

# ✱AL FM Pure

## Differentiating and Integrating Inverse Trigonometric Functions

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# Differentiating Inverse Functions

At AL you are also expected to know that, when  $y$  isn't the subject, it can sometimes be easier to find  $\frac{dx}{dy}$  and then 'turn it upside down'

## Definition

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Let  $y = \sin^{-1} x$  so we are trying to find  $\frac{dy}{dx}$

Then  $\sin y = x$  or reversing the equation

Differentiating both sides with respect to  $y$ :

Because there are no  $y$ s in the question, there should be no  $y$ s in our answer:

$\cos y =$   (because on line 2 we had  $x = \sin y$ )

$\therefore \frac{dy}{dx} =$

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# Test Your Understanding

## Problem

Prove  $\frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

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## Solution

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Then  $\cos y = x$  or reversing the equation  $x = \cos y$

Differentiating both sides with respect to  $y$ :  $\frac{dx}{dy} = -\sin y$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sin y}$$

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Prove  $\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$

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## Solution

Let  $y = \tan^{-1} x$  so we are trying to find  $\frac{dy}{dx}$

Then  $\tan y = x$  or reversing the equation  $x = \tan y$

Differentiating both sides with respect to  $y$ :  $\frac{dx}{dy} = \sec^2 y$

$\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y}$  We could write this as  $\cos^2 y$  but as we have a trig identity which relate  $\tan$  and  $\sec$ , it is simpler to leave it as it is.

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

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Given that  $y = \operatorname{arcsec} 2x$ , show that  $\frac{dy}{dx} = \frac{1}{x\sqrt{4x^2 - 1}}$

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## Solution

$$\sec y = 2x$$

$$x = \frac{\sec y}{2}$$

$$\frac{dx}{dy} = \frac{\sec y \tan y}{2}$$

$$\frac{dy}{dx} = \frac{2}{\sec y \tan y} = \frac{2}{\sec \sqrt{\sec^2 y - 1}} = \frac{2}{2x\sqrt{4x^2 - 1}}$$

$$= \frac{1}{x\sqrt{4x^2 - 1}}$$

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## Solution

Let  $u = \frac{1-x}{1+x}$ , then using the quotient rule,

$$\frac{du}{dx} = \frac{(1-x)(-1) - (1+x)(1)}{(1+x)^2} = -\frac{2}{(1+x)^2}$$

Therefore, by the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1 + \left(\frac{1-x}{1+x}\right)^2} \times -\frac{2}{(1+x)^2} \\ &= -\frac{2}{(1+x)^2 + (1-x)^2} \\ &= -\frac{2}{2 + 2x^2} \\ &= -\frac{1}{1 + x^2}\end{aligned}$$

# Integrating Inverse Functions

## Definition

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C, \quad |x| < a$$

The restriction  $|x| < a$  is to ensure that there is no negative square root

Think about what a sensible substitution might be so that you end up with something squared under the square root. Also looking at the answer gives a hint about the substitution.

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Think about what a sensible substitution might be so that you end up with something squared under the square root. Also looking at the answer gives a hint about the substitution.

## Solution

Let  $x = a \sin u$  then  $\frac{dx}{du} = a \cos u$  so  $dx = a \cos u \, du$ .

$$\begin{aligned} \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{1}{\sqrt{a^2 - a^2 \sin^2 u}} a \cos u \, du \\ &= \int \frac{1}{a \sqrt{1 - \sin^2 u}} a \cos u \, du \\ &= \int \frac{1}{a \cos u} a \cos u \, du \\ &= \int 1 \, du \\ &= u + C \end{aligned}$$

But rearranging  $x = a \sin u$  gives  $\frac{x}{a} = \sin u$  and  $u = \sin^{-1} \left( \frac{x}{a} \right)$

$$\text{Therefore } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C$$



# Integrating Inverse Functions

## Definition

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$$

Again look at the answer to think about what might be a good substitution.

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## Solution

Let  $x = a \tan u$  then  $\frac{dx}{du} = a \sec^2 u$  so  $dx = a \sec^2 u du$

$$\begin{aligned} \int \frac{1}{1+x^2} dx &= \int \frac{1}{a^2 + a^2 \tan^2 u} a \sec^2 u du \\ &= \int \frac{1}{a^2(1 + \tan^2 u)} a \sec^2 u du \\ &= \int \frac{1}{a^2 \sec^2 u} a \sec^2 u du \\ &= \frac{1}{a} \int 1 du \\ &= \frac{1}{a} u + C \end{aligned}$$

But rearranging  $x = a \tan u$  gives  $\frac{x}{a} = \tan u$  and  $u = \tan^{-1} \left( \frac{x}{a} \right)$ .

Therefore  $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$ .

# Test Your Understanding

## Problem

Find  $\int \frac{4}{5+x^2} dx$

## Problem

Find  $\int \frac{1}{25+9x^2} dx$

Hint: Factorise out  $\frac{1}{9}$  so that you get left with a single  $x^2$  in the integral

## Problem

Showing full working, find  $\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}} dx$

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## Problem

Find  $\int \frac{4}{5+x^2} dx$

## Solution

$$\int \frac{4}{5+x^2} dx = 4 \int \frac{1}{5+x^2} dx = \frac{4}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

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Find  $\int \frac{1}{25+9x^2} dx$

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## Solution

$$\int \frac{1}{25+9x^2} dx = \frac{1}{9} \int \frac{1}{\frac{25}{9}+x^2} dx = \frac{1}{9} \times \frac{1}{\frac{5}{3}} \arctan\left(\frac{\frac{x}{5}}{\frac{3}{5}}\right) + C = \frac{1}{15} \arctan\left(\frac{3x}{5}\right) + C$$

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## Solution

$$\int_{-\sqrt{3}/4}^{\sqrt{3}/4} \frac{1}{\sqrt{3-4x^2}} dx = \int_{-\sqrt{3}/4}^{\sqrt{3}/4} \frac{1}{\sqrt{4}\sqrt{\frac{3}{4}-x^2}} dx = \frac{1}{2} \left[ \arcsin\left(\frac{2x}{\sqrt{3}}\right) \right]_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} = \frac{\pi}{12} - \left(-\frac{\pi}{12}\right) = \frac{\pi}{6}$$

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## Solution

$$\int \frac{x+4}{\sqrt{1-4x^2}} dx = \int \frac{x}{\sqrt{1-4x^2}} dx + 4 \int \frac{1}{\sqrt{1-4x^2}} dx$$

Deal with each in turn:

$$\int \frac{x}{\sqrt{1-4x^2}} dx = \int x (1-4x^2)^{-\frac{1}{2}} dx = \frac{1}{4} \sqrt{1-4x^2} + C$$

$$4 \int \frac{1}{\sqrt{1-4x^2}} dx = 4 \int \frac{1}{\sqrt{4(\frac{1}{4}-x^2)}} dx = 2 \int \frac{1}{\sqrt{\frac{1}{4}-x^2}} dx = 2 \arcsin 2x + C$$

$$\therefore \int \frac{x+4}{\sqrt{1-4x^2}} dx = \frac{1}{4} \sqrt{1-4x^2} + 2 \arcsin 2x + C$$



# Quadratic Denominator that can't be Factorised

If you have a rational function where the denominator is a quadratic that can be factorised, you should factorise and use **partial fractions** as in A Level.

## Definition

When the denominator is a quadratic that can't be factorised, you should complete the square and use

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \text{ formula.}$$

## Problem

Find the exact value of  $\int_{-2}^{\sqrt{3}-3} \frac{1}{x^2 + 6x + 12} dx$

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$$\begin{aligned} \int_{-2}^{\sqrt{3}-3} \frac{1}{x^2 + 6x + 12} dx &= \int_{-2}^{\sqrt{3}-3} \frac{1}{(x+3)^2 + 3} dx \\ &= \int_{-2}^{\sqrt{3}-3} \frac{1}{(\sqrt{3})^2 + (x+3)^2} dx \\ &= \frac{1}{\sqrt{3}} \left[ \arctan\left(\frac{x+3}{\sqrt{3}}\right) \right]_{-2}^{\sqrt{3}-3} \\ &= \frac{1}{\sqrt{3}} \left( \arctan 1 - \arctan\left(-\frac{1}{\sqrt{3}}\right) \right) \\ &= \frac{1}{\sqrt{3}} \left( \frac{\pi}{4} - \left(-\frac{\pi}{6}\right) \right) \\ &= \frac{\pi\sqrt{3}}{36} \end{aligned}$$

If you can't see this step in one go, you could substitute  $u = x + 3$ . However when integrating, for a linear function of  $x$  you can do what you would do for  $x$  but divide by the derivative of that linear function.

Here the linear function is  $x + 3$  so the derivative is 1 so you don't need to divide.

# Partial Fractions

We can now integrate with partial fractions where one of the factors of the denominator is a quadratic that cannot be factorised.

Recall from A Level that when you write as partial fractions, you must ensure you have the most general possible non-top heavy fraction, i.e. the 'order' (i.e. maximum power) of the numerator is one less than the denominator.

i.e. You should initially write  $\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$

## Problem

Show that  $\int \frac{1+x}{x^3+9x} dx = A \ln \left( \frac{x^2}{x^2+1} \right) + B \arctan \left( \frac{x}{3} \right) + C$ , where  $A$  and  $B$  are constants.

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## Solution

$$\int \frac{1+x}{x^3+9x} dx = \int \frac{1+x}{x(x^2+9)} dx = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$1+x \equiv A(x^2+9) + x(Bx+C)$$

$$x=0 \Rightarrow A = \frac{1}{9} \text{ and } x=3 \Rightarrow 4 = 2 + 9B + 3C \text{ and } x=-3 \Rightarrow -2 = 2 + 9B - 3C$$

Solving the simultaneous equations gives  $B = -\frac{1}{9}$  and  $C = 1$

$$\begin{aligned} \int \frac{1+x}{x^3+9x} dx &= \frac{1}{9} \int \frac{1}{x} dx - \frac{1}{9} \int \frac{x}{x^2+9} dx + \int \frac{1}{x^2+9} dx \\ &= \frac{1}{9} \ln x - \frac{1}{18} \ln(x^2+9) + \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C \\ &= \frac{1}{18} \ln\left(\frac{x^2}{x^2+1}\right) + \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C \end{aligned}$$

# Partial Fractions

As in A Level, if the numerator has degree at least that of the denominator then you will have to do long division first.

## Problem

\* Express  $\frac{x^4 + x}{x^4 + 5x^2 + 6}$  as partial fractions.

\* Hence find  $\int \frac{x^4 + x}{x^4 + 5x^2 + 6} dx$ .

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## Solution

Using algebraic long division:  $x + 4 = (x^4 + 5x^2 + 6)1 - 5x^2 + x - 6$

$$\therefore \frac{x^4 + x}{x^4 + 5x^2 + 6} = 1 + \frac{-5x^2 + x - 6}{x^4 + 5x^2 + 6}$$

$$\text{Since } x^4 + 5x^2 + 6 = (x^2 + 2)(x^2 + 3): \frac{-5x^2 + x - 6}{x^4 + 5x^2 + 6} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{x^2 + 3}$$

$$\text{Then } -5x^2 + x - 6 = (Ax + B)(x^2 + 3) + (Cx + D)(x^2 + 2)$$

Use  $x = 0, 1, 2, 3$  to get four simultaneous equations which you can then solve on your calculator to get  $A = 1, B = 2, C = -1$  and  $D = -9$

$$\therefore \frac{x^4 + x}{x^4 + 5x^2 + 6} = 1 + \frac{x + 4}{x^2 + 2} - \frac{x + 9}{x^2 + 3}$$

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$$\therefore \frac{x^4 + x}{x^4 + 5x^2 + 6} = 1 + \frac{x + 4}{x^2 + 2} - \frac{x + 9}{x^2 + 3}$$

## Solution

$$\begin{aligned} \int 1 + \frac{x + 4}{x^2 + 2} - \frac{x + 9}{x^2 + 3} dx &= \int 1 + \frac{x}{x^2 + 2} + \frac{4}{x^2 + 2} - \frac{x}{x^2 + 3} - \frac{9}{x^2 + 3} dx \\ &= \dots = x + \frac{1}{2} \ln \left| \frac{x^2 + 2}{x^2 + 3} \right| + 2\sqrt{2} \arctan \left( \frac{x}{\sqrt{2}} \right) - 3\sqrt{3} \arctan \left( \frac{x}{\sqrt{3}} \right) + C \end{aligned}$$