AL FM Discrete

Linear Programming: Simplex Algorithm, (TeX)

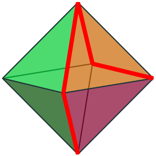
May 30, 2021

We have already seen that when we have a linear programming problem with two variables we can represent the feasible region on a 2D graph and then find the optimal solution (which will usually occur at a corner of the feasible region).

In theory, we could develop this idea with more variables but the problem is that there would need to be an axis for each variable and therefore the graph would no longer be 2D.

Definition

Instead we use an algorithmic method which involves starting at a vertex of the feasible region and then making improvements until we find an optimal solution.



Simplex Algorithm: Slack Variables

To start with we shall apply the Simplex Algorithm to a 2D problem. You could solve this by drawing a graph but you might also be asked to specifically use the Simplex Algorithm.

Problem

$$\begin{aligned} &\textit{Maximise} && I = x + 0.8y \\ &\textit{subject to} && x + y \leq 1000 \\ & && 2x + y \leq 1500 \\ & && 3x + 2y \leq 2400 . \end{aligned}$$

Definition

Step 1;

Replace the inequalities with *slack variables* in order to replace them with equalities.

$$\begin{array}{rlllll} \text{Maximise} & I & & & & \\ \text{where} & I & -x - 0.8y & & & = 0 \\ \text{subject to} & & x + y & +s_1 & & = 1000 \\ & & 2x + y & + s_2 & & = 1500 \\ & & 3x + 2y & & +s_3 & = 2400 . \end{array}$$

The slack variables $s_1, s_2, s_3 \geq 0$ in order to ensure that the $\leq$ inequalities are satisfied.

The AQA syllabus states that you will only be assessed on problems with $\leq$ inequalities.

Simplex Algorithm: Simplex Tableau

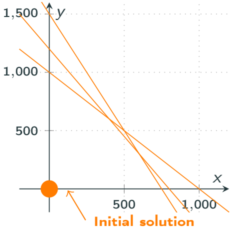
Definition

Step 2;

Find an initial solution (often when everything is zero) to create a simplex tableau.

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

This is for the initial solution; $I = 0, x = 0, y = 0, s_1 = 1000, s_2 = 1500, s_3 = 2400$.



It is not necessary to draw the graph but I include here to give a picture of what is going on.

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Note we have reformulated the objective function from $I = x + 0.8y$ to $I - x - 0.8y = 0$

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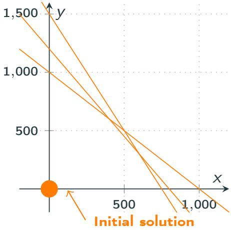

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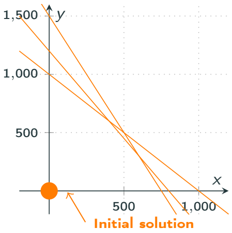


The s_1, s_2, s_3 are how much 'slack' we have in each of the constraints. We are going to use this slack to increase the value of the objective function.

It is not necessary to draw the graph but I include here to give a picture of what is going on.

Simplex Algorithm: Tableau Interpretation

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

The variables correspond to the other columns and are called non-basic or free.

The variables which make up the identity matrix in the constraint rows are called basic variables.

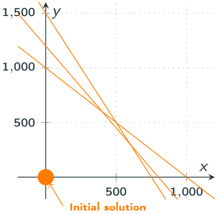
Definition

At each stage of the algorithm the current values of the basic variables is the values on the right-hand side and the free variables **have value 0**.

This is important as it allows you to read off the current solution and each point and, when you get to Step 7, the final solution.

In our initial solution s_1, s_2 and s_3 are basic which means we have used up as much of them as possible. x and y are free meaning we are not using any of them and can potentially use some more.

So we can see that the initial solution is $I = 0, x = 0, y = 0, s_1 = 1000, s_2 = 1500, s_3 = 2400$.



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0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

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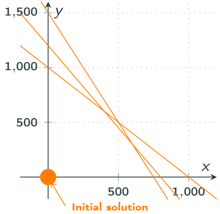

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Simplex Algorithm: Tableau Interpretation

I	x	y	s_1	s_2	s_3	RHS
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0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

The variables correspond to the other columns and are called non-basic or free.

The variables which make up the identity matrix in the constraint rows are called basic variables.

Definition

At each stage of the algorithm the current values of the basic variables is the values on the right-hand side and the free variables **have value 0**.

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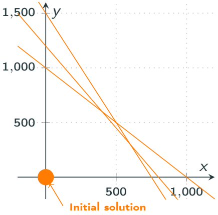
In our initial solution s_1, s_2 and s_3 are basic which means we have used up as much of them as possible. x and y are free meaning we are not using any of them and can potentially use some more.

So we can see that the initial solution is $I = 0, x = 0, y = 0, s_1 = 1000, s_2 = 1500, s_3 = 2400$.



We are going to increase the objective function by using up as much of the x and y as possible and less of the slack variables s_1, s_2, s_3.

It is not necessary to draw the graph but I include here to give a picture of what is going on.

Simplex Algorithm: Find the Pivot Element

Definition

Step 3;

The pivot column is the one with the most negative entry in the objective row.

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	2	1	0	1	0	1500
0	3	2	0	0	1	2400

Definition

Step 4; The pivot element is the entry in the pivot column with smallest **positive** value for $\frac{\text{RHS}}{\text{variable}}$.

I	x	y	s_1	s_2	s_3	RHS	$\frac{\text{RHS}}{x}$
1	-1	-0.8	0	0	0	0	
0	1	1	1	0	0	1000	$\frac{1000}{1} = 1000$
0	2	1	0	1	0	1500	$\frac{1500}{2} = 750$
0	3	2	0	0	1	2400	$\frac{2400}{3} = 800$

750 is the smallest positive value so the pivot is the element in this row in the pivot column.

The reason we choose the minimum is that this is the limiting constraint. This will make more sense on the next slide.

Simplex Algorithm: Pivoting

Definition

Step 5; Divide everything in the pivot row by the pivot element.

I	x	y	s_1	s_2	s_3	RHS
1	-1	-0.8	0	0	0	0
0	1	1	1	0	0	1000
0	1	0.5	0	0.5	0	750
0	3	2	0	0	1	2400

Definition

Step 6; Do row operations to make everything all the other elements in the pivot column zero.

I	x	y	s_1	s_2	s_3	RHS	
1	0	-0.3	0	0.5	0	750	← Objective Row + Pivot Row in table above.
0	0	0.5	1	-0.5	0	250	← First Constraint Row − Pivot Row in table above.
0	1	0.5	0	0.5	0	750	
0	0	0.5	0	-1.5	1	150	← Third Constraint Row $-3\times$ Pivot Row in table above.

Simplex Algorithm: Pivoting Interpretation

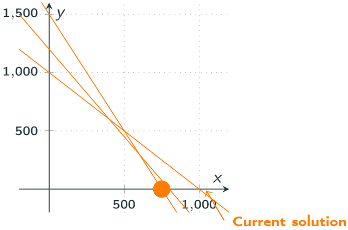
I	x	y	s_1	s_2	s_3	RHS
1	0	-0.3	0	0.5	0	750
0	0	0.5	1	-0.5	0	250
0	1	0.5	0	0.5	0	750
0	0	0.5	0	-1.5	1	150

In our latest tableau we can see that x, s_1 and s_3 are basic and y and s_2 are free. Therefore in our current situation we are using up as much of x, s_1 and s_3 as possible and none of y and s_2.

Definition

Pivoting will make one free variable basic and one basic variable free.

So our current solution is $I = 750, x = 750, y = 0, s_1 = 0, s_3 = 150$.



So what has happened on this pivot is we've used as much x as we can and we had to stop at the green line which was the second constraint (which was indicated to us by having the lowest ratio).

Simplex Algorithm: Pivoting Again

Definition

Step 7;

Repeat until everything in the objective row is non-negative.

Note it says *non-negative* which is different to positive (because 0 is non-negative).

I	x	y	s_1	s_2	s_3	RHS	$\frac{\text{RHS}}{x}$
1	0	-0.3	0	0.5	0	750	
0	0	0.5	1	-0.5	0	250	$\frac{250}{0.5} = 500$
0	1	0.5	0	0.5	0	750	$\frac{750}{0.5} = 1500$
0	0	0.5	0	-1.5	1	150	$\frac{150}{0.5} = 300$

This time y is the pivot column

Divide the pivot row by 0.5

I	x	y	s_1	s_2	s_3	RHS
1	0	-0.3	0	0.5	0	750
0	0	0.5	1	-0.5	0	250
0	1	0.5	0	0.5	0	750
0	0	1	0	-3	2	300

Row operations to make other entries in pivot column zero

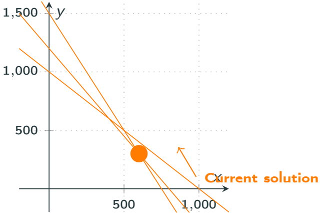
I	x	y	s_1	s_2	s_3	RHS
1	0	0	0	-0.4	0.6	840
0	0	0	1	1	-1	100
0	1	0	0	2	-1	600
0	0	1	0	-3	2	300

Simplex Algorithm: Second Pivot Interpretation

I	x	y	s_1	s_2	s_3	RHS
1	0	0	0	-0.4	0.6	840
0	0	0	1	1	-1	100
0	1	0	0	2	-1	600
0	0	1	0	-3	2	300

The basic variables are now x, y and s_1 and the free variables are s_2 and s_3.

So our current solution is $I = 840, x = 600, y = 300, s_1 = 100, s_2 = 0, s_3 = 0$.



Because there is still a negative entry in the objective row our solution is not yet optimal and we must continue...

Simplex Algorithm: Pivoting Again

I	x	y	s_1	s_2	s_3	RHS	$\frac{\text{RHS}}{s_2}$
1	0	0	0	-0.4	0.6	840	
0	0	0	1	1	-1	100	$\frac{100}{1} = 100$
0	1	0	0	2	-1	600	$\frac{600}{2} = 300$
0	0	1	0	-3	2	300	$\frac{300}{-3} = -100$

Remember we are looking for the smallest positive value.

Here the pivot element is 1 so dividing by it leaves the row unchanged.

I	x	y	s_1	s_2	s_3	RHS
1	0	0	0	-0.4	0.6	840
0	0	0	1	1	-1	100
0	1	0	0	2	-1	600
0	0	1	0	-3	2	300

Row operations to make other entries in pivot column zero.

I	x	y	s_1	s_2	s_3	RHS
1	0	0	0.4	0	0.2	880
0	0	0	1	1	-1	100
0	1	0	-2	0	1	400
0	0	1	3	0	-1	600

A company repairs and sells computer hardware, including monitors, hard drives and keyboards.

Each monitor takes 3 hours to repair and the cost of components is 40 pounds.

Each hard drive takes 2 hours to repair and the cost of components is 20 pounds.

Each keyboard takes 1 hour to repair and the cost of components is 5 pounds.

Each month, the business has 360 hours available for repairs and 2500 pounds available to buy components.

Each month, the company sells all of its repaired hardware to a local computer shop.

Each monitor, hard drive and keyboard sold gives the company a profit of 80, 35 and 15 pounds respectively.

The company repairs and sells x monitors, y hard drives and z keyboards each month.

The company wishes to maximise its total profit.

Problem

- *Find five inequalities involving x, y and z for the company's problem.*
- *Find how many of each type of computer hardware the company should repair and sell each month.*
- *Explain how you know that you had reached the optimal solution.*
- *The local computer shop complains that they are not receiving one of the types of*

Past Paper Solution

Past Paper Solution

Solution

$3x + 2y + z \leq 360$

$40x + 20y + 5z \leq 2500$

$x \geq 0, y \geq 0, z \geq 0.$

Past Paper Solution

Solution

$3x + 2y + z \leq 360$

$40x + 20y + 5z \leq 2500$

$x \geq 0, y \geq 0, z \geq 0.$

Solution

P	x	y	z	s	t	value
1	−80	−35	−15	0	0	0
0	3	2	1	1	0	360
0	40	20	5	0	1	2500
1	0	5	−5	0	2	5000
0	0	0.5	0.625	1	−0.075	172.5
0	1	0.5	0.125	0	0.025	62.5
1	0	9	0	8	1.4	6380
0	0	0.8	1	1.6	−0.12	276
0	1	0.4	0	−0.2	0.04	28

To maximise profit the company should repair and sell 28 monitors, 0 hard drives and 276 keyboards each month.

Past Paper Solution

Solution

$3x + 2y + z \leq 360$

$40x + 20y + 5z \leq 2500$

$x \geq 0, y \geq 0, z \geq 0.$

Solution

The objective row of the final tableau being non-negative numbers.

Solution

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1	−80	−35	−15	0	0	0
0	3	2	1	1	0	360
0	40	20	5	0	1	2500
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Past Paper Solution

Solution

$3x + 2y + z \leq 360$

$40x + 20y + 5z \leq 2500$

$x \geq 0, y \geq 0, z \geq 0.$

Solution

The objective row of the final tableau being non-negative numbers.

Solution

As $y = 0$, enforce some hard drives to be repaired by requiring that, for instance, $y \geq 10$.

Solution

P	x	y	z	s	t	value
1	-80	-35	-15	0	0	0
0	3	2	1	1	0	360
0	40	20	5	0	1	2500
1	0	5	-5	0	2	5000
0	0	0.5	0.625	1	-0.075	172.5
0	1	0.5	0.125	0	0.025	62.5
1	0	9	0	8	1.4	6380
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0	1	0.4	0	-0.2	0.04	28

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