EQUIVALENT ENUMERATIVE GEOMETRY

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ABSTRACT. We formulate an *equivariant conservation of number*, which proves that a generalized Euler number of a complex equivariant vector bundle can be computed as a sum of local indices of an arbitrary section. This involves an expansion of the Pontryagin–Thom transfer in the equivariant setting. We leverage this result to commence a study of enumerative geometry in the presence of a group action. As an illustration of the power of this machinery, we prove that any smooth complex cubic surface defined by a symmetric polynomial has 27 lines whose orbit types under the S_4-action on $\mathbb{CP}^3$ are given by $[S_4/C_2]+[S_4/C_2']+[S_4/D_8]$, where C_2 and C_2' denote two non-conjugate cyclic subgroups of order two. As a consequence we demonstrate that a real symmetric cubic surface can only contain 3 or 27 lines.

CONTENTS

1. INTRODUCTION

Enumerative geometry poses geometric questions of the form "how many?" and expects integral answers. Over two millennia ago Apollonius asked how many circles are tangent to any three generic circles drawn on the plane. In the mid-1800's Salmon

Date: May 5, 2023.

and Cayley famously proved that there are 27 lines on a smooth cubic surface over the complex numbers, and it is a classical result that there are 2,875 lines on a general quintic threefold. The power of enumerative geometry lies in the principle of *conservation of number* — that enumerative answers are conserved under changes in initial parameters: there are eight circles tangent to *any* three generic circles, 27 lines on *any* smooth cubic surface, and 2,875 lines on *any* general quintic threefold.

In this work we propose solving enumerative problems in the presence of a group action. Related works include [Dam91, Bet20], but these differ in perspective. We formulate and prove a version of conservation of number in this context which allows us to compute answers to equivariant enumerative problems valued in the Burnside ring of a group.

Theorem 1.1. *(Equivariant conservation of number)* Let G be any finite compact Lie group, and let $p\colon E \to M$ be an equivariant complex vector bundle of rank n over a smooth proper G-equivariant n-manifold, and let A be a complex oriented $\mathrm{RO}(G)$-graded cohomology theory. Let $\sigma\colon M \to E$ be any equivariant section with isolated simple zeros. Then we have a well-defined Euler number valued in $\pi_0^G A$, computed by:

$$n(E) = \sum_{G\cdot x \subseteq Z(\sigma)} \mathrm{Tr}_{G_x}^G(1).$$

Working in homotopical complex bordism MU_G, as a corollary we may see that given two such sections σ, σ', there is an isomorphism of G-sets between their zero loci $Z(\sigma) \cong Z(\sigma')$ (Theorem 5.17). In other words, the G-action on the solutions to an enumerative problem is conserved.

Our result is more general, admitting local indices for more general zero loci than isolated simple points (see Lemma 5.3). However the context stated above is sufficient to being doing computations.

To illustrate the power of this machinery, consider the case of a smooth cubic surface $X = V(F) \subseteq \mathbb{CP}^3$. We will say that X is S_4-*symmetric* (or just *symmetric* for short) if it is fixed under the S_4-action on $\mathbb{CP}^3$ by permuting coordinates (equivalently, $F(x_0, x_1, x_2, x_3)$ is a symmetric homogeneous polynomial). We know classically that there are 27 lines on X, however under the S_4-action lines on X are mapped to other lines on X. It is natural then to inquire whether the S_4-orbits of the lines on X are conserved as the symmetric cubic surface varies. It turns out that this question admits an answer that doesn't depend upon the choice of S_4-symmetric cubic surface.

Theorem 1.2. On *any* smooth symmetric cubic surface over the complex numbers, the 27 lines come in the following orbits:

$$[S_4/C_2] + [S_4/C_2'] + [S_4/D_8],$$

where C_2 and C_2' are non-conjugate subgroups of S_4 of order two. Explicitly, there are 12 lines in an orbit with isotropy group $C_2 = \langle (1\ 2) \rangle$, 12 lines in an orbit with isotropy group $C_2' = \langle (1\ 3)(2\ 4) \rangle$, and three lines in an orbit with isotropy group D_8.

On the famous *Clebsch surface*, which is symmetric, all 27 lines are in fact defined over the reals. The 27 lines collected into their orbits can be visualized as in Figure 1.

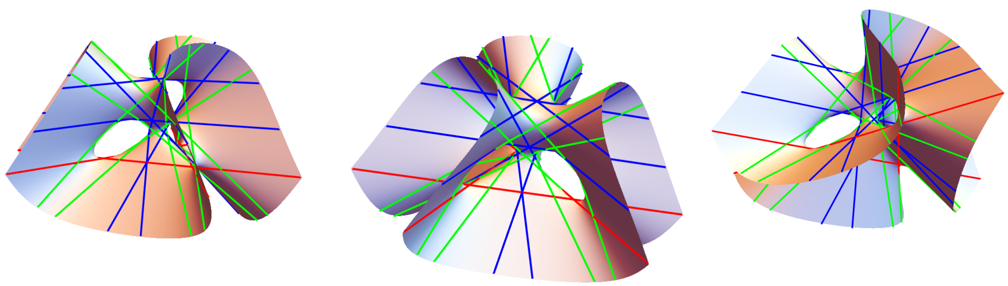

FIGURE 1. The 27 lines on the Clebsch surface, grouped into S_4-orbits according to color, pictured from a few different angles. An animated version is available on the author's webpage.

Given a real cubic surface, as it admits 27 lines after base changing to the complex numbers it is natural to ask how many of these are defined over the reals. Schläfli's Theorem tells us that a smooth real cubic surface can contain 3, 7, 15, or 27 lines, and all of these possibilities do occur [Sch58]. Under the presence of symmetry we can refine this result.

Theorem 1.3. A smooth real symmetric cubic surface can contain either 3 or 27 lines, and both of these possibilities do occur.

1.1. **Outline.** In Section 2 we discuss the theory of equivariant retractive spaces and parametrized spectra. We extend the theory of duality as laid out in [Hu03], and discuss dualizing objects in terms of cotangent complexes. This allows us to define Thom transformations analogous to those found in the motivic setting, and to flesh out the six functors formalism for genuine orthogonal parametrized G-spectra.

In Section 3, we extend the scope of the Pontryagin–Thom transfer from [Hu03, MS06, ABG18] to hold for morphisms which factor as a smooth immersion followed by a smooth proper family of manifolds. This induces Gysin pushforwards of cohomology classes along a larger classes of maps which we call *smoothable proper morphisms*.

In Section 4 we provide a broad definition of compactly supported equivariant cohomology, twisted by a perfect complex, valued in any equivariant ring spectrum. This culminates in the important result that, under certain orientation assumptions, cohomology classes twisted by a vector bundle can be pushed forward and expressed as sums of local contributions coming from the components of the zero locus of a section of a bundle.

In Section 5 we discuss refined Euler classes in the parametrized equivariant setting. We recap the theory of equivariant complex orientations, and state and prove equivariant conservation of number (Theorem 1.1, as Theorem 5.17). We use this to commence a study of enumerative geometry in the equivariant setting.

In Section 6, we provide an application of equivariant conservation of number by investigating the orbits of the 27 lines on a smooth symmetric cubic surface and proving that they are independent of the choice of symmetric cubic surface (Theorem 1.2, as Theorem 6.2). We argue that both the field of definition and the topological type (Definition 6.8) of a line are preserved under the group action. This allows us to eliminate the possibilities of seven or 15 real lines on a real symmetric cubic, refining Schläfli's Theorem in the symmetric setting (Theorem 1.3, as Theorem 6.11).

1.2. **Acknowledgements.** Thank you to Mona Merling and Kirsten Wickelgren for inspiring and supporting this work, to Cary Malkiewich for countless conversations about details big and small, and to Markus Hausmann for helpful correspondence about equivariant orientations. Thank you also to Frank Sottile, William Balderrama, and Mike Hopkins for fruitful conversation about this work and related topics. The author is supported by an NSF Graduate Research Fellowship (DGE-1845298)

2. Retractive spaces and parametrized spectra, equivariantly

In this section we will establish technical machinery with the ultimate goal of obtaining well-defined Euler numbers for equivariant sections of complex bundles over smooth proper G-manifolds. In direct analogy to the theory of Euler classes in motivic homotopy theory, we will want to work over a base space, i.e. in a parametrized way. A suitable setting for our study is the category $G\mathcal{OS}(X)$ of genuine orthogonal G-spectra parametrized over a G-space or G-spectrum X [Hu03, Mal20]. Such a theory admits a six functor formalism which allows us to translate data across equivariant maps.

An advantage of working in this setting is the presence of *Thom transformations*, which we will define as certain auto-equivalences in the stable setting. Explicitly, when working over a G-manifold M, we can take an equivariant vector bundle $E \to M$, and smash over M with the fiberwise Thom space $\mathrm{Th}_M(E)$. We use these transformations to define twisted cohomology classes valued in any genuine ring spectrum, and develop the theory of their pushforwards. In particular we will see that we have a well-defined *Euler class*, which pushes forward to an *Euler number* in complex oriented cohomology theories. This number can be interpreted, and will serve as our main tool for solving equivariant enumerative problems.

Assumption 2.1. All spaces and all maps will be assumed to be equivariant with respect to the action of a group G unless otherwise explicitly stated.

2.1. **Basic definitions.** In attempting to form a theory of G-spaces "over" a space X, one might naively consider the slice category $G\mathtt{Top}/X$. This category admits a wealth of nice properties, but misses quite a few. For example it does not admit a zero object, and therefore we cannot make sense of phenomena like suspension and thus stabilization. In order to rectify this, we slice under the terminal object. Just as pointed topological spaces are a slice category $*/\mathtt{Top}$, we obtain a pointed category $\mathcal{R}_G(X) := \left(X \xrightarrow{\mathrm{id}} X\right) / (G\mathtt{Top}/X)$.

Definition 2.2. The category $\mathcal{R}_G(X)$ of *retractive G-spaces over X* has as objects commutative diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\mathrm{id}"'] & Y \arrow[d] \\ & X. \end{tikzcd}$$

That is, the category of spaces which equivariantly retract onto X. The morphisms are equivariant maps $Y \to Y'$ which commute with the inclusion and projection maps.

Example 2.3. The category of retractive G-spaces over a point $\mathcal{R}_G(*)$ is the category of based G-spaces $G\mathtt{Top}_*$.

Example 2.4. For any subgroup $H \subseteq G$, there is an equivalence of categories $\mathcal{R}_G(G/H) \simeq H\mathtt{Top}_*$.

Example 2.5. Let Y be any G-space equipped with a map $f\colon Y \to X$. Denote by $Y_+ \in \mathcal{R}_G(X)$ the retractive space $Y \amalg X$, with inclusion given by mapping X to itself, and projection given by f and the identity:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\mathrm{id}"'] & Y \amalg X \arrow[d, "f \amalg \mathrm{id}"] \\ & X. \end{tikzcd}$$

This is exactly what we mean by "pointing" the slice category $G\mathtt{Top}/X$. We remark that the process of adjoining a point (i.e. a copy of X) to a space over X defines a left adjoint to the forgetful functor:

$$(-)_+ \colon G\mathtt{Top}/X \leftrightarrows \mathcal{R}_G(X) \colon U.$$

Example 2.6. Denote by $S^0_X := X_+ = X \amalg X$ the 0-sphere in $\mathcal{R}_G(X)$. Note that if we are thinking of X as a point, then X_+ in $\mathcal{R}_G(X)$ is analogous to S^0.

Example 2.7. More generally, any G-representation V has an associated *representation sphere* S^V, which is the associated one-point compactification, based at infinity. Denote by $S^V_X = X \times S^V$ the *fiberwise representation sphere*. This has a natural projection to X, and by convention the fiber over x is based at the point at infinity in S^V.

Example 2.8. If $p \colon E \to X$ is a G-equivariant vector bundle, then the zero section endows it with the structure of an X-retractive space.

Example 2.9. Given an equivariant vector bundle $p \colon E \to X$, denote by $\mathrm{Th}_X(E)$ the *fiberwise Thom space*, where the fibers E_x have each been compactified to some point at infinity:

$$\begin{tikzcd} S^{E_x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Th}_X(E) \arrow[d] \\ \{x\} \arrow[r, hook] & X. \end{tikzcd}$$

There is an inclusion of X into the fiberwise Thom space sending every x to the point ∞ living in its fiber. This endows $\mathrm{Th}_X(E)$ with the structure of a retractive X-space, and we see that it is in fact equivariant by letting G act on the new points at infinity by moving them around fibers.

The base change functors from slice categories translate into this based setting.

Definition 2.10. Let $f \colon X \to Y$ be a G-map. There is a *forgetful* functor

$$f_\sharp \colon \mathcal{R}_G(X) \to \mathcal{R}_G(Y),$$

given by sending a retractive space S over X to the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}"] & \\
S \arrow[r] \arrow[dr, bend right] & f_\sharp S \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Warning 2.11. There is competing notation for the six functors appearing in parametrized homotopy theory, so we should clarify our notational choices before proceeding. May and Sigurdsson refer to the functor described in Definition 2.10 as $f_!$. We use $f_\sharp$ for this functor, as does [Hu03], since it aligns with the six functors notation as developed by Grothendieck, Ayoub, and others. As in [Hu03], we reserve the shriek notation for the *exceptional adjunction*, which we will define in Definition 2.47.

Notation 2.12. For any G-space X, denote by $\pi_X\colon X \to *$ the structure map sending X to the one-point space.

Example 2.13. Observe how some retractive X-spaces behave under the forgetful functor to a point.

(1) Forgetting the zero-sphere yields $(\pi_X)_\sharp\, S^0_X = X \amalg * = X_+$ in $\mathtt{Top}_*$.
(2) Applying the forgetful functor $(\pi_X)_\sharp$ to the fiberwise Thom space of a vector bundle $\mathrm{Th}_X(E)$ collapses the basepoint copy of X, meaning that it glues all the points at infinity together to one. This recovers the ordinary (non-fiberwise) Thom space $\mathrm{Th}(E)$.
(3) Given any fiberwise representation sphere S^V_X, forgetting along π_X glues all the points at infinity together. This yields $S^V \times X/\infty \times X$, which is the half-smash product $X_+ \wedge S^V$.

Definition 2.14. Any G-map $f\colon X \to Y$ induces a *pullback* functor

$$f^*\colon \mathcal{R}_G(Y) \to \mathcal{R}_G(X),$$

given by sending a retractive space T over Y to the pullback

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, dashed] \arrow[drr, bend left, "\mathrm{id}"] & & \\
Y \arrow[dr, bend right] & f^*T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& T \arrow[r] & Y.
\end{tikzcd}$$

Example 2.15. Pulling back a representation sphere S^V along a structure map $\pi_X : X \to *$ yields a fiberwise representation sphere $\pi_X^* S^V = S^V \times X = S_X^V$.

Proposition 2.16. It is straightforward to check that there is an adjunction $f_\sharp \dashv f^*$.

2.2. **Model structures on parametrized G-spectra.** We endow the category of retractive G-spaces with the *q-model structure*, which first appeared in the non-equivariant setting in an unpublished preprint of May [May00], was fleshed out in the equivariant setting by Hu [Hu03], and was recently made explicit by Malkiewich [Mal20].

Let $f : S \to T$ be a map in $\mathcal{R}_G(X)$. We say it is a *weak equivalence* if it is a weak equivalence when viewed as a morphism in $G\mathtt{Top}$; explicitly, if $f^H : S^H \to T^H$ is a weak homotopy equivalence for every subgroup $H \leq G$. Similarly we define f to be a *fibration* if f^H is a Serre fibration for every $H \leq G$. Cofibrations in this model structure are given by retracts of relative G-cell complexes.

Notation 2.17. Given any G-space X, and any complex G-representation V, denote by ε_X^V the G-vector space $X \times V \to X$. We call this the *trivial* bundle associated to the representation V.

Following [Hu03], we define a *prespectrum* over X to be a sequence of X-retractive spaces A_V for each real orthogonal representation V, together with structure maps for each pair of real representations V, W:

$$\varepsilon_X^{W-V} \wedge_X A_V \to A_W.$$

We say that $A = (A_V)_{V \in \mathrm{RO}(G)}$ is a *spectrum* if the adjoints to the structure maps are homeomorphisms over X. Following the notation in [Mal20], we denote by $G\mathcal{OS}(X)$ the category of genuine orthogonal G-spectra parametrized over X. Denote by $\mathbf{1}_X \in G\mathcal{OS}(X)$ the sphere spectrum, which is the unit for the symmetric monoidal structure induced by the fiberwise smash product.

Given a G-map $f : X \to Y$, we obtain the associated pullback map $f^* : G\mathcal{OS}(Y) \to G\mathcal{OS}(X)$ by applying f^* levelwise to the component spaces of the associated spectra, and we obtain $f_* : G\mathcal{OS}(X) \to G\mathcal{OS}(Y)$ similarly. In order to define pushforward $f_\sharp : G\mathcal{OS}(X) \to G\mathcal{OS}(Y)$, we first apply $f_\sharp$ spacewise to a spectrum in $G\mathcal{OS}(X)$ and then spectrify the resulting prespectrum over Y.

The q-model structure we outlined for $\mathcal{R}_G(X)$ can be extended to a model structure on $G\mathcal{OS}(X)$ by defining weak equivalences and fibrations componentwise [Hu03, Definition 3.3]. This forms a closed model structure [Hu03, Proposition 3.4], [Mal20, Theorem 1.0.1]. We denote by $[-,-]_X$ homotopy classes of maps in $G\mathcal{OS}(X)$.

Proposition 2.18. For any G-map $f : X \to Y$, the adjunction

$$f_\sharp : G\mathcal{OS}(X) \leftrightarrows G\mathcal{OS}(Y) : f^*$$

is a Quillen adjunction [Hu03, §3].

2.3. **Projection and exchange.** Two key techniques frequently used in settings where a six functors formalism appears are a *projection formula* and *exchange transformations*. Projection describes the interaction of the forgetful functor with smash products, while exchange describes how functors channel data from commutative diagrams of base objects (in this case retractive G-spaces).

Theorem 2.19. *(Projection)* [Hu03, 4.7] Let $f\colon X \to Y$ be a G-map of spaces, and take $S \in G\mathcal{OS}(X)$ and $T \in G\mathcal{OS}(Y)$. Then there is an isomorphism of retractive Y-spaces, natural in both S and T:

$$T \wedge_Y f_\sharp(S) \simeq f_\sharp \left(f^*T \wedge_X S\right).$$

Example 2.20. In the case where $S = S^0_X$ is the zero-sphere over X, projection takes the form

$$f_\sharp f^*(T) \simeq T \wedge_Y f_\sharp(S^0_X).$$

That is, applying $f_\sharp f^*(-)$ has the effect of smashing fiberwise with $f_\sharp(S^0_X)$.

Theorem 2.21. *(Exchange)* [Mal20, 2.3.10] For any commutative square of G-spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q"] \\ C \arrow[r, "p"'] & D, \end{tikzcd} \tag{22}$$

there is an associated *exchange transformation* $\mathrm{Ex}^*_\sharp\colon g_\sharp f^* \to p^* q_\sharp$. This is an isomorphism if the square is cartesian.

2.4. **Thom transformations.** If A is an ordinary spectrum, we define A-cohomology by mapping into A up to homotopy. The nth cohomology groups of A are defined by mapping into A smashed with a sphere $[-, \Sigma^n A]$. We may rewrite this less concisely as $[-, \mathrm{Th}(\mathbb{R}^n) \wedge A]$ by considering S^n to be the Thom space of a rank n bundle over a point. Passing to the parametrized setting over a base space X, it might make sense then, for any vector bundle $E \to X$, to define the *Eth cohomology group* $A^E(-)$ by $[-, \mathrm{Th}_X(E) \wedge_X A]_X$. We will make such a definition in Section 4, but first we explore the process of smashing fiberwise with a Thom space of a vector bundle. This defines an invertible endofunctor on $G\mathcal{OS}(X)$ which we call a *Thom transformation*.

Definition 2.23. Let $E \to X$ be a G-equivariant vector bundle. We define the associated *Thom transformation*, denoted by Σ_X^E, to be the endofunctor defined by smashing fiberwise with the fiberwise Thom space of E.

$$\begin{aligned} \Sigma_X^E : G\mathcal{OS}(X) &\to G\mathcal{OS}(X) \\ S &\mapsto S \wedge_X \mathrm{Th}_X(E). \end{aligned}$$

Example 2.24. The easiest possible example is when $E = \varepsilon_X^V$ is the trivial bundle associated to any G-representation V. Applying the associated Thom transformation yields

$$\Sigma_X^{\varepsilon_X^V}(-) = \mathrm{Th}_X(\varepsilon_X^V) \wedge_X (-) = (X \times S^V) \wedge_X (-) = S_X^V \wedge_X (-).$$

That is, it is the same as suspending by the parametrized V-sphere over X, which is invertible in the world of parametrized G-spectra over X, since $G\mathcal{OS}(X)$ is stable with respect to suspension by representation spheres. We will use Σ_X^V instead of the more cumbersome notation $\Sigma_X^{\varepsilon_X^V}$.

Proposition 2.25. The Thom transformations convert short exact sequences to addition, in the sense that for any short exact sequence of equivariant bundles over X:

$$0 \to A \to B \to C \to 0,$$

there is an isomorphism $\Sigma_X^A \Sigma_X^C \cong \Sigma_X^B$, which is unique in the homotopy category.

Proof. Since every short exact sequence of equivariant bundles is split, we have an isomorphism $B \cong A \oplus C$, inducing a homeomorphism $\mathrm{Th}_X(B) \cong \mathrm{Th}_X(A) \wedge_X \mathrm{Th}_X(C)$. This homeomorphism is a basic property enjoyed by fiberwise Thom spaces (c.f. [Mal20, p.100]). As the choice of such splittings forms a contractible space, it is clear that this isomorphism is well-defined up to homotopy. □

Thom transformations are not just invertible on $G\mathcal{OS}(X)$ for trivial bundles, this is true for all bundles. This uses the crucial fact that equivariant vector bundles admit stable inverses.

Proposition 2.26. [Seg68, 2.4] Let $E \to X$ be an equivariant vector bundle bundle. Then there is a representation V and a G-bundle $E^\perp \to X$ so that $E \oplus E^\perp \cong \varepsilon_X^V$.

Proposition 2.27. Let $E \to X$ be a G-bundle. Then the Thom transformation Σ_X^E admits a homotopy inverse Σ_X^{-E}, which is defined by $(\Sigma_X^V)^{-1}\Sigma_X^{E^\perp}$, for any trivial bundle V and complementary bundle $E \oplus E^\perp \cong V$.

Proof. This follows directly from Proposition 2.25. □

With this notion of Thom transformations associated to virtual bundles, we can extend the definition of Thom transformations to hold for perfect complexes over our domain space. As in [Seg68], denote by $K_G(X)$ the abelian group of isomorphism classes of virtual complex G-vector bundles over X.

Corollary 2.28. The Thom transformations induce a group homomorphism from the group of isomorphism classes of virtual complex vector bundles over X:

$$\begin{aligned} K_G(X) &\to \mathrm{Aut}(\mathrm{Ho}(G\mathcal{OS}(X))) \\ [E] &\mapsto \Sigma_X^E. \end{aligned}$$

Following Segal [Seg68, §3], we define a *complex of G-vector bundles* on X to be a sequence of G-vector bundles E_i and equivariant vector bundle maps over X:

$$\cdots \xrightarrow{d} E_n \xrightarrow{d} E_{n-1} \xrightarrow{d} \cdots$$

so that $d^2 = 0$. We say that a complex $E_\bullet$ is *bounded* if $E_n = 0$ for $|n|$ sufficiently large. Let $\mathrm{Perf}(K_G(X))$ denote the collection of *perfect complexes*, meaning those which are quasi-isomorphic to bounded ones. The following definition is inspired by the motivic J-homomorphism of [BH21].

Proposition 2.29. The Thom transformations extend to perfect complexes of vector bundles on X:

$$\begin{aligned} \Sigma_X^{(-)} \colon \mathrm{Perf}(K_G(X)) &\to \mathrm{Aut}(\mathrm{Ho}(G\mathcal{OS}(X))) \\ (\cdots \to E_n \to E_{n-1} \to \cdots \to E_0) &\mapsto \Sigma_X^{(-1)^n E_n} \circ \cdots \circ \Sigma_X^{E_0}. \end{aligned}$$

Proof. It will suffice to show that the definition above is well-defined on quasi-isomorphism classes of bounded complexes. Suppose $f_\bullet \colon A_\bullet \to B_\bullet$ is a quasi-isomorphism of complexes. Considering the differential $d_n^A \colon A_n \to A_{n-1}$, we have a short exact sequence

$$0 \to \ker(d_n^A) \to A_n \to \mathrm{im}(d_n^A) \to 0,$$

which by Proposition 2.25 induces an isomorphism

$$\Sigma_X^{(-1)^n A_n} \cong \Sigma_X^{(-1)^{n+1}\ker(d_n^A)} \Sigma_X^{(-1)^{n+1}(\mathrm{im}(d_n^A))}.$$

Since $\Sigma_X^{\ker(d_n^A)-\mathrm{im}(d_{n+1}^A)} \cong \Sigma_X^{H_n(A)}$, we observe that

$$\Sigma_X^{A_\bullet} \cong \sum_n \Sigma_X^{(-1)^{n+1}H_n(A)}.$$

As A and B are quasi-isomorphic, we conclude that $\Sigma_X^{A_\bullet} \cong \Sigma_X^{B_\bullet}$. □

Corollary 2.30. Any path in K-theory between $E_\bullet$ and $F_\bullet$ in $\mathrm{Perf}(K_G(X))$ induces a canonical natural equivalence $\Sigma_X^{E_\bullet} \cong \Sigma_X^{F_\bullet}$.

As one last result, we discuss how Thom transformations behave under pullback.

Proposition 2.31. *(Thom transformations commute with pullback)* Let $f : X \to Y$ any map, and let Σ_Y^ξ be a Thom transformation over Y. Then there is a natural equivalence of functors

$$f^*\Sigma_Y^\xi \cong \Sigma_X^{f^*\xi} f^*.$$

Proof. It is straightforward to check that we have an isomorphism $f^*\Sigma_Y^E \cong \Sigma_X^{f^*E} f^*$ for any vector bundle $E \to Y$. This argument can be seen to extend to bounded complexes. □

Corollary 2.32. There is a natural isomorphism

$$f_\sharp \Sigma_X^{f^*\xi} \cong \Sigma_Y^\xi f_\sharp,$$

defined to be the mate of the isomorphism in Proposition 2.31.

Example 2.33. Given a vector bundle $E \to Y$ and a map $f : X \to Y$, the natural isomorphism Corollary 2.32 is an example of projection (Theorem 2.19). Explicitly, the natural isomorphism is of the form

$$f_\sharp \Sigma_X^{f^*E}(-) \cong \Sigma_Y^E f_\sharp(-),$$

taking a vector $v \in f^*E$ fibered over some point $x \in X$, and mapping it to the same vector v, now fibered over $f(x) \in Y$. We may verify that this is precisely the formula for the mate of the isomorphism in Proposition 2.31, as well as the formula for projection found in the proof of [Hu03, 4.7].

2.5. **Cotangent complexes and duality.** One of the key constructions in [Hu03] is that of a dualizing object C_f associated to a class of morphisms in $G\mathtt{Top}$ called *smooth proper families of G-manifolds*. With Thom transformations in hand, we are able to extend this definition to a strictly larger class of maps. Namely when $f : X \to Y$ admits a certain factorization, we define an associated *cotangent complex* $\mathcal{L}_f \in \mathrm{Perf}(K_G(X))$. The associated dualizing object will then be the Thom transformation of the cotangent complex applied to the sphere spectrum $C_f := \Sigma_X^{\mathcal{L}_f} \mathbf{1}_X$.

Definition 2.34. A G-map $f\colon X \to Y$ is said to be a *smooth proper family of G-manifolds* if the fiber over every point is a smooth proper G-manifold, varying continuously over Y. Here "proper" means that the homotopy fibers are compact [ABG18].

Remark 2.35. Duality for parametrized spectra can be checked fiberwise, in the sense that a parametrized X-spectrum is dualizable if and only if its fiber over every point in the base is a dualizable spectrum (e.g. a finite spectrum) [ABG18, Lemma 4.2]. The conditions in Definition 2.34 imply that $f_\sharp \mathbf{1}_Y$ will be an invertible spectrum over Y, and the analogous statement is true equivariantly [Hu03].

Definition 2.36. We define a map of smooth compact G-manifolds $f\colon X \to Y$ to be *smoothable proper* if it admits a factorization

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & W \arrow[d, "\pi"] \\ & Y, \end{tikzcd} \tag{37}$$

where i is a closed G-embedding and π is a smooth proper family of G-manifolds.

Given such a factorization, consider the following two short exact sequences, the first of bundles over X and the second of bundles over W:

$$\begin{gathered} 0 \to TX \to i^*TW \xrightarrow{(1)} Ni \to 0 \\ 0 \to T\pi \to TW \to \pi^*TY \to 0. \end{gathered} \tag{38}$$

Since i^* is exact, we can apply i^* to the second sequence to obtain

$$0 \to i^*T\pi \xrightarrow{(2)} i^*TW \to f^*TY \to 0. \tag{39}$$

This yields a composite

$$i^*T\pi \xrightarrow{(1)\circ(2)} Ni.$$

Definition 2.40. Let $f\colon X \to Y$ be smoothable proper with factorization $f = \pi \circ i$. Define the *cotangent complex* of f to be the two term complex

$$\mathcal{L}_f := (\cdots \to 0 \to i^*T\pi \to Ni)\,,$$

where $i^*T\pi$ is in degree zero and Ni in degree negative one.

Proposition 2.41. The cotangent complex yields a Thom transformation $\Sigma_X^{\mathcal{L}_f}$ associated to any smoothable proper morphism f, which gives a well-defined functor on the homotopy category.

Proof. Given any factorization as in Equation 37, we may use the short exact sequences in Equation 38 and Equation 39 to derive equations in $K_G(X)$:

$$\begin{aligned} [i^*T\pi] &= [i^*TW] - [f^*TY] \\ [Ni] &= [i^*TW] - [TX]. \end{aligned}$$

From this we may observe that the class of the cotangent complex can be described of the difference $[TX] - [f^*TY]$. In other words, there is an isomorphism

$$\Sigma_X^{i^*T\pi}\Sigma_X^{-Ni} \cong \Sigma_X^{TX}\Sigma_X^{f^*TY}.$$

This provides a model of the Thom transformation of the cotangent complex which is independent of the choice of factorization. $\square$

Example 2.42. The Thom transformation associated to the projection map $\pi_M : M \to *$, where M is any smooth compact manifold, is $\Sigma_M^{\mathcal{L}_{\pi_M}} \cong \Sigma_M^{TM}$.

Remark 2.43. For a smoothable proper morphism $f\colon X \to Y$, the invertible spectrum $\Sigma_X^{\mathcal{L}_f}\mathbf{1}_X$ is its associated *dualizing object*. In the setting where $f\colon X \to Y$ is a smooth proper family of G-manifolds, $\Sigma_X^{\mathcal{L}_f}\mathbf{1}_X$ agrees with Hu's dualizing object C_f as hinted at in the discussion [Hu03, pp.42—43], where $C_f = \Sigma_X^{Tf}\mathbf{1}_X$ is the Thom space of the relative tangent bundle $Tf = T_{X/Y}$. An illuminating discussion illustrating this example was laid out in [ABG18, §4.3].

Example 2.44. Let $f\colon X \to Y$ be a closed G-embedding. Then its dualizing object is the fiberwise Thom space of its inverse normal bundle $\mathrm{Th}_X(-Nf)$.

Example 2.45. Let $s\colon X \to E$ denote the zero section of a vector bundle. By Example 2.44 its cotangent complex is $Ns[-1]$, and we see that its normal bundle is precisely E, so its dualizing object is $\Sigma_X^{-E}\mathbf{1}_X$.

Proposition 2.46. Let $f : X \to Y$ and $g : Y \to Z$ be two composable smoothable proper morphisms. Then there is a natural isomorphism of functors from $\mathrm{Ho}(G\mathcal{OS}(X))$ to $\mathrm{Ho}(G\mathcal{OS}(Z))$:

$$\Sigma_X^{\mathcal{L}_{g\circ f}} \cong \Sigma_X^{\mathcal{L}_f} \circ \Sigma_X^{f^*\mathcal{L}_g}.$$

Here $f^*\mathcal{L}_g$ is defined by pulling back the two-term chain complex $\mathcal{L}_g$ along f.

Proof. We observe that there is a distinguished triangle

$$\mathcal{L}_f \to \mathcal{L}_{g\circ f} \to f^*\mathcal{L}_g.$$

This yields a path in K-theory, which induces a canonical weak equivalence by Corollary 2.30. $\square$

2.6. **The exceptional adjunction.** Using cotangent complexes and their associated Thom transformations, we can build the exceptional adjunction.

Definition 2.47. Let $f\colon X \to Y$ be smoothable proper. Define the *exceptional functors* by

$$f_! := f_\sharp\Sigma_X^{-\mathcal{L}_f} : G\mathcal{OS}(X) \leftrightarrows G\mathcal{OS}(Y) \colon \Sigma_X^{\mathcal{L}_f} f^* =: f^!$$

It is direct from the definition that these define adjoint functors.

Proposition 2.48. If $f\colon X \to Y$ is an open embedding of smooth G-manifolds, then the cotangent complex is trivial, hence $f^* \simeq f^!$ and $f_\sharp \simeq f_!$

Proof. We note that an open embedding is a smooth proper family of G-manifolds. Since the embedding is open, its differential is an isomorphism, and therefore its relative tangent bundle vanishes. □

Proposition 2.49. Let $f : X \to Y$ and $g : Y \to Z$ be smoothable proper. Then there is a natural isomorphism $(g \circ f)^! \cong f^! g^!$, and hence also $(g \circ f)_! \cong g_! f_!$.

Proof. Using Proposition 2.41, we may expand $(g \circ f)^!$ as

$$(g \circ f)^! = \Sigma_X^{\mathcal{L}_{g\circ f}} f^* g^* \cong \Sigma_X^{\mathcal{L}_f} \Sigma_X^{f^*\mathcal{L}_g} f^* g^*.$$

Commuting f^* past the Thom transformation of the cotangent complex for g via Proposition 2.31, we obtain

$$\Sigma_X^{\mathcal{L}_f} f^* \Sigma_X^{\mathcal{L}_g} f^* g^* = f^! g^!.$$

The desired equivalence for the exceptional pushforward follows then from the calculus of mates. □

Example 2.50. If $f : X \hookrightarrow Y$ is a closed immersion factoring as a closed immersion $i : X \hookrightarrow W$ followed by a smooth proper family of G-manifolds $\pi : W \to Y$, then we have that $\mathcal{L}_f = -Nf$, that $\mathcal{L}_i = -Ni$, and that $\mathcal{L}_\pi = T\pi$. The identification

$$\mathcal{L}_f = \mathcal{L}_i + i^* \mathcal{L}_\pi$$

comes from the short exact sequence of bundles

$$Nf \to Ni \to i^* T\pi.$$

Thus the isomorphism comes from this short exact sequence, and commuting Thom transformations past pullback functors.

3. Equivariant Pontryagin–Thom transfers

Given a map $f\colon X \to Y$, cohomology classes on Y can be pulled back to classes on X. A foundational question in mathematics is when cohomological data can be transmitted the other way.

Example 3.1. *(Atiyah duality)* Clasically, given a smooth compact manifold M, pushing forward cohomological data along the map $\pi_M\colon M \to *$ amounts to integrating cohomology classes in order to produce a scalar. By embedding M in Euclidean space $\mathbb{R}^n$ and then taking a one-point compactification, we obtain an embedding

$i\colon M \hookrightarrow S^n$. By collapsing S^n onto a tubular neighborhood of the embedding, we obtain, up to diffeomorphism, the Thom space of the normal bundle of the embedding $S^n \to \mathrm{Th}(Ni)$. Desuspending by n gives us a map of spectra $\mathbf{1} \to (\pi_M)_!\mathbf{1}_M$, which we think of as the *dual* of the map $M \to *$. Under the presence of a Thom isomorphism, the cohomology of $\mathrm{Th}(-TM)$ agrees with the cohomology of M up to a shift, hence cohomology classes on M can be pulled back along this dual map to cohomology classes of the sphere spectrum. We call the map $\mathbf{1} \to (\pi_M)_!\,\mathbf{1}_M$ a *transfer* (also called an *Umkehr map*), and the induced map on cohomology a *Gysin map*.

Roughly speaking, cohomological data likes to be transferred along *immersions* and along *fibrations*. With a bit of effort, one can often extend definitions of transfers to morphisms that factor as an immersion followed by a fibration. An example from the motivic setting is Gysin maps along smoothable proper morphisms of schemes as in [DJK18, Theorem 4.2.1].

In this section we explore transfers in the parametrized equivariant setting — first along closed immersions, second along smooth proper families of G-manifolds, and finally in the most general case: along smoothable proper morphisms of compact G-manifolds, as defined in Definition 2.36.

We begin first, though, with a brief recollection about duality in this context.

3.1. **Duality for parametrized spectra.** Working in the setting of parametrized spectra (and equivariant parametrized spectra), the classical story of duality does not translate in the naive way. Explicitly, it is not sufficient to look at the fibers of a parametrized spectrum over a base and verify that they are classically dualizable. This notion of *fiberwise duality* is interesting in its own right, but doesn't capture the right categorical notion of duality. The precise theory of duality in $G\mathcal{OS}(X)$ is often called *Costenoble-Waner duality*, we refer the reader to [MS06, Chapters 16—18] for details.

The category $G\mathcal{OS}(X)$ is symmetric monoidal under the fiberwise smash product, and therefore comes equipped with a categorical notion of duality. This is captured precisely by the Costenoble-Waner dual of a parametrized equivariant spectrum, defined in [MS06, 18.1.2].

Notation 3.2. Denote by $G\mathcal{OS}(X)^{\mathrm{fd}}$ the full subcategory of fully dualizable objects. If $Y \in G\mathcal{OS}(X)^{\mathrm{fd}}$, we denote by $DY \in G\mathcal{OS}(X)^{\mathrm{fd}}$ its categorical (Costenoble-Waner) dual.

As the base X varies, the categories $G\mathcal{OS}(X)$ form a closed symmetric monoidal bicategory, the general theory of which is laid out in [MS06, Chapter 16]. We highlight some specific features that are crucial for our discussion of duality here.

Proposition 3.3. Let $F : \mathscr{C} \to \mathscr{D}$ be a strong symmetric monoidal functor between symmetric monoidal categories. In this setting we have a duality functor $D : \mathscr{C}^{\mathrm{fd}} \to \mathscr{C}^{\mathrm{fd}}$ [PS14, p.6], and by abuse of notation we denote the duality operation on $\mathscr{D}^{\mathrm{fd}}$ by D as well. Suppose that F admits both left and right adjoints $L \dashv F \dashv R$.

(1) Since F is symmetric monoidal, there is a canonical isomorphism $FD(-) \cong DF(-)$.
(2) There is a natural isomorphism $LD(-) \cong DR(-)$
(3) The unit $\mathrm{id} \to RF$ is dual to the counit $LF \to \mathrm{id}$.

Proof. The first statement is a familiar property of strong monoidal functors. For the second statement, take $c \in \mathscr{C}$ and $d \in \mathscr{D}$ arbitrary. Then we see that

$$\mathrm{Hom}_{\mathscr{D}}(Fc, d) \cong \mathrm{Hom}_{\mathscr{D}}(Dd, DFc) \cong \mathrm{Hom}_{\mathscr{D}}(Dd, FDc) \cong \mathrm{Hom}_{\mathscr{C}}(LDd, Dc).$$

We similarly obtain

$$\mathrm{Hom}_{\mathscr{D}}(Fc, d) \cong \mathrm{Hom}_{\mathscr{C}}(c, Rd) \cong \mathrm{Hom}_{\mathscr{C}}(DRd, Dc).$$

This yields the natural isomorphism $LD \cong DR$.

For the last statement, we observe that, after applying the duality operation to the adjunction $F \dashv R$, we obtain an adjunction $FD \cong DF \dashv DR \cong RL$. In other words, we obtain the adjunction $L \dashv F$ after applying duals. Thus the dual of the unit and counit for the adjunction $F \dashv R$ become the counit and unit, respectively, for the adjunction $L \dashv F$ after applying duality. $\square$

3.2. **Transfers along closed immersions.** Let $i\colon X \hookrightarrow W$ be a closed G-embedding of smooth compact G-manifolds. We will define a Pontryagin–Thom transfer of the form $\mathrm{PT}(i)\colon \mathbf{1}_W \to i_!\mathbf{1}_X$. In the non-equivariant setting, this transfer was constructed by May and Sigurdsson [MS06, 18.6.3] (see also [ABG18, 4.17]).

First we will better understand the spectrum $i_!\mathbf{1}_X$.

Proposition 3.4. Let $i\colon X \hookrightarrow W$ be a closed G-embedding. Then there is a weak equivalence in $G\mathcal{OS}(W)$ of the form

$$i_!\mathbf{1}_X \simeq \Sigma^\infty C_W(W - X, W),$$

where $C_W(W - X, W)$ denotes the double mapping cylinder obtained by gluing the cylinder $(W - X) \times [0, 1]$ to two copies of W, based at the bottom copy of W, with G-action happening levelwise in each slice of the cylinder.

Proof. By [KW10, 7.2], there is a weak equivalence in $\mathcal{R}_G(W)$ of the form

$$i_\sharp \mathrm{Th}_X(Ni) \simeq C_W(W - X, W).$$

By taking suspension spectra, we would like to see that $\Sigma^\infty i_\sharp \mathrm{Th}_X(Ni) \simeq i_! \mathbf{1}_X$. That is, we must demonstrate an equivalence

$$i_\sharp \Sigma^\infty \mathrm{Th}_X(Ni) \cong \Sigma^\infty i_\sharp \mathrm{Th}_X(Ni).$$

This follows from a more general fact – that we need not spectrify when applying the forgetful functor to suspension spectra. This is a natural consequence of projection Theorem 2.19. If $T \in \mathcal{R}_G(X)$ is a retractive X-space, then the projection formula yields the following natural isomorphism (the second follows from pullback preserving spheres)

$$\varepsilon_W^{V'-V} \wedge_W i_\sharp \left(\varepsilon_X^V \wedge_X T\right) \cong i_\sharp \left(i^* \varepsilon_W^{V'-V} \wedge_X \varepsilon_X^V \wedge_X T\right) \cong i_\sharp \left(\varepsilon_X^{V'} \wedge_X T\right).$$

In other words, we have that $\left\{i_\sharp \Sigma_X^V T\right\}_{V \in \mathrm{RO}(G)}$ is already a spectrum. $\square$

Proposition 3.5. ([MS06, 18.6.5]) Given a closed G-embedding $i : X \hookrightarrow W$, the natural map $i_\sharp S_X^0 \to S_W^0$ is dualizable in $G\mathcal{OS}(W)$, with dual given by the *Pontryagin-Thom transfer*

$$\mathrm{PT}(i) \colon S_W^0 \to i_! S_X^0.$$

The construction of this transfer relies heavily on the equivariant tubular neighborhood theorem, and we refer the reader to [MS06, 18.6.3] for details about the construction. Here we will be content with the dualizability of the natural inclusion map of spheres, and the existence of a transfer.

3.3. **Transfers along smooth proper families of G-manifolds.** A foundational result of Hu states that if $\pi \colon W \to Y$ is a smooth proper family of G-manifolds, then there is a natural weak equivalence $\pi_* \simeq \pi_!$, where π_* is right adjoint to π^* [Hu03, 4.9].

Via the adjunction $\pi^* \dashv \pi_*$ and the equivalence above, we have a natural transformation $\mathrm{id} \to \pi_! \pi^*$. The component of this transformation at the sphere spectrum is of the form $\mathrm{PT}(\pi) \colon \mathbf{1}_Y \to \pi_! \mathbf{1}_X$. This is what is referred to as the *equivariant Pontryagin–Thom transfer* associated to a smooth proper family of G-manifolds. If G is the trivial group, this is consistent with the definition found in [ABG18, 4.13]. An explicit model for this transfer may be found in [Hu03, MS06, ABG18].

Proposition 3.6. The Pontryagin–Thom transfer $\mathrm{PT}(\pi) : \mathbf{1}_Y \to \pi_! \mathbf{1}_W$ is dual in the homotopy category to the natural map $\pi_\sharp S_W^0 \to S_Y^0$.

Proof. We can check that the natural map $\pi_\sharp S^0_W \to S^0_Y$ is the component of the counit of the adjunction $\pi_\sharp \dashv \pi^*$ at the sphere spectrum:

$$\pi_\sharp \pi^* S^0_W \to S^0_Y.$$

Via Proposition 3.3, this is dual to the component of the unit of the adjunction $\pi^* \dashv \pi_*$ at the sphere spectrum. This is precisely the definition of $\mathrm{PT}(\pi)$. □

3.4. **Transfers along smoothable proper morphisms.** We now combine the results from the previous two sections to argue that Pontryagin–Thom transfers along smoothable proper morphisms exist.

Definition 3.7. Suppose f is a smoothable proper morphism, factoring as a closed G-immersion i followed by a smooth proper family of G-manifolds π. Then we define the *Pontryagin–Thom transfer* $\mathrm{PT}(f)$ to be the composite of the Pontryagin–Thom transfers along π and i:

$$\mathbf{1}_Y \xrightarrow{\mathrm{PT}(\pi)} \pi_! \mathbf{1}_W \xrightarrow{\pi_! \mathrm{PT}(i)} \pi_! i_! \mathbf{1}_X \cong f_! \mathbf{1}_X,$$

where $\pi_! i_! \mathbf{1}_X \cong f_! \mathbf{1}_X$ is the canonical natural isomorphism from Proposition 2.49.

This definition is consistent with that found in [ABG18], and expands the context in which such transfers exist.

Theorem 3.8. For any smoothable proper morphism $f\colon X \to Y$ between smooth proper G-manifolds, the Pontraygin-Thom transfer $\mathrm{PT}(f) : \mathbf{1}_Y \to f_! \mathbf{1}_X$ is well-defined in $\mathrm{Ho}(G\mathcal{OS}(Y))$.

The remainder of this section is devoted to proving Theorem 3.8. Our goal will be simply to argue that, in the homotopy category, $\mathrm{PT}(f)$ is dual to the natural map $f_\sharp S^0_X \to S^0_Y$. This can be written as the composite of the natural maps

$$\pi_\sharp i_\sharp S^0_X \to \pi_\sharp S^0_W \to S^0_Y.$$

As duality will act contravariantly on composites, we will argue then that:

(1) $\mathbf{1}_Y \xrightarrow{\mathrm{PT}(\pi)} \pi_! \mathbf{1}_W$ is dual to $\pi_\sharp S^0_W \to S^0_Y$,
(2) $\pi_! \mathbf{1}_Y \xrightarrow{\pi_! \mathrm{PT}(i)} \pi_! i_! \mathbf{1}_X$ is dual (in the homotopy category) to $\pi_\sharp i_\sharp S^0_X \to \pi_\sharp S^0_W$.

This first statement is Proposition 3.6, so we are reduced to asking how duality behaves under $\pi_\sharp$, where π is a smooth proper family.

As π^* is symmetric monoidal, admitting both left and right adjoints, Proposition 3.3 implies that there is a natural isomorphism of functors

$$\pi_\sharp D \cong D \pi_* : G\mathcal{OS}(W)^{\mathrm{fd}} \to G\mathcal{OS}(Y)^{\mathrm{fd}}.$$

Combining this with the natural weak equivalence $\pi_* \simeq \pi_!$ from [Hu03, 4.9], we obtain a natural weak equivalence $\pi_\sharp D(-) \simeq D\pi_!(-)$ on dualizable objects. Theorem 3.8 follows immediately from this equivalence — we observe that the dual $D\pi_!\mathrm{PT}(i)$ agrees in the homotopy category with $\pi_\sharp D\mathrm{PT}(i)$, which is $\pi_\sharp$ applied to the natural map $i_\sharp S^0_X \to S^0_W$, as desired.

4. Cohomology

Here we develop a theory of cohomology with compact supports, twisted by perfect complexes. This theory mirrors that found in the motivic setting (c.f. [DJK18, EHK+20, BW21], etc.). The main goal is to demonstrate that cohomology classes can be pushed forward by forgetting support, or by decomposing along the clopen components of the support. In this sense, certain abstract cohomology classes can be understood in rings as sums of local contributions of data. In Section 5 we will leverage this perspective to prove conservation of number in the equivariant setting.

4.1. **Twisted cohomology.** Let $\xi \in \mathrm{Perf}(K_G(X))$ be a perfect complex of equivariant vector bundles over X, and let $A \in \mathcal{SH}(G)$ be an arbitrary genuine G-spectrum, which represents an $\mathrm{RO}(G)$-graded cohomology theory.

Definition 4.1. Define *ξ-twisted cohomology with coefficients in A* by

$$A^\xi(X) := \Big[\mathbf{1}_X, \Sigma^\xi_X \pi_X^* A\Big]_X .$$

Example 4.2. If $\xi = \varepsilon^V_X = X \times V$ is a trivial bundle for some G-representation V, then ε^V_X-twisted cohomology is of the form

$$A^{\varepsilon^V_X}(X) = \left[\mathbf{1}_X, S^V_X \wedge_X \pi_X^* A\right]_X = \Big[(\pi_X)_\sharp\, S^{-V}_X, A\Big] = \left[X_+ \wedge S^{-V}, A\right].$$

This last group is precisely the definition of $A^V(X)$, that is, the A-cohomology of X indexed over $\mathrm{RO}(G)$. This is a crucial example, since it indicates that our notation for cohomology twisted by vector bundles agrees with and subsumes existing notations for cohomology found in e.g. [LMSM86, p. 35].

Notation 4.3. For V a G-representation and $A \in \mathcal{SH}(G)$ any spectrum, Example 4.2 indicates that we can use $A^V(X)$ to refer to classical Vth A-cohomology group of X or the A-cohomology of X twisted by the trivial vector bundle ε^V_X without loss of generality. Similarly to Example 2.24, we will freely use $A^V(X)$ instead of $A^{\varepsilon^V_X}(X)$.

When $Z \subseteq X$ is a closed G-subspace, we can talk about cohomology classes that are "supported" on Z. Let $i\colon Z \hookrightarrow X$ denote the inclusion map.

Definition 4.4. For $\xi \in \mathrm{Perf}(K_G(X))$, define *$\xi$-twisted cohomology with coefficients in A and support on Z* to be

$$A_Z^\xi(X) := \left[i_!\mathbf{1}_Z, \Sigma_X^\xi \pi_X^* A\right]_X.$$

We should explain a bit why this is the right definition of cohomology supported on Z. Recall by Proposition 3.4 that $i_!\mathbf{1}_Z$ is equivalent to the double mapping cylinder $C_X(X, X-Z)$. Collapsing this space along its cylinder coordinate, we obtain the space in Figure 2. The bottom copy of X is the basepoint, which has to be sent to

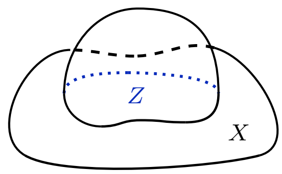

FIGURE 2. The homotopy type of the space $i_!S_Z^0$.

the basepoint in the target. What we are left with is an extra copy of Z, glued along the base, which is free to be mapped anywhere in the target. Thus we think of maps out of $C_X(X, X-Z)$ yielding cohomology classes supported on Z.

Definition 4.5. Precomposition with the Pontryagin–Thom transfer $\mathbf{1}_X \to i_!\mathbf{1}_Z$, defined in Theorem 3.8, induces a *forgetting support* map

$$A_Z^\xi(X) \to A^\xi(X).$$

Proposition 4.6. Let M be a smooth proper G-manifold, and $i : Z \hookrightarrow M$ a closed G-embedding. Then there is a canonical isomorphism

$$A_Z^{TM}(M) \cong A^{TZ}(Z).$$

Proof. We can write

$$A_Z^{TM}(M) = \left[i_!\mathbf{1}_Z, \Sigma_M^{TM}\pi_M^* A\right]_X \cong \left[\mathbf{1}_Z, i^!\Sigma_M^{TM}\pi_M^* A\right]_Z.$$

As the exceptional pullback is given by $i^! = \Sigma_Z^{\mathcal{L}_i} i^* = \Sigma_Z^{-Ni} i^*$, we may rewrite the above as

$$\left[\mathbf{1}_Z, \Sigma_Z^{-Ni} i^* \Sigma_M^{TM}\pi_M^* A\right]_Z.$$

Commuting i^* with the Thom transformation via Proposition 2.31 yields

$$\left[\mathbf{1}_Z, \Sigma_Z^{-Ni}\Sigma_Z^{i^*TM}\pi_Z^* A\right]_Z.$$

From the short exact sequence

$$0 \to TZ \to i^*TM \to Ni \to 0,$$

we have that $\Sigma_Z^{-Ni}\Sigma_Z^{i^*TM} \cong \Sigma_Z^{TZ}$, from which we can see that $A_Z^{TM}(M)$ is isomorphic to

$$\left[\mathbf{1}_Z, \Sigma_Z^{TZ}\pi_Z^* A\right]_Z = A^{TZ}(Z). \qquad \square$$

Recall classically that compactly supported cohomology classes decompose over their support. In order to make this precise, we have to be careful about what we mean by decomposing spaces equivariantly.

Terminology 4.7. Let $i\colon Z \hookrightarrow X$ be a closed G-embedding. As a topological subspace, we may decompose Z non-equivariantly into its clopen components: $Z = \amalg_i W_i$. As G acts via homeomorphisms, we see that the G-orbit of any component is both closed and open as well. Thus we may decompose Z as $Z = \amalg G \cdot W_i$, and we refer to the orbits $G \cdot W_i$ as the *equivariant clopen components* of Z in X.

By collapsing a double mapping cylinder $C_X(X - Z, X)$ down along the time axis, we obtain a "fried egg" space as in Figure 2. When Z is decomposed into its equivariant clopen components, we see that the double mapping cylinder decomposes as a wedge sum over the base copy of X, as pictured in Figure 3.

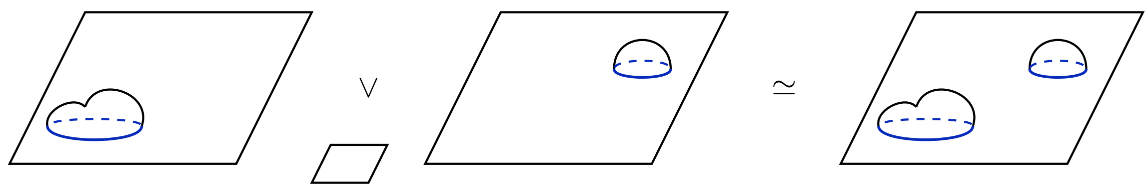

FIGURE 3. If $Z = Z_1 \amalg Z_2$, then we have that $C_X(X, X - Z_1) \vee_X C_X(X, X - Z_2) \simeq C_X(X, X - Z)$.

Proposition 4.8. Take a closed G-embedding $i\colon Z \hookrightarrow X$, let $Z = \amalg_n Z_n$ be the decomposition of Z into its equivariant clopen components, and denote by $i_n\colon Z_n \hookrightarrow X$ the composite inclusion for each n. Then there is a weak equivalence

$$i_! S_Z^0 \simeq \vee_n (i_n)_! S_{Z_n}^0.$$

Proof. Via Proposition 3.4, there is a weak equivalence $i_! S_Z^0 \simeq C_X(X - Z, X)$, and we may collapse the double mapping cylinder down. From there it is clear to see that it can be decomposed as a wedge sum along the equivariant clopen components. The stable version of this statement follows from observing that taking suspension spectra commutes with wedges. $\square$

Example 4.9. For G a finite group, if $i\colon G/H \to M$ is the closed inclusion of an orbit into a smooth manifold M, then there are weak equivalences in $G\mathcal{OS}(M)$ of the form:

$$i_! S^0_{G/H} \simeq \Sigma^\infty i_\sharp (\pi^*_{G/H} \mathrm{Th}(T_x M)) \simeq \Sigma^\infty i_\sharp \left(\mathrm{Th}_{G/H}\left(TM|_{G/H}\right)\right), \tag{10}$$

where x is any point in the orbit G/H.

Proof. By collapsing the double mapping cylinder down around the points in the orbit, we obtain the Thom spaces of the associated tangent spheres at each point in the orbit, glued along G/H to M. Note however that for a chosen point x in the orbit, its tangent space inherits an H-action,. Thus each Thom space is naturally an H-representation sphere $\mathrm{Th}(T_x M)$. The residual G-action comes from permuting the representation spheres around between points in the orbit to get $(G/H) \times \mathrm{Th}(T_x M)$. Finally, in order to obtain the collapse of the double mapping cylinders, we glue to M along the orbit G/H. This gives an equivalence

$$C_M(M, M - G/H) \simeq ((G/H) \times \mathrm{Th}(T_x M)) \cup_{G/H} M,$$

This yields the first equivalence in Equation 10. If G is further assumed to be finite, the tangent space of G/H is trivial, hence the normal bundle Ni agrees with the tangent space $TM|_{G/H}$. In particular we see that

$$\mathrm{Th}_{G/H}\left(TM|_{G/H}\right) \simeq \pi^*_{G/H} \mathrm{Th}(T_x M). \tag{11}$$

□

Corollary 4.12. Cohomology with compact supports decomposes over its support, in the sense that there is a group isomorphism

$$A^{i^*\xi}_Z(X) \cong \bigoplus_n A^{i_n^*\xi}_{Z_n}(X).$$

Proof. We see that Proposition 4.8 induces an isomorphism

$$A^{i^*\xi}_Z = \left[i_! \mathbf{1}_Z, \Sigma^\xi_X \pi^*_X A\right]_X \cong \bigoplus_n \left[(i_n)_! \mathbf{1}_{Z_n}, \Sigma^\xi_X \pi^*_X A\right]_X = \bigoplus_n A^{i_n^*\xi}_{Z_n}(X).$$

□

4.2. **Cohomological pushforward.** We can push cohomology classes forward along smoothable proper morphisms. This comes at the cost of "untwisting" by a cotangent complex.

Proposition 4.13. Let $f\colon X \to Y$ be a smoothable proper morphism of smooth manifolds. Then for any $\xi \in \mathrm{Perf}(K_G(Y))$, we have a pushforward

$$f_*\colon A^{\mathcal{L}_f + f^*\xi}(X) \to A^\xi(Y).$$

Proof. We see that $A^{\mathcal{L}_f+f^*\xi}(X) = \left[\mathbf{1}_X, f^!\Sigma^{\xi}_Y \pi^*_Y A\right]_X$. Invoking that f is smoothable proper, precomposition with the Pontryagin–Thom transfer $\mathbf{1}_Y \to f_!\mathbf{1}_X$ induces the desired pushforward. □

Example 4.14. Let M be a smooth proper G-manifold. Then there is a pushforward

$$(\pi_M)_* : A^{TM}(M) \to A^0(*) = \pi_0^G A.$$

Proposition 4.15. Let $Z \subseteq M$ be a closed subspace. Then the following diagram commutes

$$\begin{tikzcd} A^{TM}_Z(M) \arrow[r, "\text{forget}"] \arrow[d, "\cong"'] & A^{TM}(M) \arrow[d, "\text{pushforward}"] \\ A^{TZ}(Z) \arrow[r, "\text{pushforward}"'] & A^0(*). \end{tikzcd}$$

Proof. Observe that in the top left we can rewrite

$$A^{TM}_Z(M) = \left[i_!\mathbf{1}_Z, \pi^!_M A\right]_X \cong \left[(\pi_M)_!\, i_!\mathbf{1}_Z, A\right]_X .$$

The forgetful map is induced by the Pontryagin–Thom transfer $\mathbf{1}_M \to i_!\mathbf{1}_Z$ as in Definition 4.5, while the pushforward on M is precomposition with the unit $\mathbf{1} \to (\pi_M)_!\,\mathbf{1}_M$. The pushforward from Z comes from recognizing that $(\pi_M)_!\, i_! = (\pi_Z)_!$ via Proposition 2.49, and using the transfer $\mathbf{1}_M \to (\pi_Z)_!\,\mathbf{1}_Z$. The fact that the Pontryagin–Thom transfers along i and π_M compose to the unit map along π_Z follows directly from the transfer being well-defined, as in the proof of Theorem 3.8. □

Proposition 4.16. Let $Z = \amalg_n Z_n$ be a decomposition into its equivariant clopen components, following the notation in Proposition 4.8. Then the following diagram commutes:

$$\begin{tikzcd} \bigoplus_n A^{TZ_n}(Z_n) \arrow[d, "\sim"'] \arrow[dr, "\text{pushforward}"] & \\ A^{TZ}(Z) \arrow[r, "\text{pushforward}"'] & A^0(*). \end{tikzcd}$$

Proof. We remark that a cohomology class on $A^{TZ_n}(Z_n)$ can be understood by pushing forward directly, or by forgetting support and then pushing forward via Proposition 4.15. That is, for any n, the diagram commutes:

$$\begin{tikzcd} A^{TZ_n}(Z_n) \arrow[d] \arrow[dr, "\text{pushforward}"] & \\ A^{TZ}(Z) \arrow[r, "\text{pushforward}"'] & A^0(*). \end{tikzcd}$$

Applying Corollary 4.12, we see that when we sum over n, the left vertical map becomes an isomorphism. ☐

4.3. **Abstract orientation data.** As we have seen in Proposition 4.15 and Proposition 4.16, given a cohomology class in $A^{TM}_Z(M)$, we can study it in two ways — by forgetting its support and pushing it forward, or by decomposing it over its support and pushing each of the individual contributions forward then summing. We have indicated that this is an equality in $\pi_0^G A$.

We will be interested in the more general situation where we are twisting by a bundle $E \to M$ which is not the tangent bundle. To study this, we need to find a way to relate the E-twisted cohomology $A^E(-)$ with the cohomology twisted by the tangent bundle $A^{TM}(-)$.

Definition 4.17. We say that a rank n bundle $E \to M$ over a G-manifold of dimension n is *relatively A-oriented* if there is an isomorphism

$$\Sigma^E_M \pi^*_M A \simeq \Sigma^{TM}_M \pi^*_M A. \tag{18}$$

Such a choice of isomorphism we call a *relative orientation*.

In Subsection 5.2, we will see that complex oriented cohomology theories enjoy a canonical choice of relative orientations, coming from the Thom isomorphism. More generally, once we have a bundle which is relatively oriented in a ring spectrum A, we can *push forward* cohomology classes. That is, if $E \to M$ is a rank n complex bundle over a G-manifold of dimension n, then we can push forward a class in $A^E(M)$ using a relative orientation:

$$\begin{tikzcd} A^E(M) \arrow[r, "\sim"] \arrow[dr, dashed, bend right] & A^{TM}(M) \arrow[d] \\ & A^0(*). \end{tikzcd}$$

Note that if $i\colon Z \hookrightarrow M$ is the inclusion of a closed subspace, by applying i^* to Equation 18, we obtain an isomorphism of the restricted vector bundle with the tangent bundle on V:

$$\Sigma^{E|_Z}_Z \pi^*_Z A \simeq \Sigma^{TZ}_Z \pi^*_Z A.$$

In other words, a relative orientation for $E \to X$ in A descends to compactly supported cohomology groups.

Proposition 4.19. Let $E \to M$ be a rank n complex G-vector bundle over a smooth n-dimensional G-manifold, equipped with a relative A-orientation. Suppose that

$\sigma\colon M \to E$ is a section with zero locus $Z = Z(\sigma)$, which decomposes into clopen components $Z(\sigma) = \amalg_n Z_n$. Then the diagram commutes:

$$\begin{tikzcd} A^E_Z(M) \arrow[r, "\text{forget}"] \arrow[d, "\sim"'] & A^E(M) \arrow[r, "\text{or.}"] & A^{TM}(M) \arrow[d, "\text{pushforward}"] \\ \bigoplus_n A^{E|_{Z_n}}_{Z_n}(M) \arrow[r, "\text{or.}"'] & \bigoplus_n A^{TZ_n}(Z_n) \arrow[r, "\text{pushforward}"'] & A^0(*). \end{tikzcd}$$

Proof. This follows directly from Proposition 4.15 and Proposition 4.16. $\square$

Thus in the presence of a relative orientation, cohomology classes in $A^E_Z(M)$ can be studied by forgetting support and pushing forward, or decomposing, pushing foward, and then summing.

5. Equivariant conservation of number

Here we define refined Euler classes associated to sections of complex vector bundles, valued in an equivariant cohomology theory. Proposition 4.19 indicates that these can be computed as a sum over the local contributions of each of the components of the zero locus of the section. When the cohomology theory A is *complex oriented*, and the zeros are simple and isolated, we demonstrate a tractable formula for the local indices. This gives us an equality in $\pi_0^G A$ which is independent of the choice of section.

5.1. **Refined Euler classes.** Let $E \to M$ be a G-equivariant vector bundle, and let σ be an equivariant section, with zero locus $Z \subseteq X$. Then σ induces a map of pairs $(M, M - Z) \to (E, E - 0)$, which in turn yields a G-equivariant map of double mapping cylinders

$$C_M(M - Z, M) \to C_E(E - 0, E).$$

Let $i\colon Z \hookrightarrow M$ denote the inclusion of the zero locus of σ. Then the map above is of the form

$$i_! S^0_Z \to \Sigma^E_M S^0_M.$$

Definition 5.1. Given a vector bundle $E \to M$ and section σ with zero locus Z, denote by $e(E, \sigma, Z) \in A^E_Z(M)$ the *refined Euler class*, defined to be the composite

$$i_! S^0_Z \to \Sigma^E_M S^0_M \xrightarrow{\Sigma^E_M 1} \Sigma^E_M \pi^*_M A,$$

where $1\colon S^0_M \to \pi^*_M A$ is the unit associated to the ring spectrum A.

Decomposing Z into its equivariant clopen components $Z = \amalg Z_n$ (as in Terminology 4.7), we can invoke Corollary 4.12 to decompose the Euler class over its support:

$$\begin{aligned} A^E_Z(M) &\xrightarrow{\sim} \bigoplus_n A^E_{Z_n}(M) \\ e(E,\sigma,Z) &\mapsto \oplus_n e(E,\sigma,Z_n). \end{aligned}$$

Definition 5.2. When E is equipped with a relative A-orientation, the image of $e(E,\sigma,Z_n)$ under pushforward is referred to as the *local index*, denoted by $\mathrm{ind}_{Z_n}(\sigma)$:

$$\begin{aligned} A^E_{Z_n}(M) \cong A^{TZ_n}(Z_n) &\to A^0(*) \\ e(E,\sigma,Z_n) &\longmapsto \mathrm{ind}_{Z_n}(\sigma). \end{aligned}$$

We refer to the image of the (un)refined Euler class under pushforward as the *Euler number*, and denote it by $n(E,\sigma)$:

$$\begin{aligned} A^E_Z \xrightarrow{\text{forget}} A^{TM}(M) &\to A^0(*) \\ e(E,\sigma,Z) &\longmapsto n(E,\sigma). \end{aligned}$$

With this terminology in hand, we can state the following lemma.

Lemma 5.3. When a vector bundle $p\colon E \to M$ is equipped with a relative A-orientation, and $\sigma\colon M \to E$ is any section, there is an equality in $A^0(*)$:

$$n(E,\sigma) = \sum_n \mathrm{ind}_{Z_n}(\sigma).$$

Moreover, the value $n(E,\sigma) \in A^0(*)$ is independent of the choice of section σ, and only depends upon the relative orientation.

Proof. We obtain the desired equality by following the Euler class $e(E,\sigma,Z)$ in the commutative diagram of Proposition 4.19. Thus we can compute the Euler number by decomposing it over its support, and summing over the local indices. To see that this is independent of σ, we remark that $n(E,\sigma)$ was defined up to the homotopy class of σ. Since every section can be G-equivariantly homotoped to the zero section, we observe that $n(E,\sigma)$ is independent of σ. $\square$

Suppose now that $j\colon G/H \hookrightarrow Z$ is a clopen component of the zero locus. Then our local Euler class lives in

$$C_M(M, M - G/H) \to \Sigma^E_M \mathbf{1}_M.$$

By Example 4.9, we are considering the composite

$$i_\sharp j_\sharp \pi^*_{G/H}\mathrm{Th}(T_xM) \to \Sigma^E_M \mathbf{1}_M,$$

where $x \in i(j(G/H))$ is any point in the orbit. By adjunction that this is the same as

$$\pi^*_{G/H}\mathrm{Th}(T_pM) \to j^*i^*\Sigma^E_M\mathbf{1}_M = \Sigma^{E|_{G/H}}_{G/H}\mathbf{1}_{G/H}.$$

Invoking Equation 11 in the case where x is a simple zero of σ, the local index is of the form

$$(G/H \times \mathrm{Th}(T_xM)) \xrightarrow{G/H\times d_x\sigma} (G/H) \times \mathrm{Th}(E_x),$$

where $d_x\sigma$ denotes the intrinsic derivative of σ at the point x.

Remark 5.4. Here is where orientation data is needed. We have an induced map between H-representation spheres of the same dimension, but this does not canonically give a class in the H-Burnside ring. The fact is while $S^{T_xM-E_x}$ is isomorphic to S^0 when x is a finite simple zero, one must *fix an isomorphism*, and there is no canonical way to do this. To circumvent this issue, we look at the map that the intrinsic derivative $S^{T_xM} \xrightarrow{d_x\sigma} S^{E_x}$ induces on a cohomology theory A, where A comes equipped with some canonical orientation data. In particular for such a ring spectrum A, we get a composite:

$$S^{T_xM} \xrightarrow{d_x\sigma\wedge u} S^{E_x} \wedge A \xrightarrow{\text{orientation data}} S^{T_xM} \wedge A.$$

This gives us a well-defined class in $\pi_0 A$ over H, and the associated local index is obtained by transferring this up to G along the transfer available to us in the zeroth homotopy Mackey functor $\underline{\pi}_0 A$. In practice we will be concerned with complex oriented equivariant ring spectra, where this "orientation data" is the data of a Thom class arising from a universal one.

5.2. **Complex orientations in the equivariant setting.**

Definition 5.5. Let $\mathcal{U}$ denote a direct sum of infinitely many copies of each irreducible complex representation of G, and denote by

$$\mathrm{BU}_G(n) := \mathrm{Gr}(\mathbb{C}^n, \mathcal{U}),$$

the moduli space of n-dimensional subspaces of $\mathcal{U}$. Since G acts naturally on $\mathcal{U}$, it acts on $\mathrm{BU}_G(n)$ as well, and $\mathrm{BU}_G(n)$ comes equipped with a tautological bundle $\gamma_n : EU_G(n) \to \mathrm{BU}_G(n)$ which is easily seen to be equivariant.

Following tom Dieck [tD70], we may assemble the Thom spaces of the bundles $\mathrm{Th}(\gamma_n)$ into a genuine G-spectrum by setting the Vth space equal to $\mathrm{Th}(\gamma_{|V|})$, and then spectrifying (see [Sin01] for a lucid overview). This definition yields *equivariant homotopical bordism*, which we denote by MU_G.

Combining the work of tom Dieck [tD70] and Okonek [Oko82, §1], we make the following definition.

Definition 5.6. Let G be a compact Lie group, and let A be a multiplicative $\mathrm{RO}(G)$-graded cohomology theory. Define a *complex orientation* on A to be a choice, for every complex vector bundle $p\colon E \to X$ of complex rank k, of *Thom classes* $\tau(p) \in \widetilde{A}^{2k}(\mathrm{Th}(E))$ subject to the following conditions:

(0) Cupping with the Thom class $\tau(p)$ induces a *Thom isomorphism*:

$$A^*(-) \xrightarrow{\tau(p)\cup -} \widetilde{A}^{*+2k}(- \wedge \mathrm{Th}(E)).$$

(1) The assignment of Thom classes is natural in the sense that the pullback of a Thom class is the Thom class of the pullback bundle.

(2) The Thom class of a product bundle is the product of the Thom classes of the respective bundles in the product.

For any rank n representation V, viewed as an equivariant bundle over a point, its Thom class $\tau(V)$ can be thought of as a map $S^V \to \Sigma^n A$. We observe that the following composite is the Thom isomorphism, which we will also denote by $\tau(V)$:

$$A \wedge S^V \xrightarrow{1\wedge\tau(V)} A \wedge \Sigma^n A \xrightarrow{\mu} \Sigma^n A,$$

where μ denotes the multiplication on the ring spectrum. In other words, $\Sigma^V A \simeq \Sigma^{|V|}A$. This is the notion of GL-orientation one encounters e.g. in [BW21, 4.13].

Example 5.7. [Oko82] For any compact Lie group G, homotopical bordism MU_G is complex oriented.

Theorem 5.8. For any compact Lie group G, if there is a map of ring spectra $\mathrm{MU}_G \to A$, then A is complex oriented. If G is furthermore assumed to be abelian, then this is an equivalent definition of complex orientation [Oko82, Lemma 1.6] (see also [CGK02]).

Example 5.9. [Oko82, Cos87] For any compact Lie group G, complex equivariant K-theory KU_G receives a ring map $\mathrm{MU}_G \to \mathrm{KU}_G$ and is therefore complex oriented.

Counterexample 5.10. Eilenberg–Maclane spectra of Mackey functors $H\underline{M}$ are generally *not* complex oriented, in stark contrast to the non-equivariant setting. By pulling Thom classes back along the zero section, we obtain Euler classes in cohomology. If V is a G-representation of dimension n, then pulling back the Thom class of its representation sphere along the zero section $S^0 \to S^V$ yields a class in

$\pi_{-n}\mathrm{MU}_G$. This class is generally nonzero, indicating that MU_G is non-connective. All Eilenberg–MacLane spectra are integrally connective, hence in order to create a ring map $\mathrm{MU}_G \to H\underline{M}$, we would have to send Euler classes to zero, which destroys any possibility of the map preserving information about orientation.

We record an important property enjoyed by complex oriented ring spectra in the equivariant setting. Informally, the following propositions state that any isomorphism of G-representations also represents the Thom isomorphism obtained by passing between the two representations in any complex oriented cohomology theory.

Proposition 5.11. For any two complex representations V_1 and V_2 of the same rank n, the following diagram commutes:

$$\begin{tikzcd} S^{V_1} \wedge A \arrow[r, "\tau(V_1)"] \arrow[dr, "S^{V_2}\wedge\tau(V_1-V_2)"'] & \Sigma^n A \\ & S^{V_2} A. \arrow[u, "\tau(V_2)"'] \end{tikzcd}$$

Proof. This is a direct consequence of the multiplicativity of Thom classes in Definition 5.6(2). □

Proposition 5.12. Let A be a complex oriented G-ring spectrum with unit map $\mathbf{1} \xrightarrow{u} A$, let $V_1, V_2 \in R_{\mathbb{C}}[G]$ denote any two isomorphic G-representations of dimension n, and let $f\colon V_1 \xrightarrow{\sim} V_2$ denote a choice of isomorphism of representations. Then the composite

$$S^{V_1} \xrightarrow{S^f} S^{V_2} \xrightarrow{S^{V_2}\wedge 1} S^{V_2} \wedge A \xrightarrow{\tau(V_2)^{-1}\tau(V_1)} S^{V_1} \wedge A$$

is equal to $1 \in \pi_0 A$.

Proof. Precomposition with S^f induces a group homomorphism of the form

$$A^0(*) = [S^{V_1}, S^{V_2} \wedge A] \xrightarrow{-\circ f} [S^{V_1}, S^{V_2} \wedge A] = A^0(S^{V_1-V_2}).$$

This is a pullback on cohomology, which in particular preserves Thom classes, therefore this composite sends $1 \in A^0(*)$ to $\tau(V_1 - V_2)$. The Thom isomorphism on A will send $\tau(V_1 - V_2)$ back to $1 \in A^0(*)$, so in order to see these maps are inverses it suffices to observe that the group homomorphism above is a group homomorphism of free rank one $A^0(*)$-modules sending the generator to the generator. □

At no point in Proposition 5.12 did we use any specific properties of the choice of isomorphism f. This is unsuprising, due to the fact that all isomorphisms of complex representations $V_1 \xrightarrow{\sim} V_2$ are homotopic [tD70, 1.1], thus there is a single homotopy class $[S^{V_1}, S^{V_2}]$ corresponding to isomorphisms of representations. The argument above indicates roughly that after smashing with A, this homotopy class aligns with that produced by the Thom isomorphism.

We can now revisit the discussion of local indices.

Lemma 5.13. Let A be any complex oriented ring spectrum in $\mathcal{SH}(G)$, let $E \to M$ be an equivariant complex vector bundle of rank n over a compact smooth G-manifold of dimension n, and let $\sigma \colon M \to E$ be a section with an isolated simple zero at $x \in M$. Then the local index, as defined in Definition 5.2, is

$$\mathrm{ind}_{G\cdot x}\sigma = \mathrm{Tr}_{G_x}^{G}(1).$$

Proof. We must argue that the composite

$$S^{T_xM} \wedge A \xrightarrow{d_x\sigma\wedge A} S^{E_x} \wedge A \xrightarrow{\tau} S^{T_xM} \wedge A$$

is equal to $1 \in \pi_0^G A$, where τ is arising from the Thom classes provided by the equivariant complex orientation on A. As x is an isolated simple zero, the intrinsic derivative is an injective map of G-representations of the same finite dimension, and hence is an isomorphism $d_x\sigma \colon T_xM \to E_x$. Thus we find ourselves under the conditions of Proposition 5.12, from which the result follows. □

To wrap up this section, we explore a payoff of the formalism developed above, which will serve as our primary computational tool. Namely, we can develop a theory of conservation of number taking value in $\underline{\pi}_0$ of any complex oriented equivariant cohomology theory.

By Lemma 5.13, the local index at an isolated simple orbit $G \cdot x$ is the trace $\mathrm{Tr}_{G_x}^{G}(1)$ from the isotropy group of x to the entire group G, where this transfer is taking place at the level of the zeroth homotopy Mackey functor. The following key lemma should be thought of as an equivariant analogue of the Poincaré–Hopf theorem, with cohomology classes valued in complex oriented G-ring spectra.

Lemma 5.14. *(Equivariant conservation of number)* Let $E \to M$ be an equivariant complex rank n bundle over a smooth G-manifold of dimension n, and let $\sigma \colon M \to E$ be any section whose zeros are isolated and simple. Let $A \in \mathcal{SH}(G)$ be any complex oriented ring spectrum. Then there is an equality in $\pi_0^G A$:

$$n(E,\sigma) = \sum_{G\cdot x \subseteq Z(\sigma)} \mathrm{Tr}_{G_x}^{G}(1),$$

where the Euler number $n(E,\sigma)$ is independent of the choice of σ.

Example 5.15. In complex K-theory, we have that $\mathrm{KU}_{G_x}(*) = R_{\mathbb{C}}[G_x]$, and the transfer of the trivial representation 1 is the regular representation of the finite G-set G/G_x. Thus an Euler number computed as in Lemma 5.14 is given by the permutation representation $\mathbb{C}[Z(\sigma)]$ of the zero locus of a section with isolated simple zeros.

Ultimately we want to argue that an answer valued in the *Burnside ring* $A(G)$ is independent of a choice of section. The KU_G-valued Euler class as in Example 5.15 is insufficient for this purpose, due to the fact that the map $\pi_0^G \mathbb{S}_G \to \pi_0^G \mathrm{KU}_G$ from the Burnside ring to the representation ring will fail to be injective in many cases. We instead need a complex oriented cohomology theory for which the unit map is an injection on π_0^G.

We thank William Balderrama for communicating the following argument to us.

Proposition 5.16. Homotopical bordism MU_G detects nilpotence, in the sense that for any ring spectrum A equipped with a ring map $A \to \mathrm{MU}_G$, the kernel of $\pi_\star^G A \to \pi_\star^G \mathrm{MU}_G$ consists of nilpotent elements (see [BGH20, 3.20]).

Proof. Taking geometric fixed points commutes with the construction of a mapping telescope, which allows us to conclude that nilpotence can be detected at the level of geometric fixed points [BGH20, 3.17]. By [Sin01, 4.10], $\Phi^H \mathrm{MU}_G$ decomposes as a wedge sum of classical MU spectra. Finally, we can conclude by applying the classical nilpotence theorem [DHS88]. $\square$

We leverage this to prove our main result.

Theorem 5.17. *(Equivariant conservation of number)* Let $E \to M$ be an equivariant complex rank n bundle over a smooth G-manifold of dimension n, and let $\sigma, \sigma' \colon M \to E$ be any two sections whose zeros are isolated and simple. Then $Z(\sigma)$ and $Z(\sigma')$ are isomorphic as finite G-sets. In other words, the G-orbits of the zeros are independent of the choice of section.

Proof. We know for such a section σ, we can obtain an Euler class valued in $\pi_0^G \mathrm{MU}_G$ by Lemma 5.14:

$$n(E, \sigma) = \sum_{G \cdot x \subseteq Z(\sigma)} \mathrm{Tr}_{G_x}^G(1).$$

As MU_G detects nilpotence by Proposition 5.16, and the Burnside ring is reduced, we can conclude that $\pi_0^G \mathbb{S}_G \to \pi_0^G \mathrm{MU}_G$ is injective. Remarking that the map

$\underline{\pi}_0\mathbb{S}_G \to \underline{\pi}_0\mathrm{MU}_G$ is a map of Tambara functors, we can observe that $n(E,\sigma)$ admits a *unique preimage* in $A(G)$, given by $\mathrm{Tr}_{G_x}^G(1)$, where this transfer is of the form $\mathrm{Tr}_{G_x}^G : A(G_x) \to A(G)$. This is precisely the G-set $Z(\sigma)$. □

In the following section we leverage this perspective to compute the 27 lines on a symmetric smooth cubic surface.

6. The 27 lines on a smooth symmetric cubic

In this section we apply our methods to compute the orbits of lines on a smooth symmetric cubic surface. In particular in the presence of symmetry we can state further constraints about the number of lines defined on a real cubic surface.

6.1. **27 lines on a complex symmetric cubic.**

Definition 6.1. We say that a cubic surface $X = V(F) \subset \mathbb{P}^3$ is S_4-*symmetric* (or just *symmetric*) if $F(x_0, x_1, x_2, x_3)$ is a symmetric polynomial.

In particular by letting S_4 act on $\mathbb{C}\mathrm{P}^3$ by permuting the projective coordinates $[x_0 : x_1 : x_2 : x_3]$, we have that symmetric cubics are precisely those preserved under this action. The lines on such a cubic surface therefore come equipped with S_4-orbits, and we can inquire about the orbit type. By equivariant conservation of number, the answer is independent of the choice of symmetric cubic surface.

Theorem 6.2. Given any smooth symmetric complex cubic surface, its 27 lines have orbit type

$$[S_4/C_2^o] + [S_4/C_2^e] + [S_4/D_8]\,,$$

where C_2^o is a single transposition, and C_2^e is a product of two disjoint transpositions.

Proof. We remark that a symmetric complex cubic surface X induces a section of the following S_4-equivariant complex vector bundle:

$$\begin{tikzcd} \mathrm{Sym}^3\mathcal{S}^* \arrow[r] & \mathrm{Gr}_{\mathbb{C}}(1, \mathbb{C}\mathrm{P}^3), \arrow[l, bend right, "\sigma_X"'] \end{tikzcd}$$

where $\mathcal{S}$ denotes the tautological bundle on the Grassmannian. In particular $\sigma_X(\ell) = 0$ if and only if $\ell \subseteq X$ is a line on the symmetric cubic. Since the 27 lines on X are necessarily distinct (c.f. [EH16, Theorem 5.1]), the zero locus $Z(\sigma_X)$ consists of 27 points on $\mathrm{Gr}_{\mathbb{C}}(1, \mathbb{C}\mathrm{P}^3)$, each of which is a simple zero of σ_X.

By Theorem 5.17, the S_4-orbits will be independent of the choice of symmetric cubic, so it suffices to pick our favorite symmetric cubic and compute the S_4-orbits of its lines. Consider the example of the *Fermat cubic*:

$$F = \left\{[w : x : y : z] \colon w^3 + x^3 + y^3 + z^3 = 0\right\}.$$

Fix ζ to be a primitive sixth root of unity in $\mathbb{C}$, hence we have three distinct cube roots of -1, namely ζ, ζ^{-1}, and -1. The 27 lines on the Fermat are given by the following equations, where $[w : z]$ varies over $\mathbb{C}\mathrm{P}^1$:

$$\begin{array}{llll}
[w : -w : z : \zeta z] & [w : -w : z : \zeta^{-1} z] & [w : \zeta w : z : -z] & [w : \zeta^{-1} w : z : -z] \\
[w : z : \zeta w : -z] & [w : z : \zeta^{-1} w : -z] & [w : z : -w : \zeta z] & [w : z : -w : \zeta^{-1} z] \\
[w : z : -z : \zeta w] & [w : z : -z : \zeta^{-1} w] & [w : z : \zeta z : -w] & [w : z : \zeta^{-1} z : -w] \\
[w : \zeta w : z : \zeta z] & [w : \zeta w : z : \zeta^{-1} z] & [w : \zeta^{-1} w : z : \zeta z] & [w : \zeta^{-1} w : z : \zeta^{-1} z] \\
[w : z : \zeta w : \zeta z] & [w : z : \zeta w : \zeta^{-1} z] & [w : z : \zeta^{-1} w : \zeta z] & [w : z : \zeta^{-1} w : \zeta^{-1} z] \\
[w : z : \zeta z : \zeta w] & [w : z : \zeta^{-1} z : \zeta w] & [w : z : \zeta z : \zeta^{-1} w] & [w : z : \zeta^{-1} z : \zeta^{-1} w]
\end{array}$$

$$[w : -w : z : -z] \quad [w : z : -w : -z] \quad [w : z : -z : -w]$$

Thus the orbits are as follows (colors are chosen so that the orbits match the orbits of the lines on Figure 1):

Color	Generating line	Isotropy subgroup	Orbit type	# of lines
Blue	$[w : -w : \zeta z : -z]$	$\langle(1\ 2)\rangle$	S_4/C_2^o	12
Green	$[w : \zeta w : z : \zeta z]$	$\langle(1\ 3)(2\ 4)\rangle$	S_4/C_2^e	12
Red	$[w : -w : z : -z]$	$\langle(1\ 3)(2\ 4), (1\ 2), (3\ 4)\rangle$	S_4/D_8	3

□

Given a subgroup $G \subseteq S_4$, it induces a natural action on $\mathbb{C}\mathrm{P}^3$, and we can ask about the 27 lines on a G-symmetric smooth complex cubic surface. For this result, the following is key.

Proposition 6.3. Let $E \to M$ be a G-equivariant rank n complex vector bundle over a smooth compact G-manifold of dimension n. Let A be a complex oriented G-cohomology theory, let $n_G(E)$ denote the Euler number of E in the A-cohomology theory, and for a subgroup $H \subseteq G$, let $n_H(E)$ denote the Euler number in the restricted H-equivariant A-cohomology theory. Then we have that

$$n_H(E) = \mathrm{Res}^G_H n_G(E).$$

Proof. This follows directly from the computation of the Euler number along a section with isolated simple zeros being valued in the homotopy Mackey functor $\underline{\pi}_0 A$. □

Thus given any subgroup $G \subseteq S_4$, we can compute the regular representation of the orbits of its 27 lines under the associated G-action by restricting the answer for S_4.

Notation 6.4. We denote by $C_2^o := \langle(1\ 2)\rangle$ and $C_2^e := \langle(1\ 2)(3\ 4)\rangle$ the odd and even conjugacy classes of cyclic subgroups of order two in S_4. We denote by $K_4^{\triangleleft}$ the normal Klein 4-subgroup of S_4, and K_4 the non-normal one. For a Klein four group, we denote by C_2^L, C_2^R, and C_2^{Δ} the left, right, and diagonal cyclic subgroups of order two, respectively.

Corollary 6.5. For all of the conjugacy classes of subgroups $G \subseteq S_4$, we can compute the G-orbits of the 27 lines on a G-symmetric smooth complex cubic surface, where the G-action on $\mathbb{CP}^3$ is acting on coordinates. These are in Table 1.

Proof. Following Proposition 6.3, the orbits listed are the G-orbits of the 27 lines on the Fermat cubic. □

6.2. **27 lines on a real symmetric cubic.** Observe that in the proof of Theorem 6.2, the three lines in the orbit $[S_4/D_8]$, labeled in red, were in fact defined over the reals. This is true in general.

Proposition 6.6. Let F be a real smooth symmetric cubic. Then on its complexification $V(F_{\mathbb{C}})$, the lines in the orbit $[S_4/D_8]$ are all defined over the reals, and hence form an orbit $[S_4/D_8]$ on $V(F)$.

Proof. Since lines defined over $\mathbb{C}$ but not over $\mathbb{R}$ must come in complex conjugate pairs, any such orbit of lines must be of even size. Since $|S_4/D_8| = 3$, all of its lines must in fact be real. □

Subgroup $G \subseteq S_4$	G-**orbits of lines**
e	$27[e/e]$
C_2^o	$12[C_2/e] + 3[C_2/C_2]$
C_2^e	$10[C_2/e] + 7[C_2/C_2]$
C_3	$9[C_3/e]$
$K_4^{\triangleleft}$	$[K_4/e] + 4[K_4/C_2^L] + 4[K_4/C_2^R] + 2[K_4/C_2^{\Delta}] + 3[K_4/K_4]$
K_4	$4[K_4/e] + [K_4/C_2^L] + [K_4/C_2^R] + 3[K_4/C_2^{\Delta}] + [K_4/K_4]$
C_4	$5[C_4/e] + 3[C_4/C_2^o] + [C_4/C_4]$
S_3	$3[S_3/e] + 3[S_3/C_2^o]$
D_8	$[D_8/e] + 3[D_8/C_2^e] + [D_8/C_2^o] + [D_8/K_4] + [D_8/D_8]$
A_4	$[A_4/e] + 2\,[A_4/C_2^e] + [A_4/K_4]$
S_4	$[S_4/C_2^o] + [S_4/C_2^e] + [S_4/D_8]$

TABLE 1. G-orbits of the 27 lines on a cubic surface, for $G \subseteq S_4$

The study of rationality of lines on a real cubic surface is a classical problem dating back to the 19th century.

Theorem 6.7. (Schläfli, 1858) A real smooth cubic surface can only contain 3, 7, 15, or 27 real lines, and all of these possibilities do in fact occur [Sch58].

Proposition 6.6 actually implies more — by examining the possible fields of definition of the other orbits of 12 lines, we can easily eliminate the possibility of seven real lines on a real symmetric cubic surface. We can in fact do even better using some more refined information about the lines in question, namely their topological *type*.

Definition 6.8. Let ℓ be a line on a smooth real cubic surface X, and consider the map

$$\begin{aligned}\mathbb{RP}^1 \cong \ell &\to \mathrm{SO}(3)\\ x &\mapsto T_x X.\end{aligned}$$

This associates to each line ℓ on the cubic surface a loop in the frame bundle $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2 = \{\pm 1\}$. The line ℓ is said to be *hyperbolic* if the associated class is $+1 \in \mathbb{Z}/2$, and *elliptic* if its associated class is $-1 \in \mathbb{Z}/2$. We refer to this as the *type* of the line $\ell \subseteq X$.

Proposition 6.9. On a real symmetric cubic surface X, the S_4 action on $\mathbb{CP}^3$ by permuting coordinates preserves the topological type of any line.

Proof. Given a line ℓ on a real cubic X, we have that for any point $p \in \ell$, there is a uniquely determined point $q \in \ell$ so that their tangent spaces are equal: $T_p X = T_q X$. This allows us to define an involution of the line ℓ, given by sending $p \mapsto q$ for every such pair of points. The topological type of the line is equivalently defined via the discriminant of the fixed locus of this involution [FK15]. Since this involution is defined independent of coordinates, it is invariant under a change of coordinates, and therefore the S_4-action does not affect the geometric properties of the involution attached to a line on X. $\square$

This indicates that within an S_4-orbit, all lines have the same type. A classical result following from work of Segre indicates that the types of lines are constrained.

Theorem 6.10. ([Seg42, BS95, OT14, FK15, KW21]) Let X be a real smooth cubic surface. Then the following equality holds:

$$\#\{\text{real hyperbolic lines on } X\} - \#\{\text{real elliptic lines on } X\} = 3.$$

Combining this with Schläfli's result, we have the following possibilities:

Total number of lines	Number of hyperbolic lines	Number of elliptic lines
3	3	0
7	5	2
15	9	6
27	15	12.

Theorem 6.11. A real smooth symmetric cubic surface can only contain 3 or 27 real lines, and both of these possibilities do occur.

Proof. By the argument following Proposition 6.6, we have that the possibility of seven lines cannot happen, so it suffices to argue that 15 lines cannot occur as well. By Proposition 6.9, we have that the action preserves topological type. Since we only have two orbits of sizes 3 and 12, we see that we cannot possibly have 9 hyperbolic lines and 6 elliptic lines, which are the prescribed types via Segre's theorem, hence we cannot have 15 lines. To argue existence of the other solutions, we observe that the Fermat cubic is an example of a symmetric real cubic surface with three lines, while the Clebsch is a symmetric real cubic surface admitting all 27. □

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