AL FM Discrete

LP in Game Theory, (TeX)

May 25, 2021

Motivation

We have so far come across the following methods for finding the optimal mixed strategy from a pay-off matrix:

- Solving simultaneous equations (2×2)
- Drawing the graphs and find the maxmin point (2×2)
- By rewriting a $n \times 2$ pay-off in terms of the other person to reducing the problem.

Large pay-off matrices can be solved if there are dominated strategies allowing reduction to one of the smaller sizes above.

We want a method that can be extended to larger matrices and gives a programmatic way of solving the problems. We can formulate the problem of finding the largest minimum pay-off of the expected pay-off equations as a linear programming problem.

Formulation of the LP

We start with showing the method for a 2×2.

Problem

- *Formulate as an LP*
- *Use the simplex method to solve*

		Player 2	
		X	Y
Player 1	P	3	-2
	Q	1	4

Step 1

All the entries have to be positive. So add the smallest possible value to every entry so that they all become positive.

		Player 2	
		X	Y
Player 1	P	6	1
	Q	4	7

We have added 3 to each value. We do this so that we can write an inequality with probabilities summing to 1.

Formulating the LP

Step 2

We write down the objective function. This is what we want to maximise. Let V be the value of the new matrix. This implies that $V - 3$ is the value of the original game, which is what we want to maximise. Hence this is also the objective function. $P = V - 3$.

Step 3

Constraints come from the expected pay-offs being larger *or equal* to the value and the probabilities summing to 1. In the sense of an inequality being less than or equal to 1. However since all entries in the matrix are positive we know that increasing the probabilities will always increase the value, hence this the inequality suffices.

Expected pay-off if Player 2 plays strategy X is $6p + 4q$. Thus $V \leq 6p + 4q$. Hence $V - 6p - 4q \leq 0$.

And, expected pay-off if Player 2 plays strategy Y is $1p + 7q$. Hence $V - p - 7q \leq 0$.

The final constraint is simply $p + q \leq 1$. Here we would expect the slack variable to always be 0 though since all entries in the matrix are positive.

Rewriting everything using slack variables leads to:

$$\begin{aligned} &\text{Maximise} && P = V - 3 \\ &\text{Subject to} && V - 6p - 4q + s_1 = 0 \\ &&& V - p - 7q + s_2 = 0 \\ &&& p + q + s_3 = 1 \end{aligned}.$$

This leads to the following initial simplex tableau:

P	v	p	q	s_1	s_2	s_3	RHS
1	-1	0	0	0	0	0	-3
0	1	-6	-4	1	0	0	0
0	1	-1	-7	0	1	0	1
0	0	1	1	0	0	1	1

P	v	p	q	s_1	s_2	s_3	RHS
1	$\boxed{-1}$	0	0	0	0	0	-3
0	1	-6	-4	1	0	0	0
0	1	-1	-7	0	1	0	0
0	0	1	1	0	0	1	1

Rewriting for the Simplex algorithm ii

Using the indicated pivot element our second iteration is:

P	v	p	q	s_1	s_2	s_3	RHS
1	0	-6	-4	1	0	0	-3
0	1	-6	-4	1	0	0	0
0	0	$\boxed{5}$	-3	-1	1	0	0
0	0	1	1	0	0	1	1

Using the newly indicated pivot we get for the third iteration (remember the same row can't be used again.):

P	v	p	q	s_1	s_2	s_3	RHS
1	0	0	-7.6	-0.2	1.2	0	-3
0	1	0	-7.6	-0.2	1.2	0	0
0	0	1	-0.6	-0.2	0.2	0	0
0	0	0	$\boxed{1.6}$	0.2	-0.2	1	1

Finally for the fourth and final iteration we get:

Rewriting for the Simplex algorithm iii

P	v	p	q	s_1	s_2	s_3	RHS
1	0	0	0	0.75	0.25	4.75	1.75
0	1	0	0	0.75	0.25	0.45	4.75
0	0	1	0	−0.125	0.125	0.375	0.375
0	0	0	1	0.125	−0.125	0.625	0.625

The table is now optimal, giving $P = 1.75$.

Problem

For the game shown, formulate the game as a linear programming problem to find the optimal mixed strategy for Player 1.

		Player 2		
		X	Y	Z
Player 1	P	4	2	2
	Q	-3	5	1
	R	2	-1	3

First make all values positive by adding 4.

		Player 2		
1c		X	Y	Z
Player 1	P	8	6	6
	Q	1	9	5
	R	6	3	7

Formulation with a bigger matrix ii

Let each Player 1 pick their strategies with the lower-case letter probabilities.

So for player 2 choosing:

$$
\begin{aligned}
X; &\qquad \text{pay-off} = 8p + 1q + 6r \\
Y; &\qquad = 6p + 9q + 3r \\
Z; &\qquad = 6p + 5q + 7r.
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximise} &\qquad P = V - 4 \\
\text{Subject to} &\qquad V - 8p - q - 6r + s_1 = 0 \\
&\qquad V - 6p - 9q - 3r + s_2 = 0 \\
&\qquad V - 6p - 5q - 7r + s_3 = 0 \\
&\qquad p + q + r + s_4 = 1
\end{aligned}
$$

Past Paper Question

Problem

Harry and Tom play a zero-sum game. The game is represented by the following pay-off matrix for Harry.

		Tom		
		X	Y	Z
Harry	A	-3	2	3
	B	-2	3	2
	C	2	-1	-1

The game does not have a stable solution. Harry wants to know the optimal mixed strategy he should play in order to maximise his winnings from the game. He begins be defining the following variables:

$p_1 =$ *the probability of Harry playing strategy* A

$p_2 =$ *the probability of Harry playing strategy* B

$p_3 =$ *the probability of Harry playing strategy* C.

Problem

- *Formulate Harry's situation as a linear programming problem.*
- *Harry solves the linear programming problem and finds:*

$$p_1 = 0.1875$$
$$p_2 = 0.1875$$
$$p_3 = 0.625$$
$$.$$

  Find the value of the game for Tom. Fully justify your answer.
- *Tom tells Harry that he is only going to play strategy* X. *Explain which strategy or strategies Harry should play.*

Past Paper Solution

Past Paper Solution

Solution

There are no dominated strategies.

$$\begin{pmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 6 & 3 & 3 \end{pmatrix}.$$

$$\begin{aligned} p_1 + 2p_2 + 6p_3 &\geq v \\ 6p_1 + 7p_2 + 3p_3 &\geq v \\ 7p_1 + 6p_2 + 3p_3 &\geq v \\ p_1 + p_2 + p_3 &\leq 1 \\ p_1, p_2, p_3 &\geq v. \end{aligned}$$

Maximise $\quad v - 4$

Subject to

$$\begin{aligned} p_1 + 2p_2 + 6p_3 &\geq v \\ 6p_1 + 7p_2 + 3p_3 &\geq v \\ 7p_1 + 6p_2 + 3p_3 &\geq v \\ p_1 + p_2 + p_3 &\leq 1 \\ p_1, p_2, p_3 &\geq v. \end{aligned}$$

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Solution

$p_1 + 2p_2 + 6p_3$
$= 0.1875 + 2\times 0.1875 + 6 \times 0.625$

$= 4.3125 - 4$

$= 0.3125$

The value of the game for Tom is -0.3125 *because Harry's winnings* $+$ *Tom's winnings = 0 since they are playing a zero-sum game.*

Past Paper Solution

Solution

There are no dominated strategies.

$$\begin{pmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 6 & 3 & 3 \end{pmatrix}.$$

$$\begin{aligned} p_1 + 2p_2 + 6p_3 &\geq v \\ 6p_1 + 7p_2 + 3p_3 &\geq v \\ 7p_1 + 6p_2 + 3p_3 &\geq v \\ p_1 + p_2 + p_3 &\leq 1 \\ p_1, p_2, p_3 &\geq v. \end{aligned}$$

$$\begin{array}{lr} \textit{Maximise} & v - 4 \\ \textit{Subject to} & p_1 + 2p_2 + 6p_3 \geq v \\ & 6p_1 + 7p_2 + 3p_3 \geq v \\ & 7p_1 + 6p_2 + 3p_3 \geq v \\ & p_1 + p_2 + p_3 \leq 1 \\ & p_1, p_2, p_3 \geq v. \end{array}$$

Solution

$p_1 + 2p_2 + 6p_3$
$= \mathit{0.1875} + \mathit{2}\times 0.1875 + 6 \times 0.625$

$= 4.3125 - 4$

$= 0.3125$

The value of the game for Tom is -0.3125 *because Harry's winnings* $+$ *Tom's winnings* $=$ *0 since they are playing a zero-sum game.*

Solution

Harry should only play strategy C each time, because this will result in Harry winning 2 each game.

Playing strategy A or B will result in Harry losing 3 or 2 each game.