AL FM Discrete

Binary Operations, (TeX)

May 25, 2021

What is a Binary Operation?

Definition

Given a non-empty set S, a **binary operation** on S is a rule for combining any two elements $a, b \in S$ to give a unique result c where c is not necessarily an element of S.

Addition, subtraction and multiplication are all binary operations on $\mathbb{R}$ and division is a binary operation on $\mathbb{R}\backslash\{0\}$.

Square root and factorial are not binary operations (in fact both of these are unary operations as they only take one input).

Less familiar binary operations are often defined using a symbol such as $*, \Delta$ or $\odot$.

Problem

Let a binary operation on $\mathbb{Z}$ be defined by $a * b = a + 2b - 3$.

Find

- $3 * 5 = \boxed{10}$
- $3 * 0 = \boxed{}$
- $0 * 3 = \boxed{}$
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- $0 * 3 =$ $\boxed{3}$
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Test Your Understanding

Problem

The Binary operation $a \Delta b = a^2 + b^2 - 2ab$ *where* $a, b \in \mathbb{R}$.

- *Find the value of* $3\Delta 4$.
- *Find the relationship between* a *and* b *such that* $a \Delta b = 0$.

Problem

The binary operation $*$ *is given by* $M * N = MN + M - N$ *where* M *and* N *are* 2×2 *matrices. Show that, for any* M, $(M * I) * I = aM + bI$ *where* a *and* b *are integers to be determined.*

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Solution

$3\Delta 4 = 3^2 + 4^2 - 2 * 3 * 4 = 9 + 16 - 24 = 1$

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$(a - b)^2 = 0 \implies a = b$

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Solution

$$
\begin{aligned}
(M * I) * I &= (MI + M - I) * I \\
&= (M + M - I) * I \\
&= (2M - I) * I \\
&= (2M - I)(I) + (2M - I) - I
\end{aligned}
$$

Closure

Definition

A binary operation $*$ is said to be closed on a set S if $a * b \in S \forall a, b \in S$

Note that closure refers to a set *and* a binary operation

The binary operation of addition is closed on $\mathbb{Z}$ as the sum of any two integers is also an integer.

The binary operation of subtraction is not closed on $\mathbb{N}$ as $5 - 7 = -2 \notin \mathbb{N}$.

Problem

Determine whether the following binary operations are closed on $\mathbb{Z}$.

- $a * b = \frac{a+b}{a^2}$
- $a \Delta b = 2^{a+b}$
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Solution

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Which of the following sets are closed under multiplication?

- $\{a + bi | a, b \in \mathbb{Q}, b \neq 0\}$
- $\{a + bi | a, b \in \mathbb{Q}, a \neq 0\}$
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Solution

The first is not closed because $1 + i$ *and* $1 - i$ *are both in the set but* $(1 + i)(1 - i) = 2 + 0i$ *is not.*

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Solution

The second is not closed because $2 + i$ *and* $1 + 2i$ *are both in the set but* $(2 + i)(1 + 2i) = 0 + 5i$ *is not.*

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Solution

If we have two non-zero complex numbers that multiply to give zero: $zw=0$

Then since $|zw|=|z||w|$, either $|z|$ or $|w|=0$.

Therefore it is impossible to have two numbers in the set multiply to give the only number that isn't.

So the set is closed under multiplication.

Test Your Understanding

Problem

Determine whether the following function are closed on first $\mathbb{Z}$ and the on $\mathbb{Q}$.

- $a * b = a^2 - b$
- $a \circ b = \frac{a+b}{a}$
- $a \Delta b = \sqrt{a^2 b^2}$
- $a \Omega b = \frac{a+b}{a^2+1}$

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$3 \circ 2 = \frac{5}{2} \notin \mathbb{Z}$ *so not closed on* $\mathbb{Z}$.

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$\sqrt{a^2 b^2} = |ab|$. *If* $a, b \in \mathbb{Z}$ *then* $|ab| \in \mathbb{Z}$ *and if* $a, b \in \mathbb{Q}$ *then* $|ab| \in \mathbb{Q}$ *so closed on* $\mathbb{Z}$ *and* $\mathbb{Q}$.

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Solution

$1\Omega 2 = \frac{3}{2} \notin \mathbb{Z}$ *so not closed on* $\mathbb{Z}$.

Associativity

Definition

A binary operation $*$ is **associative** on a set S if $a * (b * c) = (a * b) * c \forall a, b, c \in S$.

The for all in the definition makes it easier to show that a binary operation is not associative as you only have to provide one counter example.

Problem

Determine whether the following binary operations on $\mathbb{R}$ are associative:

- $a * b = 2a + 3b$
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$(1 * 0) * 2 = 2 * 2 = 10$ *whereas* $1 * (0 * 2) = 1 * 6 = 20.$

Therefore $*$ in not associative on $\mathbb{R}$.

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Solution

$$\begin{aligned}
(a \circ b) \circ c &= (a + b + ab) \circ c \\
&= a + b + ab + c + (a + b + ab)(c) \\
&= a + b + ab + c + ac + bc + abc. \\
a \circ (b \circ c) &= a \circ (b + c + bc) \\
&= a + b + c + bc + a(b + c + bc)
\end{aligned}$$

Past Paper Question

Problem

The function $\min(a, b)$ *is defined by:*

$$\min(a, b) = \begin{cases} a, & a < b \\ b, & \text{otherwise.} \end{cases}$$

Gary claims that the binary operation Δ, *which is defined as*

$$x \Delta y = \min(x, y - 3)$$

where x *and* y *are real numbers, is associative as finding the smallest number is not affect by the order of the operation.*

Disprove Gary's claim.

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Disprove Gary's claim.

Solution

$2\Delta(1\Delta 3) = 2\Delta(0) = -3 \neq (-2)\Delta 3 = (2\Delta 1)\Delta 3.\square$

Commutativity

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A binary operation $*$ is said to be commutative on a set S if $a * b = b * a \forall a, b \in S$.

Problem

Determine whether the following operations on $\mathbb{R}$ are commutative.

- $a * b = 2a + b$
- $a \Delta b = 3^{a+b}$.

Problem

If $*$ is both associative and commutative on a set S, show that $(a * b)^2 = a^2 * b^2$.

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$3 * 2 = 2 \times 3 + 2 = 8$ *whereas* $2 * 3 = 2 \times 2 + 3 = 7$

So $*$ is not commutative on $\mathbb{R}$

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$$\begin{aligned} a \Delta b &= 3^{a+b} \\ &= 3^{b+a} \text{ (since addition is commutative on } \mathbb{R}) \\ &= b \Delta a. \end{aligned}$$

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So Δ is commutative on $\mathbb{R}$.

Problem

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Solution

$$\begin{aligned} (a * b)^2 &= (a * b) * (a * b) \\ &= a * (b * a) * b \text{ (because } * \text{ is associative on } S) \\ &= a * (a * b) * b \text{ (because } * \text{ is commutative on } S) \\ &= (a * a) * (b * b) \text{ (because } * \text{ is associative on } S) \\ &= a^2 * b^2. \end{aligned}$$

Past Paper Question

Problem

The binary operation $\blacklozenge$ *is defined as* $a \blacklozenge b = a^b$ *where a and b are non-zero real numbers.*

- *Determine whether or not* $\blacklozenge$ *is associative.*
- *Tim claims that as* $2 \blacklozenge 4 = 4 \blacklozenge 2$ *then* $\blacklozenge$ *is commutative.*
  Assess the validity of Tim's claim.

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Solution

It is not associative. $a \blacklozenge (b \blacklozenge c) = a^{(b^c)} \neq a^{bc} = (a^b)^c = (a \blacklozenge b) \blacklozenge c$

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Solution

He has only checked it for one case but commutative must hold for all $a, b \in \mathbb{R}$. It is also not commutative since $1 \blacklozenge 2 = 1 \neq 2 = 2 \blacklozenge 1$.

Identity

Definition

For a binary operation $*$ on a set S, if there exists an element e such that $x * e = e * x = x$ for all $x \in S$ then e is called the **identity** element.

This does not mean $*$ has to be commutative on S, just that e commutates.

Problem

For the following binary operations, determine whether an identity element exists in $\mathbb{R}$*:*

- $a * b = 3ab$
- $a \circ b = 3a + b$

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Solution

Using the definition we have that:

$$a * e = 3ae = a$$
$$(3e - 1)(a) = 0$$
$$e = \frac{1}{3}.$$

Easy to check that this also works for $e * a$. So an identity exists for $*$ on $\mathbb{R}$ and is $\frac{1}{3}$.

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Easy to check that this also works for $e * a$. So an identity exists for $*$ on $\mathbb{R}$ and is $\frac{1}{3}$.

Solution

$$\begin{aligned} a \circ e = 3a + e &= a \\ e &= -2a. \end{aligned}$$

Since the identity element is constant it cannot change depending on a so no identity for $\circ$ on $\mathbb{R}$ exists.

Inverse

Definition

Consider a binary operation $*$ on a set S with an identity element e. For $x \in S$, an inverse element $x^{-1} \in S$ exists if and only if $x * x^{-1} = x^{-1} * x = e$.

Note that this definition does not make sense unless there is an identity element.

For addition in $\mathbb{R}$, the inverse of the element a is $\boxed{-a}$.

For addition on the set $\{0, 1, 2, 3, \ldots\}$ there is no inverse for any element apart from 0 (even though there is an identity).

For multiplication on $\mathbb{R} \backslash \{0\}$ the inverse of an element a is $\boxed{}$.

Problem

The binary operation $\Diamond$ is given by $a \Diamond b = a + b - ab$ where $a, b \in \mathbb{Z}$.

- *Show that the identity element in the set $\mathbb{Z}$ with respect to the binary operation $\Diamond$ is 0.*
- *Find the inverse of the element 4 under the binary operation $\Diamond$.*

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For multiplication on $\mathbb{R}\backslash\{0\}$ the inverse of an element a is $\boxed{\frac{1}{a}}$.

Problem

The binary operation $\lozenge$ is given by $a \lozenge b = a + b - ab$ where $a, b \in \mathbb{Z}$.

- *Show that the identity element in the set $\mathbb{Z}$ with respect to the binary operation $\lozenge$ is 0.*
- *Find the inverse of the element 4 under the binary operation $\lozenge$.*

Solution

$a \lozenge 0 = a + 0 - a \times 0 = a$ *and* $0 \lozenge a = 0 + a - 0 \times a = a$, *so 0 is the identity element.*

Inverse

Definition

Consider a binary operation $*$ on a set S with an identity element e. For $x \in S$, an inverse element $x^{-1} \in S$ exists if and only if $x * x^{-1} = x^{-1} * x = e$.

Note that this definition does not make sense unless there is an identity element.

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Solution

If $a\lozenge 4 = 0$ *then* $a + 4 - 4a = 0$ *and* $a = \frac{4}{3}$. *This* $4^{-1} = \frac{4}{3}$.

Cayley Tables

Definition

A **Cayley table** is a square array where the elements of a set S are the headings of the rows and columns and the elements are the result of inputting the row header and column header into a binary operation $*$.

Problem

Let a binary operation $*$ on a set $\{0, 1, 2, 3\}$ be defined by $a * b = a^2 + ab$.

- *Construct the Cayley table for $*$.*
- *Is the operation $*$ closed on S?*
- *Is the operation $*$ commutative?*

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Solution

$*$	0	1	2	3
0	0	0	0	0
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2	0	6	8	10
3	0	12	15	18

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The operation $*$ is clearly not closed on S. There are elements in the Cayley table which are not in S. Take $1 * 3 = 4 \notin S$.

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Solution

The operation $*$ is clearly not closed on S. There are elements in the Cayley table which are not in S. Take $1*3=4\notin S$.

Solution

The lack of symmetry around the leading diagonal indicates that the operation $*$ is not commutative.

Test Your Understanding

Problem

Let $U_4 = \{1, i, -i, -1\}$ *and consider the operation of multiplication it* U_4.

- *Construct the Cayley table for the operation.*
- *Is the operation commutative or associative?*
- *State the identity if it exists.*
- *Find the inverse of each element, if possible.*

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i	i	-1	1	$-i$
$-i$ $-i$	1	-1	i	
-1	-1	$-i$	i	1

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Solution

The identity is 1.

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Solution

The identity is 1.

Solution

1 and -1 are both self inverse. i and $-i$ are inverse pairs.

Modular Arithmetic

Definition

Modular arithmetic is a system of arithmetic for integers where numbers 'wrap around' upon reaching a certain value called the modulus.

Definition

$+_n$ means addition modulo n.

$\times_n$ means multiplication modulo n.

You do modular arithmetic all the time when reading a clock. 10 o'clock plus three hours is 1 o'clock because we wrap around at 12. We would write $10 + 3 \equiv 1 \mod 12$ or $10 +_{12} 3 = 1$.

$5 +_{12} 10 = \boxed{3}$, $5 \times_{12} 10 = \boxed{}$, $6 \times_9 3 = \boxed{}$, $8 +_{10} 13 = \boxed{}$, $1 +_{10} 1 = \boxed{}$.

Problem

The binary operation $\blacksquare$ *is given by* $x \blacksquare y = x^2 + y + 2 \mod 16$ *where* $x, y \in \mathbb{Z}$.

- *Find the value of* $5 \blacksquare 3$.
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Solution

$5 \blacksquare 3 = 5^2 + 3 + 2 \mod 16 = 30$
$\mod 16 = 14$.

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Solution

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Solution

$a^2 + b = 14, a = 3, b = 5$

Test Your Understanding

Problem

The binary operation $*$ *is defined as* $x * y = x + y + 1 \mod 2$ *where* $x, y \in \mathbb{R}$.

- *Prove that the binary operation* $*$ *is associative.*
- *Find an identity element of the set* $\mathbb{R}$ *with respect to the binary operation* $*$.
- *Prove that the sat* $\mathbb{R}$ *has infinitely many identity elements with respect to* $*$.

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Solution

$$\begin{aligned}(a * b) * c &= (a + b + 1 \mod 2) * c \\ &= a + b + 1 + c + 1 \mod 2 \\ &= a + b + c \mod 2.\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a * (b + c + 1 \mod 2) \\ &= a + b + c + 1 + 1 \mod 2 \\ &= a + b + c \mod 2.\end{aligned}$$

$\therefore a * (b * c) = (a * b) * c$ *for all* $a, b, c \in \mathbb{R}$ *and so* $*$ *is associative.*

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Solution

For the identity element e, $e * a = a * e = a = a + e + 1 \mod 2$. *So* $e + 1 = 0 \mod 2$, *and so* $e = 1$ *works. So 1 is an identity element.*

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Solution

Anything equal to $1 \mod 2$ *is an identity element. So anything of the form* $1 + 2k, \forall k \in \mathbb{Z}$, *will work. This is identical to all odd numbers.*

Test Your Understanding

Problem

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. *Find the inverse of the element* 5 *under multiplication* $\mod 12$.

Problem

- *Construct a Cayley table far multiplication modulo 5 on* $\{1, 2, 3, 4\}$.
- *Use the table to solve the following:*
  - $2x = 1 \mod 5$
  - $4x + 3 = 4 \mod 5$
- *State the identity if it exists.*
- *Verify that* $3^{-1} = 2$.
- *State the inverse of each other element.*

Test Your Understanding

Problem

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. *Find the inverse of the element* 5 *under multiplication* $\mod 12$.

Solution

$5 \times_{12} 5 = 1$ *so* $5^{-1} = 5$. *It is self inverse.*

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Solution

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

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3	3	1	4	2
4	4	3	2	1

Solution

$x = 3$

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Solution

$x = 3$

Solution

$4x = 1 \mod 5 \implies x = 4$

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Solution

The identity is 1.

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Solution

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The identity is 1.

Solution

$3 \times_5 2 = 2 \times_5 3 = 1$

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Solution

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Solution

$4x = 1 \mod 5 \implies x = 4$

Solution

The identity is 1.

Solution

$3 \times_5 2 = 2 \times_5 3 = 1$

Solution

1 and 4 are self inverse. 2 and 3 are an inverse pair.