AL FM Discrete

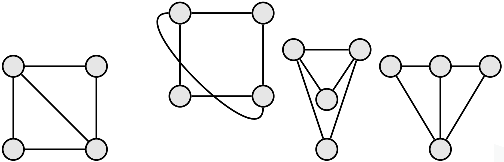
Graph Isomorphism and Planar Graphs, (TeX)

May 25, 2021

Isomorphic

Two graphs are said to be **isomorphic** if they have the same number of vertices connected in the same way.

The following graphs would all be isomorphic.



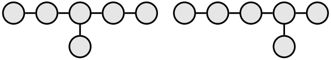
Deciding whether Graphs are Isomorphic

Deciding whether two large graphs are isomorphic is difficult and is an area of active research in Computer Science called The Graph Isomorphism Problem. For smaller graphs it can be done by inspection but there are some tricks which help:

Determining isomorphisms

- Two isomorphic graphs must have the same set of degrees
- There must be the same number of loops and multiple edges

Neither point works in reverse!



These graphs have the set of degrees but the are not isomorphic.

Problem

Which of the following is isomorphic to this graph?  *Degree sequence 1,1,1,3*

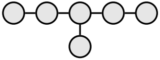
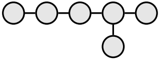
Deciding whether Graphs are Isomorphic

Deciding whether two large graphs are isomorphic is difficult and is an area of active research in Computer Science called The Graph Isomorphism Problem. For smaller graphs it can be done by inspection but there are some tricks which help:

Determining isomorphisms

- Two isomorphic graphs must have the same set of degrees
- There must be the same number of loops and multiple edges

Neither point works in reverse!

These graphs have the set of degrees but the are not isomorphic.

Problem

Which of the following is isomorphic to this graph?

Degree sequence 1,1,1,3

Different number of vertices

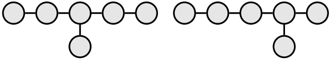
Deciding whether Graphs are Isomorphic

Deciding whether two large graphs are isomorphic is difficult and is an area of active research in Computer Science called The Graph Isomorphism Problem. For smaller graphs it can be done by inspection but there are some tricks which help:

Determining isomorphisms

- Two isomorphic graphs must have the same set of degrees
- There must be the same number of loops and multiple edges

Neither point works in reverse!



These graphs have the set of degrees but the are not isomorphic.

Problem

Which of the following is isomorphic to this graph?  *Degree sequence 1,1,1,3*

 Different number of vertices

 Degree sequence 1,1,2,4

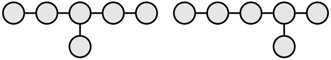
Deciding whether Graphs are Isomorphic

Deciding whether two large graphs are isomorphic is difficult and is an area of active research in Computer Science called The Graph Isomorphism Problem. For smaller graphs it can be done by inspection but there are some tricks which help:

Determining isomorphisms

- Two isomorphic graphs must have the same set of degrees
- There must be the same number of loops and multiple edges

Neither point works in reverse!



These graphs have the set of degrees but the are not isomorphic.

Problem

Which of the following is isomorphic to this graph?

Different number of vertices

Degree sequence 1,1,2,4

Degree sequence 3,3,3,3

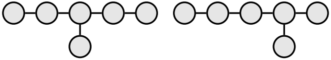
Deciding whether Graphs are Isomorphic

Deciding whether two large graphs are isomorphic is difficult and is an area of active research in Computer Science called The Graph Isomorphism Problem. For smaller graphs it can be done by inspection but there are some tricks which help:

Determining isomorphisms

- Two isomorphic graphs must have the same set of degrees
- There must be the same number of loops and multiple edges

Neither point works in reverse!



These graphs have the set of degrees but the are not isomorphic.

Problem

Which of the following is isomorphic to this graph?

Degree sequence 1,1,1,3

Different number of vertices

Degree sequence 1,1,2,4

Yes they are

Degree sequence 3,3,3,3

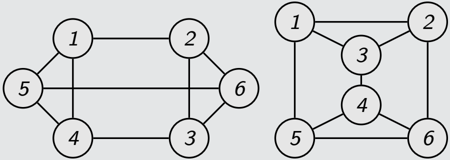
Giving the Isomorphism

Vertex mapping

The isomorphism can be given by explicitly describing which **vertex maps** to which.

Problem

Are the following graphs isomorphic? Justify your answer.



The degree sequence of both graphs is the same so they have a chance of being isomorphic.

They are isomorphic and the following would be an isomorphism:

$$1 \to A$$

$$2 \to E$$

$$3 \to F$$

$$4 \to B$$

$$5 \to C$$

$$6 \to D$$

Problem

Suppose there are three houses on a plane and each need to be connected to the gas, water and electricity companies. Without using a third dimension or sending any of the connections through another company or cottage, it there any way of making all nine connections without any of the lines crossing each other?

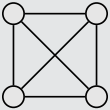
The answer is 'No'. Because no matter how you draw the graph above, it will always have some edges that cross. We can say that this is a **non-planar** graph.

Planarity

A graph is **planar** if there is no way to draw it on a 2D plane without the edges crossing.

Problem

Show that this is a planar graph.



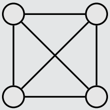
Planar Graphs

Planarity

A graph is **planar** if there is no way to draw it on a 2D plane without the edges crossing.

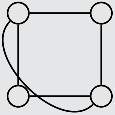
Problem

Show that this is a planar graph.



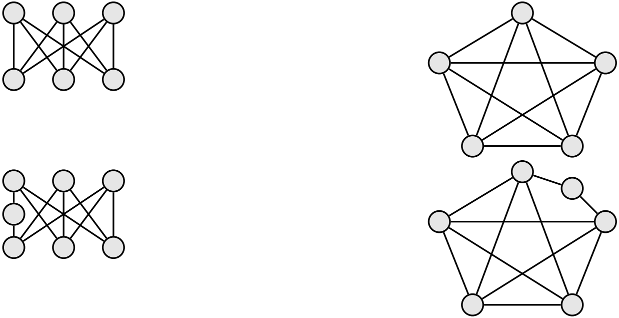
Solution

We can redraw this graph in such a way that there are no edge crossings.



Therefore it is planar.

Kuratowski's Theorem



$K_{3,3}$ cannot be drawn as a planar graph. Clearly a subdivision of it can therefore also not be drawn as a planar graph. This is also why the utilities problem cannot be solved.

K_5 cannot be drawn as a planar graph. The same reasoning for a subdivision also holds.

Recall a subdivision of a graph is when a new vertex is added on an edge so that the edge becomes two edges.

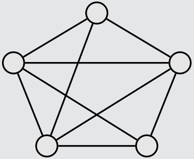
Kuratowski's Theorem

A graph is non-planar if and only if it contains a subgraph which can formed by subdividing K_5 or $K_{3,3}$.

Test Your Understanding

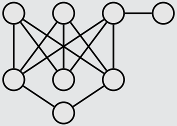
Problem

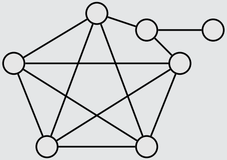
Justify whether the following graph is planar.



Problem

Determine whether or not the following graphs are planar, fully justifying your answer.

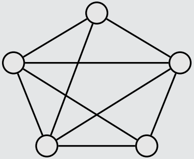




Test Your Understanding

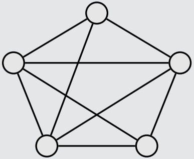
Problem

Justify whether the following graph is planar.



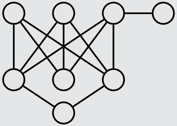
Solution

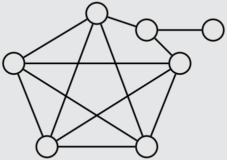
It is planar as it is K_5 with one edge removed, so it cannot have as a subgraph $K_{3,3}$ or K_5 or subdivisions thereof, thus the conclusions follows by Kuratowski's theorem. A suitable rearrangement is below.



Problem

Determine whether or not the following graphs are planar, fully justifying your answer.

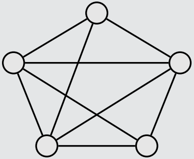




Test Your Understanding

Problem

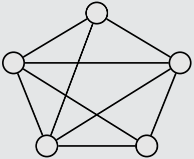
Justify whether the following graph is



planar.

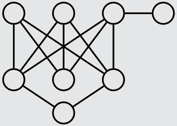
Solution

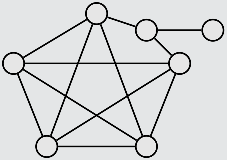
It is planar as it is K_5 *with one edge removed, so it cannot have as a subgraph* $K_{3,3}$ *or* K_5 *or subdivisions thereof, thus the conclusions follows by Kuratowski's theorem. A suitable rearrangement is below.*



Problem

Determine whether or not the following graphs are planar, fully justifying your answer.





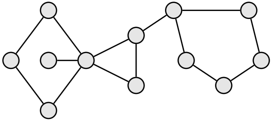
Solution

The first one contain $K_{3,3}$ *as a subgraph. Hence by Kuratowski's it is non-planar. The second contains a subdivision of* K_5 *as a subgraph hence again by Kuratowski's it is non-planar.*

Faces

Faces

- A **face** of a planar graph is an area enclosed by edges
- We include the 'outside area' as a face



A graph with four faces.

Problem

Explain why a tree only has one face.

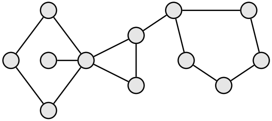
Problem

Bob claims that two graphs with the same number of vertices, edges and faces must be isomorphic. Find a counter example to Bob's statement.

Faces

Faces

- A **face** of a planar graph is an area enclosed by edges
- We include the 'outside area' as a face



A graph with four faces.

Problem

Explain why a tree only has one face.

Solution

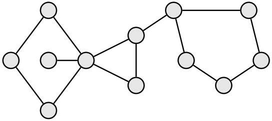
A tree has no cycles. The edges surrounding a face make a cycle. Therefore the only face of a tree is the area around the outside of the graph.

Problem

Bob claims that two graphs with the same number of vertices, edges and faces must be isomorphic. Find a counter example to Bob's statement.

Faces

- A **face** of a planar graph is an area enclosed by edges
- We include the 'outside area' as a face



A graph with four faces.

Problem

Explain why a tree only has one face.

Solution

A tree has no cycles. The edges surrounding a face make a cycle. Therefore the only face of a tree is the area around the outside of the graph.

Problem

Bob claims that two graphs with the same number of vertices, edges and faces must be isomorphic. Find a counter example to Bob's statement.

Solution

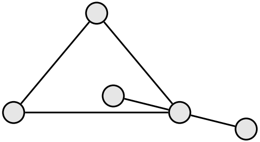
These graphs both have 4 vertices, 3 edges and 1 face but are not isomorphic.

Euler's Formula

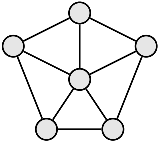
Euler's Formula

For a *connected* planar graph, $v - e + f = 2$.

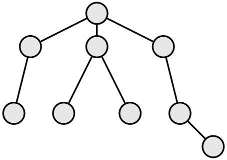
- v is the number of vertices
- e is the number of edges
- f is the number of faces



$v = 5, e = 5, f = 2$



$v = 6, e = 10, f = 6$



$v = 9, e = 8, f = 1$

Proof of Euler's Formula

We do induction on the number of edges.

If $e = 0$ then because the graph is connected, $v = 1$ and $f = 1$ so

$$v - e + f = 1 - 0 + 1 = 2.$$

Assume true for $e = k$

If we have a graph with $e = k + 1$ edges, v vertices and f faces then remove an edge which keeps the graph connected.

If that edge was part of a cycle, the new graph has v vertices, $e - 1 = k$ edges and $f - 1$ faces.

By the induction hypothesis we have that $v - (e - 1) + (f - 1) = 2$ since $v - e + f = 2$.

If the edge is 'on an end', in order to keep the graph connected we also have to remove the vertex on the end.

Hence the new graph has $v - 1$ vertices, $e - 1 = k$ edges and still f faces.

By the induction hypothesis we have that $(v - 1) + (e - 1) + f = 2$ since $v - e + f = 2$.

Problem

Determine the number of edges in a connected planar graph with 42 vertices and 40 faces.

Problem

A connected planar graph has $3x + 1$ *vertices,* x^2 *edges and* $7x + 1$ *faces. What is* x*?*

Problem

How many faces are there in a connected planar graph with 24 vertices, each having degree 3.

Test Your Understanding

Problem

Determine the number of edges in a connected planar graph with 42 vertices and 40 faces.

Solution

Euler's formula gives $v - e + f = 2$. *Hence* $(42) - e + (40) = 2$ *thus* $e = 80$.

Problem

A connected planar graph has $3x + 1$ *vertices,* x^2 *edges and* $7x + 1$ *faces. What is* x*?*

Problem

How many faces are there in a connected planar graph with 24 vertices, each having degree 3.

Test Your Understanding

Problem

Determine the number of edges in a connected planar graph with 42 vertices and 40 faces.

Solution

Euler's formula gives $v - e + f = 2$. *Hence* $(42) - e + (40) = 2$ *thus* $e = 80$.

Problem

A connected planar graph has $3x + 1$ *vertices,* x^2 *edges and* $7x + 1$ *faces. What is* x*?*

Solution

Using Euler's formula gives, $(3x + 1) - (x^2) + (7x + 1) = 2$. *This rearranges to* $x(x - 10) = 0$. *Hence* $x = 1$ *or* 10.

Problem

How many faces are there in a connected planar graph with 24 vertices, each having degree 3.

Test Your Understanding

Problem

Determine the number of edges in a connected planar graph with 42 vertices and 40 faces.

Solution

Euler's formula gives $v - e + f = 2$. *Hence* $(42) - e + (40) = 2$ *thus* $e = 80$.

Problem

A connected planar graph has $3x + 1$ *vertices,* x^2 *edges and* $7x + 1$ *faces. What is* x*?*

Solution

Using Euler's formula gives, $(3x + 1) - (x^2) + (7x + 1) = 2$. *This rearranges to* $x(x - 10) = 0$. *Hence* $x = 1$ *or* 10.

Problem

How many faces are there in a connected planar graph with 24 vertices, each having degree 3.

Solution

If a vertex has degree 3 it means that 3 edges leave/enter that vertex. So since this will count each edge twice there will be $12 \times 3 = 36$ *edges. Then using Euler's Formula gives,* $(24) - (36) + f = 2$. *This gives* $f = 14$.

Polyhedra

Euler's Formula also means that $v - e + f = 2$ for convex polyhedra as they can be 'projected' into a graph.

We can use Euler's formula to prove that there are only five Platonic solids (solids with a regular polygon on each face).