Nonparametric Bayesian Methods: Models, Algorithms, and Applications

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Associate Professor
MIT

http://www.tamarabroderick.com/tutorials.html

Nonparametric Bayes

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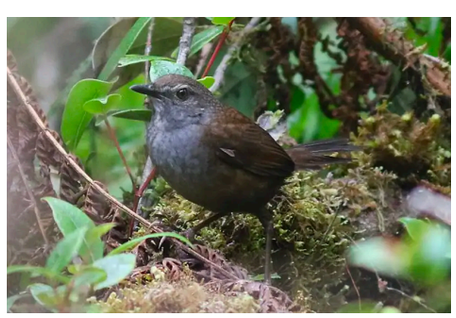

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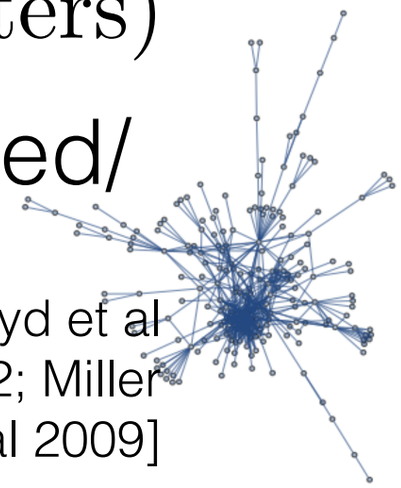

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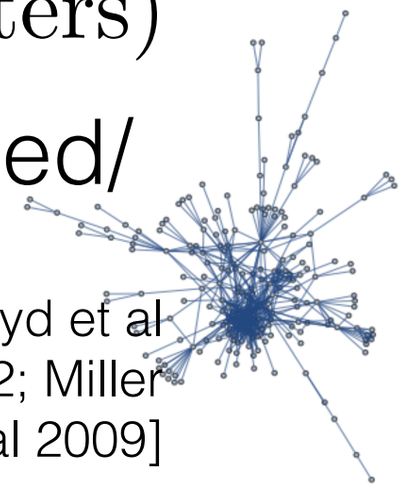

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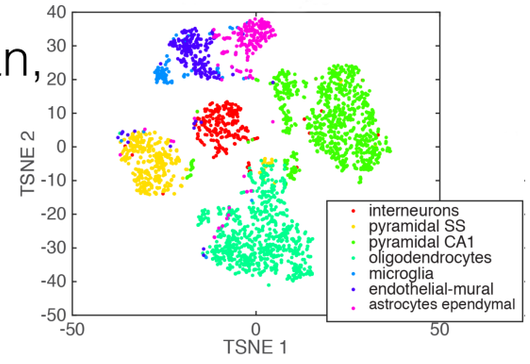

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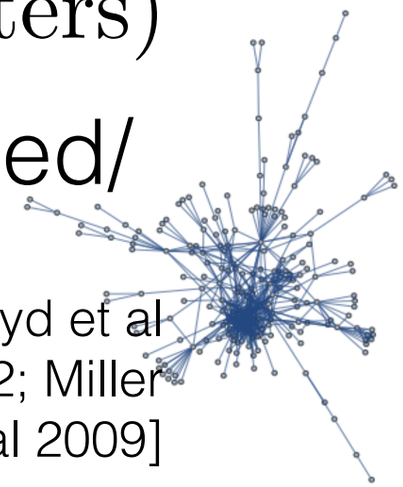

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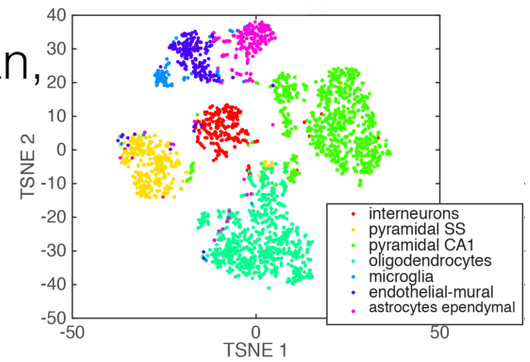

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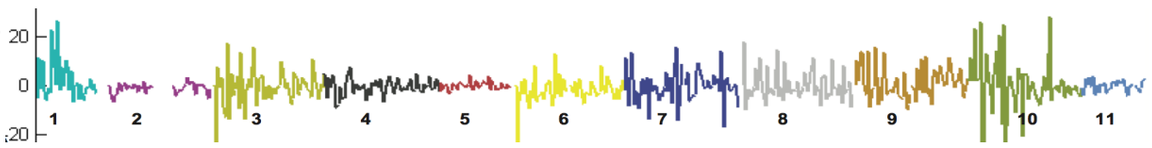

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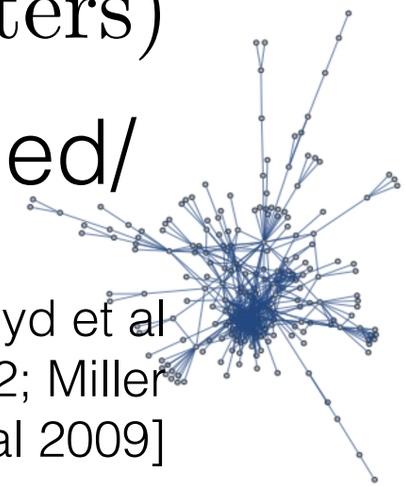

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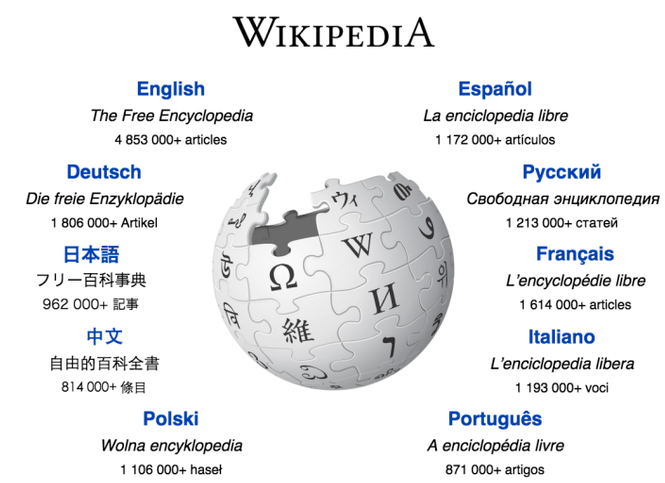

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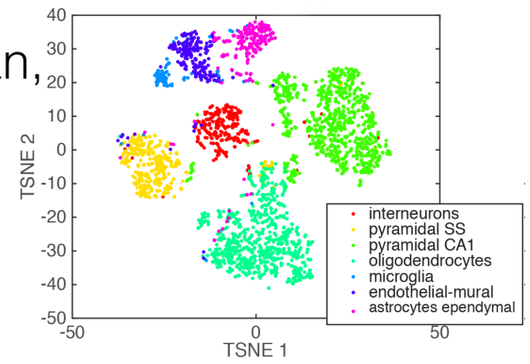

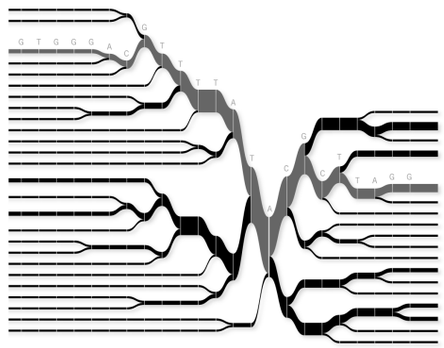

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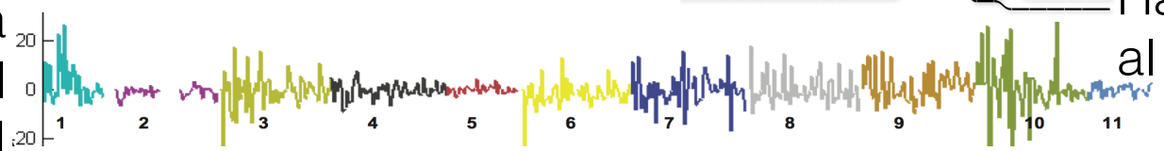

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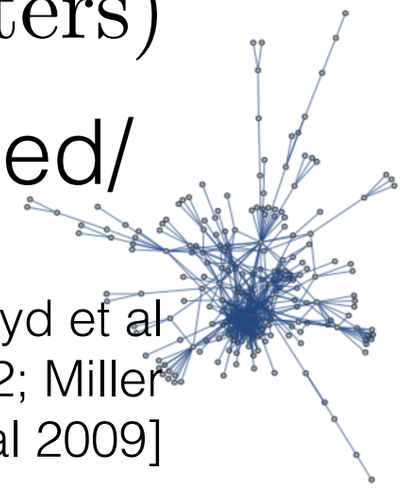

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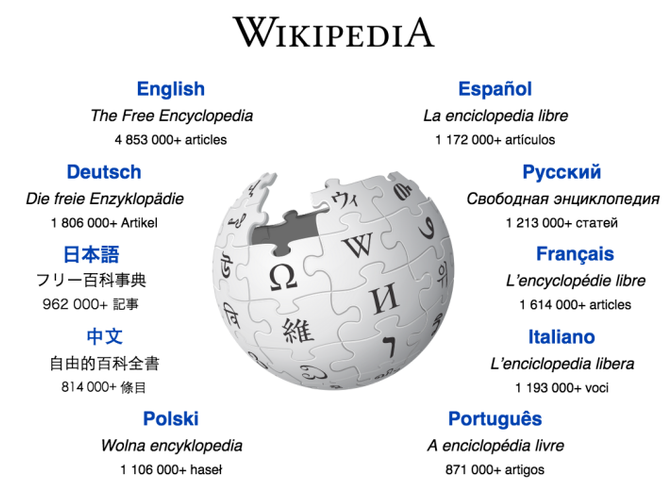

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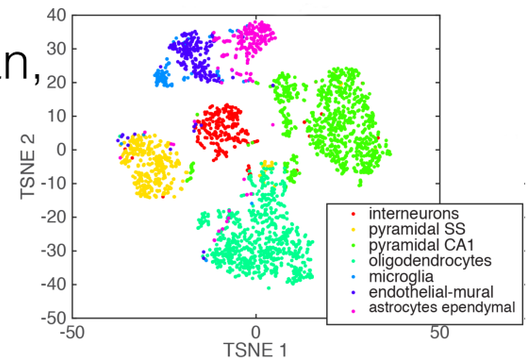

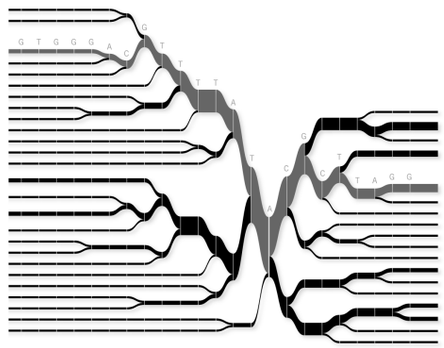

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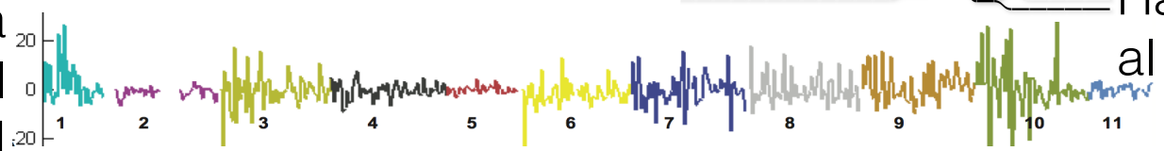

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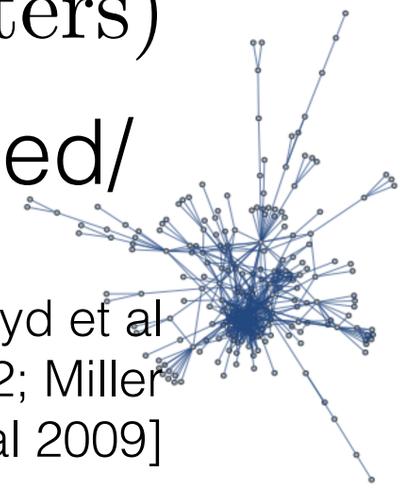

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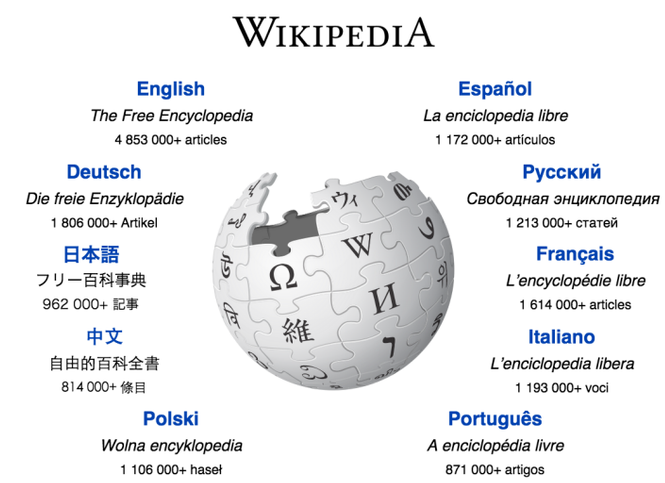

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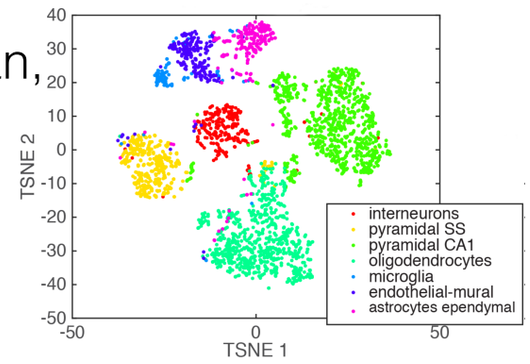

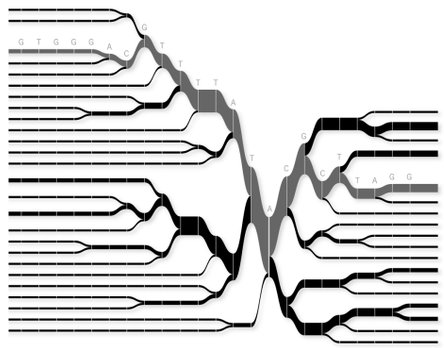

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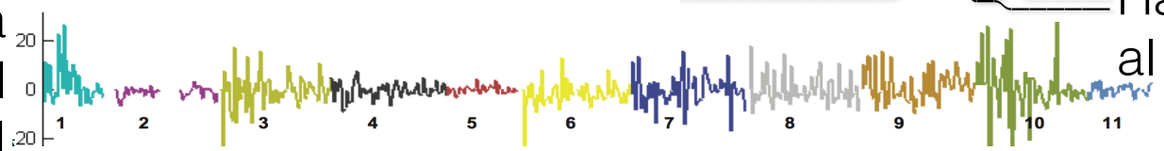

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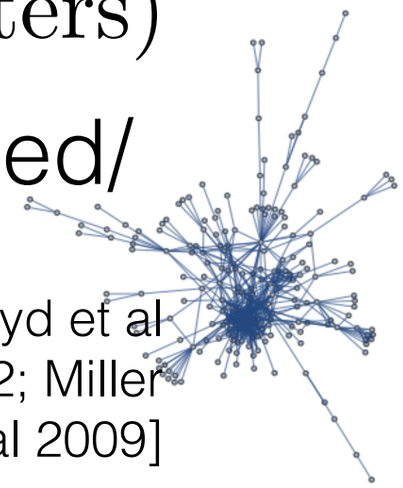

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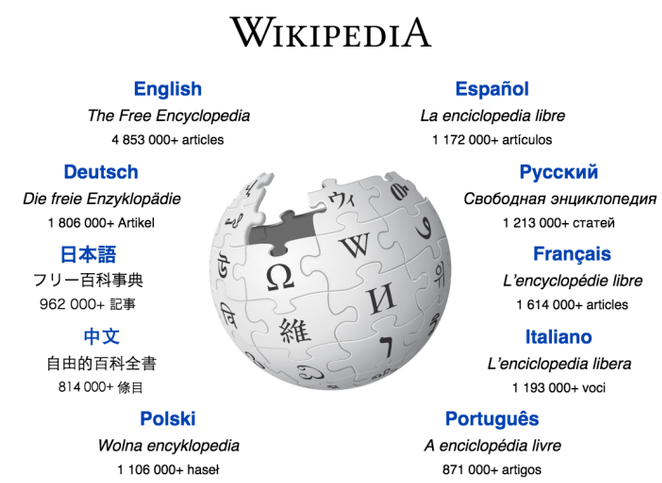

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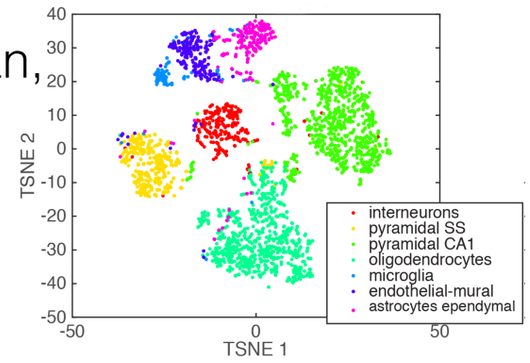

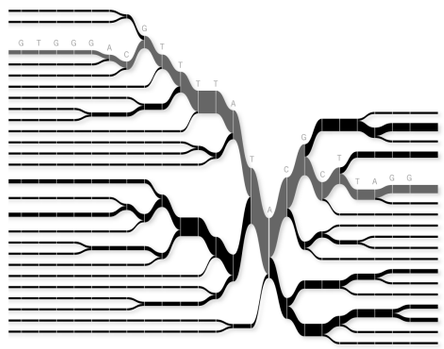

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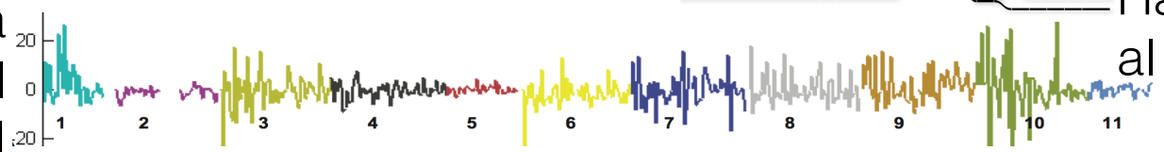

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  - "Nonparametric Bayesian" priors

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Roadmap

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- Example problem: clustering

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- Example NPBayes model: Dirichlet process

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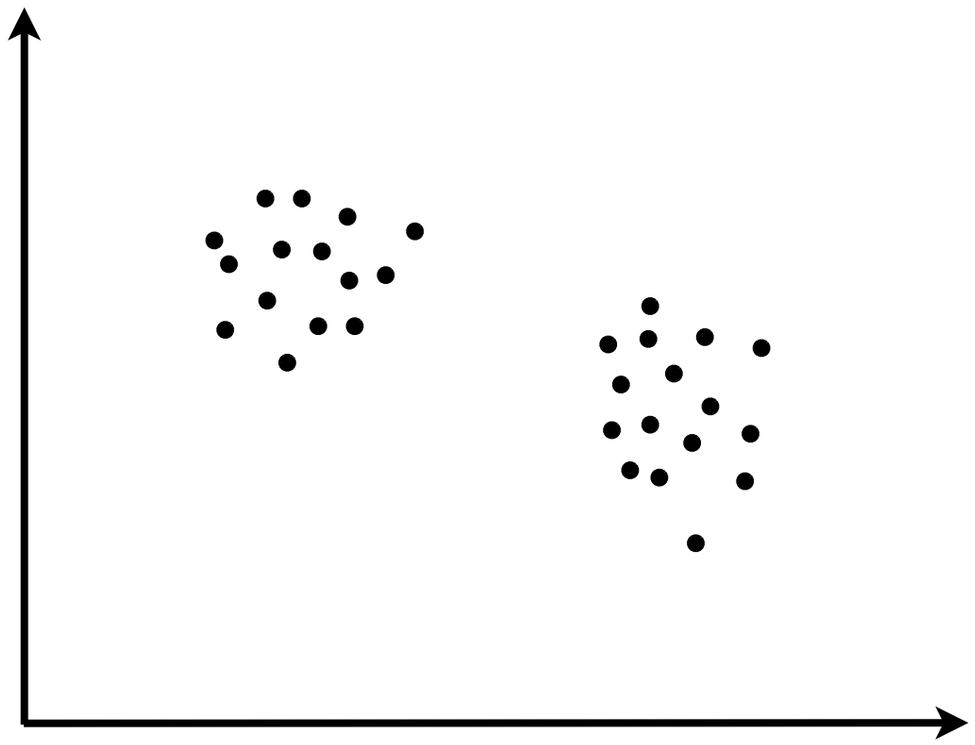
  - Why is NPBayes challenging but practical?

Generative model



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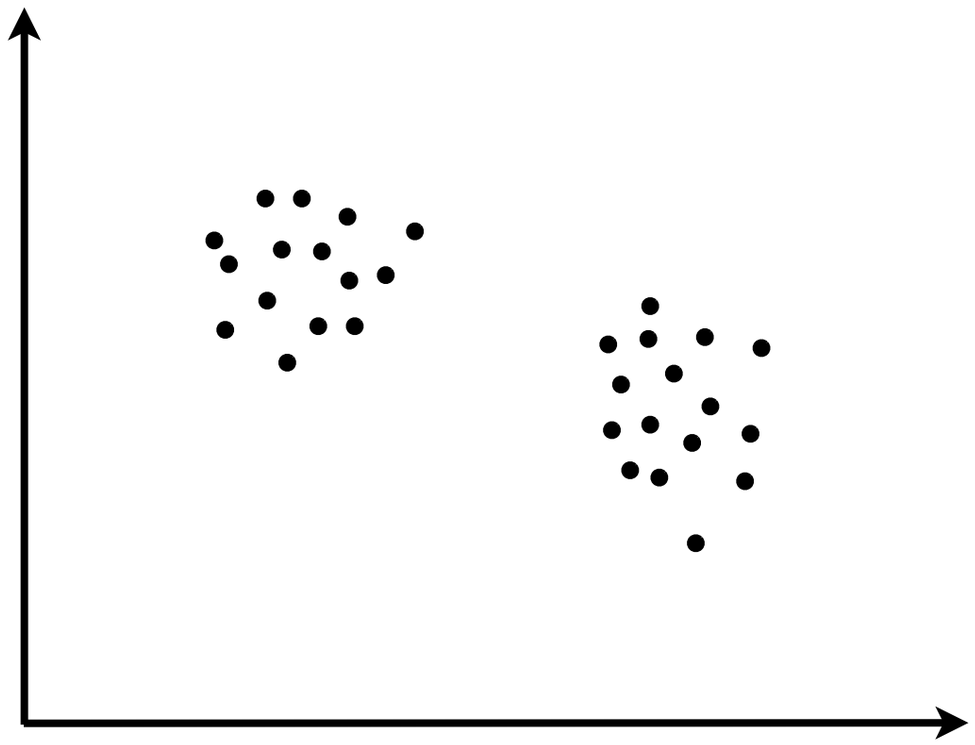

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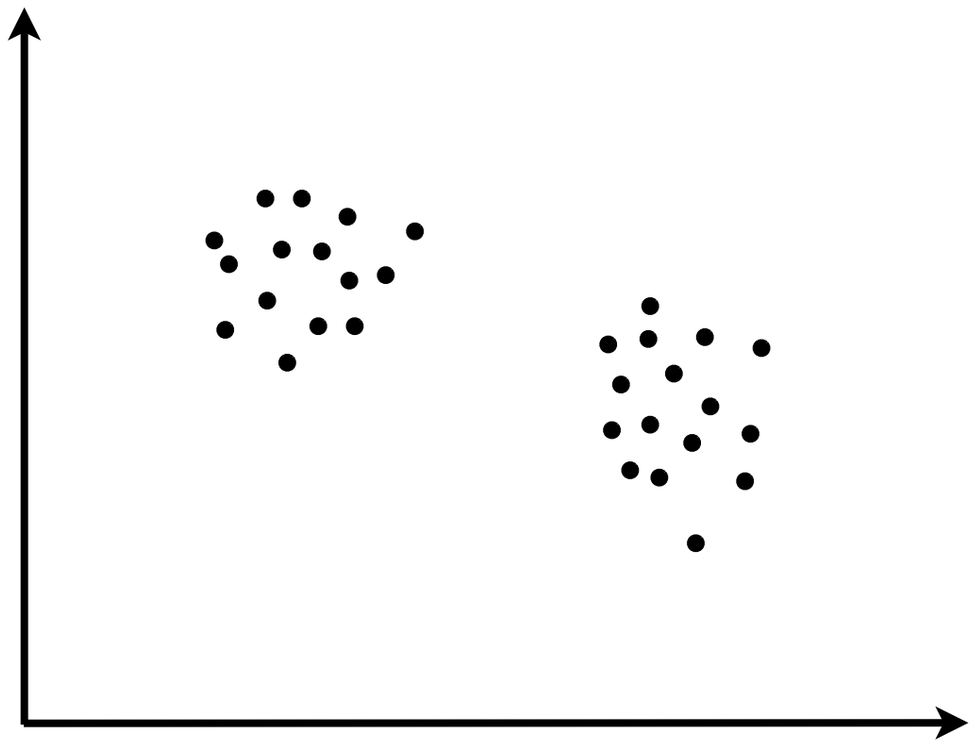

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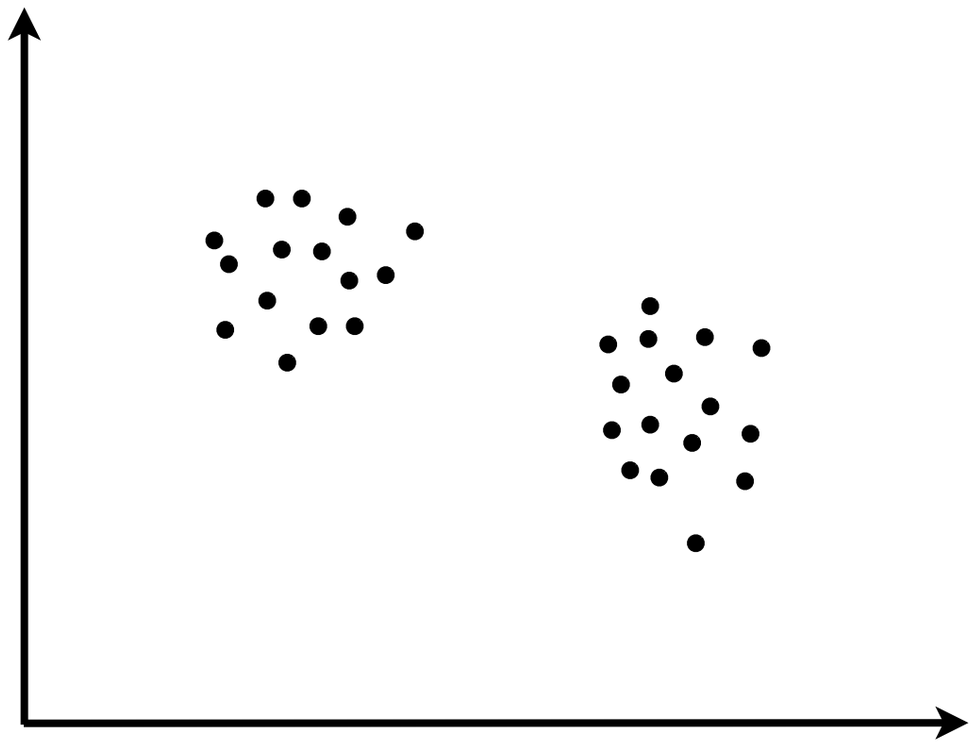

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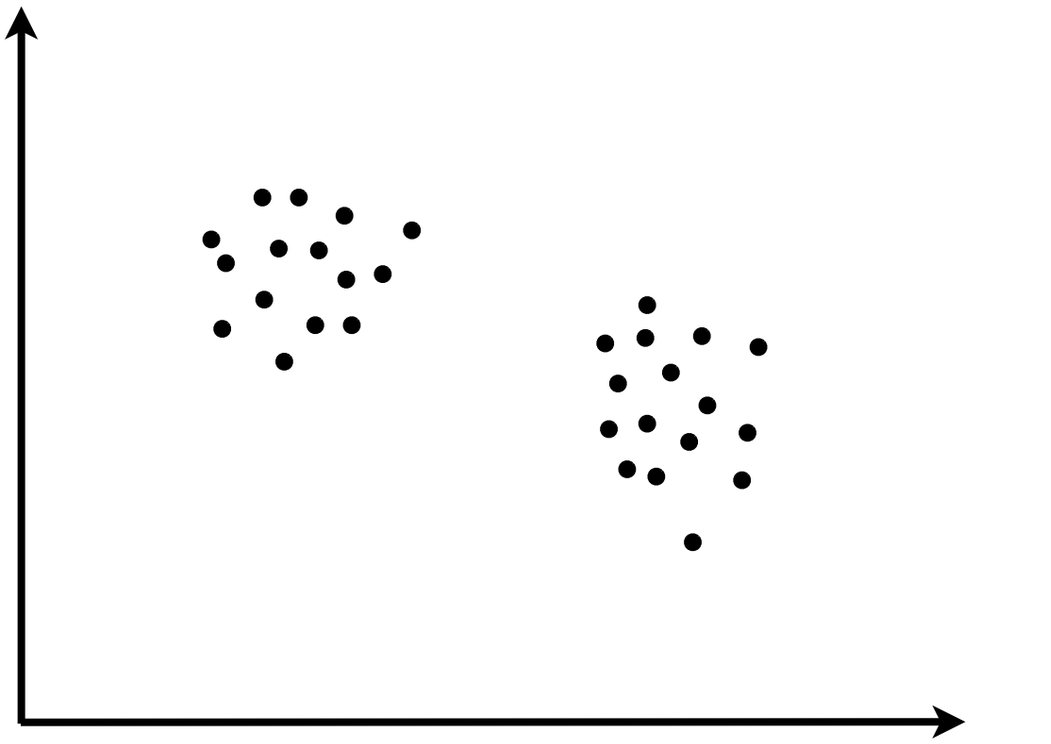

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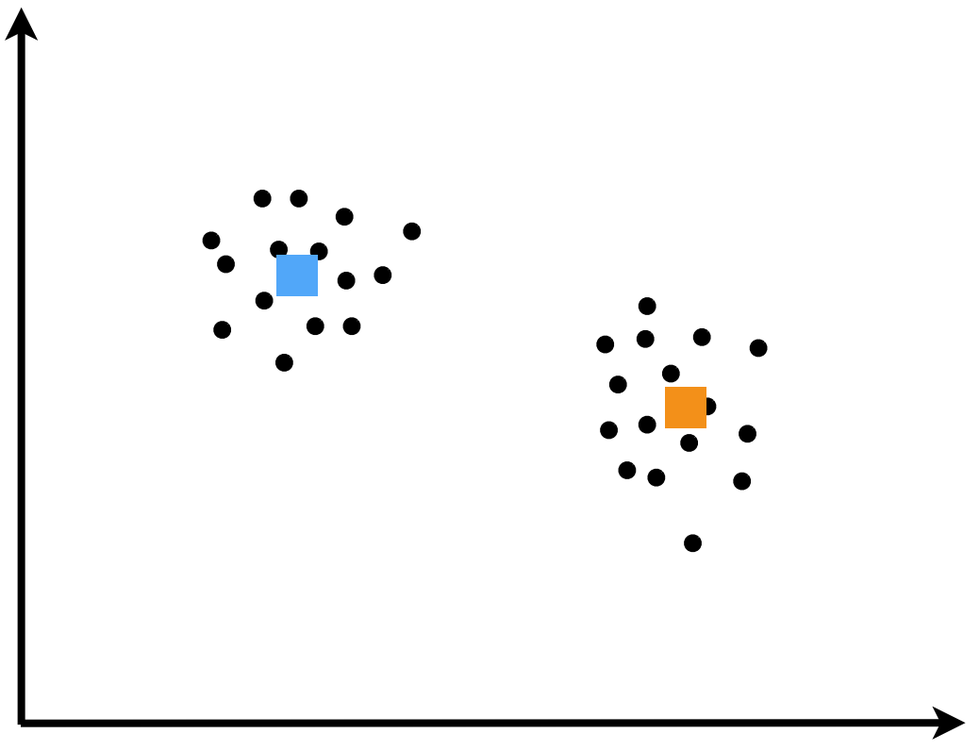

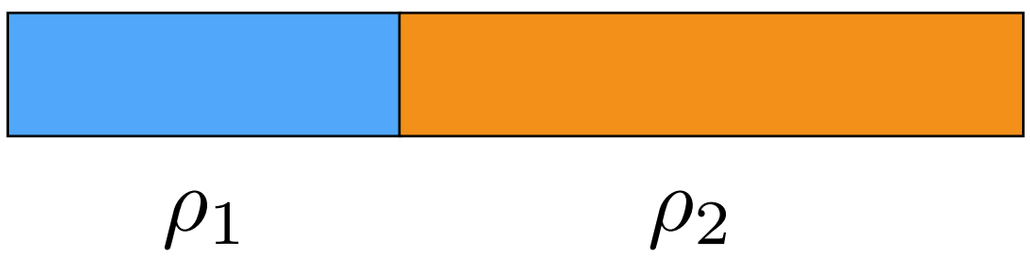

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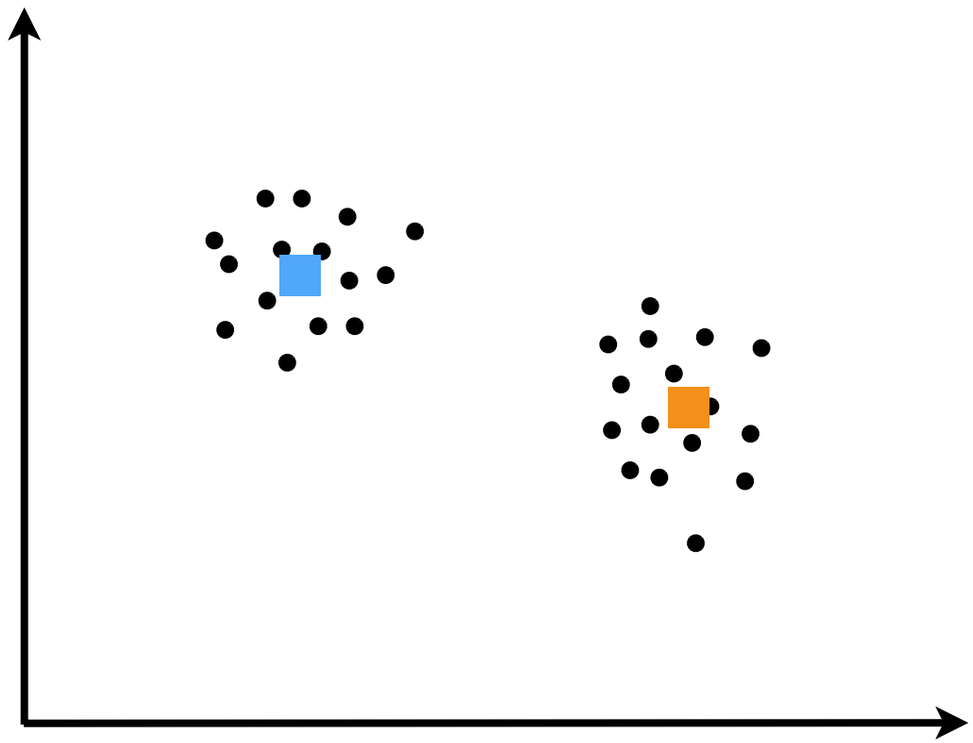

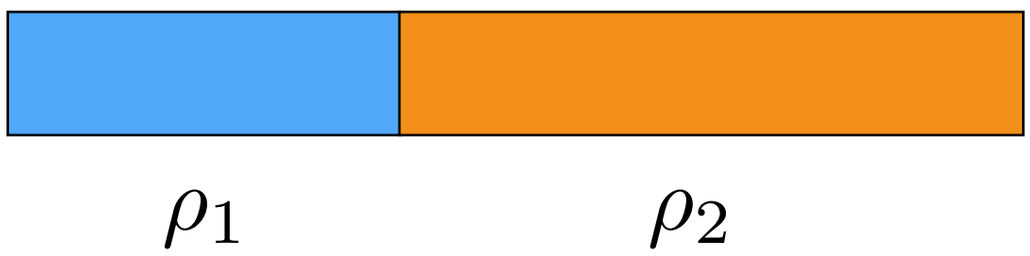

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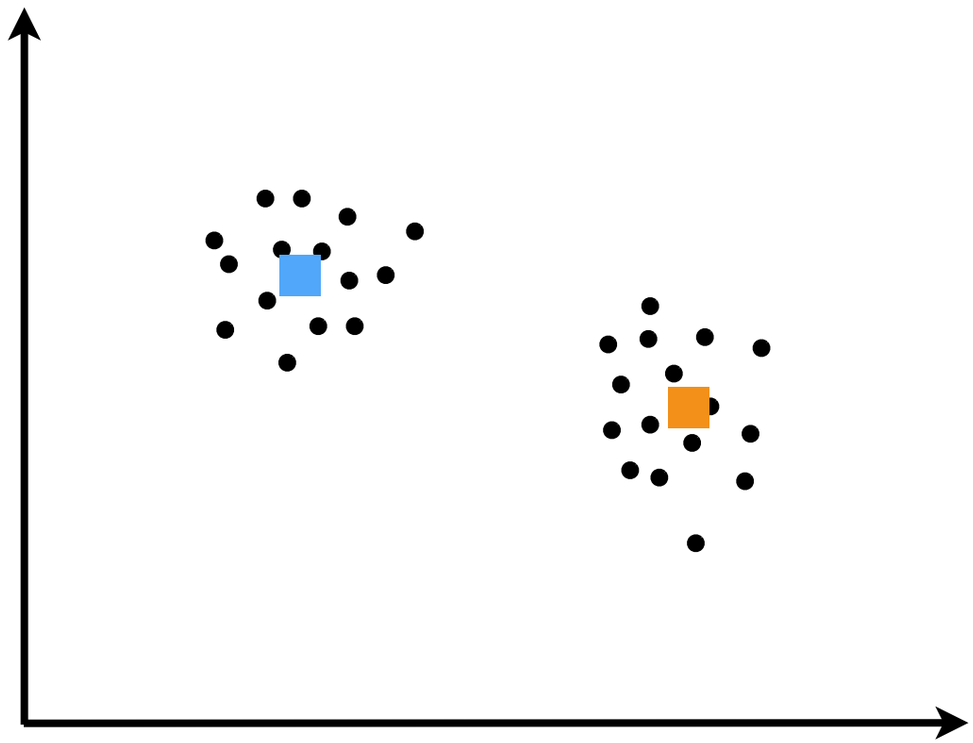

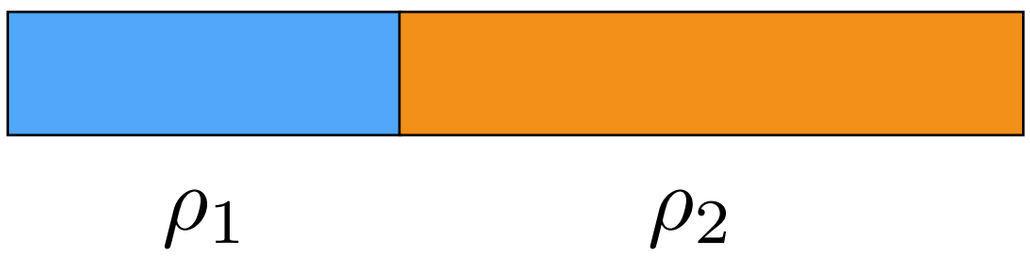

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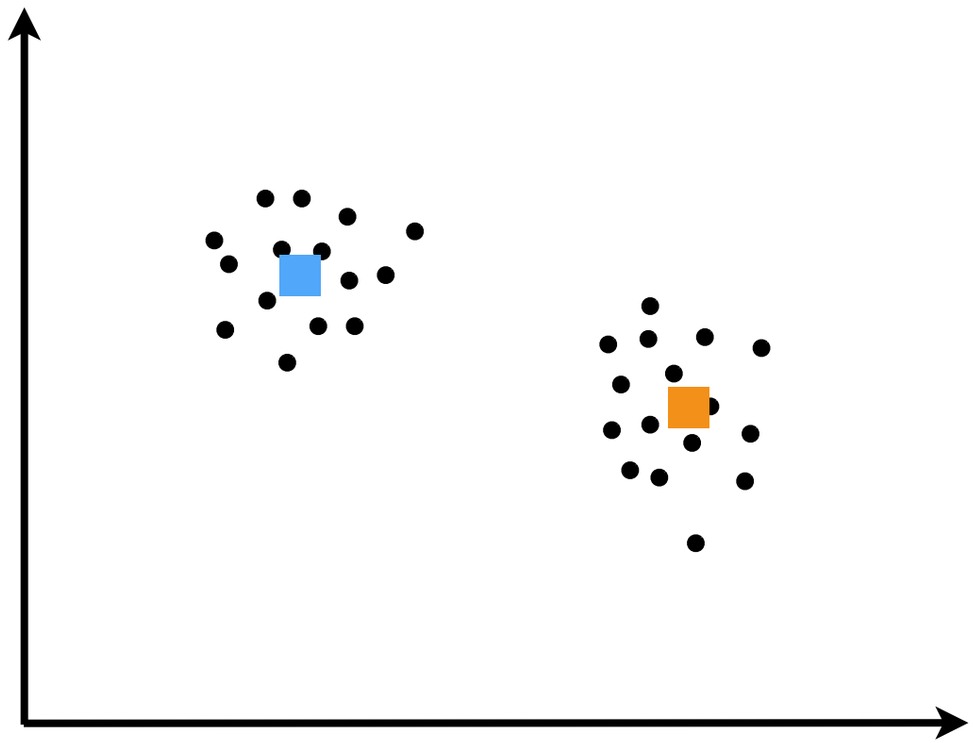

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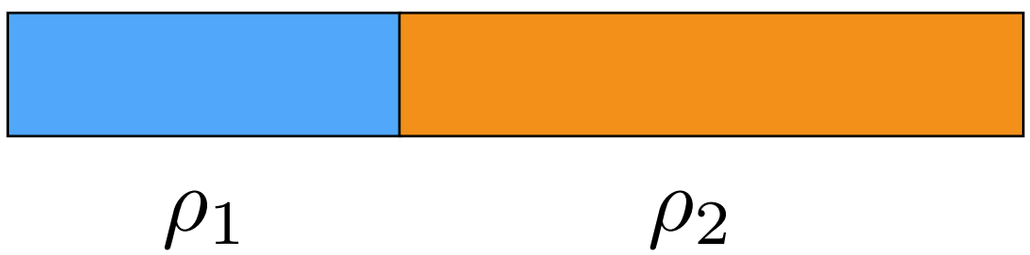

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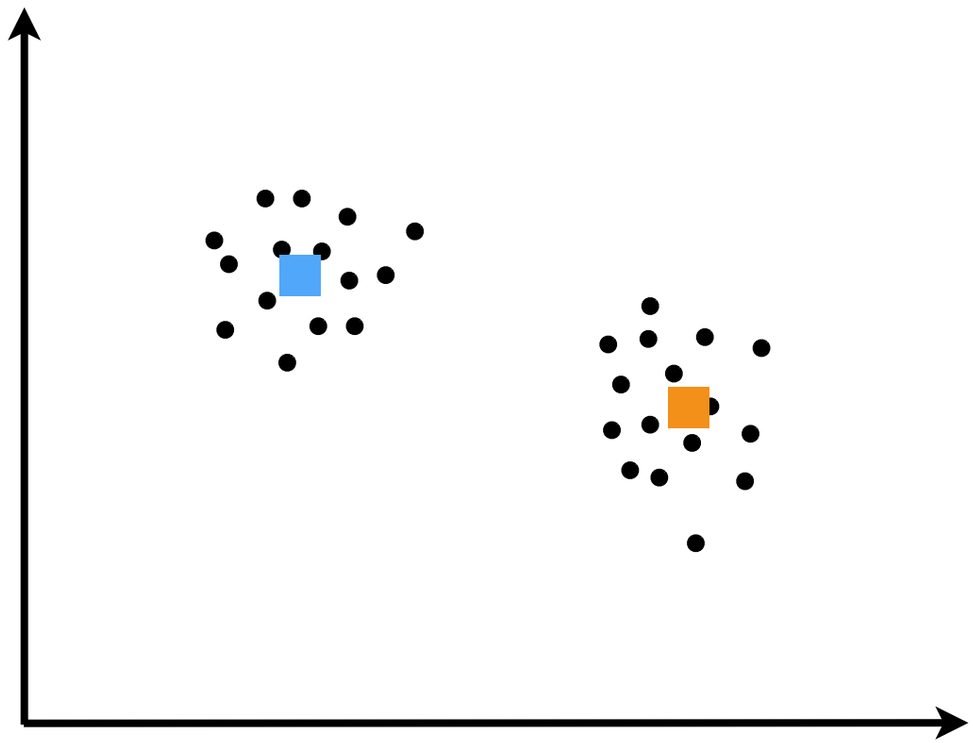

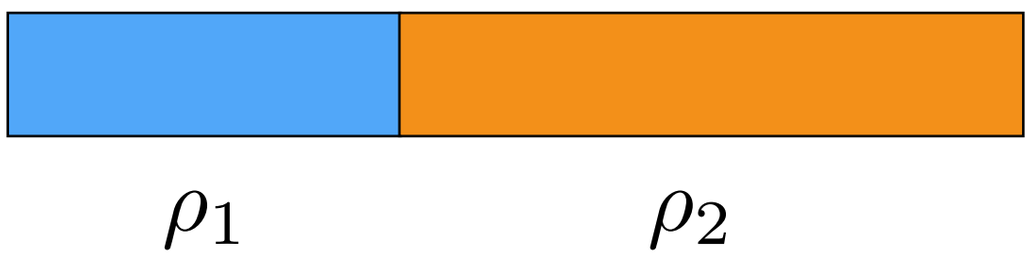

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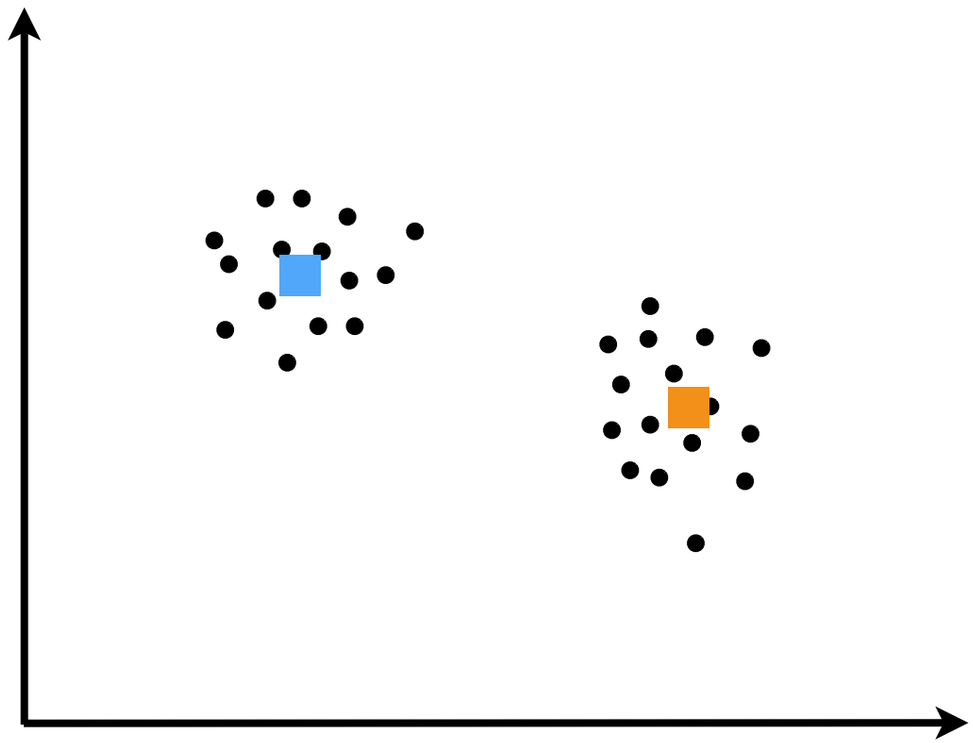

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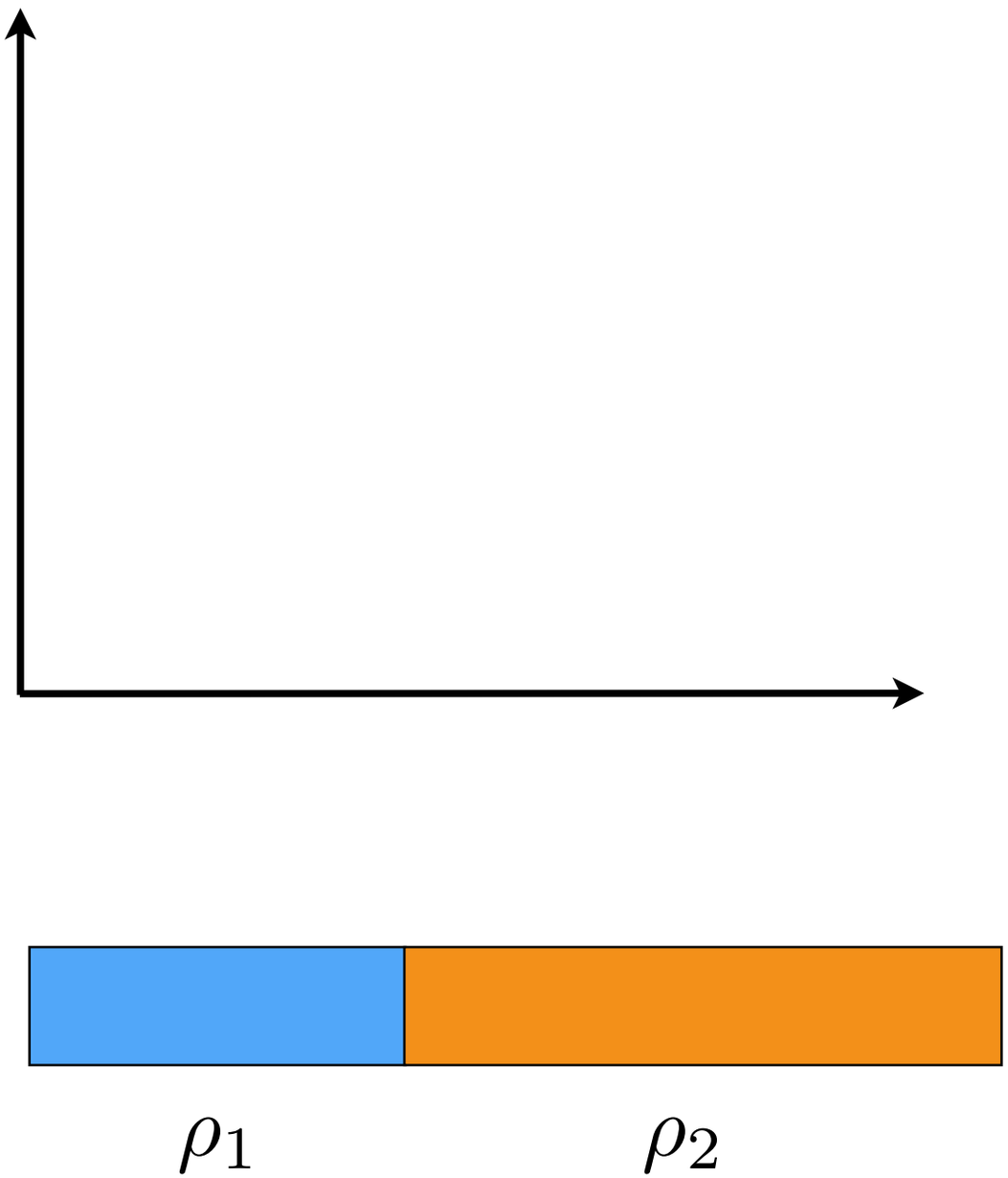

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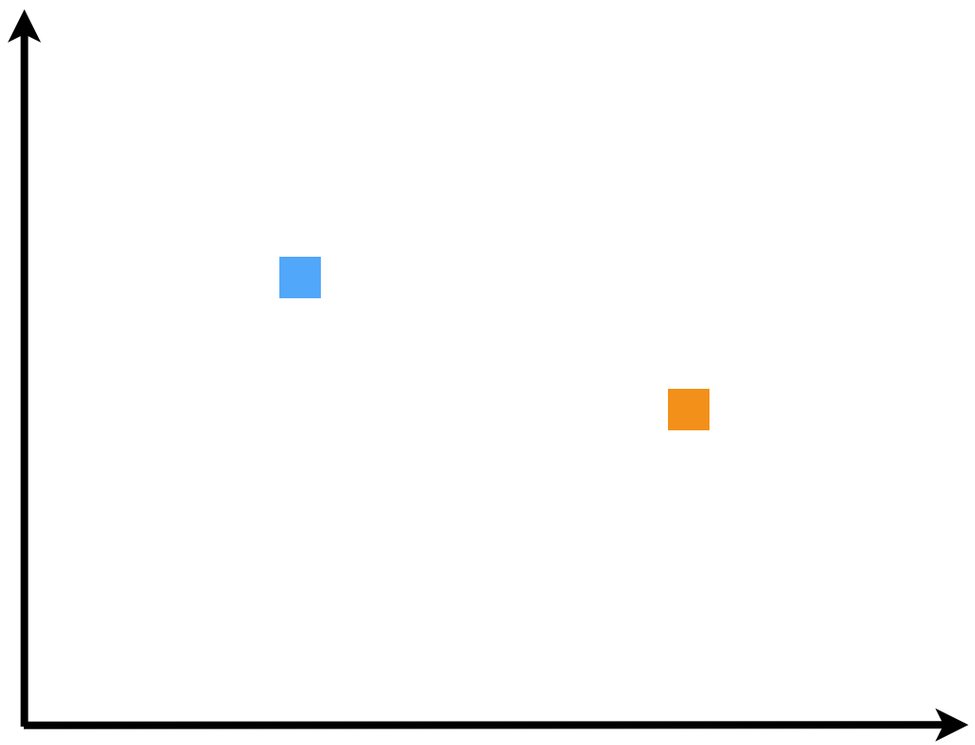

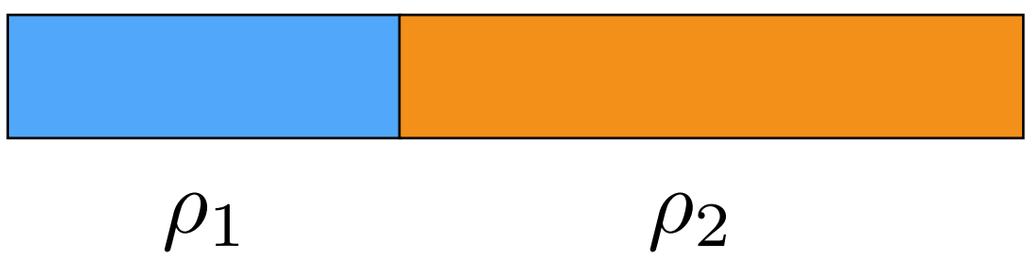

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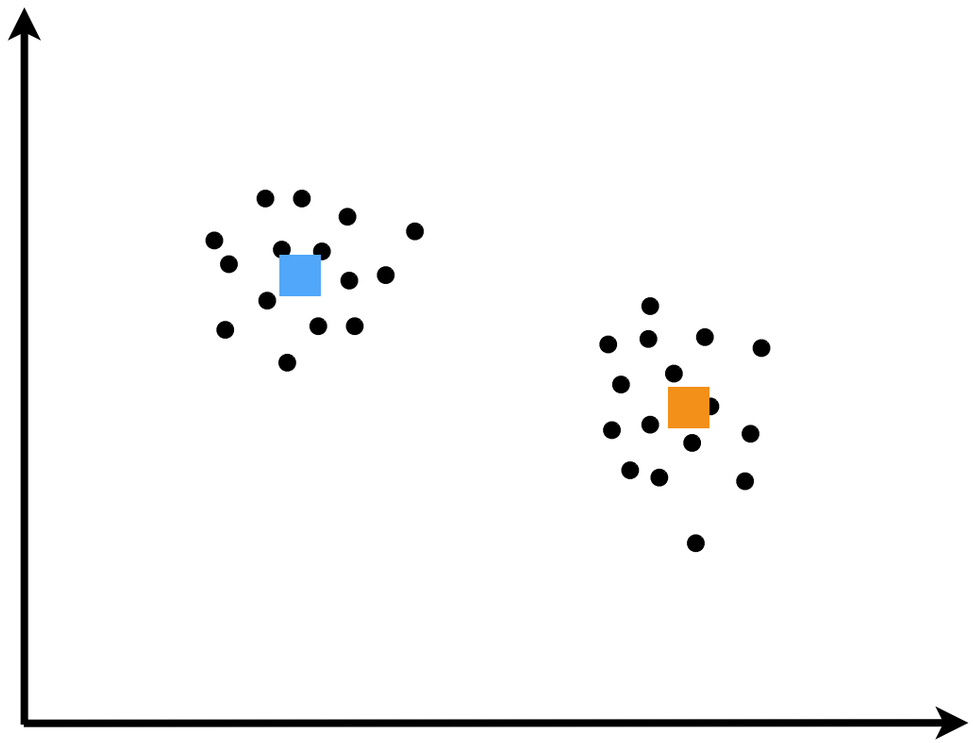

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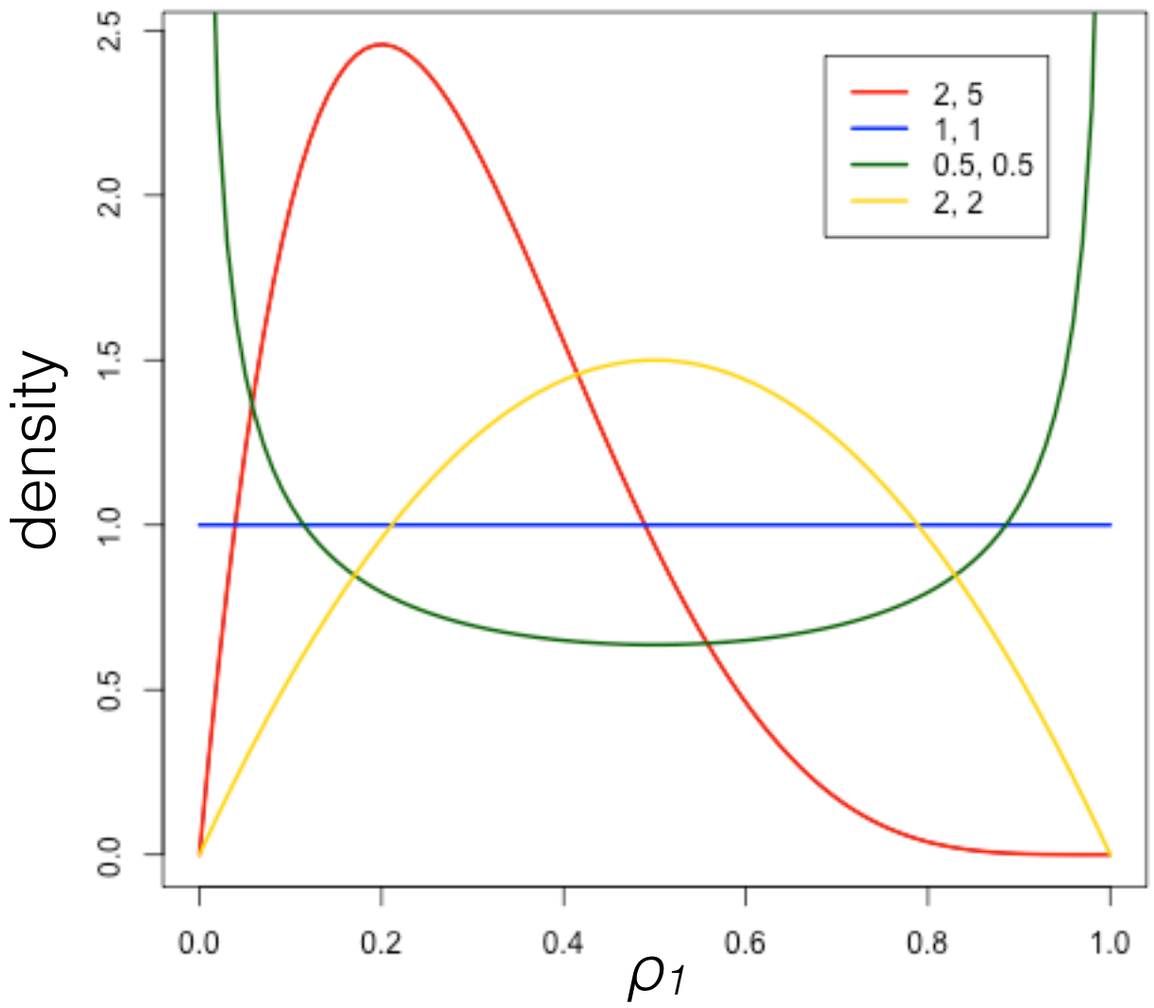

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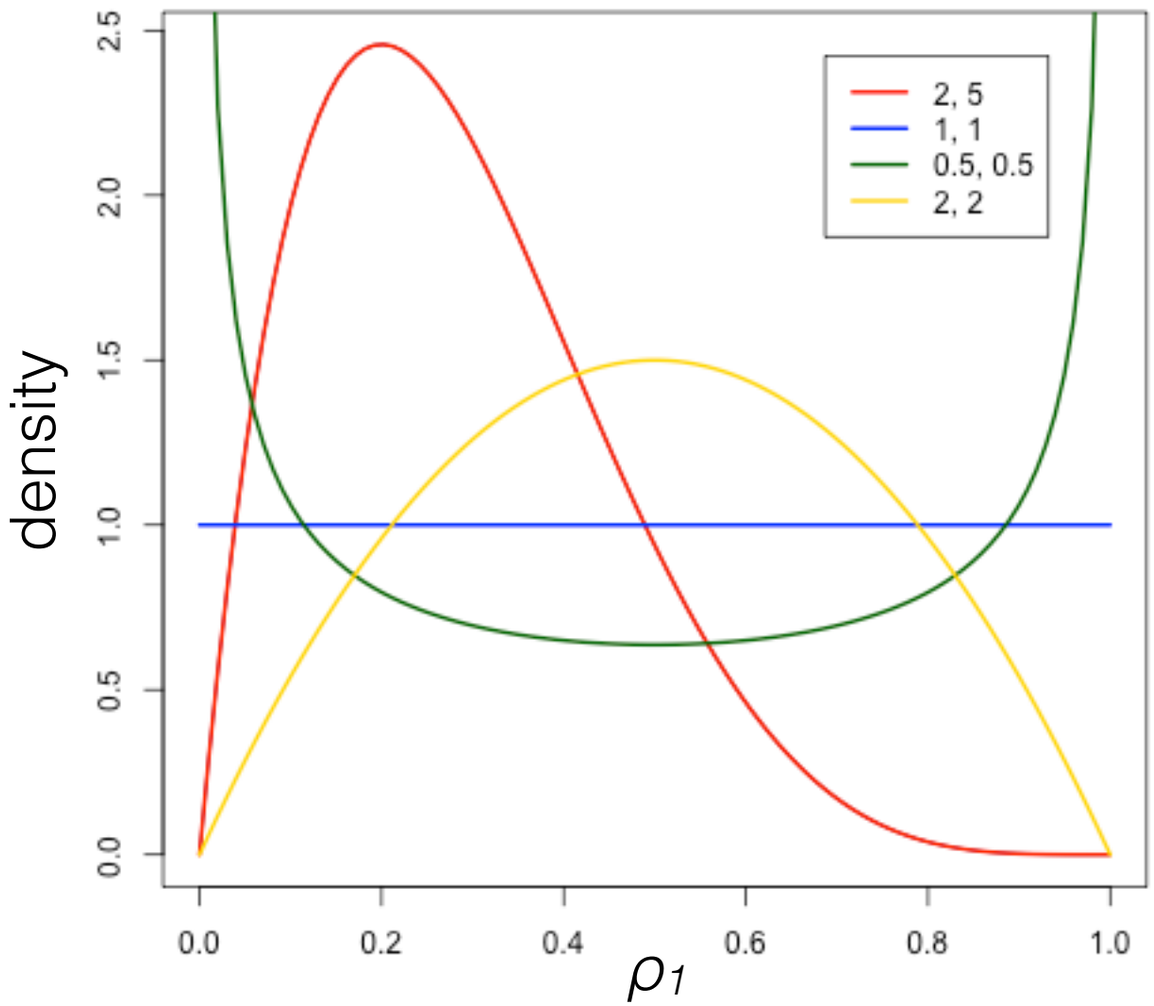

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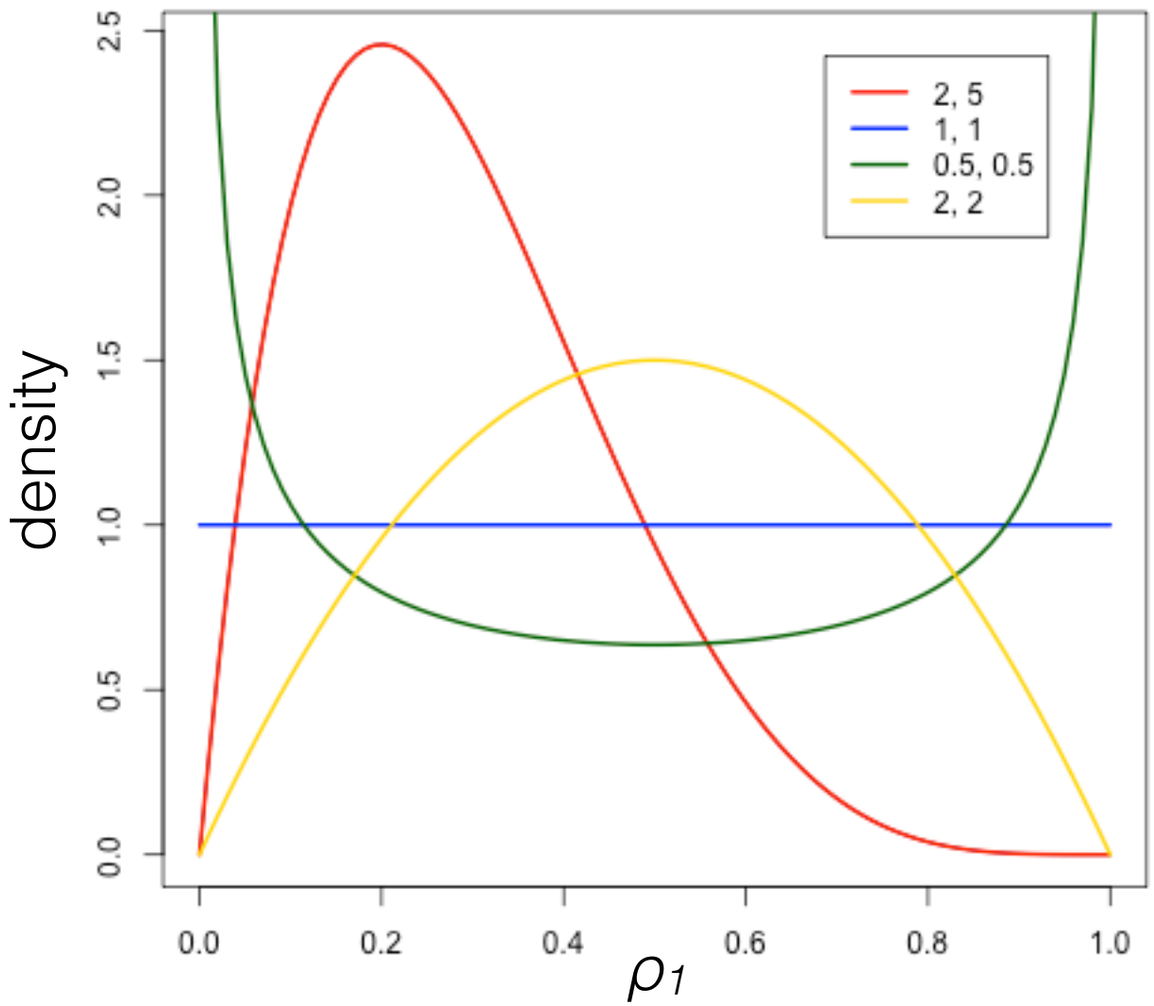

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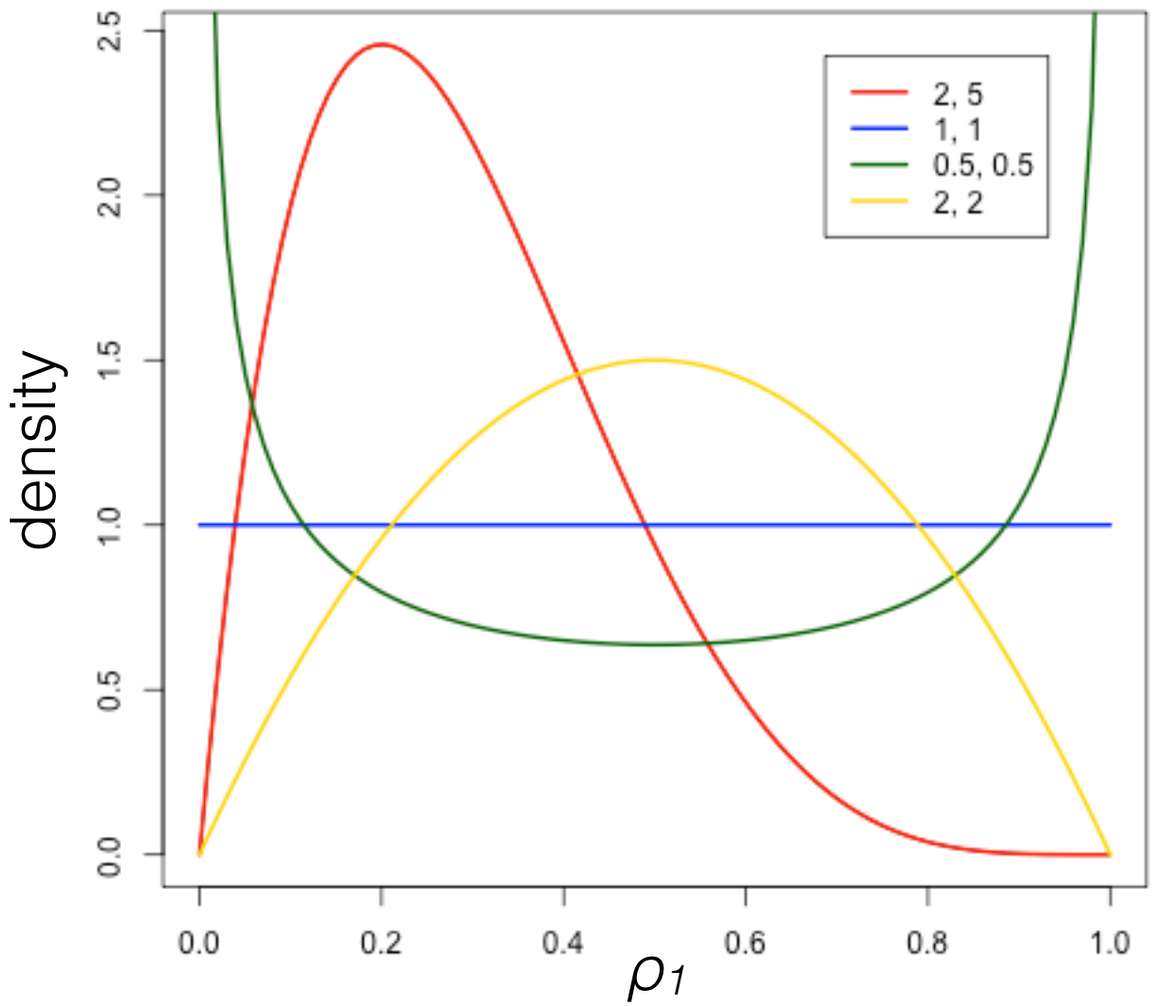

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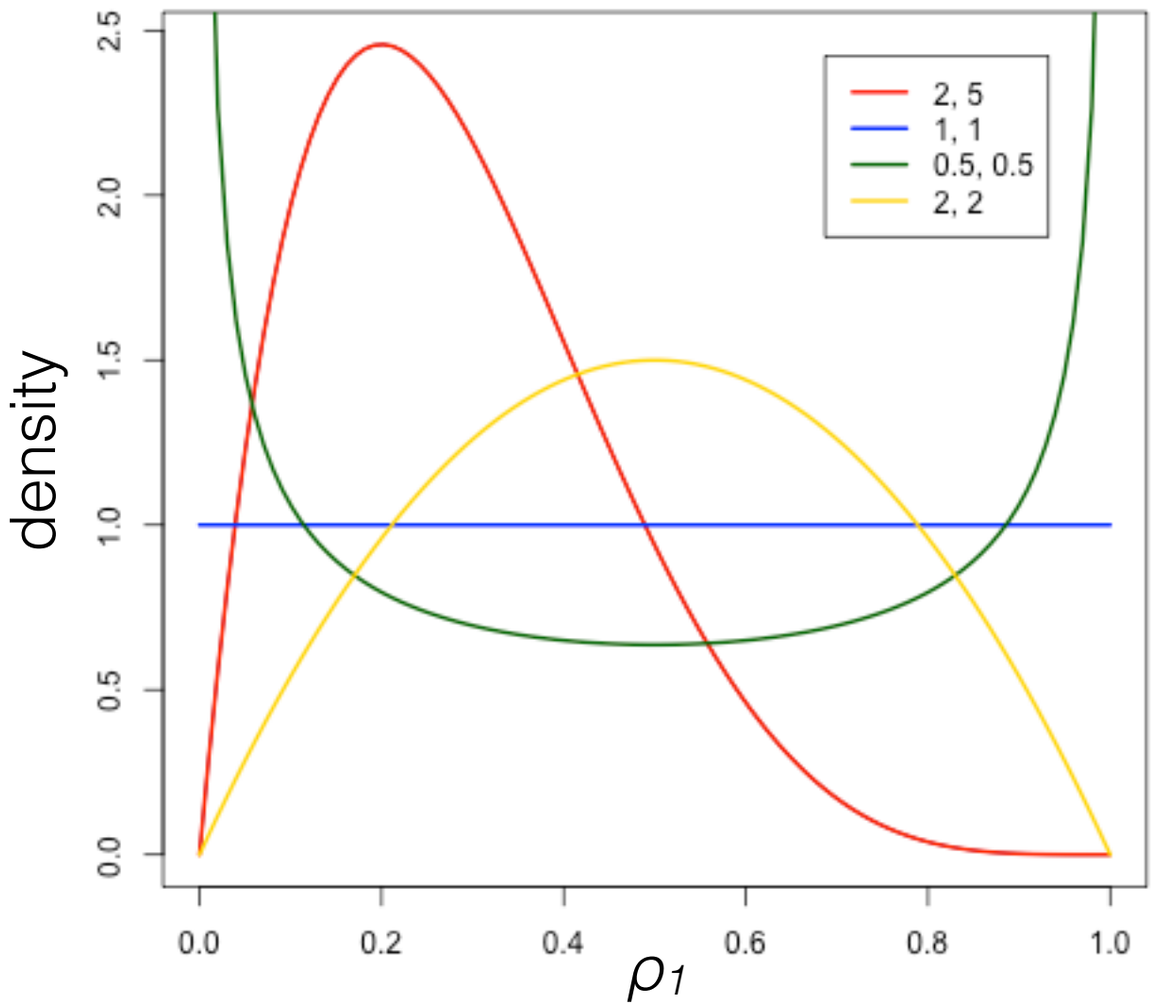

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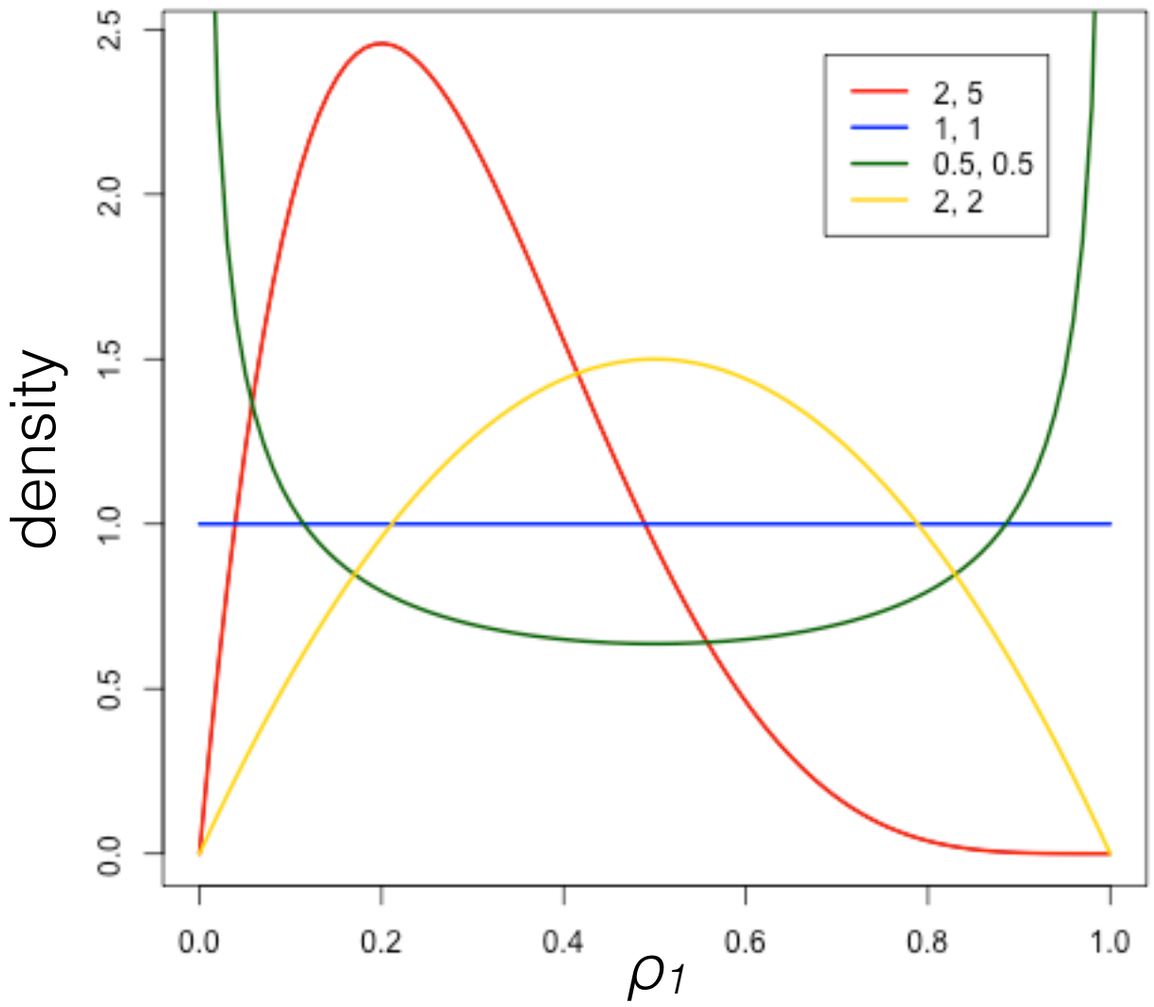

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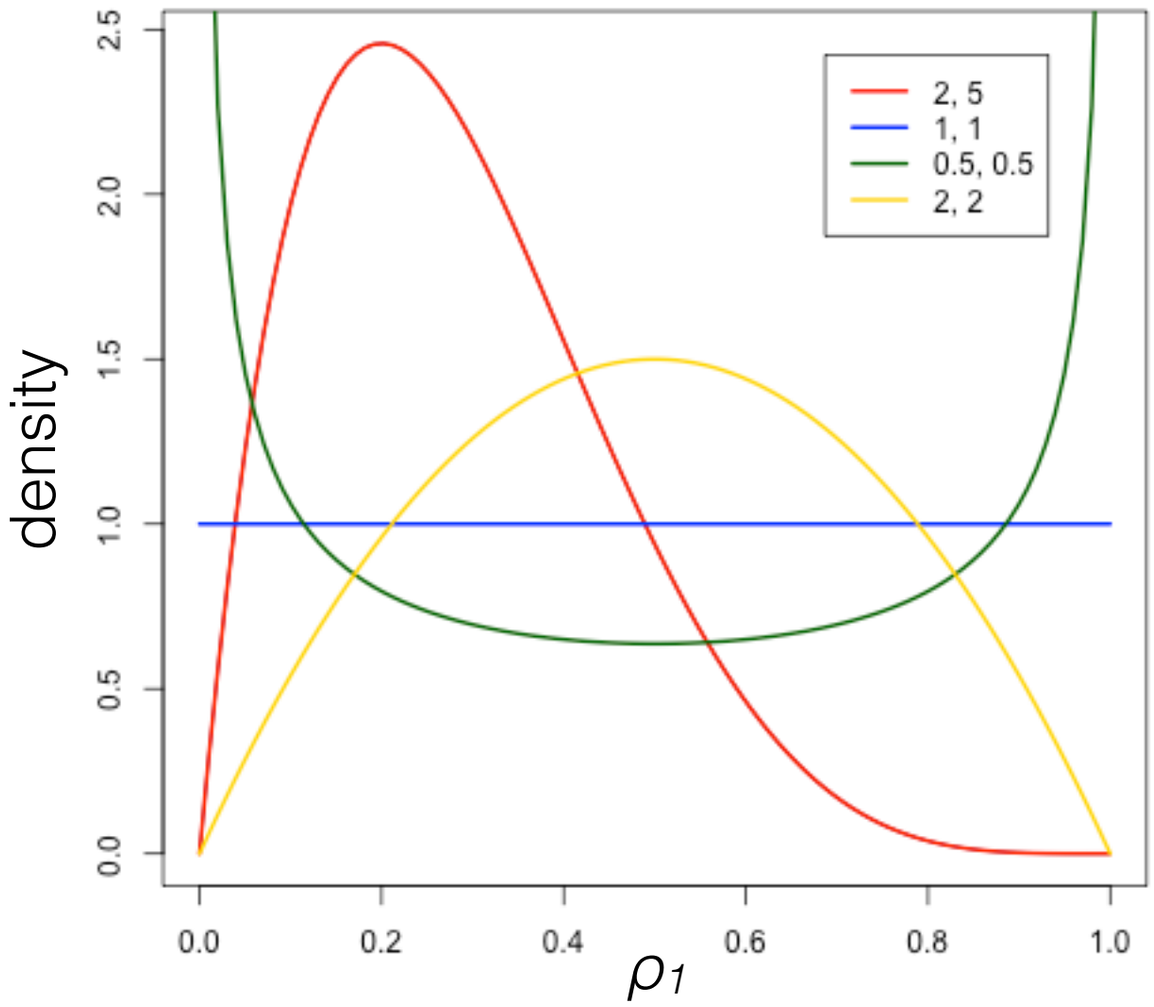

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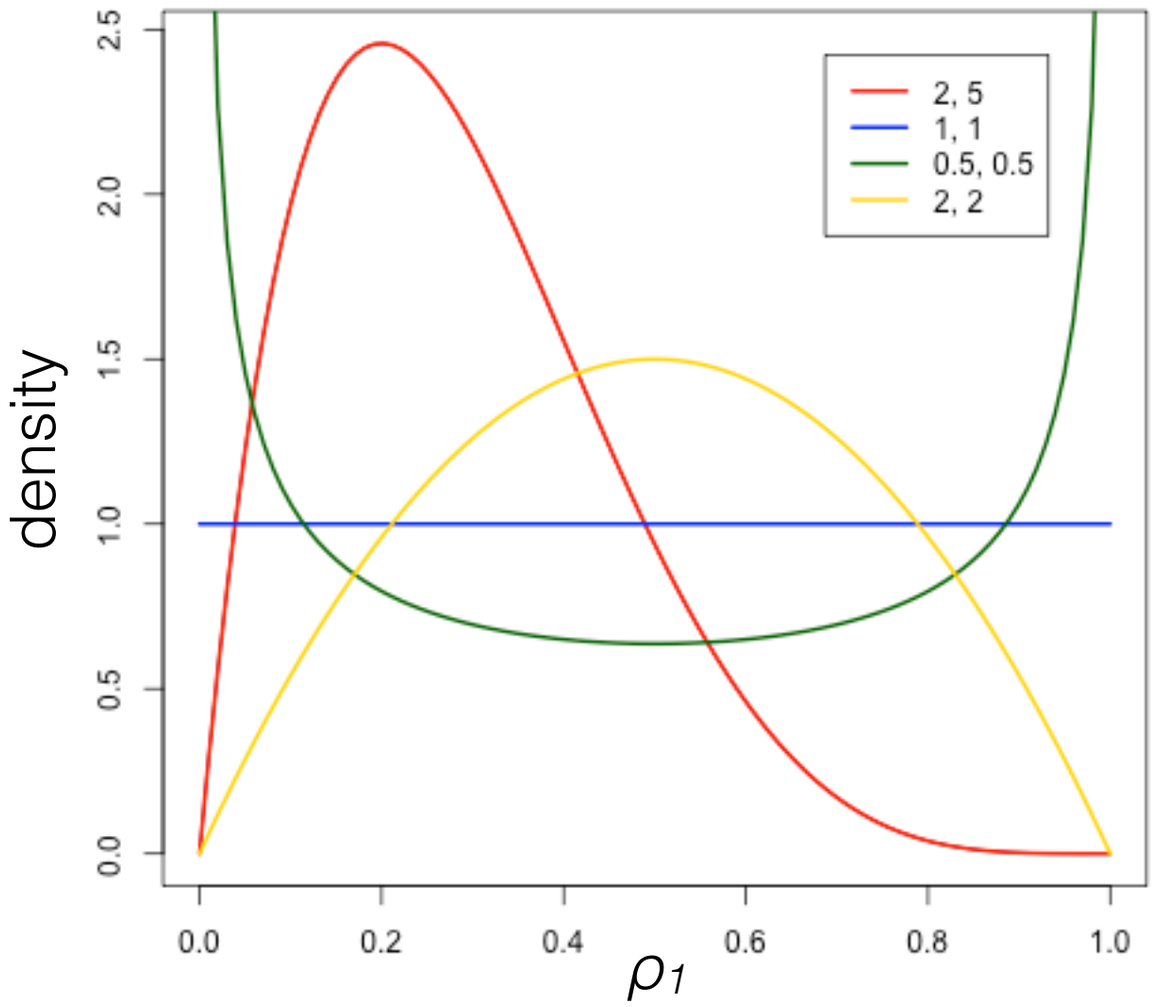

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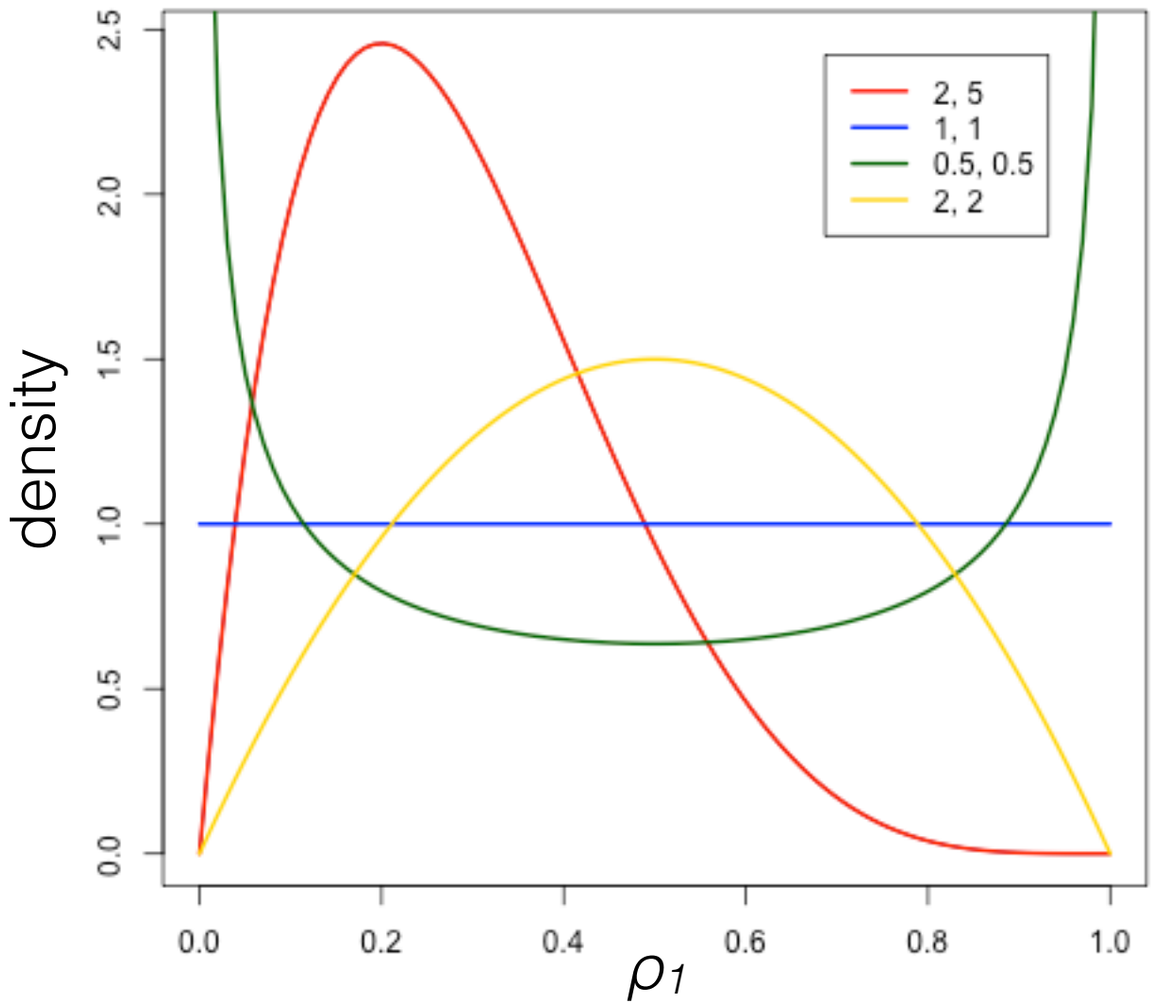

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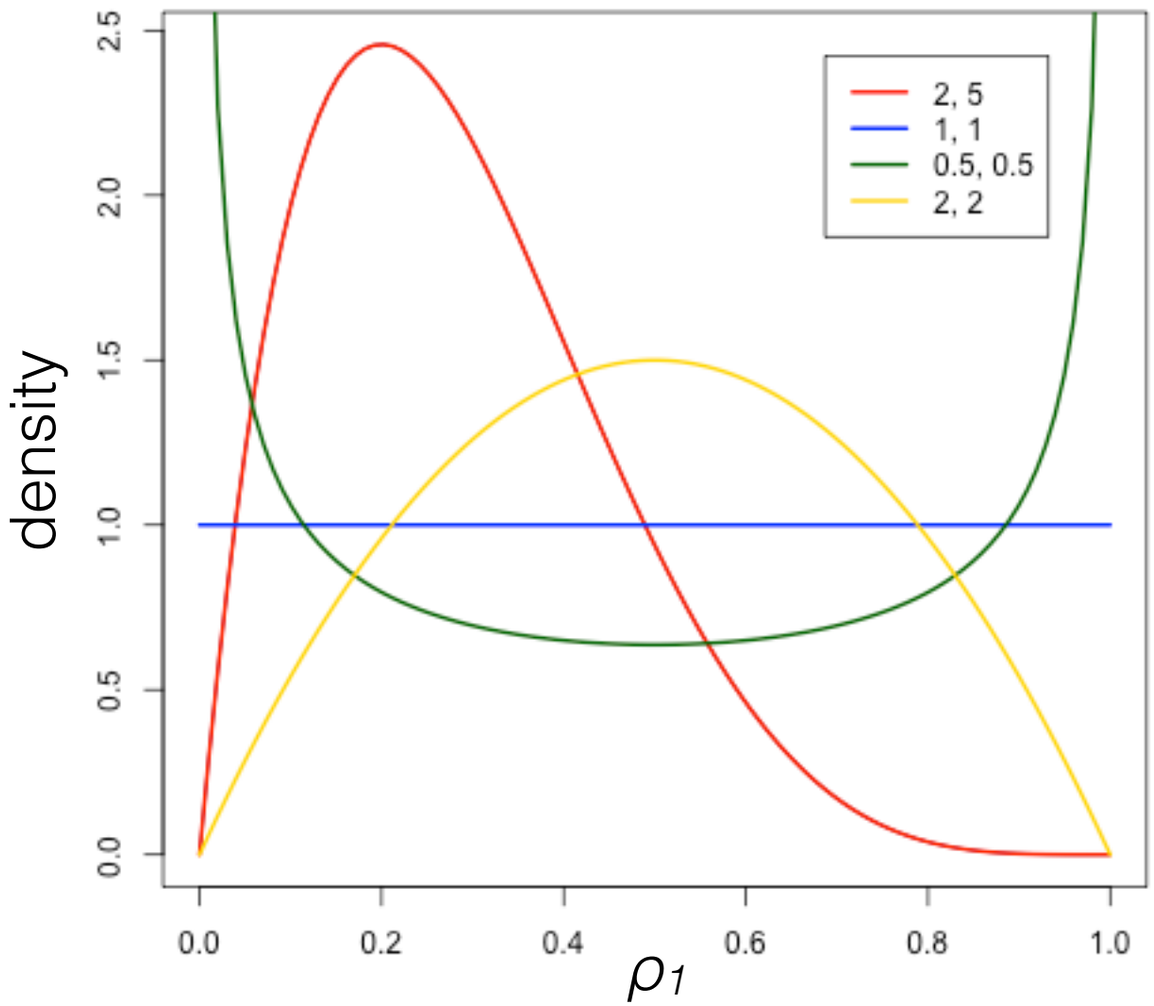

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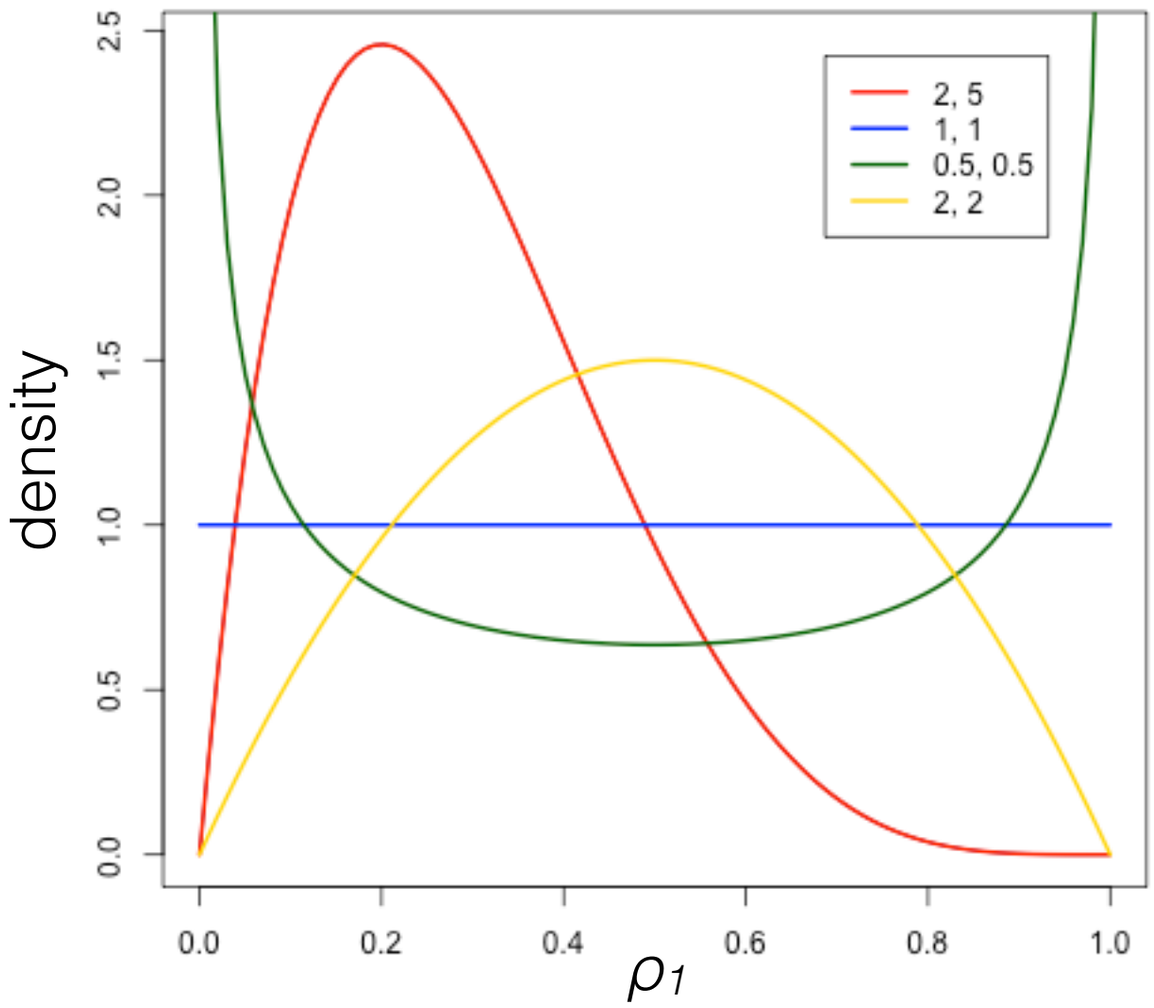

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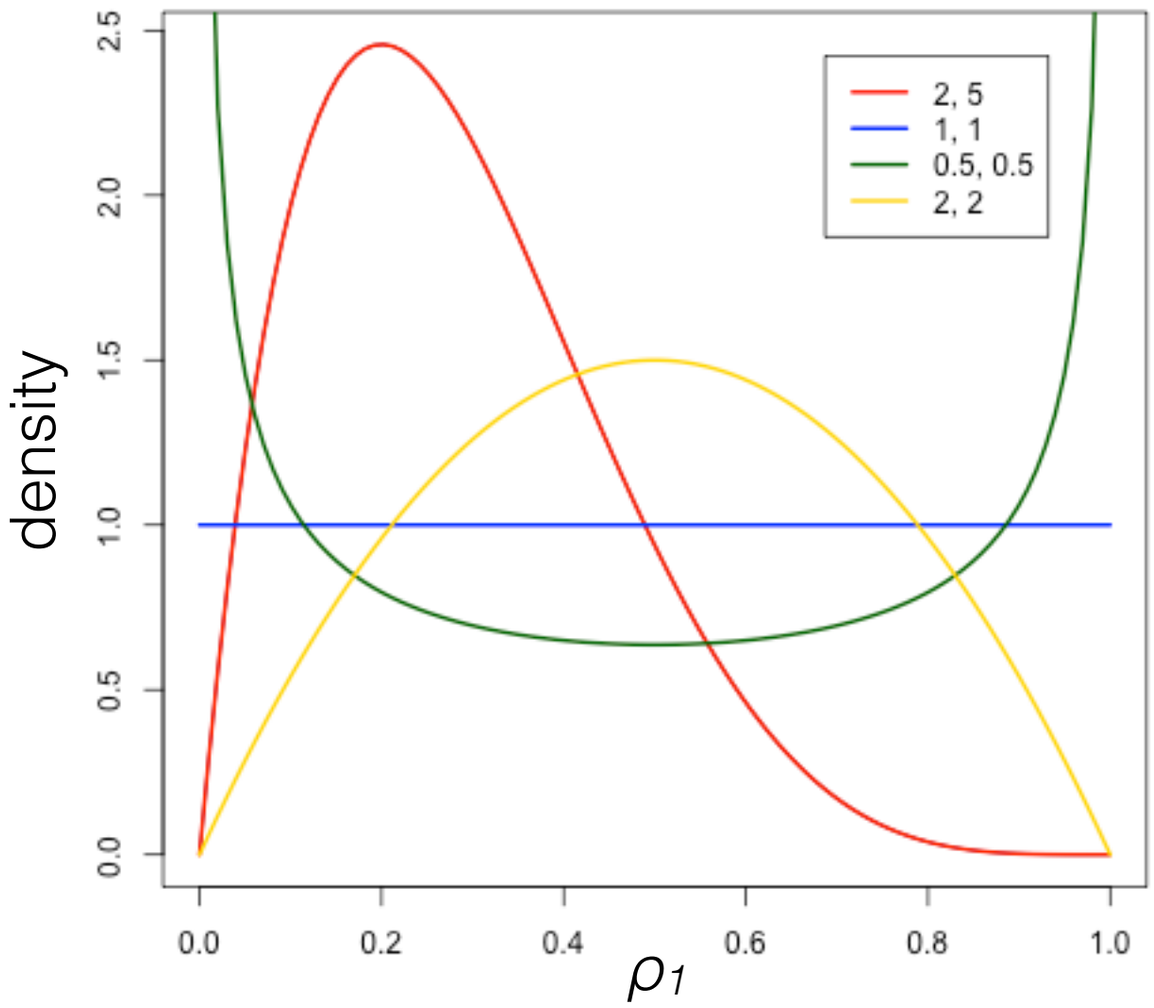

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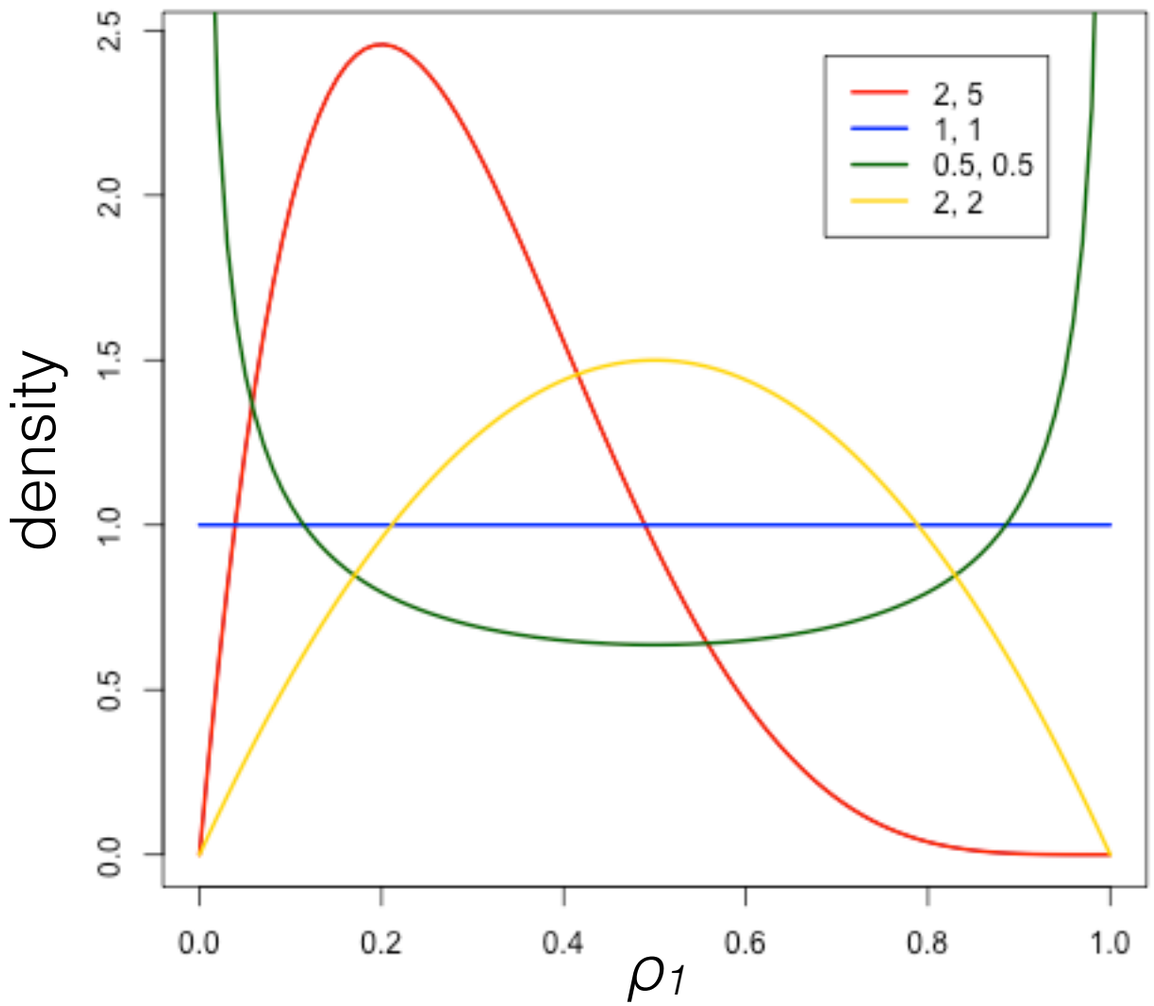

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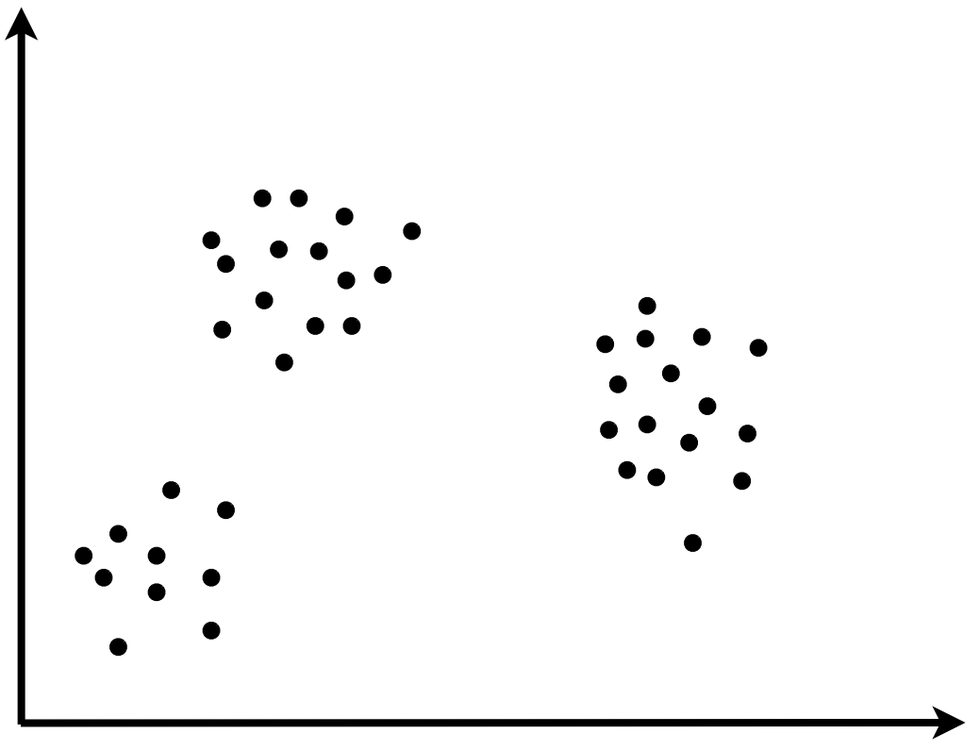

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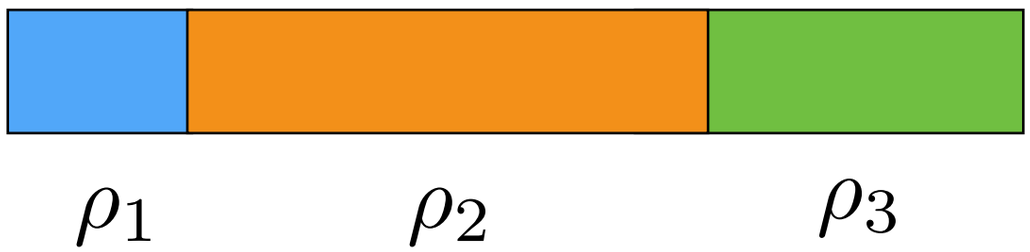

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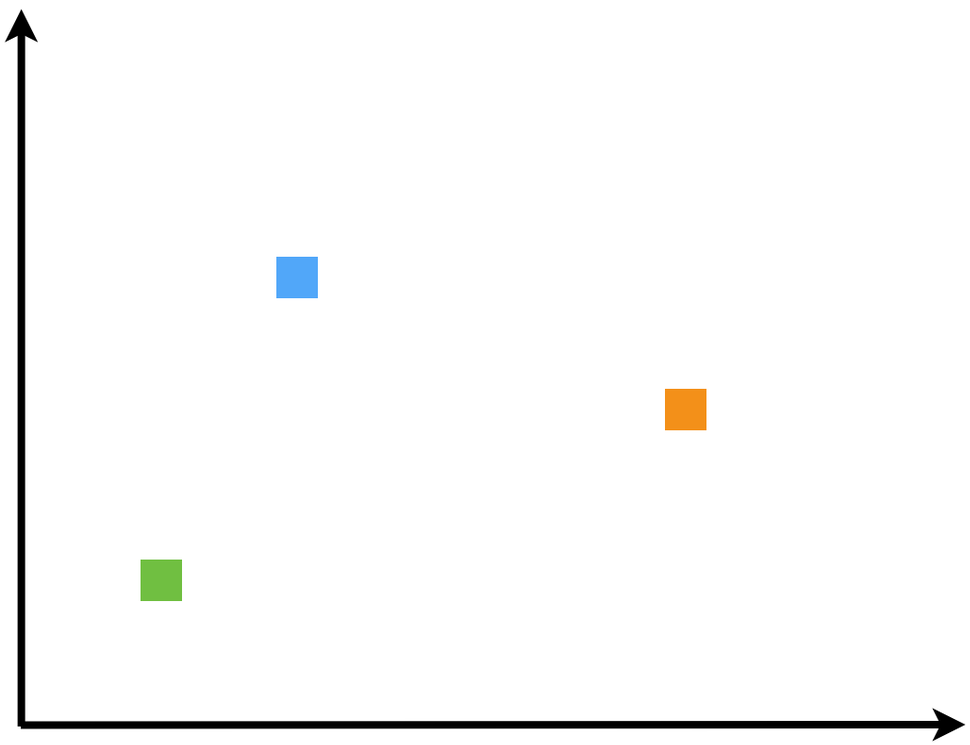

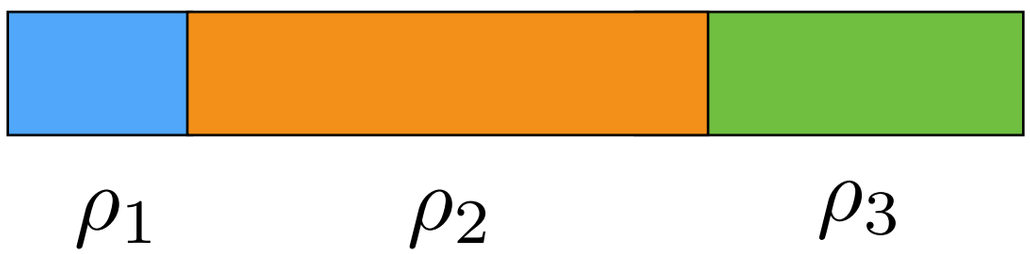

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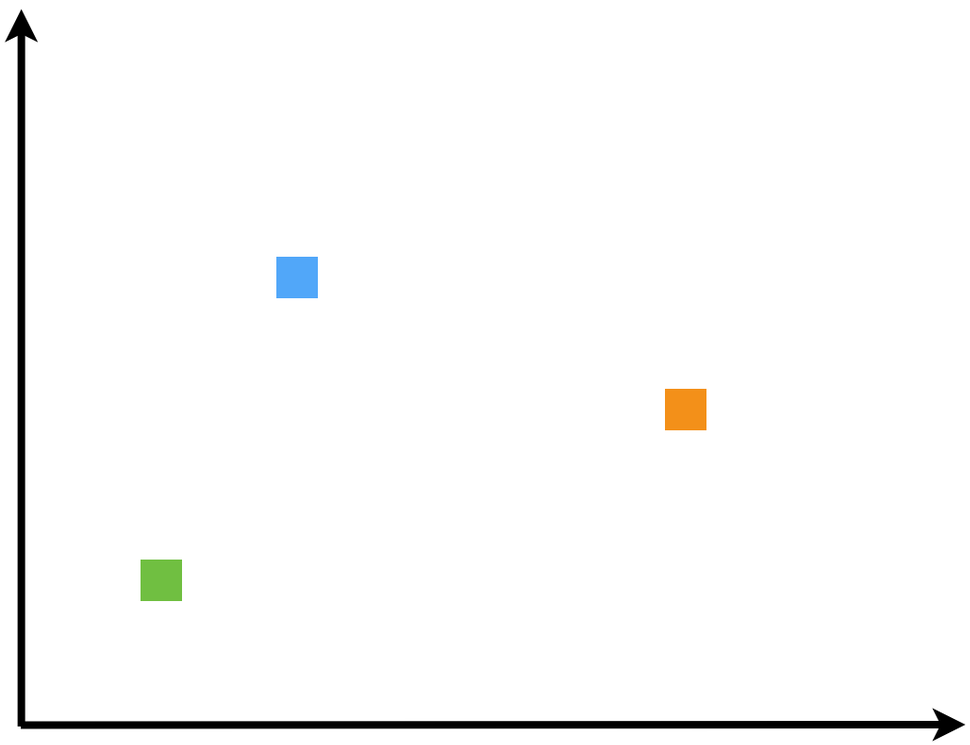

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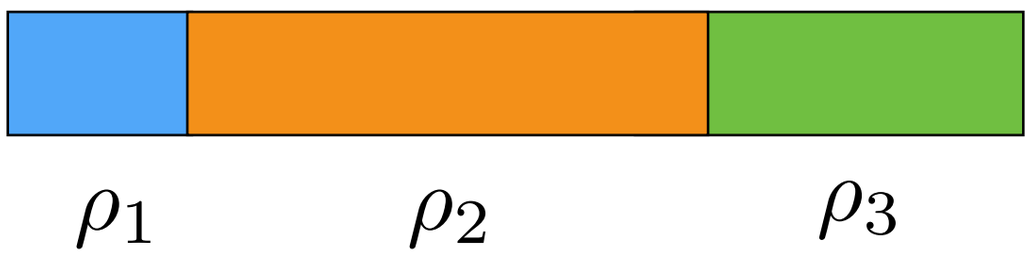

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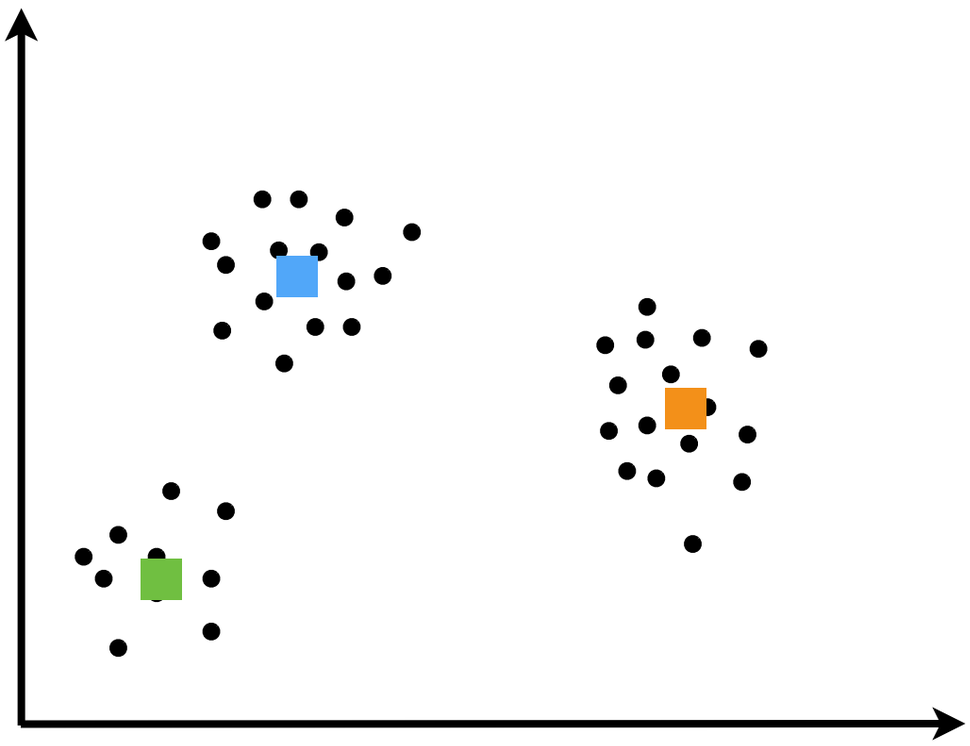

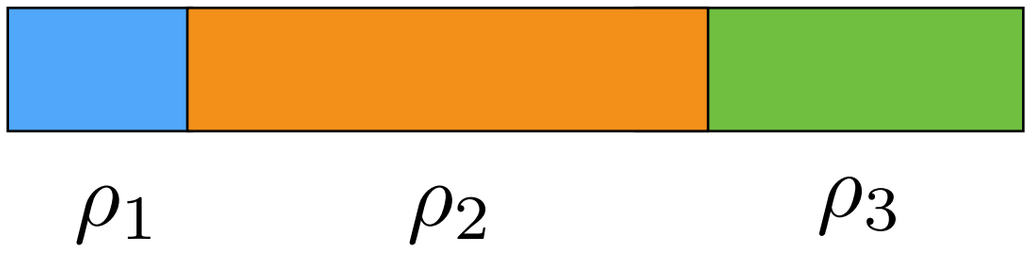

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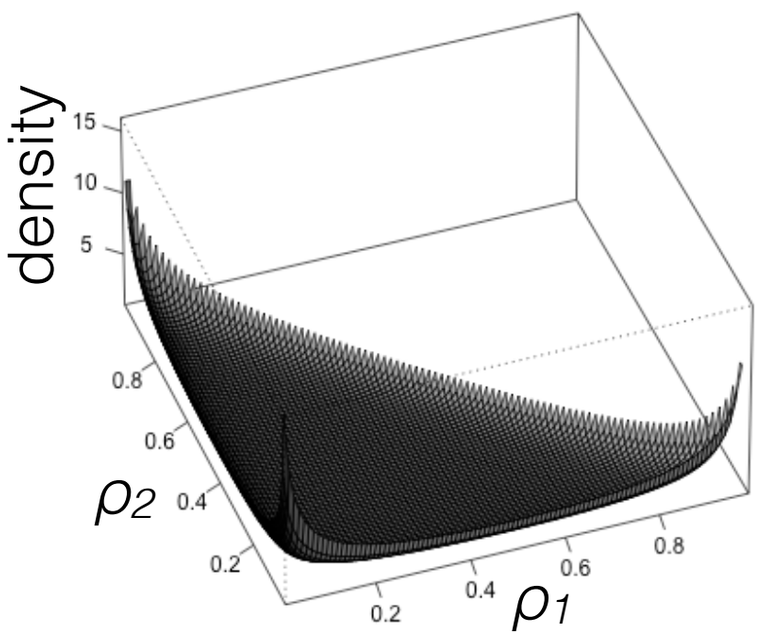

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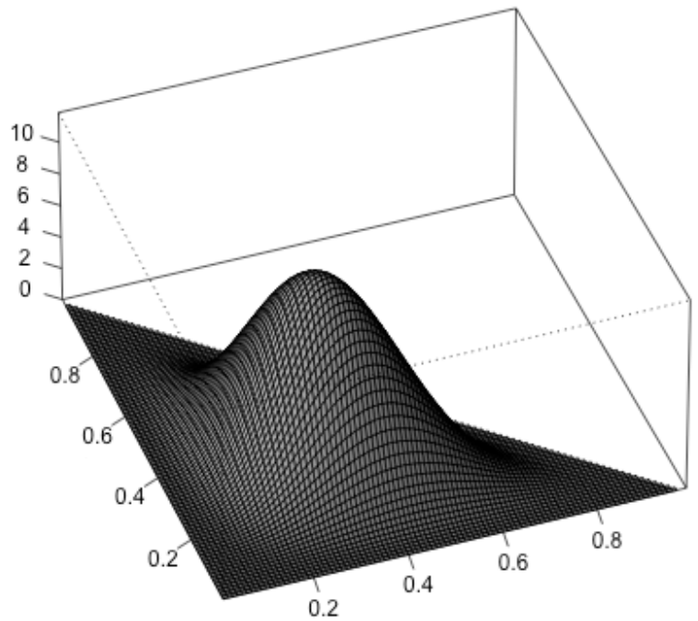

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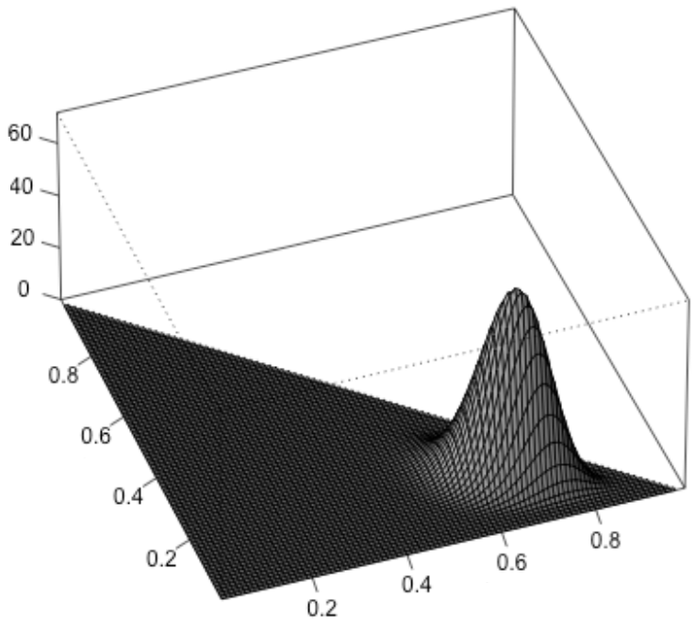

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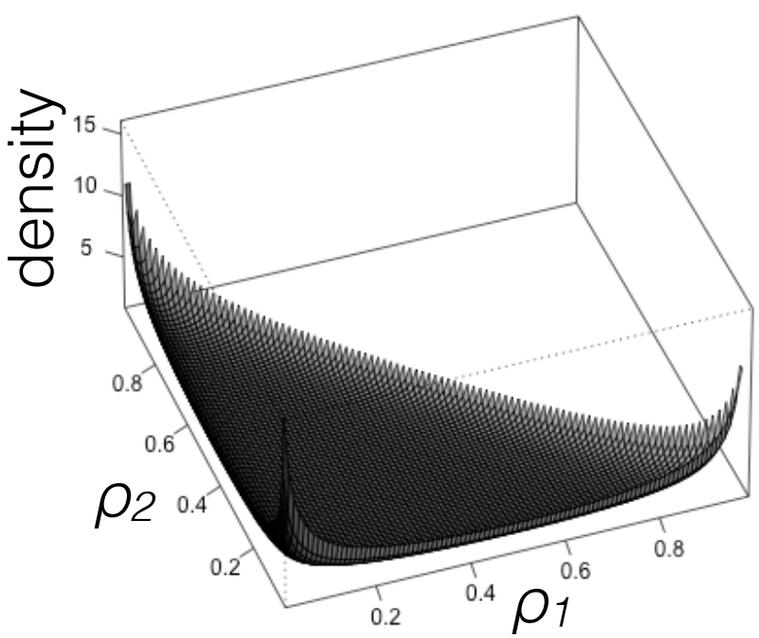

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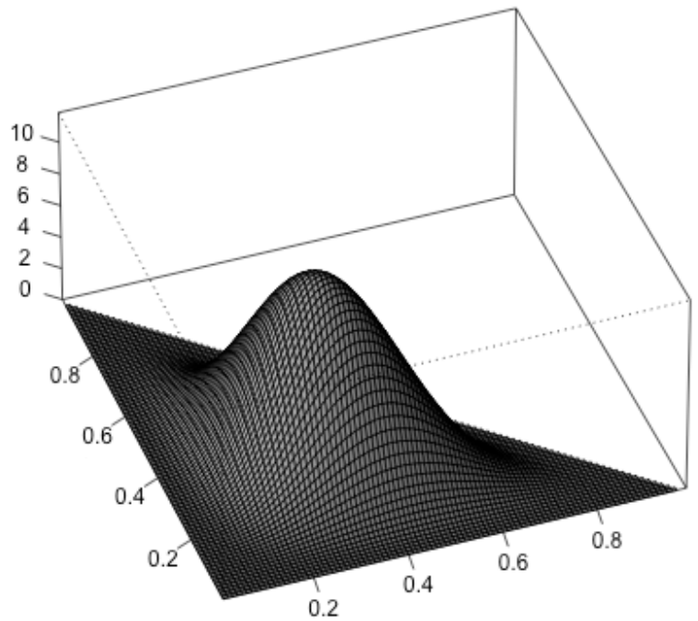

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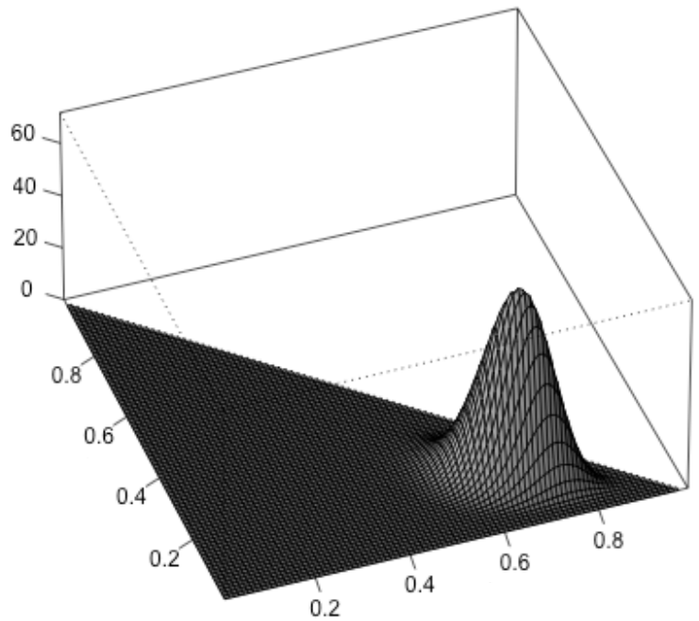

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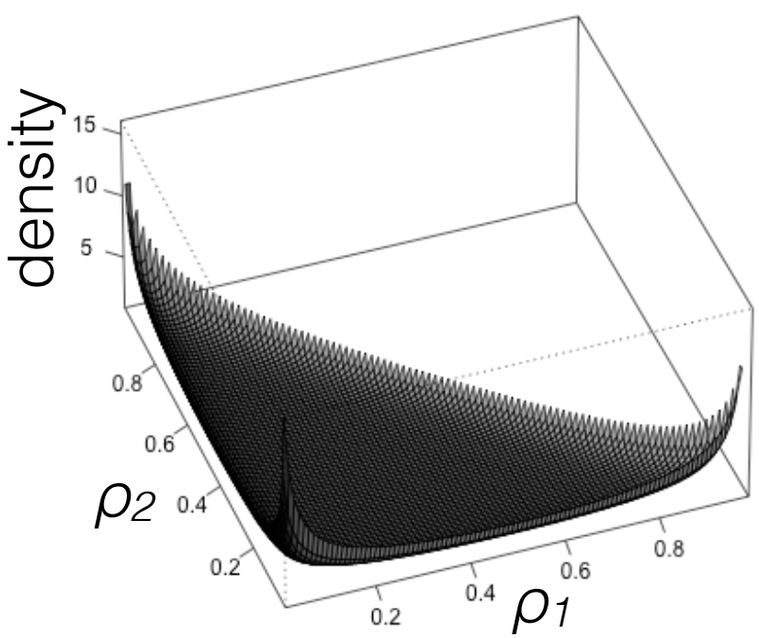

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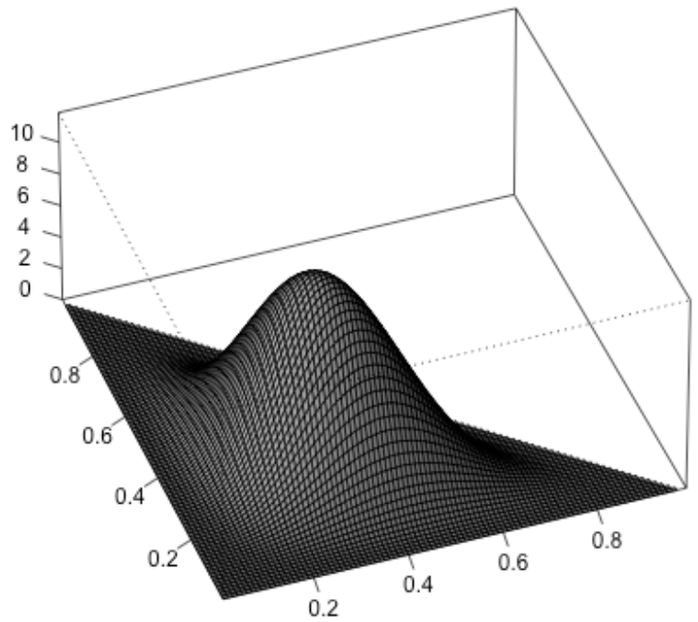

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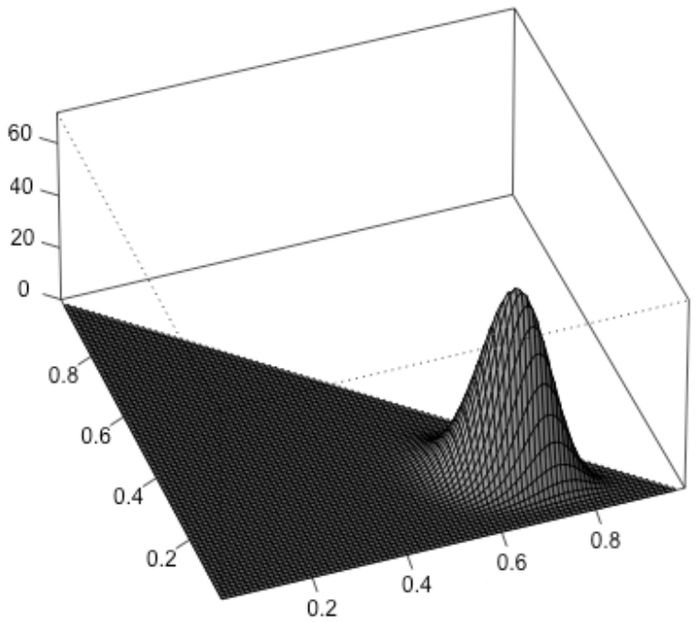

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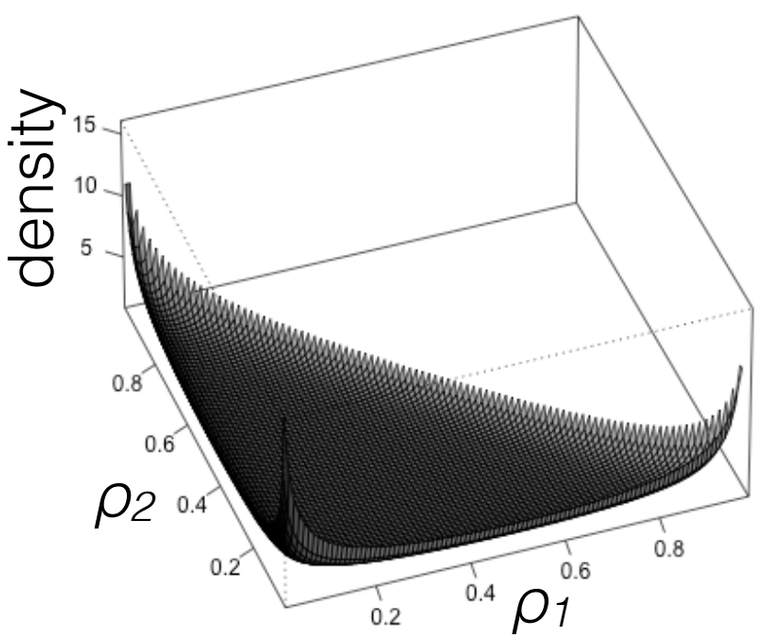

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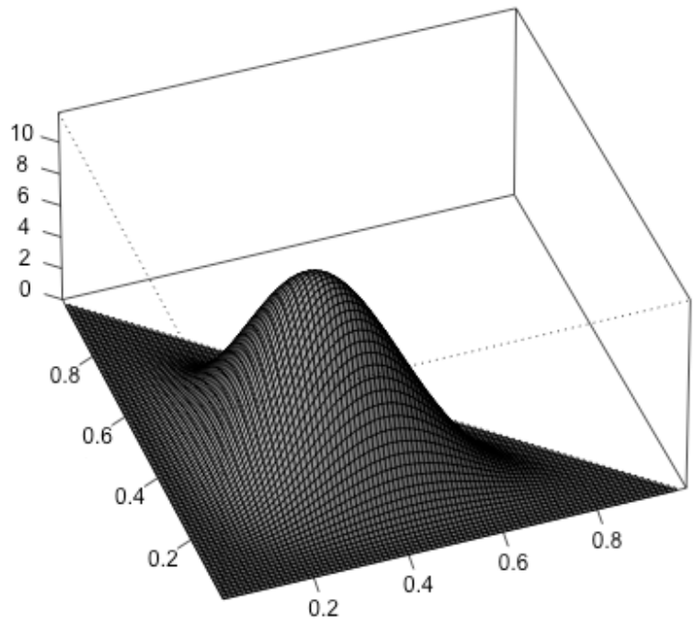

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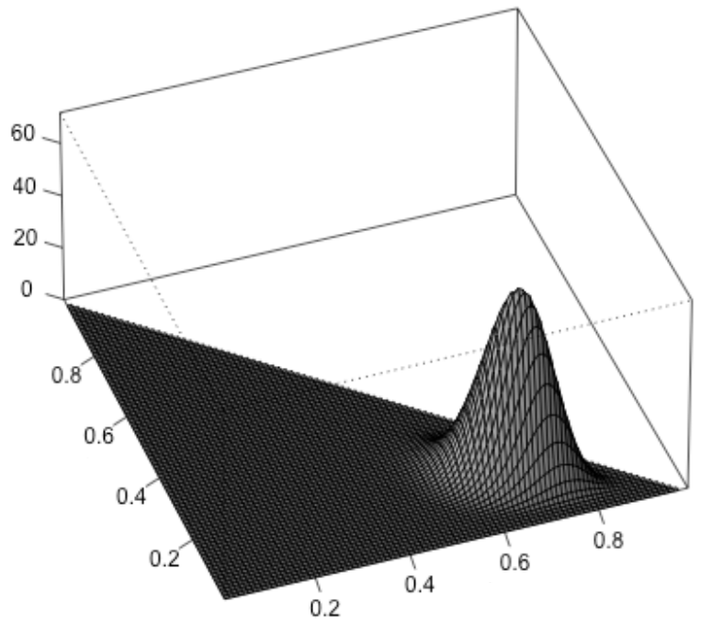

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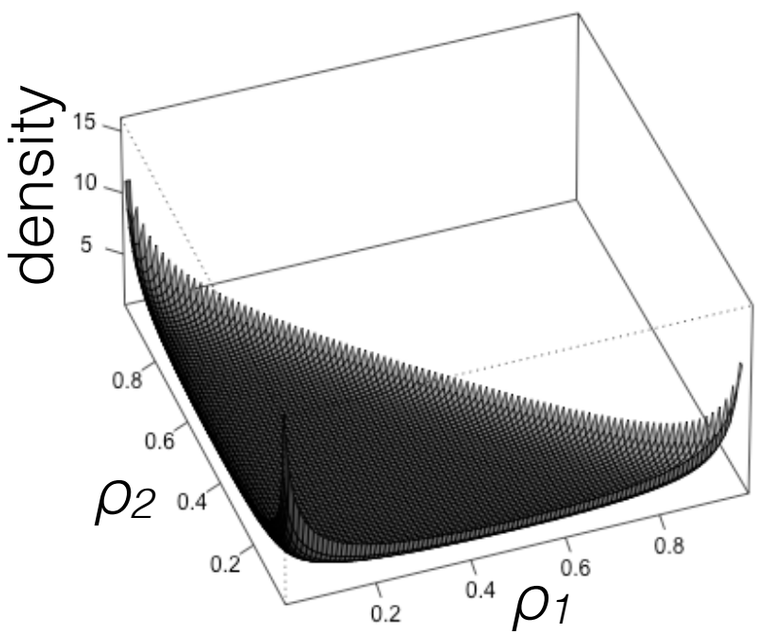

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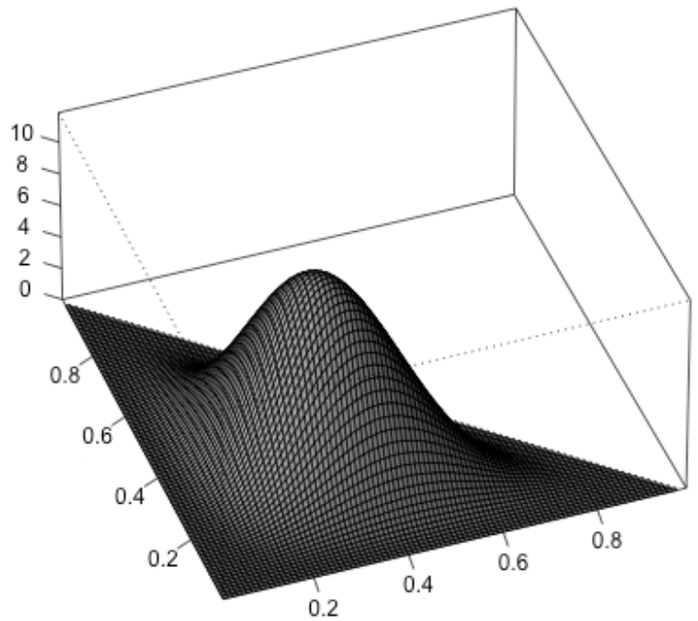

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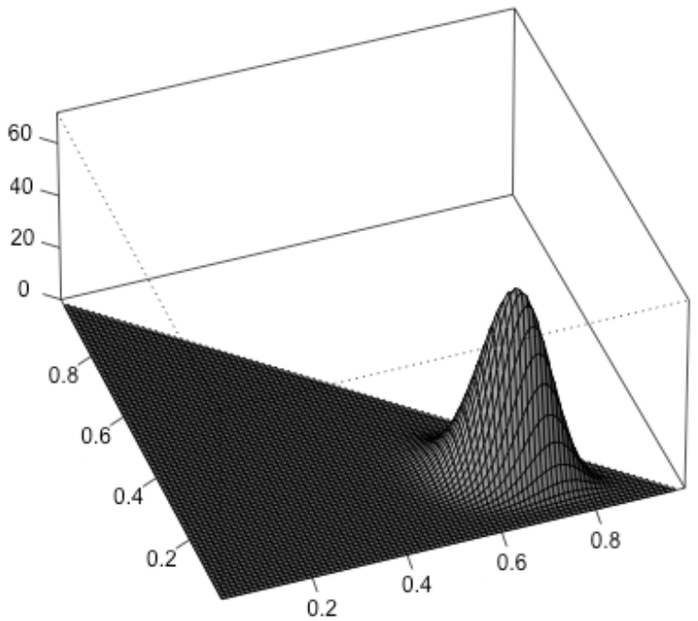

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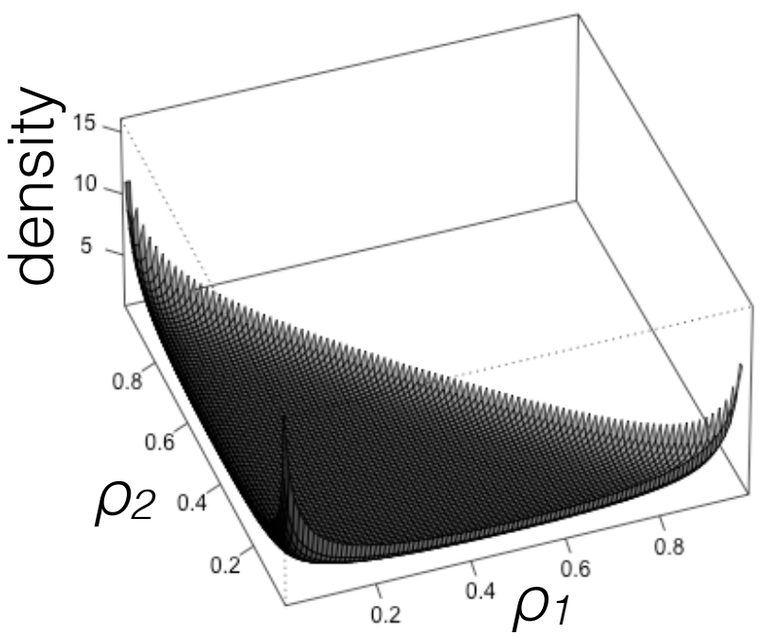

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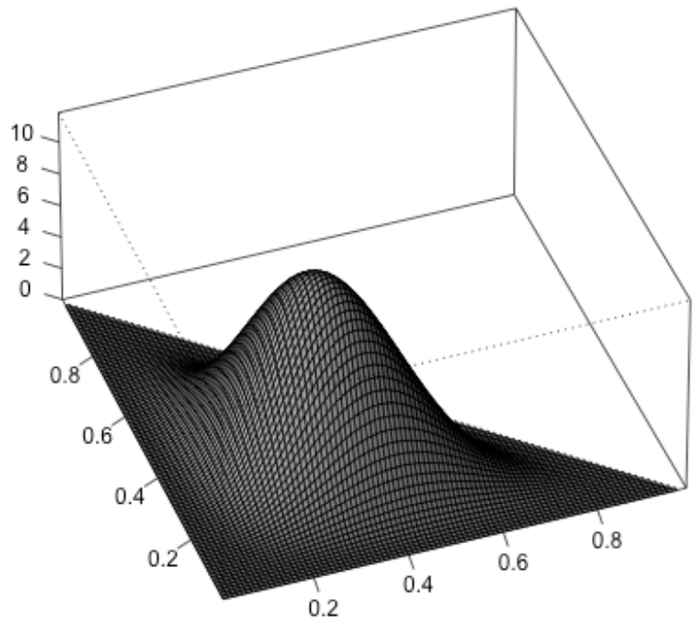
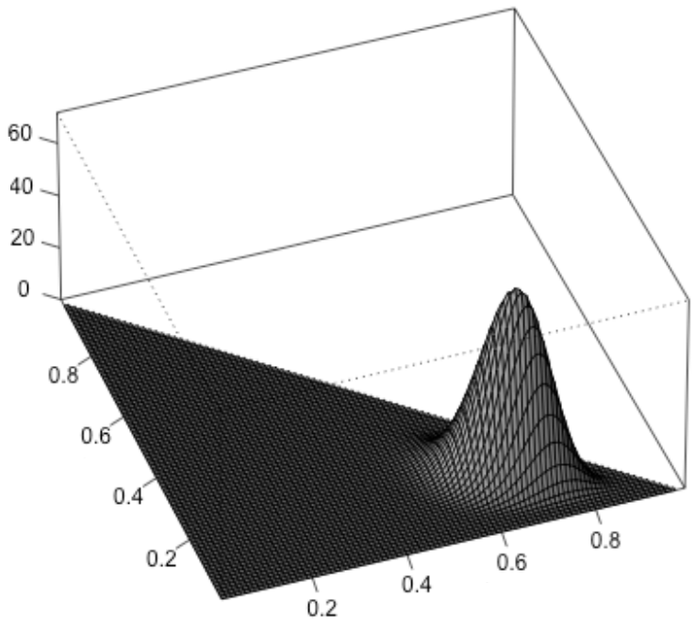


a = (40,10,10)



- What happens? $a = a_k = 1$ $a = a_k \to 0$ $a = a_k \to \infty$
- Dirichlet is conjugate to Categorical [demo]

Dirichlet distribution review

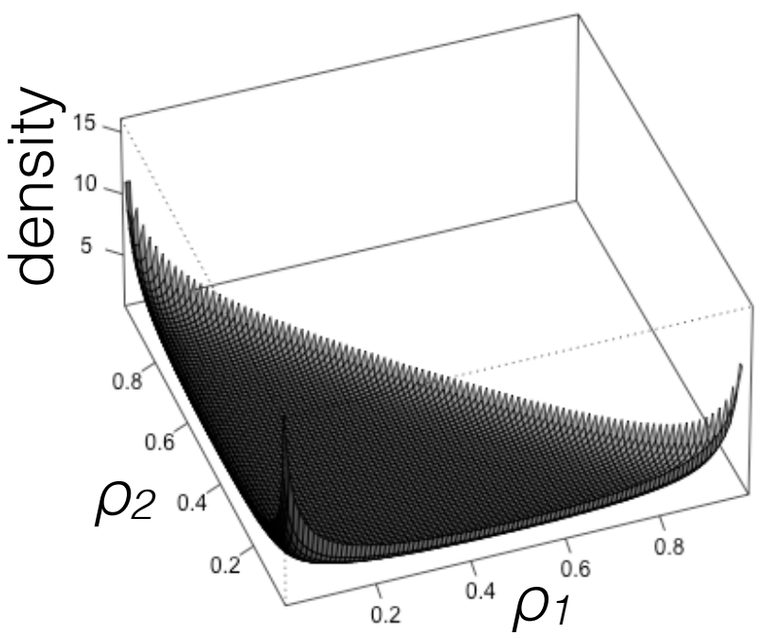
$$\text{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^K a_k)}{\prod_{k=1}^K \Gamma(a_k)} \prod_{k=1}^K \rho_k^{a_k-1}$$

$a_k > 0$

$\rho_k \in (0,1)$

$\sum_k \rho_k = 1$

a = (0.5,0.5,0.5)



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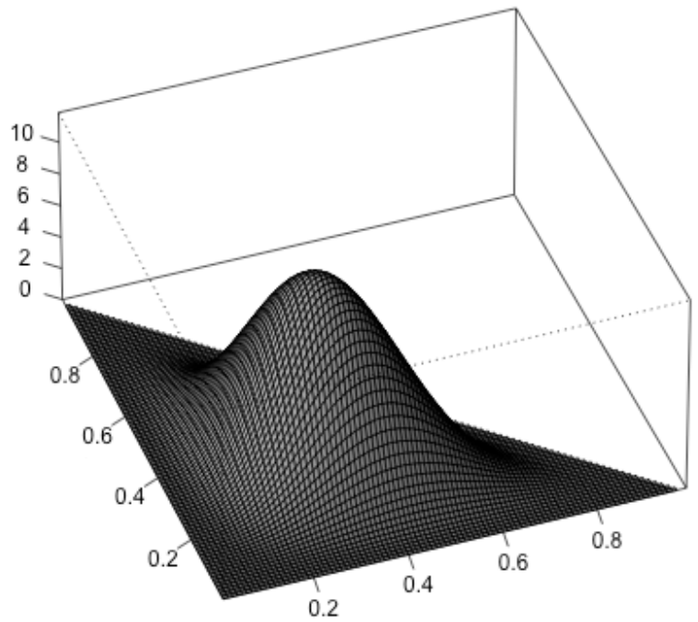

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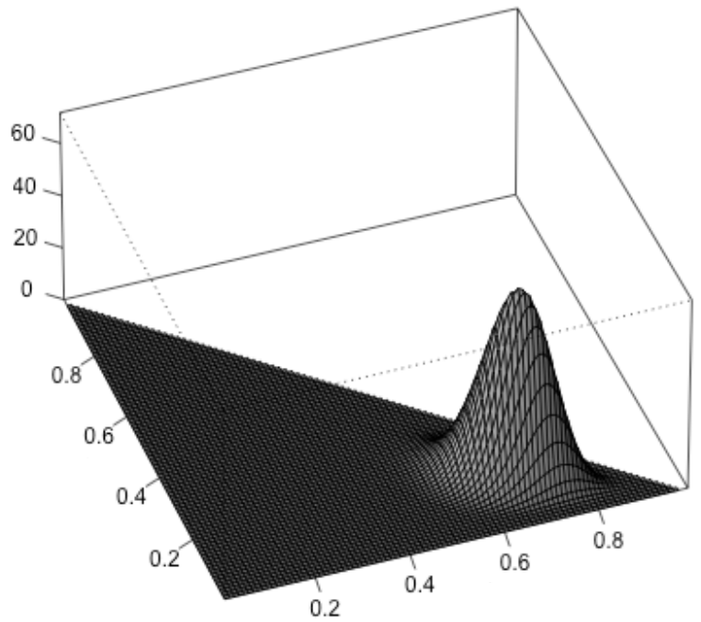

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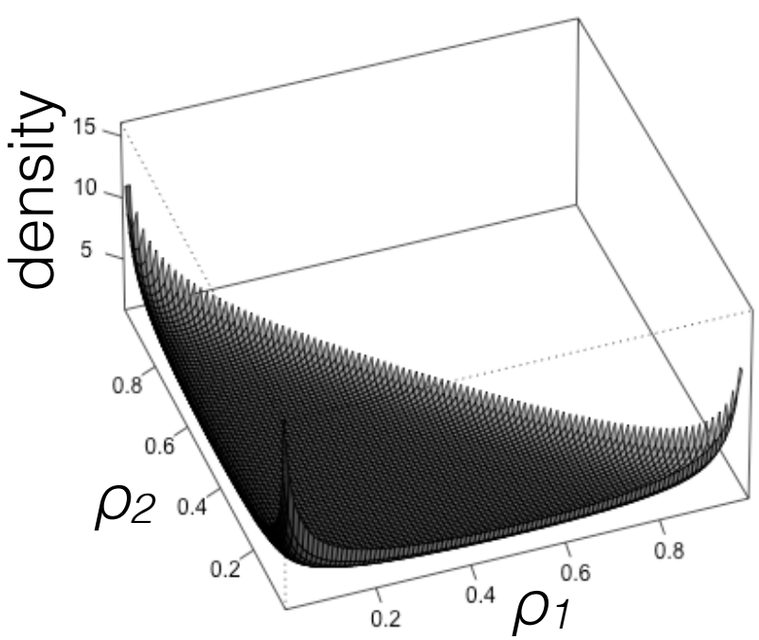

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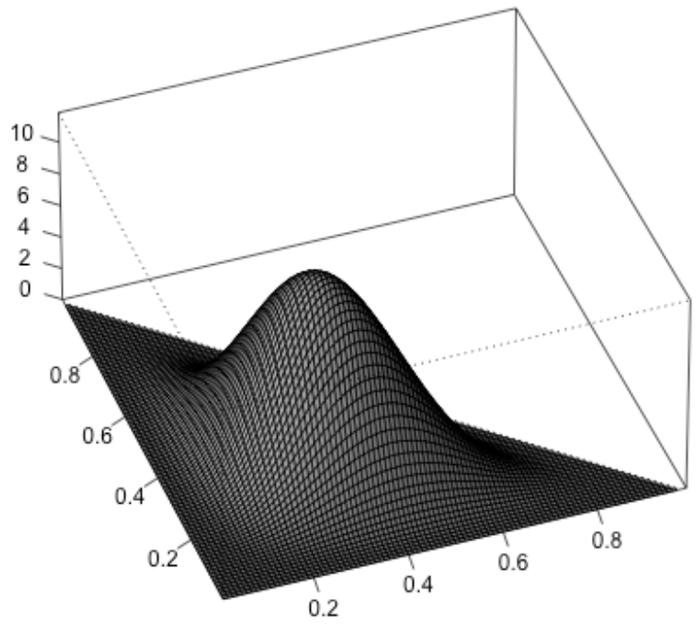

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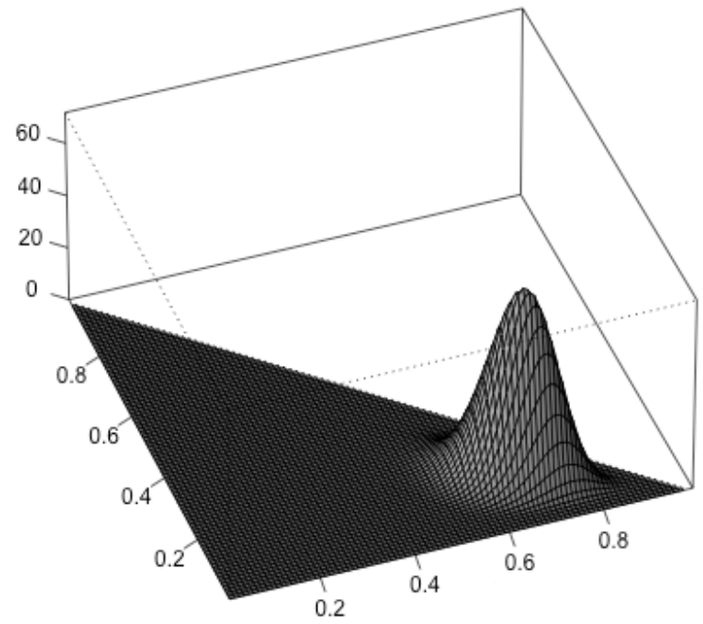

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What if $K > N$?

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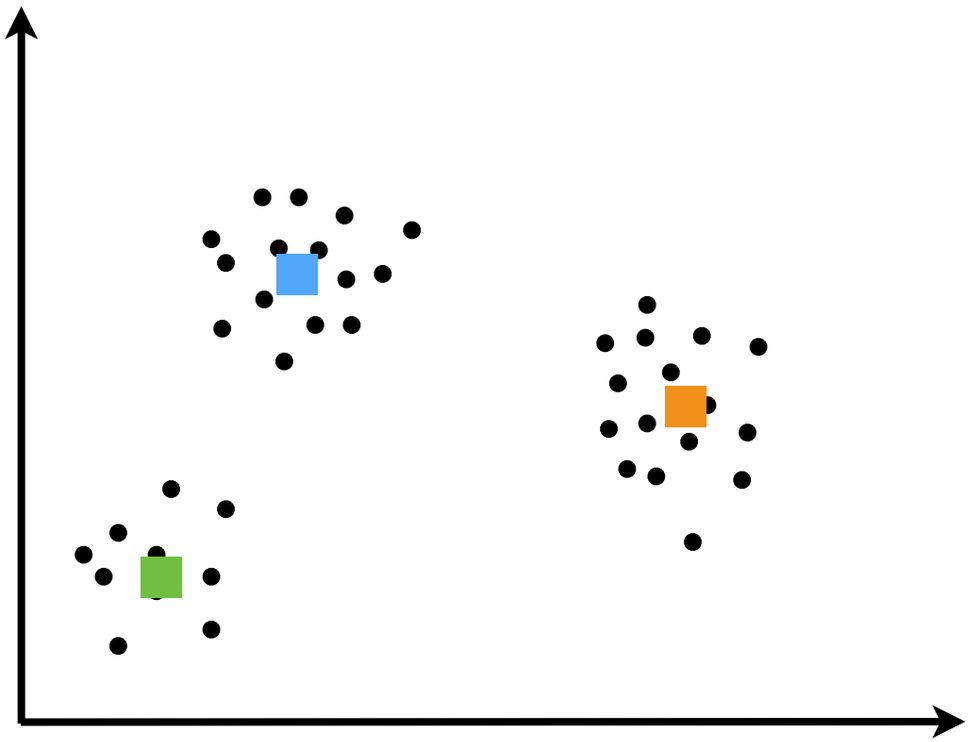

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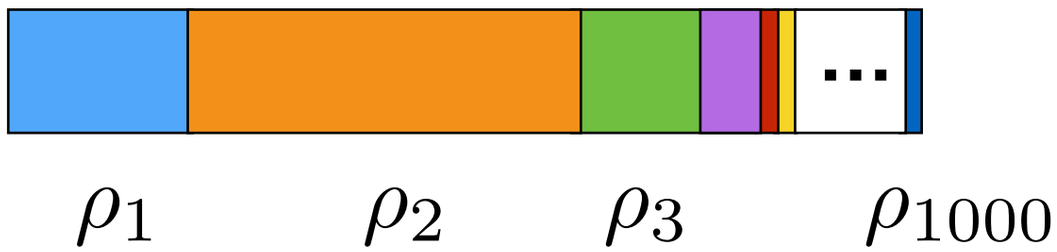

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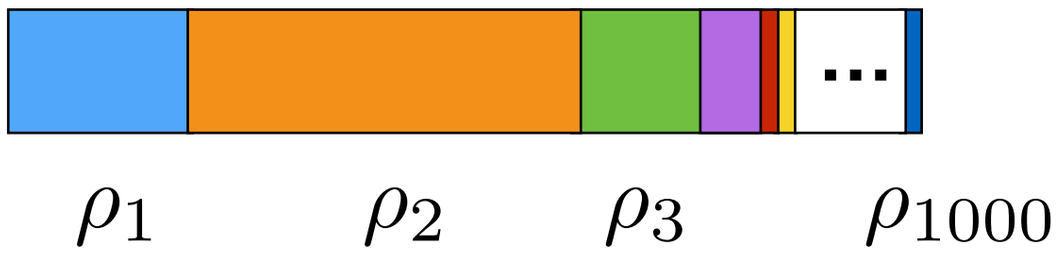
- e.g. species sampling, topic modeling, groups on a social network, etc.



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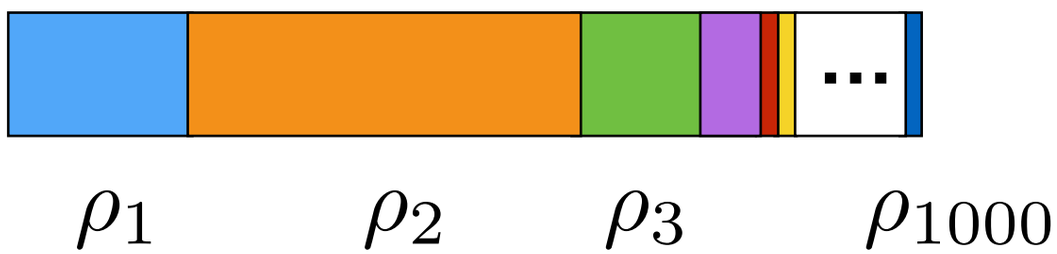

- Components: number of latent groups

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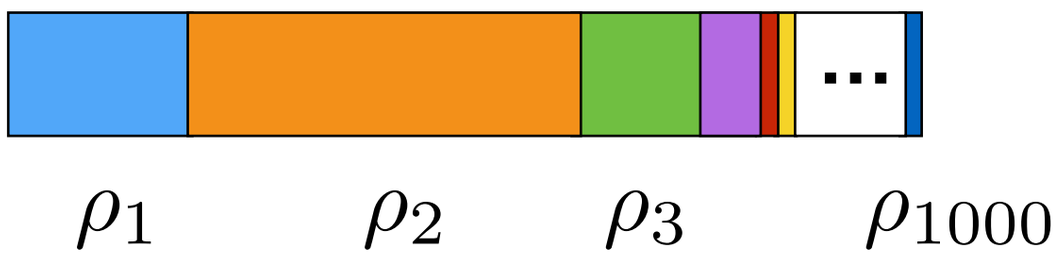

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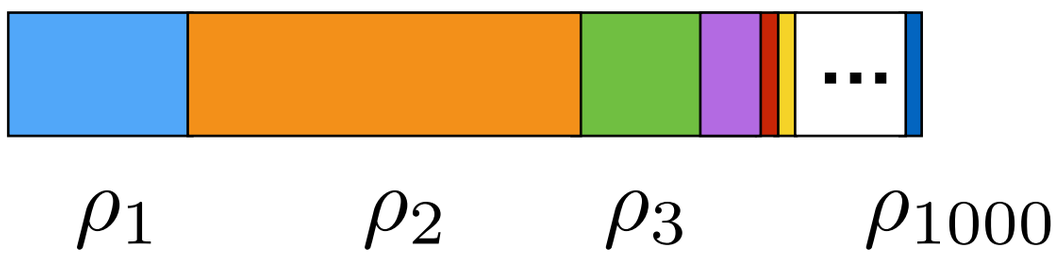

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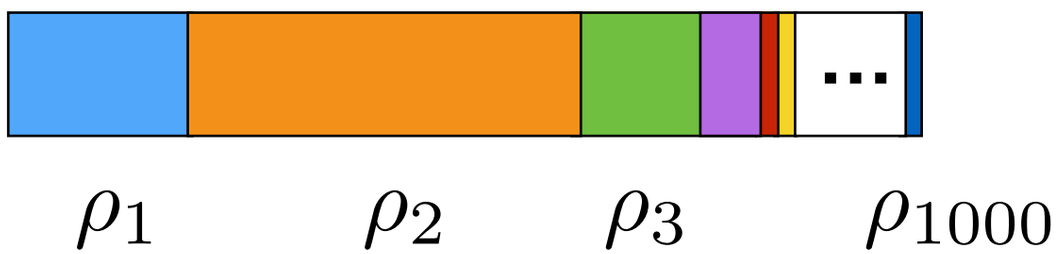

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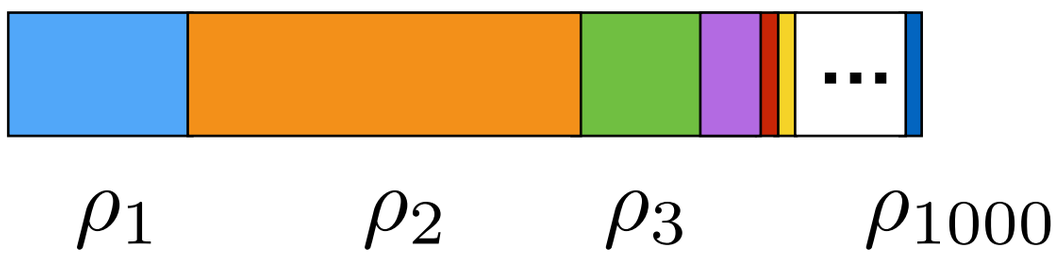

- Components: number of latent groups
- Clusters: number of components represented in the data
- [demo 1, demo 2]
- Number of clusters for N data points is $\leq K$ and random
- Number of clusters grows with N

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- "Stick breaking"

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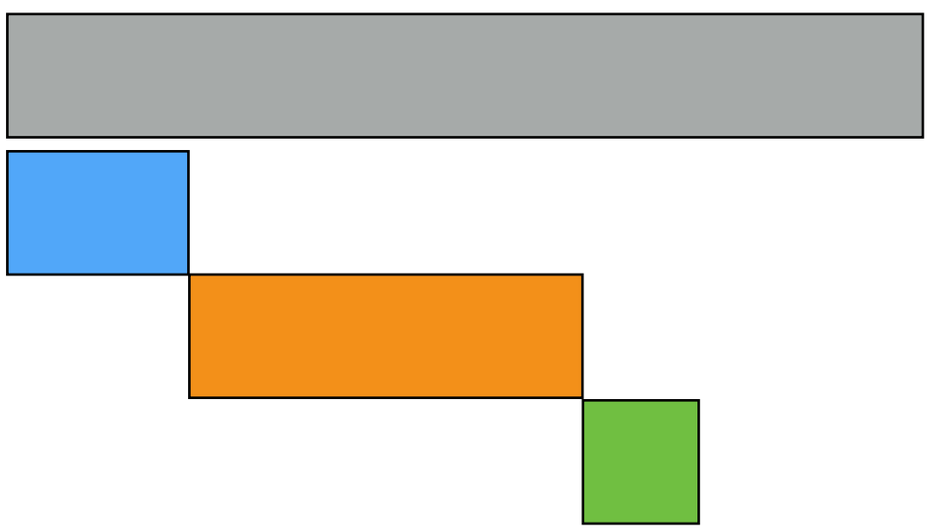

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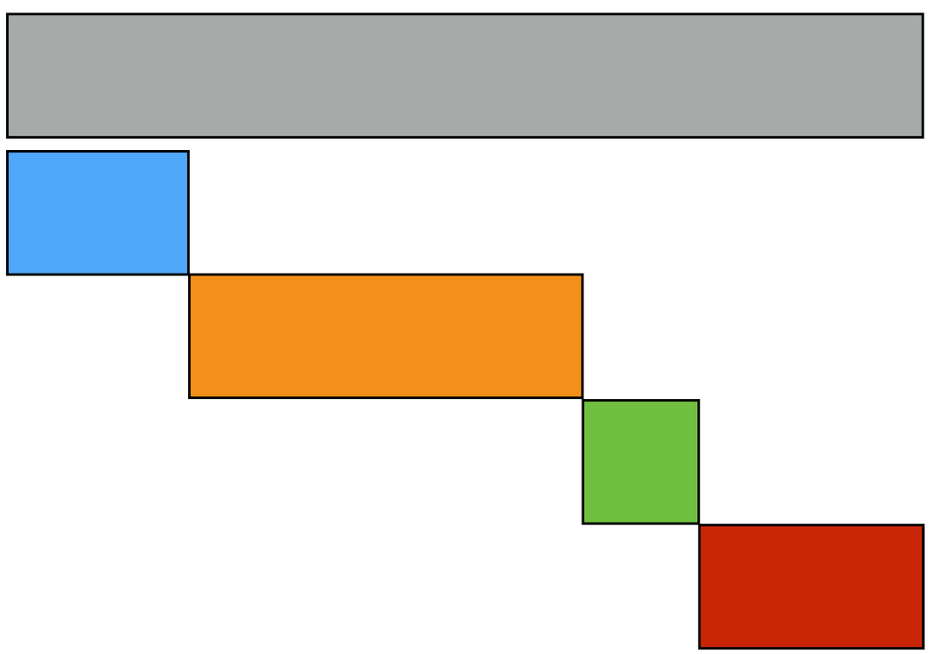

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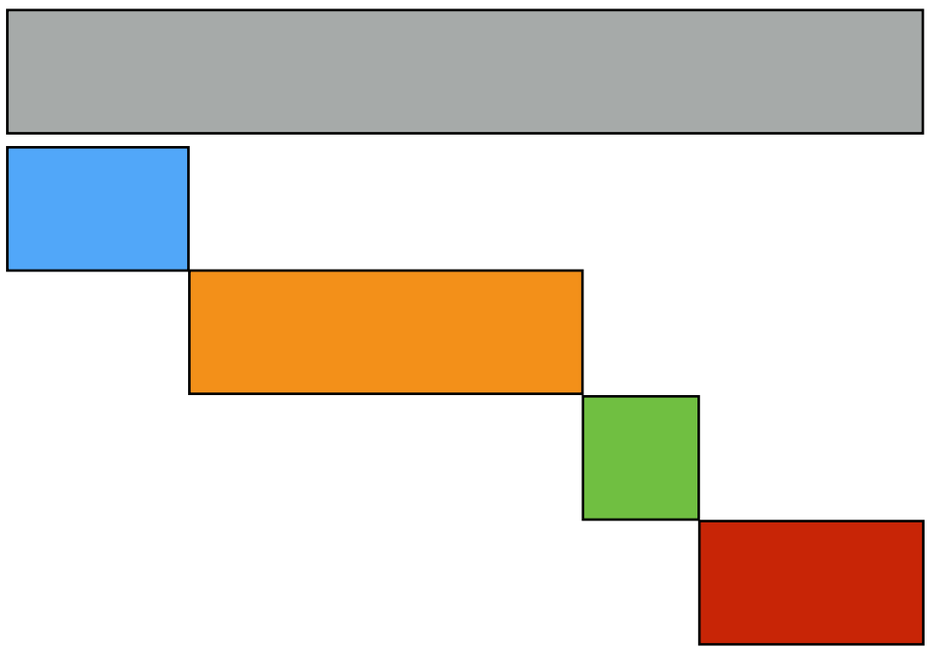

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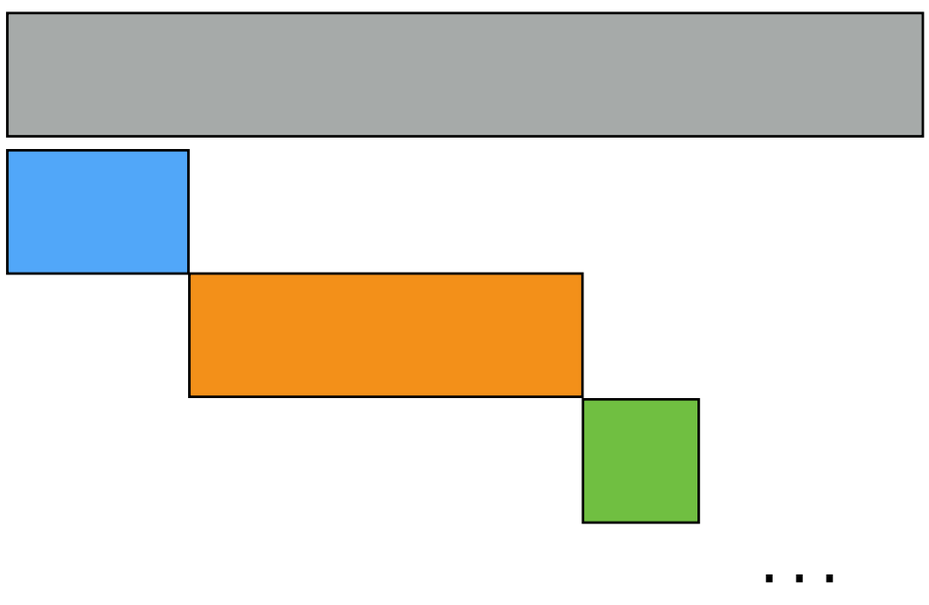

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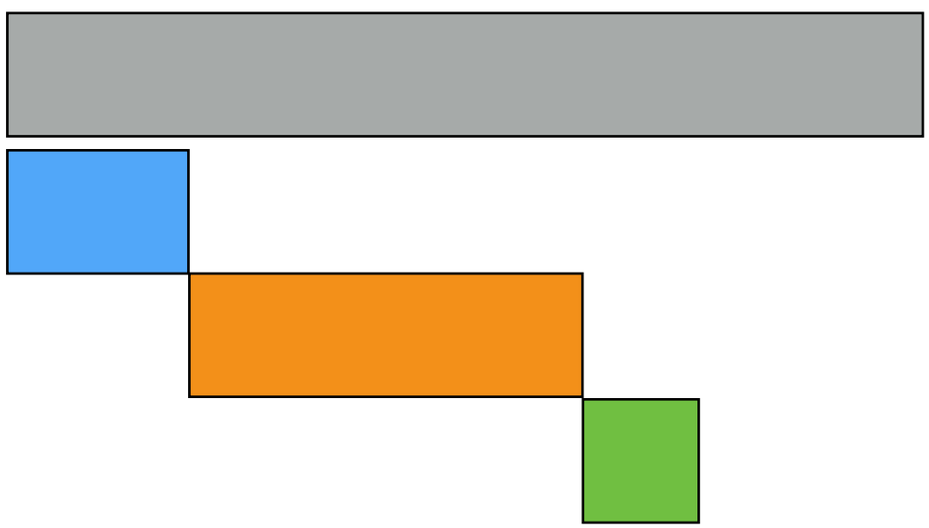

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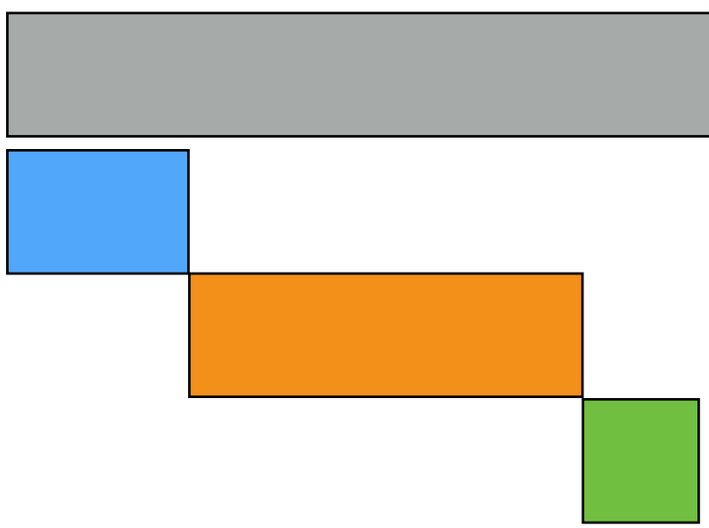

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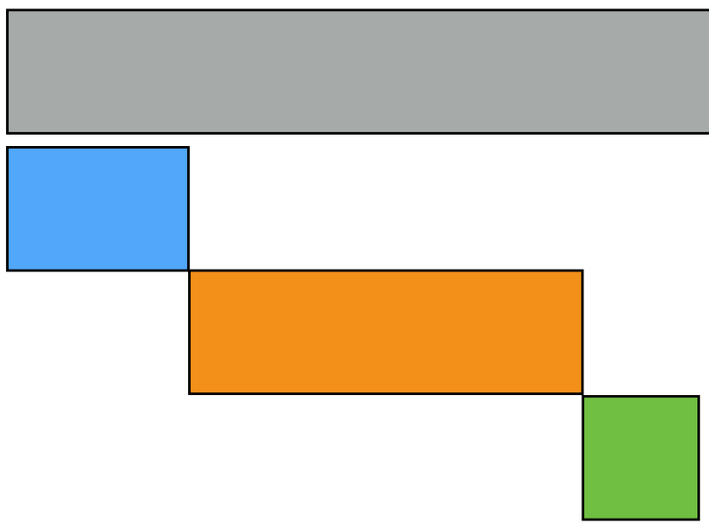

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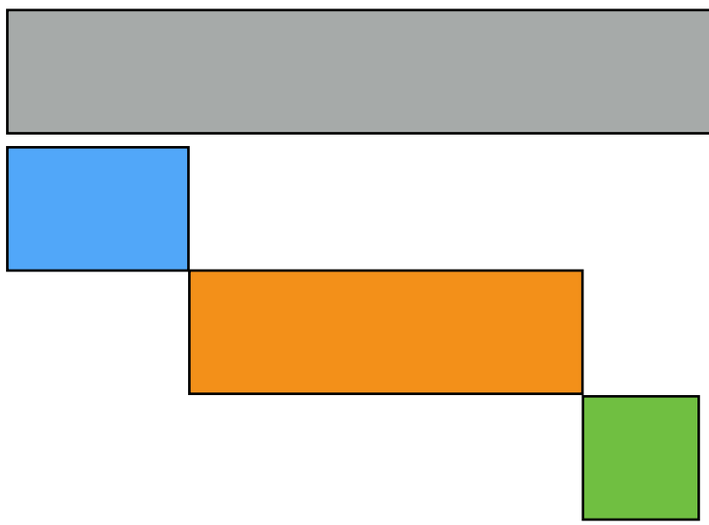

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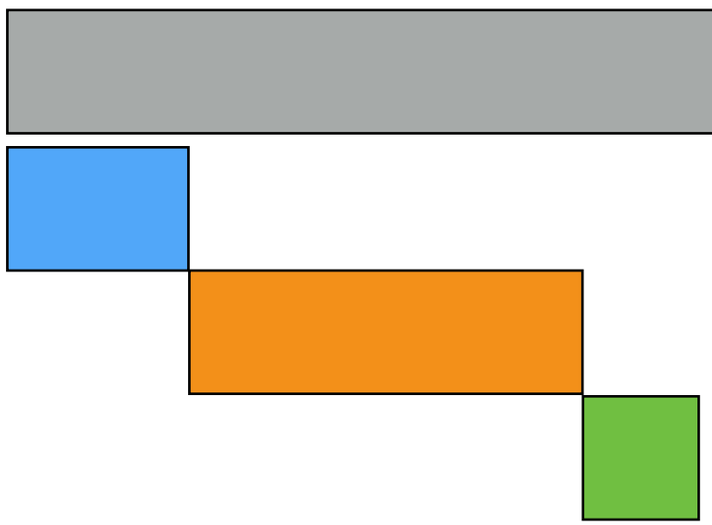

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[van der Vaart, Ghosal 2017]

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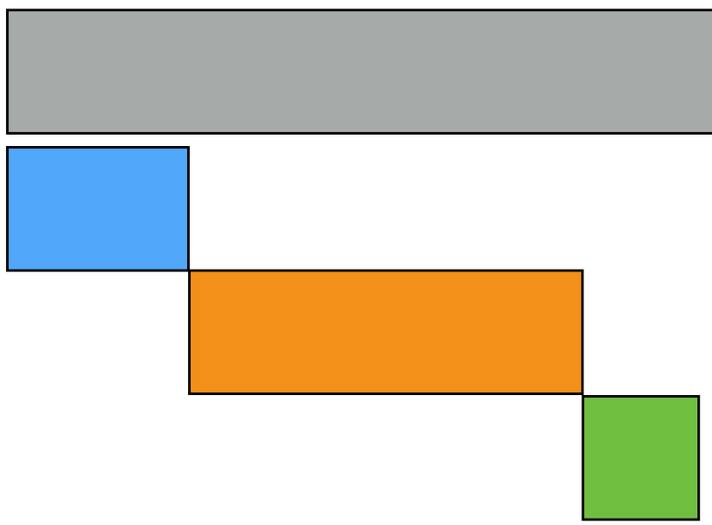

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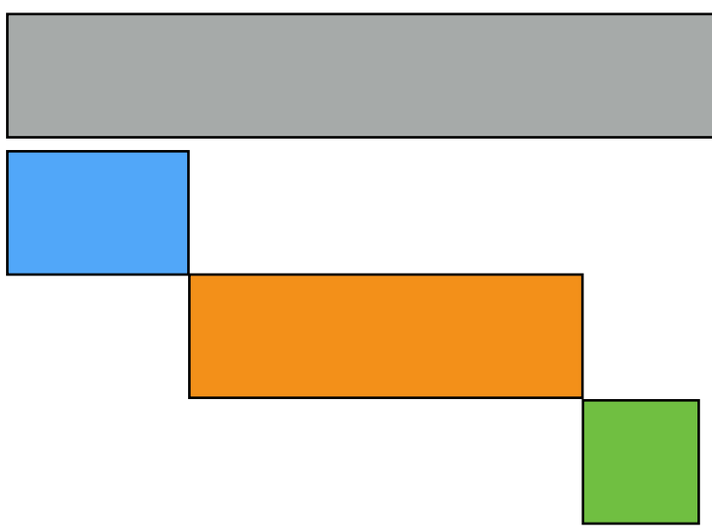

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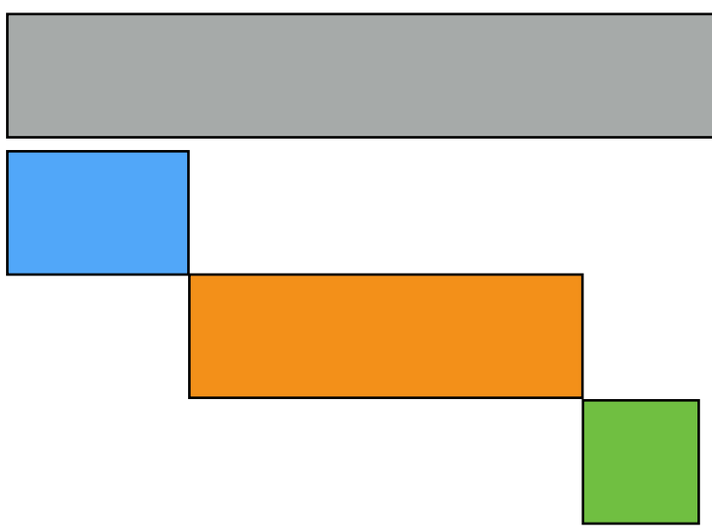

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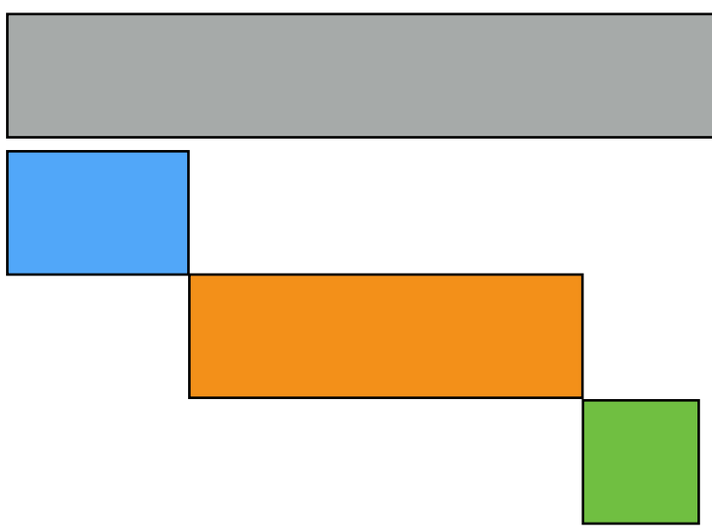

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[demo]

[McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; van der Vaart, Ghosal 2017]

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- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

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  - What does an infinite/growing number of parameters really mean (in NPBayes)?
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References

A full reference list is provided at the end of the “Part III” slides.