AL FM Discrete

Fundamentals of Game Theory

May 25, 2021

Zero-Sum Games

Definition

A **zero-sum game** is when the gain of one player is equal to the loss of the other, regardless of what strategy each of the players choose.

Definition

We represent the outcomes of such a game in a **payoff matrix.**

The person at the top (whoose startegies are given by the columns) is **Player 2**.

		Carrie		
		X	Y	Z
Gary	P	4	2	2
	Q	-3	5	1
	R	2	-1	3

The person on the left (whoose strategies are given by the rows) is **Player 1** .

The numbers in the payoff matrix correspond to the gains for Player 1

A negative number indicates a loss for Player 1 and hence, because it is a zero-sum game, a gain for Player 2.

In this example, if Carrie chooses Strategy Y and Gary chooses Strategy C then Gary looses 2 and, therefore, Carrie gains 2.

Prisoner's Dilemma

The Prisoner's Dilemma is the most famous example of Game Theory. *The study of it is not on the AL FM course* (as it is not a zero-sum game). However, given its notoriety, we briefly mention it on this slide.

		Alice: C	Alice: D
Bob	C	$(-1, -1)$	$(-3, 0)$
Bob	D	$(0, -3)$	$(-2, -2)$

- If Alice testifies (Cooperates) against Bob, Alice goes free and Bob spends three years in jail.
- Similarly if Bob testifies against Alice, Bob goes free and Alice spends three years in jail.
- If neither of them testify against each other, then they both spend one year in jail.
- If they both testify against each other, then they both spend two years in jail.

Each prisoner must make their decision without communicating with the other. Therefore they pick the play-safe strategy that will get them the highest payoff assuming that the other player's action is fixed.

Bob reasons like this:

- If Alice cooperates, then I spend 1 year in jail if I cooperate and 0 if I don't, so I should not cooperate.
- If Alice doesn't cooperate, then I spend 3 years in jail if I cooperate and 2 if I don't, so I should not cooperate.

Alice reasons something similar. This situation is a Nash equilibrium. It's optimal to defect, as under either situation the optimal solution for "me" is to defect. However, this equilibrium doesn't capture the notion of the "best" outcome. Both players cooperating is better for both players.

The Prisoner's Dilemma was used as a premise for the TV show Goldenballs. Examples can be seen here and here.

Rewriting the pay of matrix.

If you need to rewrite the pay-off matrix to be from the perspective of the other player (i.e. with them as Player 1) then you need to reflect in the leading diagonal *and* negate all the entries as all gains are now in terms of the other player.

Problem

Write down a pay-off matrix from the perspective of Carrie.

$$\begin{array}{cc|ccc} & & & \textit{Carrie} & \\ & & X & Y & Z \\ \hline & P & 4 & 2 & 2 \\ \textit{Gary} & Q & -3 & 5 & 1 \\ & R & 2 & -1 & 3 \end{array}$$

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Write down a pay-off matrix from the perspective of Carrie.

		Carrie		
		X	Y	Z
Gary	P	4	2	2
	Q	-3	5	1
	R	2	-1	3

Solution

		Gary		
		P	Q	R
Carrie	X	-4	3	-2
	Y	-2	-5	1
	Z	-2	-1	-3

Past Paper Question

Problem

Two players, Jen and Jeff, play a zero-sum game. The pay-off matrix for the game is shown below.

		Jeff I	Jeff II	Jeff III
Jen	I	1	4	−2
Jen	II	−1	0	1
Jen	III	1	−1	0

- *State what is meant by a two-player zero-sum game.*
- *On four consecutive trials of the game, Jen plays strategy III and Jeff plays strategy II. Find the gain or loss for Jen.*

Past Paper Question

Problem

Two players, Jen and Jeff, play a zero-sum game. The pay-off matrix for the game is shown below.

		Jeff *I*	Jeff *II*	Jeff *III*
Jen	*I*	1	4	-2
Jen	*II*	-1	0	1
Jen	*III*	1	-1	0

- *State what is meant by a two-player zero-sum game.*
- *On four consecutive trials of the game, Jen plays strategy III and Jeff plays strategy II. Find the gain or loss for Jen.*

Solution

For each pair of strategies (1 mark)

one player's gain is equal to the other player's loss (1 mark)

Past Paper Question

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		Jeff I	Jeff II	Jeff III
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Jen	III	1	-1	0

- *State what is meant by a two-player zero-sum game.*
- *On four consecutive trials of the game, Jen plays strategy III and Jeff plays strategy II. Find the gain or loss for Jen.*

Solution

For each pair of strategies (1 mark)

one player's gain is equal to the other player's loss (1 mark)

Solution

For each trial Jen loses 1 so after four consecutive trials of that strategy she will have lost 4.

Play-Safe Strategies

Definition

A **play-safe** strategy seeks to minimise possible losses.

Method

We seek to find the maximum minimum gain for each player for each strategy.

- For Player 1 we need to find the maximum out of the minimum pay-offs of each row
- For Player 2 we need to find the minimum out of the maximum payoffs of each column

Read the second line again. It's the other way round. *This is because the numbers represent losses for Player 2.*

Problem

Find the play-safe strategy for each player.

		Carrie		
		X	Y	Z
Gary	P	4	2	2
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Read the second line again. It's the other way round. *This is because the numbers represent losses for Player 2.*

Problem

Find the play-safe strategy for each player.

		Carrie		
		X	*Y*	*Z*
Gary	*P*	4	2	2
	Q	−3	5	1
	R	2	−1	3

Solution

		Carrie			*Minimum*
		X	*Y*	*Z*	
Gary	*P*	4	2	2	2
	Q	−3	5	1	−3
	R	2	−1	3	$\boxed{-1}$
	Maximum	3	4	$\boxed{2}$	

Smallest so play-safe for Carrie is Z

Largest so Gary's play-safe is C

Unstable solution

Consider the following pay-off matrix:

		Ben			Minimum
		X	Y	Z	
Amina	P	4	5	-1	-1
	Q	2	1	3	1
	Maximum	4	5	3	

If Amina knows that Ben will use his play-safe strategy of Z, then Amina should still play her play-safe strategy of Q, since in the Z column this gives her a better result.

However, if Ben knows that Amina will use her play-safe strategy of Q, then Ben can do better by choosing strategy Y.

Definition

If knowing the other players play-safe strategy in advance means that a player can upgrade their play-safe strategy then the solution is called *unstable*.

Value of a Stable Solution

Definition

If neither player can improve their strategy if the other plays safe, then the game has a **stable solution**.

Consider this game where only one value has changed from the previous example (the 5 in the PY strategy has become a 0):

			Ben		Minimum
		X	Y	Z	
Amina	P	4	0	-1	-1
	Q	2	1	3	1
	Maximum	4	1	3	

If Amina chooses Q, Ben cannot improve on his play-safe strategy of Y. If Ben chooses Y, Amina cannot improve on her play-safe strategy of $\boxed{Q}$. We say the game has a stable solution.

Definition

The **value** of a game is the expected gain for Player 1 for each game played.

Definition

When the game is stable, the value of the game is the pay-off in the position according to the row and column of the play-safe strategies.

The value of game above is $\boxed{}$.

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Definition

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		Ben			Minimum
		X	Y	Z	
Amina	P	4	0	-1	-1
	Q	2	1	3	1
	Maximum	4	1	3	

If Amina chooses Q, Ben cannot improve on his play-safe strategy of Y. If Ben chooses Y, Amina cannot improve on her play-safe strategy of $\boxed{Q}$. We say the game has a stable solution.

Definition

The **value** of a game is the expected gain for Player 1 for each game played.

Definition

When the game is stable, the value of the game is the pay-off in the position according to the row and column of the play-safe strategies.

The value of game above is $\boxed{1}$.

Stable Solution

Consider again the pay-off matrix from the previous example:

		Ben			Minimum
		X	Y	Z	
Amina	P	4	0	-1	-1
	Q	2	$\boxed{1}$	3	1
	Maximum	4	1	3	

We have already argued that the indicated value is stable.

If Amina could have improved her gain by switching to strategy to P (while Ben still follows Y) then the value at PY must have been bigger than one and therefore the maximum outcome of Y would increase.

Similarly if Ben could have improved his gain by switching strategy to X or Z then the value at QX or QZ must have been smaller than 1 and therefore the minimum outcome of Q would decrease.

In either of these circumstances, once we make the game unstable, either the 1 on the minimum outcome for Amina or the maximum outcome for Ben changes.

Condition for a Stable Solution

		Ben			Minimum
		X	Y	Z	
Amina	P	4	0	-1	-1
	Q	2	$\boxed{1}$	3	1
	Maximum	4	1	3	

The maximum minimum of the rows equals the minimum maximum of the columns

This works both ways and therefore can be used to identify whether a two-player zero-sum game is stable or unstable:

Method

- If a stable solution exists then the maximum minimum of the rows equals the minimum maximum of the columns.
- If the maximum minimum of the rows *does not equal* the minimum maximum of the columns then a stable solution *does not* exist.

Past Paper Question

Problem

Two people, Adam and Bill, play a zero-sum game. The game is represented by the following pay-off matrix for Adam.

		Bill		
		B_1	B_2	B_3
Adam	A_1	-6	-1	-5
	A_2	5	2	-3
	A_3	-5	4	-4
	A_4	2	1	-4

- *Show that this game has a stable solution.*
- *Find the play-safe strategies for each player.*
- *State the value of the game for Bill.*

Past Paper Question

Problem

Two people, Adam and Bill, play a zero-sum game. The game is represented by the following pay-off matrix for Adam.

		Bill B_1	B_2	B_3
Adam	A_1	-6	-1	-5
	A_2	5	2	-3
	A_3	-5	4	-4
	A_4	2	1	-4

- *Show that this game has a stable solution.*
- *Find the play-safe strategies for each player.*
- *State the value of the game for Bill.*

Solution

Row min: $-6, -3, -5, -5$

Max(row min)$= -3$

Col max $5, 4, -3$

Min(col max)$= -3$

max(row min)=min(col max) $= -3$

$\Longrightarrow$ *game has a stable solution.*

Explanation is needed!

Past Paper Question

Problem

Two people, Adam and Bill, play a zero-sum game. The game is represented by the following pay-off matrix for Adam.

		Bill		
		B_1	B_2	B_3
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	A_2	5	2	-3
	A_3	-5	4	-4
	A_4	2	1	-4

- *Show that this game has a stable solution.*
- *Find the play-safe strategies for each player.*
- *State the value of the game for Bill.*

Solution

Row min: $-6, -3, -5, -5$

Max(row min)$= -3$

Col max 5,4, -3

Min(col max)$= -3$

max(row min)=min(col max) $= -3$

$\Longrightarrow$ *game has a stable solution.*

Explanation is needed!

Solution

Adam plays A_2 *and Bill plays* B_3.

Past Paper Question

Problem

Two people, Adam and Bill, play a zero-sum game. The game is represented by the following pay-off matrix for Adam.

		Bill		
		B_1	B_2	B_3
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	A_2	5	2	-3
	A_3	-5	4	-4
	A_4	2	1	-4

- *Show that this game has a stable solution.*
- *Find the play-safe strategies for each player.*
- *State the value of the game for Bill.*

Solution

Row min: $-6, -3, -5, -5$

Max(row min)$= -3$

Col max 5,4, -3

Min(col max)$= -3$

max(row min)=min(col max) $= -3$

$\implies$ *game has a stable solution.*

Explanation is needed!

Solution

Adam plays A_2 and Bill plays B_3.

Solution

Value of game for bill is $+3$.

Question says Bill!

Past Paper Question

Problem

Romeo and Juliet play a zero-sum game. The game is represented by the following pay-off matrix for Romeo:

		Juliet		
		D	*E*	*F*
Romeo	*A*	4	−4	0
	B	−2	−5	3
	C	2	1	−2

- *Find the play-safe strategy for each player.*
- *Show that there is no stable solution.*

Past Paper Question

Problem

Romeo and Juliet play a zero-sum game. The game is represented by the following pay-off matrix for Romeo:

		Juliet		
		D	*E*	*F*
Romeo	*A*	4	−4	0
	B	−2	−5	3
	C	2	1	−2

- *Find the play-safe strategy for each player.*
- *Show that there is no stable solution.*

Solution

R min: $-4, -5, -2$ *plays C*

J max: $4, 1, 3$ *plays E*

Past Paper Question

Problem

Romeo and Juliet play a zero-sum game. The game is represented by the following pay-off matrix for Romeo:

			Juliet	
		D	*E*	*F*
Romeo	*A*	4	−4	0
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	C	2	1	−2

- *Find the play-safe strategy for each player.*
- *Show that there is no stable solution.*

Solution

R min: $-4, -5, -2$ *plays C*

J max: $4, 1, 3$ *plays E*

Solution

maximin $R = -2 \neq 1 =$*minimax J*

Past Paper Question

Problem

Two players, Gary and Carrie, play a zero-sum game. The pay-off matrix for the game is shown below,

		Carrie		
		X	*Y*	*Z*
Gary	*A*	3	-3	-3
	B	-1	k	1
	C	1	-4	2

A stable solution for the game exists.

Find the allowed value(s) of k.

Past Paper Question

Problem

Two players, Gary and Carrie, play a zero-sum game. The pay-off matrix for the game is shown below,

		Carrie		
		X	*Y*	*Z*
Gary	*A*	3	−3	−3
	B	−1	*k*	1
	C	1	−4	2

A stable solution for the game exists.

Find the allowed value(s) of k.

Solution

Case 1: $k > 2$

Row min: $-3, -1, -4$

Col Max: $3, k, 2$

$2 \neq -1 \implies$ *no stable soln.*

Past Paper Question

Problem

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		Carrie		
		X	Y	Z
Gary	A	3	-3	-3
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Solution

Case 1: $k > 2$

Row min: $-3, -1, -4$

Col Max: $3, k, 2$

$2 \neq -1 \implies$ *no stable soln.*

Solution

Case 2: $-1 < k < 2$

Row min: $-3, -1, -4$

Col Max: $3, k, 2$

$k > -1 \implies$ *no stable soln.*

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Problem

Two players, Gary and Carrie, play a zero-sum game. The pay-off matrix for the game is shown below,

		Carrie		
		X	Y	Z
Gary	A	3	-3	-3
	B	-1	k	1
	C	1	-4	2

A stable solution for the game exists.

Find the allowed value(s) of k.

Solution

Case 3: $-3 < k \leq -1$

Row min: $-3, k, -4$

Col max: $3, k, 2$

$k = k \implies$ *Stable soln.*

Solution

Case 1: $k > 2$

Row min: $-3, -1, -4$

Col Max: $3, k, 2$

$2 \neq -1 \implies$ *no stable soln.*

Solution

Case 2: $-1 < k < 2$

Row min: $-3, -1, -4$

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Past Paper Question

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		Carrie X	Carrie Y	Carrie Z
Gary	A	3	-3	-3
Gary	B	-1	k	1
Gary	C	1	-4	2

A stable solution for the game exists.

Find the allowed value(s) of k.

Solution

Case 1: $k > 2$

Row min: $-3, -1, -4$

Col Max: $3, k, 2$

$2 \neq -1 \implies$ *no stable soln.*

Solution

Case 2: $-1 < k < 2$

Row min: $-3, -1, -4$

Col Max: $3, k, 2$

$k > -1 \implies$ *no stable soln.*

Solution

Case 3: $-3 < k \leq -1$

Row min: $-3, k, -4$

Col max: $3, k, 2$

$k = k \implies$ *Stable soln.*

Solution

Case 4: $k \leq -3$

Row min: $-3, k, -4$

Col Max: $3, -3, 2$

$k = -3 \implies$ *Stable soln.*

Past Paper Question

Problem

Two players, Gary and Carrie, play a zero-sum game. The pay-off matrix for the game is shown below,

		Carrie X	Carrie Y	Carrie Z
Gary	A	3	-3	-3
Gary	B	-1	k	1
Gary	C	1	-4	2

A stable solution for the game exists.

Find the allowed value(s) of k.

Solution

Case 1: $k > 2$

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Col Max: $3, k, 2$

$2 \neq -1 \implies$ *no stable soln.*

Solution

Case 2: $-1 < k < 2$

Row min: $-3, -1, -4$

Col Max: $3, k, 2$

$k > -1 \implies$ *no stable soln.*

Solution

Case 3: $-3 < k \leq -1$

Row min: $-3, k, -4$

Col max: $3, k, 2$

$k = k \implies$ *Stable soln.*

Solution

Case 4: $k \leq -3$

Row min: $-3, k, -4$

Col Max: $3, -3, 2$

$k = -3 \implies$ *Stable soln.*

Solution

So $-3 \leq k \leq -1$ *gives a stable soln.*

Dominated Strategies

Definition

A **dominated strategy** is one which it never makes sense for a player to choose.

If the outcomes for a particular row are always smaller than the corresponding outcomes for another row, then clearly it would not make sense for the first player to choose the option with the smaller outcomes, regardless of the other player's strategy.

Similarly, if the outcomes for a particular column are always larger than the corresponding outcomes for another column, then clearly it would not make sense for the second player to choose the option with the larger outcomes, regardless of the other player's strategy.

		Player 2		
		X	Y	Z
Player 1	P	5	3	2
	Q	6	2	-3
	R	4	-1	1

In this pay-off matrix the entries in Row R are always less than those in Row P so it is always better for player 2 to choose Strategy P over Strategy R. Therefore Row R can be ignored.

		Player 2		
		X	Y	Z
Player 1	P	2	1	3
	Q	-3	2	3
	R	1	-2	-1

The entries in Column Z are always more than those in Column Y so it always better for Player 2 to choose Strategy Z over Strategy Y. Therefore Column Z can be ignored.

Remember the pay-off matrix shows how much Player 1 wins.

Past Paper Questions

Problem

Two people, Roz and Colum, play a zero-sum game. The game is represented by the following pay-off matrix for Roz.

		Colum		
		C_1	C_2	C_3
Roz	R_1	-2	-6	-1
	R_2	-5	2	-6
	R_3	-3	3	-4

Show that the matrix can be reduced to a 2 by 3 matrix, giving the reason for deleting one of the rows.

Problem

Two people Rhona and Colleen, play a zero-sum game. The game is represented by the following pay-off matrix for Rhona.

		Colleen		
		C_1	C_2	C_3
Rhona	R_1	2	6	4
	R_2	3	-3	-1
	R_3	x	$x+3$	3

It is given that $x < 2$.

Explain why Rhona should never play strategy R_3.

Past Paper Questions

Problem

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		Colum		
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	R_2	-5	2	-6
	R_3	-3	3	-4

Show that the matrix can be reduced to a 2 by 3 matrix, giving the reason for deleting one of the rows.

Solution

Delete R_2.

Since R_3 dominates R_2.

Problem

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It is given that $x < 2$.

Explain why Rhona should never play strategy R_3.

Solution

$x < 2, x+3 < 6, 3 < 4$

$\Longrightarrow$ R_1 *dominates* R_3.