AL FM Pure

Differentiating and Integrating Inverse Trigonometric Functions

Yavuz

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Differentiating Inverse Functions

At AL you are also expected to know that, when y isn't the subject, it can sometimes be easier to find $\frac{dx}{dy}$ and then *'turn it upside down'*

Definition

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Let $y = \sin^{-1} x$ so we are trying to find $\boxed{\frac{dy}{dx}}$

Then $\sin y = x$ or reversing the equation

Differentiating both sides with respect to y:

Because there are no ys in the question, there should be no ys in our answer:

$\cos y =$ (because on line 2 we had $x = \sin y$)

$\therefore \frac{dy}{dx} =$

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$\cos y = \boxed{\sqrt{1-\sin^2 y} = \sqrt{1-x^2}}$ (because on line 2 we had $x = \sin y$)

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$\cos y = \boxed{\sqrt{1-\sin^2 y} = \sqrt{1-x^2}}$ (because on line 2 we had $x = \sin y$)

$$\therefore \frac{dy}{dx} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

Test Your Understanding

Problem

Prove $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$

Test Your Understanding

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Prove $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$

Solution

Let $y = \cos^{-1} x$ *so we are trying to find* $\frac{dy}{dx}$

Then $\cos y = x$ *or reversing the equation* $x = \cos y$

Differentiating both sides with respect to y*:* $\frac{dx}{dy} = -\sin y$

$\therefore \frac{dy}{dx} = \frac{-1}{\sin y}$

$\sin y = \sqrt{1 - \cos y} = \sqrt{1 - x^2}$ *(because on line 2 we had* $x = \cos y$*)*

$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

Test Your Understanding

Problem

Prove $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

Test Your Understanding

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Prove $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

Solution

Let $y = \tan^{-1} x$ *so we are trying to find* $\frac{dy}{dx}$

Then $\tan y = x$ *or reversing the equation* $x = \tan y$

Differentiating both sides with respect to y*:* $\frac{dx}{dy} = \sec^2 y$

$\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y}$

We could write this as $\cos^2 y$ *but as we have a trig identity which relate* tan *and* sec*, it is simpler to leave it as it is.*

$\sec^2 y = 1 + \tan^2 y = 1 + x^2$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$

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Given that $y = \operatorname{arcsec} 2x$*, show that* $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{4x^2 - 1}}$

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Solution

$\sec y = 2x$

$x = \dfrac{\sec y}{2}$

$\dfrac{dx}{dy} = \dfrac{\sec y \tan y}{2}$

$\dfrac{dy}{dx} = \dfrac{2}{\sec y \tan y} = \dfrac{2}{\sec\sqrt{\sec^2 y - 1}} = \dfrac{2}{2x\sqrt{4x^2 - 1}}$

$= \dfrac{1}{x\sqrt{4x^2 - 1}}$

Test Your Understanding

Problem

Given that $y = \arctan\left(\dfrac{1-x}{1+x}\right)$, *find* $\dfrac{dy}{dx}$.

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Given that $y = \arctan\left(\frac{1-x}{1+x}\right)$, *find* $\frac{dy}{dx}$.

Solution

Let $u = \frac{1-x}{1+x}$, *then using the quotient rule,*

$$\frac{du}{dx} = \frac{(1-x)(-1)-(1-x)(1)}{(1+x)^2} = -\frac{2}{(1+x)^2}$$

Therefore, by the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \times -\frac{2}{(1+x)^2} \\ &= -\frac{2}{(1+x)^2+(1-x)^2} \\ &= -\frac{2}{2+2x^2} \\ &= -\frac{1}{1+x^2}\end{aligned}$$

Integrating Inverse Functions

Definition

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad |x| < a$$

The restriction $|x| < a$ is to ensure that there is $\boxed{\text{no negative square root}}$

Think about what a sensible substitution might be so that you end up with something squared under the square root. Also looking at the answer gives a hint about the substitution.

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Solution

Let $x = a \sin u$ *then* $\frac{dx}{du} = a \cos u$ *so* $dx = a \cos u \; du.$

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2 - x^2}} \; dx &= \int \frac{1}{\sqrt{a^2 - a^2 \sin^2 u}} a \cos u \; du \\
&= \int \frac{1}{a\sqrt{1 - sin^2 u}} a \cos u \; du \\
&= \int \frac{1}{a \cos u} a \cos u \; du \\
&= \int 1 \; du \\
&= u + C
\end{aligned}$$

But rearranging $x = a \sin u$ *gives* $\frac{x}{a} = \sin u$ *and* $u = \sin^{-1}(\frac{x}{a})$

Therefore $\int \frac{1}{\sqrt{a^2 - x^2}} \; dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

Integrating Inverse Functions

Definition

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Again look at the answer to think about what might be a good substitution.

Integrating Inverse Functions

Definition

$$\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Again look at the answer to think about what might be a good substitution.

Solution

Let $x = a\tan u$ *then* $\dfrac{dx}{du} = a\sec^2 u$ *so* $dx = a\sec^2 u\ du$

$$\begin{aligned}
\int \frac{1}{1+x^2}\ dx &= \int \frac{1}{a^2+a^2\tan^2 u} a\sec^2 u\ du \\
&= \int \frac{1}{a^2(1+\tan^2 u)} a\sec^2\ du \\
&= \int \frac{1}{a^2\sec^2 u} a\sec^2\ du \\
&= \frac{1}{a}\int 1\ du \\
&= \frac{1}{a}u + C
\end{aligned}$$

But rearranging $x = a\tan u$ *gives* $\frac{x}{a} = \tan u$ *and* $u = \tan^{-1}(\frac{x}{a})$.

Therefore $\int \dfrac{1}{a^2+x^2}\ dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C.$

Test Your Understanding

Problem

Find $\int \frac{4}{5+x^2}\ dx$

Problem

Find $\int \frac{1}{25+9x^2}\ dx$

Hint: Factorise out $\frac{1}{9}$ so that you get left with a single x^2 in the integral

Problem

Showing full working, find $\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}}\ dx$

Test Your Understanding

Problem

Find $\int \frac{4}{5+x^2}\,dx$

Solution

$$\int \frac{4}{5+x^2}\,dx = 4\int \frac{1}{5+x^2}\,dx = \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

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Hint: Factorise out $\frac{1}{9}$ so that you get left with a single x^2 in the integral

Solution

$$\int \frac{1}{25+9x^2}\,dx = \frac{1}{9}\int \frac{1}{\frac{25}{9}+x^2}\,dx = \frac{1}{9}\times\frac{1}{\frac{5}{1}}\arctan\left(\frac{\frac{x}{5}}{3}\right) + C = \frac{1}{15}\arctan\left(\frac{3x}{5}\right) + C$$

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Problem

Showing full working, find $\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx$

Solution

$$\int_{-\sqrt{3}/4}^{\sqrt{3}/4} \frac{1}{\sqrt{3-4x^2}}\,dx = \int_{-\sqrt{3}/4}^{\sqrt{3}/4} \frac{1}{\sqrt{4}\sqrt{\frac{3}{4}-x^2}}\,dx = \frac{1}{2}\left[\arcsin\left(\frac{2x}{\sqrt{3}}\right)\right]_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} = \frac{\pi}{12} - \left(-\frac{\pi}{12}\right) = \frac{\pi}{6}$$

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Problem

Find $\int \frac{x+4}{\sqrt{1-4x^2}}\ dx$

Hint: Split up the numerator

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Solution

$$\int \frac{x+4}{\sqrt{1-4x^2}} dx = \int \frac{x}{\sqrt{1-4x^2}}\ dx + 4\int \frac{1}{\sqrt{1-4x^2}}\ dx$$

Deal with each in turn:

$$\int \frac{x}{\sqrt{1-4x^2}}\ dx = \int x\left(1-4x^2\right)^{-\frac{1}{2}}\ dx = \frac{1}{4}\sqrt{1-4x^2} + C$$

$$4\int \frac{1}{\sqrt{1-4x^2}}\ dx = 4\int \frac{1}{\sqrt{4(\frac{1}{4}-x^2)}}\ dx = 2\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\ dx = 2\arcsin 2x + C$$

$$\therefore \int \frac{x+4}{\sqrt{1-4x^2}}\ dx = \frac{1}{4}\sqrt{1-4x^2} + 2\arcsin 2x + C$$

Quadratic Denominator that can't be Factorised

If you have a rational function where the denominator is a quadratic that can be factorised, you should factorise and use partial fractions as in A Level.

Definition

When the denominator is a quadratic that can't be factorised, you should complete the square and use $\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$ formula.

Problem

Find the exact value of $\int_{-2}^{\sqrt{3}-3} \dfrac{1}{x^2+6x+12}\ dx$

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Problem

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Solution

$$\begin{aligned}
\int_{-2}^{\sqrt{3}-3} \frac{1}{x^2+6x+12}\,dx &= \int_{-2}^{\sqrt{3}-3} \frac{1}{(x+3)^2+3}\,dx \\
&= \int_{-2}^{\sqrt{3}-3} \frac{1}{\left(\sqrt{3}\right)^2+(x+3)^2}\,dx \\
&= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{x+3}{\sqrt{3}}\right)\right]_{-2}^{\sqrt{3}-3} \\
&= \frac{1}{\sqrt{3}}\left(\arctan 1 - \arctan\left(-\frac{1}{\sqrt{3}}\right)\right) \\
&= \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \frac{\pi\sqrt{3}}{36}
\end{aligned}$$

If you can't see this step in one go, you could substitute $u = x+3$. However when integrating, for a linear function of x you can do what you would do for x but divide by the derivative of that linear function.

Here the linear function is $x+3$ so the derivative is 1 so you don't need to divide.

Partial Fractions

We can now integrate with partial fractions where one of the factors of the denominator is a quadratic that cannot be factorised.

Recall from A Level that when you write as partial fractions, you must ensure you have the most general possible non-top heavy fraction, i.e. the 'order' (i.e. maximum power) of the numerator is one less than the denominator.

i.e. You should initially write $\dfrac{1}{x\left(x^2+1\right)} \equiv \dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$

Problem

Show that $\int \dfrac{1+x}{x^3+9x}\,dx = A\ln\left(\dfrac{x^2}{x^2+1}\right)+B\arctan\left(\dfrac{x}{3}\right)+C$*, where* A *and* B *are constants.*

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Problem

Show that $\int \dfrac{1+x}{x^3+9x}\, dx = A\ln\left(\dfrac{x^2}{x^2+1}\right) + B\arctan\left(\dfrac{x}{3}\right) + C$, *where* A *and* B *are constants.*

Solution

$$\int \frac{1+x}{x^3+9x}\, dx = \int \frac{1+x}{x(x^2+9)}\, dx = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$1 + x \equiv A\left(x^2+9\right) + x\left(Bx+C\right)$

$x = 0 \Rightarrow A = \dfrac{1}{9}$ *and* $x = 3 \Rightarrow 4 = 2 + 9B + 3C$ *and* $x = -3 \Rightarrow -2 = 2 + 9B - 3C$

Solving the simultaneous equations gives $B = -\frac{1}{9}$ *and* $C = 1$

$$\begin{aligned}\int \frac{1+x}{x^3+9x}\, dx &= \frac{1}{9}\int \frac{1}{x}\, dx - \frac{1}{9}\int \frac{x}{x^2+9}\, dx + \int \frac{1}{x^2+9}\, dx \\ &= \frac{1}{9}\ln x - \frac{1}{18}\ln\left(x^2+9\right) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C \\ &= \frac{1}{18}\ln\left(\frac{x^2}{x^2+1}\right) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C\end{aligned}$$

Partial Fractions

As in A Level, if the numerator has degree at least that of the denominator then you will have to do long division first.

Problem

- *Express* $\dfrac{x^4+x}{x^4+5x^2+6}$ *as partial fractions.*
- *Hence find* $\displaystyle\int \frac{x^4+x}{x^4+5x^2+6}\,dx$.

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- *Hence find* $\int \dfrac{x^4+x}{x^4+5x^2+6}\,dx$.

Solution

Using algebraic long division: $x+4 = (x^4+5x^2+6)1 - 5x^2 + x - 6$

$\therefore \dfrac{x^4+x}{x^4+5x^2+6} = 1 + \dfrac{-5x^2+x-6}{x^4+5x^2+6}$

Since $x^4+5x^2+6 = (x^2+2)(x^2+3)$*:* $\dfrac{-5x^2+x-6}{x^4+5x^2+6} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{x^2+3}$

Then $-5x^2+x-6 = (Ax+B)(x^2+3) + (Cx+D)(x^2+2)$

Use $x = 0, 1, 2, 3$ *to get four simultaneous equations which you can then solve on your calculator to get* $A = 1, B = 2, C = -1 and D = -9$

$\therefore \dfrac{x^4+x}{x^4+5x^2+6} = 1 + \dfrac{x+4}{x^2+2} - \dfrac{x+9}{x^2+3}$

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- *Hence find* $\int \dfrac{x^4+x}{x^4+5x^2+6}\,dx$.

Solution

Using algebraic long division: $x+4=(x^4+5x^2+6)1-5x^2+x-6$

$$\therefore \frac{x^4+x}{x^4+5x^2+6}=1+\frac{-5x^2+x-6}{x^4+5x^2+6}$$

Since $x^4+5x^2+6=(x^2+2)(x^2+3)$:$\dfrac{-5x^2+x-6}{x^4+5x^2+6}=\dfrac{Ax+B}{x^2+2}+\dfrac{Cx+D}{x^2+3}$

Then $-5x^2+x-6=(Ax+B)(x^2+3)+(Cx+D)(x^2+2)$

Use $x=0,1,2,3$ *to get four simultaneous equations which you can then solve on your calculator to get* $A=1, B=2, C=-1 and D=-9$

$$\therefore \frac{x^4+x}{x^4+5x^2+6}=1+\frac{x+4}{x^2+2}-\frac{x+9}{x^2+3}$$

Solution

$$\int 1+\frac{x+4}{x^2+2}-\frac{x+9}{x^2+3}\,dx=\int 1+\frac{x}{x^2+2}+\frac{4}{x^2+2}-\frac{x}{x^2+3}-\frac{9}{x^2+3}\,dx$$

$$=\ldots=x+\frac{1}{2}\ln\left|\frac{x^2+2}{x^2+3}\right|+2\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)-3\sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right)+C$$