AL FM Discrete

Linear Programming: Simplex Algorithm, (TeX)

May 29, 2021

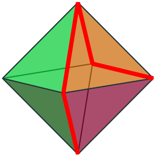
Overview

We have already seen that when we have a linear programming problem with two variables we can represent the feasible region on a 2D graph and then find the optimal solution (which will usually occur at a corner of the feasible region).

In theory, we could develop this idea with more variables but the problem is that there would need to be an axis for each variable and therefore the graph would no longer be 2D.

Definition

Instead we use an algorithmic method which involves starting at a vertex of the feasible region and then making improvements until we find an optimal solution.



Simplex Algorithm: Slack Variables

To start with we shall apply the Simplex Algorithm to a 2D problem. You could solve this by drawing a graph but you might also be asked to specifically use the Simplex Algorithm.

Problem

$$\begin{aligned} &\textit{Maximise} && I = x + 0.8y \\ &\textit{subject to} && x + y \leq 1000 \\ &&& 2x + y \leq 1500 \\ &&& 3x + 2y \leq 2400 \end{aligned}$$

Definition

Step 1;

Replace the inequalities with *slack variables* in order to replace them with equalities.

$$\begin{array}{rlllll} \text{Maximise} & I & & & & \\ \text{where} & I & -x - 0.8y & & & = 0 \\ \text{subject to} & & x + y & +s_1 & & = 1000 \\ & & 2x + y & & + s_2 & = 1500 \\ & & 3x + 2y & & +s_3 & = 2400 \end{array} .$$

The slack variables $s_1, s_2, s_3 \geq 0$ in order to ensure that the $\leq$ inequalities are satisfied.

The AQA syllabus states that you will only be assessed on problems with $\leq$ inequalities.

Simplex Algorithm: Simplex Tableau

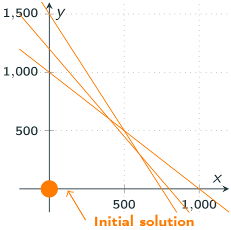
Definition

Step 2;

Find an initial solution (often when everything is zero) to create a simplex tableau.

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

This is for the initial solution; $I = 0, x = 0, y = 0, s_1 = 1000, s_2 = 1500, s_3 = 2400$.



It is not necessary to draw the graph but I include here to give a picture of what is going on.

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0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

Note we have reformulated the objective function from $I = x + 0.8y$ to $I - x - 0.8y = 0$

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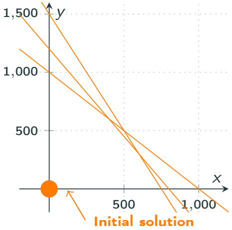

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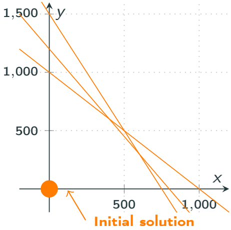


The s_1, s_2, s_3 are how much 'slack' we have in each of the constraints. We are going to use this slack to increase the value of the objective function.

It is not necessary to draw the graph but I include here to give a picture of what is going on.

Simplex Algorithm: Tableau Interpretation

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

The variables correspond to the other columns and are called non-basic or free.

The variables which make up the identity matrix in the constraint rows are called basic variables.

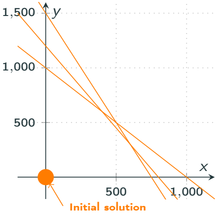
Definition

At each stage of the algorithm the current values of the basic variables is the values on the right-hand side and the free variables **have value 0**.

This is important as it allows you to read off the current solution and each point and, when you get to Step 7, the final solution.

In our initial solution s_1, s_2 and s_3 are basic which means we have used up as much of them as possible. x and y are free meaning we are not using any of them and can potentially use some more.

So we can see that the initial solution is $I = 0, x = 0, y = 0, s_1 = 1000, s_2 = 1500, s_3 = 2400$.



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Simplex Algorithm: Tableau Interpretation

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
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0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

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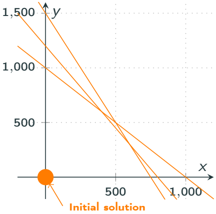

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Simplex Algorithm: Tableau Interpretation

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

The variables correspond to the other columns and are called non-basic or free.

The variables which make up the identity matrix in the constraint rows are called basic variables.

Definition

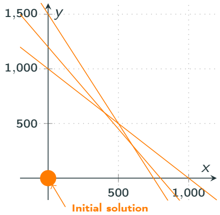
At each stage of the algorithm the current values of the basic variables is the values on the right-hand side and the free variables **have value 0**.

This is important as it allows you to read off the current solution and each point and, when you get to Step 7, the final solution.

In our initial solution s_1, s_2 and s_3 are basic which means we have used up as much of them as possible. x and y are free meaning we are not using any of them and can potentially use some more.

So we can see that the initial solution is $I = 0, x = 0, y = 0, s_1 = 1000, s_2 = 1500, s_3 = 2400$.

We are going to increase the objective function by using up as much of the x and y as possible and less of the slack variables s_1, s_2, s_3.



It is not necessary to draw the graph but I include here to give a picture of what is going on.

Simplex Algorithm: Find the Pivot Element

Definition

Step 3;

The pivot column is the one with the most negative entry in the objective row.

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

Definition

Step 4; The pivot element is the entry in the pivot column with smallest **positive** value for $\frac{\text{RHS}}{\text{variable}}$.

I	x	y	s_1	s_2	s_3	RHS	$\frac{\text{RHS}}{x}$
1	-1	-0.8	0	0	0	0	
0	1	1	1	0	0	1000	$\frac{1000}{1} = 1000$
0	2	1	0	1	0	1500	$\frac{1500}{2} = 750$
0	3	2	0	0	1	2400	$\frac{2400}{3} = 800$

750 is the smallest positive value so the pivot is the element in this row in the pivot column.

The reason we choose the minimum is that this is the limiting constraint. This will make more sense on the next slide.

Simplex Algorithm: Find the Pivot Element

Definition

Step 5; Divide everything in the pivot row by the pivot element.

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	1	0.5	0	0.5	0	750
0	3	2	0	0	1	2400

Definition

Step 6; Do row operations to make everything all the other elements in the pivot column zero.

I	x	y	s_1	s_2	s_3	RHS	
1	0	-0.3	0	0.5	0	750	Objective Row + Pivot Row in table above.
0	0	0.5	1	-0.5	0	250	First Constraint Row − Pivot Row in table above.
0	1	0.5	0	0.5	0	750	
0	0	0.5	0	-1.5	1	150	Third Constraint Row $-3\times$ Pivot Row in table above.