AL FM Discrete

Linear Programming: Graphical Solution, (TeX)

May 28, 2021

Linear Programming

Definition

Linear Programming is a mathematical technique for maximising or minimising a linear function subject to some constraints.

In this first topic our linear function will only have two variables and we will find its maximum or minimum using a graph.

There will be three steps in this process:

- **Formulating the Problem**
  We will typically be given a written description of the problem which we need to formulate mathematically.
- **Graphing the Constraints**
  Without any constraints then there is typically no problem because we could make our linear function as big, or as small, as we like. The importance of linear programming is that we are maximising or minimising subject to some constraints and so we need to identify where feasible solutions to our problem could lie before finding which of these is the best.
- **Maximising/Minimising the Objective Function**
  We identify which of our possible feasible solutions give make the function we are trying to maximise/minimise as big/small as possible.

Definition

The **objective function** is the thing we are trying to maximise. Typically this will be something like profit or income and will be an expression rather than an equation or inequality.

Formulating the Problem

Problem

Ed is buying some cows and sheep for his farm. Cows cost £120 each and sheep £200 each.

- *He wants at least 10 animals in total.*
- *He wants more sheep than cows.*
- *He has a maximum of £1800 to spend.*

Ed can sell the produce of each cow for £150 and the produce of each sheep for £450. How many of each should he sell to maximise his income?

Let c be the number of cows and s be the number of sheep.

It is really important that when you start formulating the problem you write down precisely what each variable represents. For example, "Let c be the number of cows", *not* "c =cows".

Writing inequalities for the constraints:

Objective function:

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Writing inequalities for the constraints:

$c + s \geq 10$
$s > c$
$120c + 200s \leq 1800$

⟵ Make sure that you read the question carefully to work out whether or not the inequalities are strict. For instance, here "at least" implies $\geq$ and "more than" implies $>$.

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Objective function:

$$\boxed{150c + 450s}$$

Note that this is not an equation nor an inequality. It is the thing we are trying to maximise.

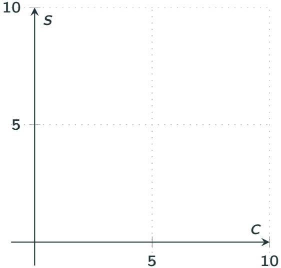
Graphing the Feasible Region

Problem

Sketch the feasible region for the inequalities on the previous slide:

$c + s \geq 10$
$s > c$
$120c + 200s \leq 1800$

Note that as c was the number of cows and s the number of sheep than the inequalities $c \geq 0$ and $s \geq 0$ were also implicit. However, here the feasible region from the other three inequalities is already entirely within the positive quadrant so these aren't needed.



If it is a strict inequality we usually use a dotted line. This means that anything on the line is not a feasible solution because it does not satisfy all the constraints of the feasible region.

Unshaded area is feasible region. Any solution in here satisfies the constraints and we want to find the one which maximises/minimises the objective function.

Definition

By convention we shade the area which is **not** included. This means that, once all the inequalities have been graphed, the unshaded area is the feasible region.

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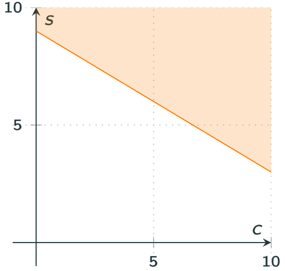

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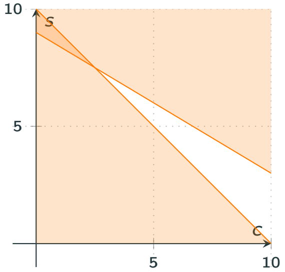

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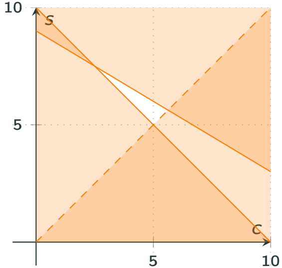

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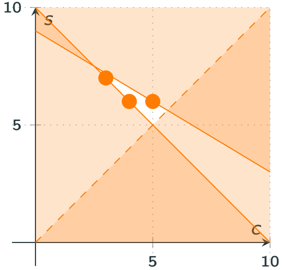
Finding the Optimal Solution

Problem

Maximise the objective function $150c + 450s$ for the feasible region below.



In this instance where the variables have to be integers, there are only three feasible solutions. We can enumerate the objective function for each individually.

c	s	$150c + 450s$
3	7	3600
4	6	3300
5	6	3450

Therefore the maximum profit is purchasing 3 cows and 7 sheep.

More Feasible Solutions

Problem

A factory produces two types of drink, an 'energy' drink and a 'refresher' drink. The day's output is to be planned. Each drink
requires syrup, vitamin supplement and concentrated flavouring, as shown in the table.

	Syrup	Vitamin Supplement	Concentrated Flavouring
1 litre of energy drink	0.25 litres	0.4 units	6 cc
1 litre of refresher drink	0.25 litres	0.2 units	4 cc
Availabilities	250 litres	300 units	4.8 litres

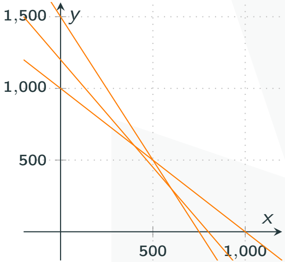
Energy drink sells at £1 per litre and refresher drink at 80p per litre. How much of each drink should be made to maximise the income?

Let x represent the number of litres of energy drink.
Let y represent the number of litres of refresher drink.

Constraints;

Objective function;

This time we clearly cannot check every solution in the feasible region as there are far too many.



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$$0.25x + 0.25y \leq 250$$
$$0.4x + 0.2y \leq 300$$
$$6x + 4y \leq 4800$$

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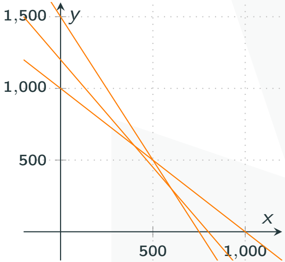

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$$\boxed{\begin{aligned} 0.25x + 0.25y &\leq 250 \\ 0.4x + 0.2y &\leq 300 \\ 6x + 4y &\leq 4800 \end{aligned}}$$

Objective function; $\boxed{\text{Maximise } x + 0.8y}$

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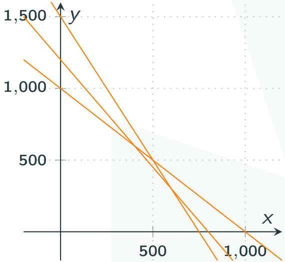
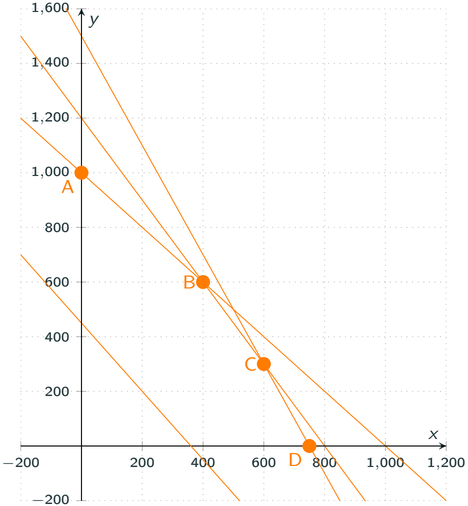


Solution using the objective Line

Definition

Draw the straight line with equation $f(x, y) = k$ where $f(x, y)$ is the objective function and move it until it reaches the edge of the feasible region.

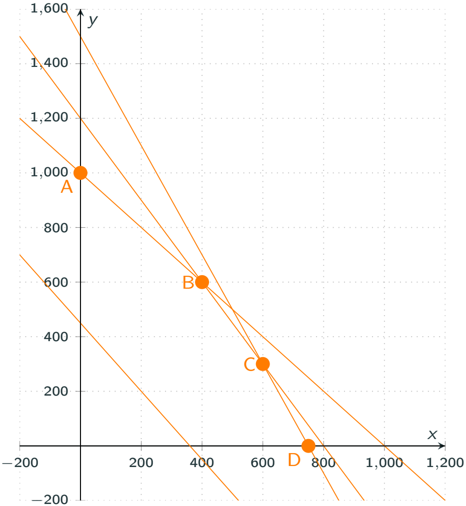


The furthest you can move the objective line is B.
Therefore, because B is at (400,600), the maximum value of the objective function is $400 + 0.8 \times 600 = 880$ which is achieved with 400 litres of energy drink and 600 litres of refresher drink.
Depending on the accuracy of the graph, you should consider using your calculator to solve the simultaneous equations to find the coordinates of the vertex.

Solution using Vertex Method

Definition

Evaluate the objective function at each of the vertices of the objective region.



Objective function is $x + 0.8y$.
At A; $0 + 0.8 \times 1000 = 800$
At B; $400 + 0.8 \times 600 = 880$
At C; $600 + 0.8 \times 300 = 840$
At D; $750 + 0 = 750$
Therefore the maximum is at vertex B where 400 litres of energy drink and 600 litres of refresher drink give an income of £880.
You *must* state the solution in the context of the problem.

Optimal Vertex does not have Integer Coordinates

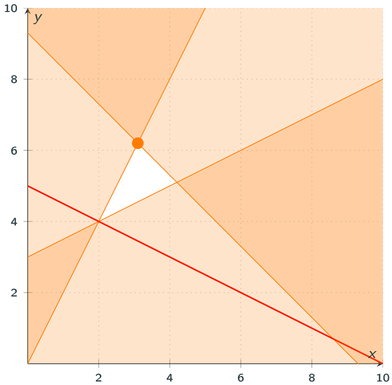
Definition

Evaluate the objective function at nearby sets of integer coordinates.

Sometimes the nearest point will be 'obvious'

Problem

Maximise x+2y.

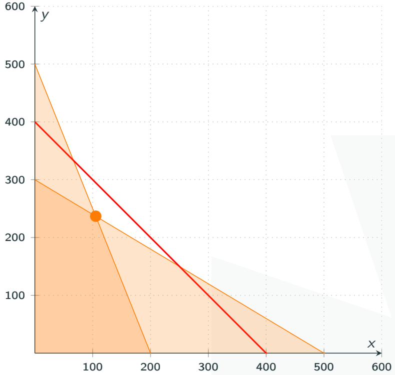


The closest point is 'obviously' (3,6).

Sometimes it will be less so...

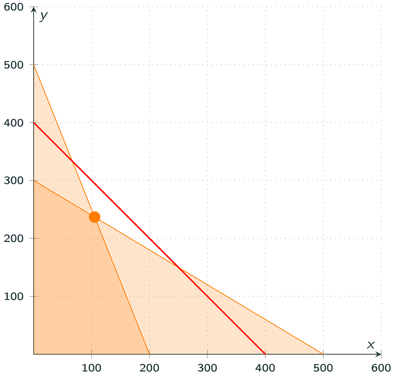
Problem

Minimise x+y.



The closest point is NOT obvious because of the scale of the graph.

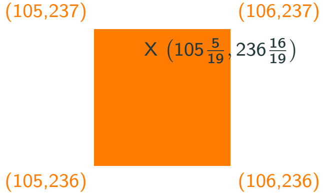
Optimal Vertex does not have Integer Coordinates



Using calculator:

Vertex is at $\left(105\frac{5}{19}, 236\frac{16}{19}\right)$

Check the nearest integer coordinates



This method is not guaranteed to give the optimum solution (particularly if the objective line is almost parallel to one of the constraints). However, if more searching is required some direction should be given in the question.

Point	$5x + 2y \geq 1000$	$3x + 5y \geq 1500$	In R?	$C = x + y$
(105,236)	$997 \ngeq 1000$			
(105,237)	$999 \ngeq 1000$			
(106,236)	$1002 \geq 1000$	$1498 \ngeq 1500$		
(106,237)	$1004 \geq 1000$	$1503 \geq 1500$	yes	343

A common mistake is to forget to check whether the nearby points are in the feasible region.

Past Paper Question

Problem

A family business makes and sells two kinds of kitchen table.

Each pine table takes 6 hours to make and the cost of materials is £30.

Each oak table takes 10 hours to make and the cost of materials is £70.

Each month, the business has 360 hours available for making the tables and £2100 available for the materials.

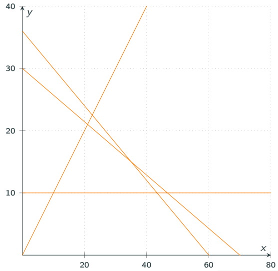
Each month, the business sells all of its tables to a wholesaler.

The wholesaler specifies that it requires at least 10 oak tables per month and at least as many pine tables as oak tables.

Each pine table sold gives the business a profit of £40 and each oak table sold gives the business a profit of £75.

Use a graphical method to find the number of each type of table the business should make each month, in order to maximise its total profit.

Show clearly how you obtain your answer.



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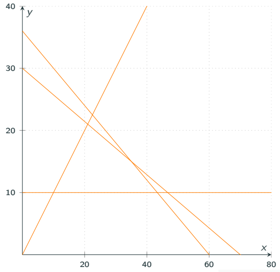

Solution

$x =$ *number of pine tables.*

$y =$ *number of oak tables.*

$30x + 70y \leq 2100$

$6x + 10y \leq 360$

$y \leq x, y \geq 10, x \geq 0$

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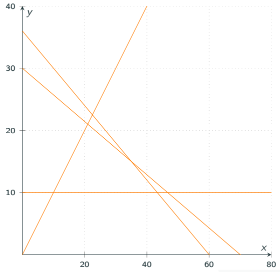

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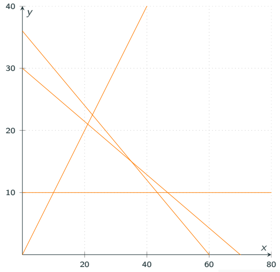

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Solution

The objective line has a gradient of $-\frac{8}{15}$.

35 Pine tables and 15 Oak Tables (Profit=£2525)

Make sure you get the marks for defining and state the variables and result in context