15 = {ideal (x  - x , x  + x , x  + x )}
               3    4   2    4   1    4

o15 : List
```

By inspection, we can see that the points of intersection are given by

$$[-1:-1:1:1] \quad \text{and} \quad [-1:1:-1:1],$$

which in the $[s:t]$ coordinates on $\mathbb{P}^1_{\mathbb{Q}}$ defined on our basis $\{-e_2+e_3, -e_1+e_4\}$ are the points $[1:1]$ and $[-1:1]$, respectively. One can find the involution of $\mathrm{PGL}_2(\mathbb{Q})$ swapping these two points is given by

$$\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{Q})$$

and compute its determinant.

```
i16 : M = matrix(QQ, {{3, 5},{-5,-3}});

                 2        2
o16 : Matrix QQ  <--- QQ

i17 : det M

o17 = 16

o17 : QQ
```

This agrees with the topological type of the line previously given by the form β as defined in [7, Definition 9] (see also the discussion following [7, Definition 9] for the interpretation as the determinant of the involution matrix).

```
i18 : beta' = diagonalForm(QQ, 16);

i19 : gwIsomorphic(beta, beta')

o19 = true
```

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