AL FM Pure

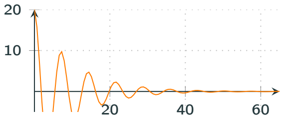
Limits, (TeX)

May 22, 2021

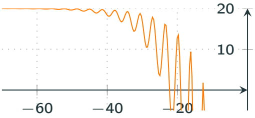
What are Limits?

A limit is the value a function gets closer to as the input gets closer to a given value

We talk about the limit as x tends to ∞ (or $-\infty$) as being the value that the functions is getting closer to as the x gets larger and larger (or smaller and smaller).

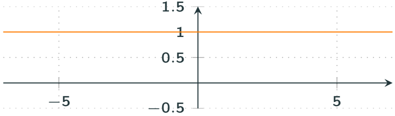


$$\lim_{x\to\infty} f(x) = \boxed{0}.$$



$$\lim_{x\to-\infty} f(x) = \boxed{}.$$

However, there may also be gaps in a function where it is undefined but the function is getting closer to a certain value as it approaches that gap.



The function $f(x) = \frac{x}{x}$ is undefined at $x = 0$ (because you can't compute $\frac{0}{0}$ but is 1 everywhere else. It should be clear to see that $\lim_{x\to 0} \frac{x}{x} = \boxed{}$.

You have seen such a limiting process in AL differentiation from first principles.

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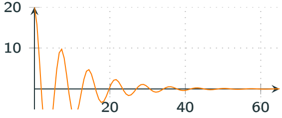

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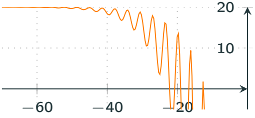

$$\lim_{x\to-\infty} f(x) = \boxed{20}.$$

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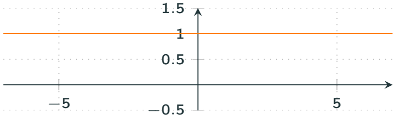

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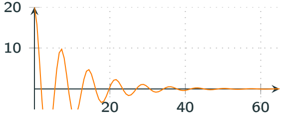

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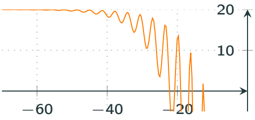

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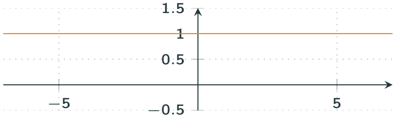

The function $f(x) = \frac{x}{x}$ is undefined at $x = 0$ (because you can't compute $\frac{0}{0}$ but is 1 everywhere else. It should be clear to see that $\lim_{x\to 0} \frac{x}{x} = \boxed{1}$.

You have seen such a limiting process in AL differentiation from first principles.

Using Series Expansions to find a Limit

Problem

Find $\lim_{x \to 0} \frac{x}{1-\sqrt{1-x}}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \quad (|x| < 1, n \in \mathbb{Q})$$

Using Series Expansions to find a Limit

Problem

Find $\lim_{x \to 0} \frac{x}{1-\sqrt{1-x}}$

Solution

Using the binomial expansion,

$$\begin{aligned}\sqrt{1-x} &= (1-x)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}(-x)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!}(-x)^3 + \dots \\ &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots\end{aligned}$$

Hence,

$$\begin{aligned}f(x) &= \frac{x}{1-(1-\frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots} \\ &= \frac{x}{\frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots} \\ &= \frac{1}{\frac{1}{2} + \frac{1}{8}x + \frac{1}{16}x^2 + \dots}\end{aligned}$$

.

It can now be seen that when $x \to 0$, all the terms in the denominator of the above expression, except the first, tend to zero. Hence,

$$\lim_{x \to 0} f(x) = \frac{1}{\frac{1}{2}} = 2.$$

Notice that the key step in this question was that after we had used the binomial expansion then we cancelled an x from the top and the bottom. Before this the value of the function at $x = 0$ was $\frac{0}{0}$ but afterwards we got something that could be evaluated.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1.2\dots r}x^r + \dots \quad (|x| < 1, n \in \mathbb{Q})$$

Using Series Expansions to find a Limit

Problem

Find $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{\cos x - \cos 2x}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all} \quad x, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all} \quad x.$$

Using Series Expansions to find a Limit

Problem

Find $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{\cos x - \cos 2x}$.

Solution

$$\begin{aligned}\frac{2\sin x - \sin 2x}{\cos x - \cos 2x} &= \frac{2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots\right)}{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right)} \\ &= \frac{\left(2x - \frac{1}{3}x^3 + \frac{1}{60}x^5 - \dots\right) - \left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots\right)}{\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right) - \left(1 - 2x^2 + \frac{2}{3}x^4 - \dots\right)} \\ &= \frac{x^3 + \textit{terms in } x^5 \textit{ and higher powers}}{\frac{3}{2}x^2 + \textit{terms in } x^4 \textit{ and higher powers}} \\ &= \frac{x + \textit{terms in } x^3 \textit{ and higher powers}}{\frac{3}{2} + \textit{terms in } x^2 \textit{ and higher powers}}.\end{aligned}$$

As $x \to 0$, *the numerator tends to zero and the denominator tends to* $\frac{3}{2}$. *Hence,*

$$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{\cos x - \cos 2x} = \frac{0}{\frac{3}{2}} = 0.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all} \quad x, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all} \quad x.$$

The mathematical notation for this is $O(x^5)$.

Past Paper Question

Problem

- *Find the first three non-zero in the expansion of* $\frac{x}{\ln(1+x)}$ *as a series in ascending powers of* x.
- *Hence find* $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right)$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \quad (-1 < x \leq 1).$$

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Solution

$$\begin{aligned}
\frac{x}{\ln(1+x)} &= \frac{x}{x - \frac{x^2}{2} + \frac{x^3}{3} - \dots} \\
&= \frac{1}{1 - \frac{x}{2} + \frac{x^2}{3} - \dots} \\
&= \left(1 - \frac{x}{2} + \frac{x^2}{3} - \dots\right)^{-1} \\
&= 1 - \left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right) + \frac{-1 \times -2}{2!}\left(-\frac{x}{2} + \frac{x^2}{3}\right)^2 + \dots \\
&= 1 + \frac{x}{2} - \frac{x^2}{3} + \frac{x^2}{4} + \dots \\
&= 1 + \frac{x}{2} - \frac{x^2}{12} + \dots.
\end{aligned}$$

Using the binomial expansion for x^{-1} *in the formula booklet but with* $\ln(1+x)$ *instead of* x

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1}\frac{x^r}{r} + \dots \quad (-1 < x \le 1).$$

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$$\begin{aligned}
\frac{x}{\ln(1+x)} &= \frac{x}{x - \frac{x^2}{2} + \frac{x^3}{3} - \dots} \\
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&= 1 + \frac{x}{2} - \frac{x^2}{3} + \frac{x^2}{4} + \dots \\
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\end{aligned}$$

Using the binomial expansion for x^{-1} *in the formula booklet but with* $\ln(1+x)$ *instead of* x

Solution

$$\begin{aligned}
\frac{1}{\ln(1+x)} - \frac{1}{x} &= \frac{1}{x}\left(\frac{x}{\ln(1+x)} - 1\right) \\
&= \frac{1}{x}\left(1 + \frac{x}{2} - \frac{1}{12}x^2 + \dots - 1\right) \\
&= \frac{1}{x}\left(\frac{x}{2} - \frac{1}{12}x^2 + \dots\right) \\
&= \frac{1}{2} - \frac{1}{12}x + \dots \\
&\to \text{as} \quad x \to 0.
\end{aligned}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1}\frac{x^r}{r} + \dots \quad (-1 < x \le 1).$$

L'Hôpital's Rule

L'Hôpital's Rule can be used to find limits for function of the form $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ either both tend to zero or both tend to infinity.

Definition

L'Hôpital's Rule

If $f(c) = g(c) = 0$ or $\pm\infty$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ Not in formula booklet

In other words if, when you plug in the limit, you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then you can differentiate the numerator and the denominator and you will still get the same limit.

Definition

If after applying l'Hôpital's Rule we get something that is getting infinitely large as we get closer and closer to the limit, we say that the limit does not exist.

Find $\lim_{x \to 0} \frac{\sin x}{x^2}$

We can use l'Hôpital's rule because $f(0) = \sin 0 = 0$ and $g(0) = 0^2 = 0$

$$\frac{f'(x)}{g'(x)} = \frac{\cos x}{2x}$$

As $x \to 0$ then $\frac{\cos x}{2x}$ is getting larger and larger (tending to $\frac{1}{0}$) and so we say that the limit does not exist.

L'Hôpital's Rule

Problem

Evaluate $\lim_{x \to 1} \frac{5 \ln x}{x-1}$.

Here we can use L'Hôpital's rule because as $x \to 1$ both the numerator and denominator tend to 0.

Problem

Evaluate $\lim_{x \to \infty} \frac{4x}{e^x}$.

Here we can use L'Hôpital's rule because as $x \to \infty$ both the numerator and denominator tend to ∞

Problem

Evaluate $\lim_{x \to \frac{\pi}{2}} (3 \sec x - 3 \tan x)$.

Sometimes you might have to manipulate the expression to make it in a form suitable for use in L'Hôpital's Rule

Problem

Evaluate $\lim_{x \to 1} \frac{5 \ln x}{x-1}$.

Here we can use L'Hôpital's rule because as $x \to 1$ both the numerator and denominator tend to 0.

Solution

Differentiating the numerator and denominator: $\lim_{x \to 1} \frac{5 \ln x}{x-1} = \lim_{x \to 1} \frac{\frac{5}{x}}{1} = 5$.

Problem

Evaluate $\lim_{x \to \infty} \frac{4x}{e^x}$.

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L'Hôpital's Rule

Problem

Evaluate $\lim_{x\to 1} \frac{5\ln x}{x-1}$.

Here we can use L'Hôpital's rule because as $x \to 1$ both the numerator and denominator tend to 0.

Solution

Differentiating the numerator and denominator: $\lim_{x\to 1} \frac{5\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{5}{x}}{1} = 5$.

Problem

Evaluate $\lim_{x\to\infty} \frac{4x}{e^x}$.

Here we can use L'Hôpital's rule because as $x \to \infty$ both the numerator and denominator tend to ∞

Solution

Differentiating the numerator and denominator: $\lim_{x\to\infty} \frac{4x}{e^x} = \lim_{x\to\infty} = \frac{4}{e^x} = 0$.

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Solution

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Problem

Evaluate $\lim_{x\to\infty} \frac{4x}{e^x}$.

Here we can use L'Hôpital's rule because as $x \to \infty$ both the numerator and denominator tend to ∞

Solution

Differentiating the numerator and denominator: $\lim_{x\to\infty} \frac{4x}{e^x} = \lim_{x\to\infty} = \frac{4}{e^x} = 0.$

Problem

Evaluate $\lim_{x\to\frac{\pi}{2}} (3\sec x - 3\tan x)$.

Sometimes you might have to manipulate the expression to make it in a form suitable for use in L'Hôpital's Rule

Solution

$3\sec x - 3\tan x = \frac{3}{\cos x} - \frac{3\sin x}{\cos x} = \frac{3-3\sin x}{\cos x}$. *We can use L'Hôpital's Rule to evaluate* $\lim_{x\to\frac{\pi}{2}} \frac{3-3\sin x}{\cos x}$ *as numerator and denominator both tend to 0 as* $x \to \frac{\pi}{2}$.

Differentiating numerator and denominator: $\lim_{x\to\frac{\pi}{2}} \frac{3-3\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{-3\cos x}{-\sin x} = \frac{0}{-1} = 0.$

Sometimes you might have to use L'Hôpital's Rule more than once.

Problem

Evaluate $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right)$.

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Problem

Evaluate $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right)$.

Solution

$\frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{x-\ln(1+x)}{x\ln(1+x)}$.

We can use L'Hôpital's Rule to evaluate $\lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)}$ *as numerator and denominator both tend to zero as* $x \to 0$.

Differentiating numerator and denominator:

$\lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)} = \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{\ln(1+x)+\frac{x}{1+x}} = \lim_{x\to 0}\frac{x}{(1+x)\ln(1+x)+x}$.

We can use L'Hôpital's Rule again to evaluate $\lim_{x\to 0}\frac{x}{(1+x)\ln(1+x)+x}$ *as numerator and denominator both tend to zero as* $x \to 0$.

Differentiating numerator and denominator:

$\lim_{x\to 0}\frac{x}{(1+x)\ln(1+x)+x} = \lim_{x\to 0}\frac{1}{1+\ln(1+x)+1} = \frac{1}{2}$.

Past Paper Question

Problem

- *Find the value of A for which we can use l'Hôpital's rule to evaluate the limit*

$$\lim_{x\to 2} \frac{x^2 + Ax - 2}{x - 2}.$$

- *For the value of A, give the value of the limit.*

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Solution

We need $\lim_{x \to 2} x^2 + Ax - 2 = 0$ *so* $4 + 2A - 2 = 0$ *so* $A = -1$.

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Solution

We need $\lim_{x\to 2} x^2 + Ax - 2 = 0$ *so* $4 + 2A - 2 = 0$ *so* $A = -1$.

Solution

Differentiating numerator and denominator: $\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{2x-1}{1} = 3$.