AL FM Discrete

First Order Differential Equations

April 18, 2021

Differential Equations

Definition

Differential Equations are equations which include the derivatives of the variable.

Typically we will start with an equation which involves derivatives and aim to get the equation which just relates the two variables to each other.

Example

- The rate of temperature loss is proportional to the current temperature.
- The rate of population change is proportional to $P\left(1-\frac{P}{M}\right)$ where P is the current population and M is the limiting size of the population. (the Verhulst-Pearl Model)
- Suppose x is GDP (Gross Domestic Product). Rate of change of GDP is proportional to current GDP.

- $$\frac{dT}{dt} = -kT.$$
- $$\frac{dP}{dt} = -kP\left(1-\frac{P}{M}\right).$$
- $$\frac{dx}{dt} = kx.$$

They are used a lot in physics and engineering, including modelling radioactive decay, mixing fluids, cooling materials, shock absorbance and bodies falling under gravity against resistance.

Definition

A **first order differential equation** mean the equation contain the first derivative, $\frac{dy}{dx}$, but not the second derivative nor beyond.

x and y are said to be *separable* because there is just one product of a function of x and a function of y.

$$\frac{dy}{dx} = f(x)g(y)$$
$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$
$$\frac{1}{g(y)}dy = f(x)dx$$
$$\int \frac{1}{g(y)}dy = \int f(x)dx.$$

Divide through by $g(y)$ and then multiply by dx.

Put an integral symbol on the front!

Exceptions

However, we would not be able to use this trick on the differential equation:

$$x^3\frac{dy}{dx} + 3x^2y = \sin(x).$$

General Solution

Definition

If we are just given a differential equation then there is a family of equations that could be the final answer which we call the **general solution** .

Problem

Find general solutions to $\frac{dy}{dx} = 2$.

$\boxed{y = 2x + C}$, Why is it called the general solution?

Problem

Find general solutions to $\frac{dy}{dx} = -\frac{x}{y}$.

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Solution

$$\int y dy = \int -x dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$x^2 + y^2 = 2C$$

$$\textit{If we let } r^2 = 2C$$

$$x^2 + y^2 = r^2.$$

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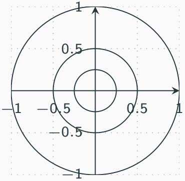
So the 'family of circles' satisfies this D.E.



Test Your Understanding

Problem

Find general solution to $\frac{dy}{dx} = xy + x$.

Problem

Find general solutions to $\frac{dy}{dx} = -\frac{y}{x}$.

Test Your Understanding

Problem

Find general solution to $\frac{dy}{dx} = xy + x$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= x(y+1) \\ \int \frac{1}{y+1} dy &= \int x dx \\ \ln(y+1) &= \frac{1}{2}x^2 + C \\ y + 1 &= e^{\frac{1}{2}x^2 + C} \\ y &= Ae^{\frac{1}{2}x^2} - 1. \end{aligned}$$

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Solution

$$\int \frac{1}{y} dy = \int -\frac{1}{x} dx$$
$$\ln|y| = -\ln x + C$$
$$\ln x + \ln|y| = C$$
$$\ln xy = C$$
$$|xy| = e^C = A$$
$$y = \pm\frac{A}{x}.$$

A family of rectangular hyperbola.

Particular Solutions

Definition

In an A-level question we usually only end up with one solution because...

They give us the details of one point (sometimes called *initial condition*) which must be satisfied by the solution. We can the substitute this into the general solution and this gives us a **particular solution** .

Problem

Find the particular solution for $\frac{dx}{dt} = \sqrt{x}$ *which satisfy the initial conditions* $x = 9$ *when* $t = 0$.

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Solution

$$\int \frac{1}{\sqrt{x}} dx = \int 1 dt$$

$$2\sqrt{x} = t + c$$

$$2(3) = 0 + c \implies c = 6$$

$$2\sqrt{x} = t + 6$$

$$x = \left(\frac{t+6}{2}\right)^2 = \frac{(t+6)^2}{4}.$$

Some people (including myself) like to sub in as soon as possible before rearranging.

Reversing the Product Rule

Problem

Find general solutions of the equation $x^3 \frac{dy}{dx} + 3x^2 y = \sin x$.

We can't $\boxed{\text{separate}}$ the variables. But do you notice anything about the LHS?

Quickfire Questions:

$\frac{d}{dx}(x^2 y) =$

$\frac{d}{dx}(y \sin(x)) =$

$x^4 \frac{dy}{dx} + 4x^3 y =$

$e^x \frac{dy}{dx} + e^x y =$

$(\ln x) \frac{dy}{dx} + \frac{y}{x} =$

So it appears whatever term ends up on front of the $\frac{dy}{dx}$ will be on front of the y in the integral.

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$\frac{d}{dx}\left(y \sin(x)\right) = \boxed{}$

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$\frac{d}{dx}\left(x^2y\right) = \boxed{2xy + x^2\frac{dy}{dx}.}$

$\frac{d}{dx}\left(y\sin(x)\right) = \boxed{}$

$x^4\frac{dy}{dx} + 4x^3y = \boxed{}$

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Differential Equations by Reversing the Product Rule

Problem

Find general solutions of the equation $x^3 \frac{dy}{dx} + 3x^2 y = \sin x$.

Problem

Find general solutions of the equation $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = e^x$.

Problem

Find general solution of the equation $4xy\frac{dy}{dx} + 2y^2 = x^2$.

Differential Equations by Reversing the Product Rule

Problem

Find general solutions of the equation $x^3 \frac{dy}{dx} + 3x^2 y = \sin x$.

Solution

$$\frac{d}{dx}(x^3 y) = \sin x$$
$$x^3 y = \int \sin x dx = -\cos x + C$$
$$y = \frac{C - \cos x}{x^3}.$$

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$$\begin{aligned} \frac{d}{dx}\left(\frac{y}{x}\right) &= e^x \\ \frac{y}{x} &= e^x + c \\ y &= x(e^x + c). \end{aligned}$$

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Solving $\frac{dy}{dx} + Py = Q$

Even if the coefficient of y isn't the derivative by the coefficient of $\frac{dy}{dx}$ there is a cunning trick which means that this becomes the case and we can reverse Th product rule.

Definition

We can multiply through by the **integrating factor**, $e^{\int Pdx}$. This then produces an equation where we can use the previous reverse product rule trick.

Problem

Find the general solution of $\frac{dy}{dx} - 4y = e^x$.

Integrating Factor $=$

Then multiplying through the original D.E. by the integrating factor:

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$\boxed{\text{Integrating Factor}} = \boxed{e^{\int -4dx} = e^{-4x}}$

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Then we can solve the usual way:

Solution

$$\begin{aligned}\frac{d}{dx}\left(ye^{-4x}\right) &= e^{-3x}\\ ye^{-4x} &= -\frac{1}{3}e^{-3x} + C\\ y &= -\frac{1}{3}e^{x} + Ce^{4x}.\end{aligned}$$

Proof that the integrating factor method works

Problem

Solve the general equation $\frac{dy}{dx} + Py = Q$ where P, Q are function of x.

Suppose $f(x)$ is the I.F. As usual we'd multiply by it:

$$\boxed{f(x)\frac{dy}{dx} + f(x)Py = f(x)Q}.$$

If we can use the reverse trick on the LHS, then it would be of the form:

$$\boxed{}.$$

Thus comparing the coefficients of the two LHSs:

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Dividing by $f(x)$ and integrating:

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Dividing by $f(x)$ and integrating:

Solution

$$\int \frac{f'(x)}{f(x)}dx = \int Pdx$$

$$\ln f(x) = \int Pdx$$

$$f(x) = e^{\int Pdx}.$$

Problem

Find the general solution of $\cos x \frac{dy}{dx} + 2y \sin x = \cos^4 x$.

What shall we do first so that we have an equation like before? Divide by $\cos x$

Step 1

Divide by anything on front of $\frac{dy}{dx}$.

Step 2

Determine the integrating factor.

Step 3

Multiply through by I.F. and use product rule backwards.

Step 4

Integrate and simplify.

When the Coefficient of $\frac{dy}{dx}$ isn't 1

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Solution

$$I.F. = e^{\int 2\tan x dx} = e^{2\ln\sec x} = \sec^2 x.$$

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Solution

$$\sec^2 x \frac{dy}{dx} + 2y \sec^2 x \tan x = \cos x.$$
$$\frac{d}{dx}(y \sec^2 x) = \cos x.$$

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$$\frac{d}{dx}(y \sec^2 x) = \cos x.$$

Solution

$$y \sec^2 x = \sin x + c.$$
$$y = \cos^2 x(\sin x + c).$$

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Divide by anything on front of $\frac{dy}{dx}$.

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Past Paper Question

Problem

Find the general solution of the differential equation

$$x\frac{dy}{dx} + 5y = \frac{\ln x}{x}, \quad x > 0.$$

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Find the general solution of the differential equation

$$x\frac{dy}{dx} + 5y = \frac{\ln x}{x}, \quad x > 0.$$

Solution

$$\begin{aligned}
\frac{dy}{dx} + \frac{5}{x}y &= \frac{\ln x}{x^2} \\
x^5\frac{dy}{dx} + 5x^4 y &= \ln(x)x^3 \\
x^5 y &= \int \ln(x)x^3 \; dx \\
x^5 y &= \frac{1}{4}x^4 \ln x - \frac{1}{16x^4} + c \\
y &= \frac{1}{4x}(\ln x - \frac{1}{4} + \frac{c}{5}).
\end{aligned}$$

Making a Substitution

Sometimes making a substitution will turn a more complex differential equation into one like we've seen before.

Problem

Use the substitution $z = \frac{y}{x}$ *to transform the differential equation* $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}, x > 0$ *into a differential equation in* z *and* x*. By first solving this new equation, find the general solution of the original equation, giving* y^2 *in terms of* x*.*

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Solution

$z = \frac{y}{x} \implies y = xz$

$\frac{dy}{dx} = x\frac{dz}{dx} + z$ (*Product rule*)

- Make y the subject so that we can find $\frac{dy}{dx}$ and sub y out.

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Solution

$z = \frac{y}{x} \implies y = xz$

$\frac{dy}{dx} = x\frac{dz}{dx} + z$ (Product rule)

Solution

$x\frac{dz}{dx} + z = \frac{x^2+3x^2z^2}{2x^2z} = \frac{1}{2z} + \frac{3}{2}z$

- Make y the subject so that we can find $\frac{dy}{dx}$ and sub y out.
- Sub in y and $\frac{dy}{dx}$.

Making a Substitution

Sometimes making a substitution will turn a more complex differential equation into one like we've seen before.

Problem

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}, x > 0$ into a differential equation in z and x. By first solving this new equation, find the general solution of the original equation, giving y^2 in terms of x.

Solution

$z = \frac{y}{x} \implies y = xz$

$\frac{dy}{dx} = x\frac{dz}{dx} + z$ *(Product rule)*

Solution

$x\frac{dz}{dx} + z = \frac{x^2+3x^2z^2}{2x^2z} = \frac{1}{2z} + \frac{3}{2}z$

Solution

$x\frac{dz}{dx} = \frac{1}{2z} + \frac{1}{2}z = \frac{1+z^2}{2z}$

$\int \frac{2z}{1+z^2}\,dz = \int \frac{1}{x}\,dx$?

$1 + z^2 = Ax$

- Make y the subject so that we can find $\frac{dy}{dx}$ and sub y out.
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- Simplify to either get a separable case or one with I.F.

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- Sub back y in at the end.

Solution

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Solution

$y^2 = x^2(Ax - 1)$

Substitution then Integrating Factor

Problem

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Solution

$$
\begin{aligned}
y = \frac{1}{z}, \frac{dy}{dx} &= -\frac{1}{z^2}\frac{dz}{dx} \\
-\frac{1}{z^2}\frac{dz}{dx} + \frac{x}{z} &= \frac{x}{z^2} \\
\frac{dz}{dx} - xz &= -x \\
I.F. = e^{\int -x \ dx} &= e^{-\frac{1}{2}x^2} \\
\frac{d}{dx}\left(ze^{-\frac{1}{2}x^2}\right) &= -xe^{-\frac{1}{2}x^2} \\
ze^{-\frac{1}{2}x^2} &= e^{-\frac{1}{2}x^2} + c \\
z &= 1 + ce^{\frac{1}{2}x^2} \\
y &= \frac{1}{1 + ce^{\frac{1}{2}x^2}}.
\end{aligned}
$$

Test Your Understanding

Problem

A particle is moving along the x-axis and its displacement, x meters, is modelled using the differential equation

$$t\frac{dx}{dt} + x = 2t^3x^2, \quad 0 < t < 1.5.$$

Where t is the time in seconds.

- *Use the substitution $u = xt$ to show that the differential equation can be expressed as $\frac{du}{dt} = 2u^2t$.*
- *Hence show that the general solution to the differential equation is $x = \frac{1}{t(A-t^2}$, where A is an arbitrary constant.*
- *Given that $x = 1$ when $t = 0.5$, find the displacement after 1.2 seconds.*

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$\frac{du}{dt} = t\frac{dx}{dt} + x$

$x = \frac{u}{t}$

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$\frac{du}{dt} = 2t^3\left(\frac{u}{t}\right)^2 = 2u^2t$

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Solution

$\int \frac{1}{u^2}du = \int 2tdt$

$-\frac{1}{u} = t^2 + c$

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$xt = \frac{1}{A-t^2}$

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Solution

$x = 1$ *when* $t = 0.5$ *:*

$1 = \frac{1}{0.5(A-0.25)} \implies A = 2.25$

$x = \frac{1}{1.2(2.25-1.2^2}$

The displacement after 1.2s is 1.03m (3s.f)

Past Paper Question

Problem

- *Show that the substitution $y = vx$ transforms the differential equation*
$$3xy^2\frac{dy}{dx} = x^3 + y^3.$$
Into the differential equation
$$3v^2x\frac{dv}{dx} = 1 - 2v^3.$$
- *By solving the second differential equation, find a general solution to the first differential equation in the form $y = f(x)$.*
- *Given that $y = 2$ at $x = 1$, find the value of $\frac{dy}{dx}$ at $x = 1$.*

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Solution

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$3x(vx)^2(v + x\frac{dv}{dx}) = x^3 + (vx)^3$

$3(v)^2(v + x\frac{dv}{dx}) = 1 + (v)^3$

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$1 - 2v^3 = \frac{A}{x^2}$

$x^3 - 2y^3 = Ax$

$y = (\frac{1}{2}(x^3 - Ax))^{\frac{1}{3}}$

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$3v^2x\frac{dv}{dx} = 1 - 2v^3$

Solution

$3(1)(2)^2\frac{dy}{dx} = (1)^3 + (2)^3$

$12\frac{dy}{dx} = 1 + 8$

Solution

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