K-THEORY OF INFINITY-CATEGORIES

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ABSTRACT. Notes from an expository talk given in the algebraic K-theory seminar at UPenn, spring 2020.

0. About

Notes from an expository talk given in the algebraic K-theory seminar at UPenn, spring 2020.

Note: There are much better resources than this pdf since the time of writing it. At some point I'd like to come back and clean it up, but for now I'd recommend Akhil's notes on dualizable categories and localizing invariants as well as the papers of Efimov.

0.1. References.

- ▷ Barwick, *On the Algebraic K-Theory of Higher Categories*
- ▷ Blumberg, Gepner, Tabuada (BGT), *A Universal Characterization of Higher Algebraic K-Theory*
- ▷ Brasca, K-theory of Waldhausen categories
- ▷ Lurie, MATH281, Lectures 14, 16
- ▷ Lurie, *Higher Topos Theory*
- ▷ Lurie, *Kerodon*
- ▷ Joyal, *The Theory of Quasi-Categories and its Applications*

1. Infinity Categories

1.1. Intuition. Let Cat be the category of all small categories. This had the following data:

small categories	objects
functors	morphisms
natural transformations	morphisms between morphisms.

If we consider natural transformations as part of the data of Cat, we have considerably more data than an ordinary category (often called a 1-category). Here we call Cat a *2-category*, and we can call the natural transformations *2-morphisms*.

1.2. Enrichment. The reason we were able to reasonably talk about 2-morphisms in Cat is due to the following observation:

For any $\mathscr{C}, \mathscr{D} \in \mathrm{Cat}$, we have that $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ is a 1-category, whose objects are functors and whose morphisms are natural transformations.

This leads to the following ad hoc definition: a *2-category* is any category whose homs are 1-categories.

To make this more explicit we would need the notion of enriched categories, which we won't discuss here. However to soup this up to a legitimate definition, we would say a *2-category* is any category enriched in a category of 1-categories.

1.3. **Infinity categories.** Inductively, we think about *n-categories* as being any category enriched in a category of $(n-1)$-categories. That means homs in an n-category are $(n-1)$-categories, and we think about n-morphisms as being morphisms between $(n-1)$-morphisms.

If we have n-morphisms for every n, then we say that we have an *∞-category*. We remark though that we can always view any category as an ∞-category by just letting all the higher morphisms be the identity.

Terminology 1.1. An (n, r)-category is a category for which all k-morphisms with $k > n$ are trivial, and all k-morphisms with $k > r$ are equivalences (will come back to a definition of this).

We could inductively define an $(n+1, r+1)$-category to be any category enriched in a category of (n, r)-categories. In particular an $(\infty, 1)$-category is any category enriched in a category of $(\infty, 0)$-categories.

Examples 1.2.

- ▷ a $(1, 1)$-category is an ordinary category
- ▷ a $(1, 0)$-category is a groupoid
- ▷ an "∞-category" generally refers to an $(\infty, 1)$-category, that is a category with higher morphisms above degree n invertible.

1.4. **Spaces are ∞-groupoids.**

Definition 1.3. An *∞-groupoid* is an $(\infty, 0)$-category.

For example, any topological space canonically determines an $(\infty, 0)$-groupoid as follows:

0-morphisms (objects)	points
1-morphisms	directed paths between points
2-morphisms	homotopies of paths
3-morphisms	homotopies of homotopies
$\vdots$	$\vdots$

1.5. **Various models.** Referring to something as an "infinity-category" is a bit loaded. In order to get a good handle on infinity categories, we should have some type of *model* of what a good theory of infinity categories should look like.

In particular a model should be a home for infinity categories, i.e. *a category whose objects are infinity categories.* In our previous example, for instance, we saw that Top was a good model for ∞-groupoids.

Here are a few models of $(\infty, 1)$-categories:

- ▷ quasi-categories
- ▷ simplicially enriched categories
- ▷ topologically enriched categories
- ▷ Segal categories
- ▷ complete Segal spaces

These are all "equivalent" in the sense that they have model structures and Quillen equivalences between them.

For the remainder of this talk, an $(\infty, 1)$-category will mean a quasi-category.

1.6. **Quasi-categories.** Recall that we have a standard n-simplex $\Delta^n \in \text{sSet}$. Its boundary is denoted by $\partial\Delta^n$.

Definition 1.4. The *ith horn*, denoted Λ_i^n, is the boundary $\partial\Delta^n$ minus the face opposite the ith vertex.

$$\begin{tikzcd} & \{1\} \arrow[dr, dashed] \arrow[d, phantom, "\Lambda^2_0"] & \\ \{0\} \arrow[ur] \arrow[rr] & {} & \{2\} \end{tikzcd} \qquad \begin{tikzcd} & \{1\} \arrow[dr] \arrow[d, phantom, "\Lambda^2_1"] & \\ \{0\} \arrow[ur] \arrow[rr, dashed] & {} & \{2\} \end{tikzcd} \qquad \begin{tikzcd} & \{1\} \arrow[dr] \arrow[d, phantom, "\Lambda^2_2"] & \\ \{0\} \arrow[ur, dashed] \arrow[rr] & {} & \{2\} \end{tikzcd}$$

Definition 1.5. We say that a simplicial set X is a *quasicategory* if any inclusion of a horn Λ^n_i, with $0 < i < n$, extends to an inclusion of the n-simplex:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

We denote by qCat the full subcategory of sSet containing all quasi-categories.

1.7. **Horn filling.** Morally, what does it mean for the dashed arrow to exist:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Consider the smallest example: Λ^2_1.

$$\begin{tikzcd} & \{1\} \arrow[dr] \arrow[d, phantom, "\Lambda^2_1"] & \\ \{0\} \arrow[ur] \arrow[rr, dashed] & {} & \{2\} \end{tikzcd}$$

The inclusion of this into the simplicial set X corresponds to the selection of three 0-cells (which we consider to be objects) and two composable 1-cells (which we consider to be morphisms). The existence of a "filling" of this horn (an extension to the 2-simplex) means that you can compose these 1-cells in a way that *maybe doesn't commute strictly*, but commutes up to some 2-cell.

Analogously, filling a horn Λ^n_i for $0 < i < n$ means that for any composable collection of $(n-1)$-morphisms in a quasi-category X, there is a way to compose them weakly in X, where the composition is witnessed by some n-cell.

1.8. **Nerves of categories: Λ^2_1.** Suppose $X = N(\mathscr{C})$ is the simplicial set obtained as the nerve of some small category $\mathscr{C}$ (remember the nerve had as n-cells strings of n-composable morphisms in $\mathscr{C}$).

For any inclusion $\Lambda^2_1 \to N\mathscr{C}$, this specifies two morphisms f and g in $\mathscr{C}$ which are composable. We remark that this horn can be filled uniquely by the composite $g \circ f$, and the 2-cell witnessing this composition is the identity.

1.9. **Nerves of categories: Λ^3_2.** The image of $\Lambda^3_2 \to N\mathscr{C}$ looks like:

$$\begin{tikzcd}
& \bullet \arrow[dd, "g"] \arrow[dr, "hg"] & \\
\bullet \arrow[ur, "f"] \arrow[rr] \arrow[dr, "gf"'] & & \bullet \\
& \bullet \arrow[ur, "h"'] &
\end{tikzcd}$$

where the back face is missing. The bottom face exists, so the back unlabelled arrow must be equal to the composite of h and gf, giving the arrow $h \circ (gf)$. In order to fill the back face, we must see that the back arrow commutes up to some higher cell, that is, there is a 2-cell witnessing the composite $(hg) \circ f \Rightarrow h \circ (gf)$.

However *because $\mathscr{C}$ was a category*, we have associativity of morphisms. Thus the back face fills, and the entire 3-cell in the center fills, corresponding to the fact that the following composites are equal:

$$h \circ g \circ f = h \circ (g \circ f) = (h \circ g) \circ f.$$

1.10. **Nerves of categories: higher horns.** As you might imagine, filling other horns in $N\mathscr{C}$ has analogous interpretations, corresponding to various ways to compose n-composable morphisms. Moreover since the composition is strict (higher cells witnessing this composition are the identity) we have the following result.

Proposition 1.6. The nerve of any small 1-category is a quasi-category; moreover, horns fill uniquely: for $0 < i < n$ we have a unique dashed map

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X \\
\Delta^n \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

This condition is actually sufficient to recognize when a simplicial set arises as the nerve of a category.

Proposition 1.7. We have that a simplicial set X is the nerve of a category $\mathscr{C}$ if and only if for all $0 < i < n$, the inclusion of any horn $\Lambda^n_i \to X$ extends *uniquely* to the inclusion of an n-simplex.

1.11. **Outer horns.** For quasi-categories, we said that we wanted filling for horns Λ^n_i where $0 < i < n$. Why shouldn't we expect filling for $i = 0, n$? Consider the following example:

$$\begin{tikzcd}
& \{1\} \arrow[dr, dashed] & \\
\{0\} \arrow[ur, "f"] \arrow[rr, "g"'] & & \{2\}.
\end{tikzcd}$$

Say we were mapping $\Lambda^2_0 \to N\mathscr{C}$ to the nerve of a category. In order for the dashed map to exist, it must be equal to gf^{-1}, that is, f must be an isomorphism in the category $\mathscr{C}$. In general there is no way to guarantee this. However if all maps in $\mathscr{C}$ were isomorphisms, then we would have this filling.

Proposition 1.8. The nerve of a groupoid admits horn filling for all Λ^n_i, where $0 \leq i \leq n$.

Definition 1.9. If a simplicial set X admits horn filling for all Λ^n_i for $0 \leq i \leq n$, we say it is a *Kan complex.*

The category Kan of Kan complexes serves as a model for $(\infty, 0)$-categories.

1.12. **Hom-sets.** The full subcategory $\mathrm{qCat} \subseteq \mathrm{sSet}$ serves as a model for $(\infty, 1)$-categories. In particular for $C, D \in \mathrm{qCat}$, we define an ∞-*functor* $F : C \to D$ to just be any morphism in the ambient category of simplicial sets.

Let $X \in \mathrm{qCat}$, then for two vertices $a, b \in X_0$ (remember these are supposed to be objects) we should describe $\mathrm{Hom}_X(a, b) =: X(a, b)$. By our discussion of enrichment, we should expect this object to be a Kan complex.

Consider the source and target maps

$$(s, t) : \mathrm{Hom}_{\mathrm{qCat}}(\Delta^1, X) \to \mathrm{Hom}_{\mathrm{qCat}}(\Delta^0 \amalg \Delta^0, X) = X \times X,$$

and let $X(a, b)$ denote the fiber of this map over the pair (a, b). We define this to be the hom-object $\mathrm{Hom}_X(a, b)$.

Proposition 1.10. [Lur09, 1.2.2.3] If $X \in \mathrm{qCat}$ then $X(a, b) \in \mathrm{Kan}$ for any $a, b \in X_0$.

1.13. **Adjunctions.** Let $\mathscr{C}$ and $\mathscr{D}$ be quasi-categories, and let $a \in \mathscr{C}$ and $b \in \mathscr{D}$. We say that ∞-functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are *adjoint* if we have a natural weak equivalence of Kan complexes

$$\mathrm{Hom}_{\mathscr{D}}(Fa, b) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(a, Gb).$$

Here a weak equivalence of Kan complexes means a weak equivalence after geometric realization.

1.14. **Terminal objects.** Let $\mathscr{C} \in \mathrm{qCat}$.

Definition 1.11. We say that $x \in \mathscr{C}_0$ is *terminal* if, for every $a \in \mathscr{C}_0$, the Kan complex $\mathscr{C}(a, x)$ is contractible (meaning its geometric realization is contractible). Similarly, $x \in \mathscr{C}_0$ is *initial* if $\mathscr{C}(x, a)$ is contractible for all $a \in \mathscr{C}_0$.

We say $\mathscr{C}$ is *pointed* if it has a *zero object*, which is an object that is both initial and terminal.

In general limits and colimits are hard to construct, see Higher Topos Theory Chapter 4 for more detail.

1.15. **Cofibers.** Suppose that $\mathscr{C}$ is a quasi-category which has a zero object, denoted $*$, and pushouts.

Definition 1.12. The *cofiber* of a morphism $f : X \to Y$ in $\mathscr{C}$ is define to be the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "!"'] & Y \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{cofib}(f) \end{tikzcd}$$

We refer to a sequence $X \xrightarrow{f} Y \to \mathrm{cofib}(f)$ as a *cofiber sequence.*

Definition 1.13. The *suspension* of X is defined to be the cofiber of the unique map $X \xrightarrow{!} *$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X. \end{tikzcd}$$

2. K_0 FOR INFINITY CATEGORIES

2.1. $K_0(\mathscr{C})$**.** Let $\mathscr{C}$ be a pointed ∞-category admitting pushouts. Then define $K_0(\mathscr{C})$ to be the free abelian group $[X]$ on objects of $\mathscr{C}$ modulo that a cofiber sequence

$$Z \to X \to Y$$

gives the relation $[Z] + [Y] = [X]$.

Exercise 2.1. $K_0(\mathscr{C})$ is abelian.

Exercise 2.2. We have that $[*] = 0$.

Exercise 2.3. We have that $[\Sigma X] = -[X]$.

Warning: If $\mathscr{C}$ admits infinite coproducts, then any object X fits into a cofiber sequence

$$\amalg_{n \geq 1} X \to \amalg_{n \geq 0} X \to X,$$

for which we see $[X] = 0$.

2.2. **Functoriality of** $K_0(\mathscr{C})$**.** Suppose $\mathscr{C}$ and $\mathscr{D}$ are pointed ∞-categories with pushouts. What conditions do we need on a functor $F : \mathscr{C} \to \mathscr{D}$ to induce a group homomorphism $K_0(\mathscr{C}) \to K_0(\mathscr{D})$?

Clearly we need F to preserve the zero object. Moreover we need that $[F(X \amalg Y)] = [F(X)] + F[Y]$, that is, since $X \to X \amalg Y \to Y$ is a cofiber diagram, so must be $F(X) \to F(X \amalg Y) \to F(Y)$. Therefore we should require F to preserve cofiber sequences as well.

Example 2.4. We remark that Σ was a colimit itself, thus it preserves all finite colimits. The functor $\Sigma : \mathscr{C} \to \mathscr{C}$ induces the multiplication by (-1) map on $K_0(\mathscr{C})$.

2.3. **Stable ∞-categories.**

Definition 2.5. We say an ∞-category $\mathscr{C}$ is *stable* if it is pointed, has pushouts, and so that the endofunctor $\Sigma : \mathscr{C} \to \mathscr{C}$ is an equivalence of categories.

Properties of stable ∞-categories:

(1) $\mathscr{C}$ has all finite limits and colimits
(2) A square in $\mathscr{C}$ is a pullback square if and only it is a pushout square
(3) A functor between stable ∞-categories preserves the zero object and cofibers if and only if it preserves all finite colimits.

Example 2.6. The ∞-category of spectra is stable.

For any category, it admits a *stabilization*, that is a functor to a stable infinity category, initial among such functors. This is given by the *Spanier-Whitehead category* $\mathrm{SW}(\mathscr{C})$, defined as the colimit:

$$\mathrm{SW}(\mathscr{C}) := \operatorname{colim} \left(\mathscr{C} \xrightarrow{\Sigma} \mathscr{C} \xrightarrow{\mathscr{C}} \xrightarrow{\Sigma} \cdots \right).$$

Remark 2.7. We see that Σ preserves all colimits, therefore

$$K_0(\mathscr{C}) \simeq K_0(\mathrm{SW}(\mathscr{C})).$$

So without loss of generality, for K_0, we can assume we are working with stable ∞-categories (this will be true in general).

3. Constructing higher K-theory

3.1. **Reminder: K-theory of a Waldhausen category.** Briefly, we had a category $\mathscr{C}$ with cofibrations and weak equivalences.

- ▹ we built categories $S_n\mathscr{C}$, whose objects were these "inverted staircase" diagrams of pushouts
- ▹ this gave a bisimplicial set $S_\bullet\mathscr{C}$, for which we could take any type of geometric realization, which all yielded equivalent spaces
- ▹ the fundamental group of this space was $K_0(\mathscr{C})$, which is shifted from what we want, so we loop the space to define $K(\mathscr{C})$.

3.2. **Waldhausen K-theory of ∞-categories. Goal**: to replicate the construction of Waldhausen K-theory for ∞-categories in order to define the higher K-theory of ∞-categories.

This will proceed as follows:

- ▹ starting with an ∞-category $\mathscr{C}$, we get the abelian group $K_0(\mathscr{C})$
- ▹ build categories $S_n\mathscr{C}$, which under the nerve functor are considered as ∞-categories
- ▹ take the geometric realization of this bisimplicial set
- ▹ again, take the loop space to arrive at $K(\mathscr{C})$

3.3. **Objects as paths: 2-simplices.** As in the Waldhausen construction for Waldhausen categories, we want to build a based space W, where each $[X] \in K_0(\mathscr{C})$ corresponds to a path p_X in W beginning and ending at the base point $*$.

For a cofiber sequence $X' \to X \to X''$ we want the paths $p_{X'} \circ p_{X''}$ and p_X to be homotopic, in order to encode the relations on $K_0(\mathscr{C})$ as relations in $\pi_1(W)$. That is, we need a 2-simplex:

$$\begin{tikzcd} & * \arrow[dr, "p_{X'}"] & \\ * \arrow[ur, "p_{X''}"] \arrow[rr, "p_X"'] & & * \end{tikzcd}$$

3.4. **3-simplices.** What can we say for an arbitrary pair of maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, not necessarily forming a cofiber diagram?

Proposition 3.1. We have that

$$[Z] = [X] + [Y/X] + [Z/Y].$$

Proof 1. Use the cofiber diagrams

$$\begin{aligned} X \to Z \to Z/X \quad &\rightsquigarrow \quad [Z] = [X] + [Z/X] \\ (Y/X) \to (Z/X) \to (Z/Y) \quad &\rightsquigarrow \quad [Z/X] = [Y/X] + [Z/Y]. \end{aligned}$$

□

Proof 2. Use the cofiber diagrams

$$\begin{aligned} Y \to Z \to Z/Y \quad &\rightsquigarrow \quad [Z] = [Y] + [Z/Y] \\ X \to Y \to Y/X \quad &\rightsquigarrow \quad [Y] = [X] + [Y/X]. \end{aligned}$$

□

We can compile all of this into the following 3-simplex:

$$
\begin{tikzcd}
& * \arrow[dr, "p_{Y/X}"] \arrow[ddr, "p_Y"'] & \\
& & * \arrow[d, "p_X"] \\
* \arrow[uur, "p_{Z/Y}"] \arrow[urr, dashed, "p_{Z/X}"] \arrow[rr, "p_Z"'] & & *
\end{tikzcd}
$$

Analogous information is available for any string of composable morphisms — how do we encode this information simplicially?

3.5. **3-simplices, continued.**

$$
\begin{tikzcd}
& * \arrow[dr, "p_{Y/X}"] \arrow[ddr, "p_Y"'] & \\
& & * \arrow[d, "p_X"] \\
* \arrow[uur, "p_{Z/Y}"] \arrow[urr, dashed, "p_{Z/X}"] \arrow[rr, "p_Z"'] & & *
\end{tikzcd}
$$

We could also encode this information via the following diagram, where we stipulate that all rectangles in sight are pushout diagrams:

$$
\begin{tikzcd}
* \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\
& * \arrow[r] & Y/X \arrow[r] \arrow[d] & Z/X \arrow[d] \\
& & * \arrow[r] & Z/Y \arrow[d] \\
& & & *
\end{tikzcd}
$$

When looking for higher analogs for how to encode the cofiber relations induced by a composite $X_1 \to X_2 \to \cdots \to X_n$ of maps, we obtain the correct notion by generalizing this diagram above.

3.6. **Higher simplices.** Thus when building our space W, we should adjoin an n-simplex for every diagram of the form

$$
\begin{tikzcd}
* \arrow[r] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X_n \arrow[d] \\
& * \arrow[r] & \cdots \arrow[r] \arrow[d] & X_n/X_1 \arrow[d] \\
& & \cdots \arrow[r] & \cdots \arrow[d] \\
& & & *
\end{tikzcd}
$$

Let's formalize this— let $[n]$ be the ordered set $\{0 < 1 < \cdots < n\}$, and let

$$[n]^{(2)} := \{(i,j) \in [n] \times [n] \ : \ i \leq j\}\,.$$

Then, by associating $[n]^{(2)}$ with its nerve, which is an ∞-category, we should view our n-simplices as objects of the ∞-functor category

$$\mathrm{Fun}(N([n]^{(2)}), \mathscr{C}).$$

3.7. **Gapped objects.** Explicitly we define an $[n]$-*gapped object of* $\mathscr{C}$ to be a functor $X : N([n]^{(2)}) \to \mathscr{C}$ so that

(i) for each $i \in [n]$ we have that $X(i,i) \cong *$ in $\mathscr{C}$ is the zero object
(ii) for each $i \leq j \leq k$ we have a pushout diagram

$$\begin{tikzcd} X(i,j) \arrow[r] \arrow[d] & X(i,k) \arrow[d] \\ X(j,j) \arrow[r] & X(j,k), \end{tikzcd}$$

equivalently using the previous condition, we have a cofiber sequence

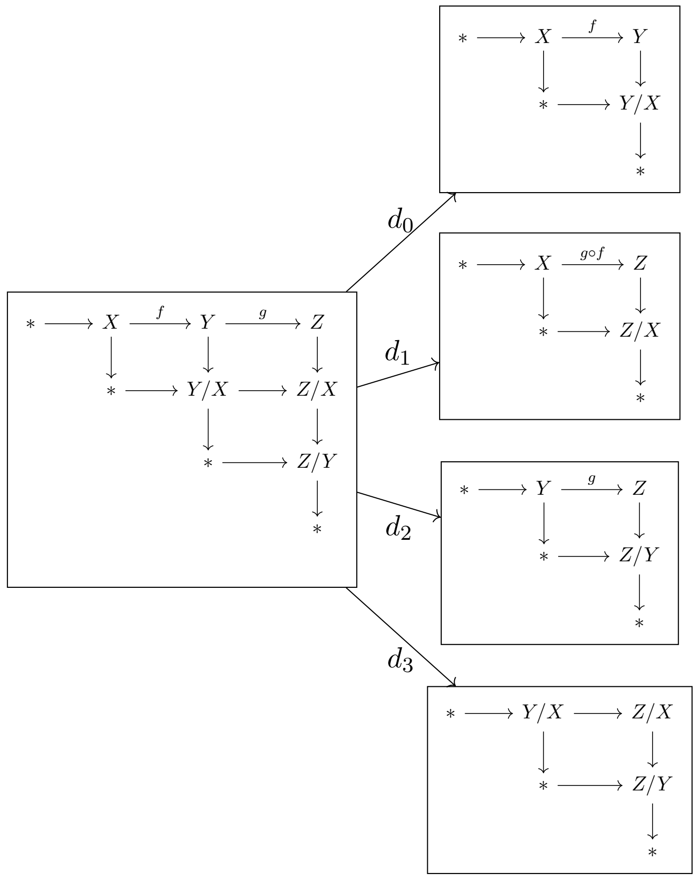
$$X(i,j) \to X(i,k) \to X(j,k).$$

We denote by $\mathrm{Gap}_{[n]}(\mathscr{C})$ the collection of all $[n]$-gapped objects. This forms an ∞-category.

Proposition 3.2. The inclusion functor Kan $\to$ qCat admits a right adjoint, which provides the largest Kan complex contained in a quasicategory.

We denote by $S_n(\mathscr{C})$ the largest Kan complex contained in $\mathrm{Gap}_{[n]}(\mathscr{C})$.

3.8. **Face and degeneracy maps.** For $S_2(\mathscr{C}) \subseteq \mathrm{Gap}_{[n]}(\mathscr{C})$, we provide the face maps:



3.9. **Degeneracy maps.** Degeneracy maps are less interesting, we simply add in an identity along the top row and add in identities horizontally going down.

3.10. **The simplicial Kan complex.** We now have a simplicial Kan complex $S_\bullet\mathscr{C}$. Regarding this as a bisimplicial set, we can take its geometric realization. We then define the K*-theory space*:

$$K(\mathscr{C}) := \Omega\,|S_\bullet\mathscr{C}|\,.$$

Properties:

- ▷ if $F : \mathscr{C} \to \mathscr{D}$ preserves finite colimits, it induces a continuous map $K(\mathscr{C}) \to K(\mathscr{D})$
- ▷ the projection functors $\mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $\mathscr{C} \times \mathscr{D} \to \mathscr{D}$ preserve finite colimits. These maps induce a homotopy equivalence

$$K(\mathscr{C} \times \mathscr{D}) \xrightarrow{\sim} K(\mathscr{C}) \times K(\mathscr{D})$$

- ▷ the coproduct functor

$$\begin{aligned} \amalg : \mathscr{C} \times \mathscr{C} &\to \mathscr{C} \\ (X, Y) &\mapsto X \amalg Y \end{aligned}$$

  preserves colimits, and therefore ascends to a monoidal structure

$$K(\mathscr{C}) \times K(\mathscr{C}) \to K(\mathscr{C}).$$

  This turns $K(\mathscr{C})$ into a grouplike E_∞-space, that is, an infinite loop space.
- ▷ The map $\mathscr{C} \to \mathrm{SW}(\mathscr{C})$ induces an equivalence

$$K(\mathscr{C}) \xrightarrow{\sim} K(\mathrm{SW}(\mathscr{C})).$$

3.11. **Examples.**

(1) If R is a ring, we can take $D^b(R)$, the derived category of the ring, which can be viewed as a stable ∞-category. We have that
$$K(D^b(R)) \cong K(R) \cong \mathrm{BGL}(R)^+,$$
therefore we recover the algebraic K-theory of the ring.

(2) Given a scheme X, we can take its category $\mathrm{Perf}(X)$ of perfect complexes, which has the structure of a stable ∞-category. Taking its K-theory we recover Thomason-Trobaugh K-theory

(3) Given a topological space X, its singular chains $\mathrm{Sing}(X)$ is a Kan complex, and therefore an ∞-category. Let $\mathscr{C} \subseteq \mathrm{Fun}(\mathrm{Sing}(X), \mathrm{Sp})$ be the subcategory on *compact objects* (see nLab). Then $K(\mathscr{C}) \simeq A(X)$ is the *A-theory* of the space X.

4. Additivity

4.1. **Additivity theorem (classically).** Let $\mathcal{E}(\mathscr{C})$ be the category whose objects are exact sequences $(A \to B \to C)$ in a Waldhausen category $\mathscr{C}$. Then there are three functors, $s, t, q : \mathcal{E}(\mathscr{C}) \to \mathscr{C}$ respectively picking out each of the three objects in any exact sequence.

Theorem 4.1. *(Additivity)* If $F' \to F \to F''$ is an exact sequence of functors between Waldhausen categories $\mathscr{C} \to \mathscr{D}$, then $K_n(F) = K_n(F') + K_n(F'')$.

Proof. We remark that giving such an exact sequence of functors is equivalent to giving a functor $\mathscr{C} \to \mathcal{E}(\mathscr{D})$, so we can reduce to proving the statement for the triple $(s, t, q) : \mathcal{E}(\mathscr{D}) \to \mathscr{D}$. We prove that the functor
$$\begin{aligned} \mathscr{D} \times \mathscr{D} &\to \mathcal{E}(\mathscr{D}) \\ (A, B) &\mapsto (A \to A \amalg B \to B) \end{aligned}$$
is a homotopy equivalence at the level of K-theory, and the result follows. □

4.2. **Additivity for ∞-categories.** To generalize $\mathcal{E}(\mathscr{C})$ for infinity categories, we want a category whose objects are cofiber sequences. As we can see, this is given by $\mathrm{Gap}_{[2]}(\mathscr{C})$, whose objects we recall are diagrams
$$\begin{tikzcd} * \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[d] \\ & * \arrow[r] & Z \arrow[d] \\ & & * \end{tikzcd}$$
where $X \to Y \to Z$ is a cofiber sequence. Since a cofiber sequence is determined up to equivalence by the map $f : X \to Y$, we have an equivalence of ∞-categories
$$\mathrm{Fun}(\Delta^1, \mathscr{C}) \simeq \mathrm{Gap}_{[2]}(\mathscr{C}).$$

Let $F : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{C} \times \mathscr{C}$ denote the map sending a functor $\Delta^1 \to \mathscr{C}$, whose image is an arrow $X \xrightarrow{\alpha} Y$, to the pair $(X, \mathrm{cofib}(\alpha))$.

Theorem 4.2. *(Additivity)* We have that F induces a homotopy equivalence
$$K(\mathrm{Fun}(\Delta^1, \mathscr{C})) \xrightarrow{\sim} K(\mathscr{C} \times \mathscr{C}) \simeq K(\mathscr{C}) \times K(\mathscr{C}).$$

At the level of quasi-categories, we have that F admits a right homotopy inverse, given by

$$\begin{aligned}\mathscr{C} \times \mathscr{C} &\to \mathrm{Fun}(\Delta^1, \mathscr{C})\\ (X, Y) &\mapsto (X \to X \amalg Y).\end{aligned}$$

This gives the homotopy inverse at the level of K-spaces.

4.3. Corollaries of additivity.

Corollary 4.3. Given a cofiber sequence of functors $F' \to F \to F''$ between pointed ∞-categories admitting finite colimits $\mathscr{C} \to \mathscr{D}$, we have that $K(F) = K(F') + K(F'')$.

Proof. We have three functors $s, t, q : \mathrm{Fun}(\Delta^1, \mathscr{D}) \to \mathscr{D}$ given by taking $X \to Y$ to X, Y, and Y/X, respectively. One can easily see that $K(t) = K(s) + K(q)$.

We see that the natural transformation $F' \to F$ determines a functor $H : \mathscr{C} \to \mathrm{Fun}(\Delta^1, \mathscr{D})$, and we can rewrite $K(F') = K(s) \circ K(H)$, $K(F) = K(t) \circ K(H)$, and $K(F'') = K(q) \circ K(H)$. $\square$

Corollary 4.4. We have that suspension induces a group homomorphism $K(\Sigma) : K_n(\mathscr{C}) \to K_n(\mathscr{C})$ which is multiplication by -1 for every n.

Proof. Apply additivity to the cofiber sequence of morphisms $\mathrm{id} \to * \to \Sigma$. $\square$

5. Universality (Blumberg, Gepner, Tabuada)

5.1. Overview and related results.

Results in this section are from [BGT13].

We will attempt to get a handle on what type of enlightening universal property the K-theory of ∞-categories satisfies.

For example given a category $\mathscr{C}$ with some notion of short exact sequences (exact category, triangulated category, Waldhausen category), we can say that an *Euler characteristic* valued in an abelian group A is an assignment of group elements for each isomorphism class in $\mathscr{C}$ which *splits short exact sequences*, that is:

$$0 \to X' \to X \to X'' \to 0 \quad \rightsquigarrow \chi(X) = \chi(X') + \chi(X'').$$

In this sense, $K_0(\mathscr{C})$ is the *universal target group* for Euler characteristics. We would ideally like to extend this universality results to higher K-theory.

5.2. Definitions and notation.

Denote by Cat_∞ the category of small ∞-categories (e.g. quasi-categories).

Denote by $\mathrm{Cat}_\infty^{\mathrm{ex}}$ the (pointed) category of small stable ∞-categories and exact functors (functors which preserve finite limits and colimits).

An ∞-category $\mathscr{C}$ is called *idempotent-complete* if its image under the Yoneda embedding (here the Yoneda embedding is into functors valued in spaces) is closed under retracts. We denote by $\mathrm{Cat}_\infty^{\mathrm{perf}}$ the category of small idempotent-complete stable ∞-categories, so we have an inclusion

$$\mathrm{Cat}_\infty^{\mathrm{perf}} \subseteq \mathrm{Cat}_\infty^{\mathrm{ex}}.$$

This inclusion admits a left adjoint (Higher Topos Theory, 5.1.4.2), which we denote by $\mathrm{Idem} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$.

5.3. Morita equivalence.

Two rings R and S are *Morita equivalent* if the categories Mod_R and Mod_S are equivalent. This is a *weaker notion* than ring isomorphism, but it is enough to guarantee that the algebraic K-theory of R and S coincide:

$$K(R) \cong K(S).$$

We say two small stable ∞-categories $\mathscr{C}, \mathscr{D} \in \mathrm{Cat}_\infty^{\mathrm{ex}}$ are *Morita equivalent* if $\mathrm{Idem}(\mathscr{C})$ and $\mathrm{Idem}(\mathscr{D})$ are equivalent, and a morphism $\mathscr{C} \to \mathscr{D}$ is a *Morita equivalence* if it induces an equivalence of categories $\mathrm{Idem}(\mathscr{C}) \xrightarrow{\sim} \mathrm{Idem}(\mathscr{D})$.

5.4. **Exact sequences.** A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ (of small stable idempotent-complete infinity categories) is *exact* if:

- ▷ the composite is zero
- ▷ $\mathcal{A} \to \mathcal{B}$ is fully faithful
- ▷ the induced map $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence.

A sequence is *split exact* if it is exact and there exist appropriate adjoint splitting maps.

A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ (small stable ∞-categories) is (split) exact if the associated sequence

$$\mathrm{Idem}(\mathcal{A}) \to \mathrm{Idem}(\mathcal{B}) \to \mathrm{Idem}(\mathcal{C})$$

is (split) exact in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

5.5. **Additive and localizing invariants.** Let $E : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathscr{D}$ be a functor to a stable presentable* ∞-category $\mathscr{D}$. We say E is an *additive invariant* if it:

- ▷ inverts Morita equivalences
- ▷ preserves filtered colimits
- ▷ sends split exact sequences to cofiber sequences.

We say E is a *localizing invariant* if it sends all exact sequences to cofiber sequences.

Localizing invariants are additive, but the converse does not hold; a counterexample is K^{cn}, connective algebraic K-theory.

5.6. **Some more notation (sorry).** Let $\mathrm{Fun}_{\mathrm{add}}\left(\mathrm{Cat}_\infty^{\mathrm{ex}}, \mathscr{D}\right)$ denote the functor category of additive invariants valued in $\mathscr{D}$.

Let $\mathrm{Fun}^L(\mathscr{C}, \mathscr{D})$ be the ∞-category of *colimit-preserving functors*.

Let $\mathcal{S}_\infty$ denote the ∞-category of spectra.

5.7. **The universal additive invariant.** Let $\mathcal{U}_{\mathrm{add}} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{M}_{\mathrm{add}}$ denote the following composite, where $\mathcal{M}_{\mathrm{add}}$ denotes the resulting category:

- ▷ apply $\mathrm{Idem} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$
- ▷ take the Yoneda embedding y where presheaves are valued in the ∞-category of spectra $\mathcal{S}_\infty$
- ▷ restrict to the subcategory of compact objects
- ▷ if $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ is a split exact sequence, localize at maps of the form $y(\mathscr{B})/y(\mathscr{A}) \to y(\mathscr{C})$
- ▷ stabilize.

Theorem 5.1. [BGT13, 6.7, 6.10] The functor $\mathcal{U}_{\mathrm{add}}$ is an additive invariant, and moreover is the *universal additive invariant*, in the sense that, for any stable presentable ∞-category $\mathscr{D}$ we have an equivalence of ∞-categories:

$$\mathrm{Fun}^L(\mathscr{M}_{\mathrm{add}}, \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{ex}}, \mathscr{D}).$$

That is, *every additive invariant* factors through $\mathscr{M}_{\mathrm{add}}$.

5.8. **The application for K-theory.** We should view $\mathscr{M}_{\mathrm{add}}$ as some category of non-commutative motives which is the receptacle for all information about additive invariants. This turns out to be enriched in spectra.

We claim that $\mathcal{S}_\infty^\omega$, the ∞-category of compact spectra, is a stable idempotent-complete ∞-category, that is, it is an element in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Let $\mathcal{A}$ be any other element of $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

Theorem 5.2. [BGT13, 1.3] There is an equivalence of spectra

$$K(\mathscr{A}) \simeq \mathrm{Map}_{\mathscr{M}_{\mathrm{add}}}(\mathcal{U}_{\mathrm{add}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\mathrm{add}}(\mathscr{A})).$$

In particular for $n \in \mathbb{Z}$ we have an isomorphism of abelian groups

$$K_n(\mathscr{A}) \cong \mathrm{Hom}\left(\mathcal{U}_{\mathrm{add}}(\mathcal{S}_\infty^\omega), \Sigma^{-n}\mathcal{U}_{\mathrm{add}}(\mathcal{A})\right).$$

The suspension functor $\Sigma : \mathscr{M}_{\text{add}} \to \mathscr{M}_{\text{add}}$ turns out to agree with $S_\bullet$ *(BGT, 7.17)*.

5.9. **The universal localizing invariant.** An analogous construction may be made to obtain a *universal localizing invariant*

$$\mathcal{U}_{\text{loc}} : \text{Cat}_\infty^{\text{ex}} \to \mathcal{M}_{\text{loc}}.$$

This category $\mathcal{M}_{\text{loc}}$ is analogously some category of non-commutative motives which receives all information about localizing invariants. Its suspension is also given by $S_\bullet$.

Analogous results to those above can be used to describe the *non-connective K-theory* of idempotent-complete stable ∞-categories.

5.10. **Why do we care?** Algebraic K-theory of ∞-categories was *not defined* in terms of universal constructions of presheaves and localizations for infinity categories, so this provides a more universal construction.

The previous result with the Yoneda lemma provides a *total classification of natural transformations from K-theory to other additive (or localizing) invariants.*

This construction provides a tractable formulation of other interesting invariants, for example *topological Hochschild homology*, which is an additive invariant. Via the previous classification we can understand and characterize the trace map $K \to \text{THH}$, an active area of research (see [BGT13, §10]).

6. Universality (a la Barwick)

Results in this section are from [Bar16].

6.1. **Slogan.**

Algebraic K-theory is "a *universal homology theory*, which takes suitable higher categories as input and produces either spaces or spectra as output."

6.2. **Definitions and notation.** For any ∞-category $\mathscr{C}$, we denote by $\iota\mathscr{C}$ its maximal Kan subcomplex.

A *pair of ∞-categories* $(\mathscr{C}, \mathscr{C}_\dagger)$ is an ∞-category $\mathscr{C}$ along with an ∞-subcategory $\mathscr{C}_\dagger$ so that

$$\iota\mathscr{C} \subseteq \mathscr{C}_\dagger \subseteq \mathscr{C}.$$

A morphism of $\mathscr{C}_\dagger$ is called an *ingressive morphism.*

A *functor of pairs* $(\mathscr{C}, \mathscr{C}_\dagger) \to (\mathscr{D}, \mathscr{D}_\dagger)$ is a functor $\mathscr{C} \to \mathscr{D}$ sending ingressive morphisms to ingressive morphisms.

6.3. **Examples of pairs.** For any ∞-category $\mathscr{C}$, there are two trivial pairs:

(1) the *minimal pair*, denoted $\mathscr{C}^\flat$, which is the pair $(\mathscr{C}, \iota\mathscr{C})$, where we recall $\iota\mathscr{C}$ is the maximal Kan subcomplex
(2) the *maximal pair*, denoted $\mathscr{C}^\sharp$, which is the pair $(\mathscr{C}, \mathscr{C})$.

6.4. **Waldhausen ∞-categories.** We say a pair $(\mathscr{C}, \mathscr{C}_\dagger)$ is a *Waldhausen ∞-category* if the following axioms hold:

(1) $\mathscr{C}$ is pointed, and the map $0 \to X$ is ingressive for any X
(2) pushouts of ingressive morphisms exist and are ingressive.

We define a *morphism of Waldhausen ∞-categories* to be any exact functor, by which we mean it:

- ▷ preserves zero objects
- ▷ sends pushouts along an ingressive morphism to pushouts along an ingressive morphism.

We think (roughly) as the ∞-categorical structure encoding and generalizing weak equivalences, and ingressive morphisms as encoding cofibrations.

We denote by Wald_∞ the ∞-category of Waldhausen ∞-categories (*Barwick, §2*).

6.5. **Examples of Waldhausen ∞-categories.** Equipped with the minimal pair structure $\mathscr{C}^\flat = (\mathscr{C}, \iota\mathscr{C})$, we have a Waldhausen ∞-category if and only if $\mathscr{C}$ is a contractible Kan complex.

With the maximal pair structure $\mathscr{C}^\sharp = (\mathscr{C}, \mathscr{C})$, we have a Waldhausen ∞-category if $\mathscr{C}$ has a zero object and all finite colimits.

Any stable ∞-category equipped with the maximal pair structure is a Waldhausen ∞-category.

If $(\mathscr{C}, \mathscr{C}^{\mathrm{cof}})$ is an *ordinary 1-category with cofibrations*, then its nerve $(N\mathscr{C}, N\mathscr{C}^{\mathrm{cof}})$ is a Waldhausen ∞-category.

6.6. **Theories.** A functor of ∞-categories is *reduced* if it sends the zero object to the terminal object.

Let $\mathcal{E}$ be the category Kan of Kan complexes (or more generally, any ∞-topos). Then we define a *$\mathcal{E}$-valued theory* to be any reduced functor

$$\phi : \mathrm{Wald}_\infty \to \mathcal{E}$$

which preserves filtered colimits. Denote by

$$\mathrm{Thy}(\mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Wald}_\infty, \mathcal{E})$$

the full subcategory spanned by $\mathcal{E}$-valued theories.

6.7. **Examples of theories.** The easiest example of a theory $\iota \in \mathrm{Thy}(\mathcal{E})$ is the *interior functor* theory:

$$\iota : \mathrm{Wald}_\infty \to \mathrm{Kan}$$
$$(\mathscr{C}, \mathscr{C}_\dagger) \mapsto \iota\mathscr{C},$$

sending a Waldhausen ∞-category to its maximal Kan subcomplex.

Give Γ^{op}, the category of finite pointed sets, a set of cofibrations given by monomorphisms of sets with disjoint basepoints. Then $N\Gamma^{\mathrm{op}} \in \mathrm{Wald}_\infty$, and moreover this object *corepresents the interior functor* in the sense that

$$\mathrm{Fun}_{\mathrm{Wald}_\infty}(N\Gamma^{\mathrm{op}}, \mathscr{C}) \xrightarrow{\sim} \iota\mathscr{C}$$

for any $\mathscr{C} \in \mathrm{Wald}_\infty$ [Bar16, Prop. 10.5].

6.8. **Additive theories.** A theory is *additive* if it sends direct sums to products, and a few other technical axioms that are very involved to state [Bar16, 7.4, 7.5]. We think about them as the correct analog, in this setting, of functors splitting exact sequences.

In some sense we would want K-theory to be an additive theory.

Example 6.1. The interior functor $\iota : \mathrm{Wald}_\infty \to \mathrm{Kan}$ is *not additive.*

For theories that fail to be additive, can we provide some additive approximation to them?

Theorem 6.2. *(Additivization)*[Bar16, 7.8] Any theory $\phi : \mathrm{Wald}_\infty \to \mathcal{E}$ admits an additivization $D\phi$. Moreover, it is computable as

$$D\phi \simeq \mathrm{colim}_{n\to\infty} \left(\Omega^n_{\mathcal{E}} \circ \Phi \circ \Sigma^n \circ y\right),$$

where y is the map to the derived category $D(\mathrm{Wald}_\infty)$, and Φ is the derived functor of ϕ.

In the sense of Goodwillie calculus, this is the *linearization* of the functor ι.

6.9. **Algebraic K-theory of Waldhausen ∞-categories. Huge definition/theorem:** The *algebraic K-theory functor*

$$K : \mathrm{Wald}_\infty \to \mathrm{Kan}$$

is defined to be the *additivization* of the interior functor $\iota : \mathrm{Wald}_\infty \to \mathrm{Kan}$.

This admits a *canonical delooping*, so we may assume that K-theory is valued in connective spectra (see [Bar16, §7]).

6.10. **Classifying transformations out of K-theory.** Recall that ι was corepresented by $N\Gamma^{\mathrm{op}}$. Combining this fact with the universal property of additivization, we obtain a classification of natural transformations from K-theory to *any other additive theory*.

Proposition 6.3. [Bar16, 10.2, 10.5.1] For any additive theory $\phi : \mathrm{Wald}_\infty \to \mathrm{Kan}$, there is a homotopy equivalence

$$\mathrm{Map}(K, \phi) \simeq \mathrm{Map}(\iota, \phi) = \mathrm{Map}(\mathrm{Fun}_{\mathrm{Wald}_\infty}(N\Gamma^{\mathrm{op}}, -), \phi) \simeq \phi(N\Gamma^{\mathrm{op}}).$$

Corollary 6.4. *([Bar16, 10.5.2], Barratt-Priddy-Quillen)* Applying this to $\phi = K$, we get that the *endomorphisms of algebraic K-theory* are

$$\mathrm{End}(K) = K(\Gamma^{\mathrm{op}}) = QS^0 = \mathrm{colim}_n \Omega^n S^n.$$

6.11. **Relation to A-theory.** For any ∞-topos $\mathcal{E}$, we can take its ∞-category of pointed compact objects $\mathcal{E}_*^\omega$. Its algebraic K-theory

$$K(\mathcal{E}_*^\omega)$$

is called the *A-theory of $\mathcal{E}$*.

For $X \in \mathrm{Kan}$, we have an ∞-topos $\mathrm{Fun}(X, \mathrm{Kan})$, and we have that

$$K(\mathrm{Fun}(X, \mathrm{Kan})) = A(X)$$

agrees with the A-theory of X that we have seen.

7. Conclusion

7.1. **Conclusion.** Infinity categories are the most general setting for a study of algebraic K-theory.

Universal constructions of algebraic K-theory provide a framework for the analysis of K-theory and other theories like A-theory.

Representability results allow a more tangible grasp of interactions between algebraic K-theory and other related theories like THH and TC, as well as trace maps between these.

References

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