AL FM Discrete
Binary Operations

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When? April 17, 2021

Outline

What is a Binary Operation?

Definition

Given a non-empty set S, a **binary operation** on S is a rule for combining any two elements $a, b \in S$ to give a unique result c where c is not necessarily an element of S.

Addition, subtraction and multiplication are all binary operations on $\mathbb{R}$ and division is a binary operation on $\mathbb{R}\backslash\{0\}$.
Square root and factorial are not binary operations (in fact both of these are unary operations as they only take one input).
Less familiar binary operations are often defined using a symbol such as $*, \Delta$ or $\odot$.

Problem

Let a binary operation on $\mathbb{Z}$ be defined by $a * b = a + 2b - 3$.
Find

- $3 * 5 = \boxed{10}$
- $3 * 0 = \boxed{}$
- $0 * 3 = \boxed{}$
- $-5 * 0 = \boxed{}$

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Test Your Understanding

Problem *The Binary operation $a\Delta b = a^2 + b^2 - 2ab$ where $a, b \in \mathbb{R}$.*

- *Find the value of $3\Delta 4$.*
- *Find the relationship between a and b such that $a\Delta b = 0$.*

Problem *The binary operation $*$ is given by $M * N = MN + M - N$ where M and N are 2×2 matrices. Show that, for any M, $(M * I) * I = aM + bI$ where a and b are integers to be determined.*

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Solution $3\Delta 4 = 3^2 + 4^2 - 2 * 3 * 4 = 9 + 16 - 24 = 1$

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Solution

$$\begin{aligned}
(M * I) * I &= (MI + M - I) * I \\
&= (M + M - I) * I \\
&= (2M - I) * I \\
&= (2M - I)(I) + (2M - I) - I \\
&= 2M - I + 2M - I - I \\
&= 4M - 3I \text{ so } a = 4 \text{ and } b = -3.
\end{aligned}$$

Closure

Definition A binary operation $*$ is said to be closed on a set S if $a * b \in S \forall a, b \in S$

Note that closure refers to a set *and* a binary operation

The binary operation of addition is closed on $\mathbb{Z}$ as the sum of any two integers is also an integer.

The binary operation of subtraction is not closed on $\mathbb{N}$ as $5 - 7 = -2 \notin \mathbb{N}$.

Problem *Determine whether the following binary operations are closed on* $\mathbb{Z}$.

- $a * b = \frac{a+b}{a^2}$
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Solution *Since a and b are in $\mathbb{Z}$ then their sum $a + b$ and product ab are also in $\mathbb{Z}$. Therefore $a \circ b$ is closed.*

Test Your Understanding

Problem

Which of the following sets are closed under multiplication?

- $\{a + bi | a, b \in \mathbb{Q}, b \neq 0\}$
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Solution *The first is not closed because* $1 + i$ *and* $1 - i$ *are both in the set but* $(1 + i)(1 - i) = 2 + 0i$ *is not.*

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Solution *If we have two non-zero complex numbers that multiply to give zero:*
$zw = 0$
Then since $|zw| = |z||w|$, either $|z|$ or $|w| = 0$.
Therefore it is impossible to have two numbers in the set multiply to give the only number that isn't.
So the set is closed under multiplication.

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Problem

Determine whether the following function are closed on first $\mathbb{Z}$ *and the on* $\mathbb{Q}$.

- $a * b = a^2 - b$
- $a \circ b = \frac{a+b}{a}$
- $a \Delta b = \sqrt{a^2 b^2}$
- $a \Omega b = \frac{a+b}{a^2+1}$

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Solution $3 \circ 2 = \frac{5}{2} \notin \mathbb{Z}$ *so not closed on* $\mathbb{Z}$.
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Solution $\sqrt{a^2 b^2} = |ab|$. *If* $a, b \in \mathbb{Z}$ *then* $|ab| \in \mathbb{Z}$ *and if* $a, b \in \mathbb{Q}$ *then* $|ab| \in \mathbb{Q}$ *so closed on* $\mathbb{Z}$ *and* $\mathbb{Q}$.

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Solution $1\Omega 2 = \frac{3}{2} \notin \mathbb{Z}$ *so not closed on* $\mathbb{Z}$.
Since $a^2 + 1 > 0$ *then* $\frac{a+b}{a^2+1} \in \mathbb{Q}$ *so closed on* $\mathbb{Q}$.

Associativity

Definition A binary operation $*$ is **associative** on a set S if $a * (b * c) = (a * b) * c \forall a, b, c \in S$.

The for all in the definition makes it easier to show that a binary operation is not associative as you only have to provide one counter example.

Problem *Determine whether the following binary operations on $\mathbb{R}$ are associative:*

- $a * b = 2a + 3b$
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$(1 * 0) * 2 = 2 * 2 = 10$ *whereas* $1 * (0 * 2) = 1 * 6 = 20$. *Therefore* $*$ *in not associative on* $\mathbb{R}$.

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Solution

$$\begin{aligned}(a\circ b)\circ c &= (a+b+ab)\circ c\\ &= a+b+ab+c+(a+b+ab)(c)\\ &= a+b+ab+c+ac+bc+abc.\end{aligned}$$

$$\begin{aligned}a\circ(b\circ c) &= a\circ(b+c+bc)\\ &= a+b+c+bc+a(b+c+bc)\\ &= a+b+c+bc+ab+ac+abc.\end{aligned}$$

$\implies \circ$ *is associative.*

Past Paper Question

Problem

The function $\min(a, b)$ *is defined by:*

$$\min(a, b) = \begin{cases} a, & a < b \\ b, & \textit{otherwise.} \end{cases}$$

Gary claims that the binary operation Δ, *which is defined as*

$$x \Delta y = \min(x, y - 3)$$

where x *and* y *are real numbers, is associative as finding the smallest number is not affect by the order of the operation.*
Disprove Gary's claim.

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Disprove Gary's claim.

Solution $2\Delta(1\Delta 3) = 2\Delta(0) = -3 \neq (-2)\Delta 3 = (2\Delta 1)\Delta 3.\square$

Commutativity

Definition A binary operation $*$ is said to be commutative on a set S if $a * b = b * a \forall a, b \in S$.

Problem *Determine whether the following operations on $\mathbb{R}$ are commutative.*

- $a * b = 2a + b$
- $a \Delta b = 3^{a+b}$.

Problem *If $*$ is both associative and commutative on a set S, show that $(a * b)^2 = a^2 * b^2$.*

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Solution $3 * 2 = 2 \times 3 + 2 = 8$ *whereas* $2 * 3 = 2 \times 2 + 3 = 7$
So $*$ *is not commutative on* $\mathbb{R}$

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Solution

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$$\begin{aligned} (a * b)^2 &= (a * b) * (a * b) \\ &= a * (b * a) * b \text{ (because } * \text{ is associative on } S) \\ &= a * (a * b) * b \text{ (because } * \text{ is commutative on } S) \\ &= (a * a) * (b * b) \text{ (because } * \text{ is associative on } S) \\ &= a^2 * b^2. \end{aligned}$$

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Problem

The binary operation $\blacklozenge$ is defined as $a \blacklozenge b = a^b$ where a and b are non-zero real numbers.

- *Determine whether or not $\blacklozenge$ is associative.*
- *Tim claims that as $2 \blacklozenge 4 = 4 \blacklozenge 2$ then $\blacklozenge$ is commutative.*
  Assess the validity of Tim's claim.

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Solution *It is not associative.* $a \blacklozenge (b \blacklozenge c) = a^{(b^c)} \neq a^{bc} = (a^b)^c = (a \blacklozenge b) \blacklozenge c$

Solution *He has only checked it for one case but commutative must hold for all $a, b \in \mathbb{R}$. It is also not commutative since* $1 \blacklozenge 2 = 1 \neq 2 = 2 \blacklozenge 1$.

Identity

Definition For a binary operation $*$ on a set S, if there exists an element e such that $x * e = e * x = x$ for all $x \in S$ then e is called the **identity** element.

This does not mean $*$ has to be commutative on S, just that e commutes.

Problem *For the following binary operations, determine whether an identity element exists in $\mathbb{R}$:*

- $a * b = 3ab$
- $a \circ b = 3a + b$

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$$\begin{aligned} a * e = 3ae &= a \\ (3e-1)(a) &= 0 \\ e &= \frac{1}{3}. \end{aligned}$$

Easy to check that this also works for $e * a$*. So an identity exists for* $*$ *on* $\mathbb{R}$ *and is* $\frac{1}{3}$*.*

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Easy to check that this also works for $e * a$. So an identity exists for $*$ on $\mathbb{R}$ and is $\frac{1}{3}$.

Solution

$$\begin{aligned} a \circ e = 3a + e &= a \\ e &= -2a. \end{aligned}$$

Since the identity element is constant it cannot change depending on a so no identity for $\circ$ on $\mathbb{R}$ exists.

Inverse

Definition

Consider a binary operation $*$ on a set S with an identity element e. For $x \in S$, an inverse element $x^{-1} \in S$ exists if and only if $x * x^{-1} = x^{-1} * x = e$.

Note that this definition does not make sense unless there is an identity element.

For addition in $\mathbb{R}$, the inverse of the element a is $\boxed{-a}$.

For addition on the set $\{0, 1, 2, 3, \ldots\}$ there is no inverse for any element apart from 0 (even though there is an identity).

For multiplication on $\mathbb{R}\backslash\{0\}$ the inverse of an element a is $\boxed{\quad}$.

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The binary operation $\Diamond$ is given by $a \Diamond b = a + b - ab$ where $a, b\mathbb{Z}$.

- *Show that the identity element in the set $\mathbb{Z}$ with respect to the binary operation $\Diamond$ is 0.*
- *Find the inverse of the element 4 under the binary operation $\Diamond$.*

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For addition in $\mathbb{R}$, the inverse of the element a is $\boxed{-a}$.
For addition on the set $\{0, 1, 2, 3, \ldots\}$ there is no inverse for any element apart from 0 (even though there is an identity).
For multiplication on $\mathbb{R} \backslash \{0\}$ the inverse of an element a is $\boxed{\frac{1}{a}}$.

Problem

The binary operation $\lozenge$ is given by $a \lozenge b = a + b - ab$ where $a, b\mathbb{Z}$.

- *Show that the identity element in the set $\mathbb{Z}$ with respect to the binary operation $\lozenge$ is 0.*
- *Find the inverse of the element 4 under the binary operation $\lozenge$.*

Solution

$a \lozenge 0 = a + 0 - a \times 0 = a$ and $0 \lozenge a = 0 + a - 0 \times a = a$, so 0 is the identity element.

Inverse

Definition Consider a binary operation $*$ on a set S with an identity element e. For $x \in S$, an inverse element $x^{-1} \in S$ exists if and only if
$x * x^{-1} = x^{-1} * x = e$.
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Solution *If $a \lozenge 4 = 0$ then $a + 4 - 4a = 0$ and $a = \frac{4}{3}$. This $4^{-1} = \frac{4}{3}$.*

Cayley Tables

Definition A **Cayley table** is a square array where the elements of a set S are the headings of the rows and columns and the elements are the result of inputting the row header and column header into a binary operation $*$.

Problem *Let a binary operation $*$ on a set $\{0, 1, 2, 3\}$ be defined by $a * b = a^2 + ab$.*

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Solution

$*$	0	1	2	3
0	0	0	0	0
1	0	2	3	4
2	0	6	8	10
3	0	12	15	18

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Solution

The operation $*$ *is clearly not closed on* S. *There are elements in the Cayley table which are not in* S. *Take* $1 * 3 = 4 \notin S$.

Solution

The lack of symmetry around the leading diagonal indicates that the operation $*$ *is not commutative.*

Test Your Understanding

Problem *Let* $U_4 = \{1, i, -i, -1\}$ *and consider the operation of multiplication it* U_4.

- *Construct the Cayley table for the operation.*
- *Is the operation commutative or associative?*
- *State the identity if it exists.*
- *Find the inverse of each element, if possible.*

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$\times$	1	i	$-i$	-1
1	1	$-i$	i	-1
i	i	-1	1	$-i$
$-i$ $-i$	1	-1	i	
-1	-1	$-i$	i	1

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Solution *The operation is both commutative and associative as multiplication is associative and commutative on the parent group of $\mathbb{C}$.*

Solution *The identity is 1.*

Solution *1 and -1 are both self inverse. i and $-i$ are inverse pairs.*

Modular Arithmetic

Definition

Modular arithmetic is a system of arithmetic for integers where numbers 'wrap around' upon reaching a certain value called the modulus.

Definition

$+_n$ means addition modulo n.
$\times_n$ means multiplication modulo n.

You do modular arithmetic all the time when reading a clock.
10 o'clock plus three hours is 1 o'clock because we wrap around at 12.
We would write $10 + 3 \equiv 1 \mod 12$ or $10 +_{12} 3 = 1$.
$5 +_{12} 10 = \boxed{3}$, $5 \times_{12} 10 = \boxed{}$, $6 \times_9 3 = \boxed{}$, $8 +_{10} 13 = \boxed{}$, $1 +_{10} 1 = \boxed{}$.

Problem

The binary operation $\blacksquare$ *is given by* $x \blacksquare y = x^2 + y + 2 \mod 16$ *where* $x, y \in \mathbb{Z}$.

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$5 \blacksquare 3 = 5^2 + 3 + 2 \mod 16 = 30 \mod 16 = 14$.

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Solution $a^2 + b = 14, a = 3, b = 5$

Test Your Understanding

Problem *The binary operation $*$ is defined as $x * y = x + y + 1 \mod 2$ where $x, y \in \mathbb{R}$.*

- *Prove that the binary operation $*$ is associative.*
- *Find an identity element of the set $\mathbb{R}$ with respect to the binary operation $*$.*
- *Prove that the sat $\mathbb{R}$ has infinitely many identity elements with respect to $*$.*

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Solution

$$\begin{aligned}(a * b) * c &= (a + b + 1 \mod 2) * c\\ &= a + b + 1 + c + 1 \mod 2\\ &= a + b + c \mod 2.\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a * (b + c + 1 \mod 2)\\ &= a + b + c + 1 + 1 \mod 2\\ &= a + b + c \mod 2.\end{aligned}$$

$\therefore a * (b * c) = (a * b) * c$ *for all* $a, b, c \in \mathbb{R}$ *and so* $*$ *is associative.*

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Solution

For the identity element e, $e * a = a * e = a = a + e + 1 \mod 2$.
So $e + 1 = 0 \mod 2$, *and so* $e = 1$ *works.*
So 1 is an identity element.

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So $e + 1 = 0 \mod 2$, and so $e = 1$ works.
So 1 is an identity element.

Solution *Anything equal to $1 \mod 2$ is an identity element. So anything of the form $1 + 2k, \forall k \in \mathbb{Z}$, will work. This is identical to all odd numbers.*

Test Your Understanding

Problem $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. *Find the inverse of the element* 5 *under multiplication* $\mod 12$.

Problem

- *Construct a Cayley table far multiplication modulo 5 on* $\{1, 2, 3, 4\}$.
- *Use the table to solve the following:*
  - $2x = 1 \mod 5$
  - $4x + 3 = 4 \mod 5$
- *State the identity if it exists.*
- *Verify that* $3^{-1} = 2$.
- *State the inverse of each other element.*

Test Your Understanding

Problem $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. *Find the inverse of the element* 5 *under multiplication* $\mod 12$.

Solution $5 \times_{12} 5 = 1$ *so* $5^{-1} = 5$. *It is self inverse.*

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Solution

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Test Your Understanding

Problem $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. *Find the inverse of the element* 5 *under multiplication* mod 12.

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Solution $x = 3$

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4	4	3	2	1

Solution $x = 3$

Solution $4x = 1 \mod 5 \implies x = 4$

Test Your Understanding

Problem $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. *Find the inverse of the element* 5 *under multiplication* $\mod 12$.

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Solution $4x = 1 \mod 5 \implies x = 4$

Solution *The identity is 1.*

Solution $3 \times_5 2 = 2 \times_5 3 = 1$

Test Your Understanding

Problem $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. *Find the inverse of the element* 5 *under multiplication* mod 12.

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Solution $4x = 1 \mod 5 \implies x = 4$

Solution *The identity is 1.*

Solution $3 \times_5 2 = 2 \times_5 3 = 1$

Solution *1 and 4 are self inverse. 2 and 3 are an inverse pair.*