SETS, GROUPS, AND GEOMETRY
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Abstract. Course notes for MATH101: Sets, groups and geometry, taught at Harvard in Spring 2025.

1. Sets

A *set* is a collection of things, and these things are called elements. We won't give a formal definition of a set, since this gets us too deep into mathematical logic, so we'll kind of take a set as a given and build mathematics on top of it.

We denote by $\{1, 2, 3\}$ the set whose elements are the numbers 1, 2, and 3. These curly braces are used to list the elements of a set.

Example 1.1. The set

$$S = \{a, b, c, d\}$$

is a set consisting of four elements, which are *letters* a, b, c, and d.

Note 1.2. Elements are not allowed to be repeated! For instance, $\{a, b, a, c, d\}$ is not a valid set.[1]

Notation 1.3. We use the symbol $\in$ to denote if an element is in a set. So if $T = \{0, 4, 1, 6\}$, we might write

$$1 \in T$$

to mean that 1 is an element of T. We will write $\notin$ to say something is **not** an element of a set. So for instance

$$2 \notin T.$$

Example 1.4. We denote by $\mathbb{N}$ the set of all *natural numbers*, meaning counting numbers including zero:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$

We denote by $\mathbb{Z}$ the set of all *integers*[2] meaning all positive and negative counting numbers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

We denote by $\mathbb{Q}$ the set of all *rational numbers*, meaning numbers of the form $\frac{p}{q}$ where p and q are integers, and $q \neq 0$.

Example 1.5. We don't just need to have numbers and letters be elements of sets. We can really let *anything* be an element in a set. For instance

$$S = \{\bigcirc, \triangle, \square\}.$$

We can also have *sets* being elements of sets. For instance we can take

$$B = \{\mathbb{N}, \mathbb{Z}, 3, \{4\}\}.$$

[1] This is a convention that we're not allowing for repeated elements. We can build a different type of set theory where you *can* have repeated elements in sets, these are called *multisets*. The math that you build with these becomes a lot more complicated though.

[2] This letter comes from the German *Zahlen*, meaning "numbers."

This is a set with four elements – the set of natural numbers, the set of integers, the number 3, and the set with one element which is the number 4. This might feel weird but we'll get used to it soon enough.

In the above examples, we didn't list out every element of a set when we wrote it, instead we did a . . . when the pattern became clear. For instance what is the following set:

$$A = \{0, 3, 6, 9, 12, 15, \ldots\}.$$

It is the set of all multiples of three! Instead of listing it out, we might *build it*, meaning give a rule for elements to be a part of it. This is done using set builder notation:

$$A = \{3 \cdot n \colon n \in \mathbb{N}\}.$$

This means A is the set of all numbers of the form $3 \cdot n$ where n is an element of $\mathbb{N}$.[3]

A special set is the *empty set*, which has no elements. We could write it as $\{\}$ if we wanted, but we use special notation for it, namely $\varnothing$.

1.1. **Cardinality.** If A is a set, we denote by $|A|$ the *cardinality* of the set, roughly meaning its size. It is the number of elements in the set, possibly infinite.

Example 1.6. The cardinality of some sets we've discussed are:

$$\begin{aligned} |\{a, b, c, d\}| &= 4 \\ |\mathbb{N}| &= \infty \\ |\mathbb{Z}| &= \infty \\ |\{\bigcirc, \triangle, \square\}| &= 3 \\ |\{\mathbb{N}, \mathbb{Z}, 3, \{4\}\}| &= 4 \\ |\varnothing| &= 0. \end{aligned}$$

1.2. **Subsets.** Note that every element in $\mathbb{N}$ is an element of $\mathbb{Z}$. When this happens, we write $\subseteq$, and we say one set is a *subset* of the other.

Definition 1.7. Given two sets A and B, we write $A \subseteq B$ if $x \in A$ implies that $x \in B$. In words, every element in A is also an element in B. We write $A \subsetneq B$ if A is *not* a subset of B.

Example 1.8. We have that $\mathbb{N} \subseteq \mathbb{Z}$.

Question 1.9. Given two sets A and B, how would you argue that A is *not* a subset of B?

You just have to find some element in A that is not in B.

Example 1.10. To argue that $A = \{3, 6, 8, 1\}$ is not a subset of $B = \{2, 6, 8, 1, 5\}$, we see that $3 \in A$ but $3 \notin B$. Therefore $A \subsetneq B$.

Example 1.11. Let $A = \{1, 2, 3\}$. Is it true that $\varnothing \subseteq A$?

Yes! The condition that $\varnothing \subseteq A$ means that for every $x \in \varnothing$ we have that $x \in A$. Since $\varnothing$ has no elements, this is true.[4] In fact $\varnothing \subseteq S$ for *any* set S.

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[3] People who know a little CS, we might think about this as an infinite for loop (for all $n \in \mathbb{N}$, add $3 \cdot n$ to the set we're building, and let A be the resulting output). Obviously this wouldn't terminate on a computer, but we're mathematicians so we can let things happen infinitely many times and keep moving!

[4] We refer to statements like this as *vacuously true* – they're true because no elements exist to check the conditions on. For example I might say "every number which is both even and odd is equal to 7." This is a true statement, not because 7 is both even and odd, but because no numbers are both even and odd.

1.3. **Set equality.**

Question 1.12. What does it mean for two sets to be equal?

Example 1.13. We claim that $\{4, 1, 0\} = \{0, 1, 4\}$.

Answer 1.14. Two sets A and B are equal if they have the same elements. Phrased differently, $x \in A$ implies $x \in B$ and $x \in B$ implies $x \in A$. That is, $A \subseteq B$ and $B \subseteq A$.

1.4. **Operations with sets.** Given two sets A and B we denote by $A \cup B$ their *union*, meaning the set of all elements in A or in B.

$$A \cup B = \{x \colon x \in A \text{ or } x \in B\}.$$

Example 1.15. We have that

$$\{1, 2, 3\} \cup \{4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}.$$

Note we don't allow repeats, so

$$\{1, 2, 3\} \cup \{3, 4\} = \{1, 2, 3, 4\}.$$

Given two sets A and B, we denote by $A \cap B$ their *intersection*, meaning the set of all elements in both A **and** B:

$$A \cap B = \{x \colon x \in A \text{ and } x \in B\}.$$

Example 1.16. We have

$$\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}.$$

Question 1.17. What is

$$\{1, 2, 3\} \cap \{4, 5, 6\}?$$

It is the empty set! There are no elements in both sets.

Finally we denote by $A - B$ their *difference*, meaning

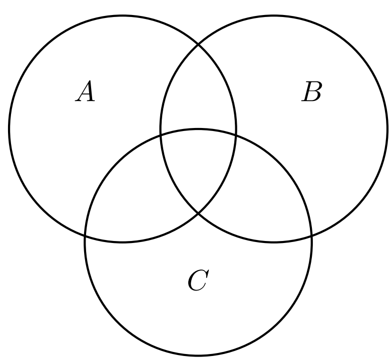
$$A - B = \{x \colon x \in A \text{ and } x \notin B\}.$$

For instance

$$\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}.$$

Note that difference depends on the order of sets! We always have that $A \cup B = B \cup A$ and $A \cap B = B \cap A$, but $A - B$ and $B - A$ might be different sets.

Venn diagrams are a great way to visualize sets and their overlaps:



1.5. **Power sets.** Given a set A, we denote by

$$\mathcal{P}(A) := \{X \colon X \subseteq A\}$$

the *power set* of A, meaning the set of all subsets of A.

Question 1.18. What is the power set of $\{1, 2\}$?

It is the set

$$\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}.$$

Don't forget that $\varnothing \subseteq S$ and $S \subseteq S$ for every set S.

Question 1.19. If S has cardinality n, what do you think the cardinality of the power set $\mathcal{P}(S)$ is? Think about this.

1.6. **The real numbers.** We denote by $\mathbb{R}$ the set of *real numbers*. These are numbers we think about as lying on the number line, but need not be rational. For instance $\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$.[5] It's not super easy to define $\mathbb{R}$ formally, so we'll come back to this later in the class.

We define *intervals* to be subsets of $\mathbb{R}$. You may have seen the notation $[0, 1]$ before. This refers to the *closed interval* between zero and one. Explicitly in terms of set builder notation, we would write:

$$[0, 1] = \{x \in \mathbb{R} \colon 0 \leq x \text{ and } x \leq 1\}.$$

We also have open intervals, denoted by (a, b). For instance

$$(2, 3) := \{x \in \mathbb{R} \colon 2 < x \text{ and } x < 3\}.$$

1.7. **Cartesian products.**

2. Axiomatic rules for sets

[5] This is not super easy to prove, but we'll see examples later of irrational numbers.