AL FM Discrete

Fundamentals of Graph Theory

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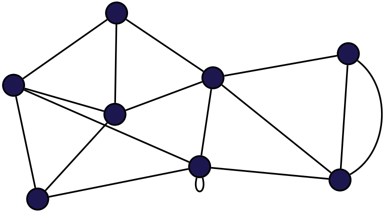
Outline

What is Graph Theory?

A graph has two meanings in Mathematics.



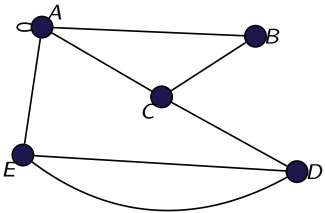
It can be a plot showing how two objects relate to each other. Or a representation of connections between objects.

Definition of a graph

A graph is made of:

- A vertex set of objects.
- An edge set where two vertices are connected if the edge between them is in the edge set.

Representing Graphs



The vertex set of this graph would be A, B, C, D, E.
And, the edge set would be
$(A, A), (A, B), (A, C), (A, E), (B, C), (C, D), (D, E), (D, E)$.
Rather than writing out the sets explicitly we usually prefer to represent the graph pictorially (as above) or use an adjacency matrix (as below).

	A	B	C	D	E
A	2	1	1	0	1
B	1	0	1	0	0
C	1	1	0	1	0
D	0	0	1	0	2
E	1	0	0	2	0

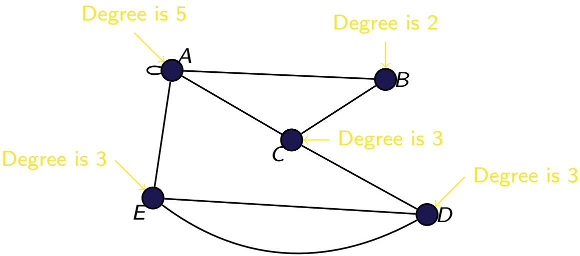
The adjacency matrix

Each entry represents the number of edges between the vertex corresponding to that row and column. The entry a_{ij}, will give the

Degree of a Vertex

Definition of Degree

The **degree** of a vertex is the number of edges that are adjacent to that vertex. Loop counts twice.



The degree can also be calculated from the adjacency matrix by adding the numbers in the corresponding row or column.

Problem

A graph has 5 vertices and 6 edges. Find the sum of the degrees of the vertices.
Answer: 12

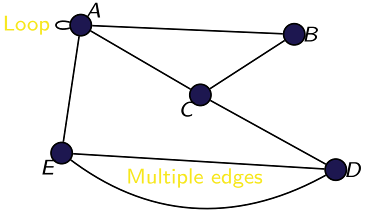
Simple Graphs

Multiple edges

Two vertices have **multiple edges** between them if there are two or more edges between them.

Loop

A **loop** is an edge which starts and ends at the same vertex.



Simple Graph

A **Simple Graph** has no loops nor multiple edges.

A simple graph will thus have only 1s in its adjacency matrix.

Connected and Complete Graphs

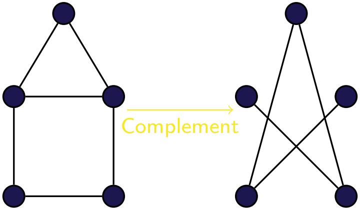
Connected A graph is **connected** if any vertex can be reached via the edges from any other vertex.

Complete A graph is **complete** if there is an edge between every pair of vertices. We denote this K_n where n is the number of vertices. The adjacency matrix will be the matrix of ones minus the identity.
It will have ${}^nC_2 = n\frac{n-1}{2}$ edges.

Graph Complements

Complement

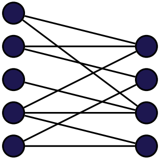
The **complement** of a graph G, denoted G', is the graph with same vertex set but where all the non-edges of G become edges of G' and all the edges of G become non-edges of G'.
Its adjacency matrix is equal to that of the correctly sized complete graph subtract the matrix of G.
Taking the union of all the edges will necessarily give you the entire complete graph.



Bipartite Graphs

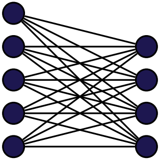
Bipartite Graph

A **bipartite graph** can have its vertices subdivided into two distinct subsets such that all the edges in the graph go from one subgroup to the other.



Complete Bipartite Graph

A **complete bipartite graph** is a bipartite graph with all the possible edges from one subset to the other. It is denoted $K_{m,n}$ where m and n are the size of the two subsets.

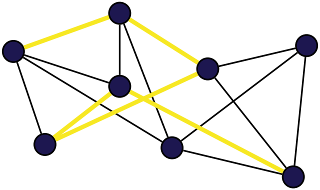


Above is $K_{5,4}$.

Trails, Circuits and Cycles

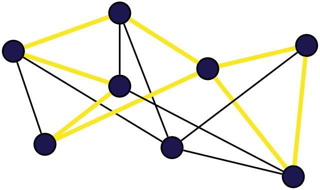
Trail

A **trail** is a sequence of edges in which the end of edge (except for the last) is the beginning of the next, and no edge is repeated. Vertex can be repeated though.



Circuit

A **circuit** is a closed trail, i.e. the end of the last edge is the start of the first.

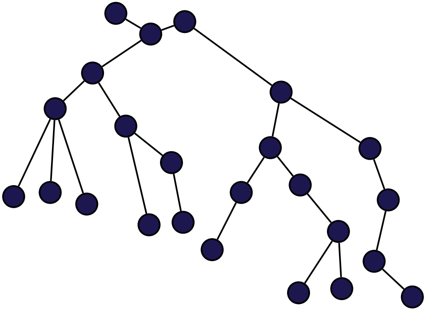


Cycle

A **cycle** is a circuit in which no vertex is repeated.

Tree

- A **tree** is a simple and connected graph with no cycles.
- A tree with n vertices will have $n - 1$ edges.
- There is only one path between any pair of vertices in a tree.

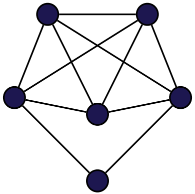


Bridges of Königsberg I

In 1736, the mathematician Leonhard Euler was asked whether it was possible to take a walk in the town of Königsberg while crossing each bridge exactly once.
In order to solve this he came up with the concept of an Eulerian graph.

Eulerian

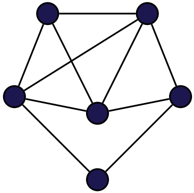
A graph is **Eulerian** if the whole graph is a circuit.
A graph is Eulerian $\equiv$ Every vertex has even degree.



Semi-Eulerian

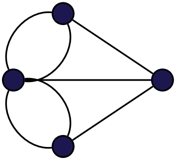
A graph is **Semi-Eulerian** if the whole graph is a trail. A graph is Semi-Eulerian $\equiv$ Exactly two vertices have odd degree.

Bridges of Königsberg II



Problem

The bridge problem The bridge problem can be reduced to the graph shown below. Since all four vertices are odd the graph is not even Semi-Eulerian, two extra bridges would have to be added for it to be Eulerian.



Hamiltonian Graphs

Hamiltonian cycle

A **Hamiltonian cycle** is a cycle which visits every vertex. Since a cycle is defined as a trail in which no vertex is repeated, a Hamiltonian cycle visits every vertex exactly once.

Hamiltonian Graph

A **Hamiltonian Graph** is one which *contains* a Hamiltonian cycle. We say contains as a Hamiltonian cycle will likely leave some edges unused despite using all the vertices.