# Edge-exchangeable graphs and sparsity

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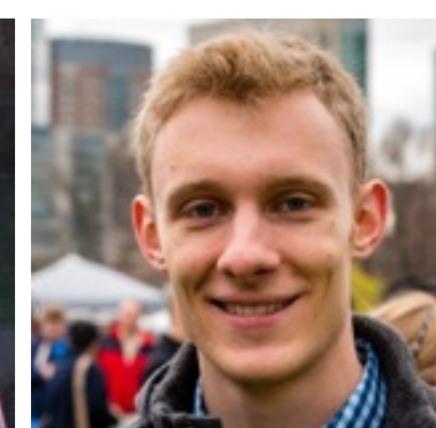
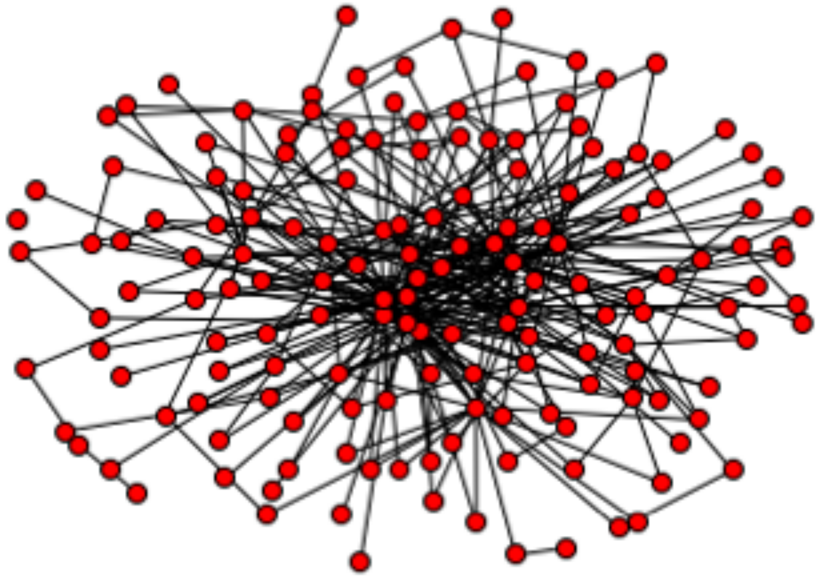
ITT Career Development
Assistant Professor,
MIT

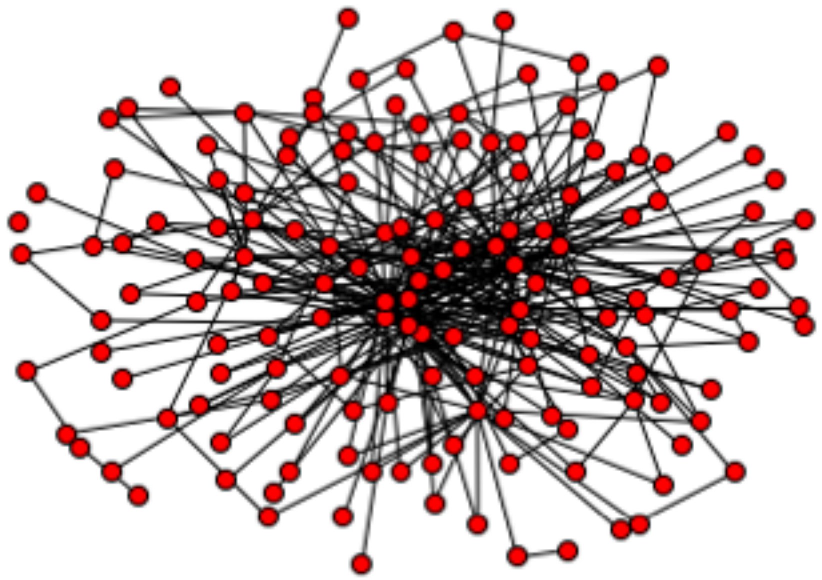
Joint work with: Diana Cai, Trevor Campbell







[Holland et al 1983; Kemp et al 2006; Xu et al 2007; Airoldi et al 2008; Lloyd et al 2012]



social: *Facebook, Twitter, email*
biological: *ecological, protein, gene*
transportation: *roads, railways*



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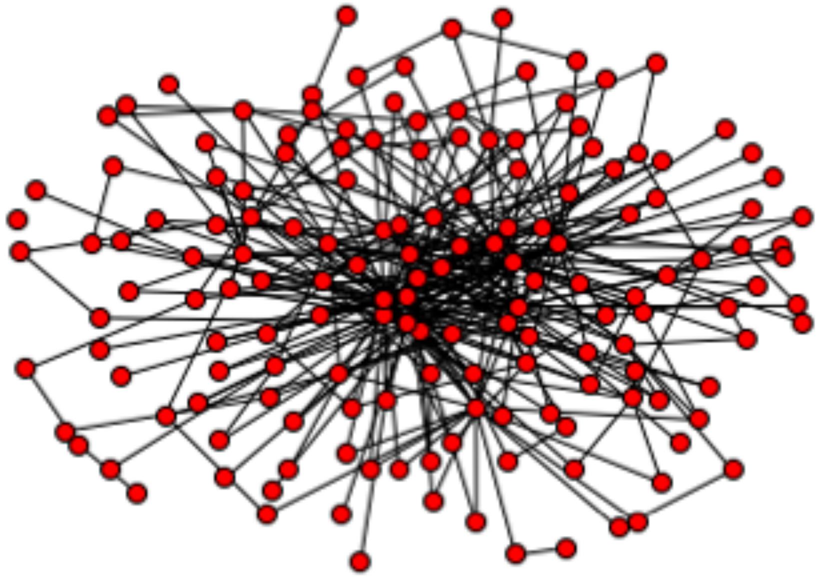
# Probabilistic models for graphs

$p($  $)$

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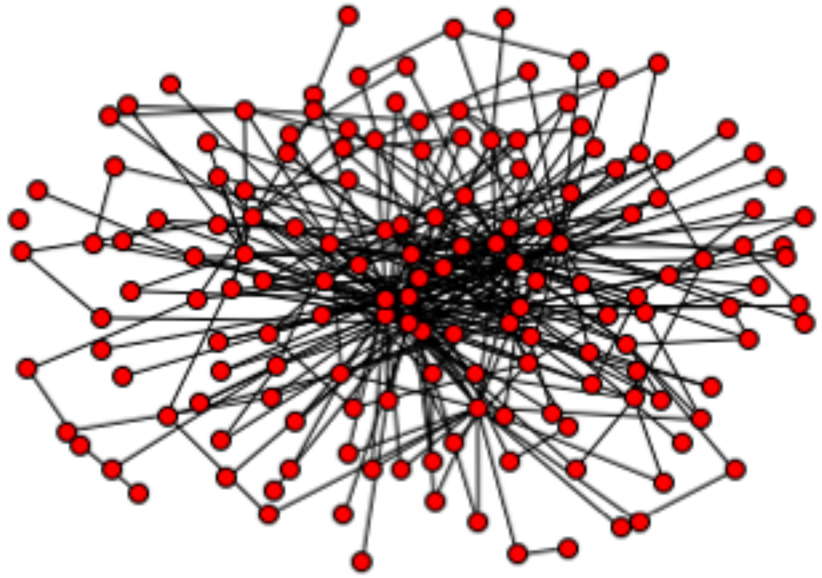
$p($  $)$

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- Rich relationships, coherent uncertainties, prior info



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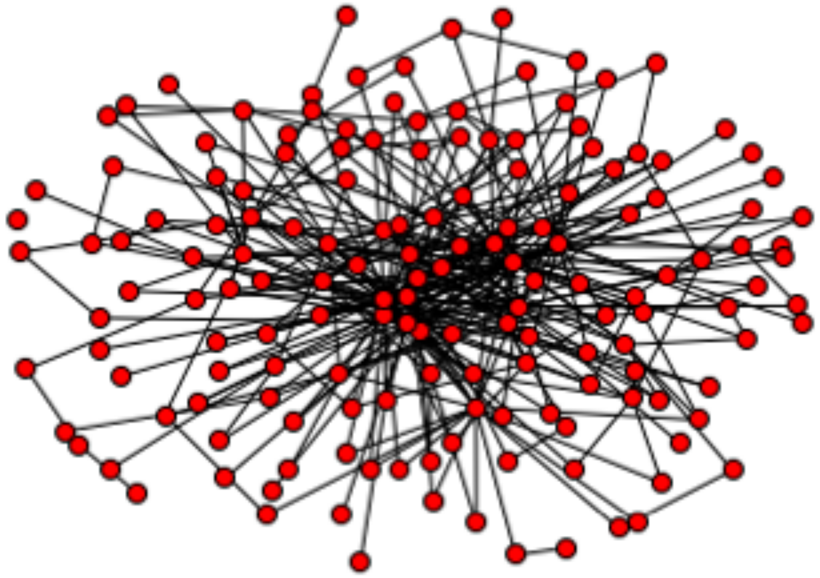
$p($  $)$

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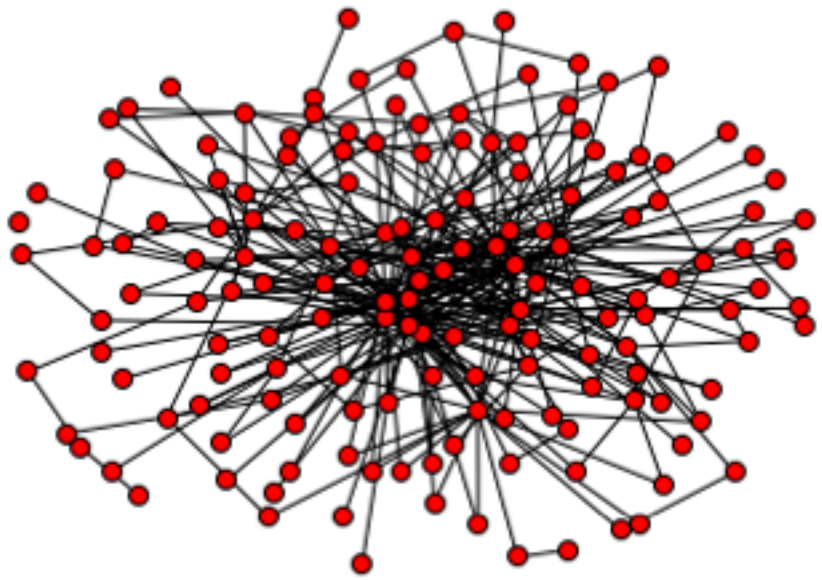
$p($  $)$

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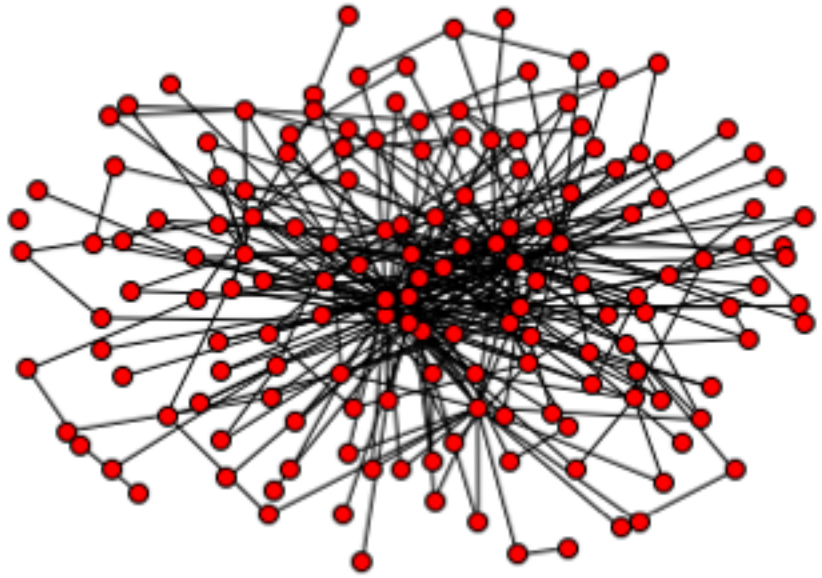
$p($  $)$

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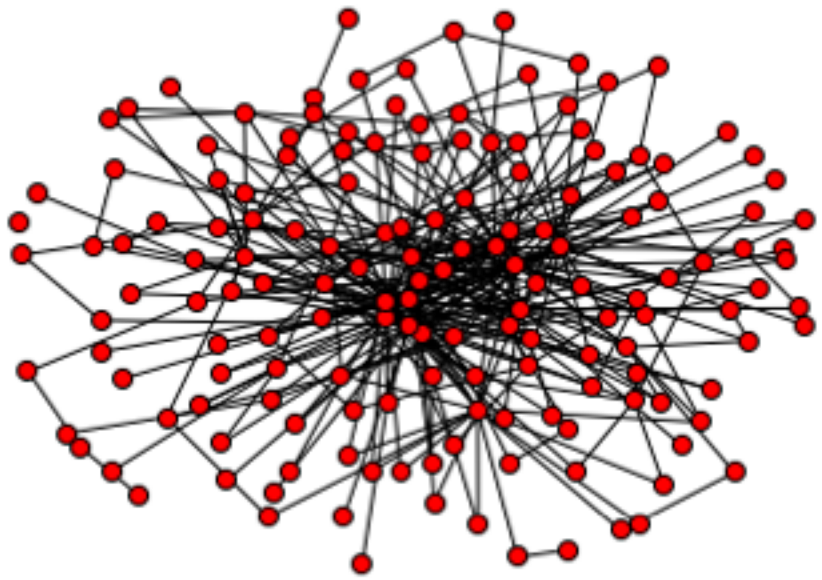
$p($  $)$

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- **Problem**: model misspecification, dense graphs



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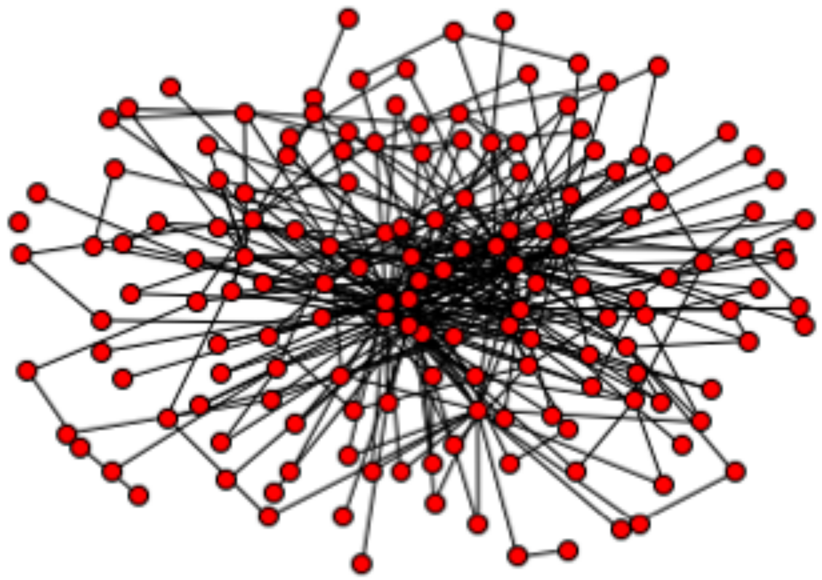
$p($  $)$

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- **Solution**: a new framework for sparse graphs



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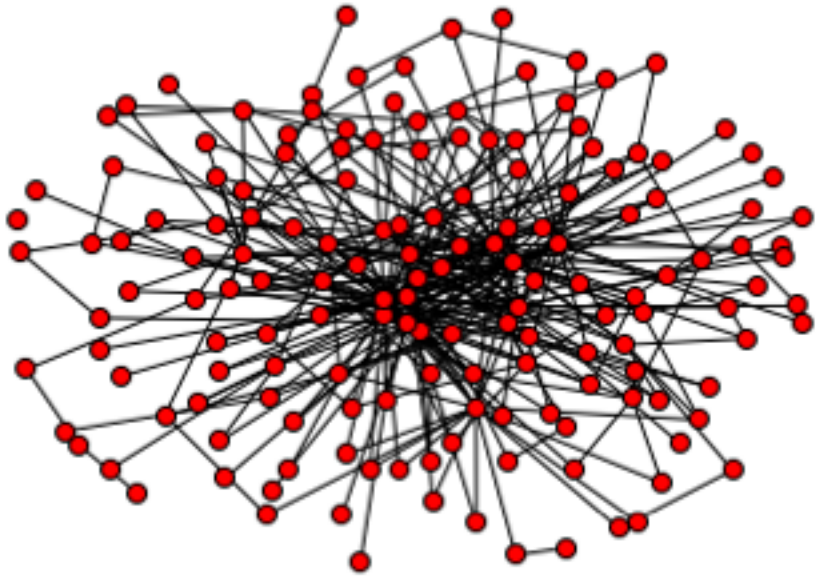
$p($  $)$

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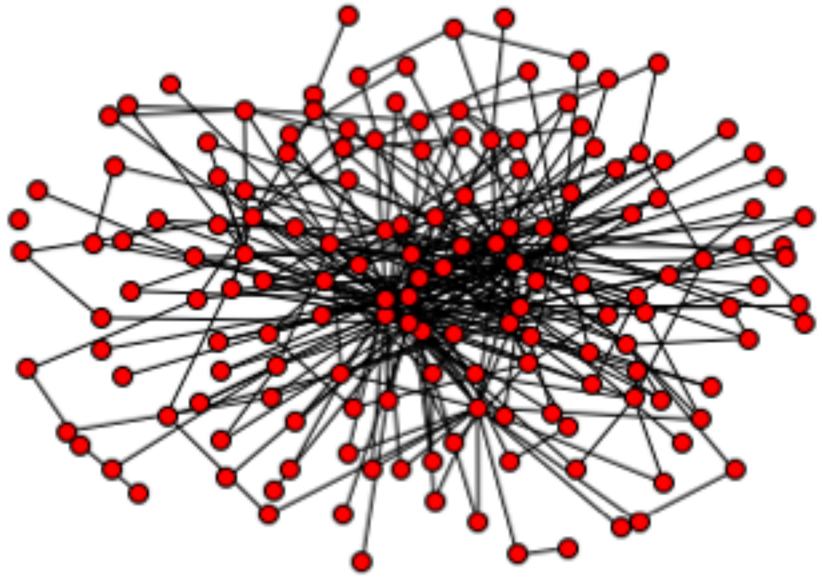

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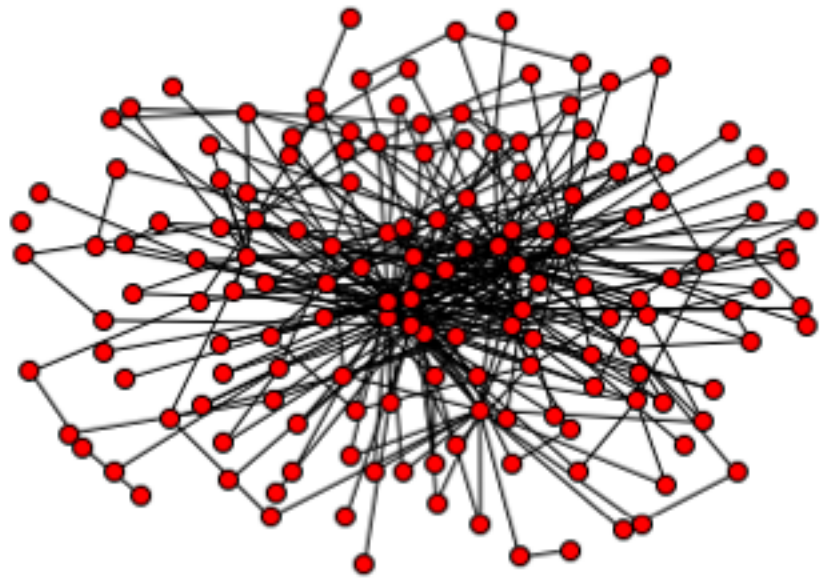
$p($  $)$

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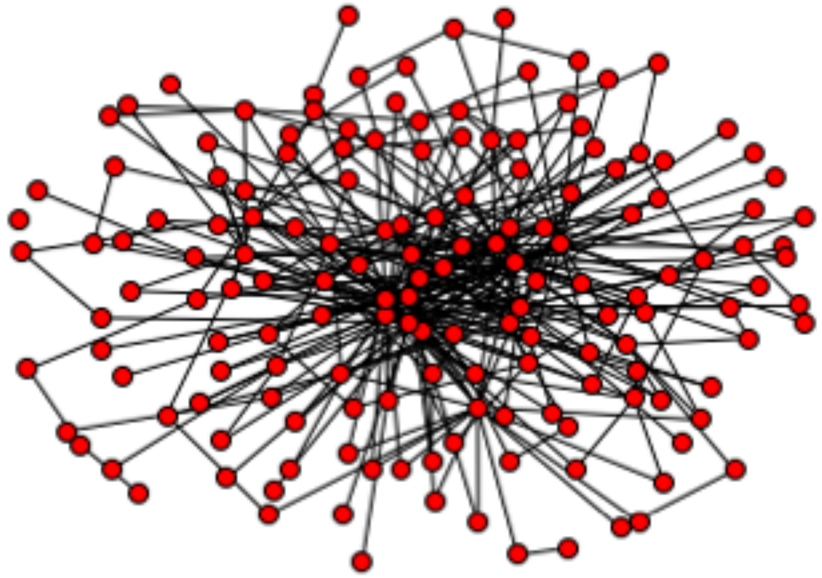
$p($  $)$

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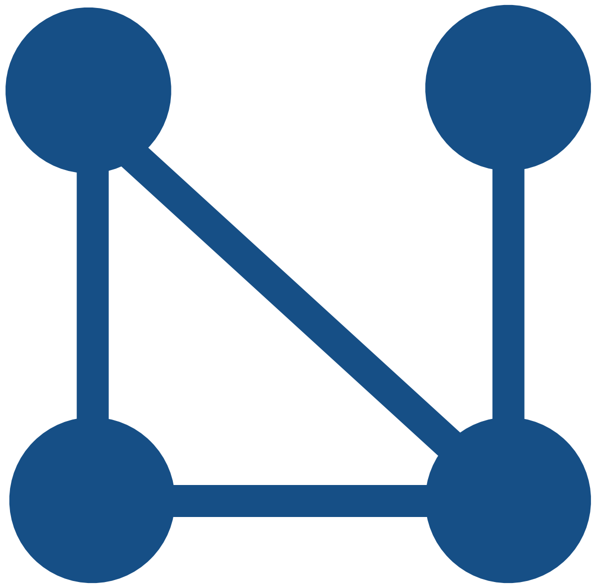
- Assume: Adding more data doesn't change distribution of earlier data (*projectivity*)
- **Problem**: model misspecification, dense graphs
- **Solution**: a new framework for sparse graphs
  - Concurrent & independent graphs work by Crane & Dempsey

[B, Cai 2015; Cai, B 2015a,b; Crane, Dempsey 2015a,b,16a,b; Cai, Campbell, B 2016; Campbell, Cai, B 2016; Williamson 2016]

# Sequence of graphs



$G$

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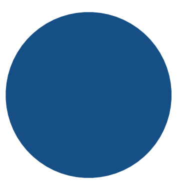
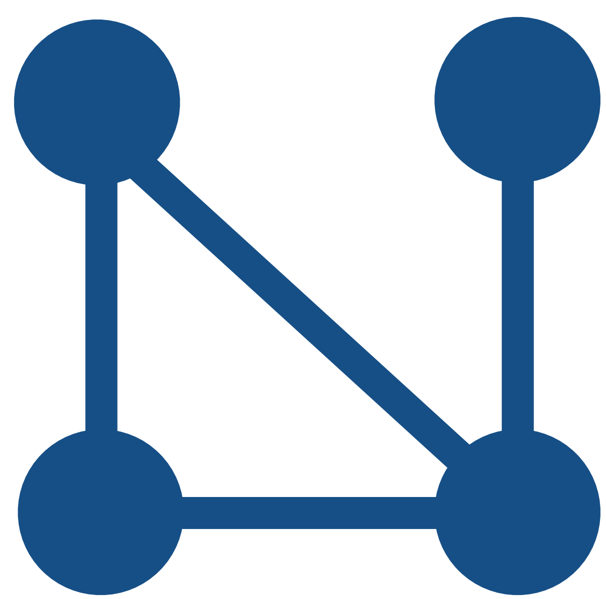


$G_1$



$G$

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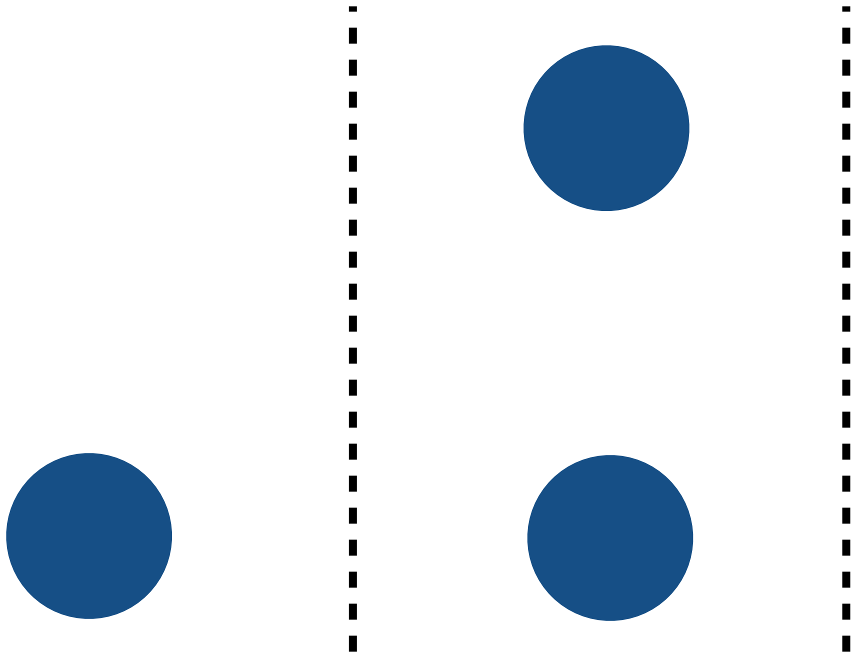
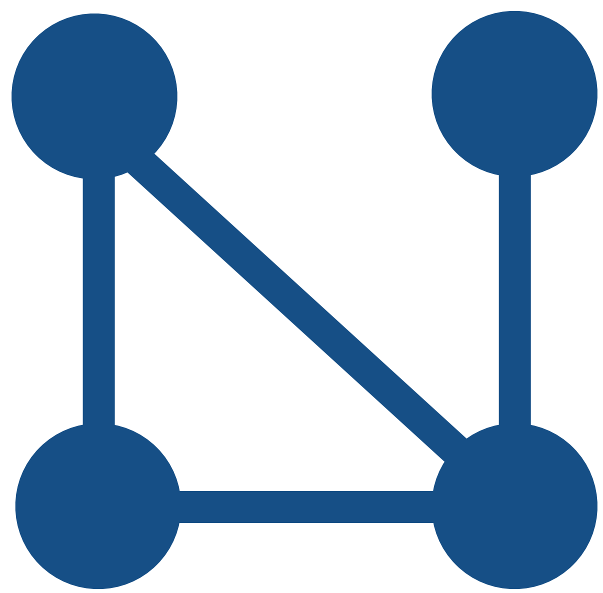


$G_1$ $G_2$



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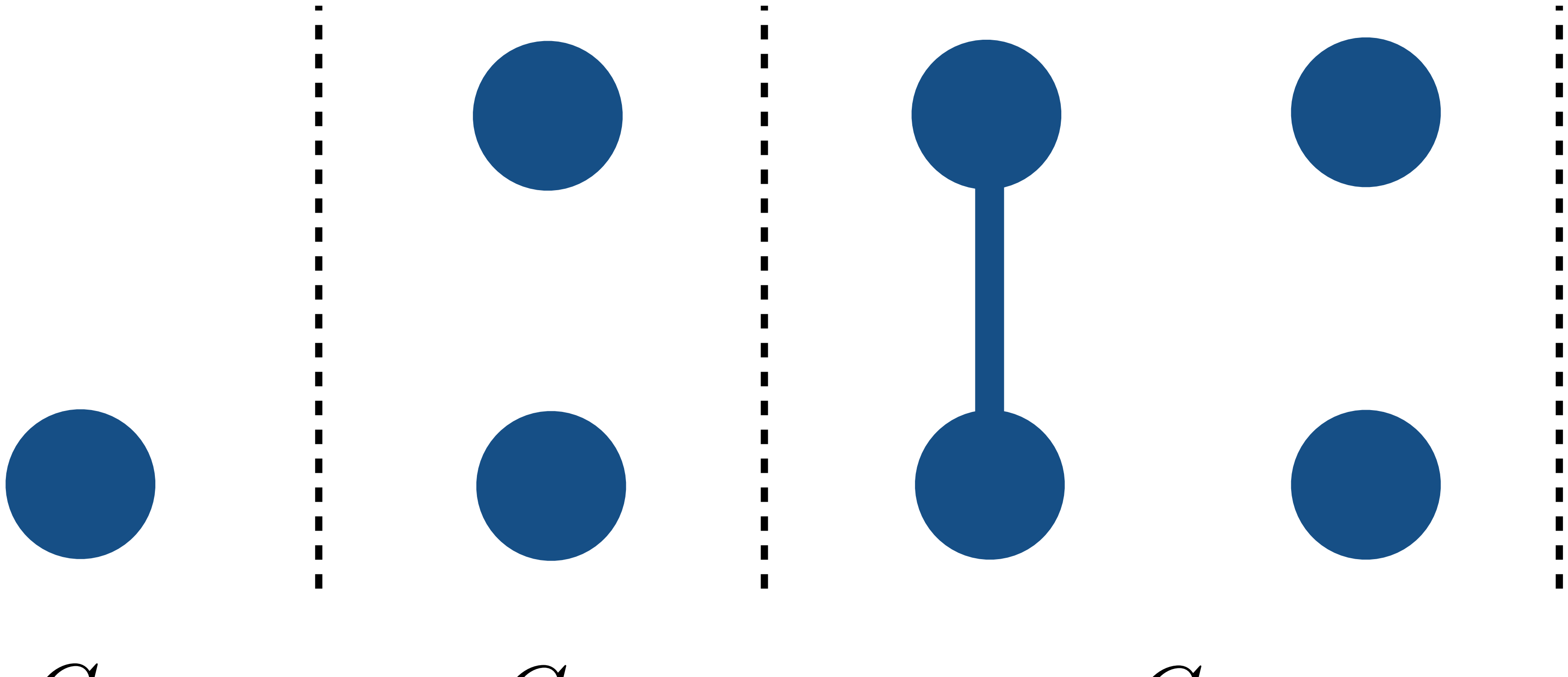
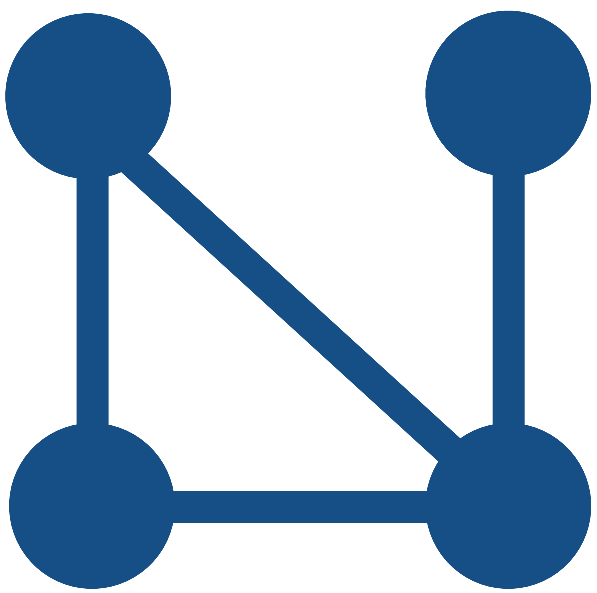


$G_1$ $G_2$ $G_3$



$G$

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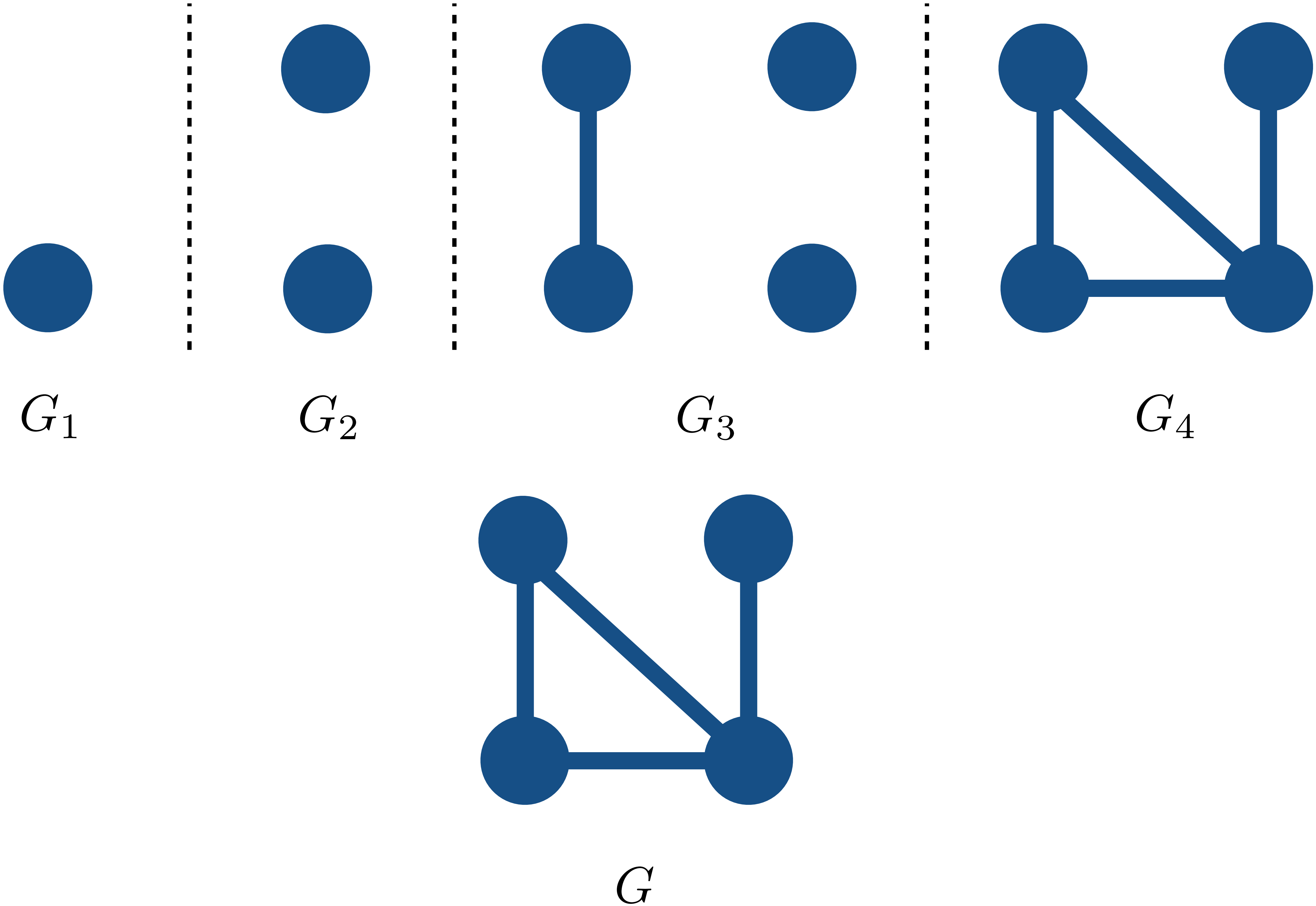

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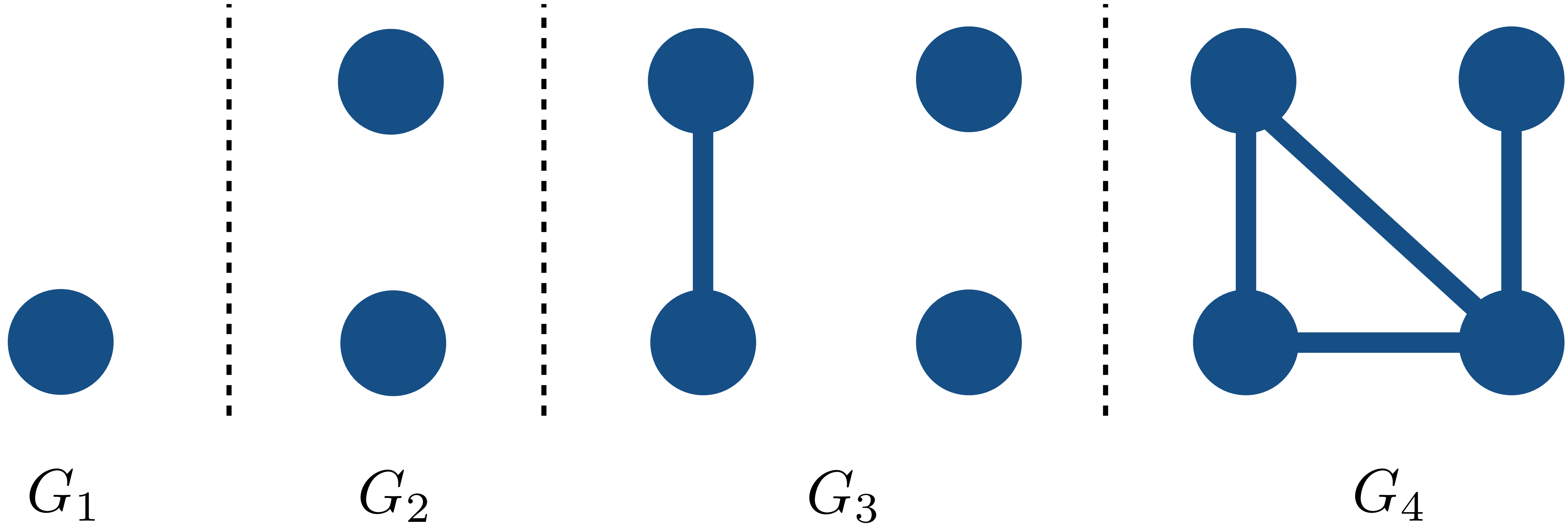

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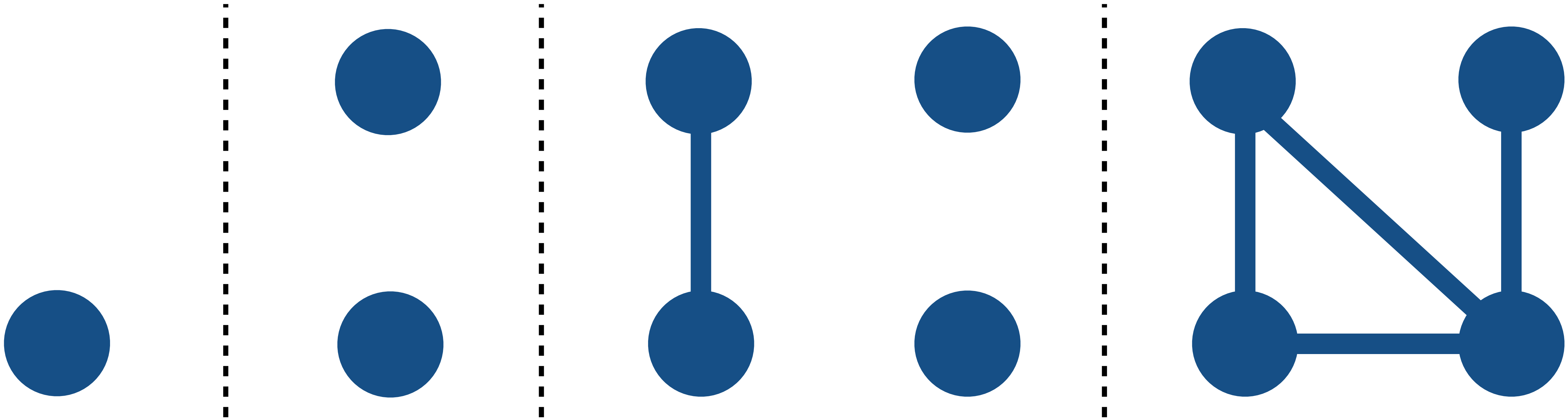
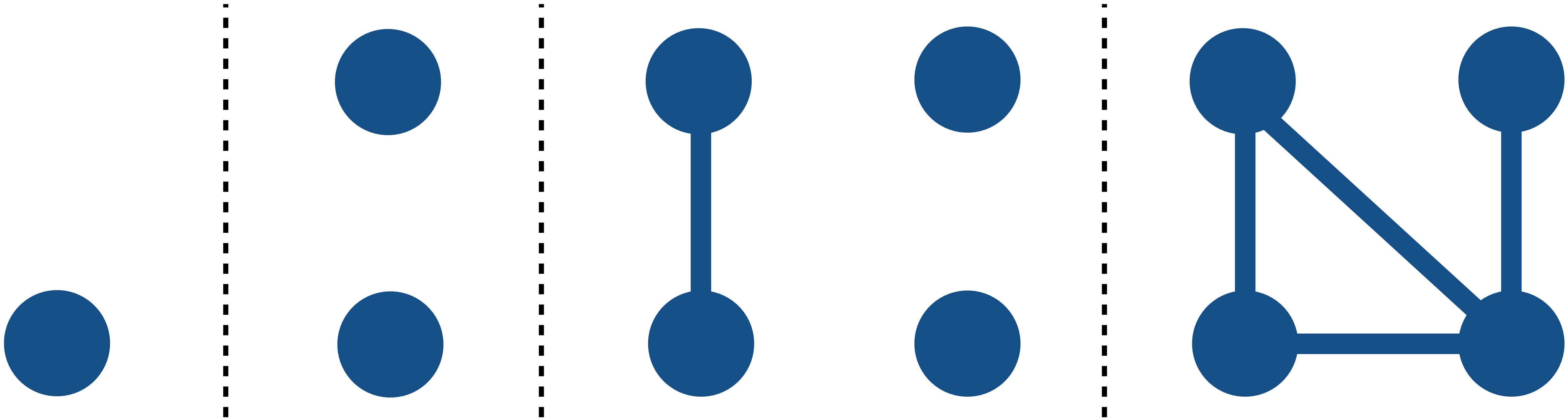
# Sequence of graphs



$G_1$ $G_2$ $G_3$ $G_4$

If $\#\text{nodes}(G_n) \to \infty$,

# Sequence of graphs



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- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$

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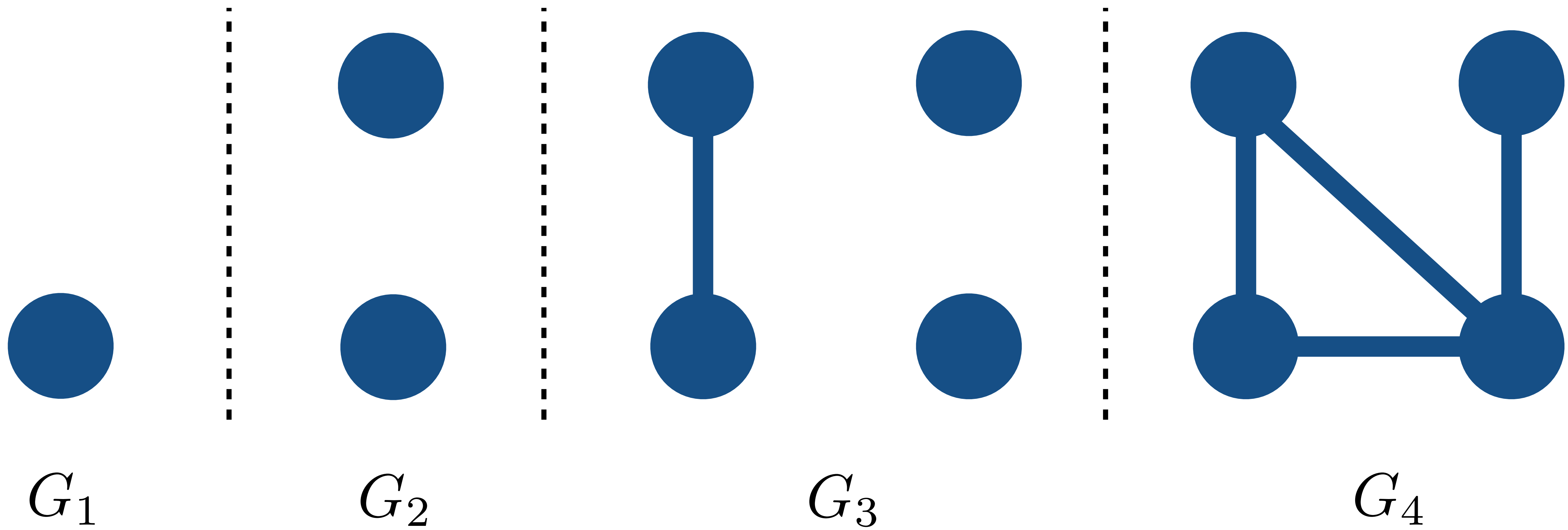
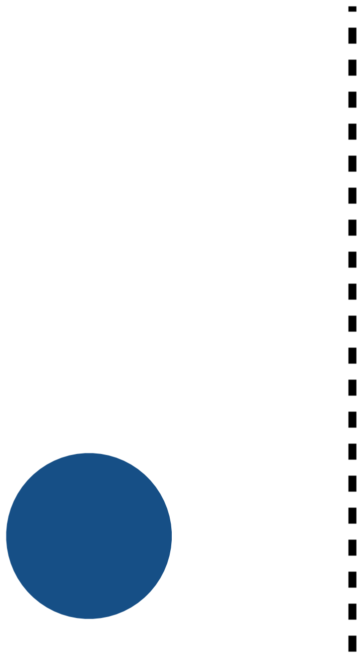


If $\#\text{nodes}(G_n) \to \infty$,

- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$
- *Sparse* graph sequence $\#\text{edges}(G_n) \in o([\#\text{nodes}(G_n)]^2)$

# The Old Way: Nodes



$G_1$



[Hoover 1979, Aldous 1981]

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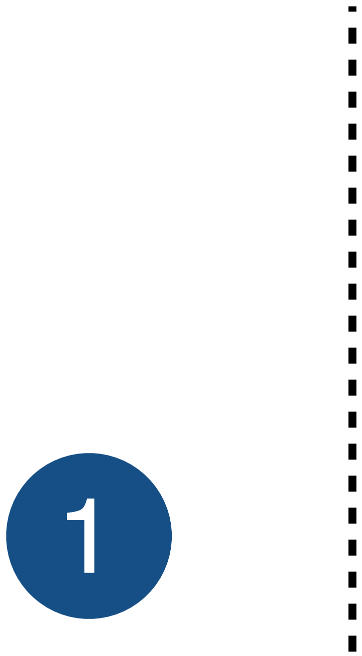


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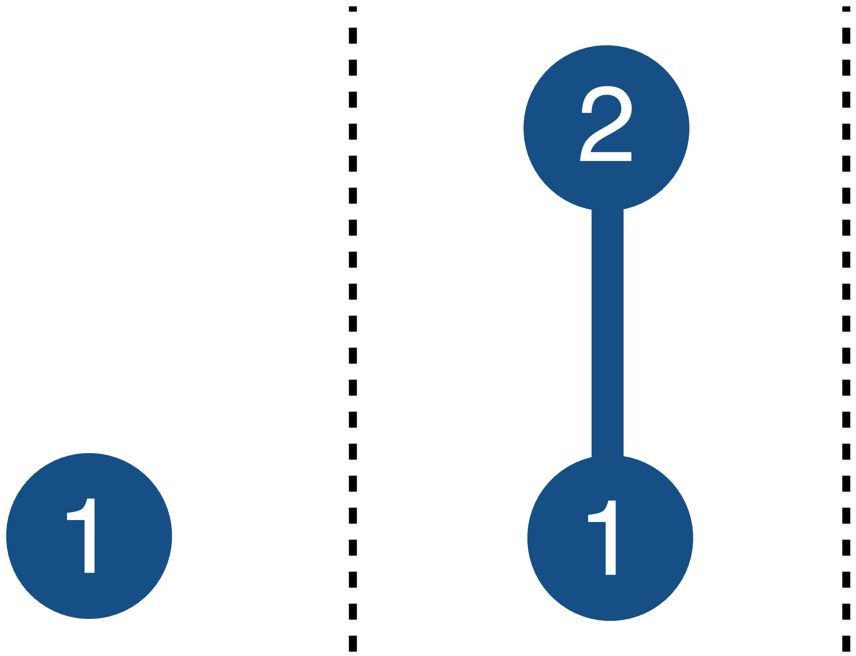


$G_1$ $G_2$



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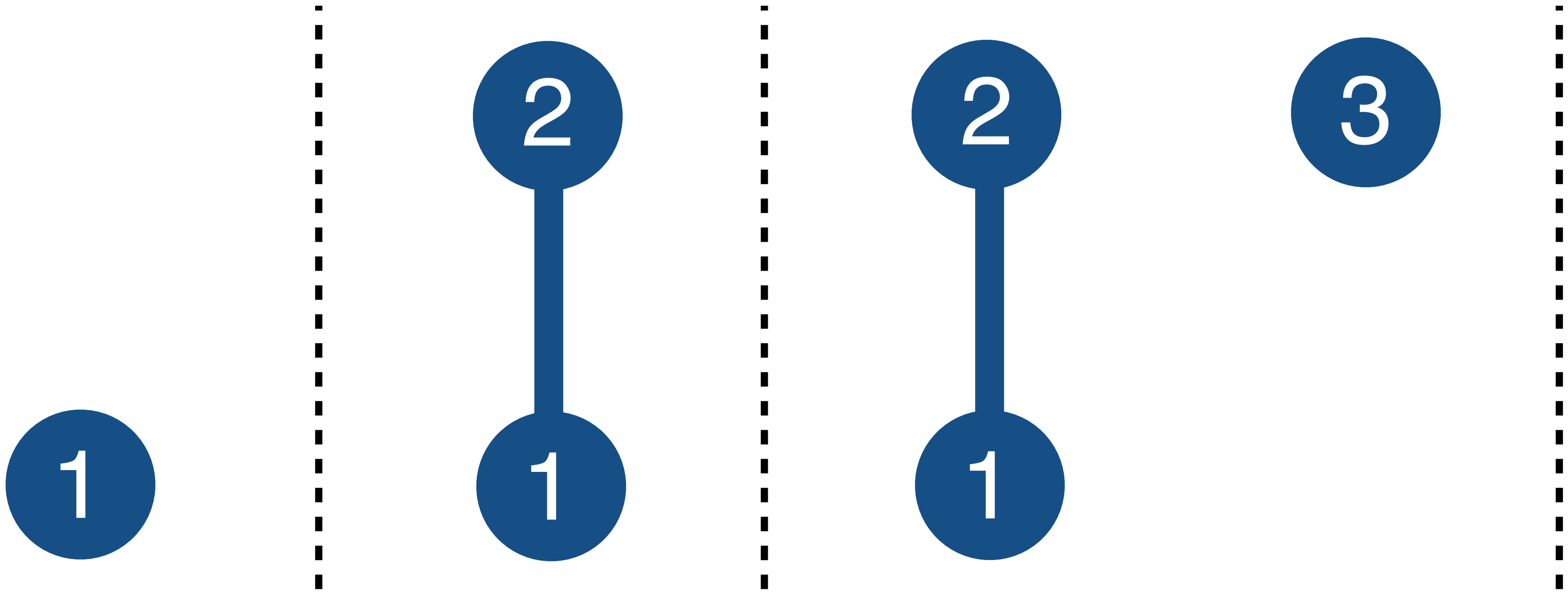


$G_1$ $G_2$ $G_3$



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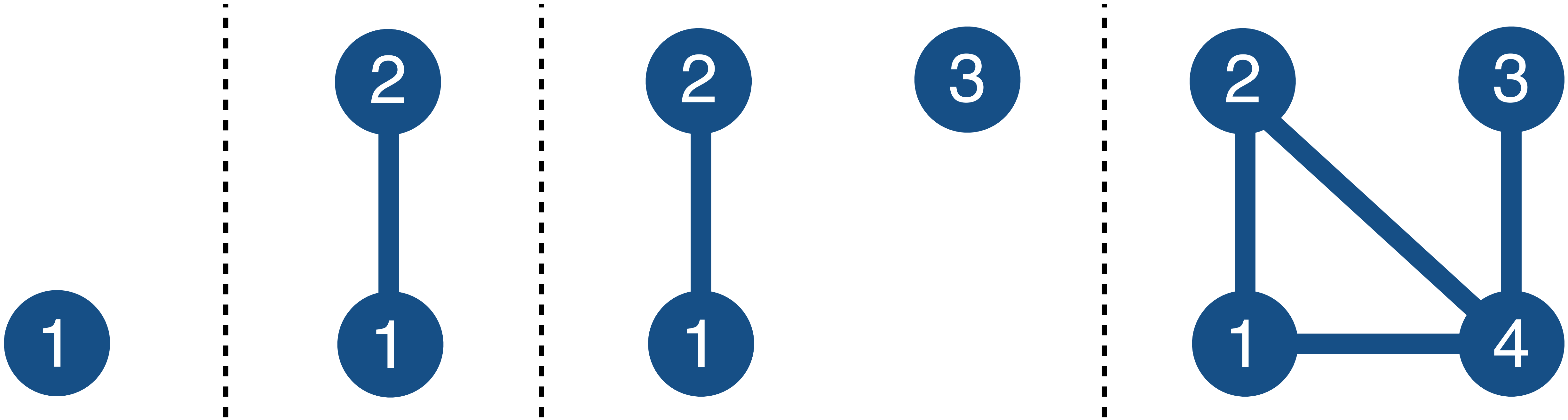


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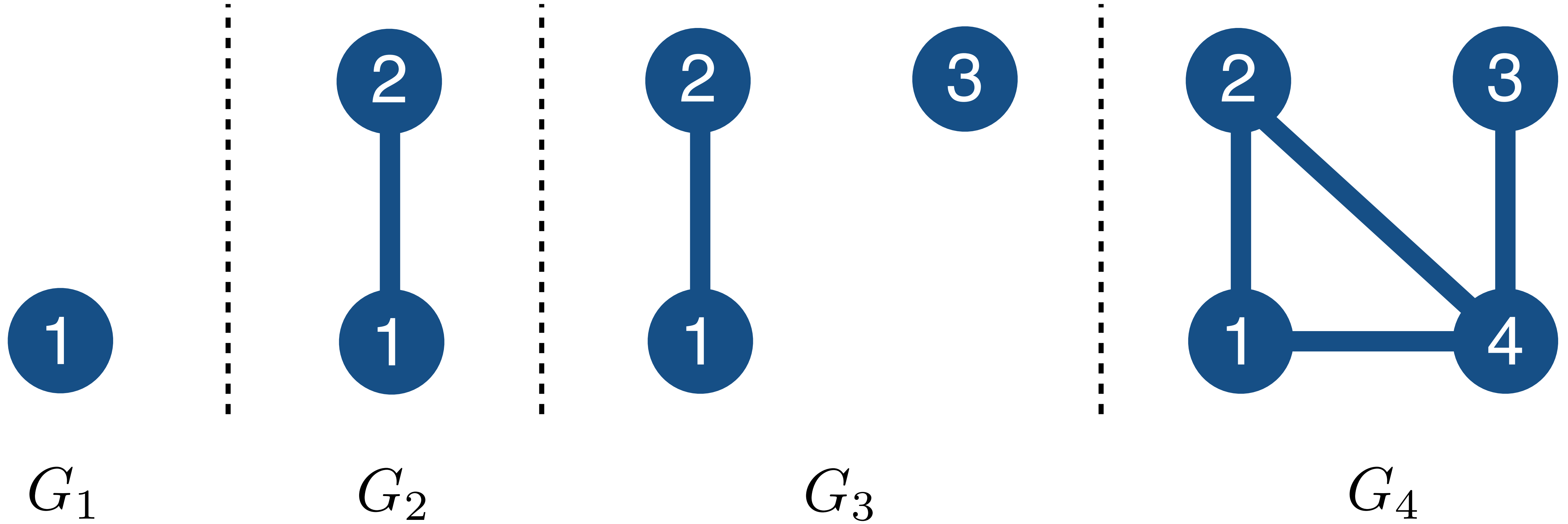




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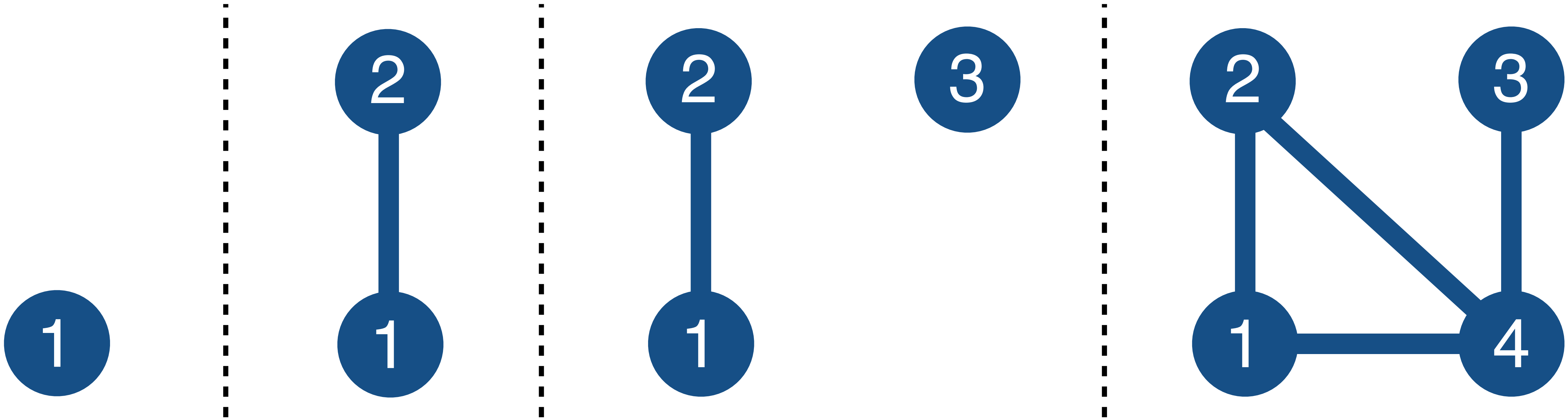


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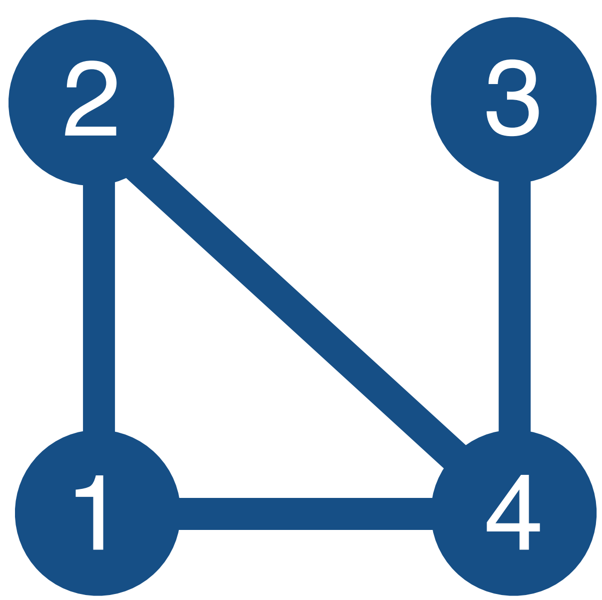




[Hoover 1979, Aldous 1981]

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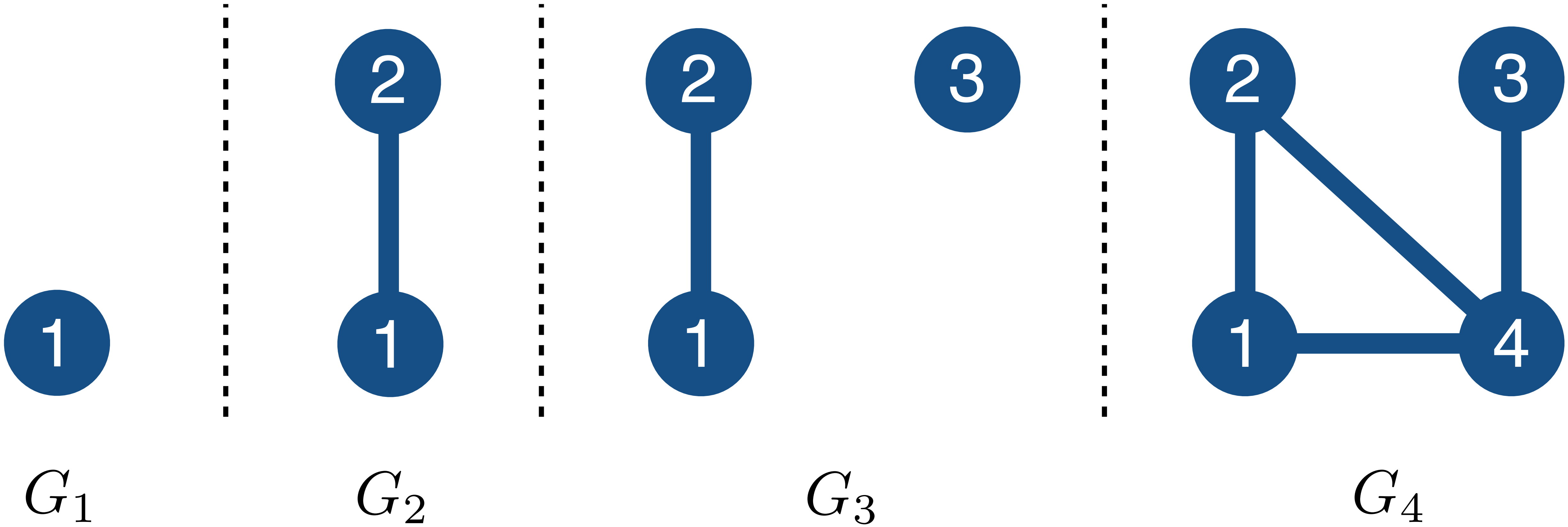
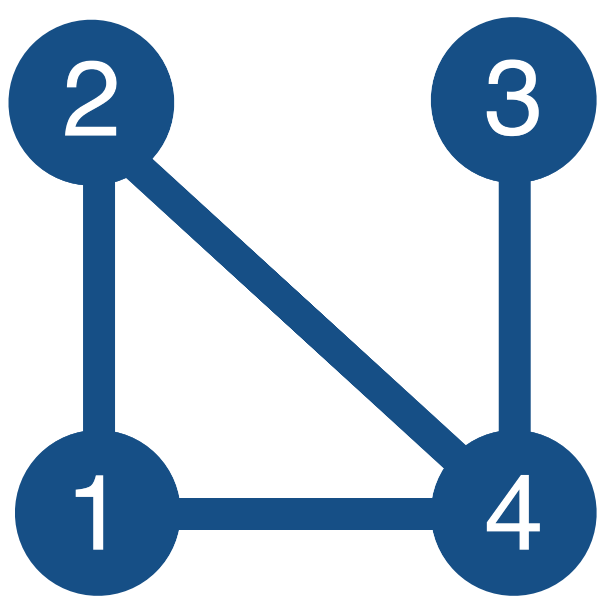



$p($  $)$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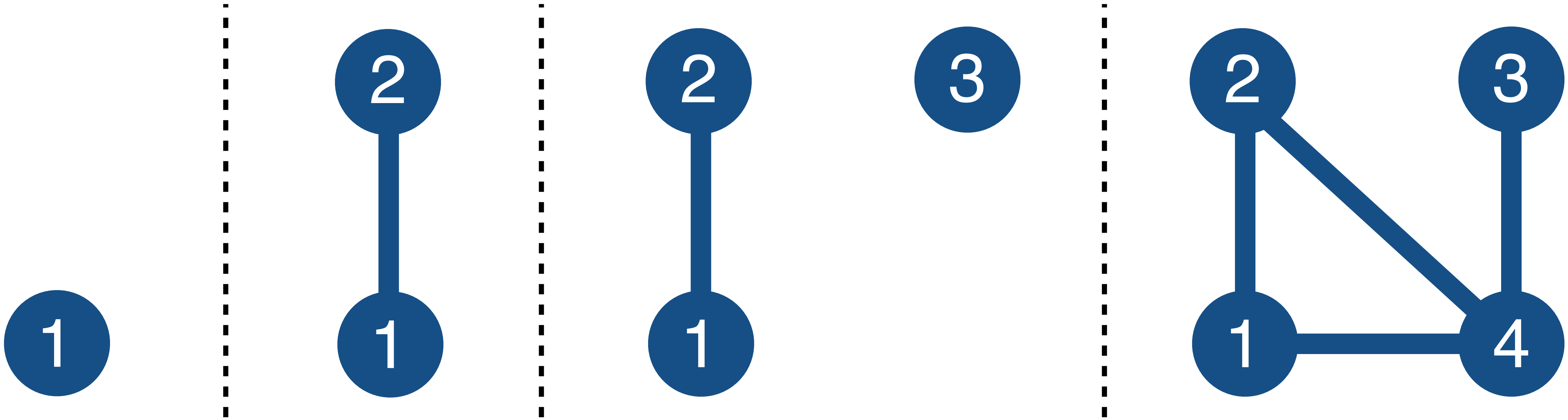


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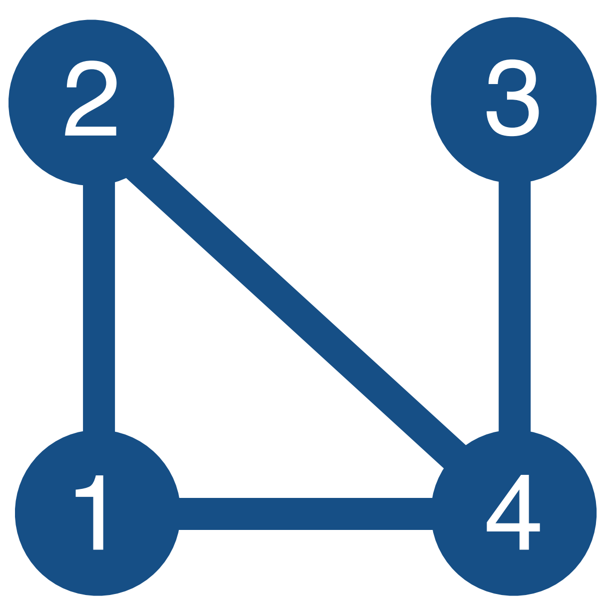
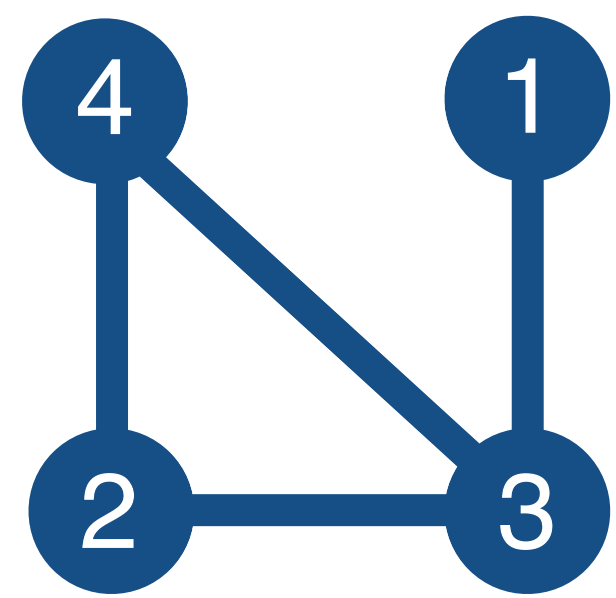
$p($ 

$) = p($ 

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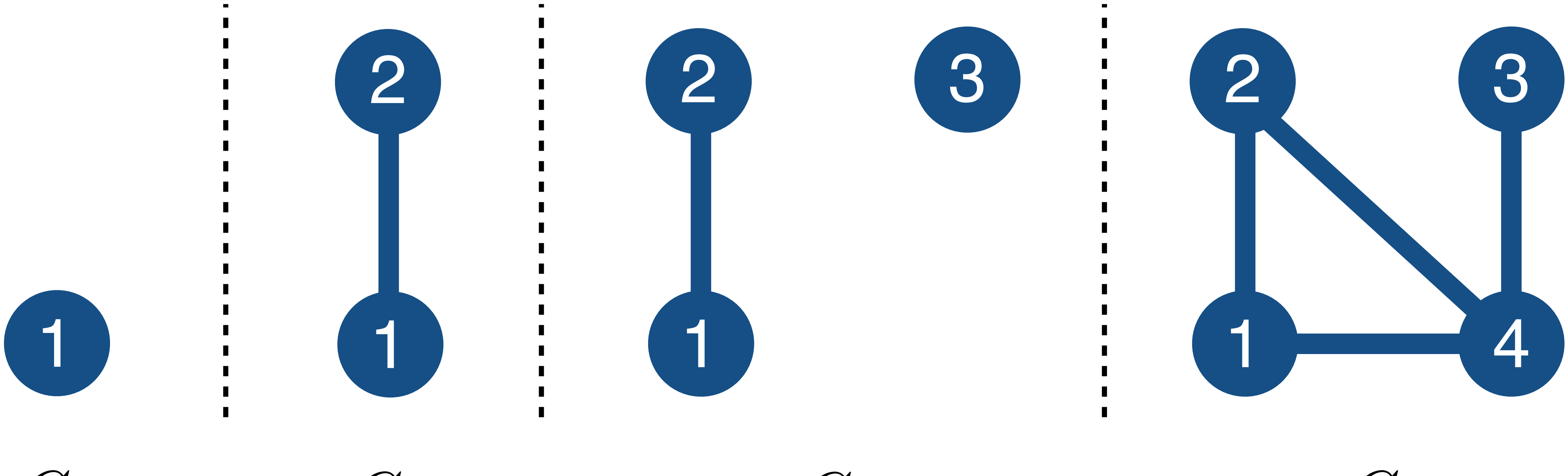
[Hoover 1979, Aldous 1981]

# The Old Way: Node exchangeability

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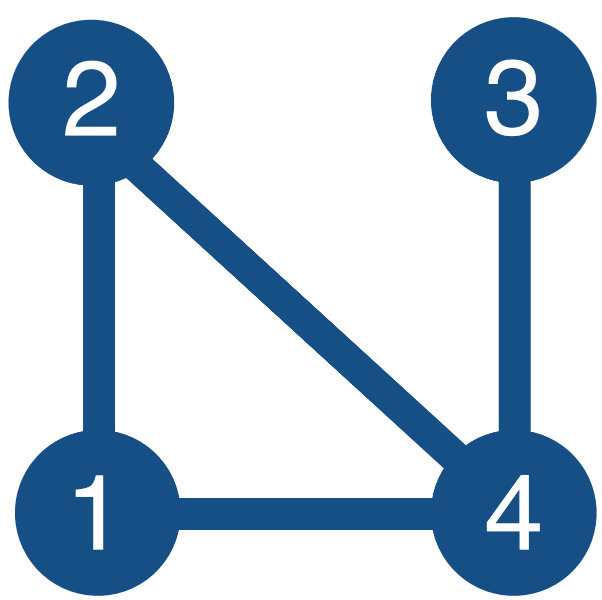
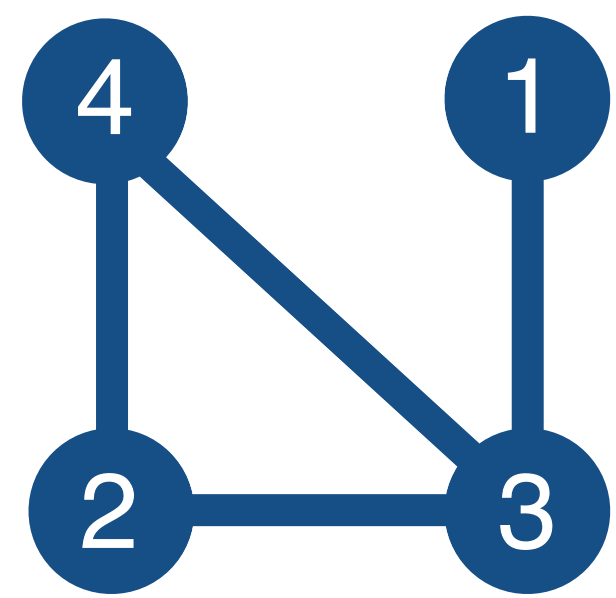
$p($ 

$) = p($ 

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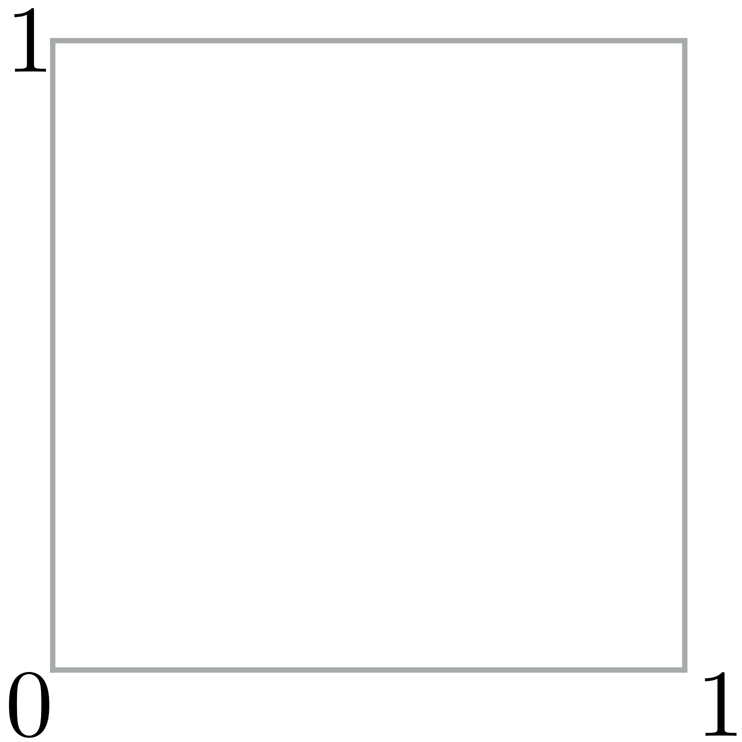
[Hoover 1979, Aldous 1981]

# Aldous-Hoover



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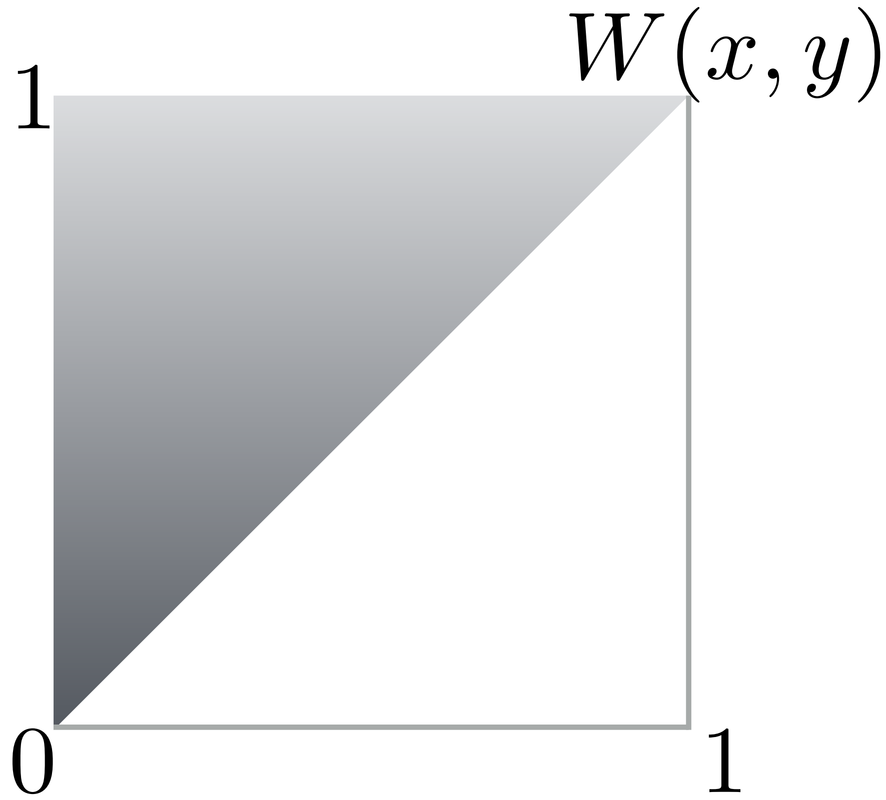




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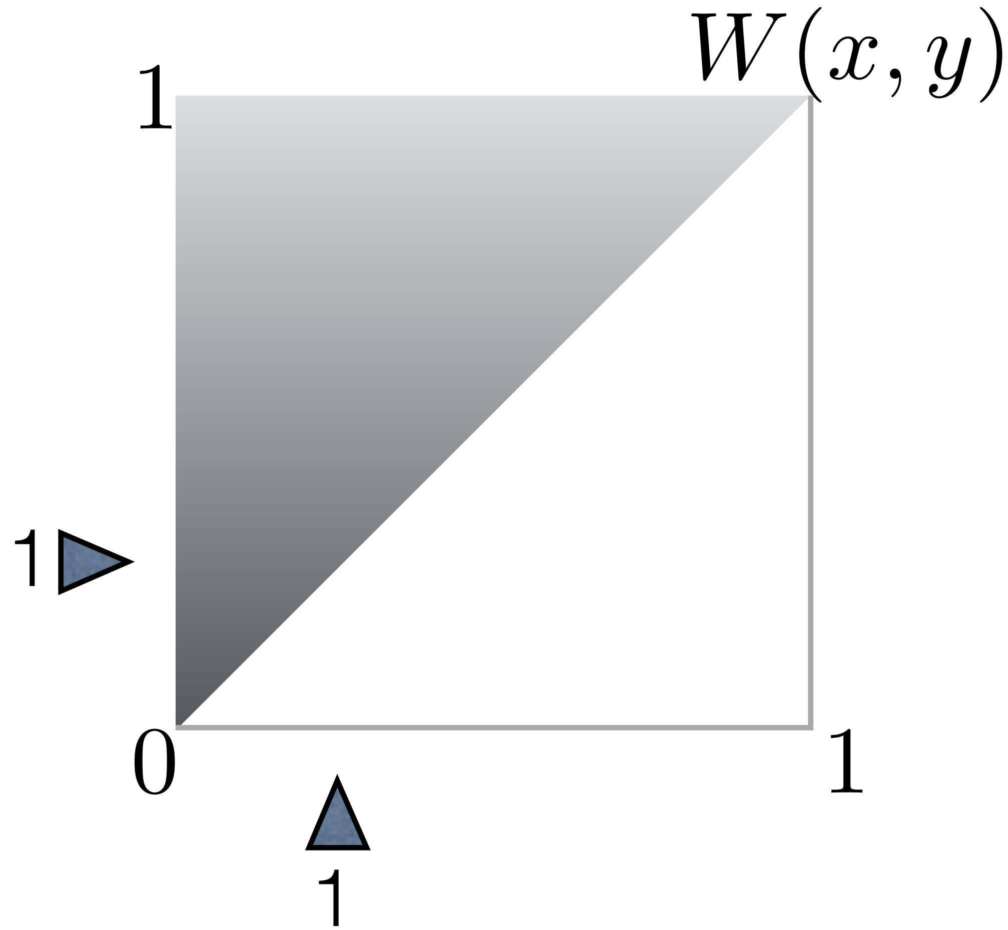





[Hoover 1979, Aldous 1981]

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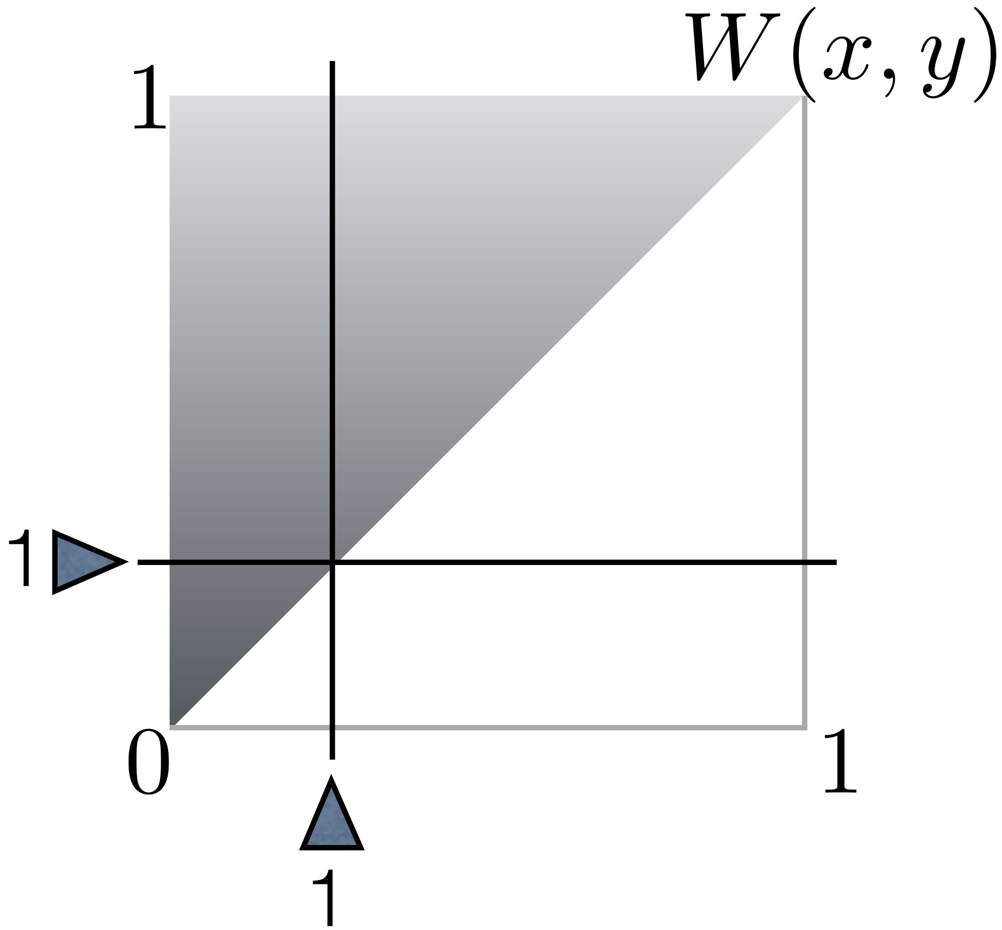





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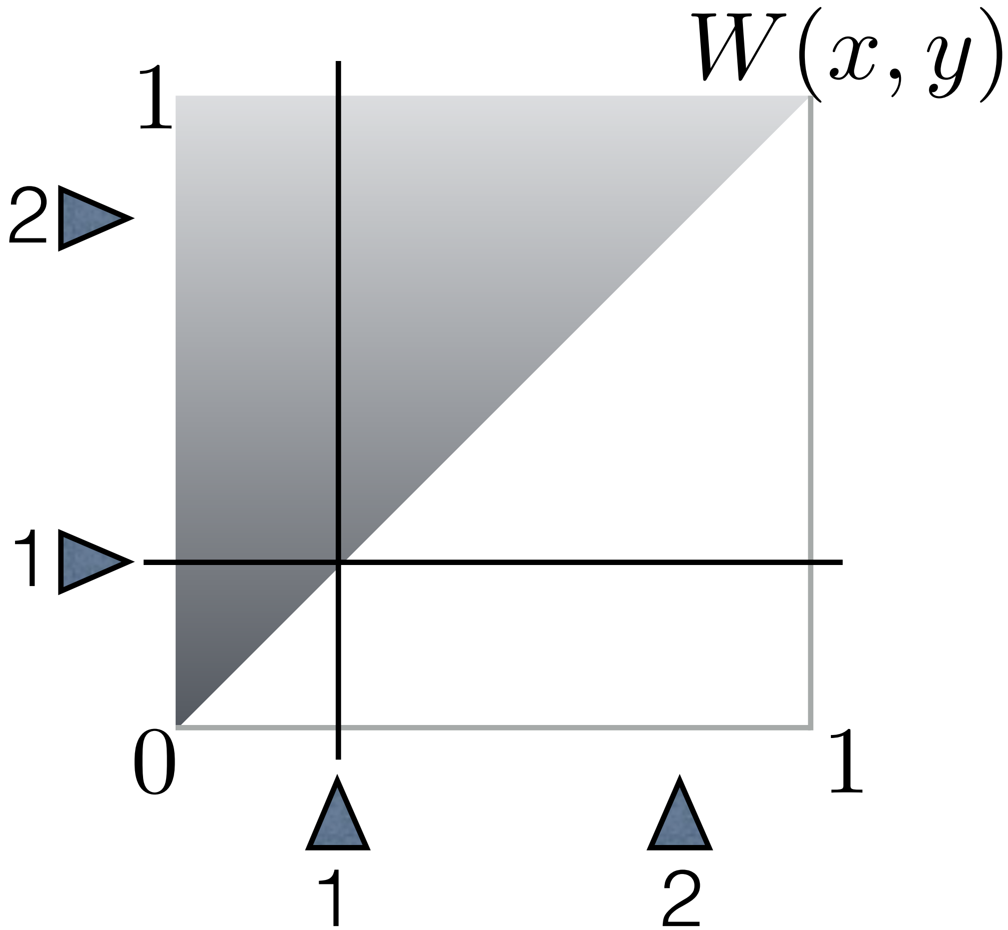





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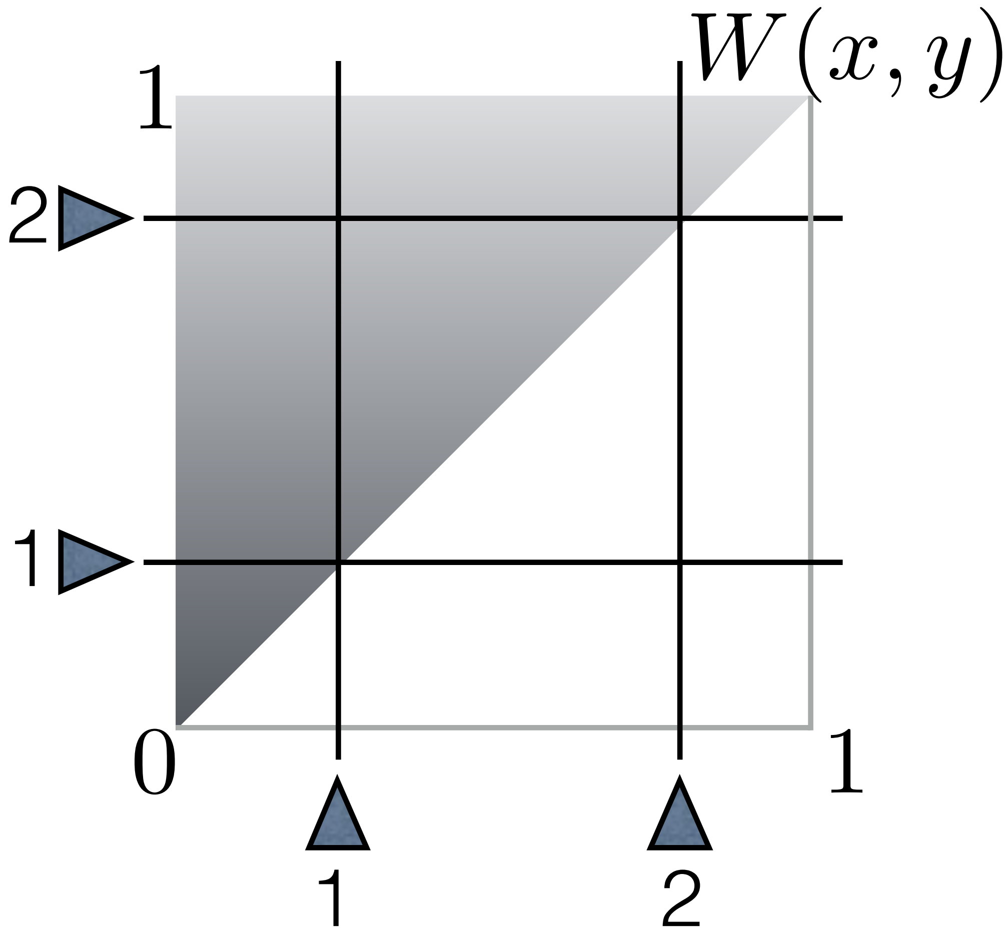





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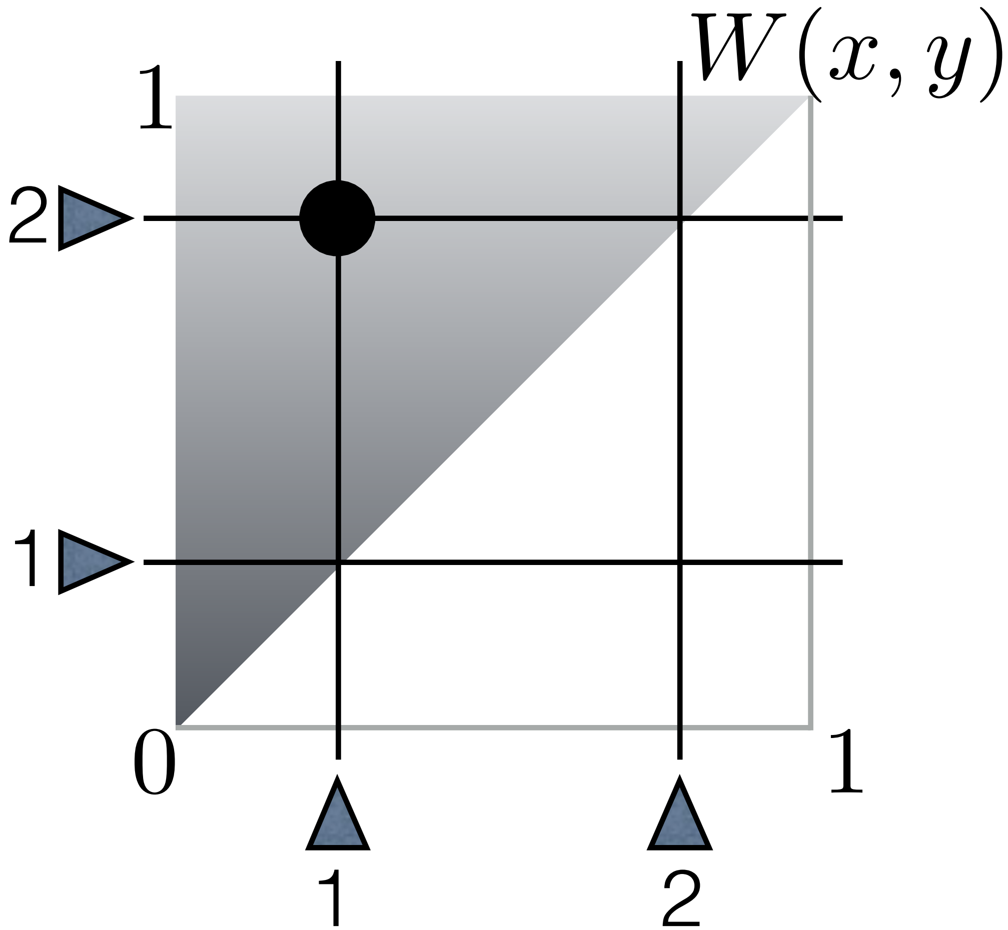


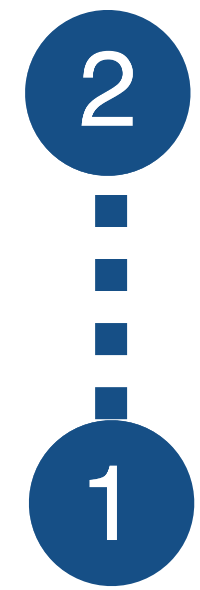




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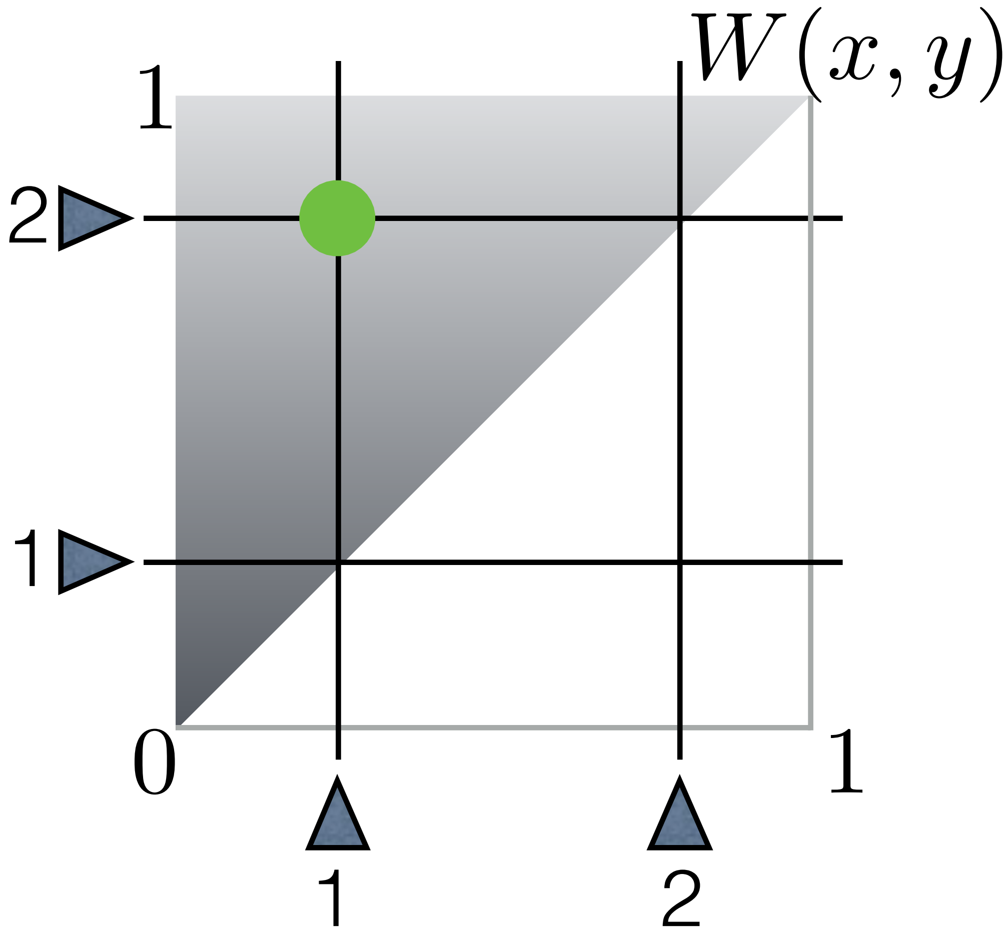


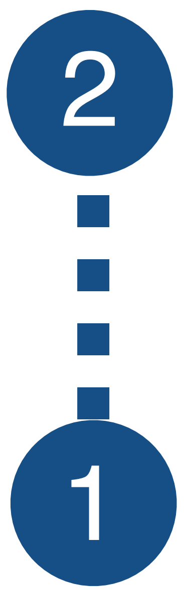




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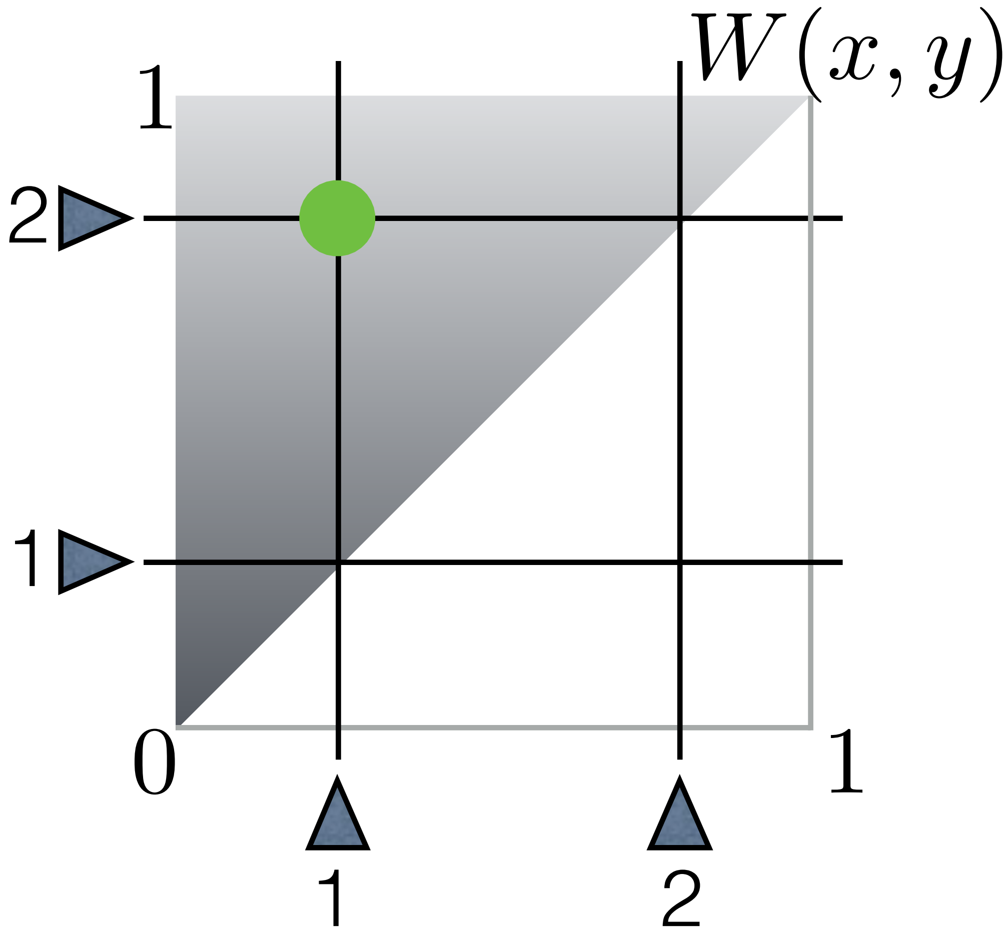





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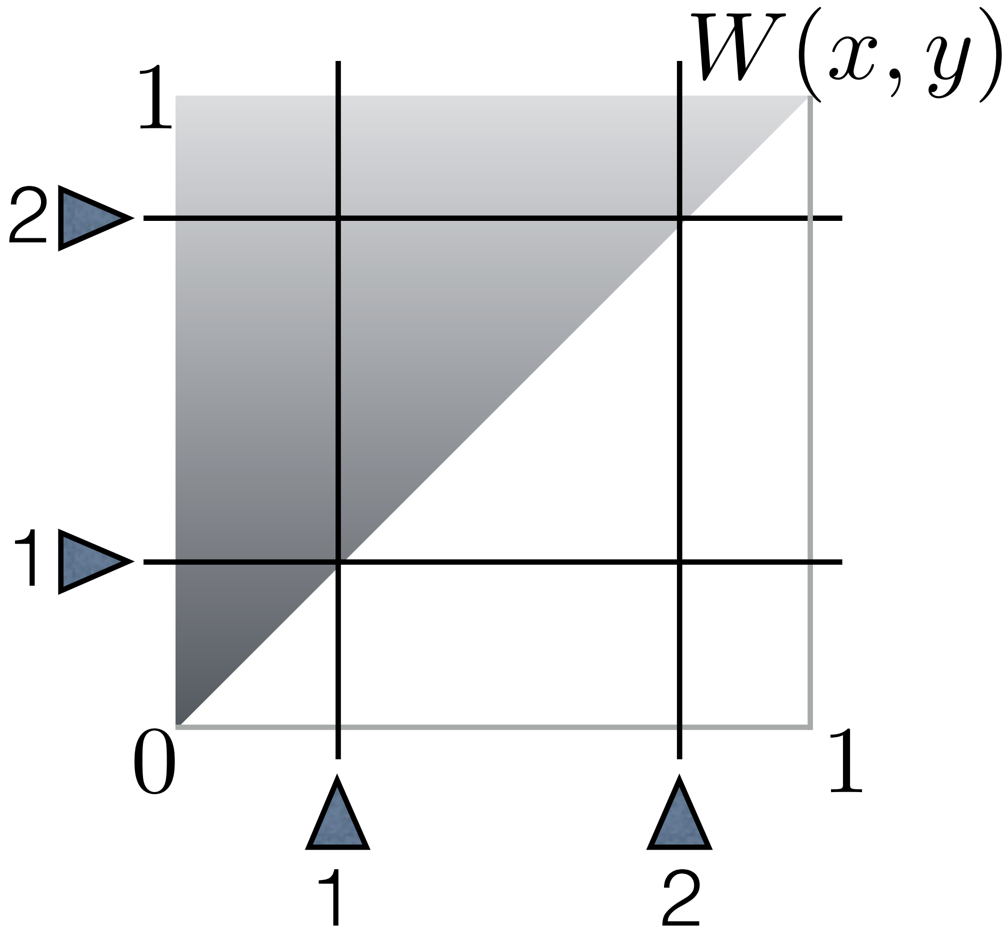
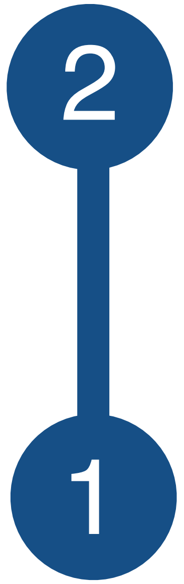
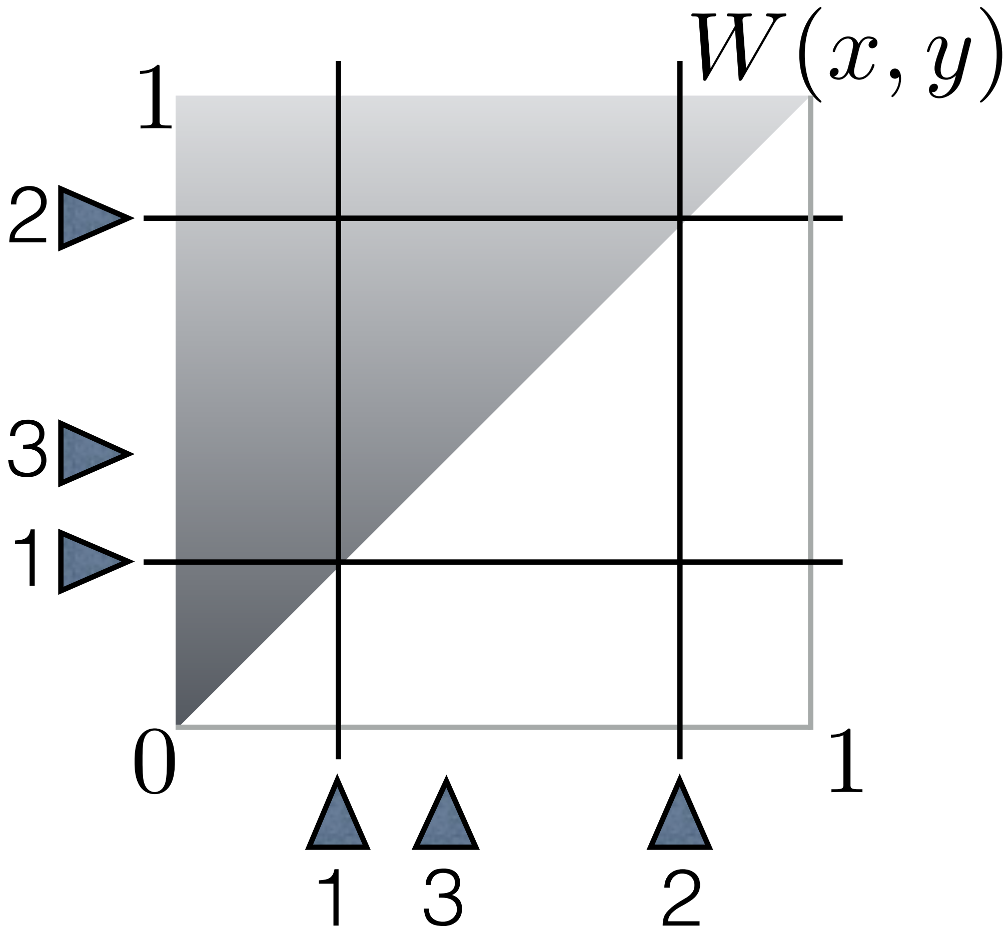


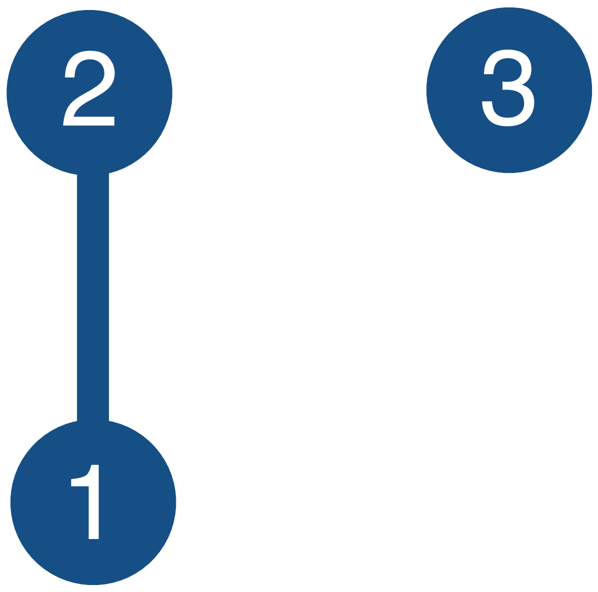






[Hoover 1979, Aldous 1981]

# Aldous-Hoover






[Hoover 1979, Aldous 1981]

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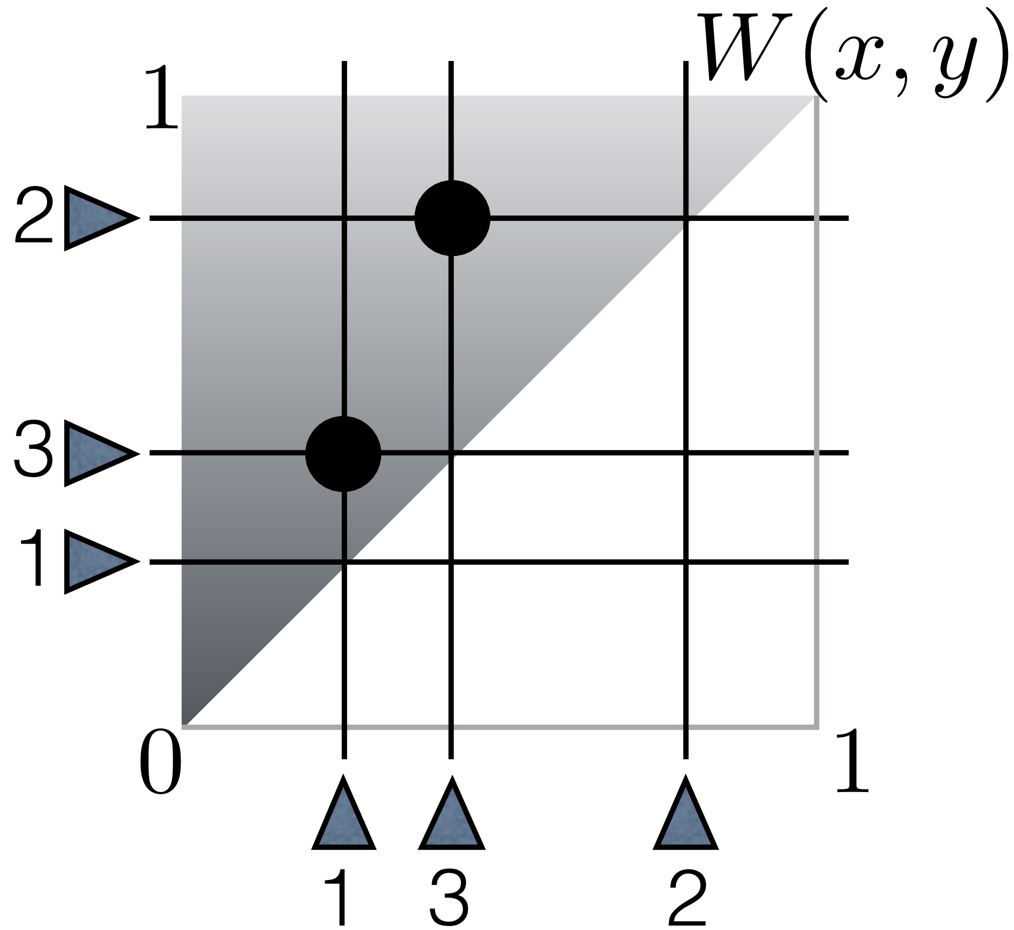


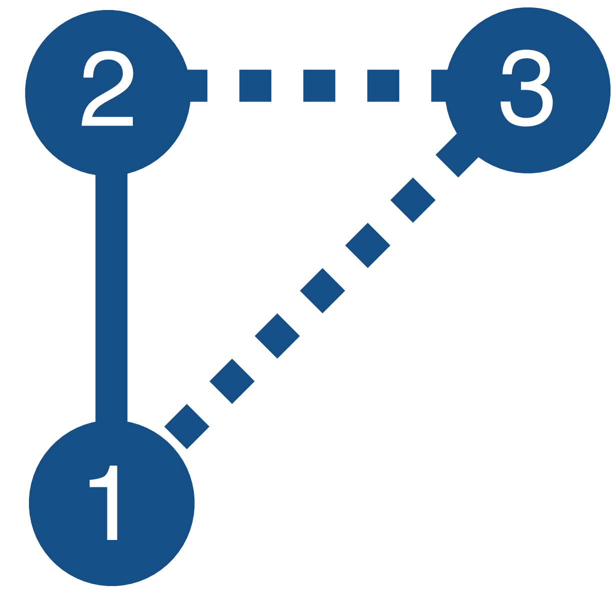




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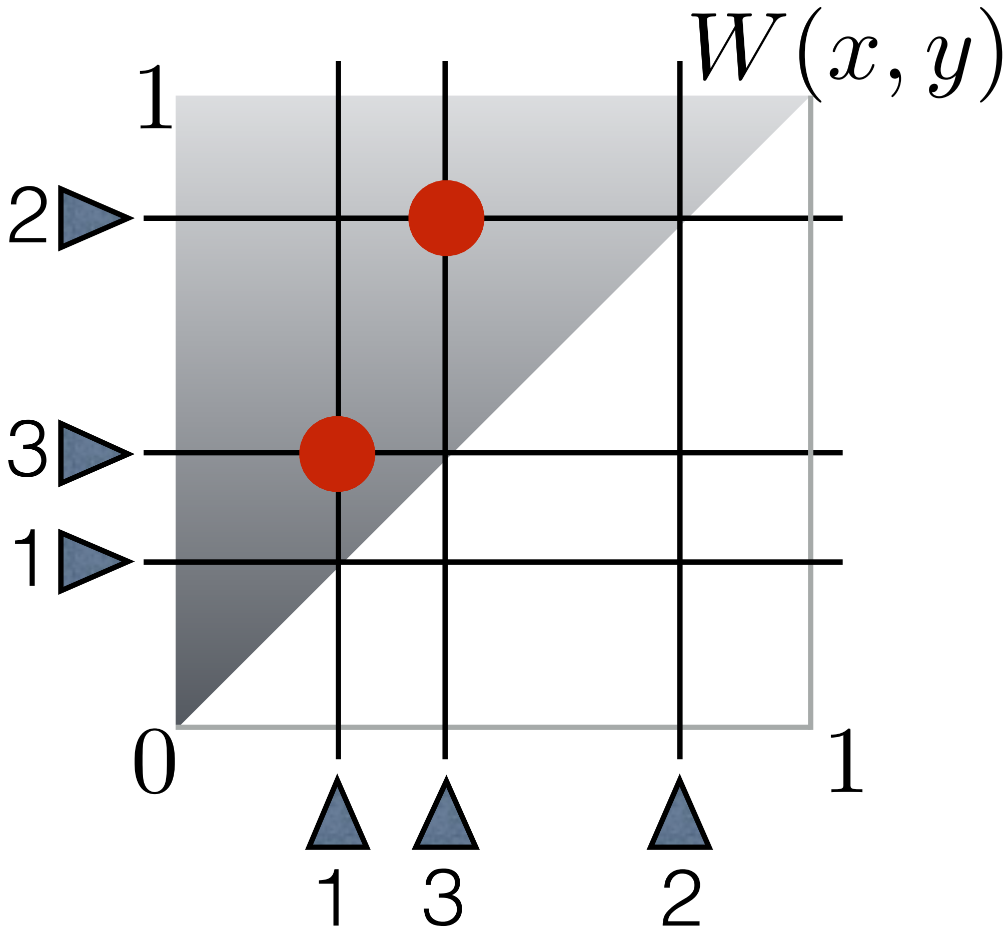


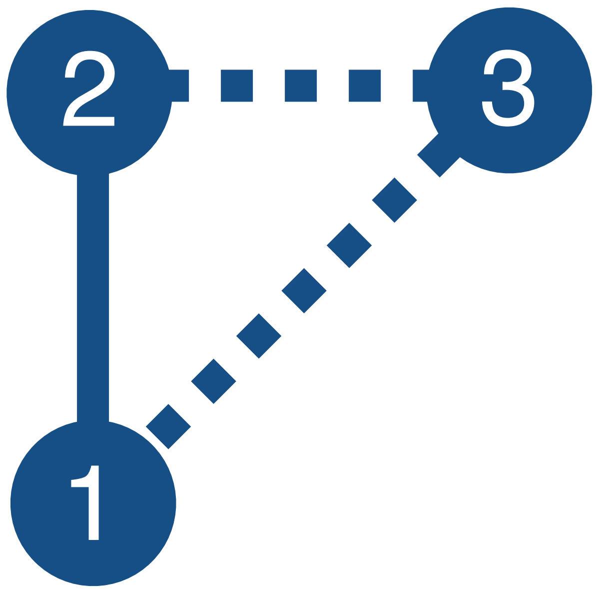




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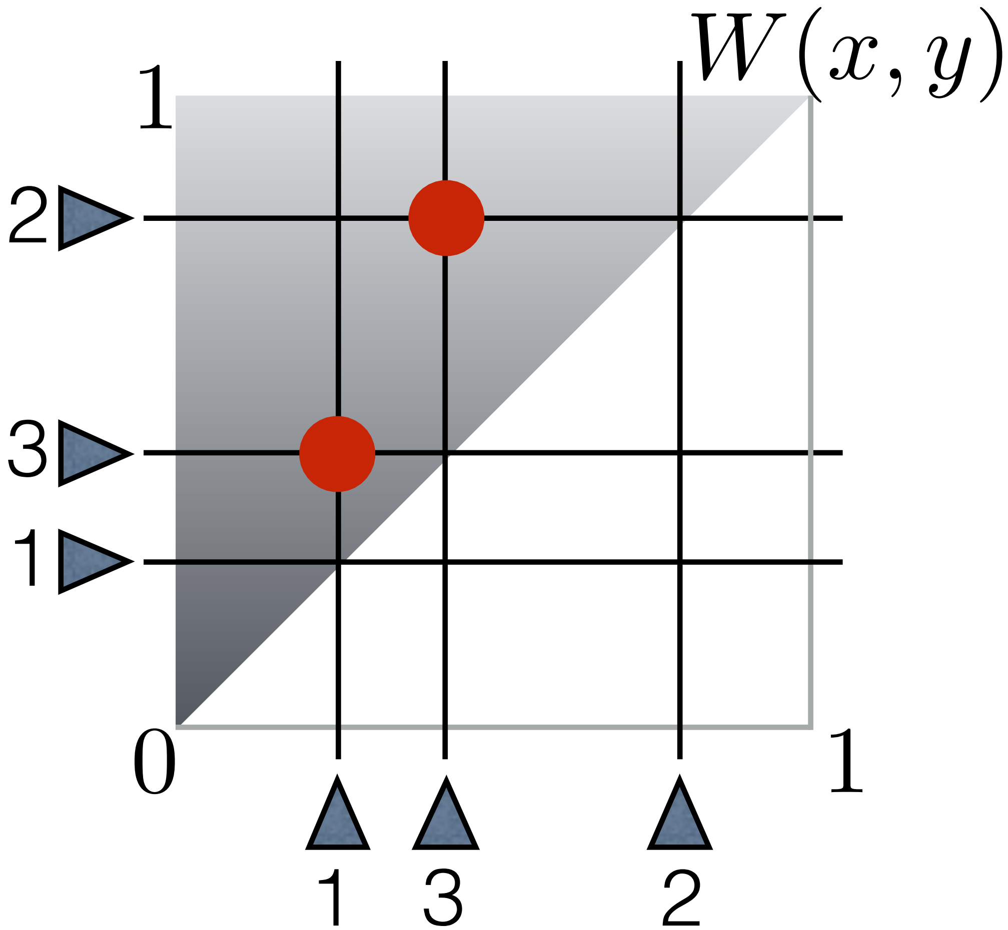


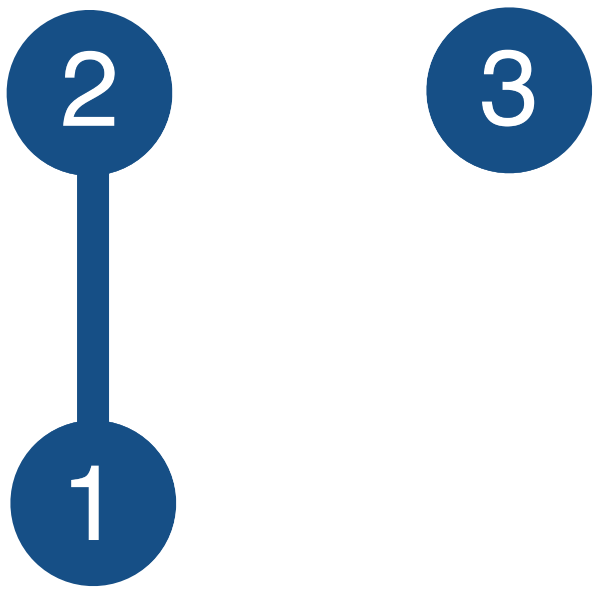




[Hoover 1979, Aldous 1981]

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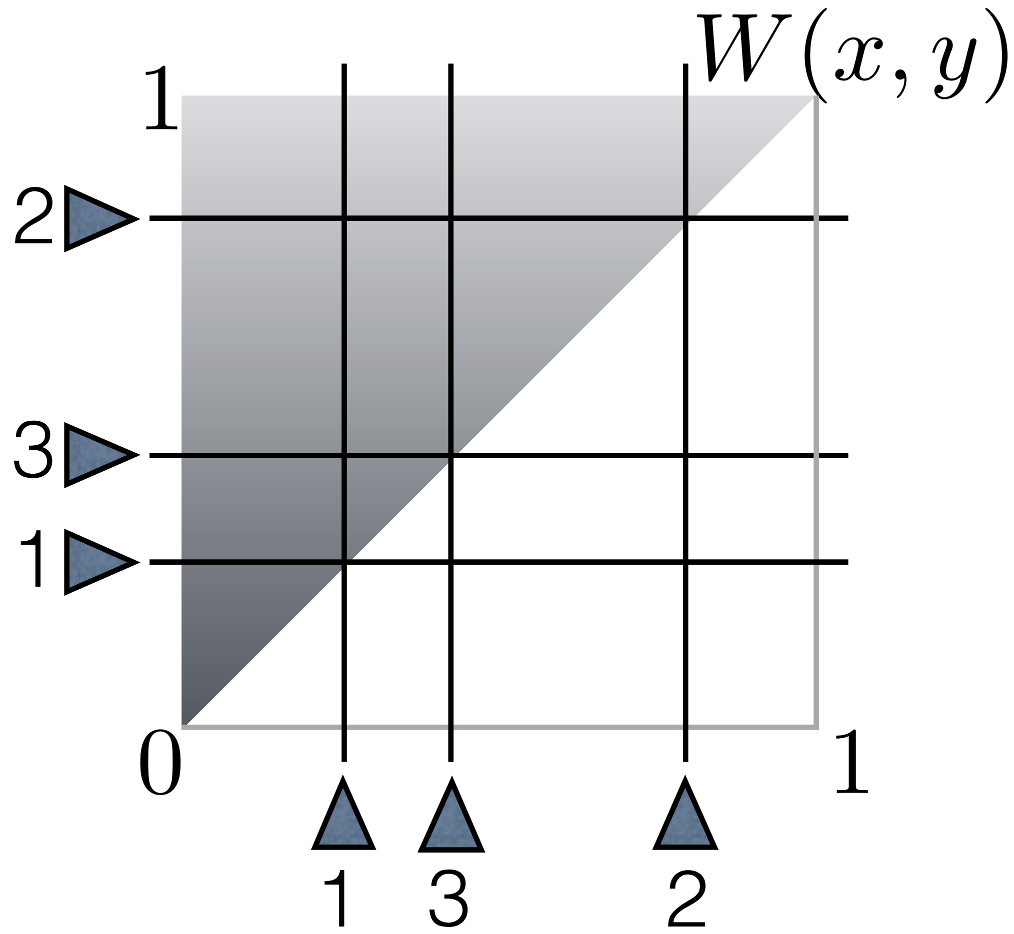


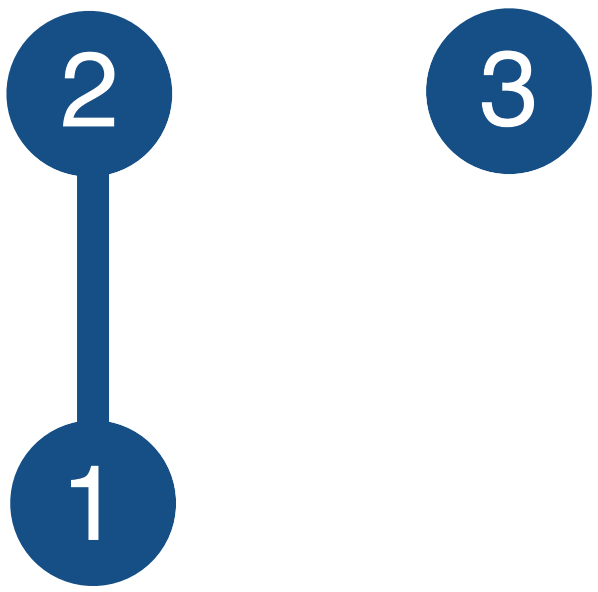




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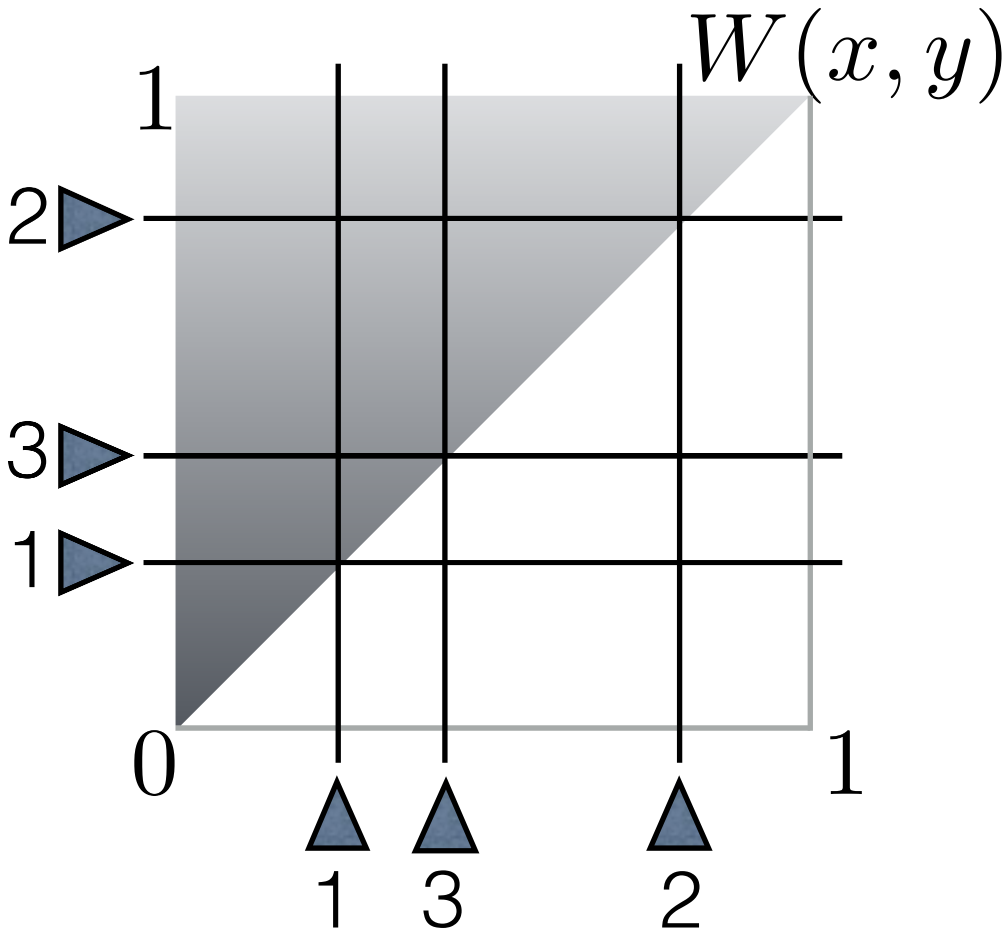


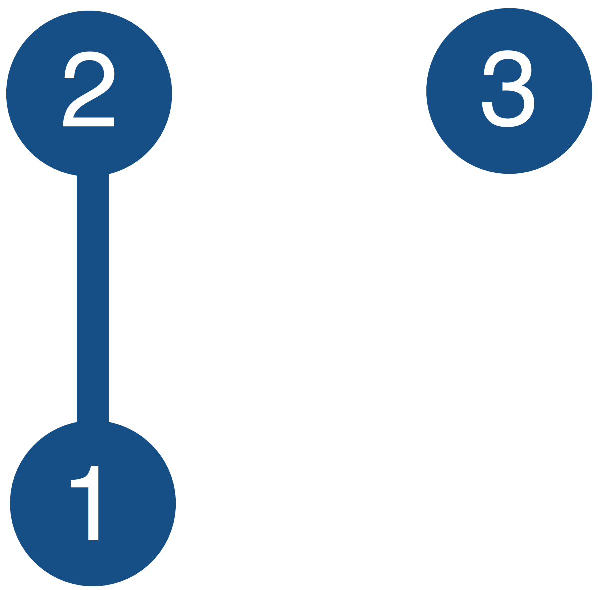


Thm (AH). Every node-exchangeable graph has a *graphon* rep



[Hoover 1979, Aldous 1981]

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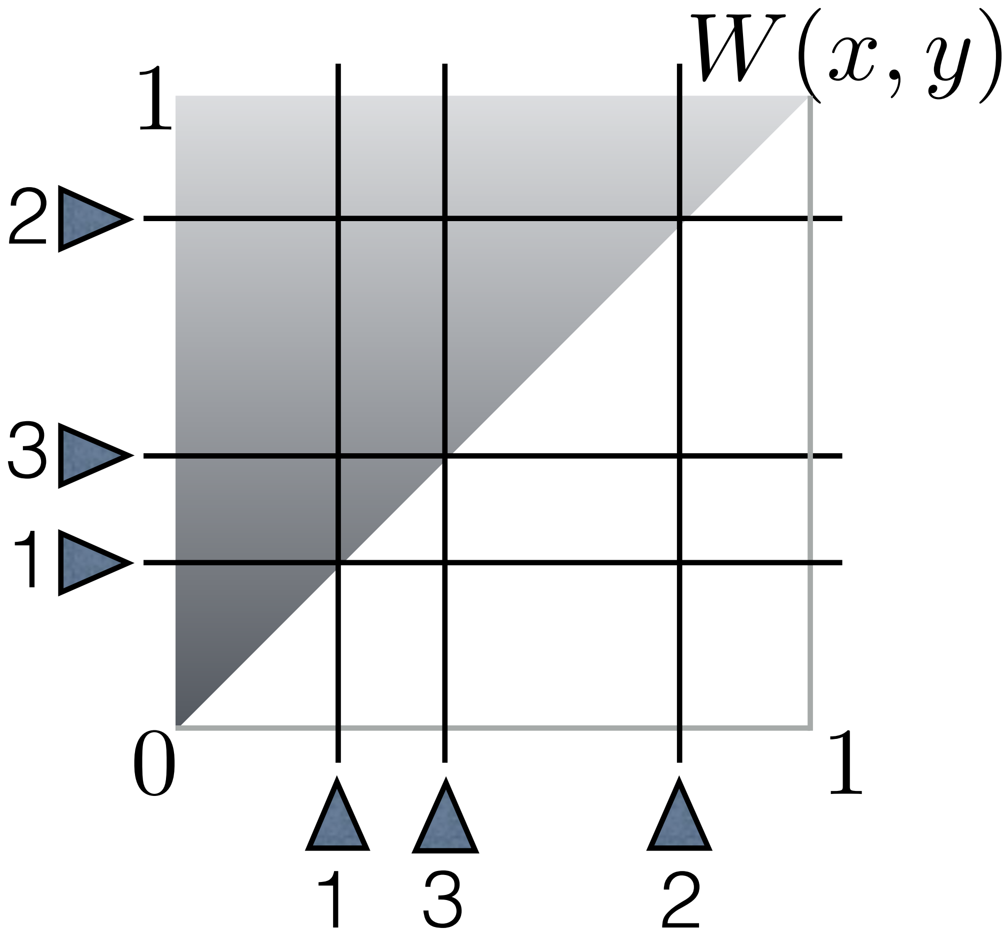


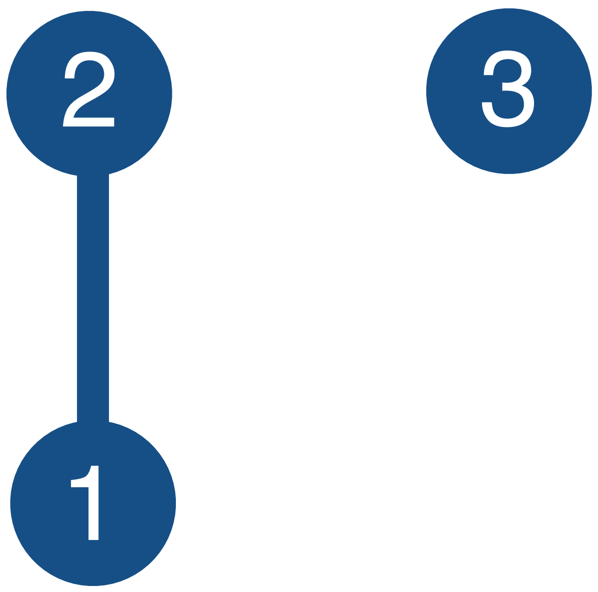


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$$\mathbb{E}[\#\text{edges}(G_n)]$$



[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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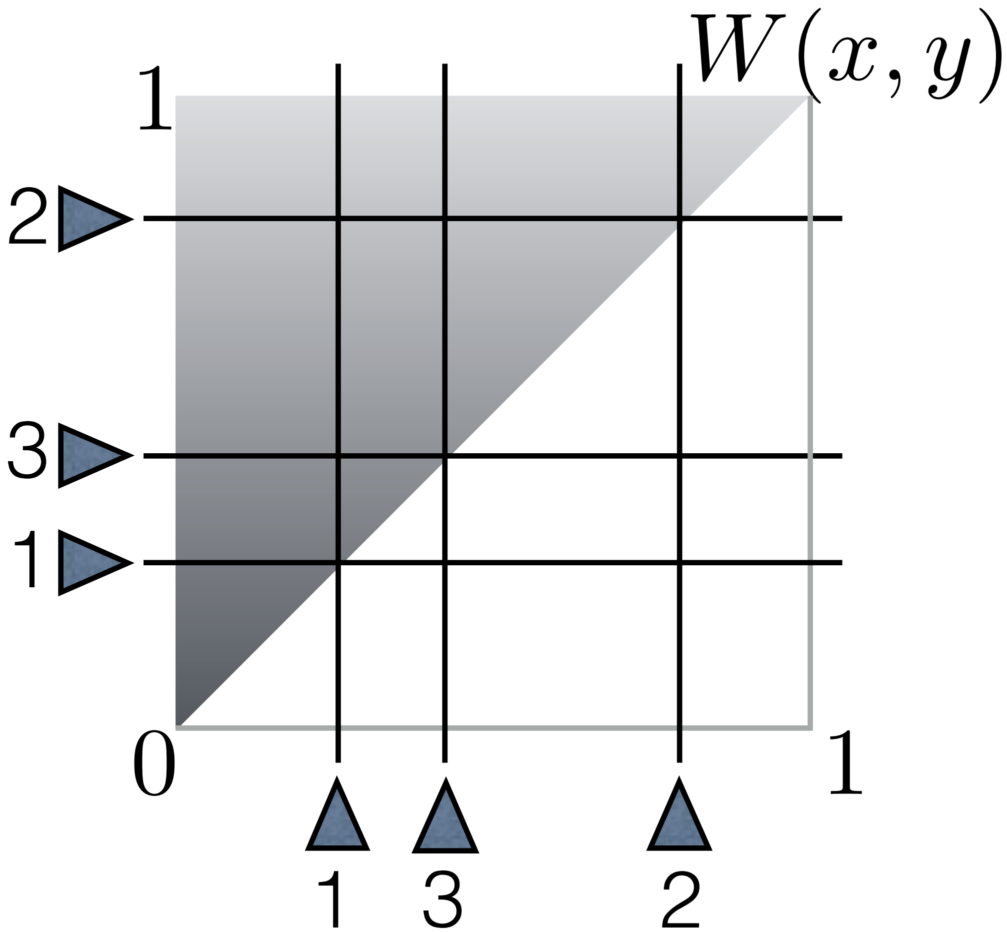


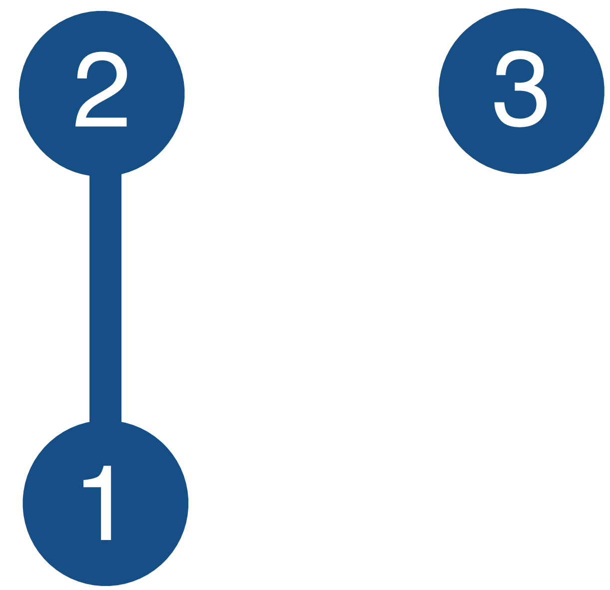


Thm (AH). Every node-exchangeable graph has a *graphon* rep

$$\mathbb{E}[\#\text{edges}(G_n)] = \mathbb{E}\left[\binom{n}{2}\frac{1}{2}\int_0^1\int_0^1 W(x,y)\,dx\,dy\right]$$



[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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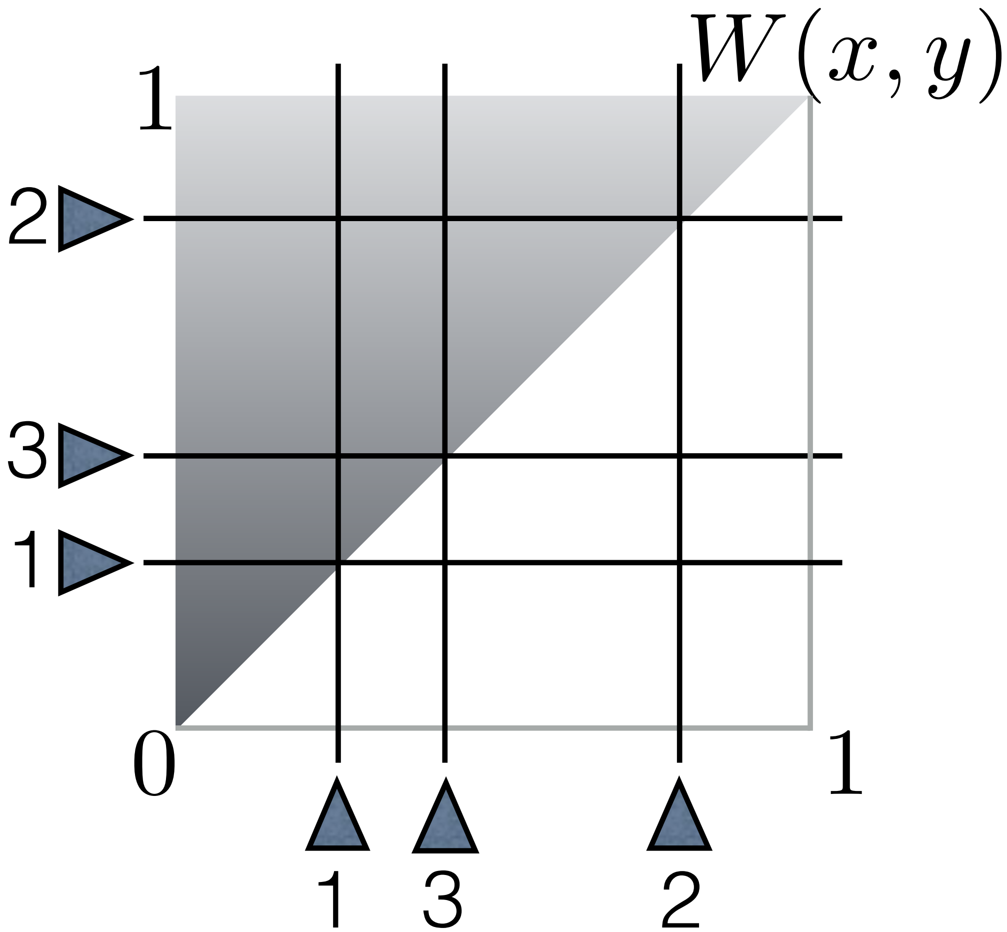


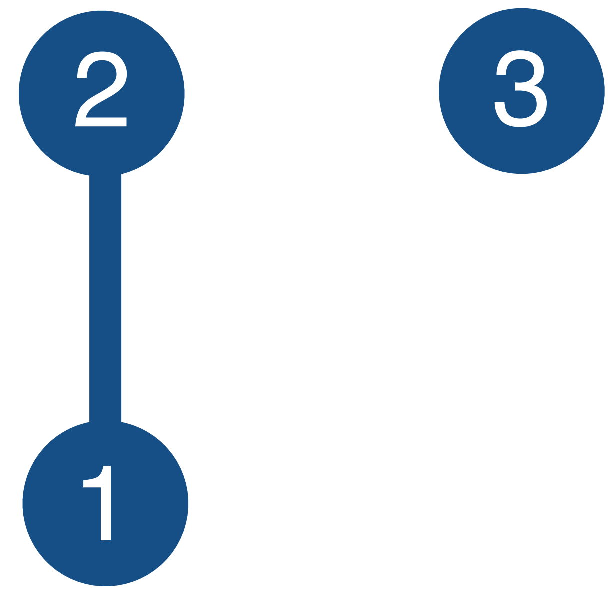


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$$\sim cn^2$$



[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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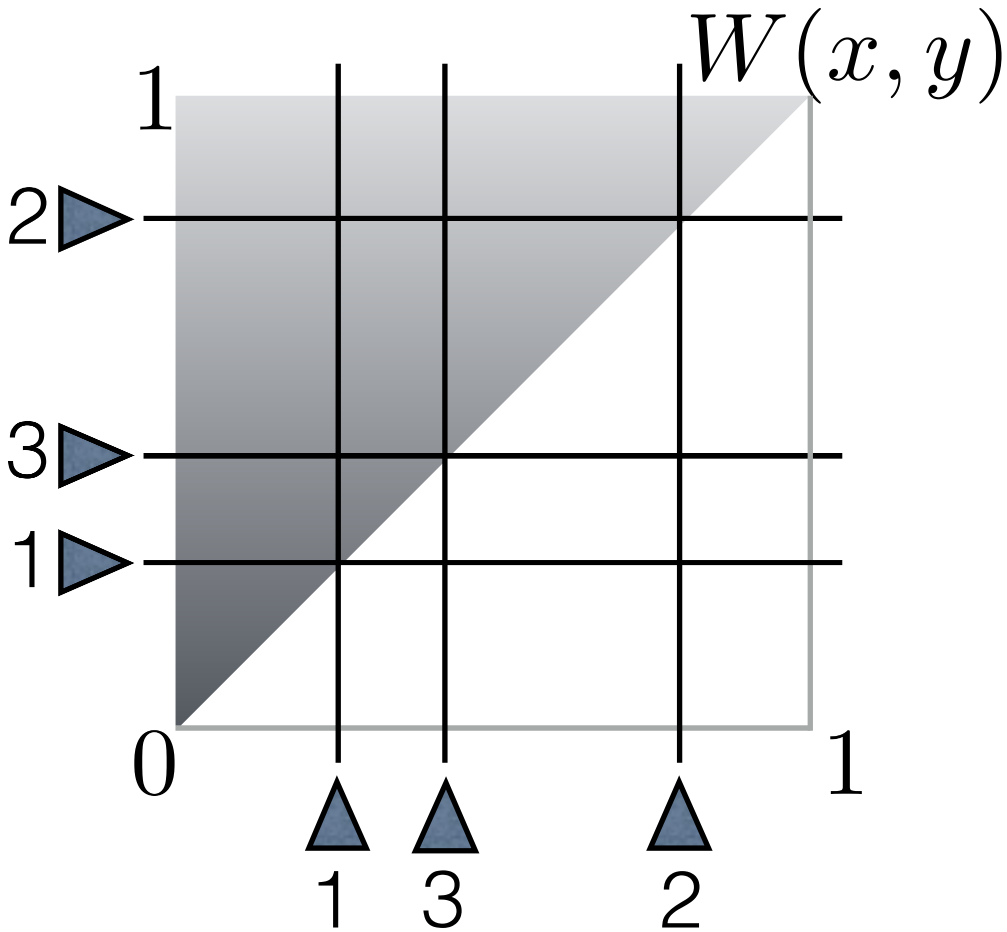


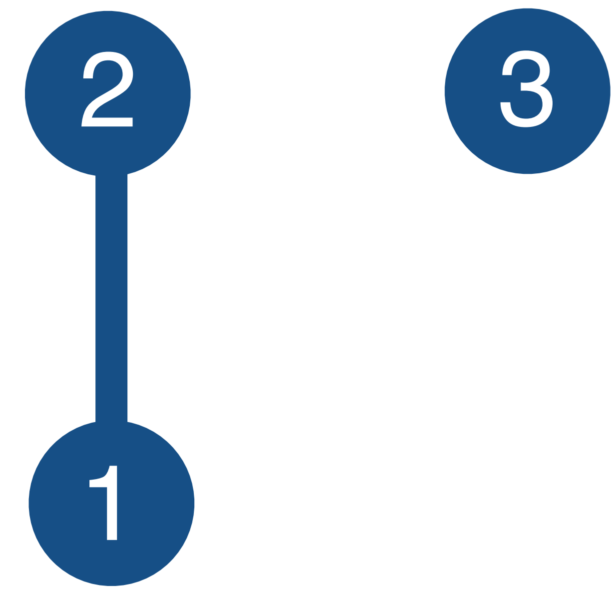


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[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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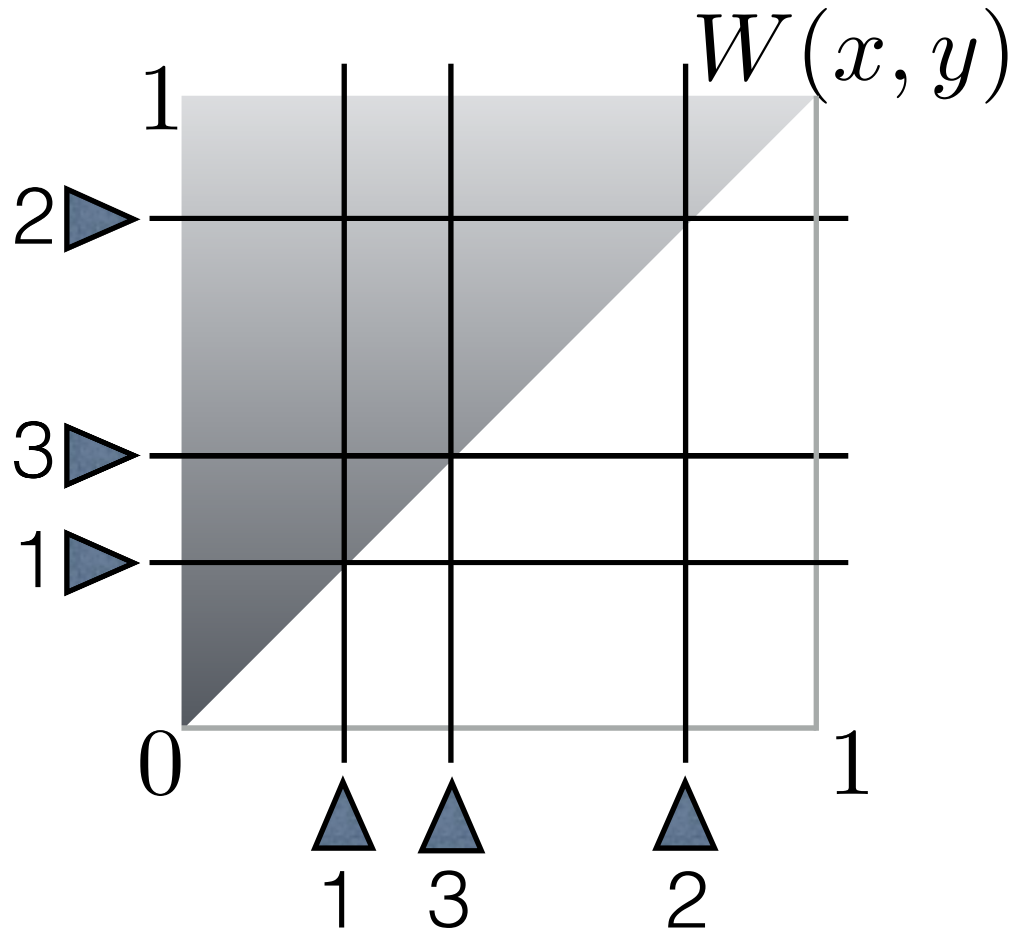


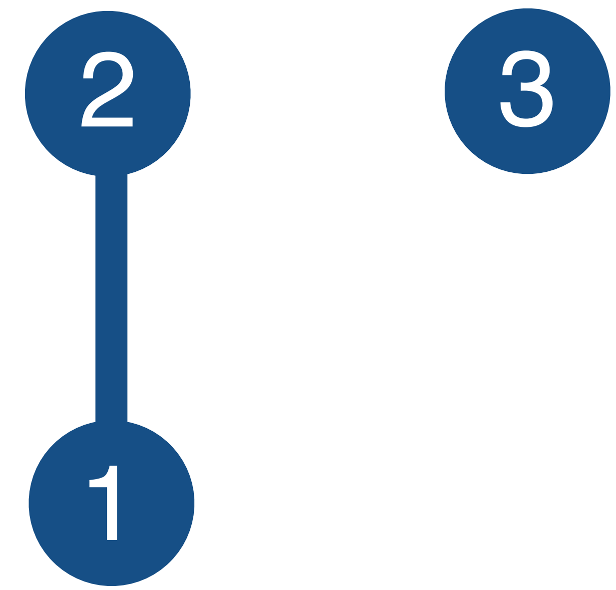


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Cor. Every node-exch graph sequence is dense (or empty) a.s.



[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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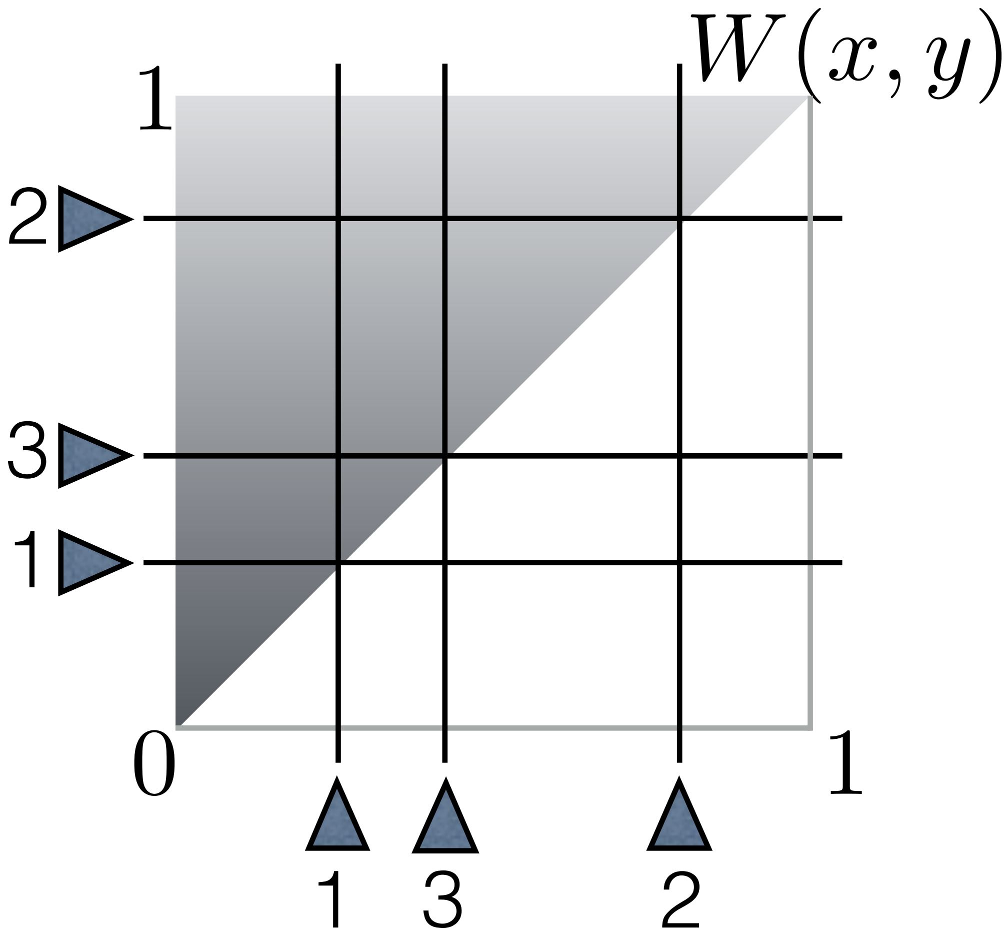


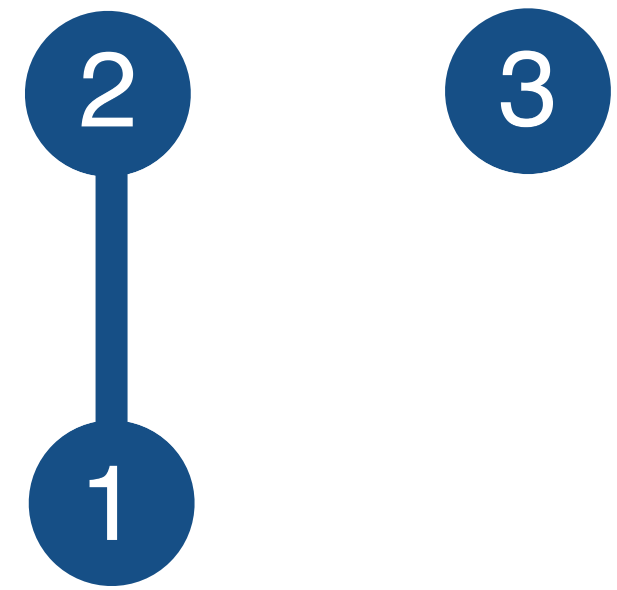


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Cor. Every node-exch graph sequence is dense (or empty) a.s.

Intuition: To a given node, all other nodes look the same.



[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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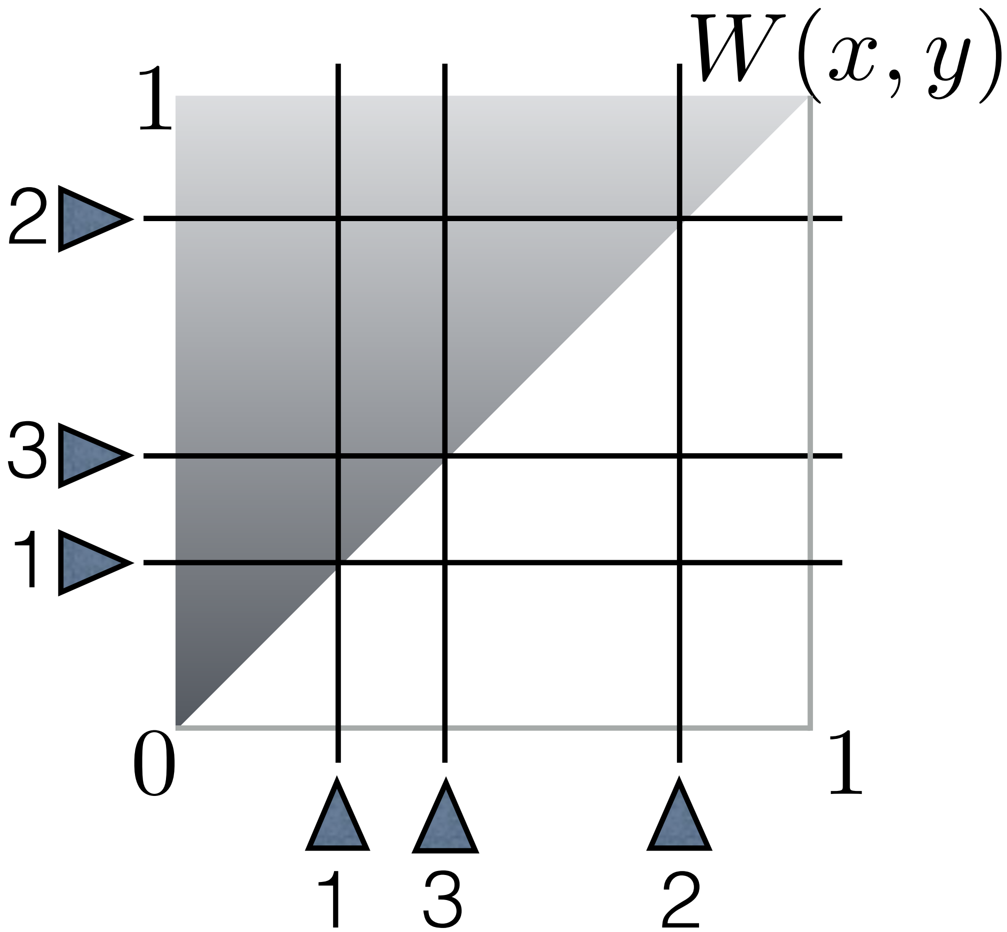


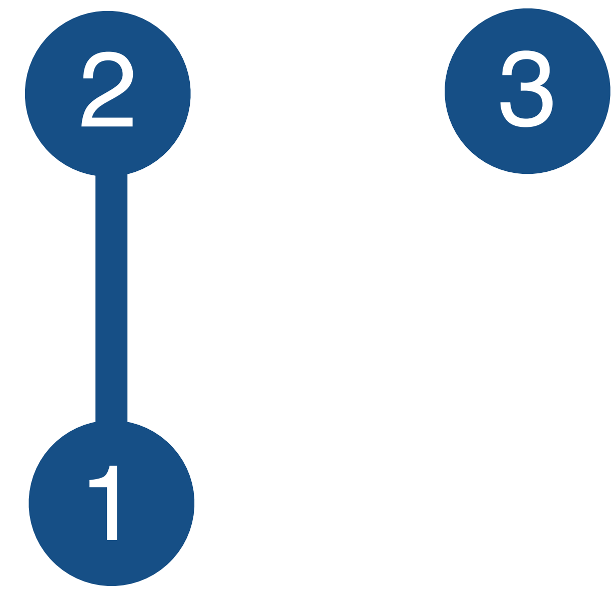


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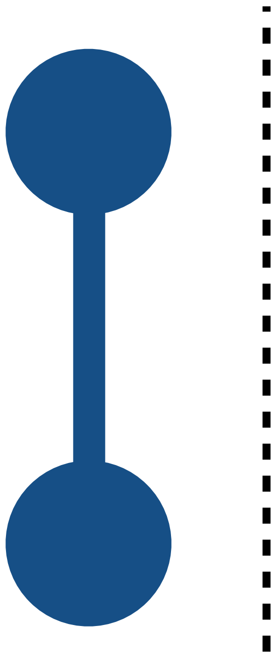
[Caron, Fox 2014; Veitch, Roy 2015; Borgs, Chayes, Cohn, Holden 2016; Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b, 2016a; Cai, Campbell, Broderick 2016]

# A New Way: Edges



$G_1$

 [Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b,16a; Cai, Campbell, Broderick 2016]

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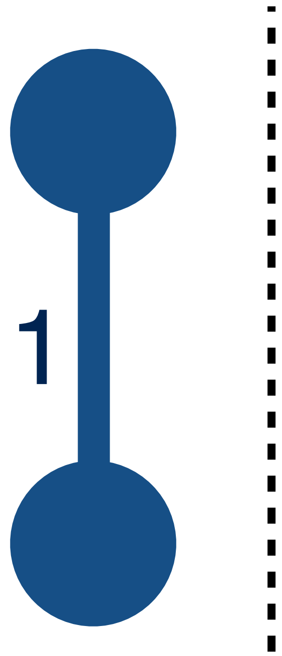


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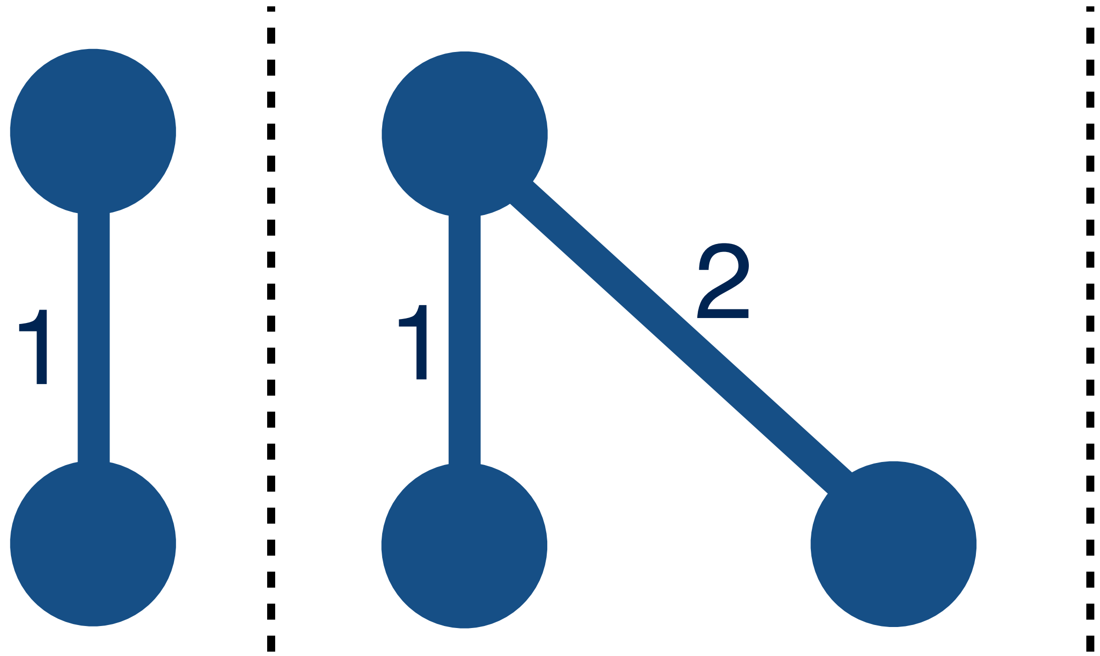


$G_1$ $G_2$

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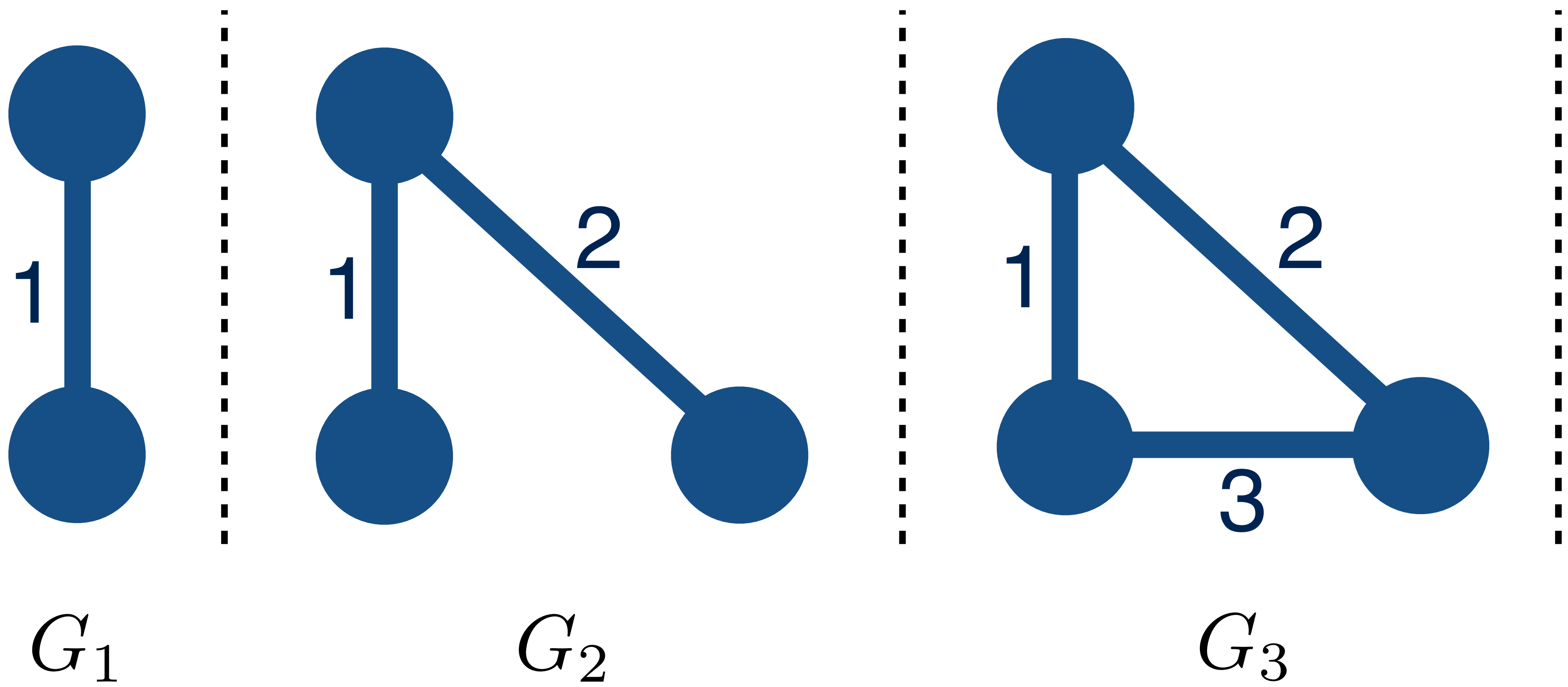




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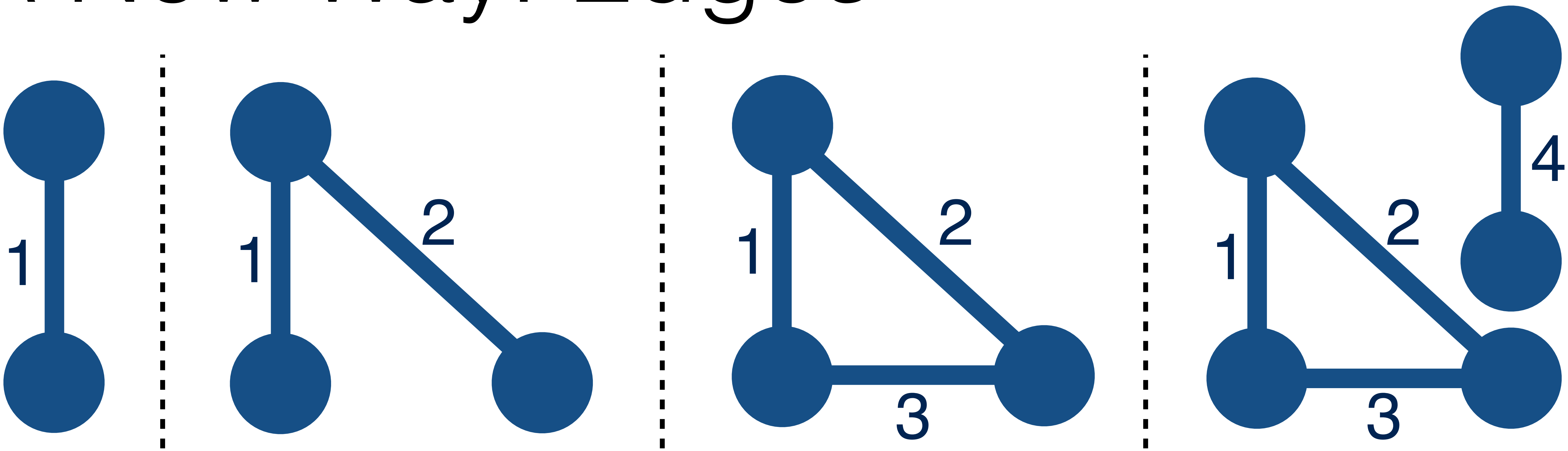
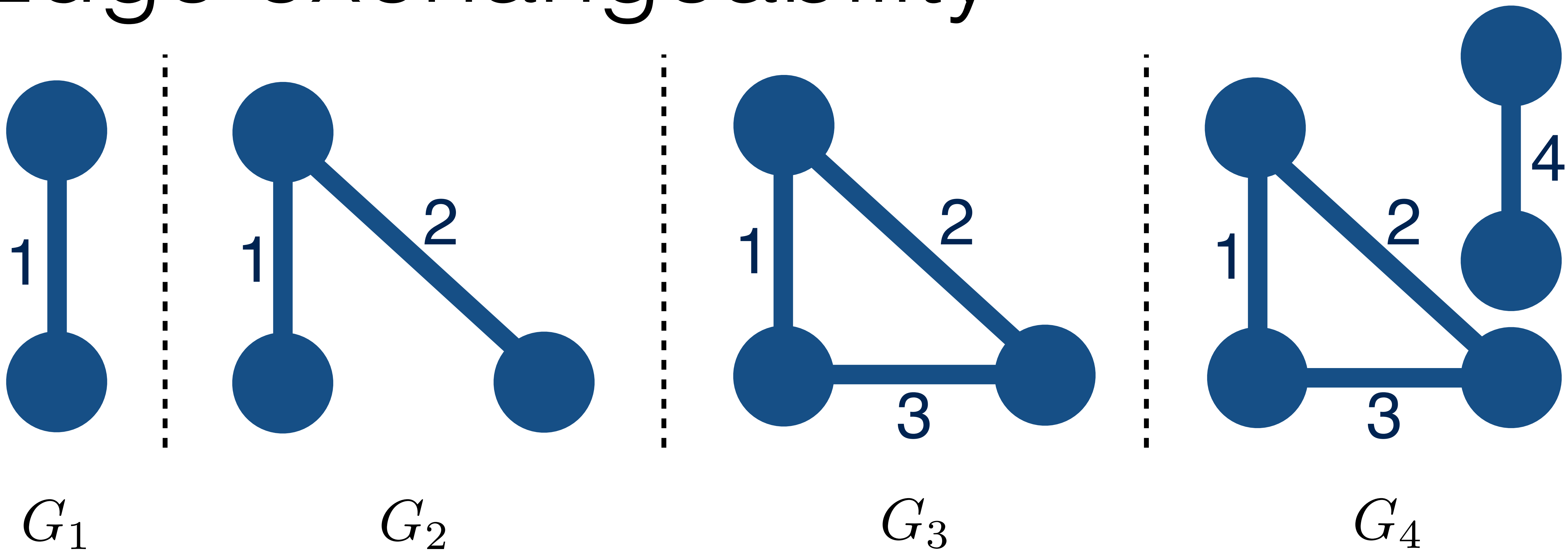



$G_1$ $G_2$ $G_3$ $G_4$


[Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b,16a; Cai, Campbell, Broderick 2016]

# Edge exchangeability




 [Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b,16a; Cai, Campbell, Broderick 2016]

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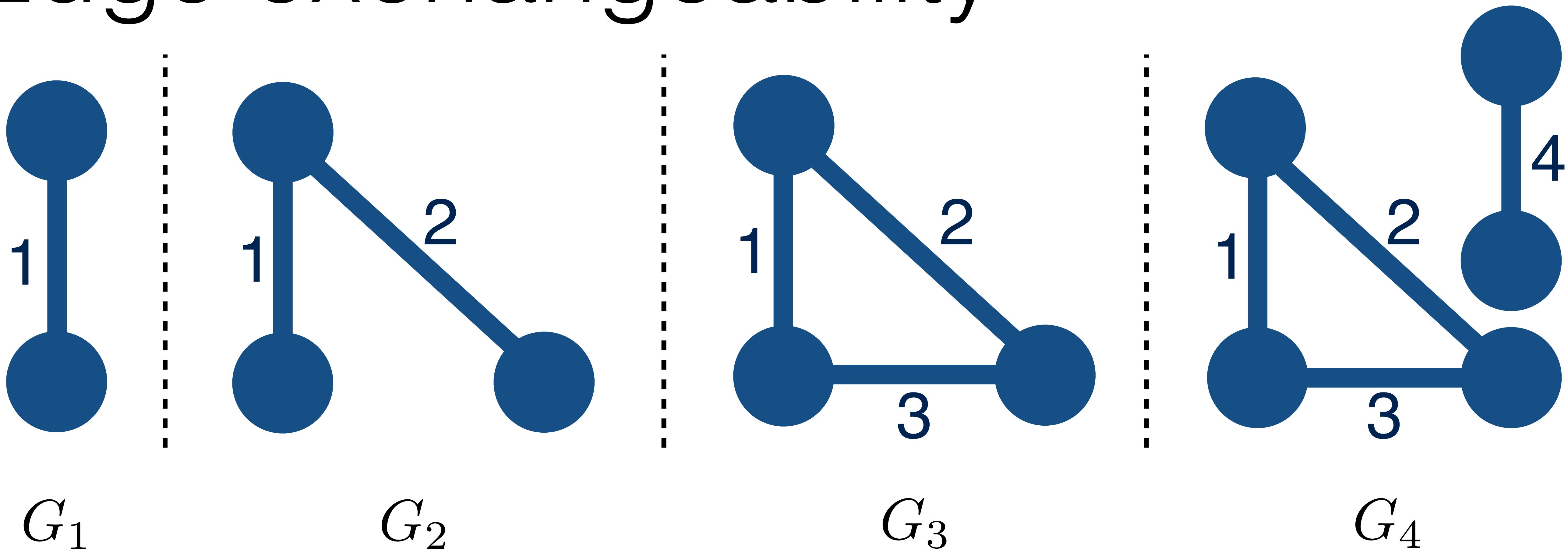


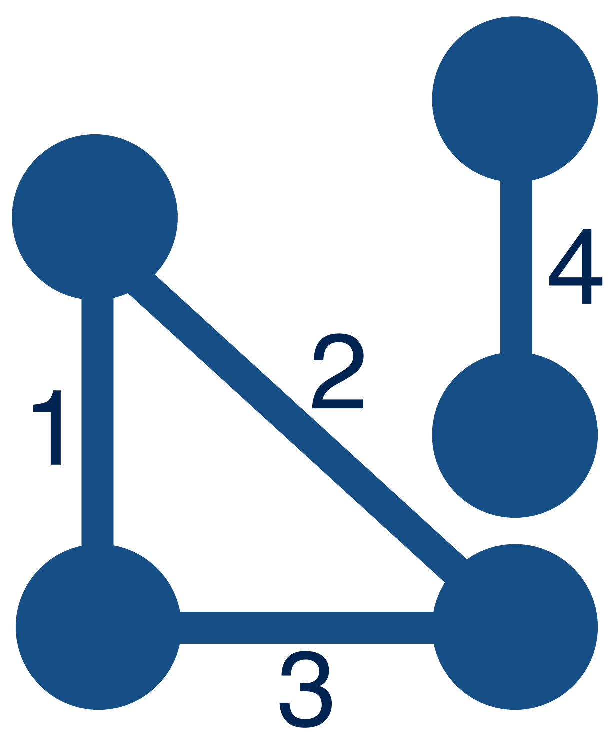


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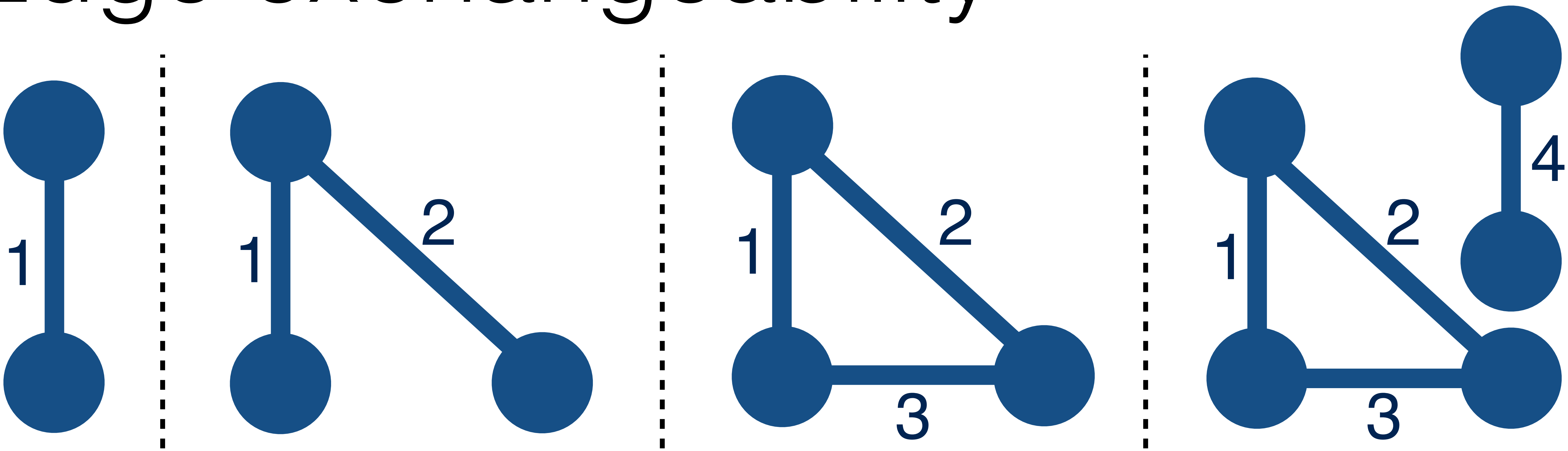
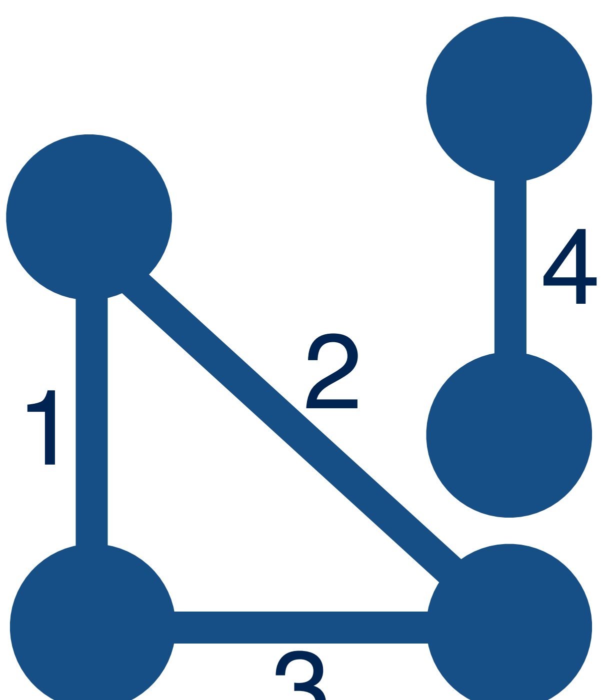



$G_1$ $G_2$ $G_3$ $G_4$

$p($  $)$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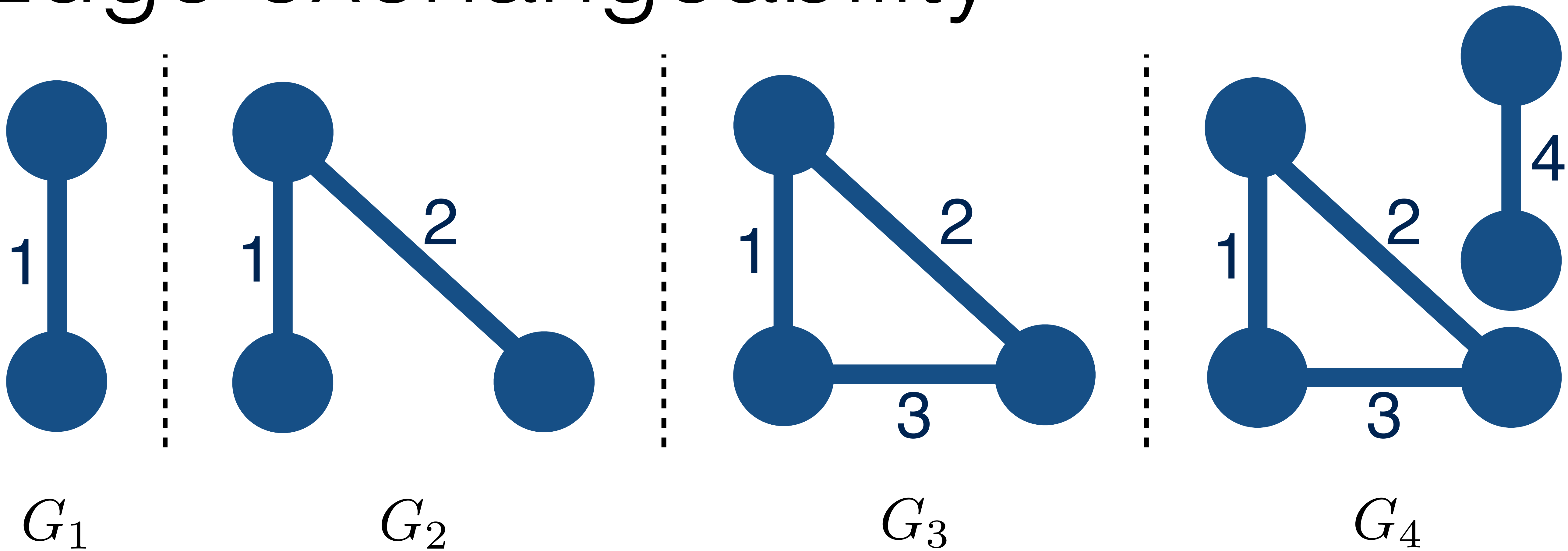


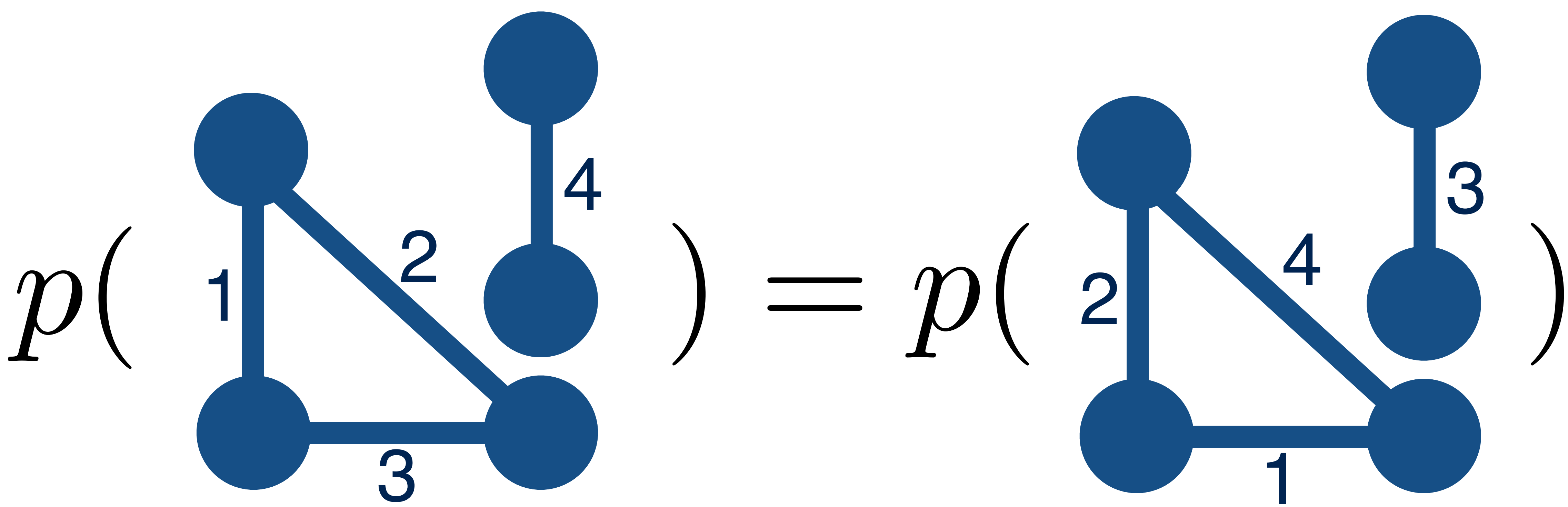


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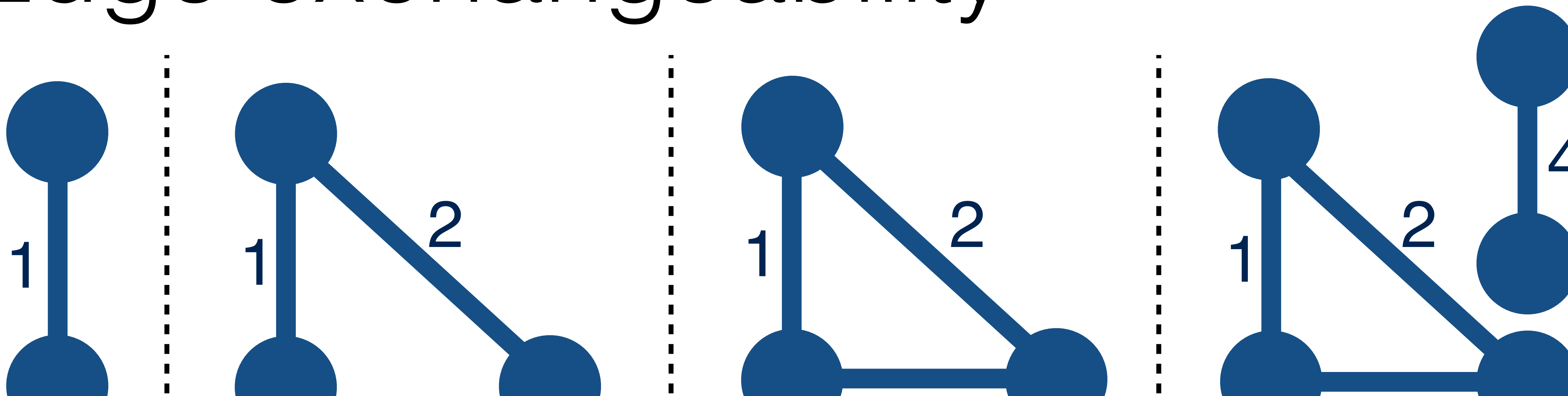
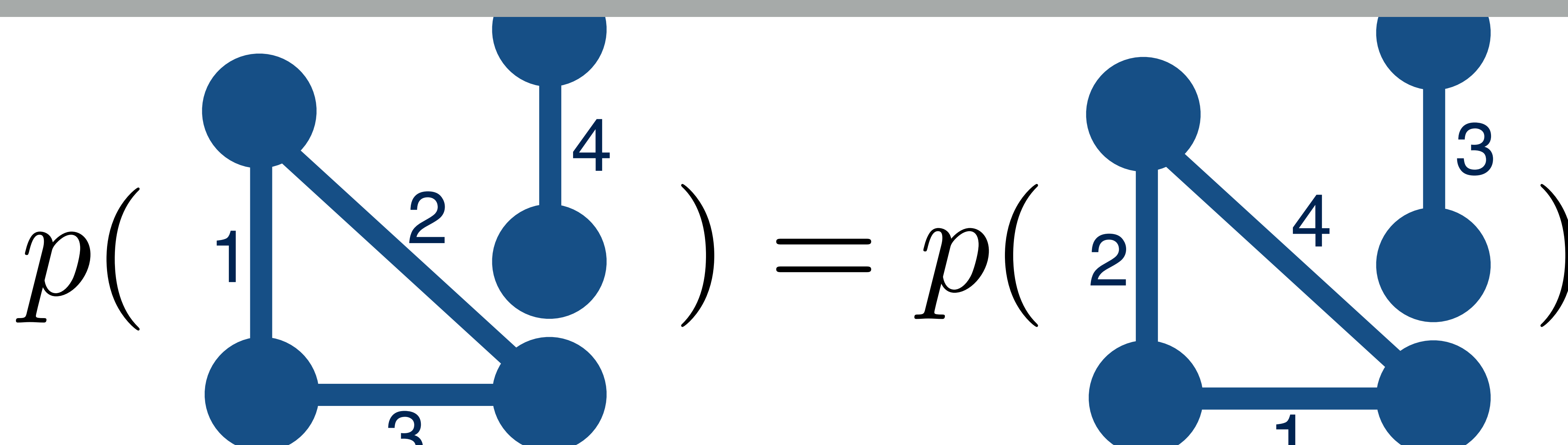
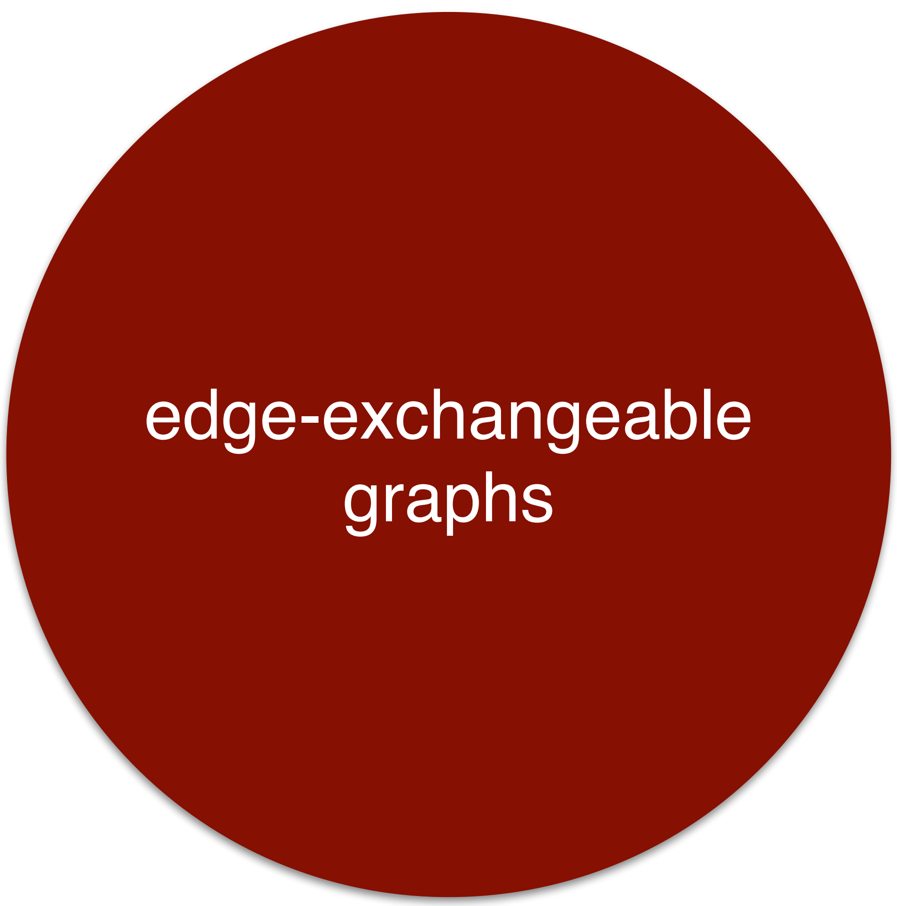



**Thm (CCB). A wide class of edge-exchangeable graph models yields sparse graph sequences.**



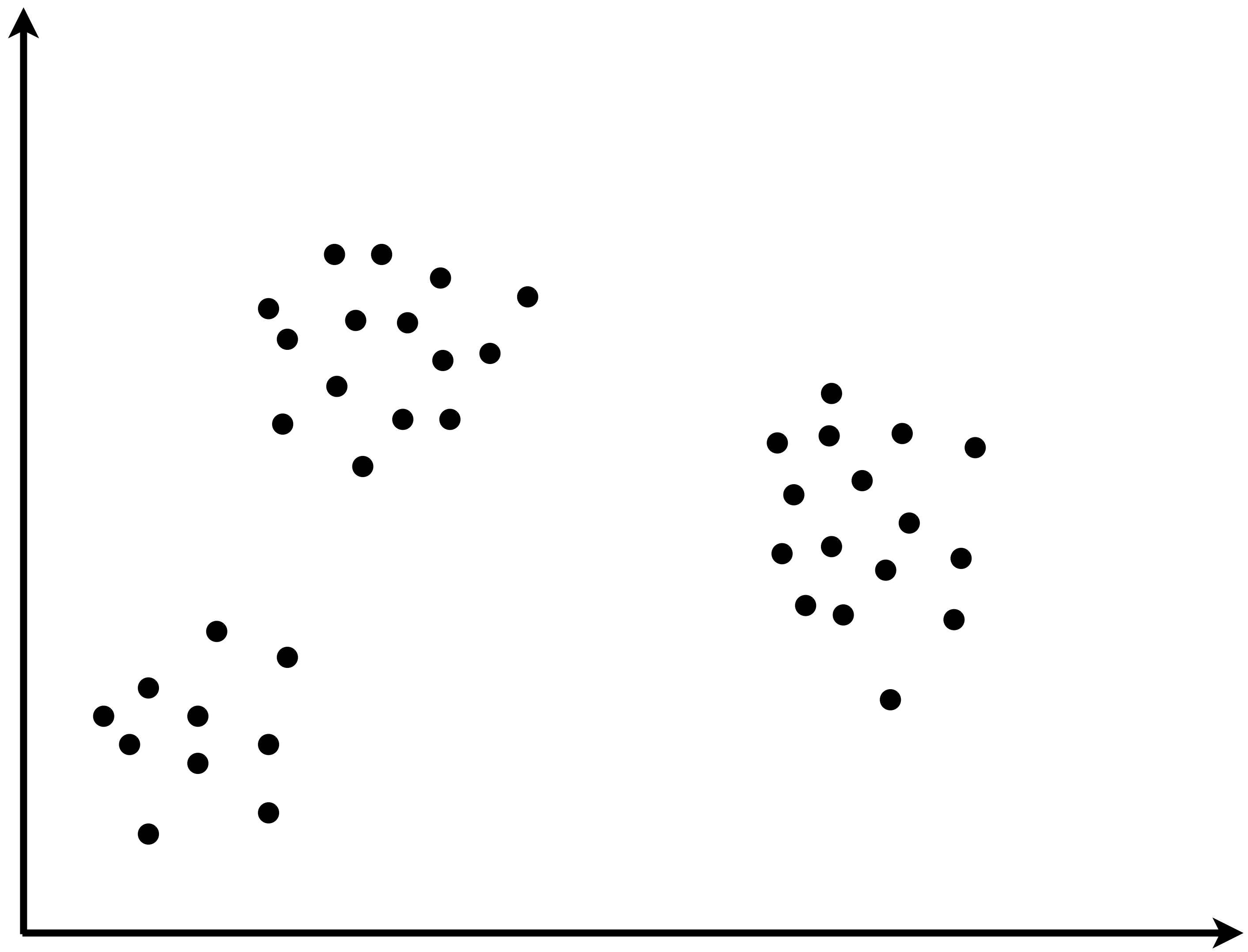

 [Broderick, Cai 2015; Cai, Broderick 2015a,b; Crane, Dempsey 2015a,b,16a; Cai, Campbell, Broderick 2016]

# What we know so far




- Goal 1: characterization theorem for edge-exchangeable graphs
- Goal 2: sparsity theorem for edge-exchangeable graphs

# Clustering



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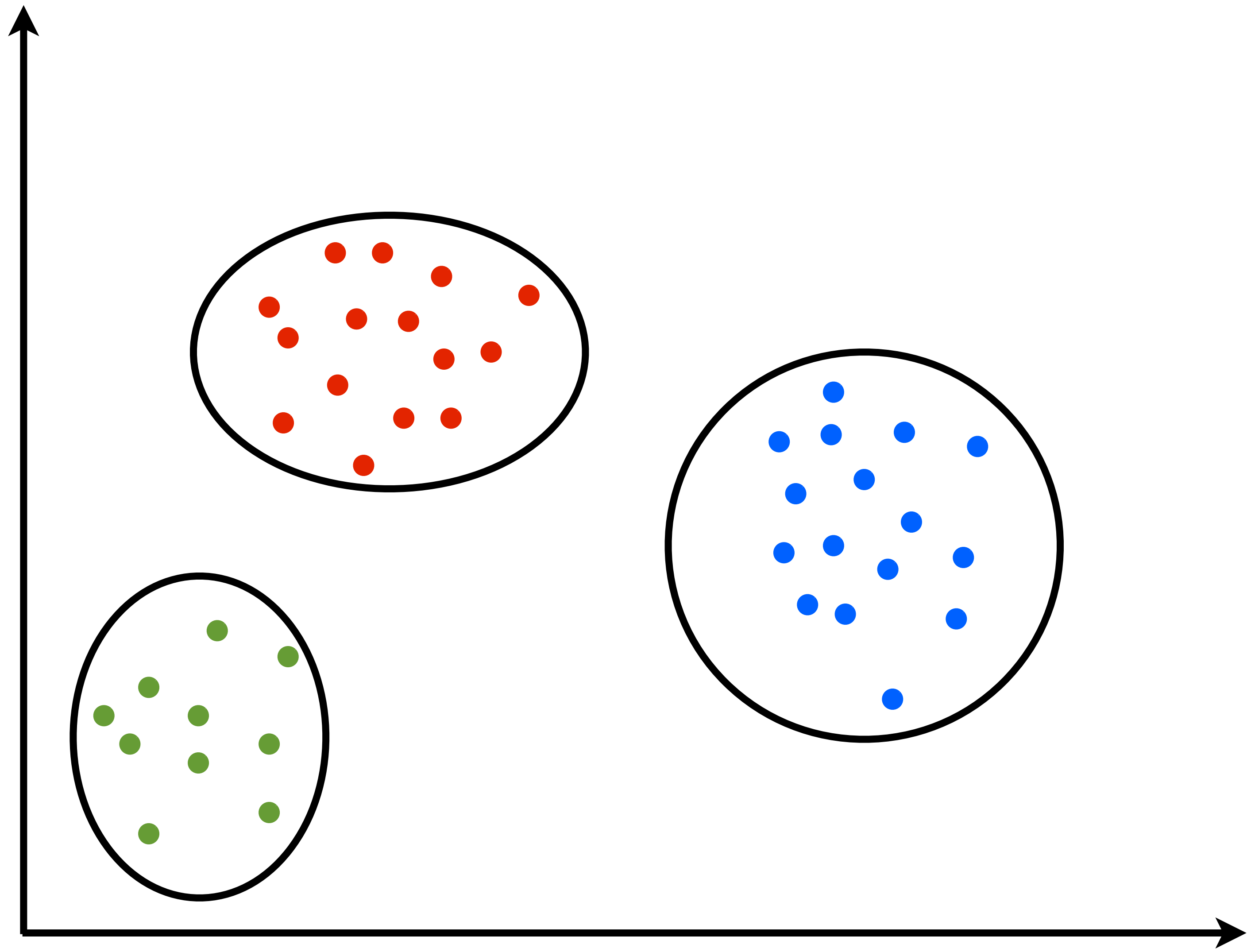
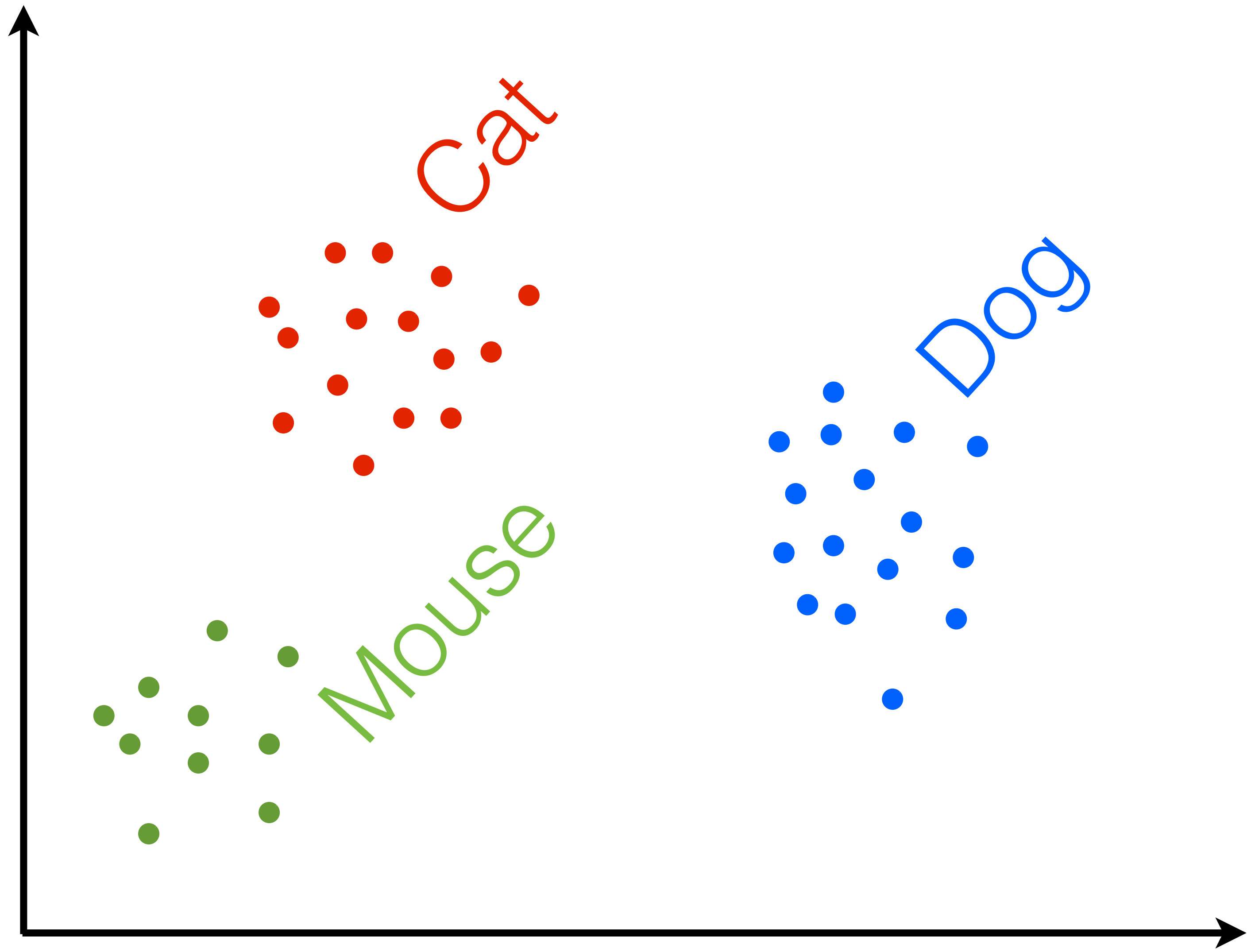


"Clusters"

# Clustering




“Clusters”

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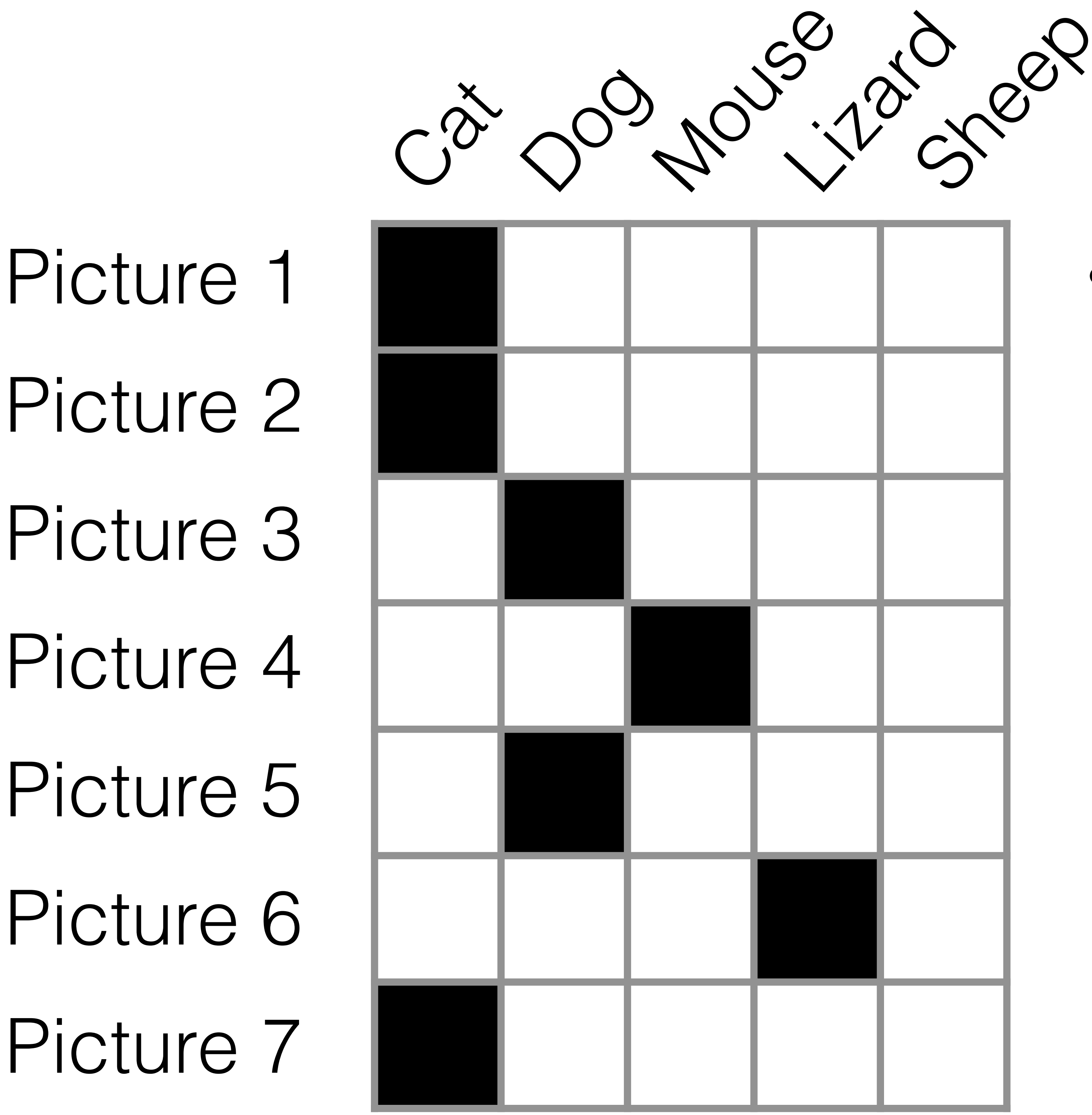
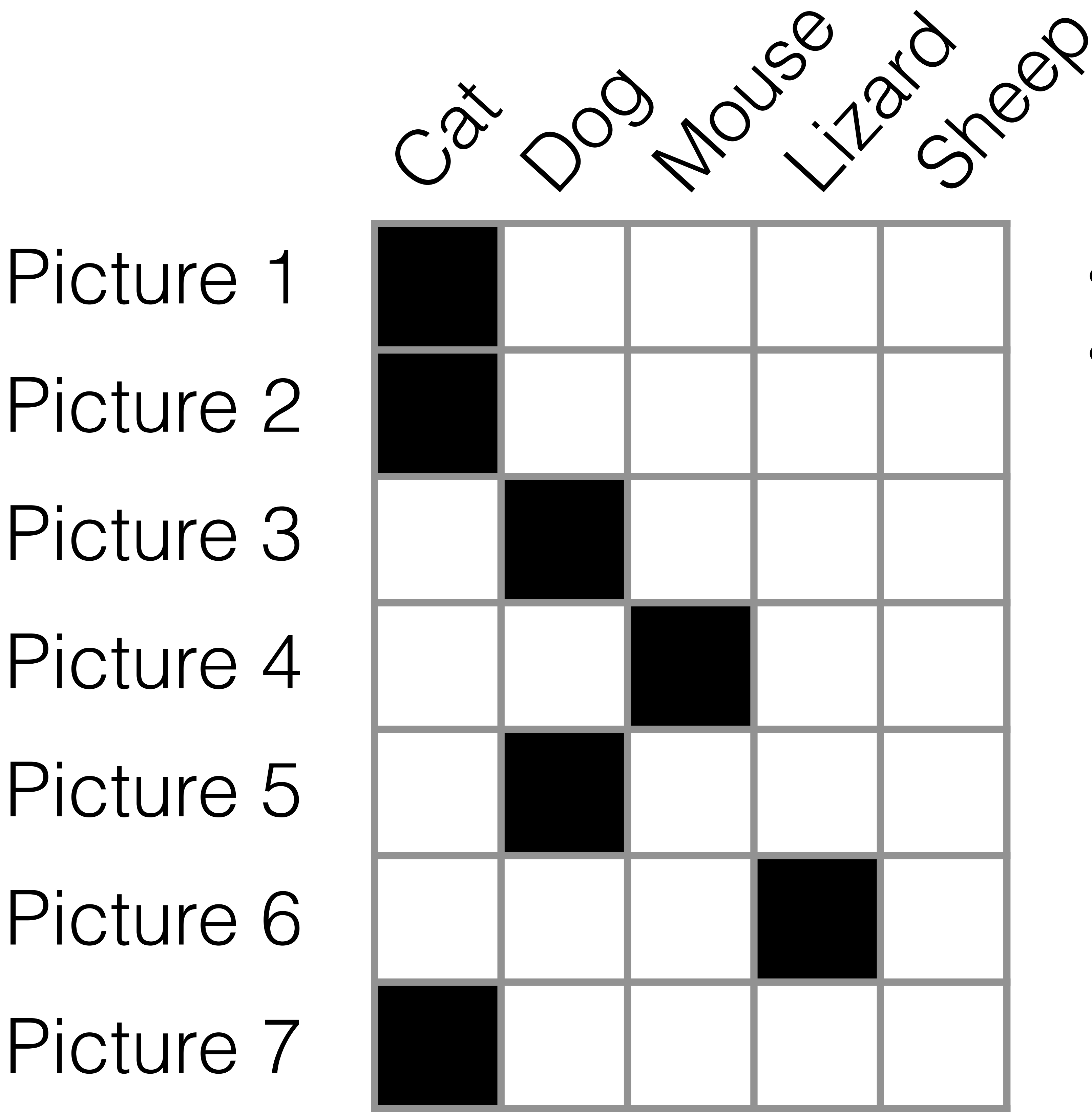


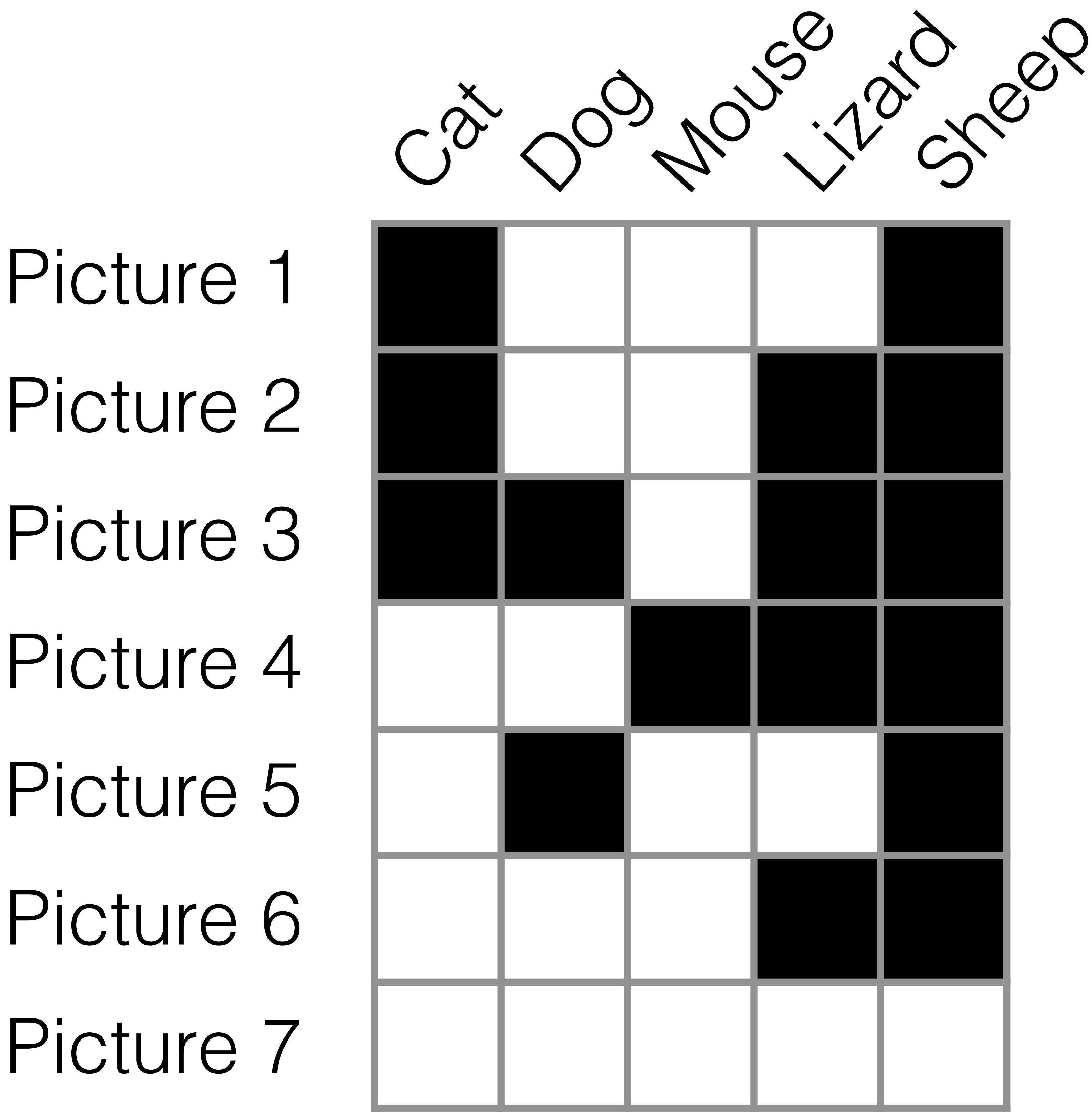

- Groups: clusters

# Clustering




- Groups: clusters
- Exchangeable

# Feature allocation



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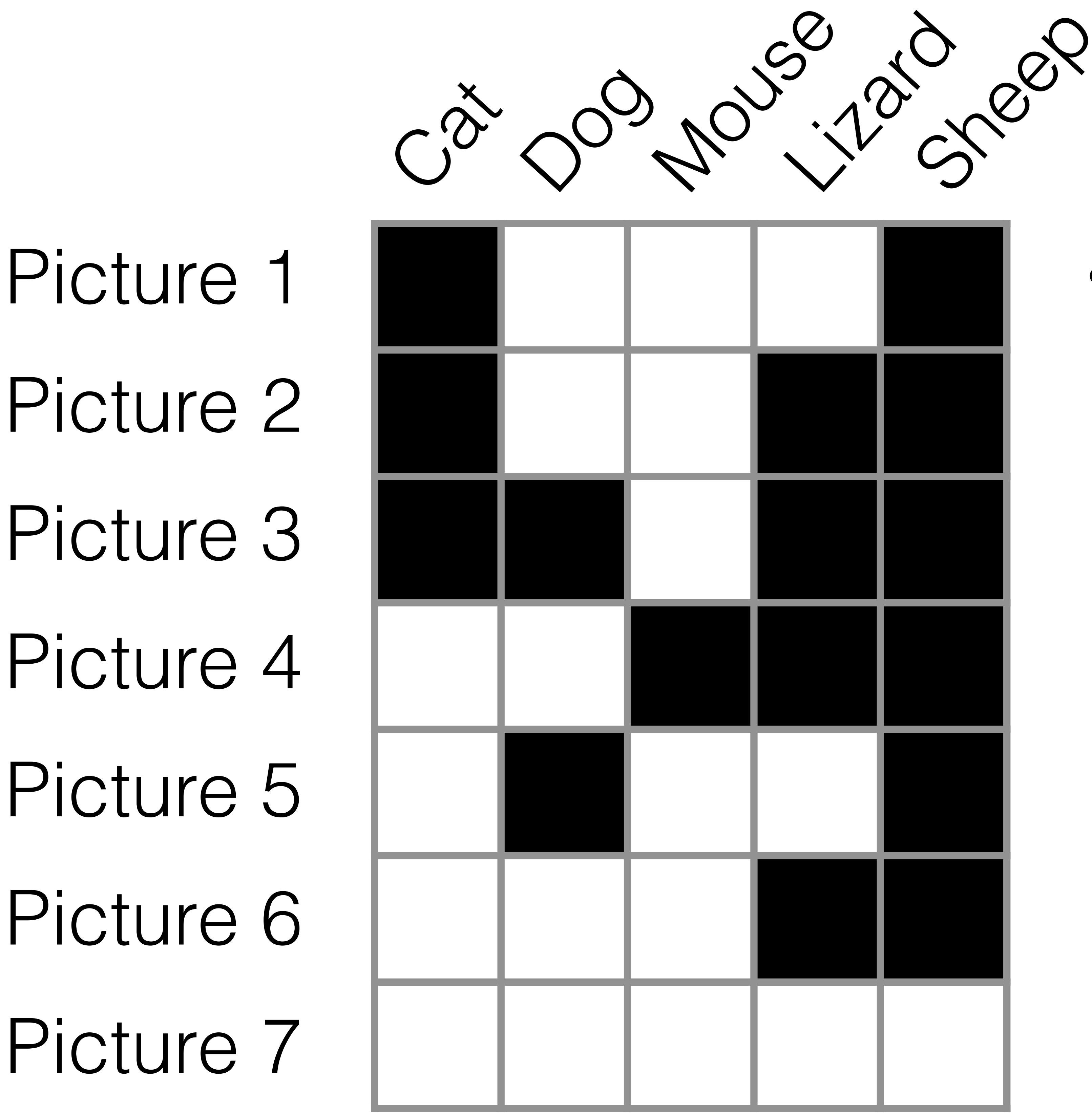
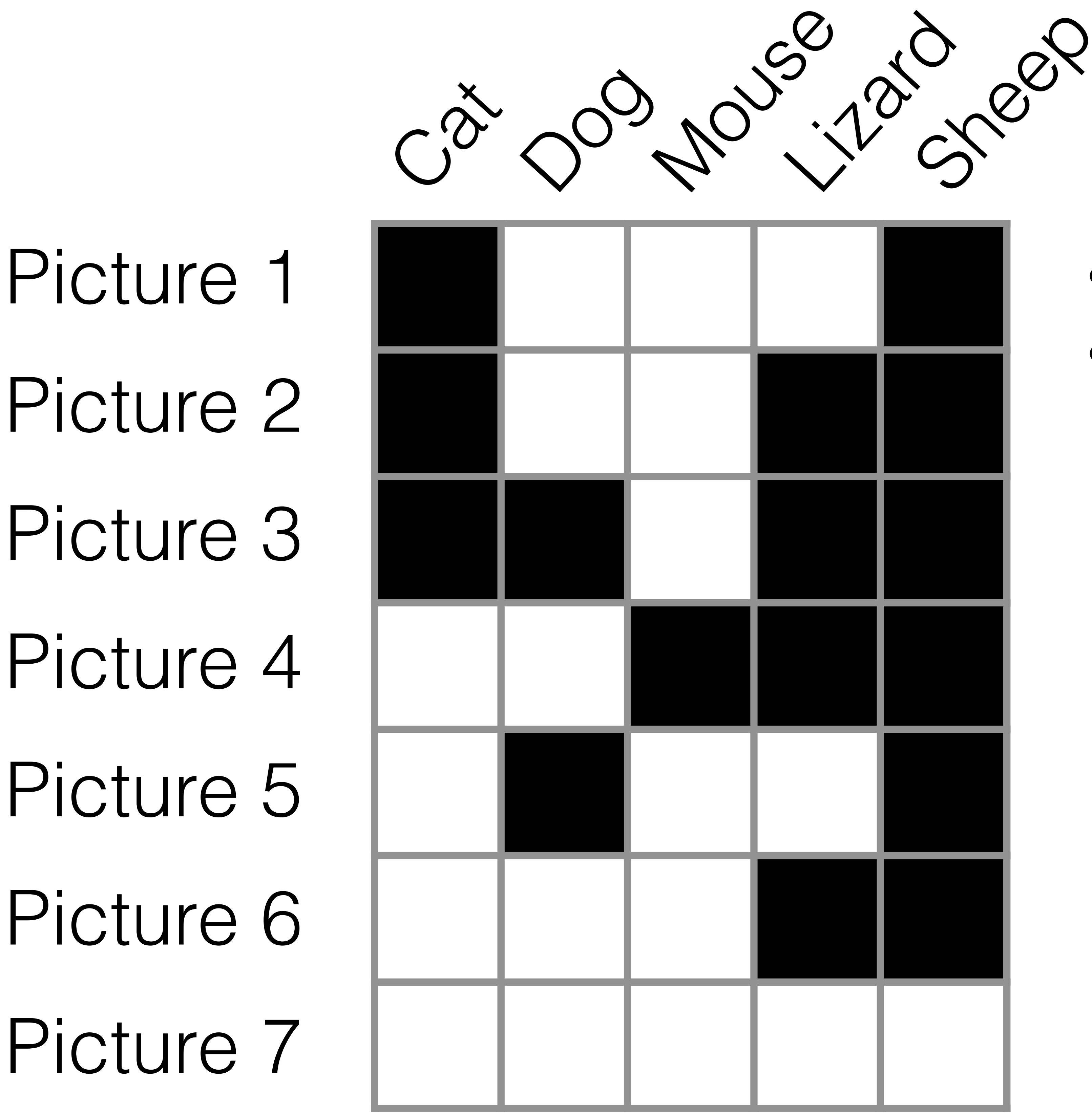



- Groups: features

# Feature allocation

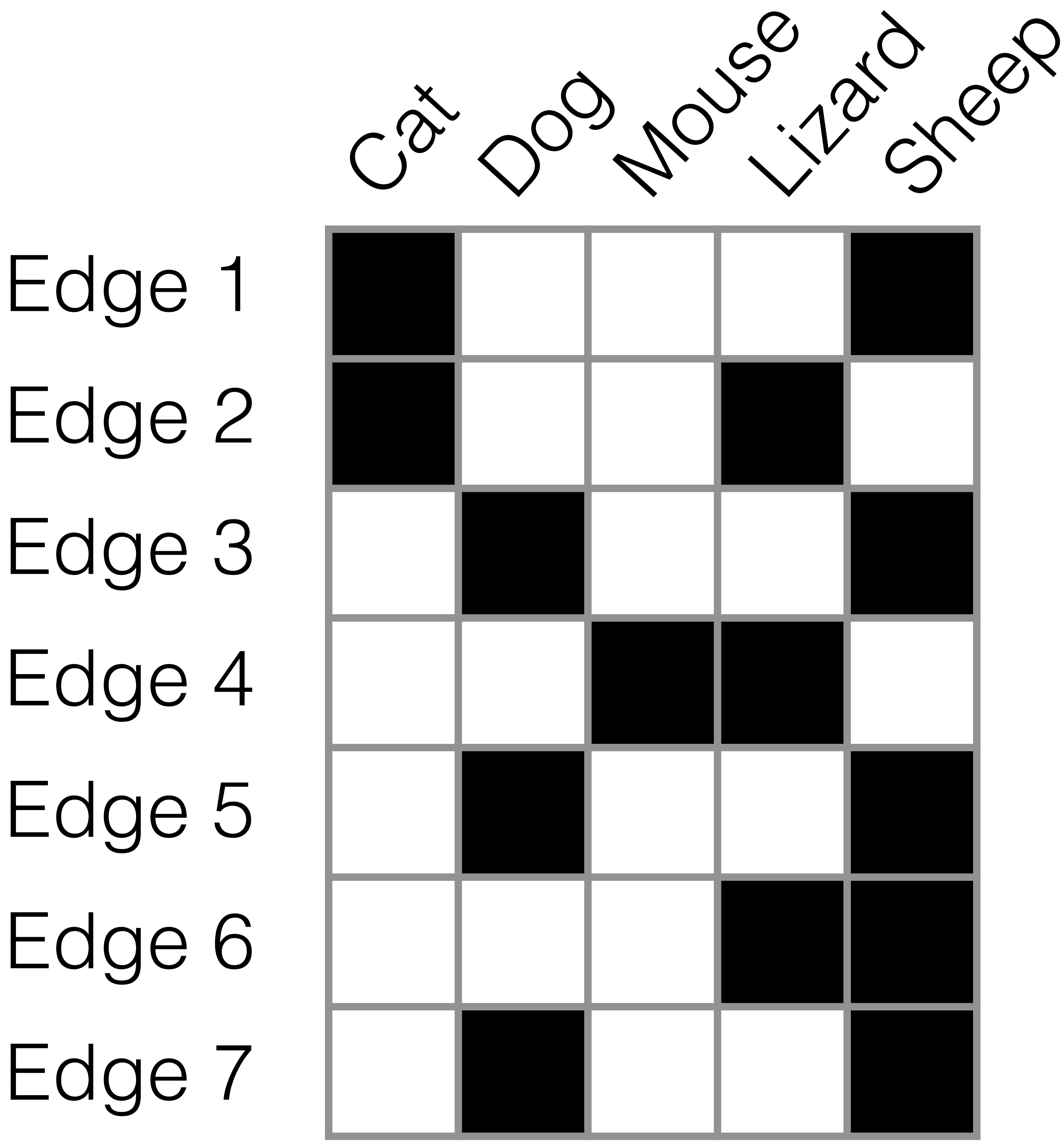



- Groups: features
- Exchangeable

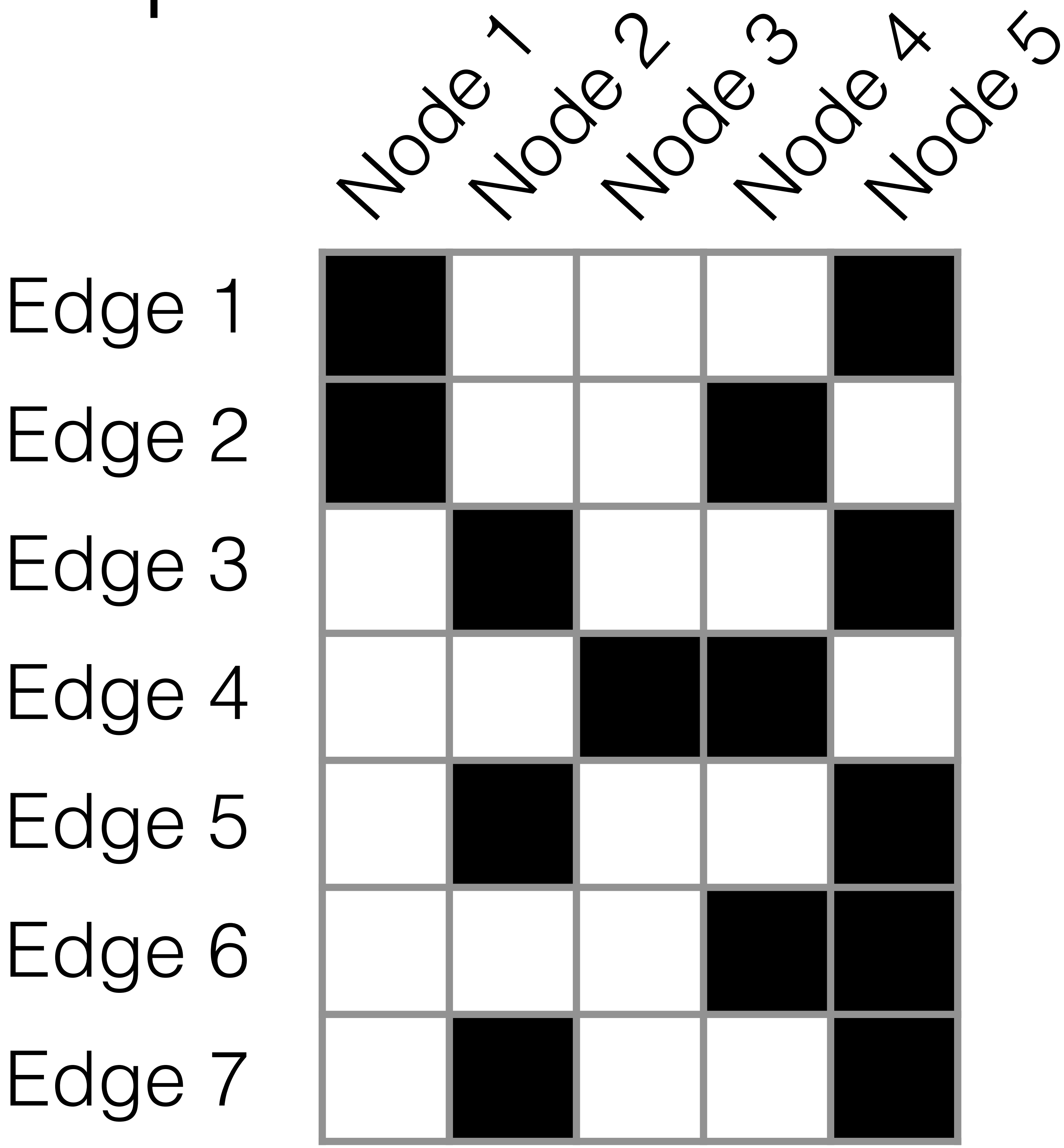


[Broderick, Jordan, Pitman 2013; Broderick, Pitman, Jordan 2013]

# Graph


Cat
Dog
Mouse
Lizard
Sheep
Edge 1
Edge 2
Edge 3
Edge 4
Edge 5
Edge 6
Edge 7

# Graph



Node 1
Node 2
Node 3
Node 4
Node 5
Edge 1
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Edge 4
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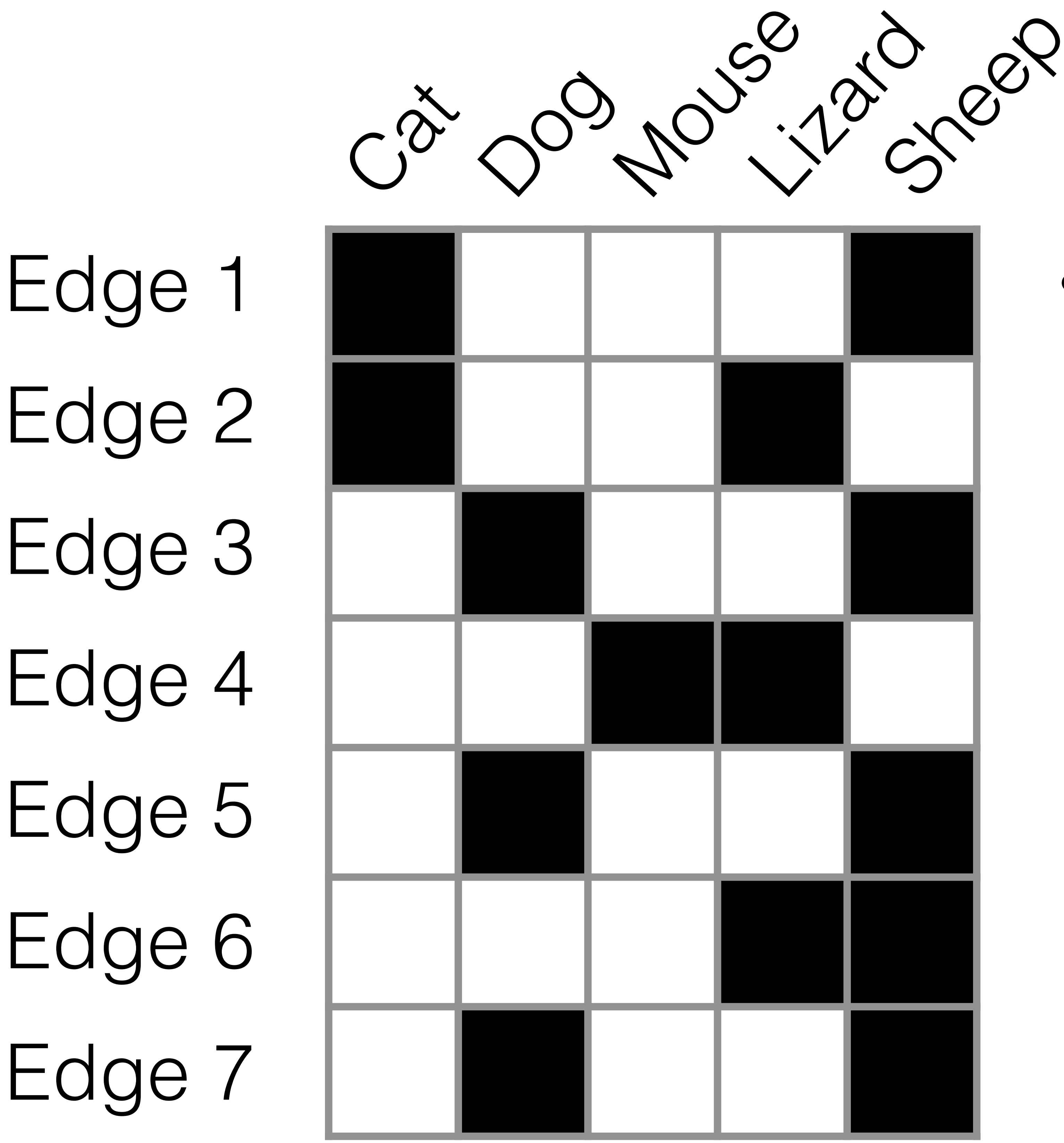
Edge 7

# Graph




- Groups: vertices

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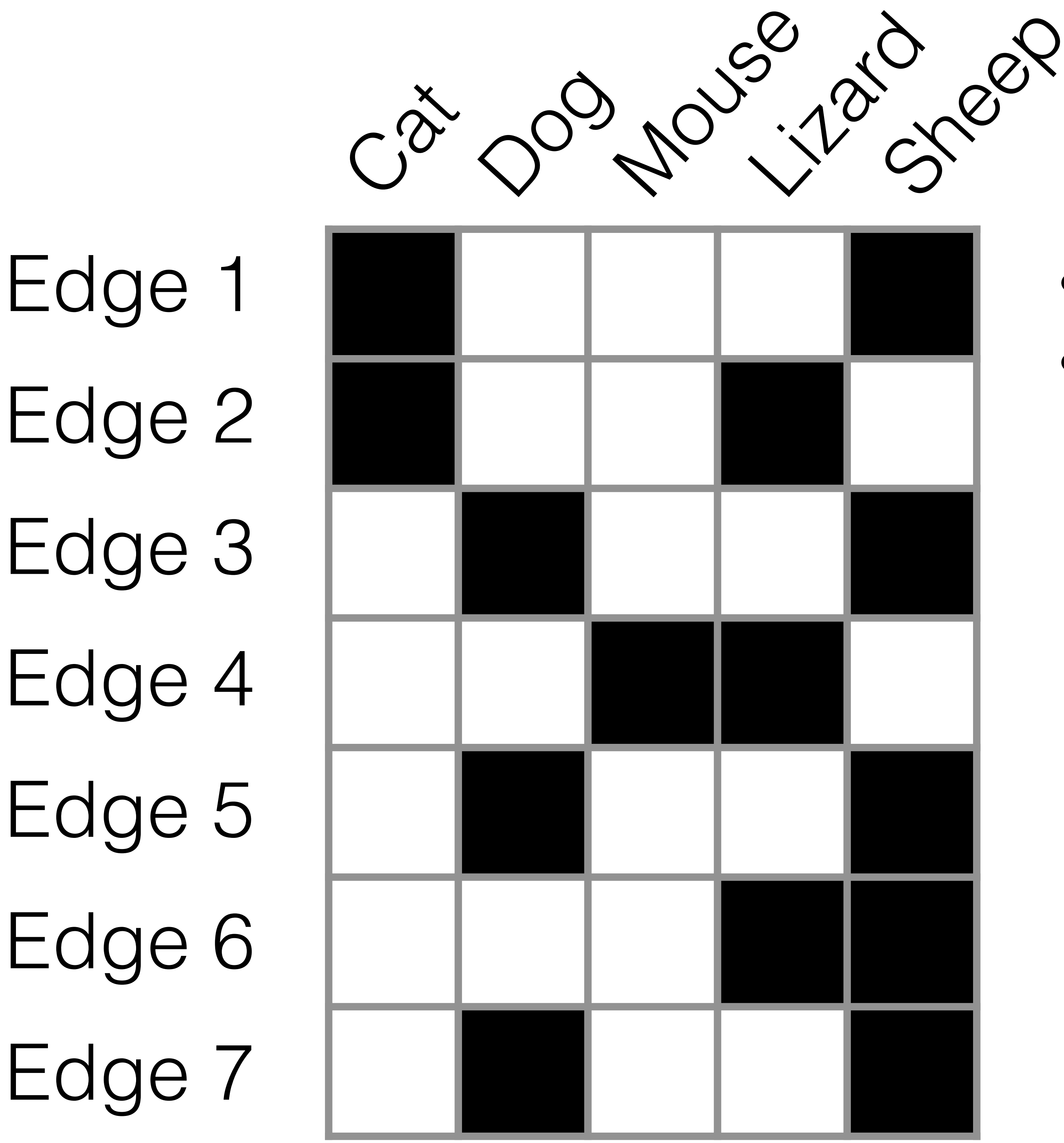
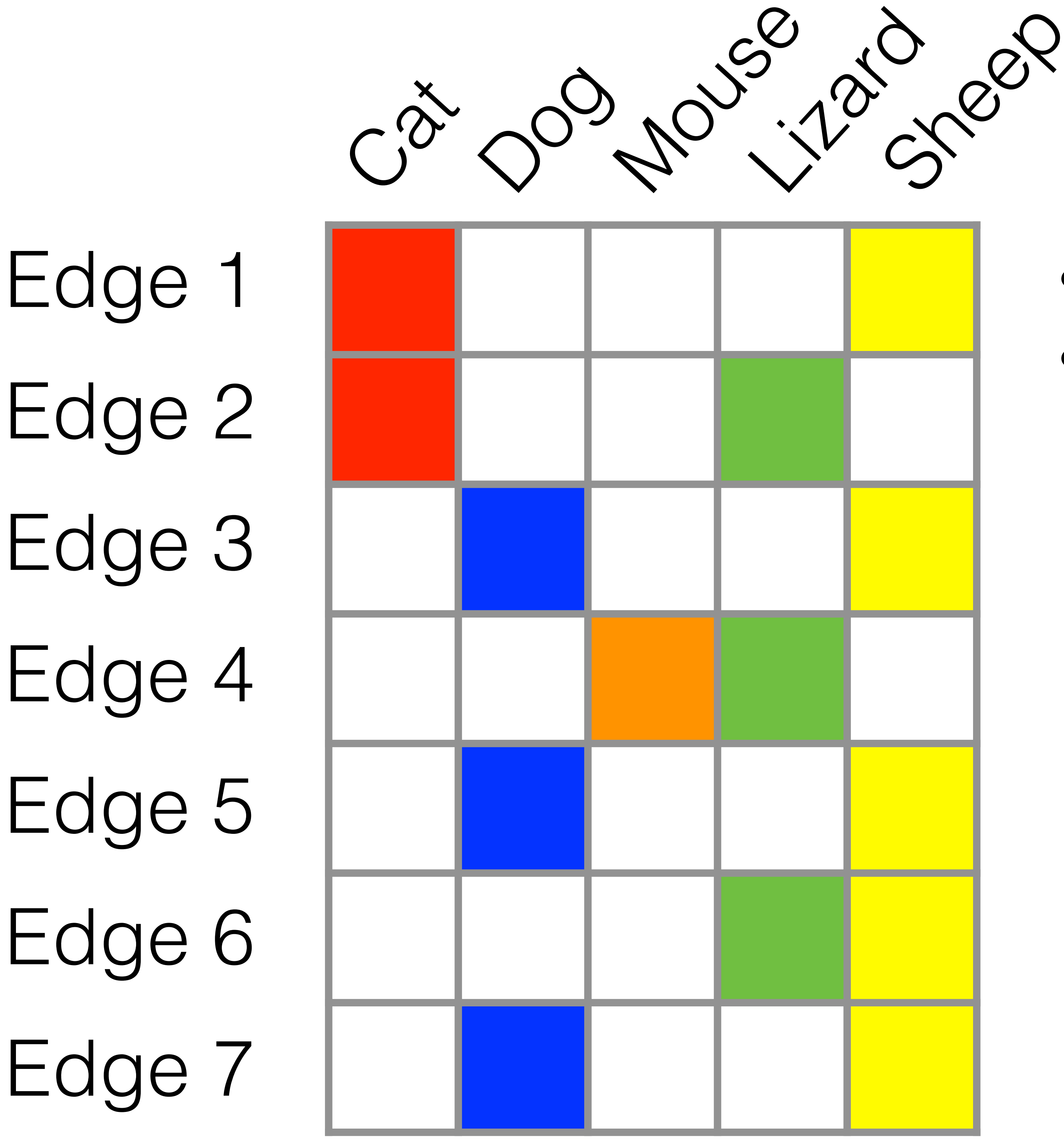



- Groups: vertices
- Edge-exchangeable

# Graph




- Groups: vertices
- Edge-exchangeable

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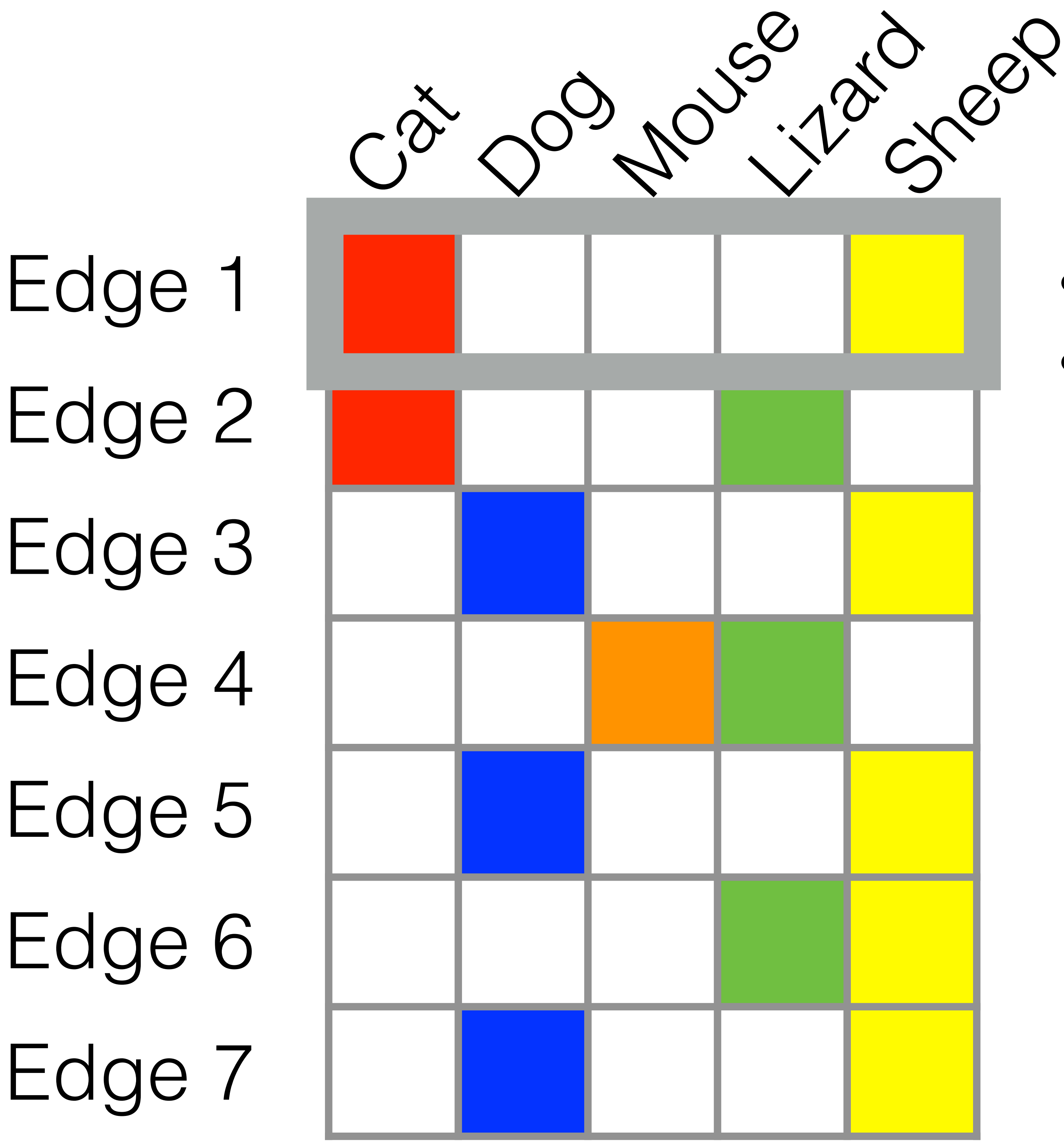


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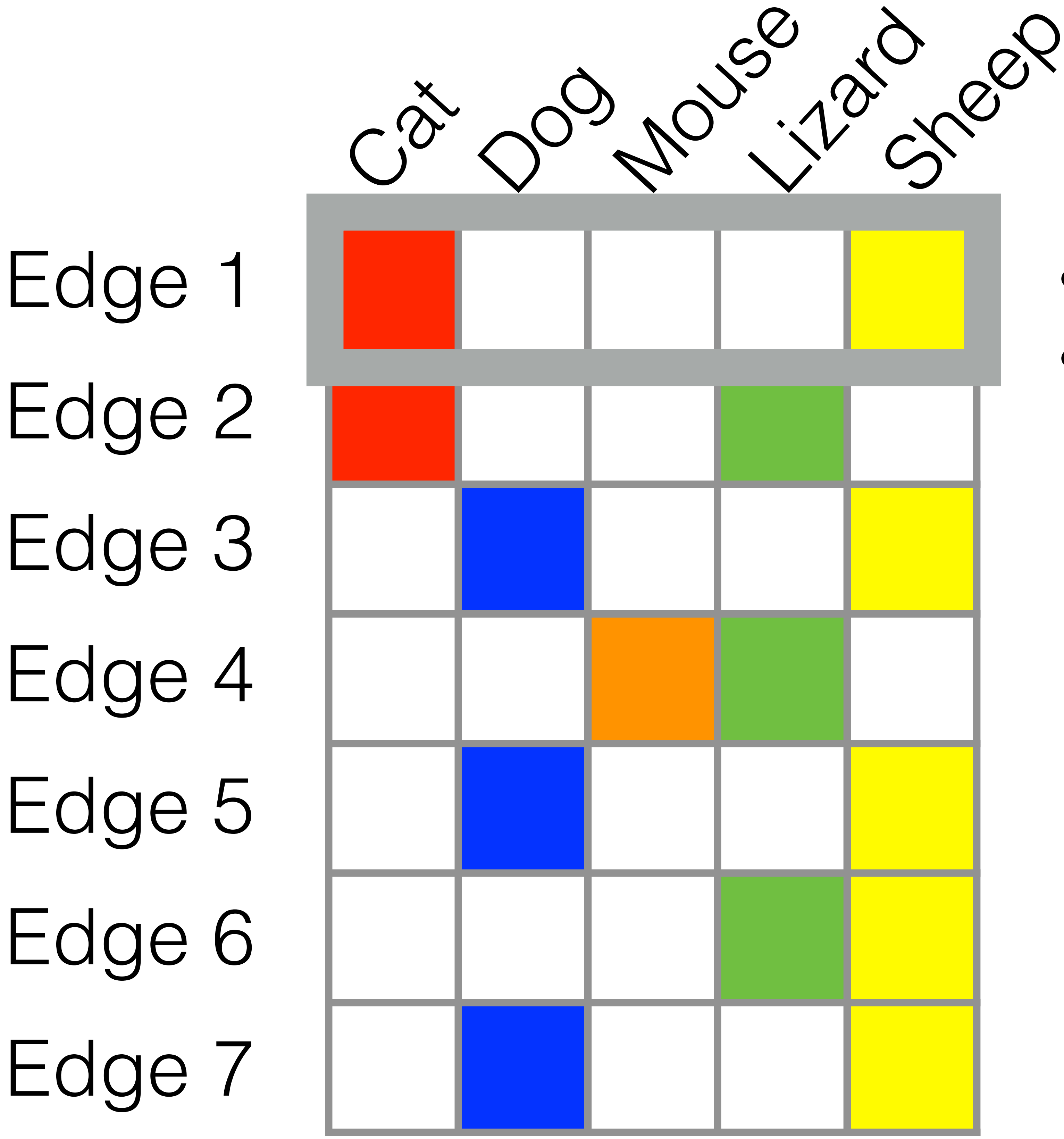


- Groups: vertices
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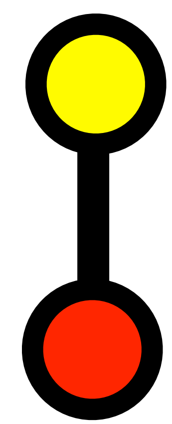

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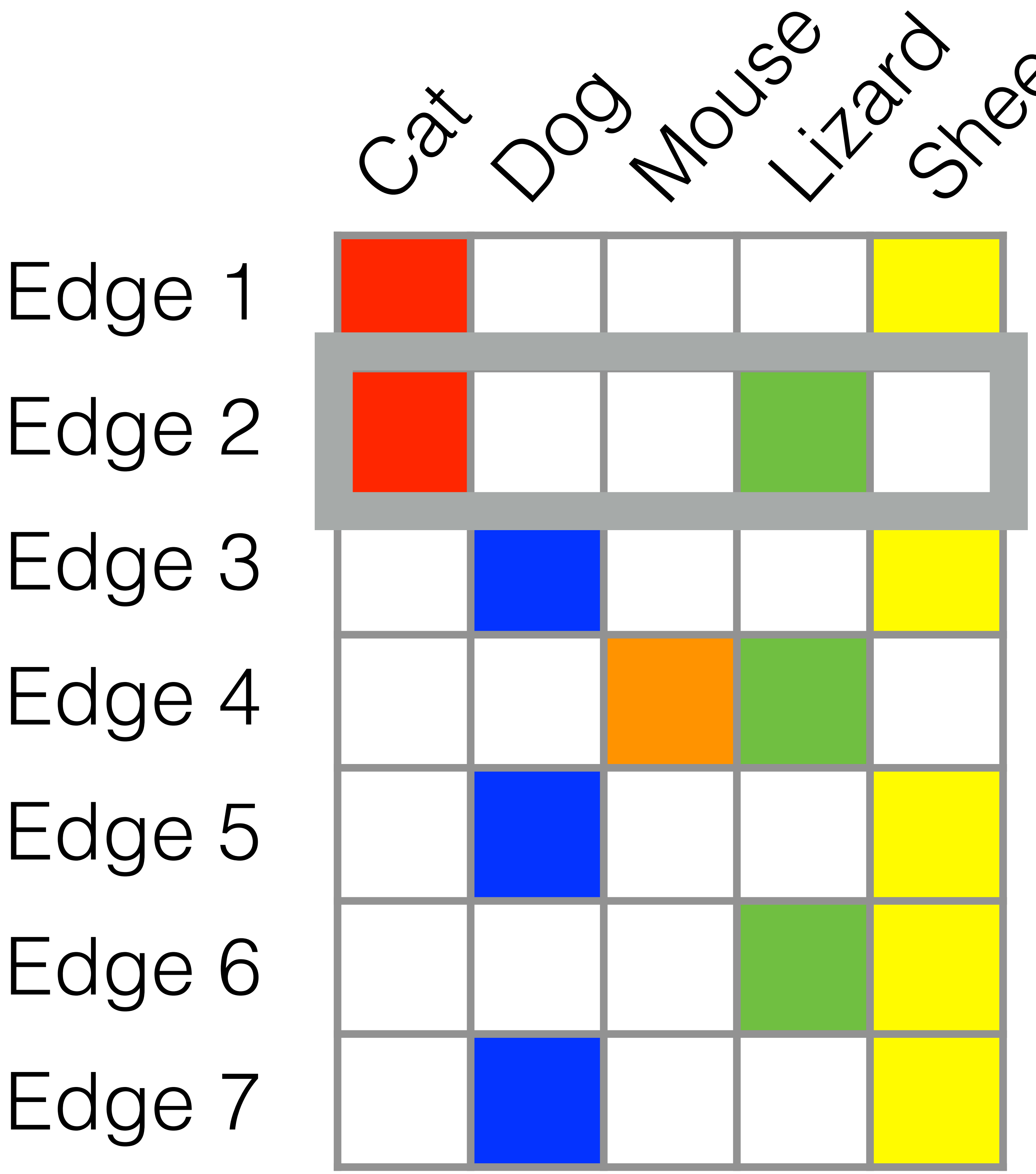


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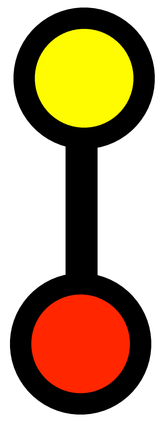

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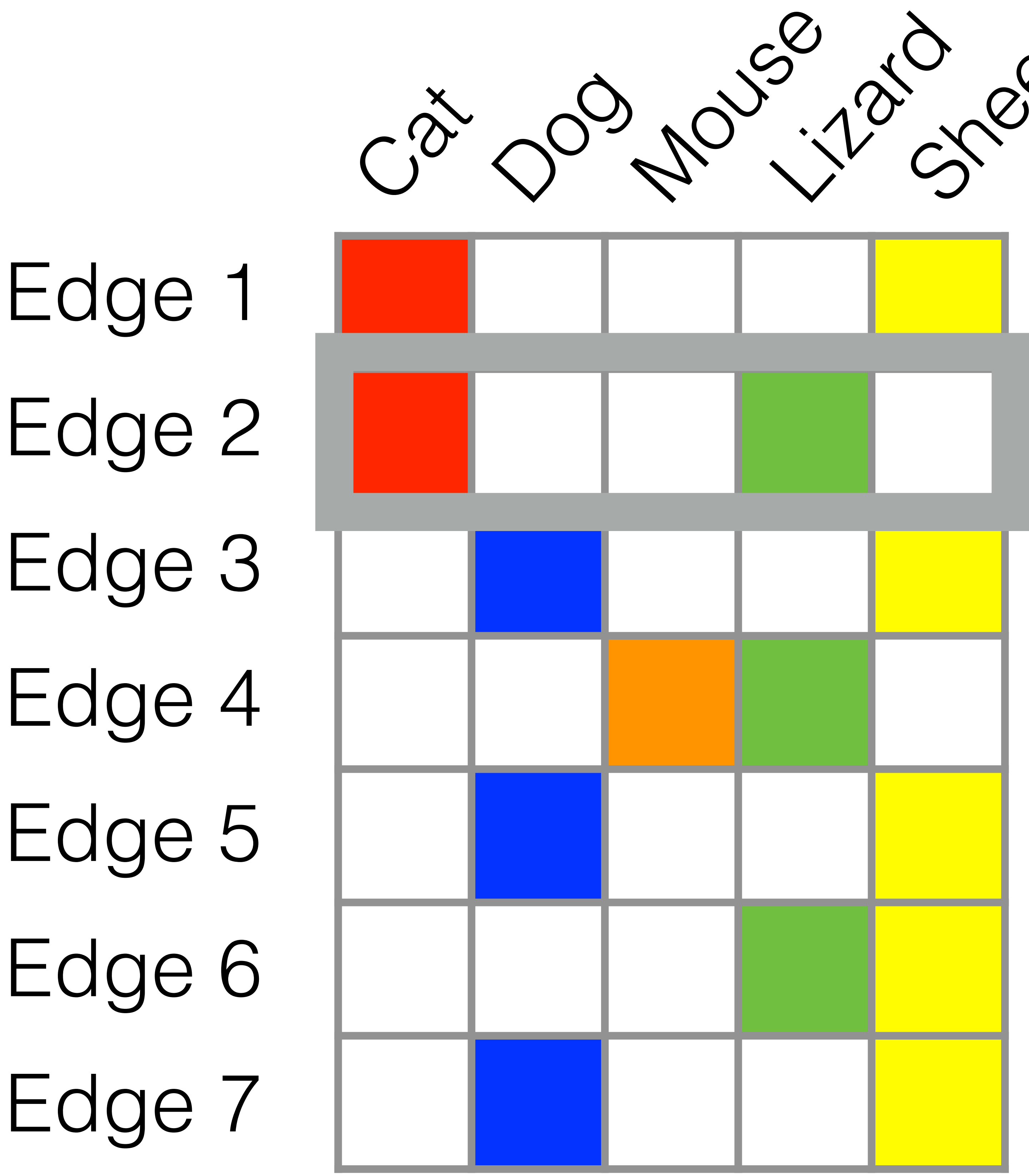


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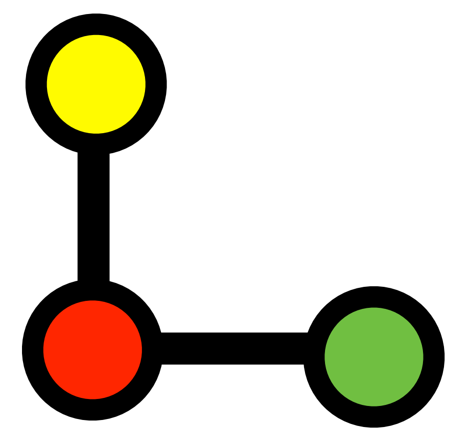

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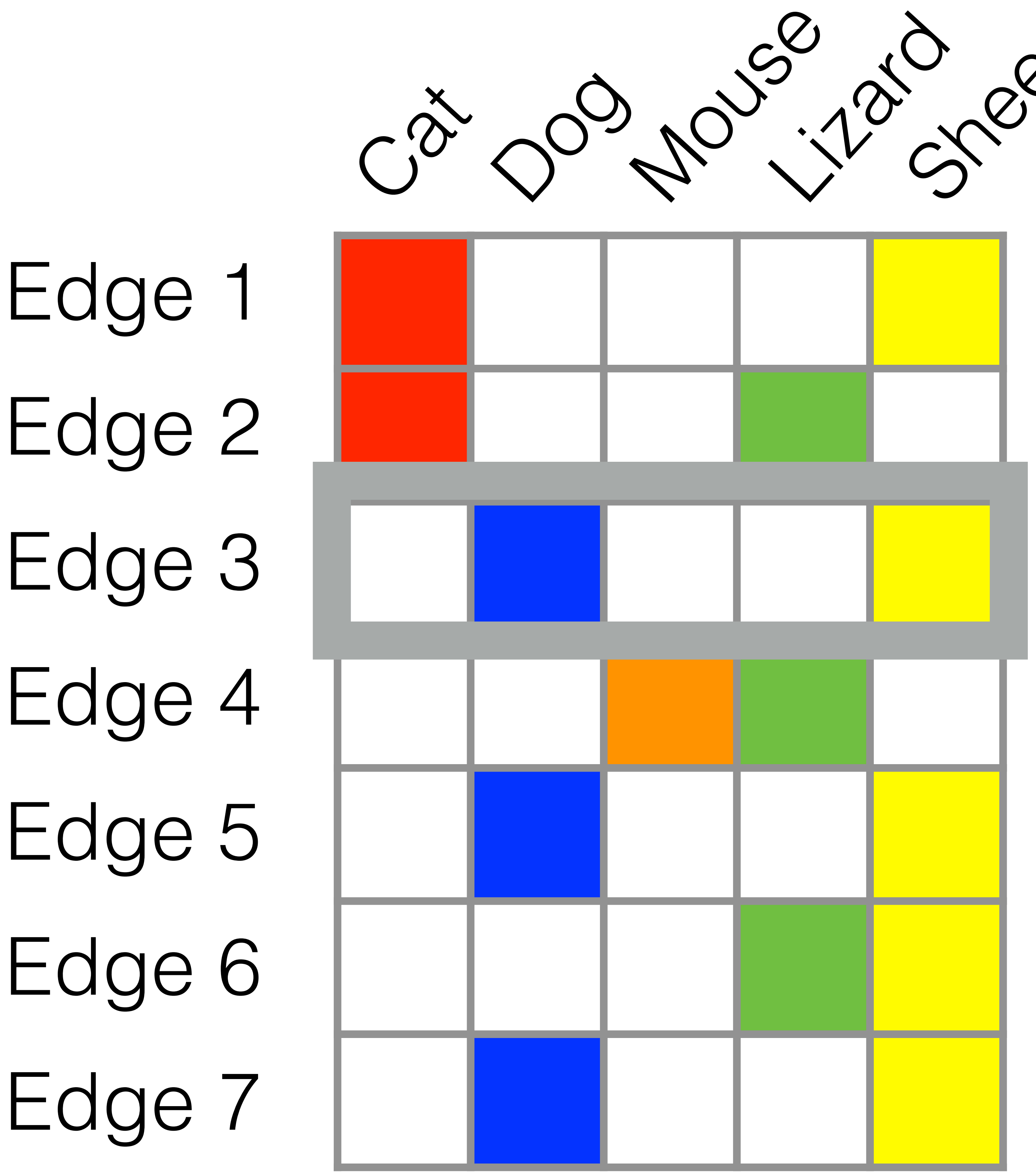


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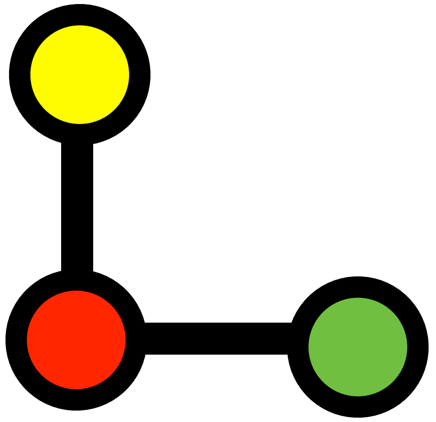

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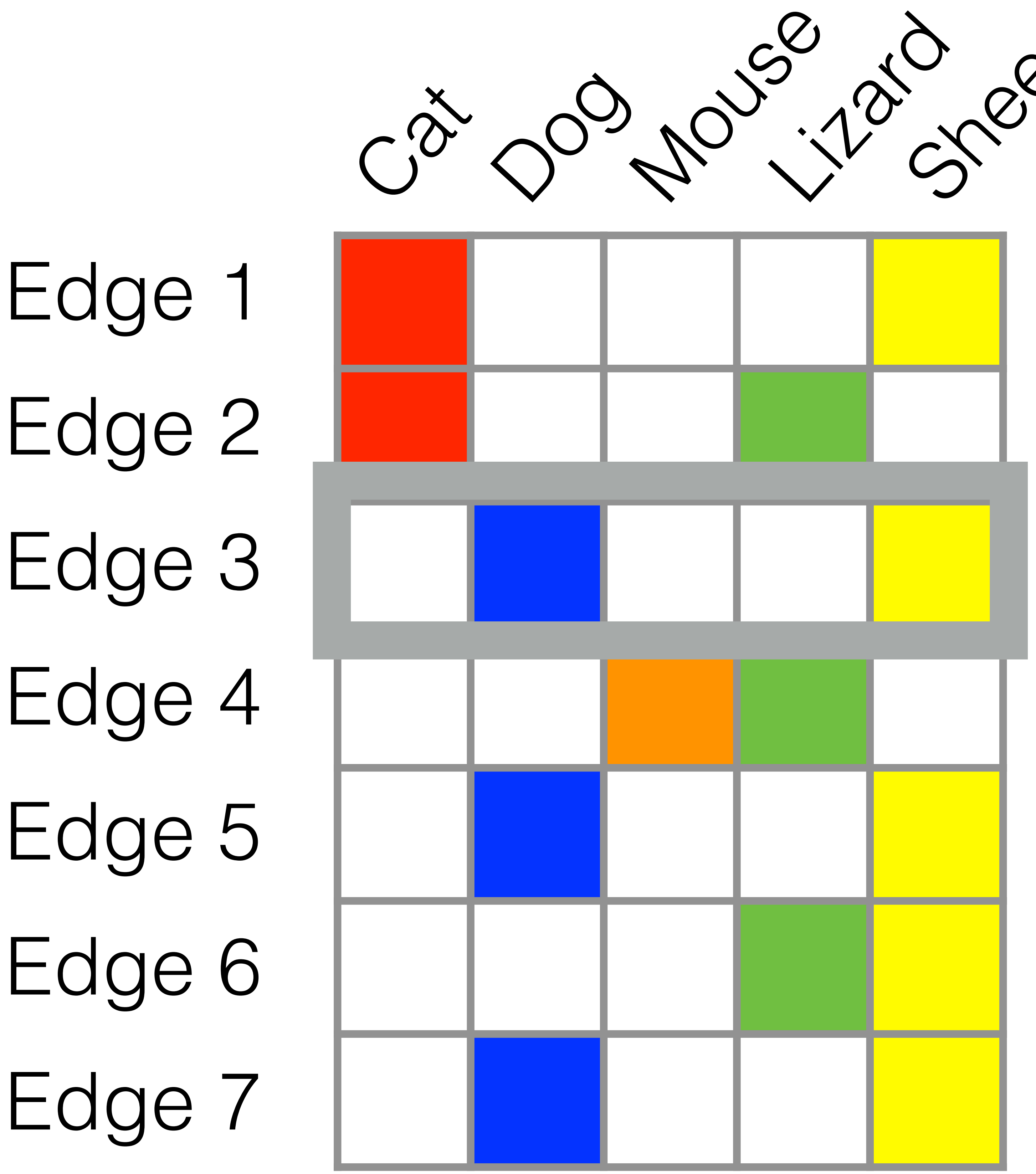


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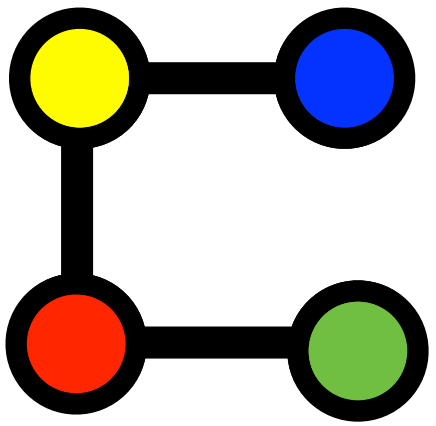

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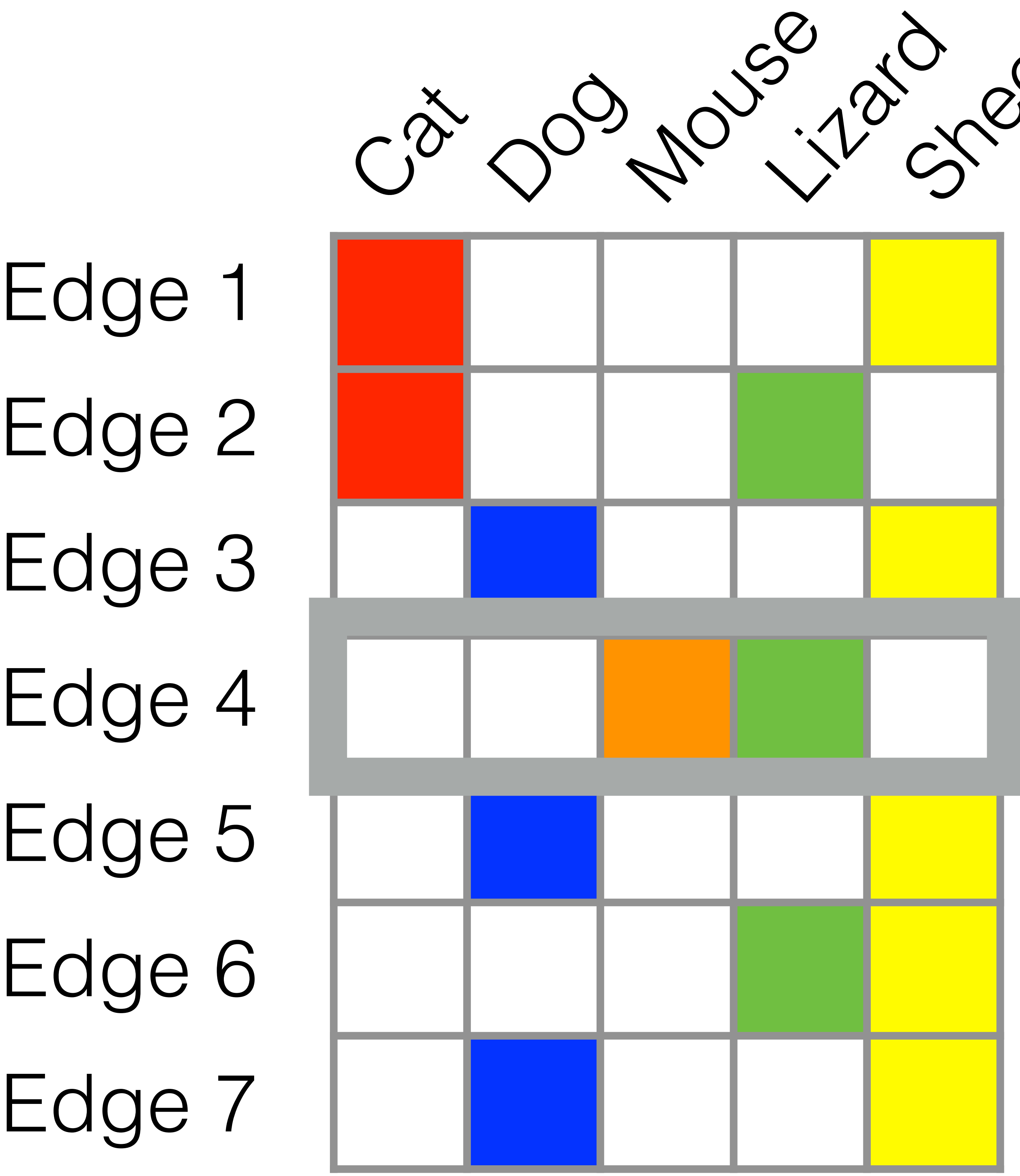


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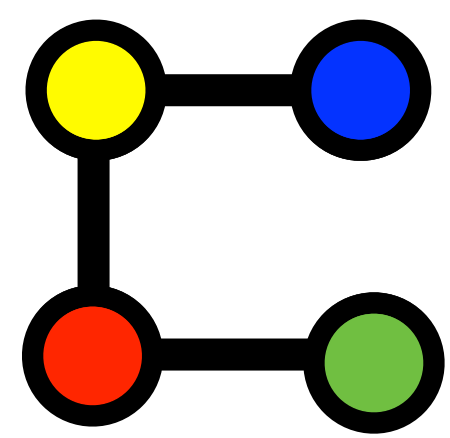

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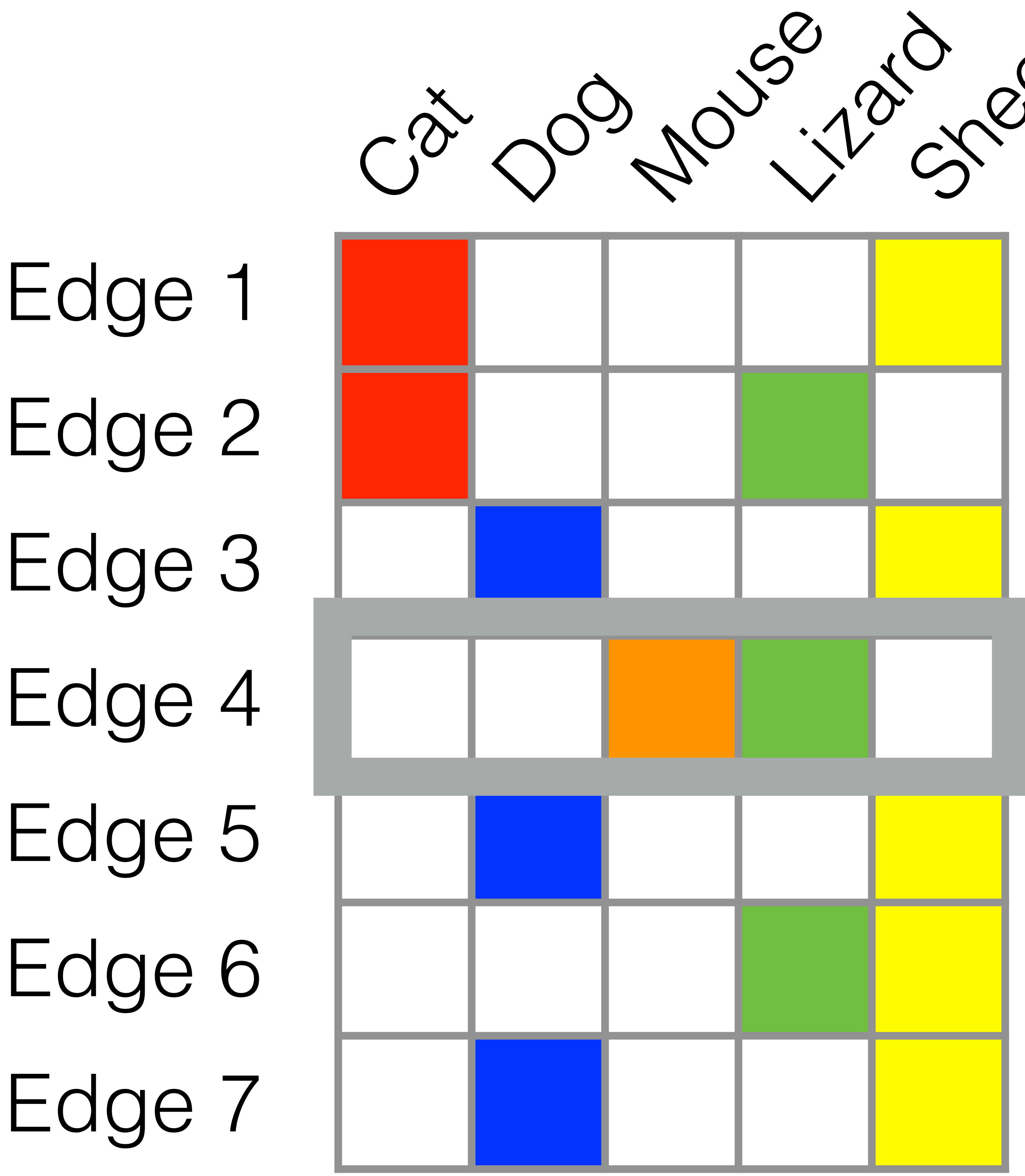


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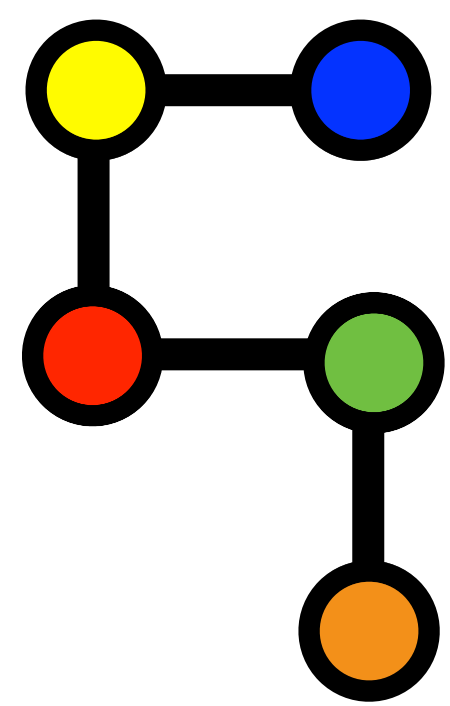

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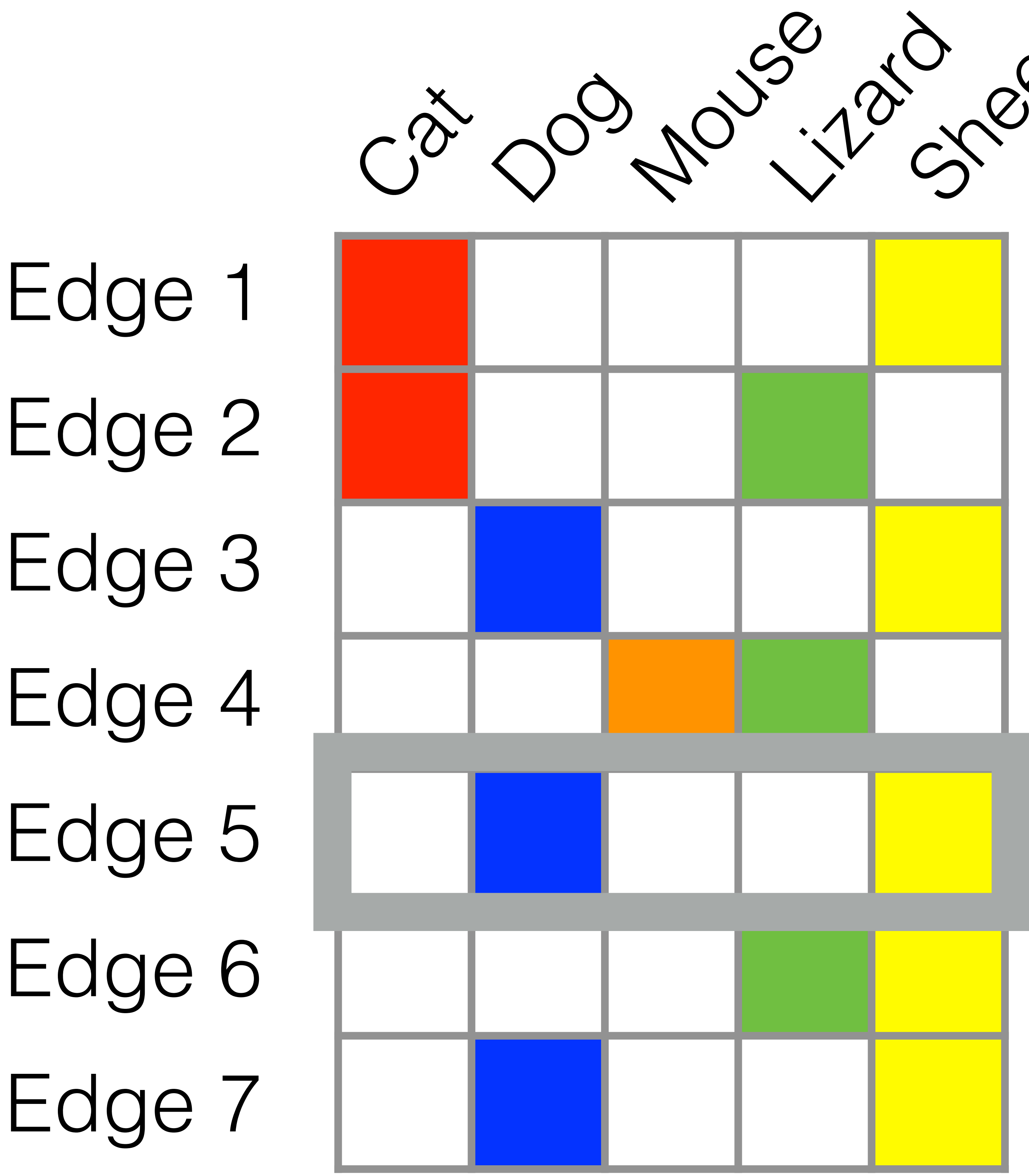


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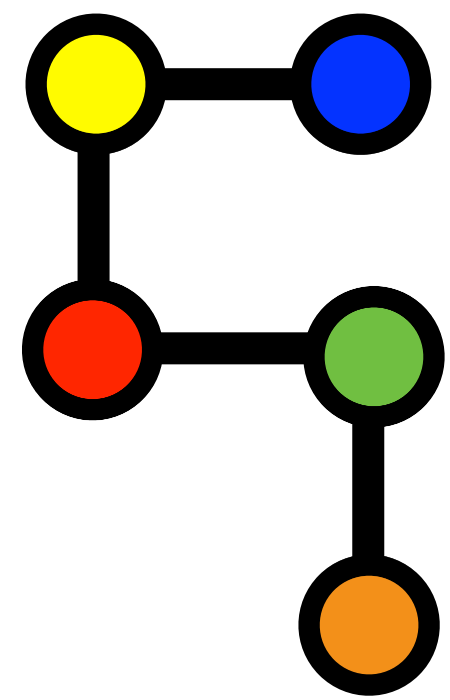

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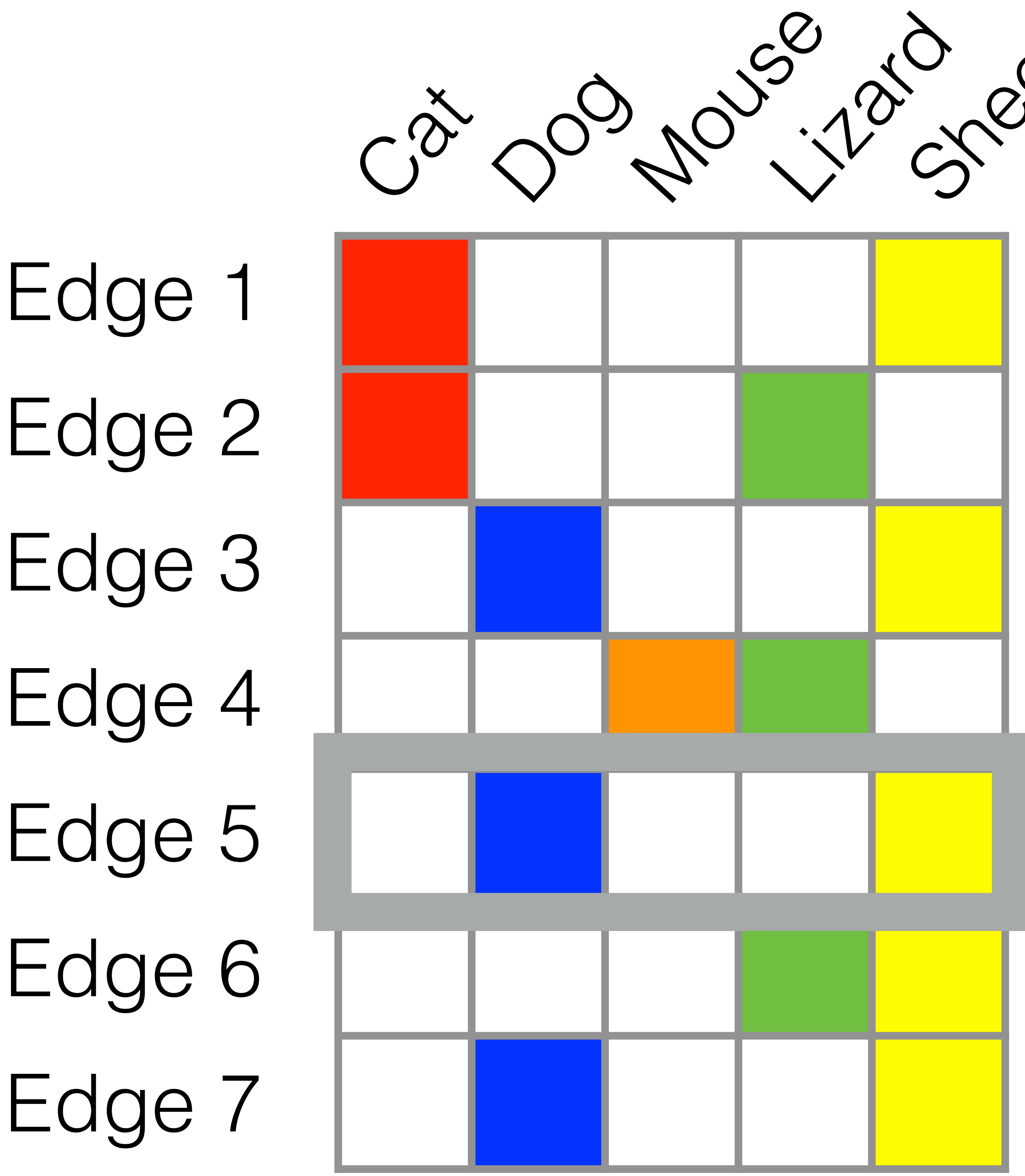


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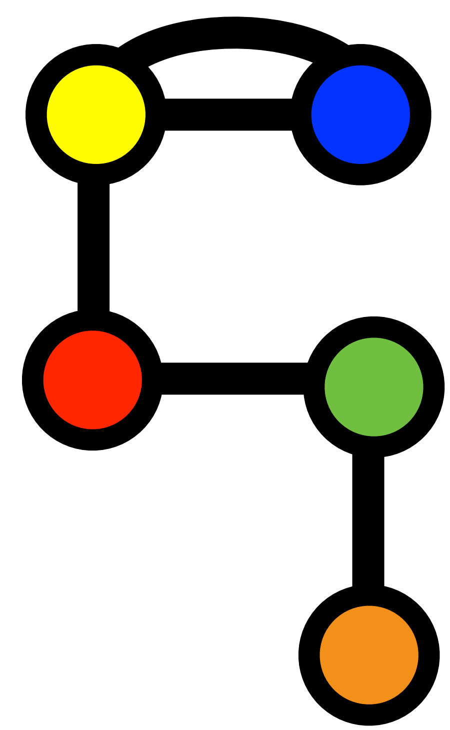

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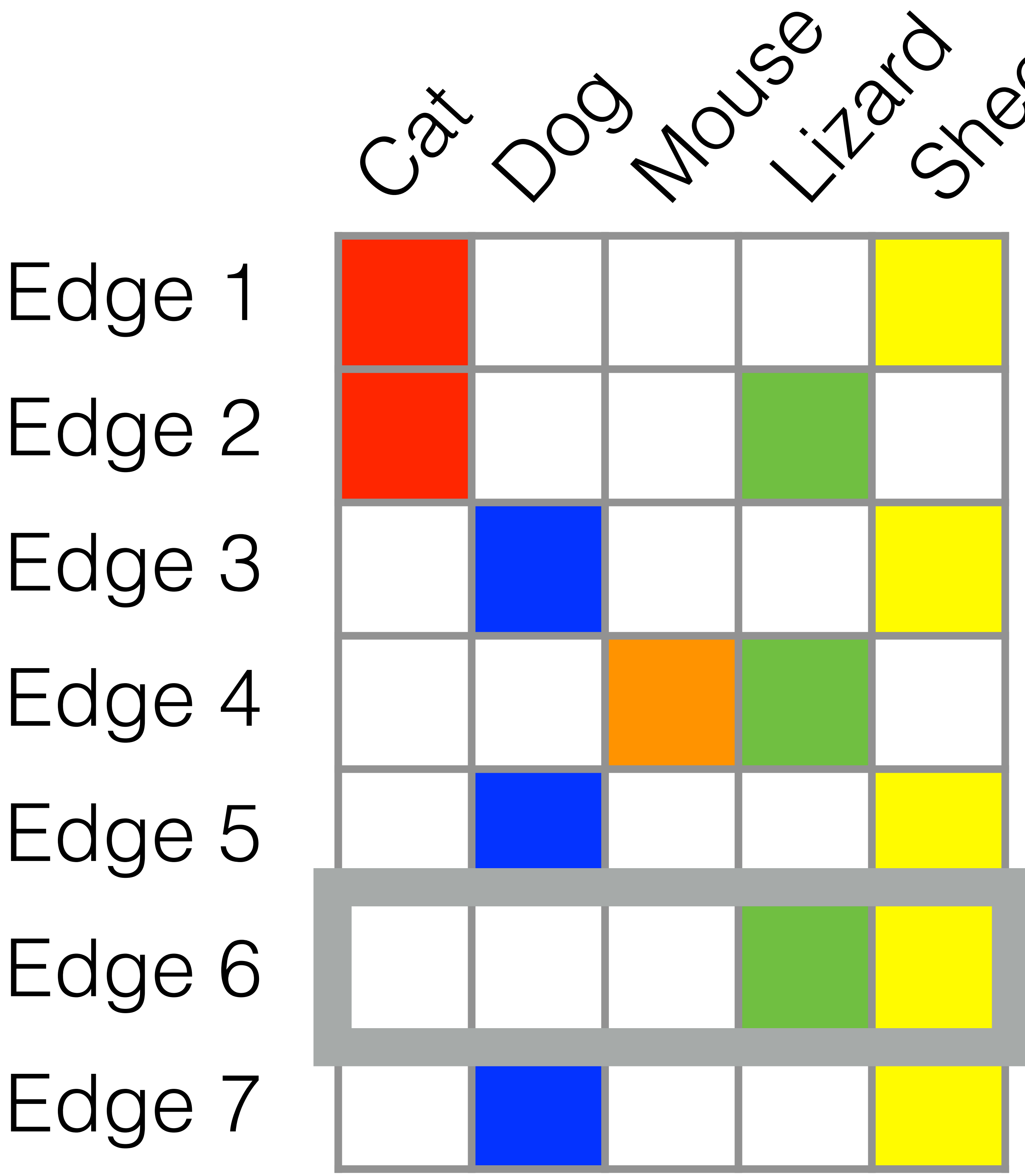


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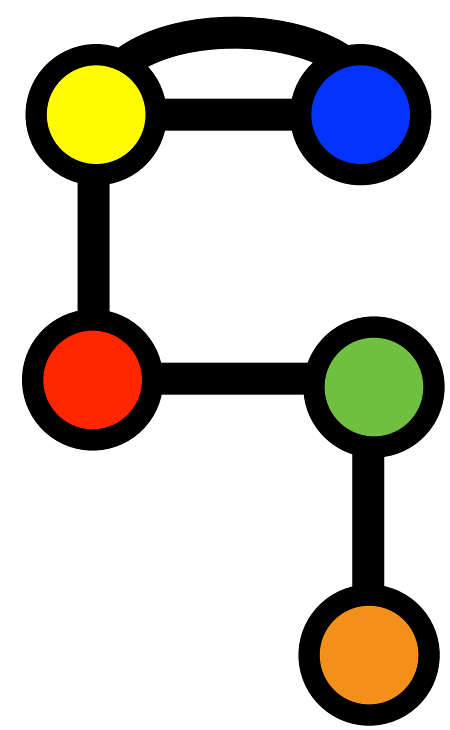

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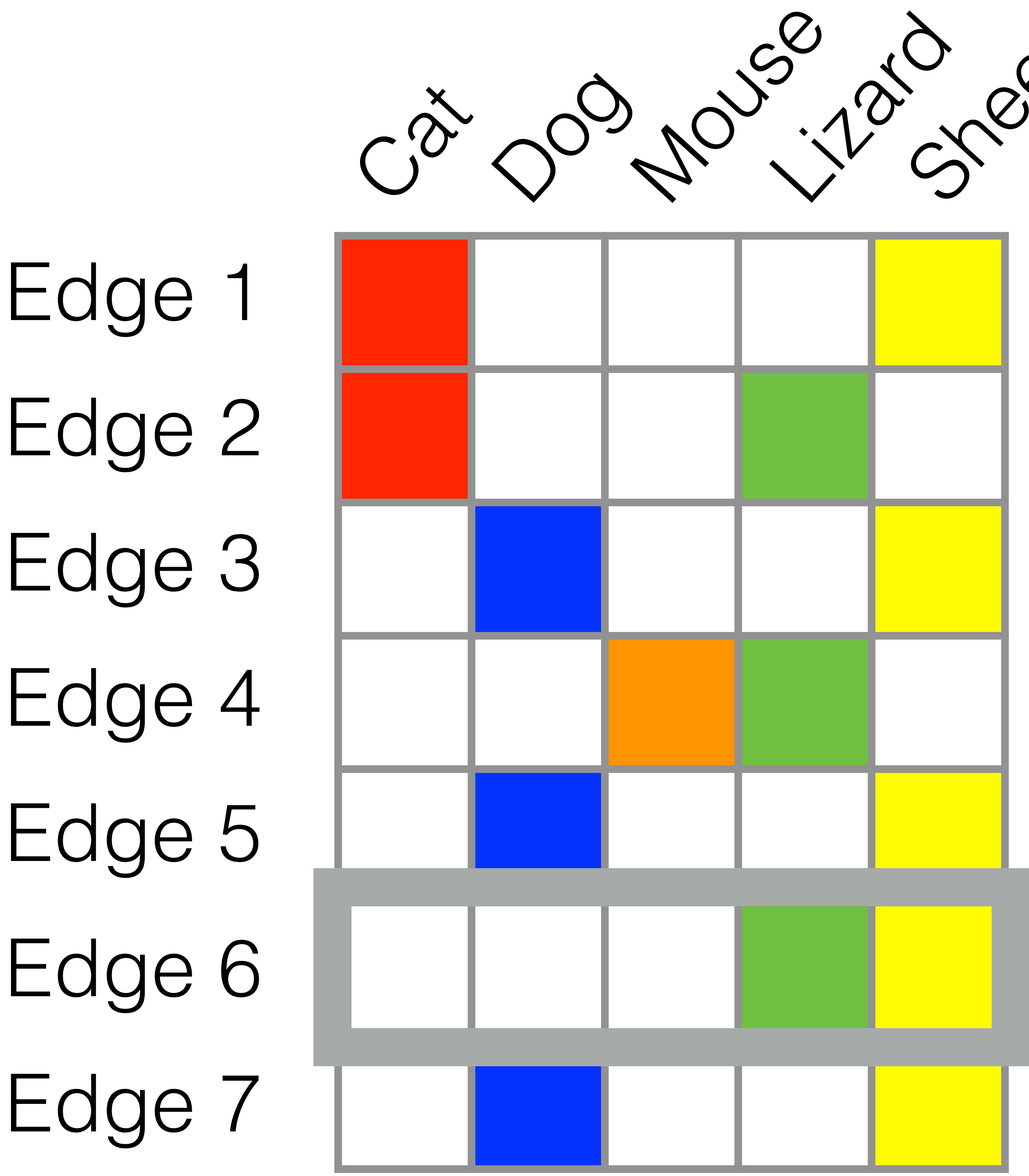


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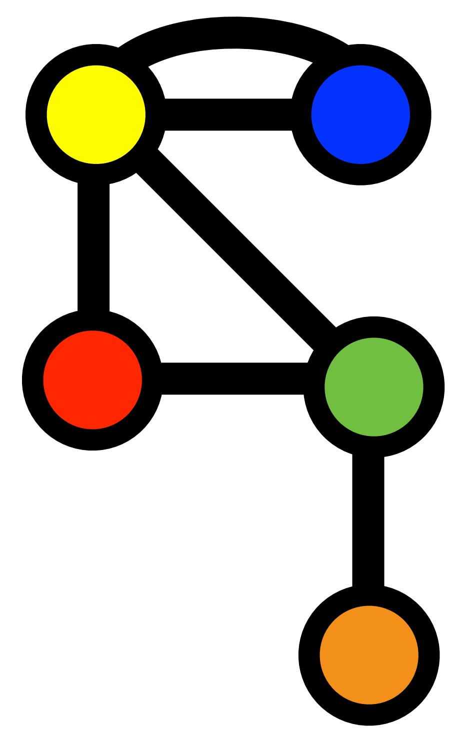

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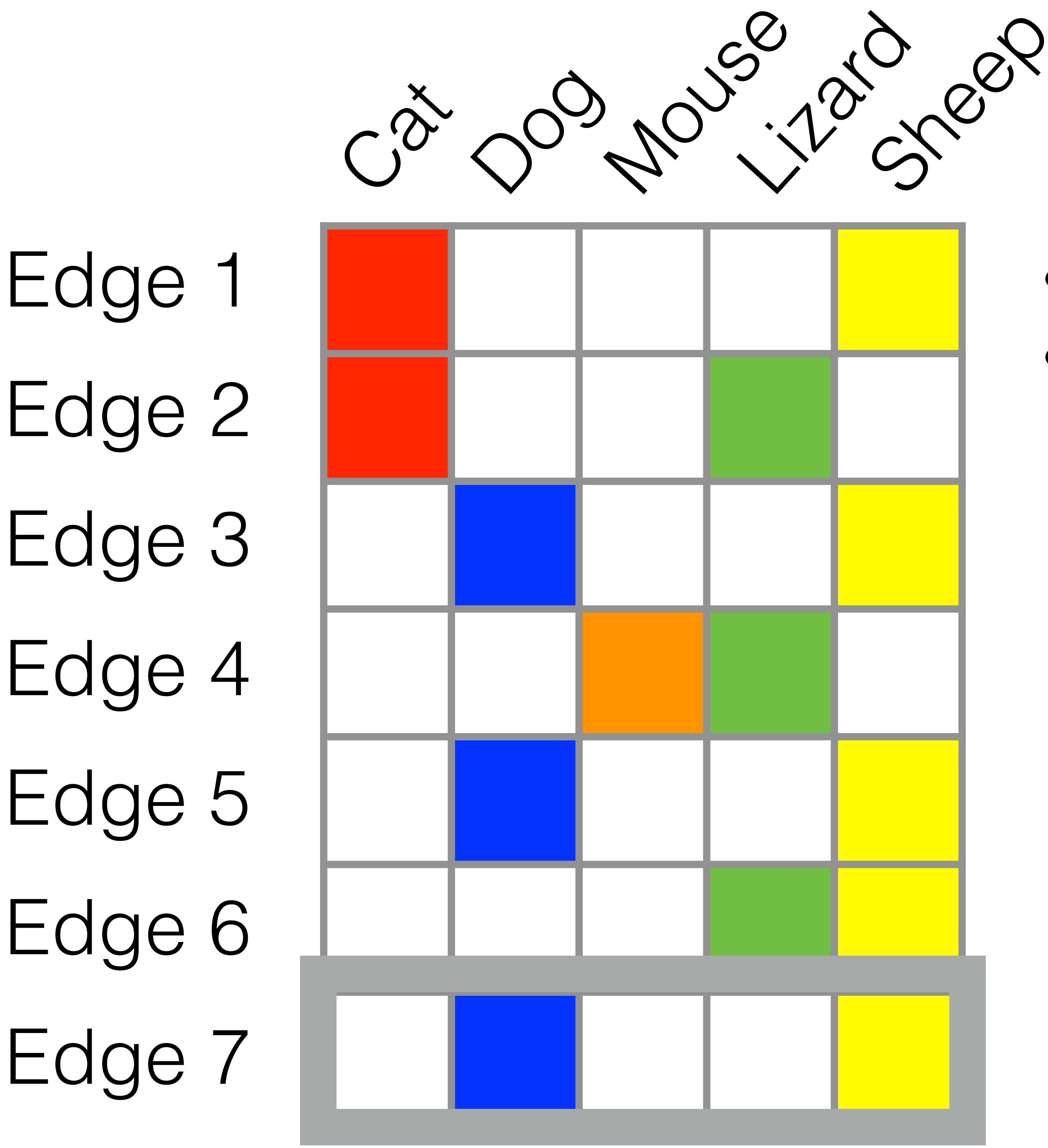


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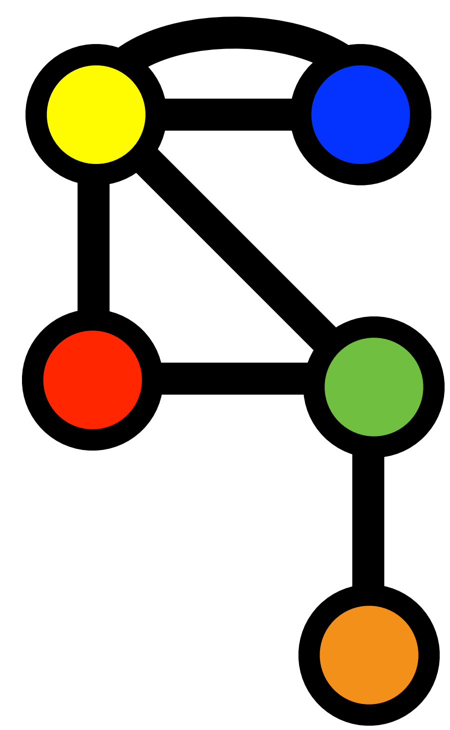

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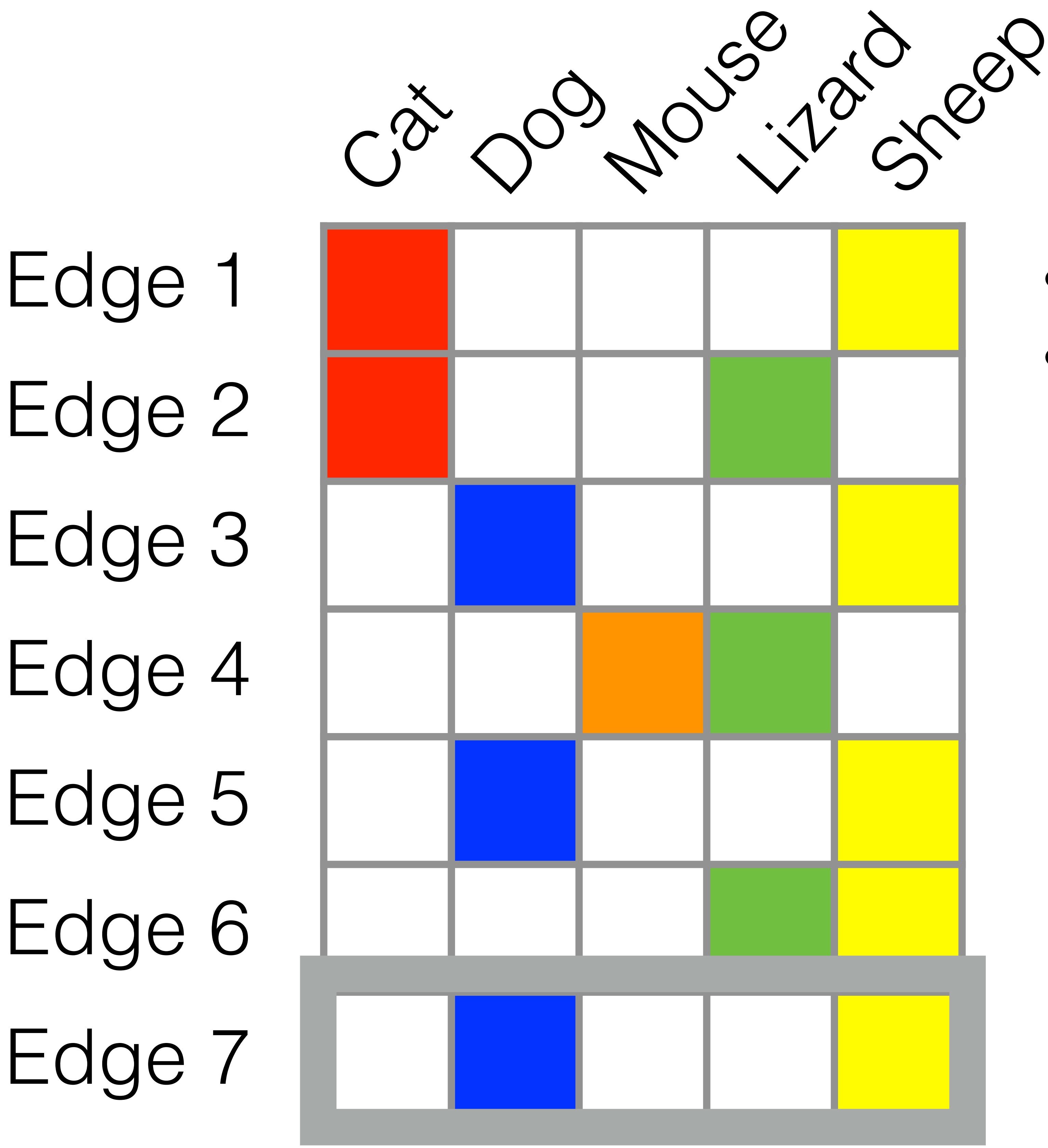


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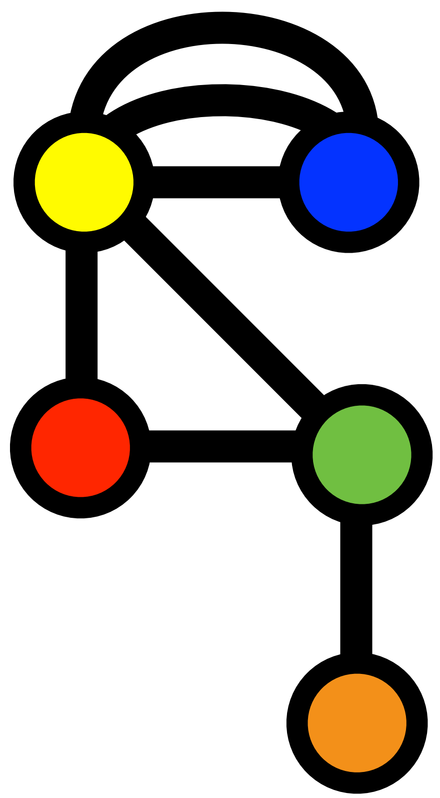


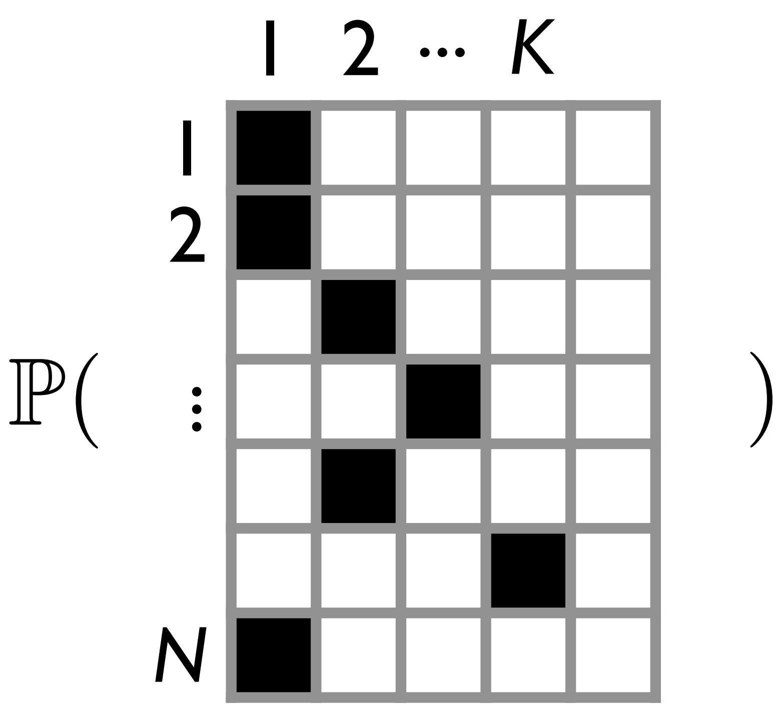
Exchangeable clustering distributions are characterized

What about:

Exchangeable feature allocations?

**Edge-exchangeable graphs?**

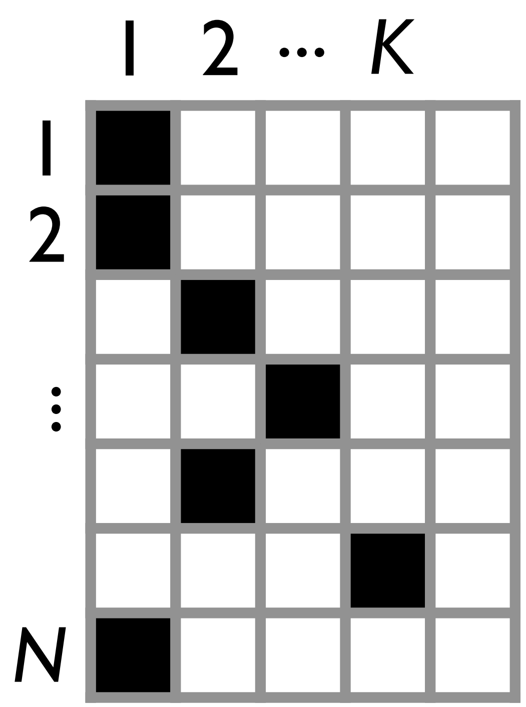
# Exchangeable probability functions






[Pitman 1995]

# Exchangeable probability functions

$\mathbb{P}($  $) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$

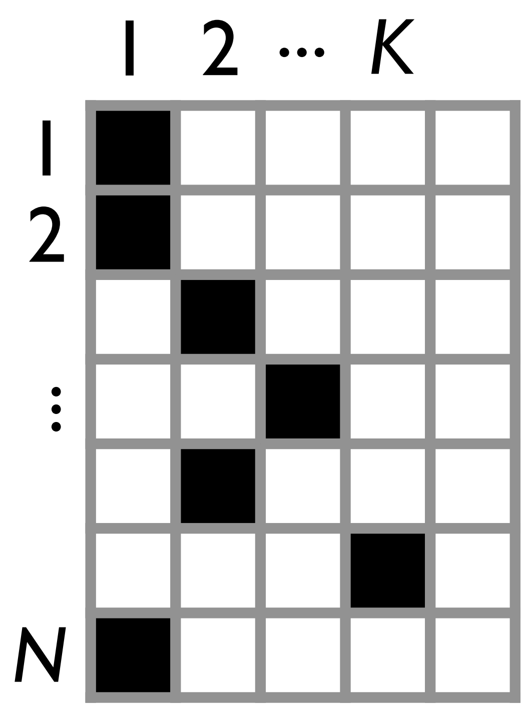




[Pitman 1995]

# Exchangeable probability functions

Size of *K*th cluster




$$\mathbb{P}(\ \cdots\ ) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$$



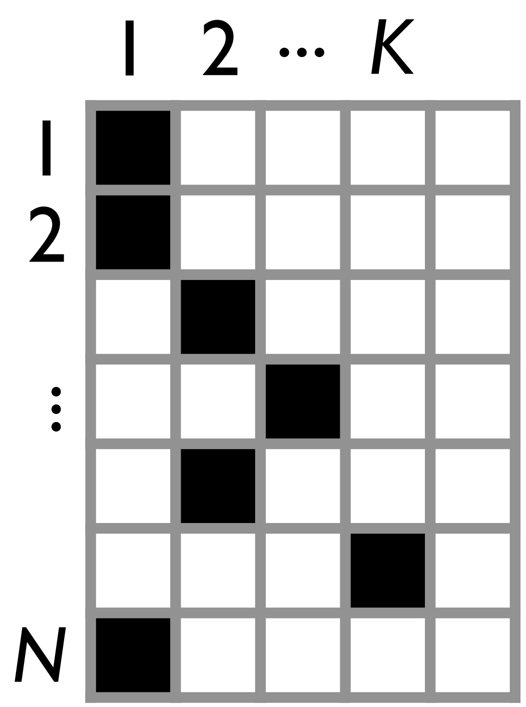
[Pitman 1995]

# Exchangeable probability functions

exchangeable **partition** probability function (E**P**PF)

Size of *K*th cluster

$\mathbb{P}($

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Thm (Pitman). Every exchangeable clustering has an exchangeable **partition** probability function (E**P**PF)

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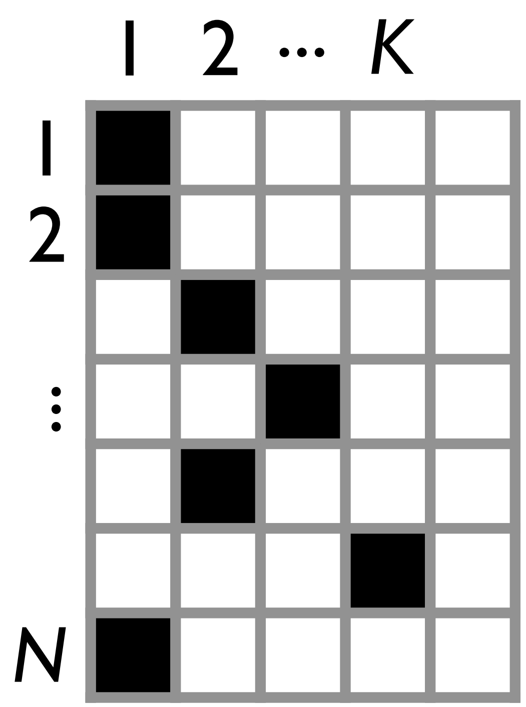
$$\mathbb{P}\Big($$

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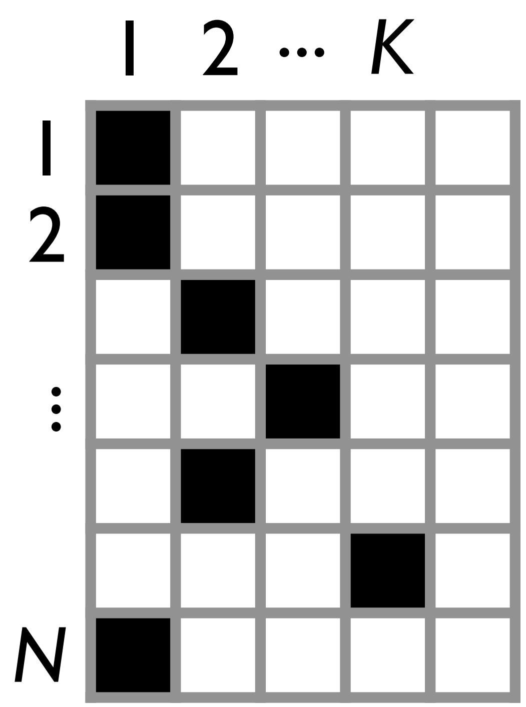
$\mathbb{P}($ 

$) = p(N; S_{N,1}, \ldots, S_{N,K})$

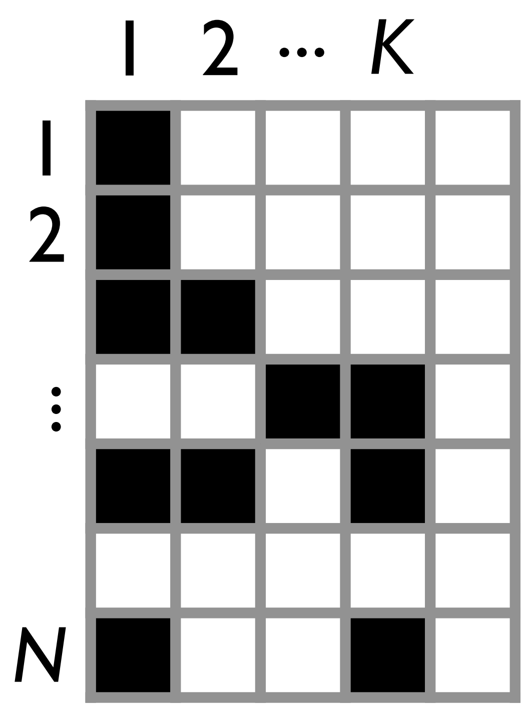


[Pitman 1995]

# Exchangeable probability functions

Thm (Pitman). Every exchangeable clustering has an exchangeable **partition** probability function (E**P**PF)

$\mathbb{P}($

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Size of *K*th cluster

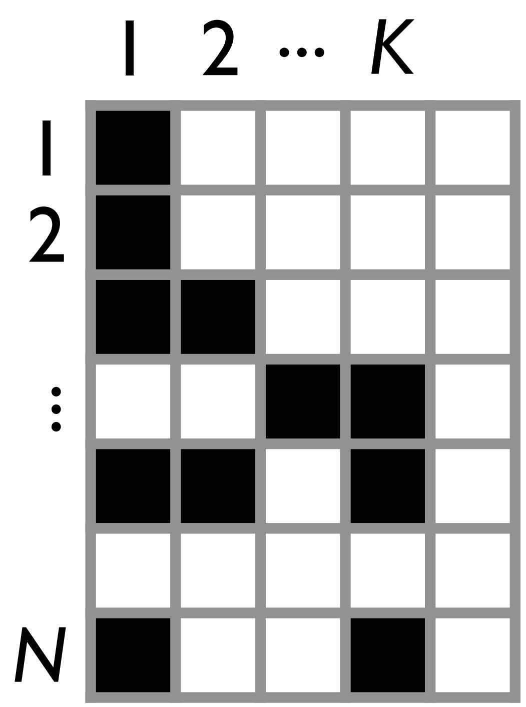


[Pitman 1995]

# Exchangeable probability functions

Thm (Pitman). Every exchangeable clustering has an exchangeable **partition** probability function (E**P**PF)

Size of *K*th feature

$\mathbb{P}($  $) = p(N; S_{N,1}, \ldots, S_{N,K})$





[Pitman 1995]

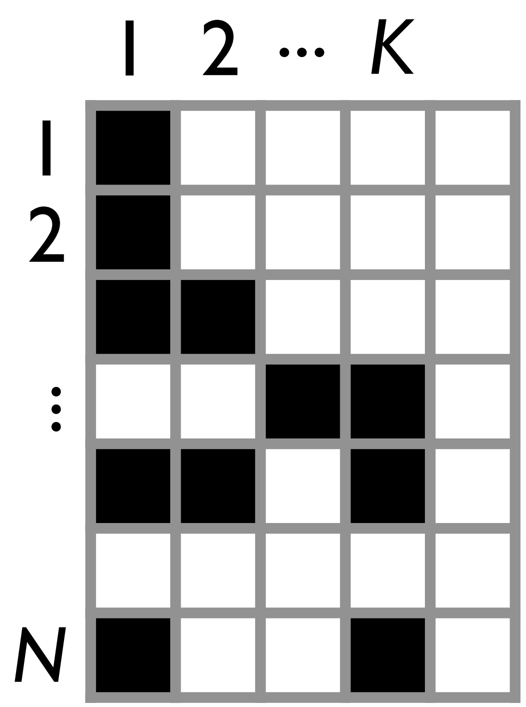
# Exchangeable probability functions

Thm (Pitman). Every exchangeable clustering has an exchangeable **partition** probability function (E**P**PF)

exchangeable **feature** probability function (E**F**PF)

Size of *K*th feature

$\mathbb{P}($




$) = p(N; S_{N,1}, \ldots, S_{N,K})$



[Pitman 1995; Broderick, Pitman, Jordan 2013]

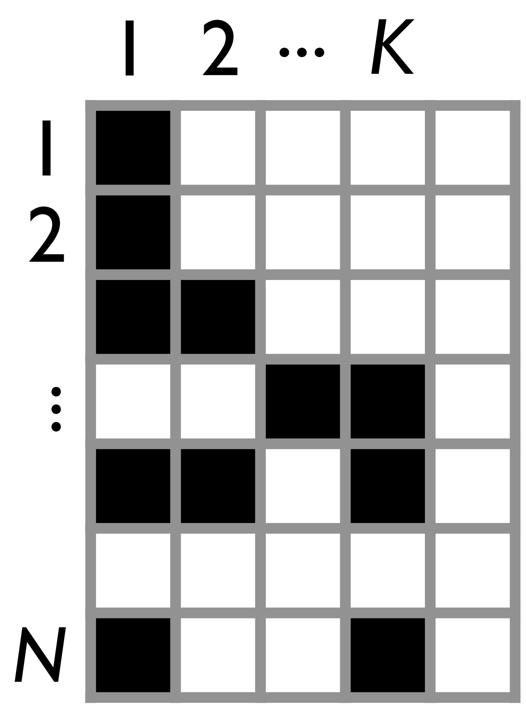
# Exchangeable probability functions

Thm (Pitman). Every exchangeable clustering has an exchangeable **partition** probability function (E**P**PF)

Prop (BPJ). Many, but not all, exchangeable feature allocations have an exchangeable **feature** probability function (E**F**PF)

Size of *K*th feature

$$\mathbb{P}\Bigg(\quad$$

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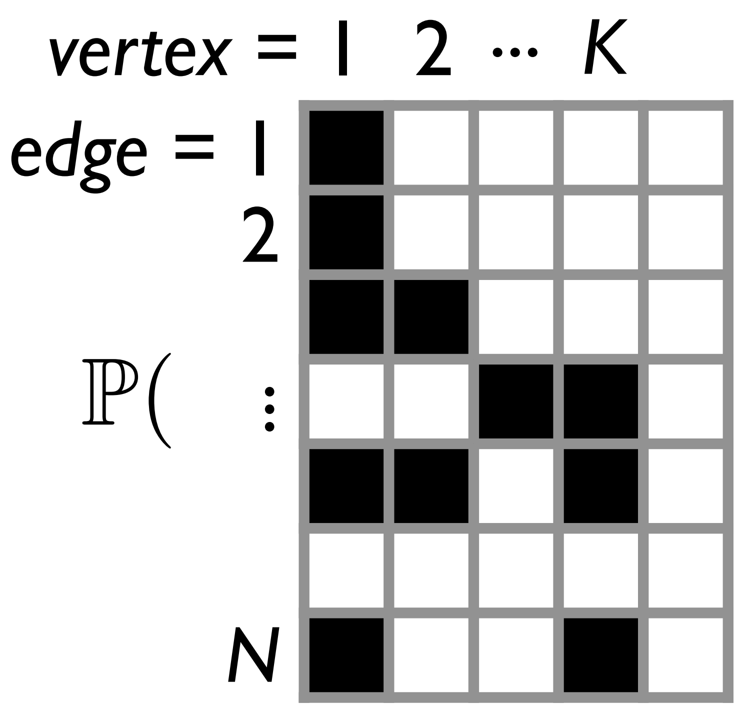
$\mathbb{P}($ 

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[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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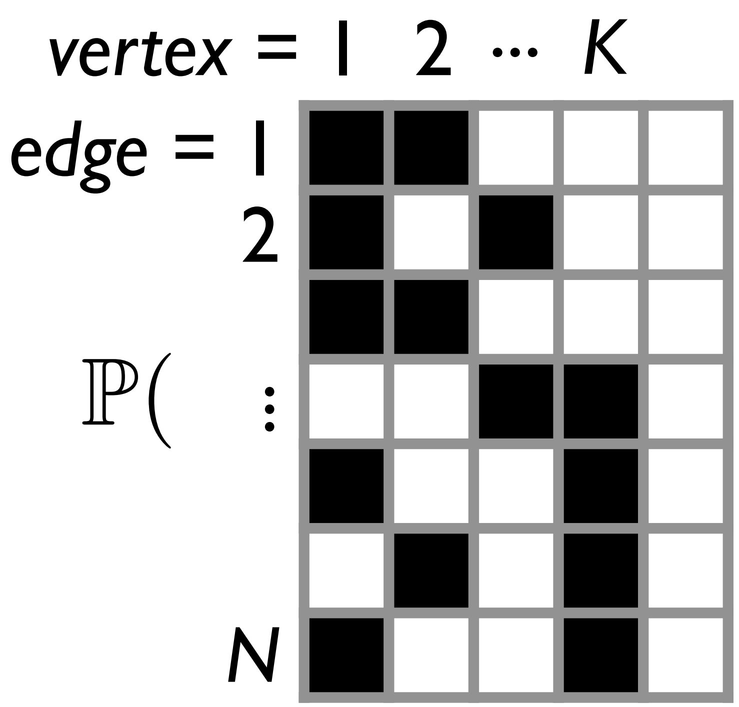


$$\mathbb{P}(\quad) = p(N; S_{N,1}, \ldots, S_{N,K})$$



[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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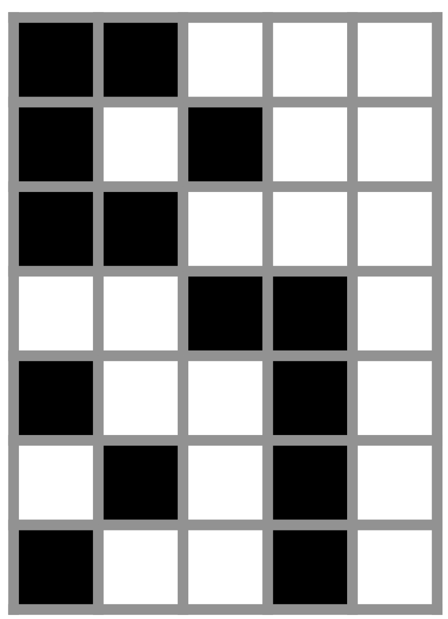
Size of *K*th vertex

*vertex* = 1 2 ··· *K*

*edge* = 1, 2, ⋮, *N*

$\mathbb{P}($  $) = p(N; S_{N,1}, \ldots, S_{N,K})$



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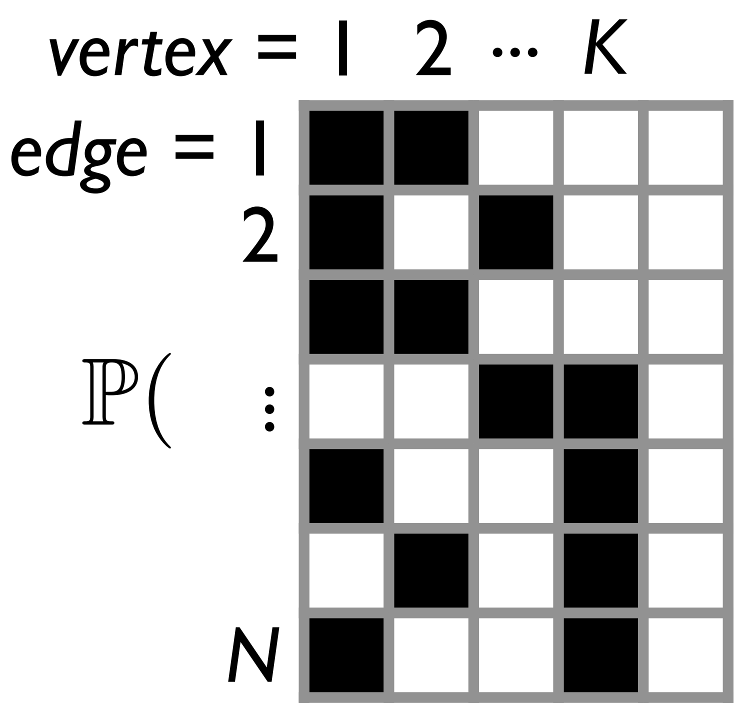


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Degree of *K*th vertex



[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

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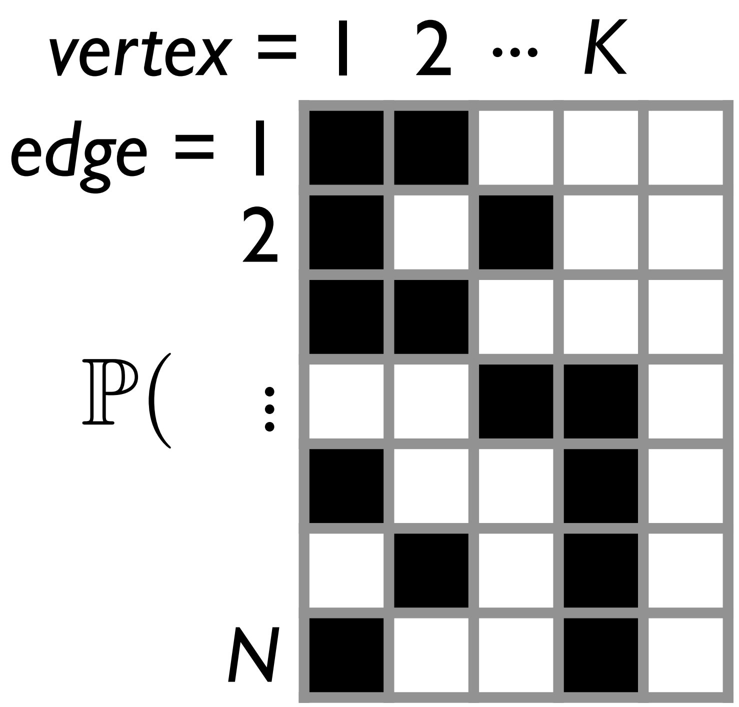


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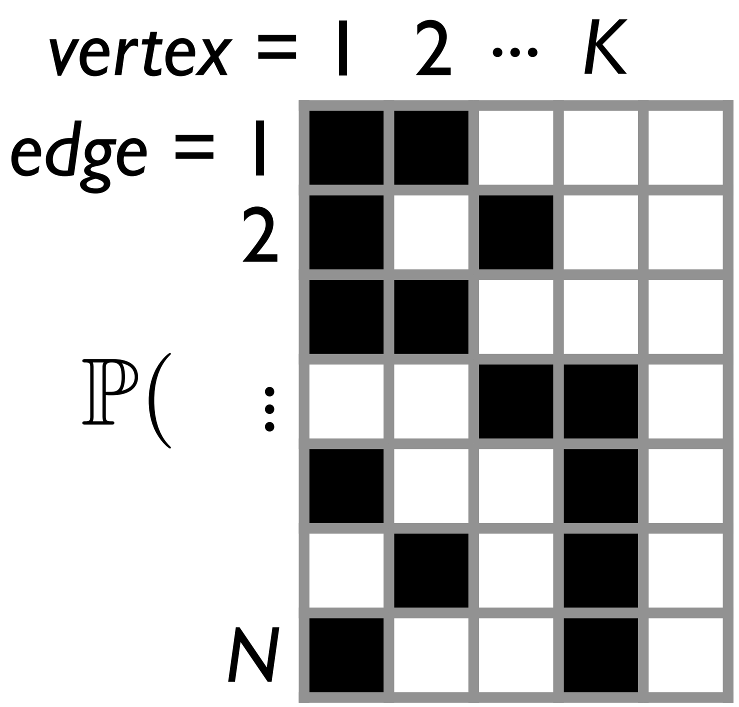


$$\mathbb{P}(\quad) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$$

Definition (CCB). Exchangeable **vertex** probability function (E**V**PF)



[Pitman 1995; Broderick, Pitman, Jordan 2013; Campbell, Cai, Broderick 2016]

# Exchangeable probability functions

Q: Does every edge-exchangeable graph have an EVPF?



[Broderick, Jordan, Pitman 2013]

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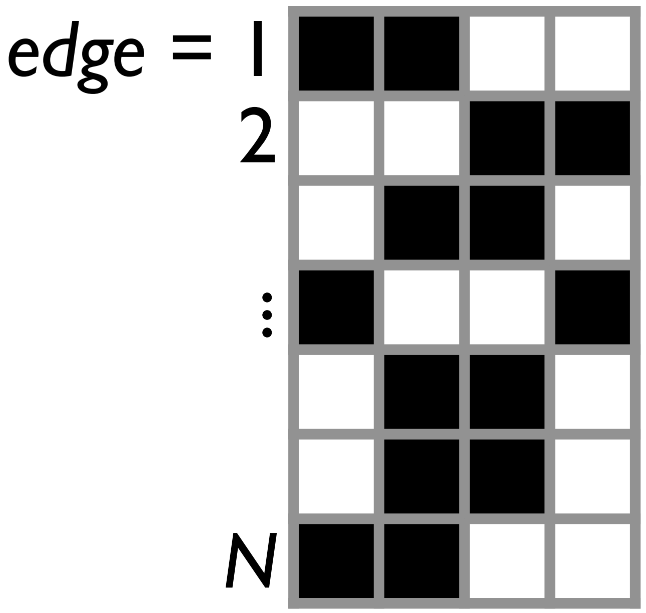




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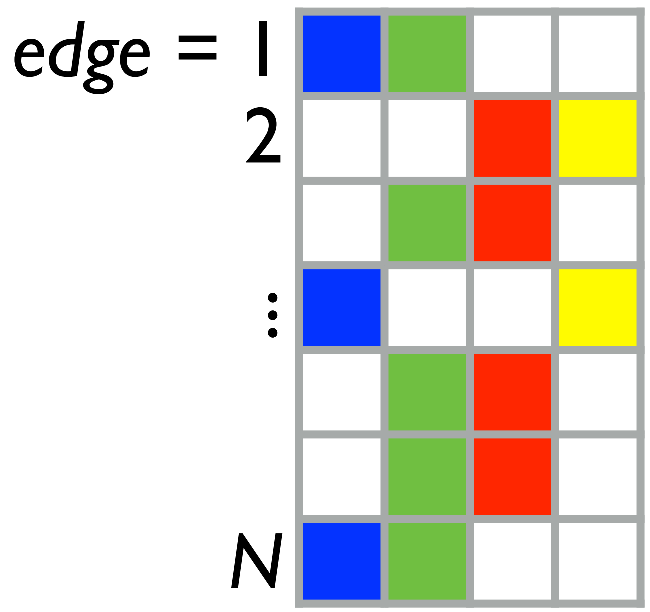




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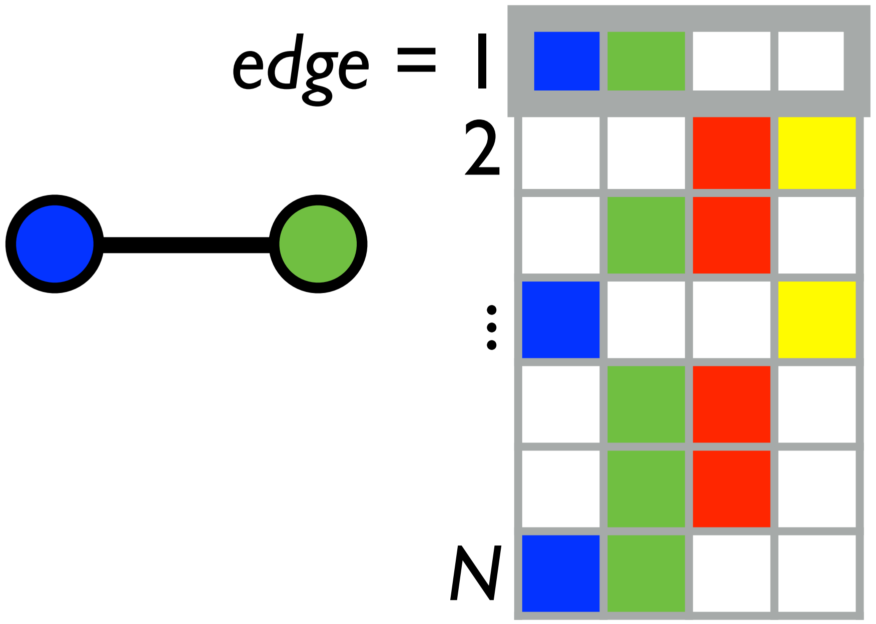




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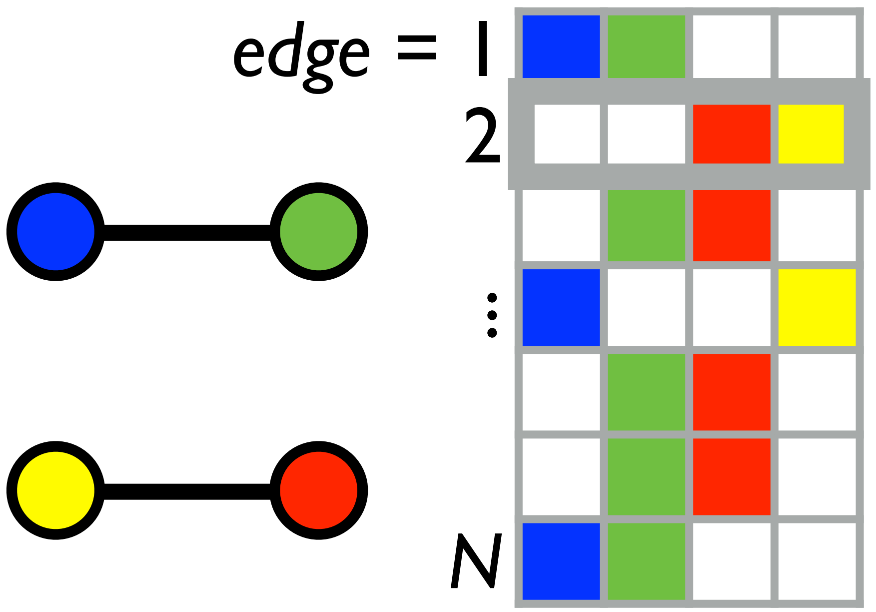




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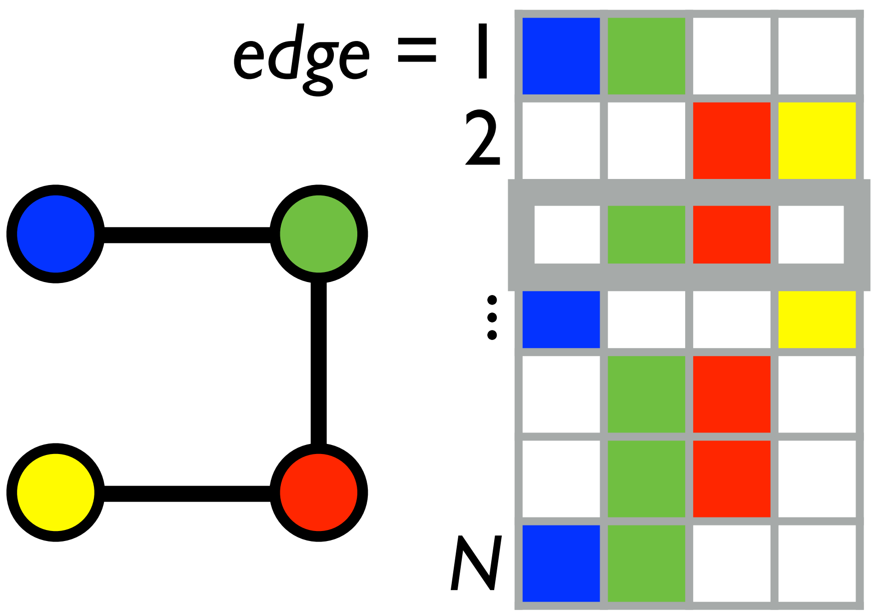




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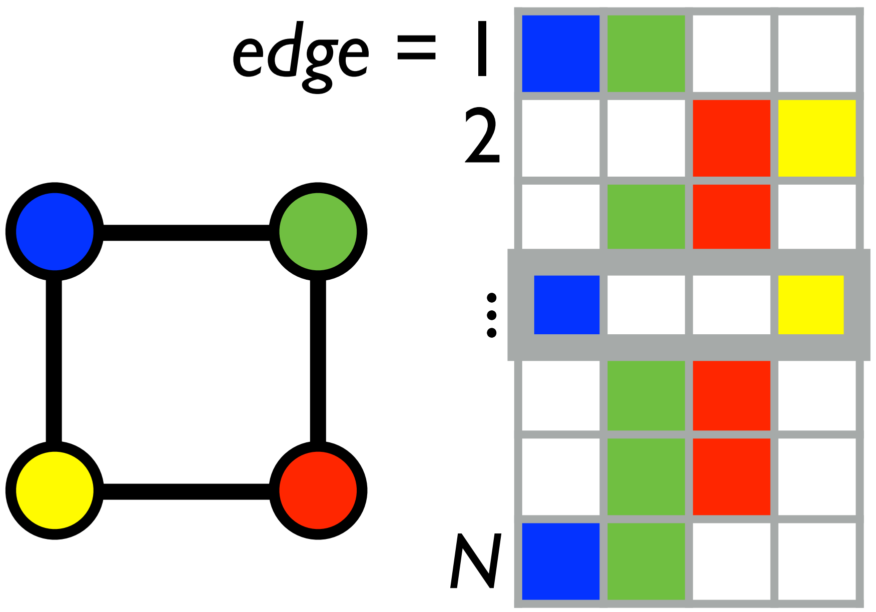




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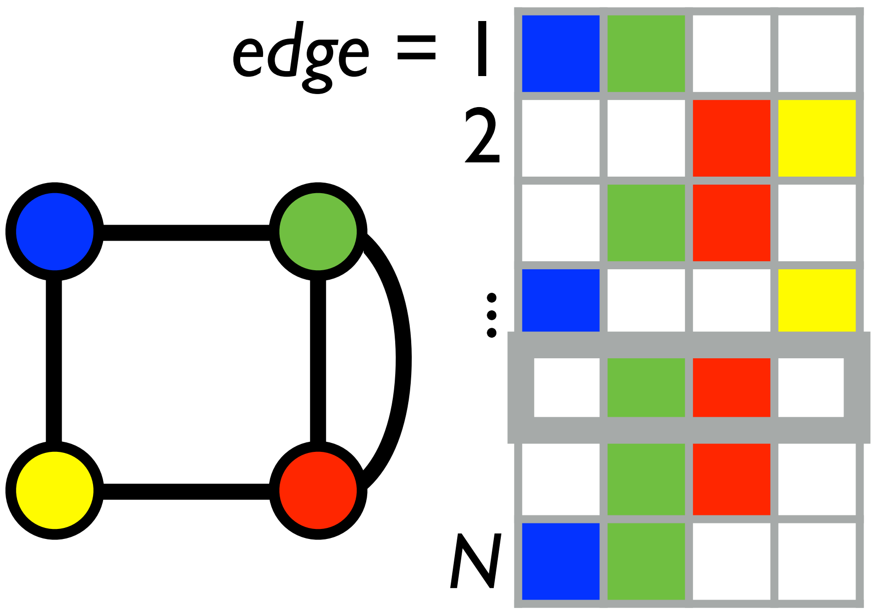




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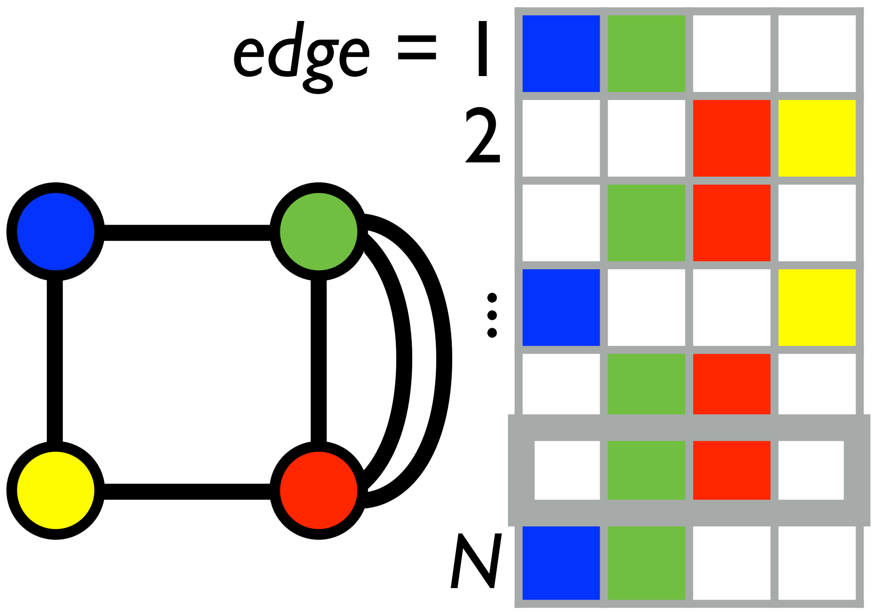




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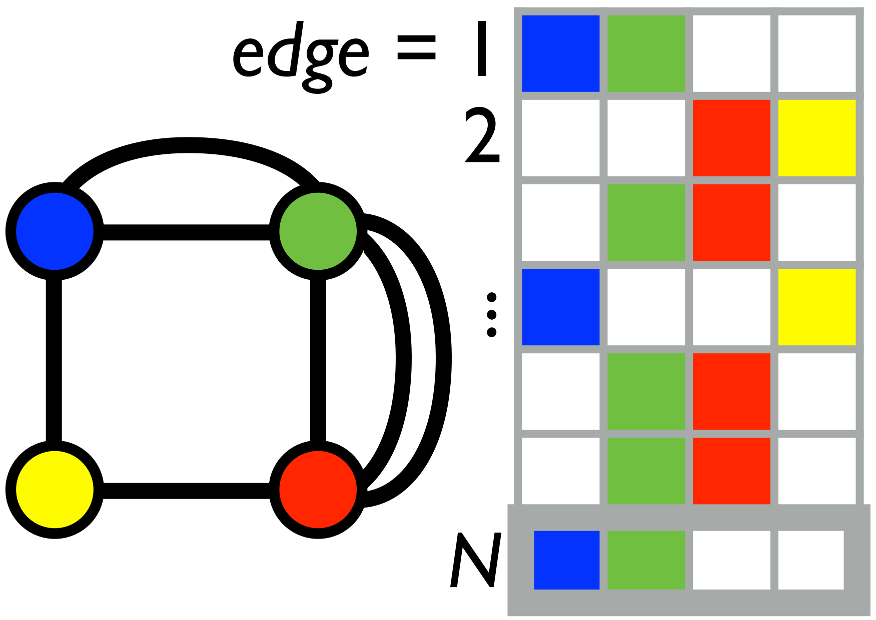




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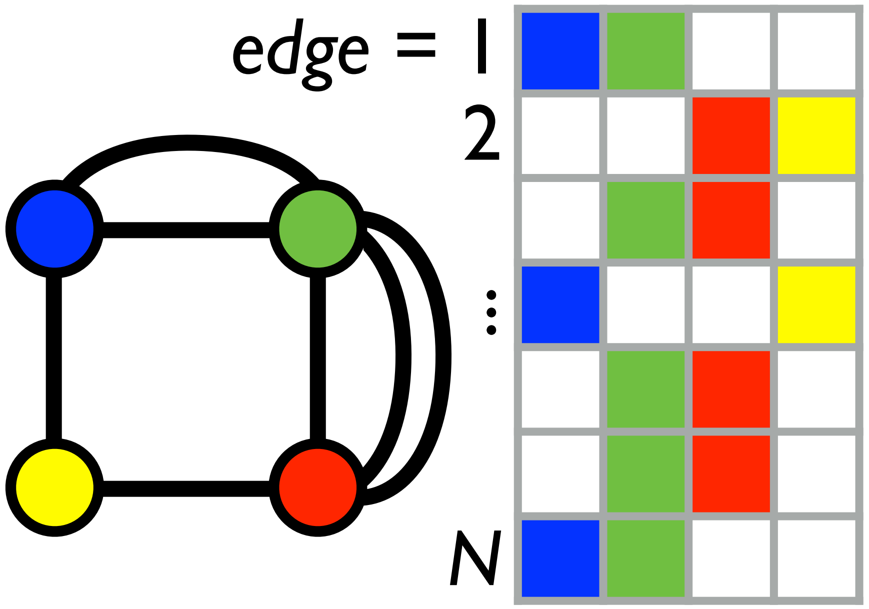




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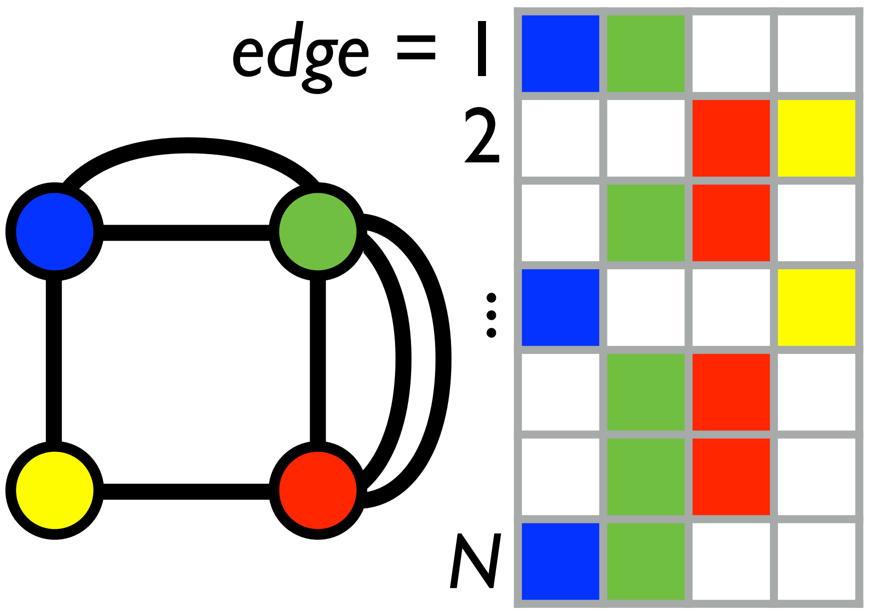
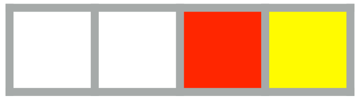
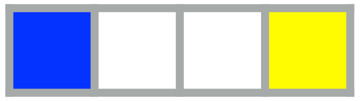



$\mathbb{P}(\mathrm{row} =$  $) = p_1$

$\mathbb{P}(\mathrm{row} =$  $) = p_2$

$\mathbb{P}(\mathrm{row} =$  $) = p_3$

$\mathbb{P}(\mathrm{row} =$  $) = p_4$



[Broderick, Jordan, Pitman 2013]

# Exchangeable probability functions

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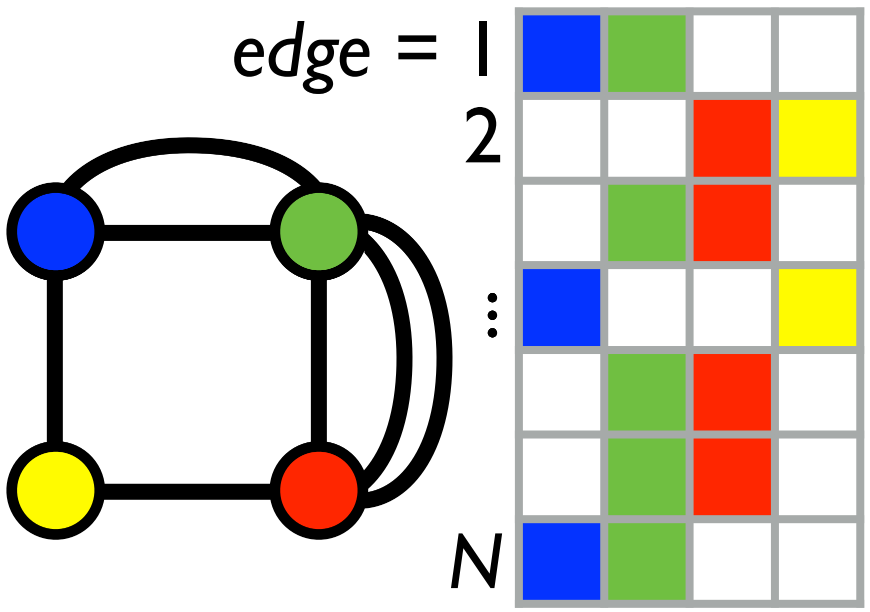


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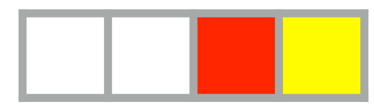
$\mathbb{P}(\text{row} =$  $) = p_2$

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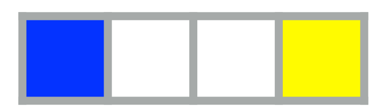
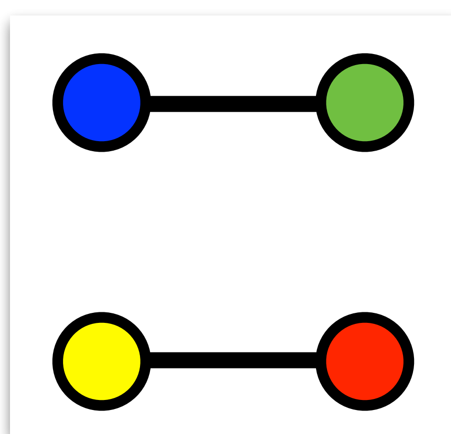
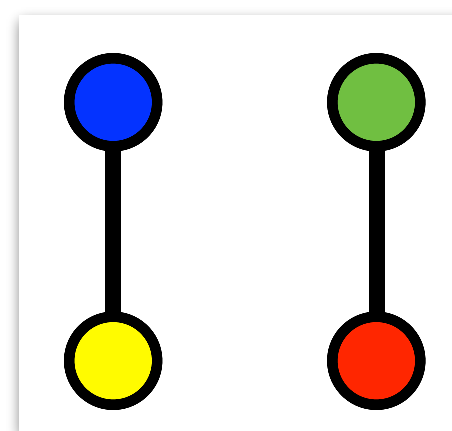
$\mathbb{P}(\text{row} =$  $) = p_4$







[Broderick, Jordan, Pitman 2013]

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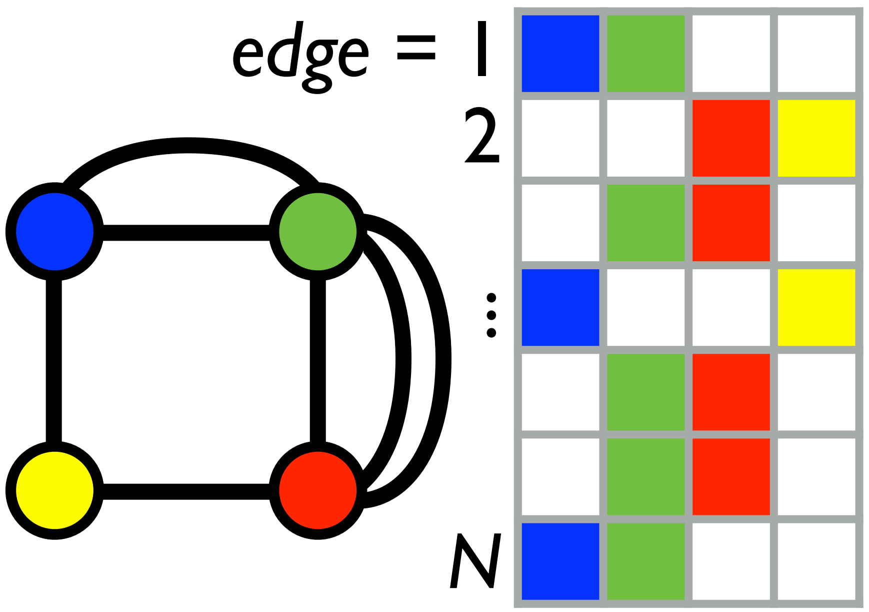


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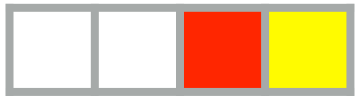
$\mathbb{P}($row $=$  $) = p_2$

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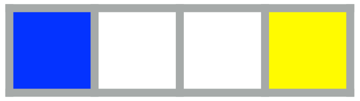
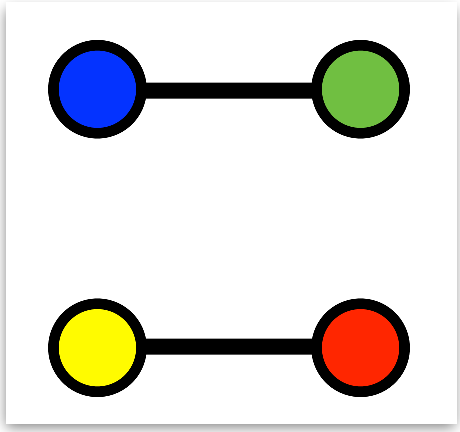
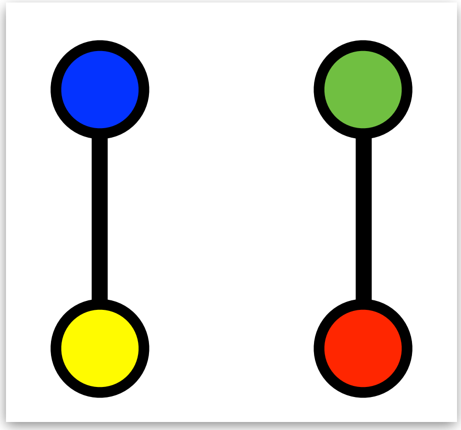
$\mathbb{P}($row $=$  $) = p_4$

$\mathbb{P}($  $)$ $\mathbb{P}($  $)$



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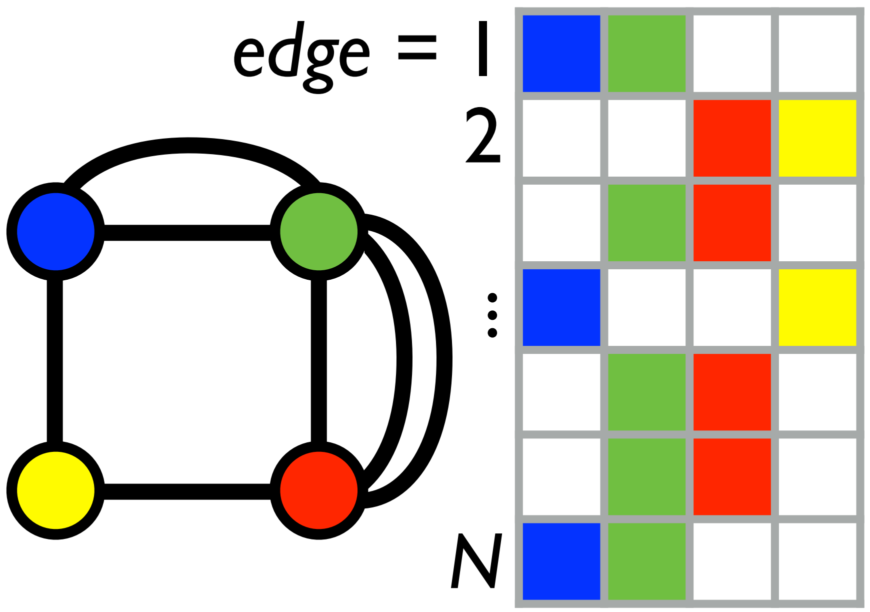


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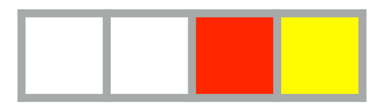
$\mathbb{P}(\text{row} =$  $) = p_2$

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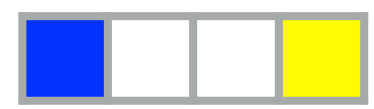
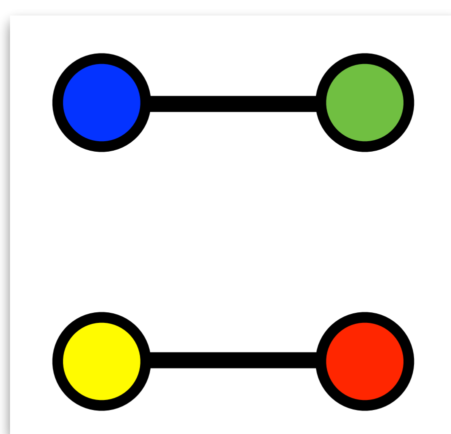
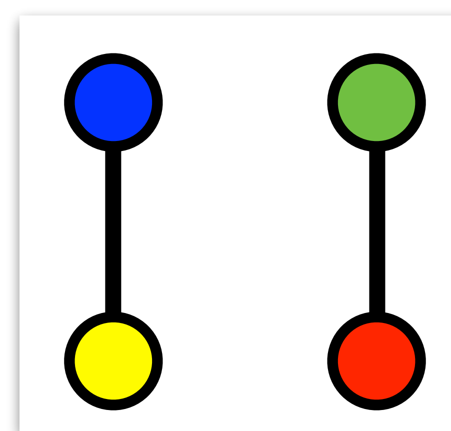
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\quad$ $\mathbb{P}($  $)$

$p_1 p_2 \qquad p_3 p_4$



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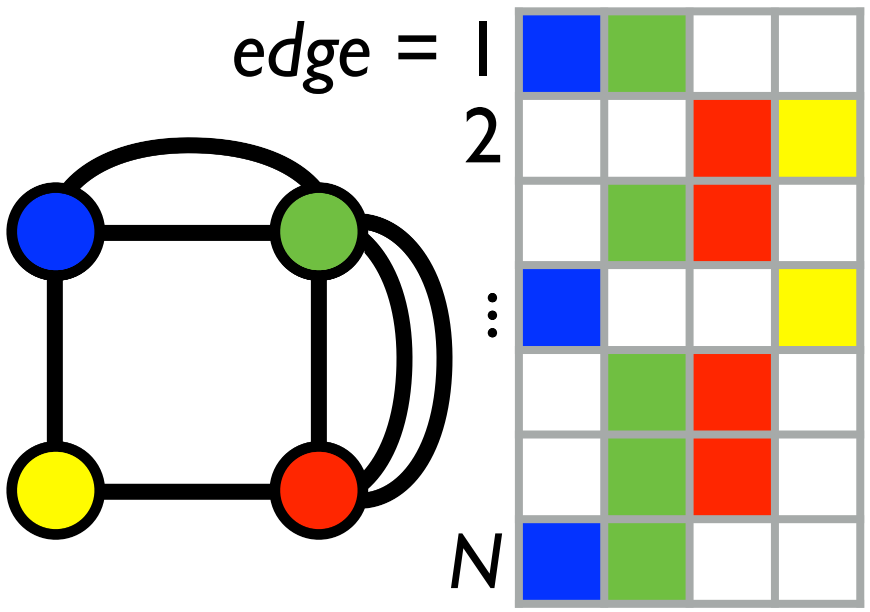


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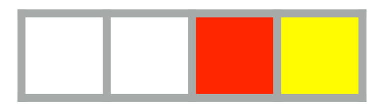
$\mathbb{P}(\text{row} =$  $) = p_2$

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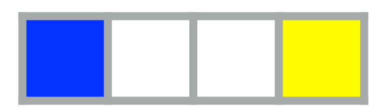
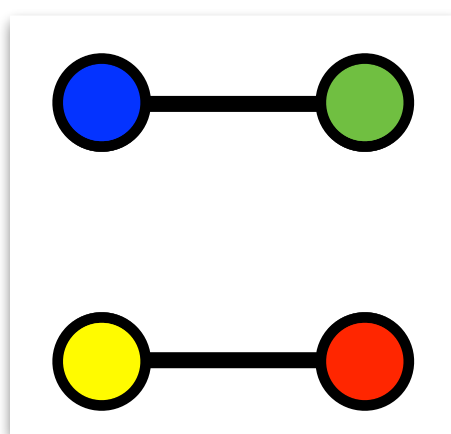
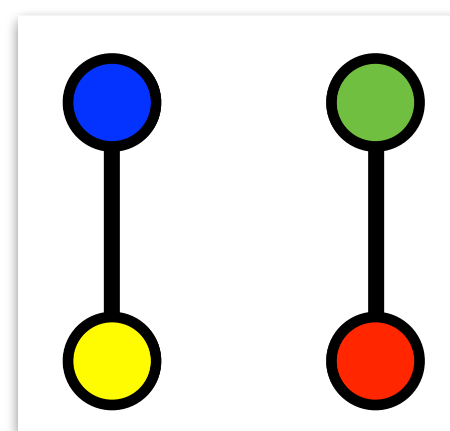
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $) \neq \mathbb{P}($  $)$

$$p_1 p_2 \neq p_3 p_4$$

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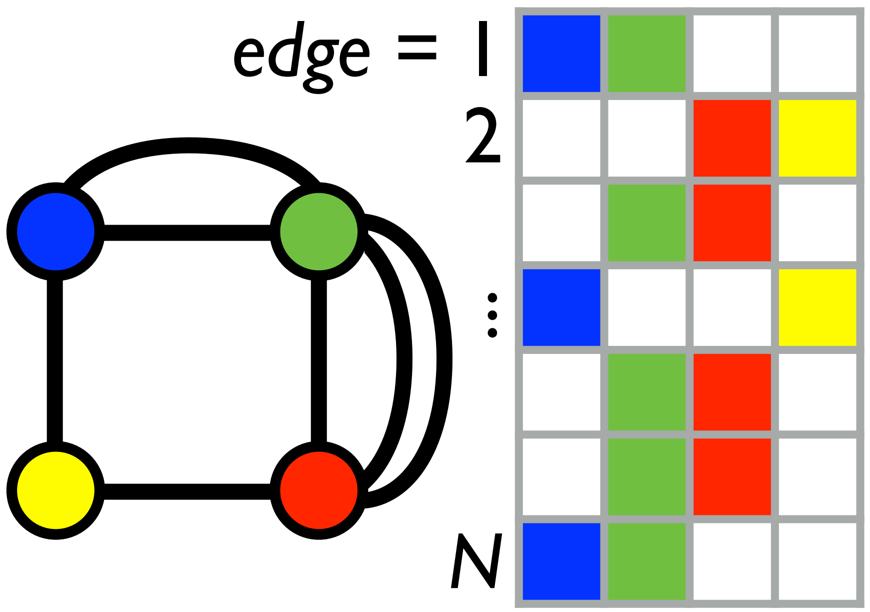


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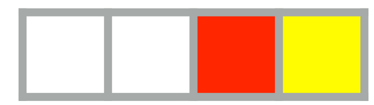
$\mathbb{P}(\text{row} =$  $) = p_2$

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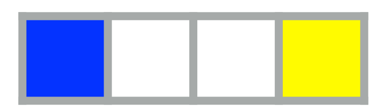
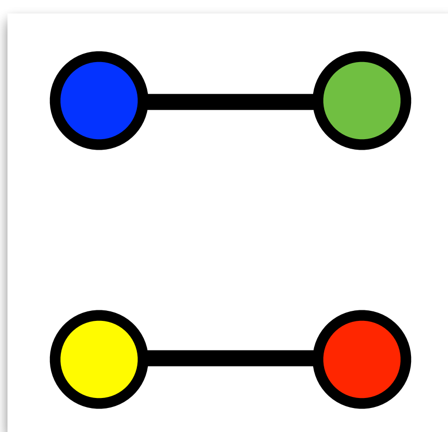
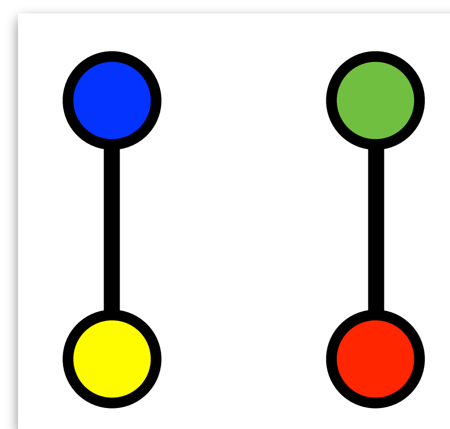
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $) \neq \mathbb{P}($  $)$

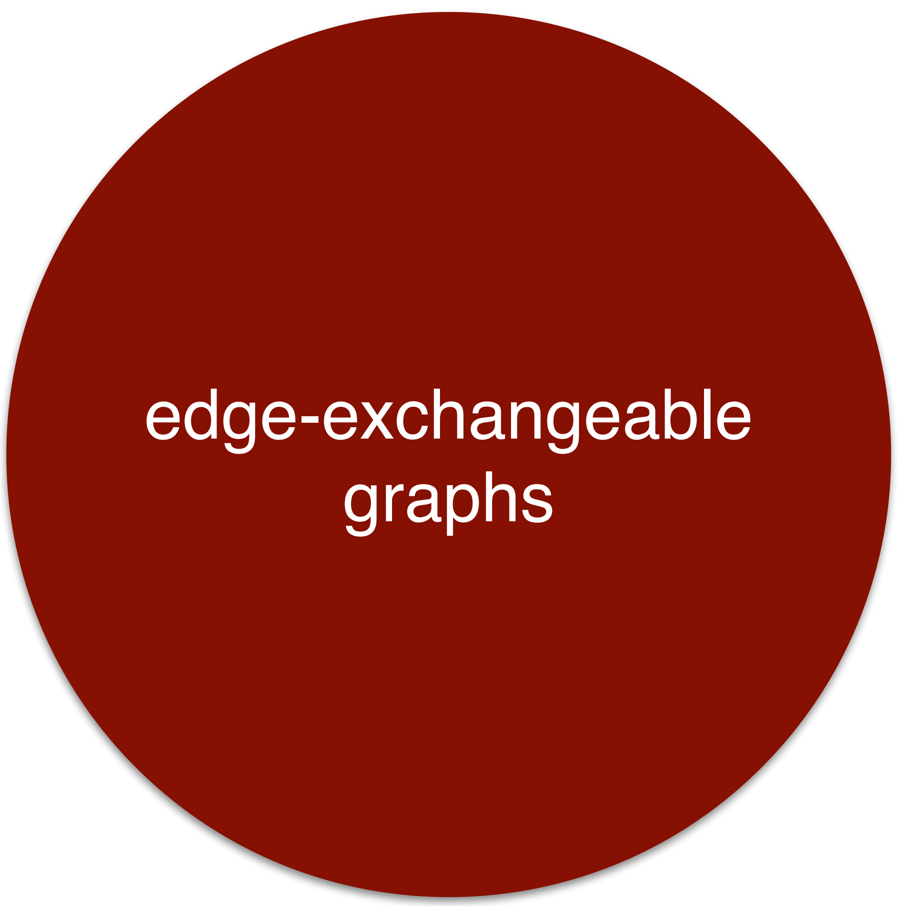
$$p_1 p_2 \neq p_3 p_4$$

Cor. Not every exchangeable feature allocation has an EFPF.



[Broderick, Jordan, Pitman 2013]

# What we know so far




- Goal 1: characterization theorem for edge-exchangeable graphs
- Goal 2: sparsity theorem for edge-exchangeable graphs

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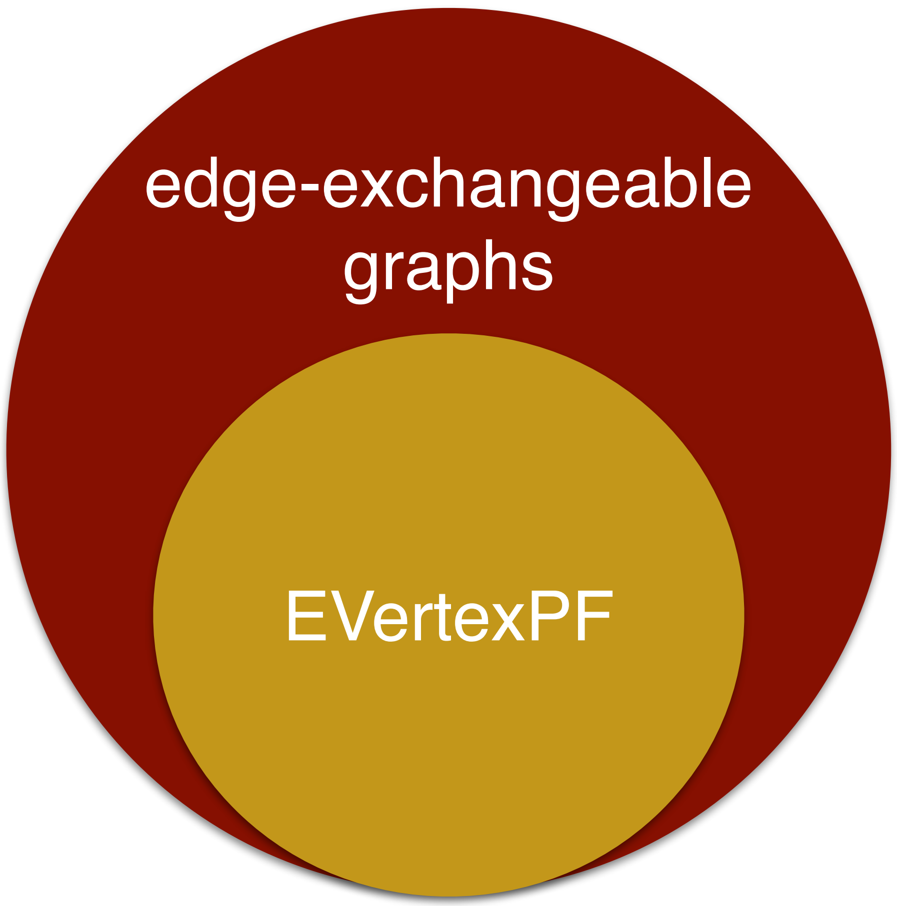


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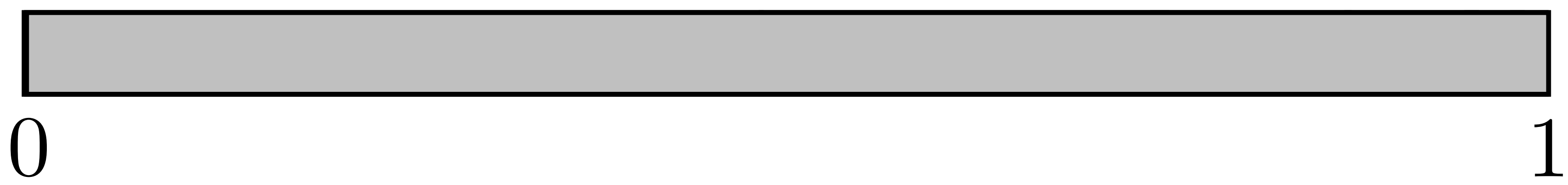
# Clustering




[Kingman 1978]

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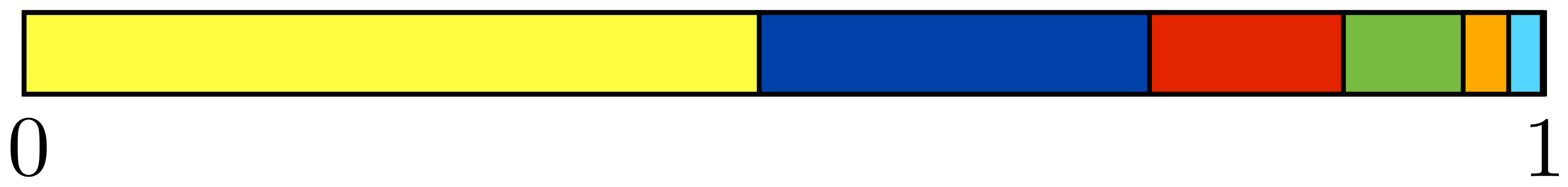




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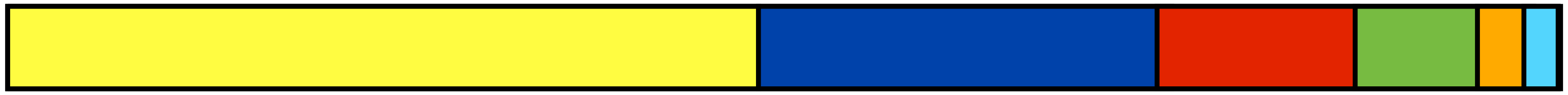

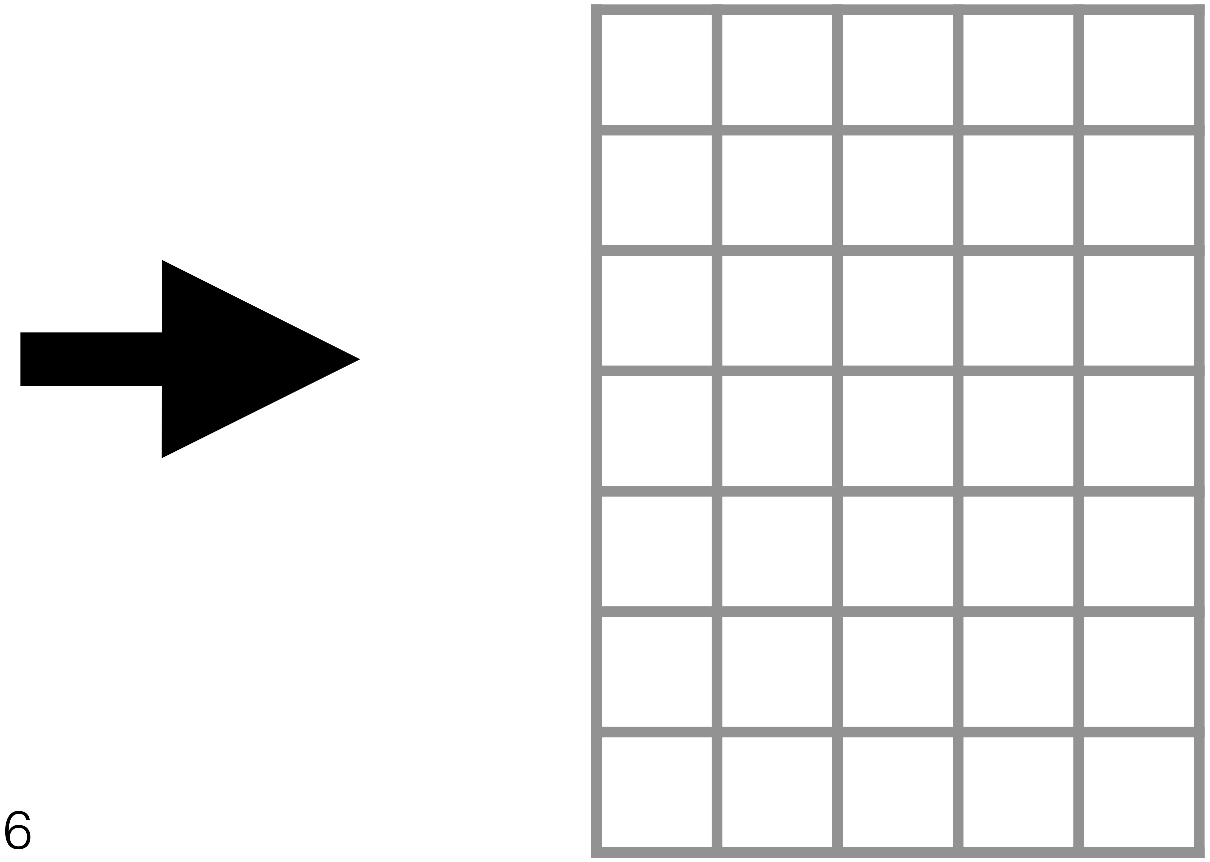



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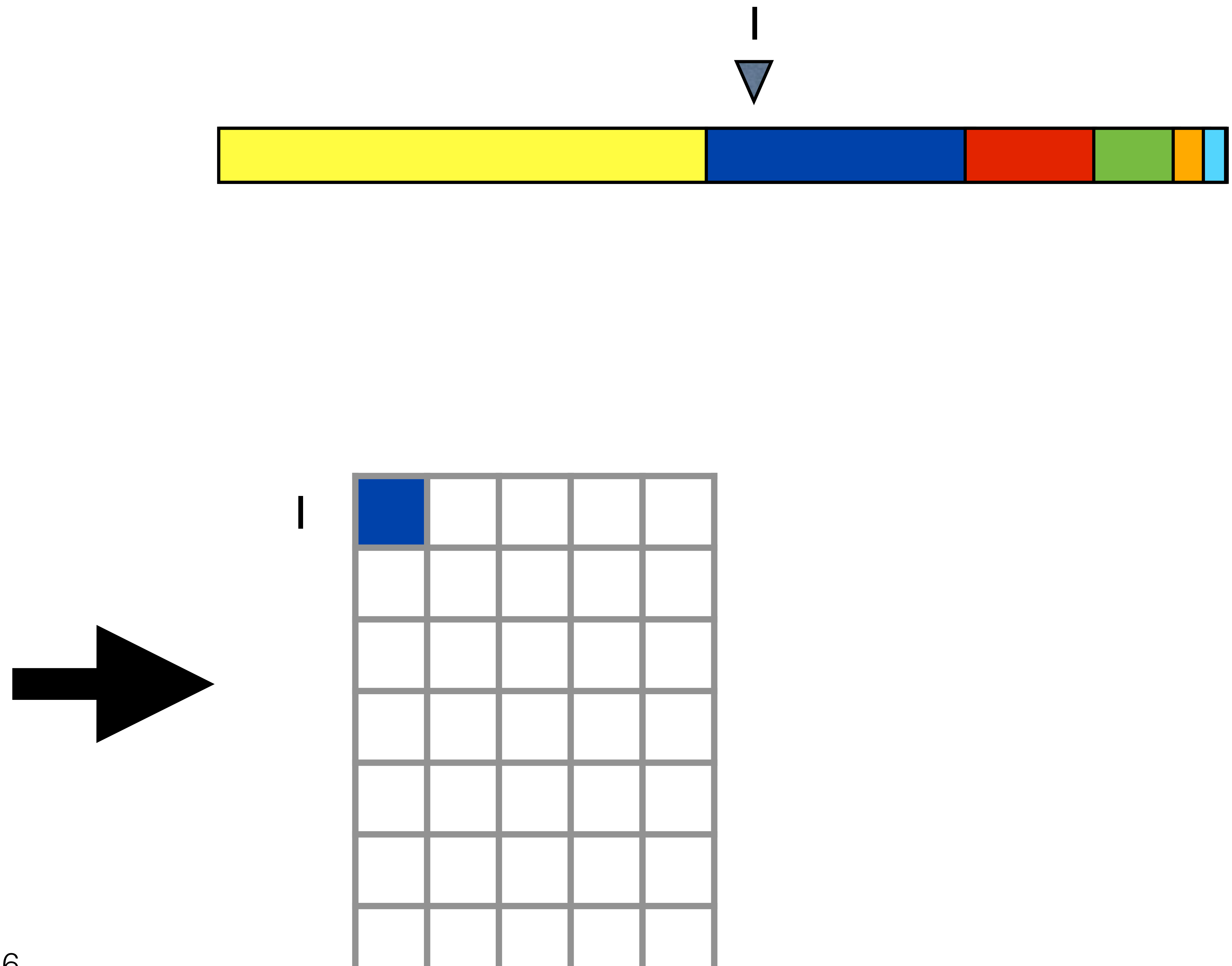




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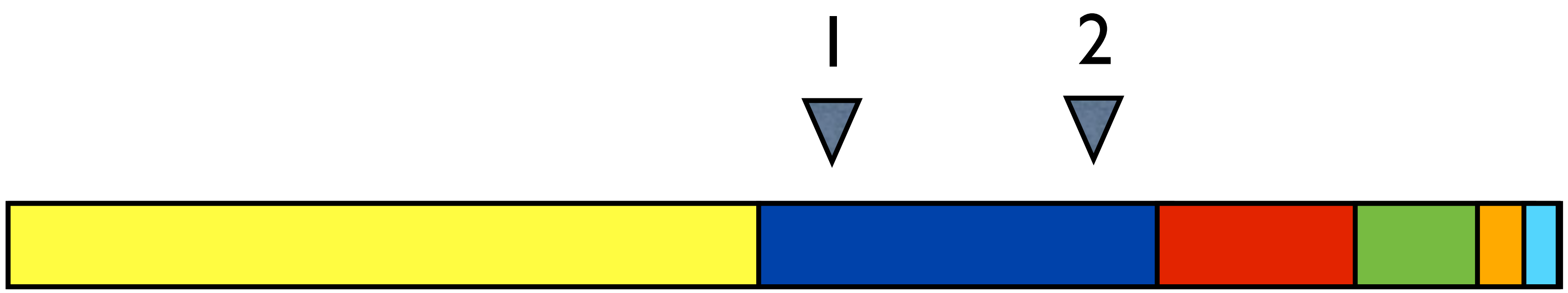


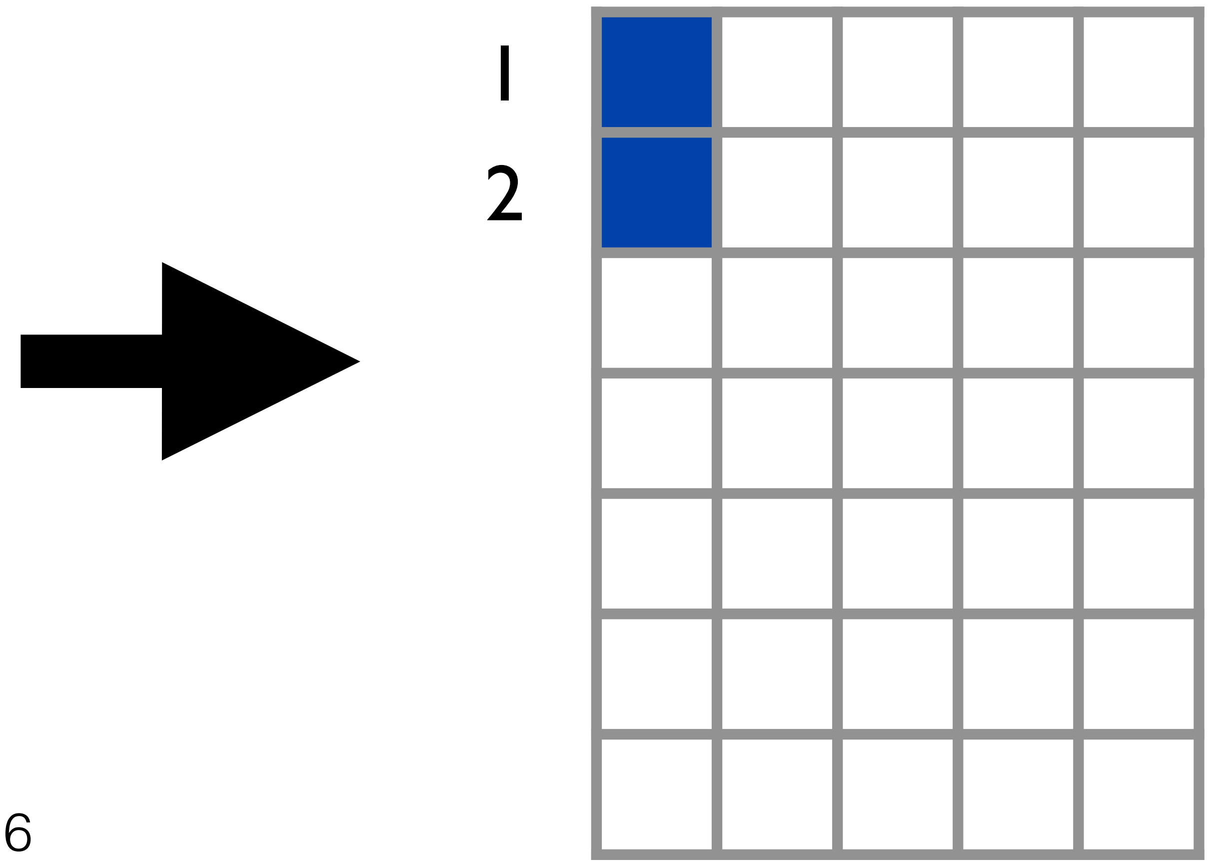




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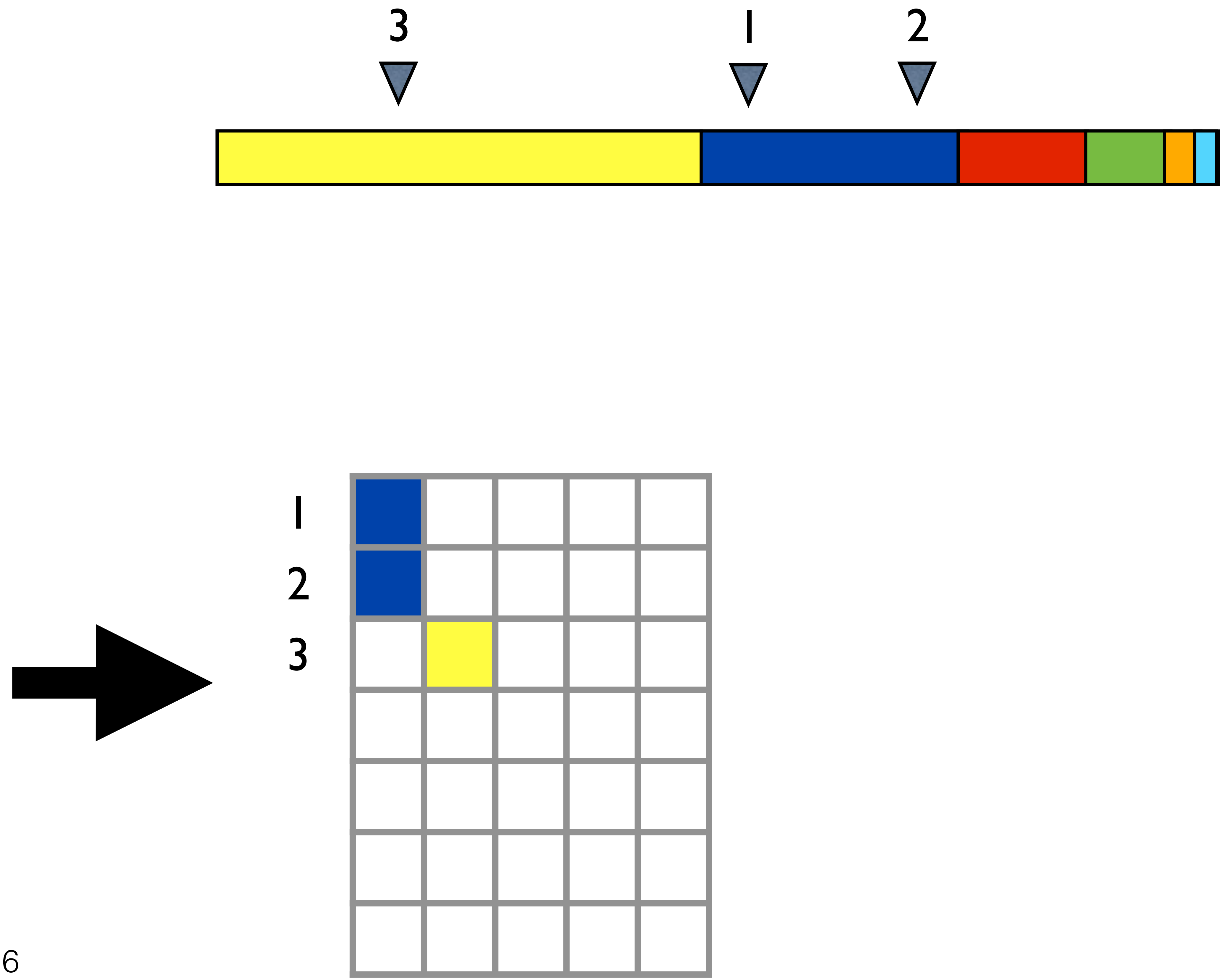




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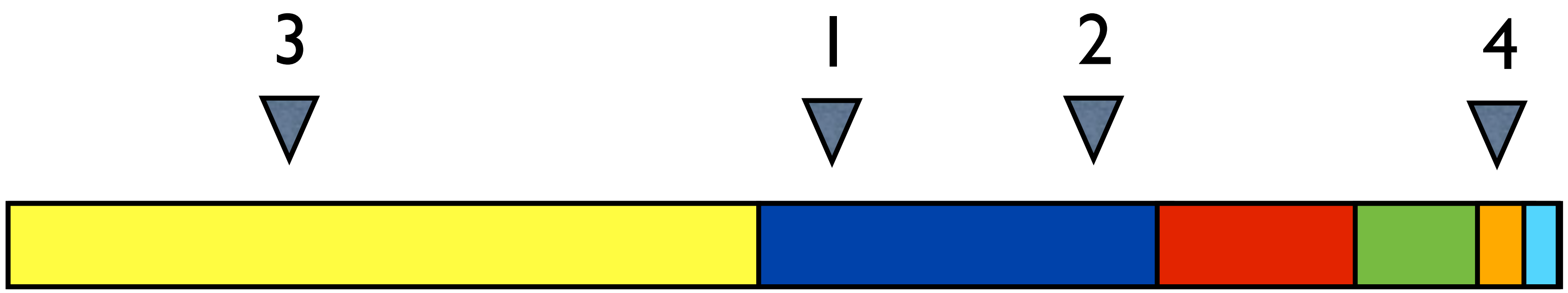


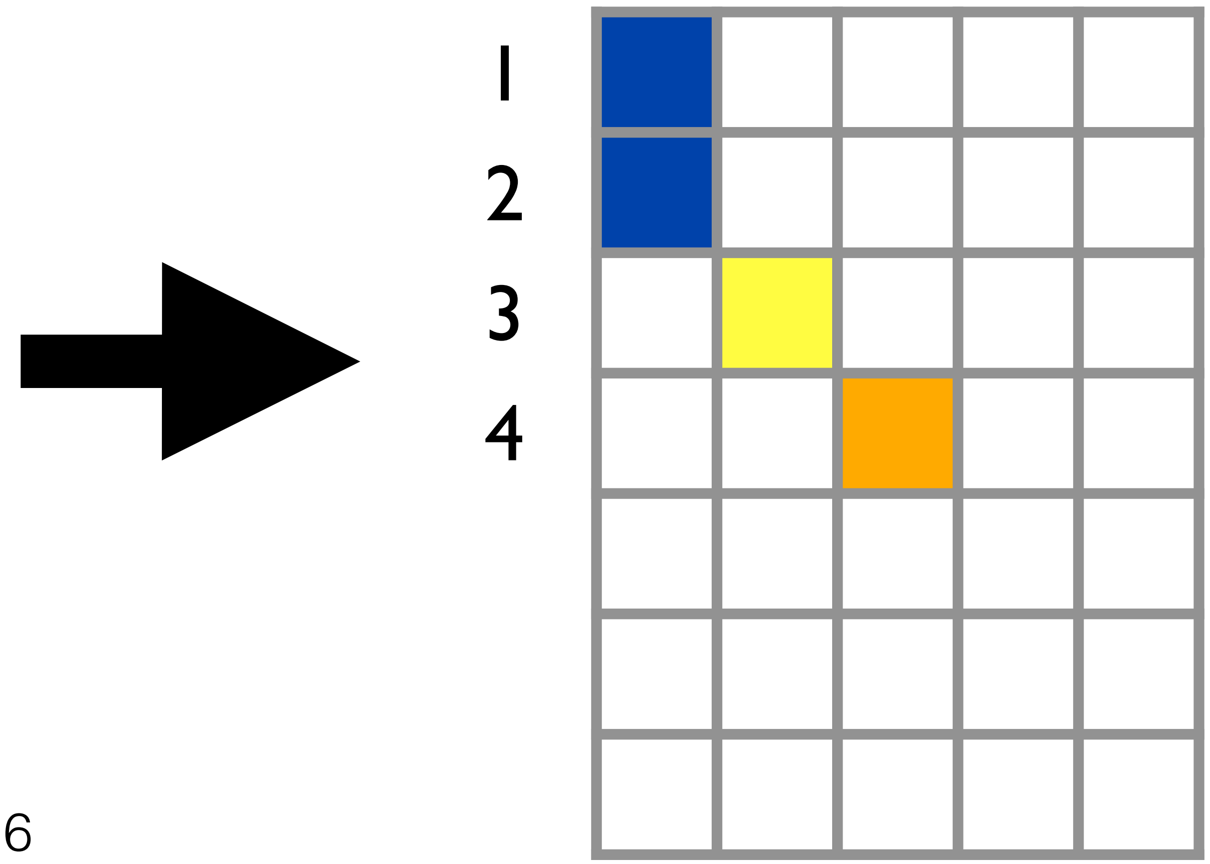




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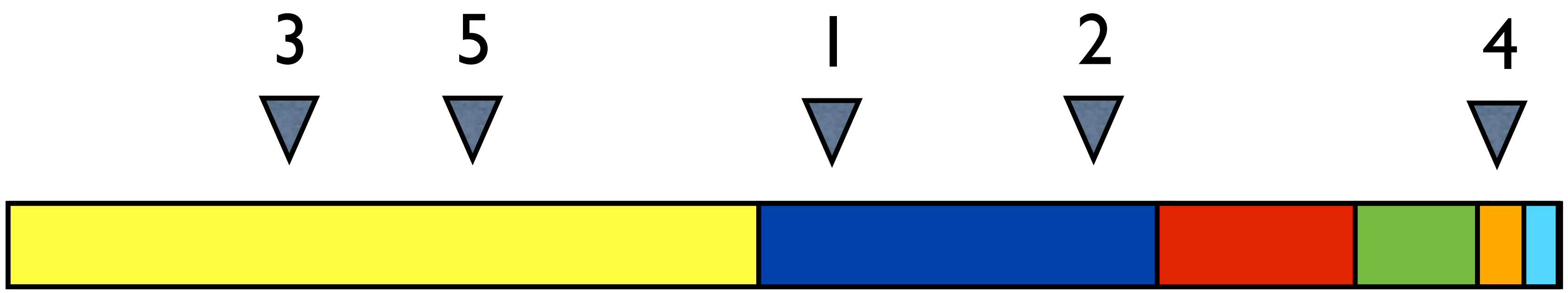


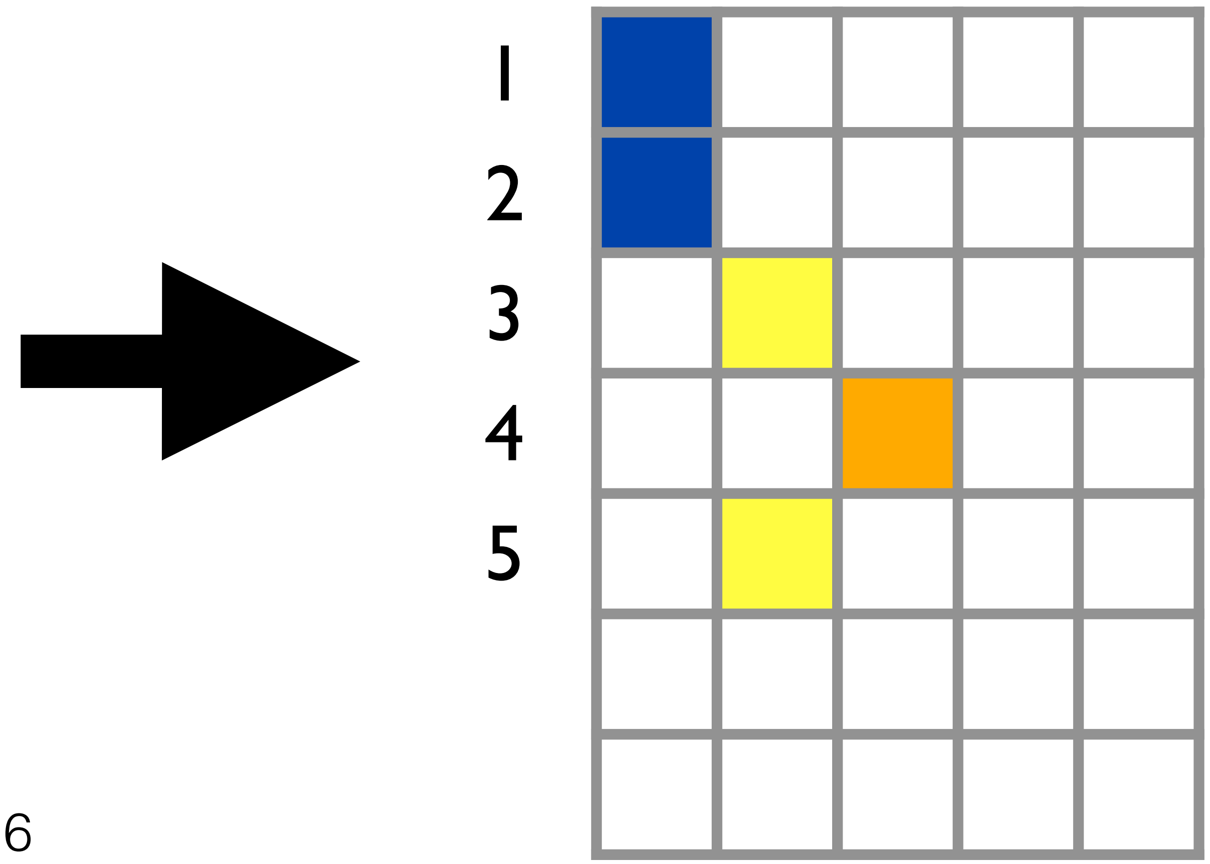




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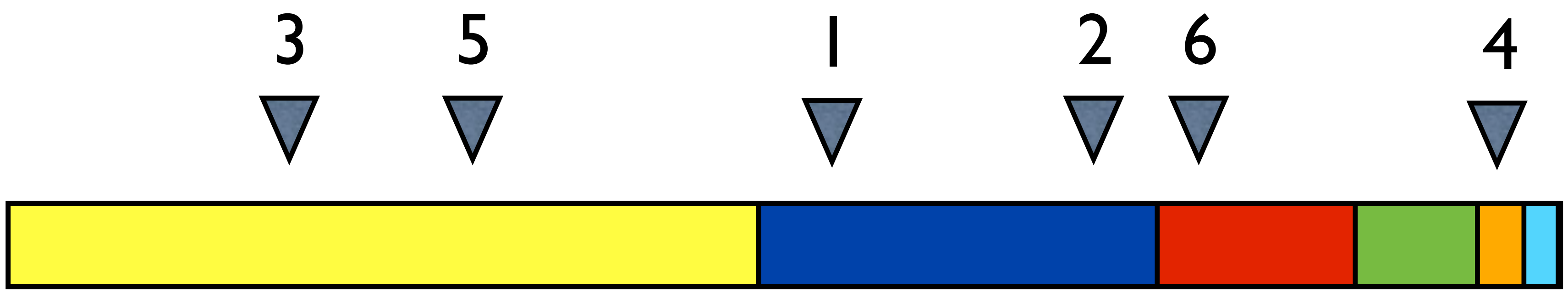


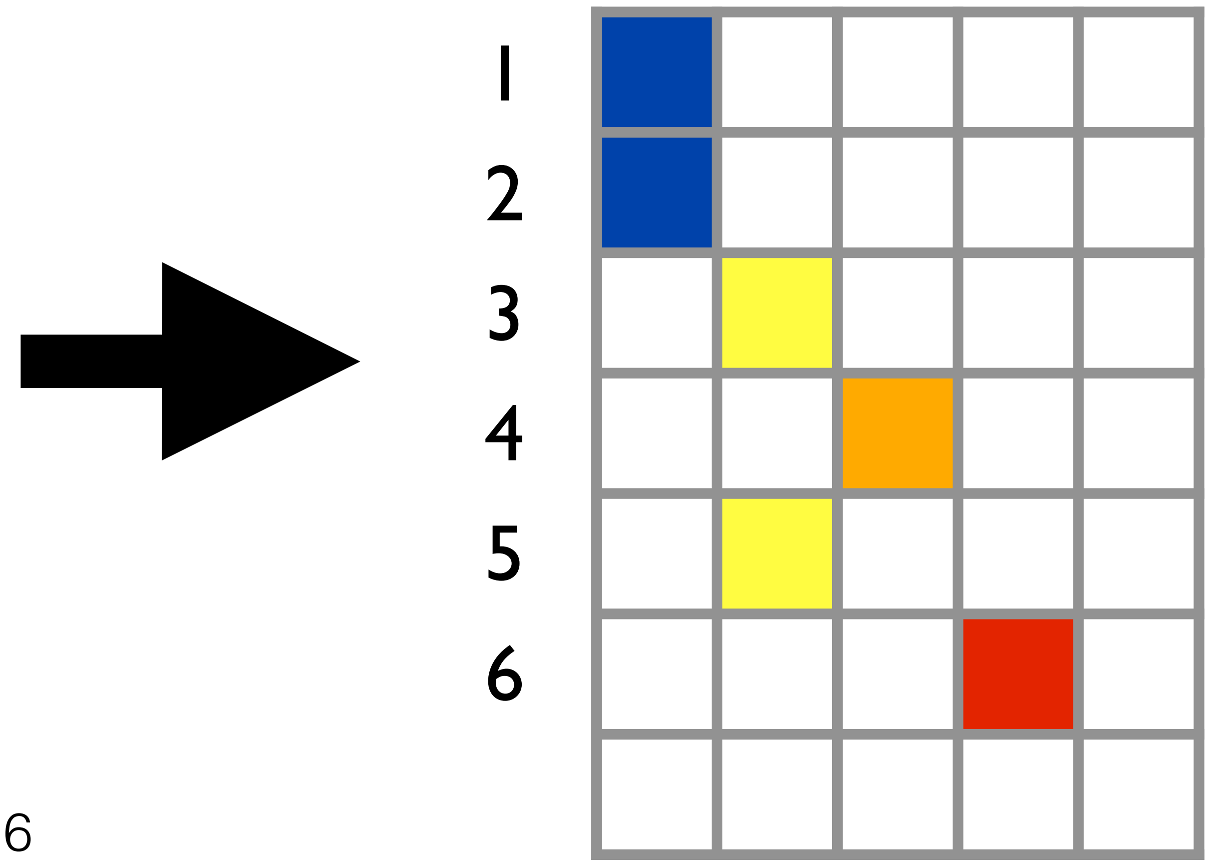




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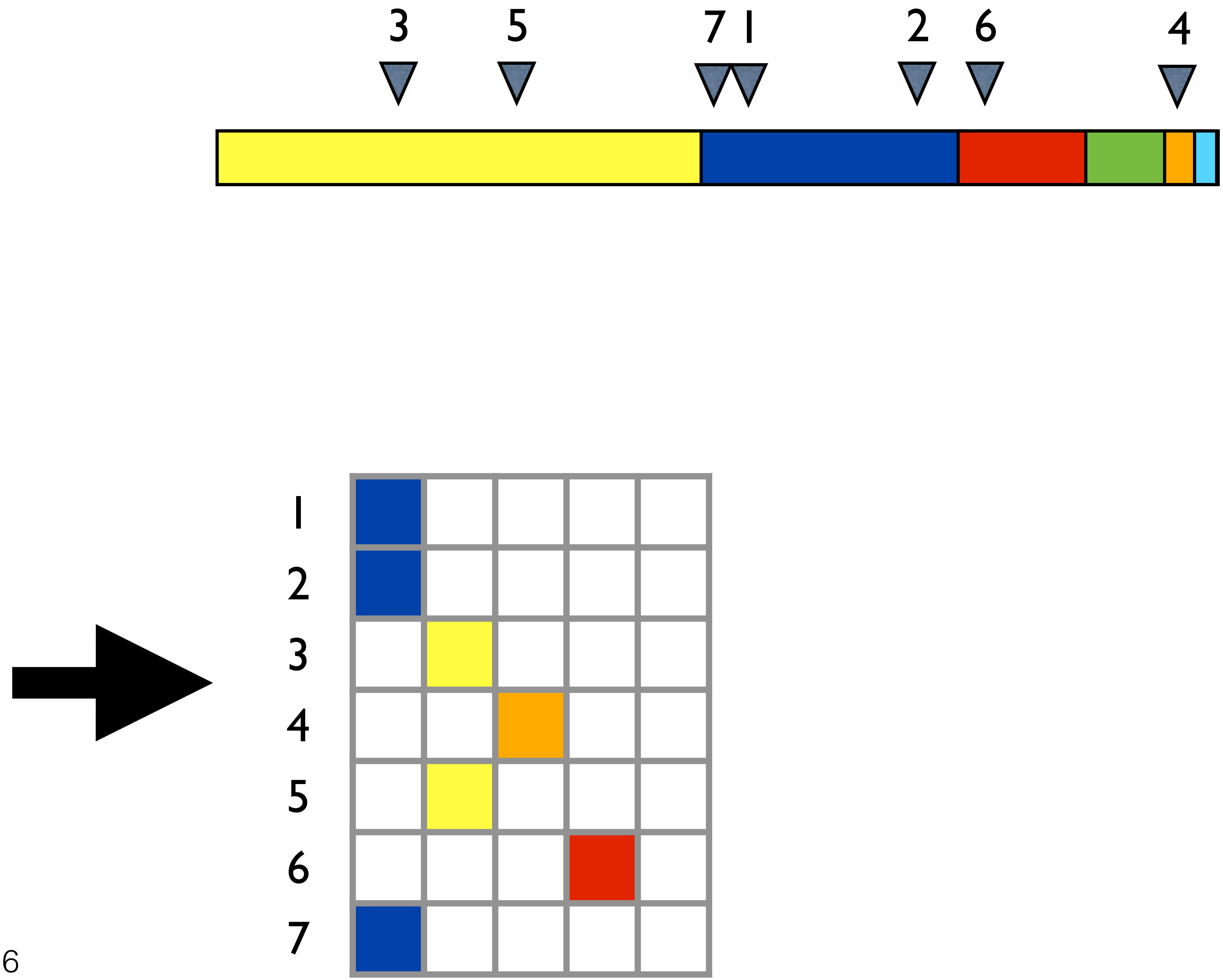




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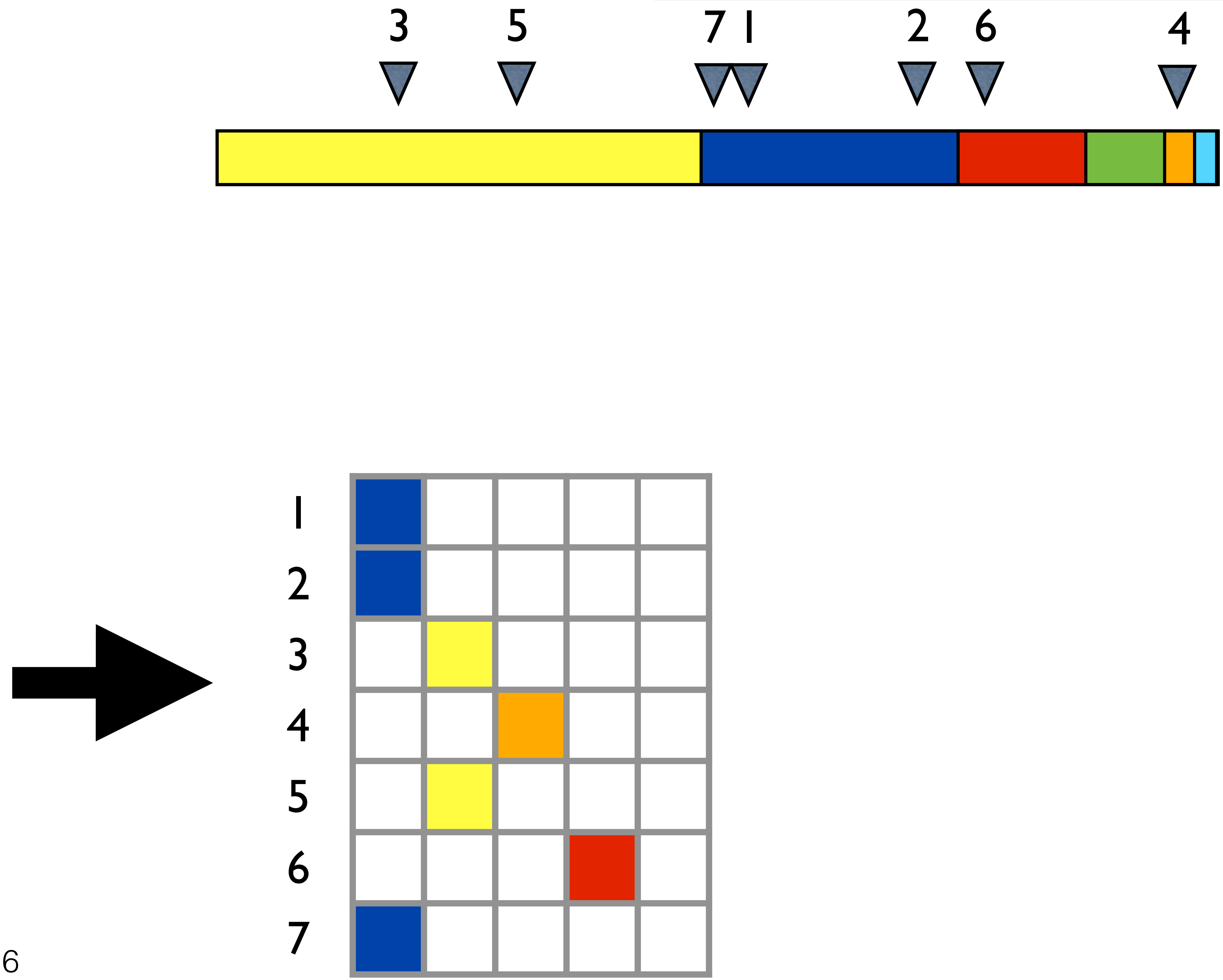
# Clustering

Thm (Kingman). A clustering is exchangeable iff it has a Kingman paintbox representation

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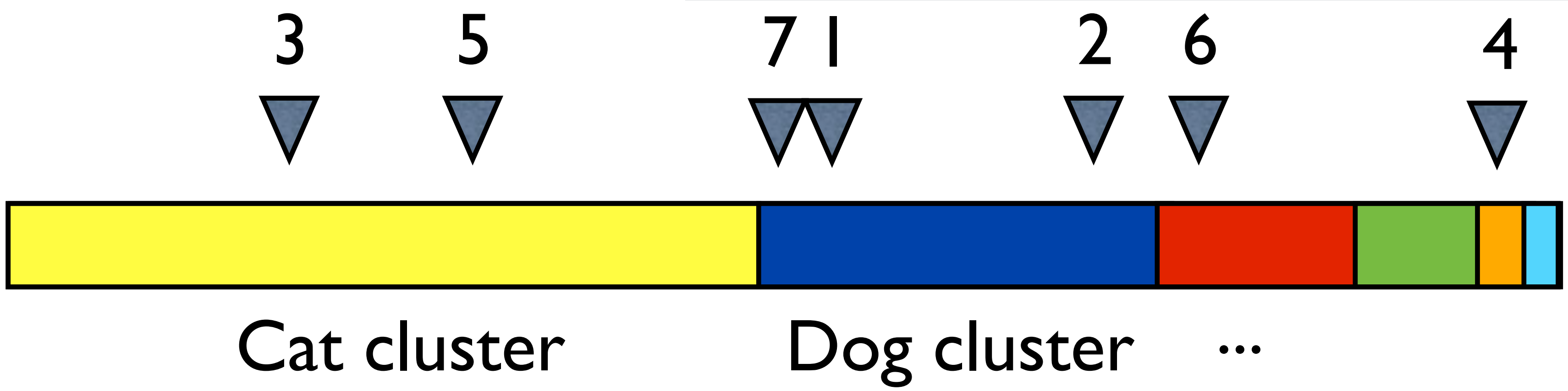


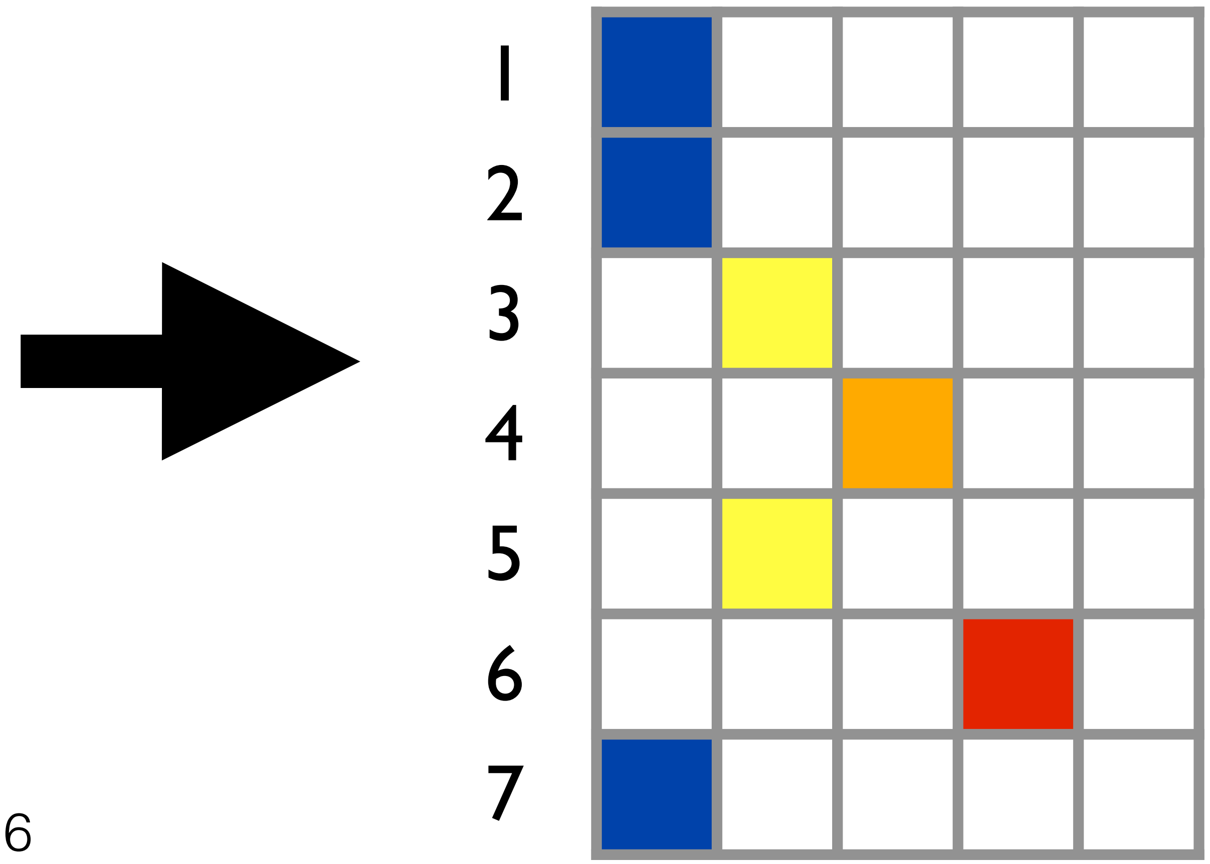




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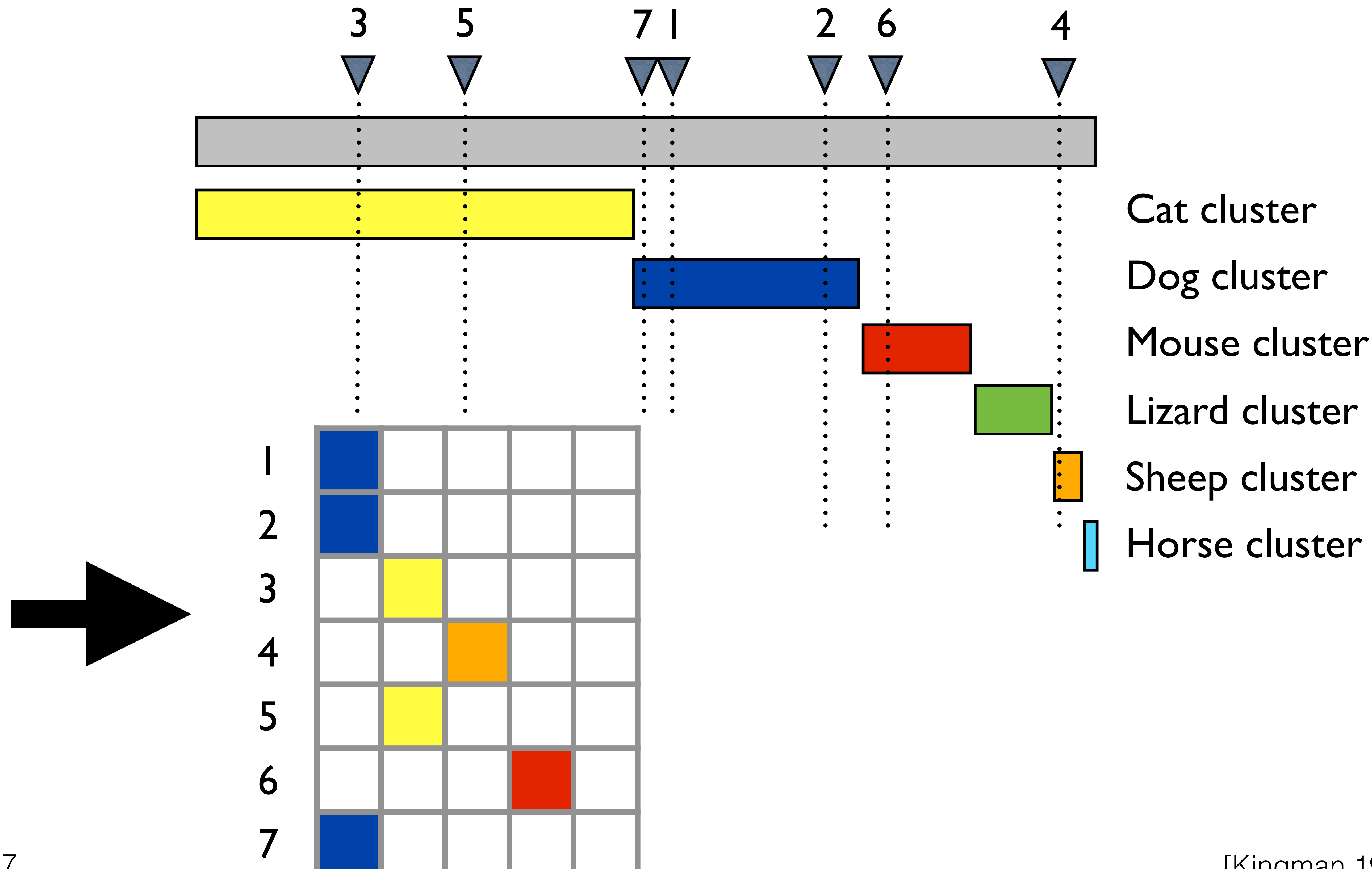




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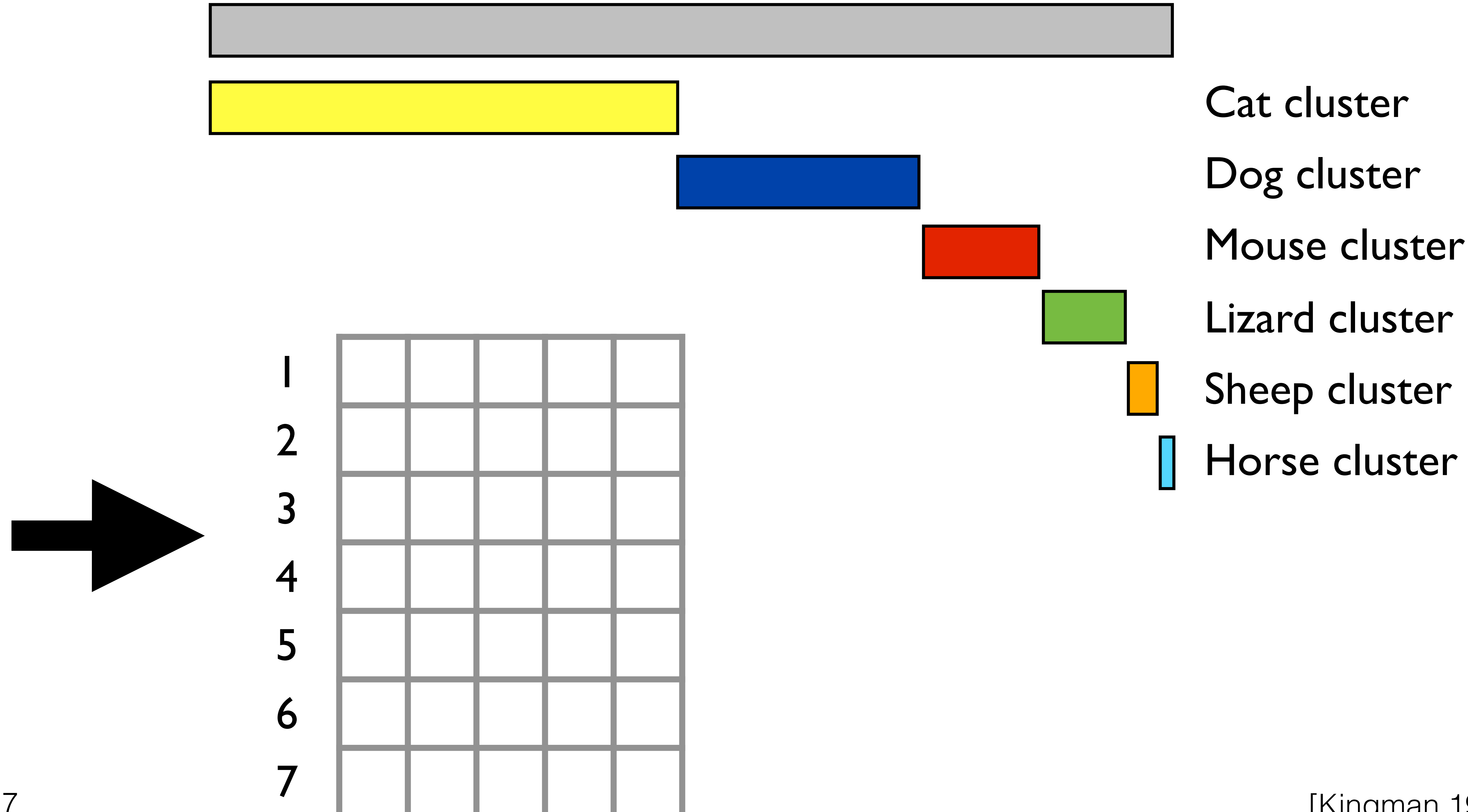




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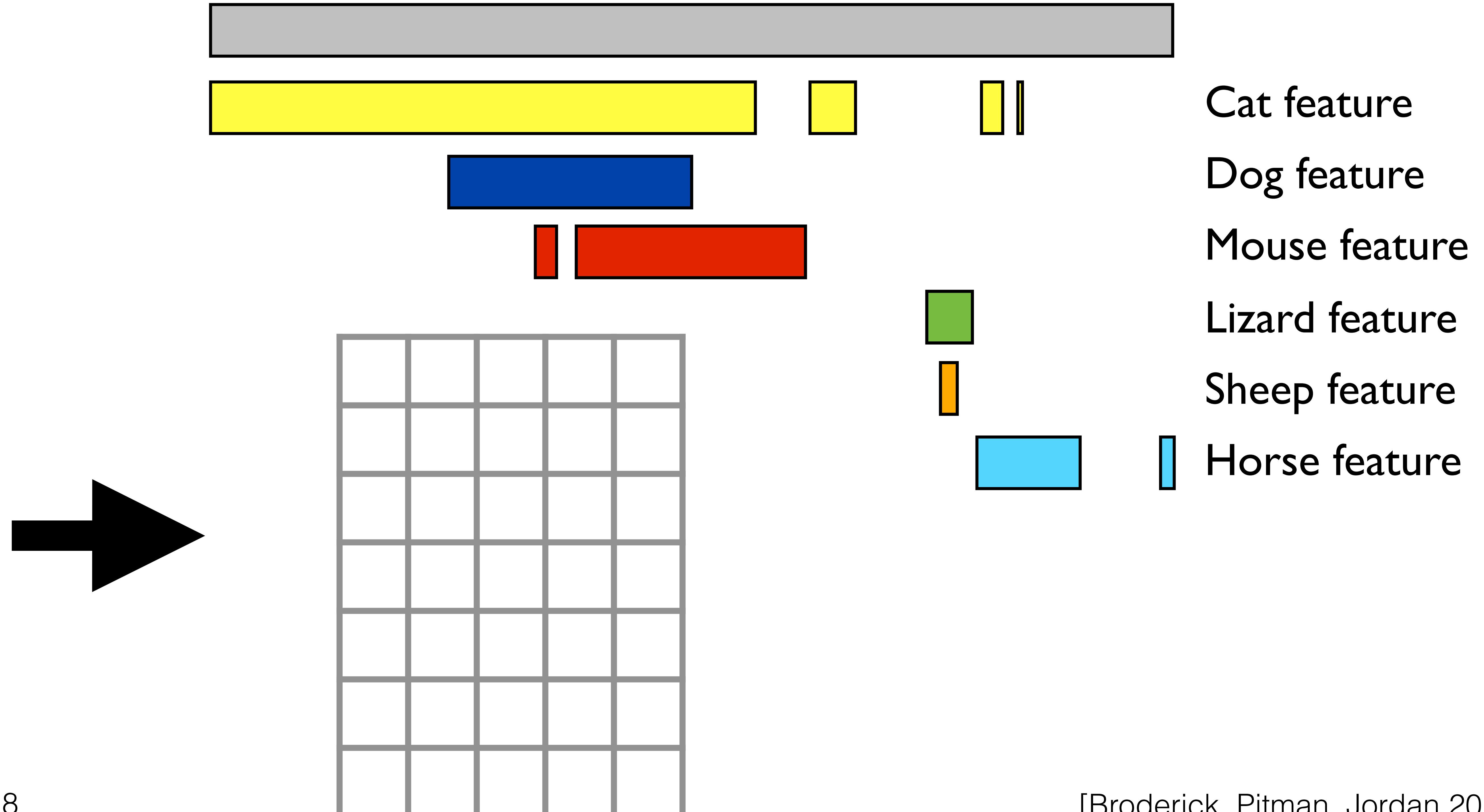




[Broderick, Pitman, Jordan 2013]

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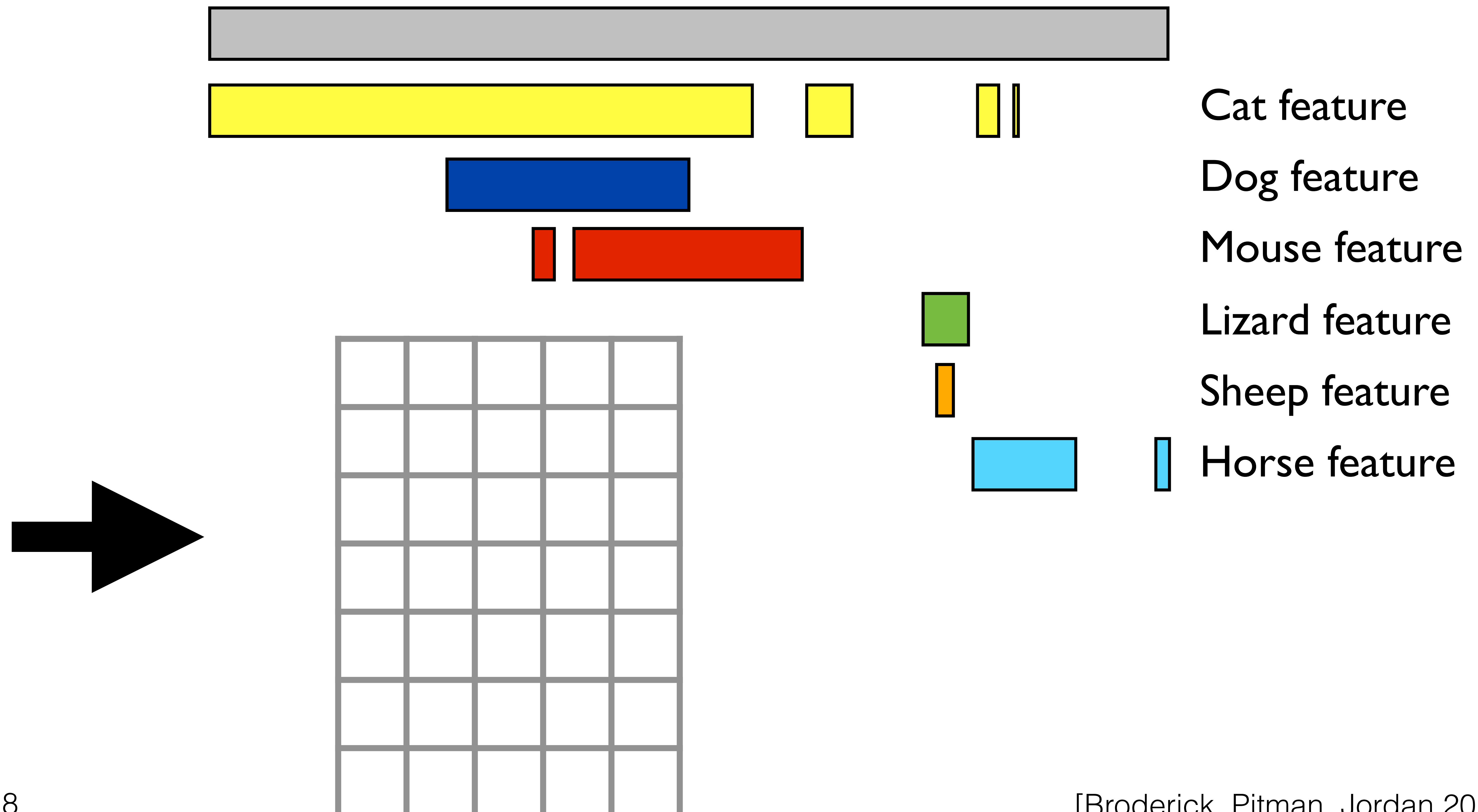


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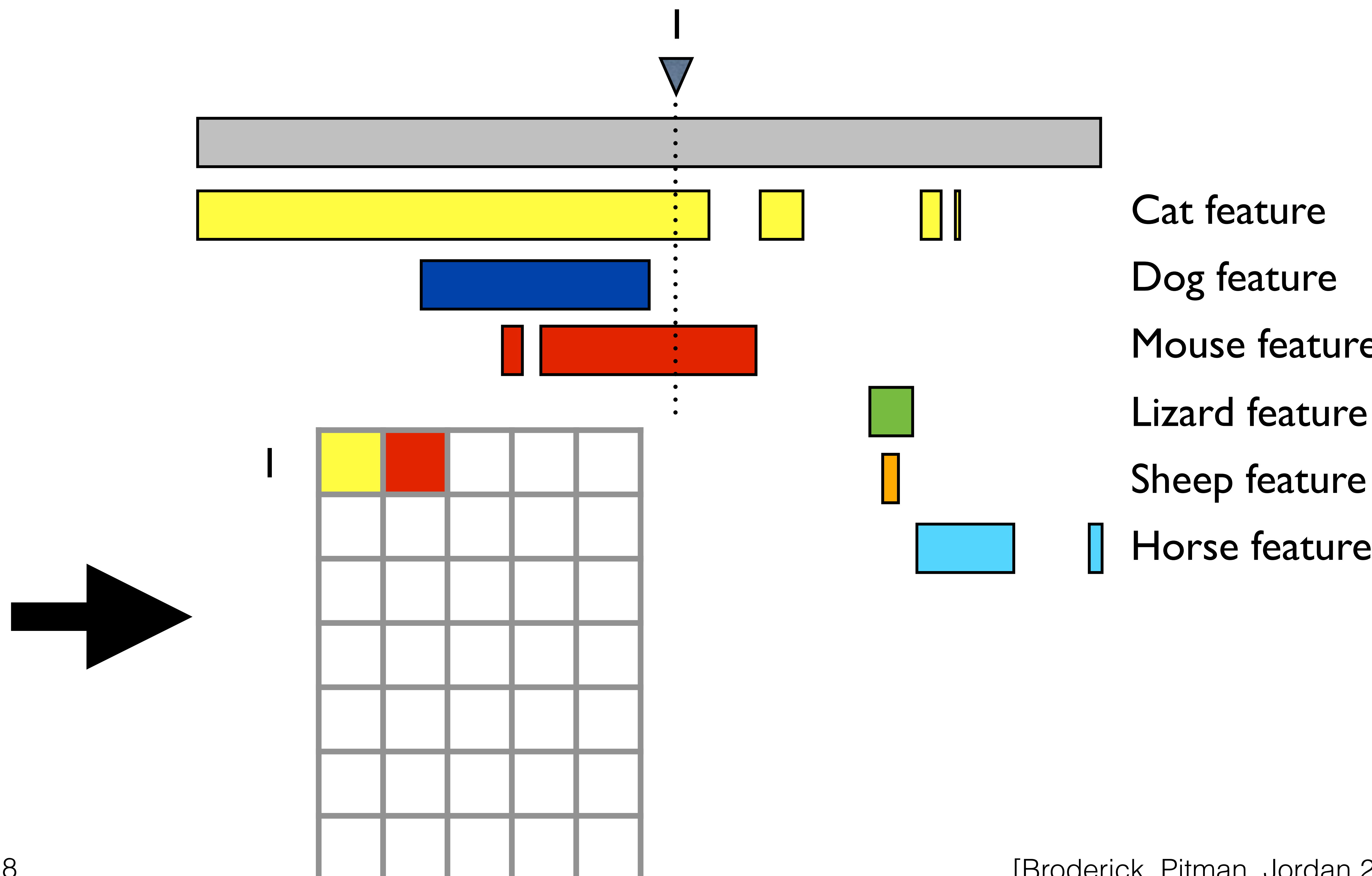




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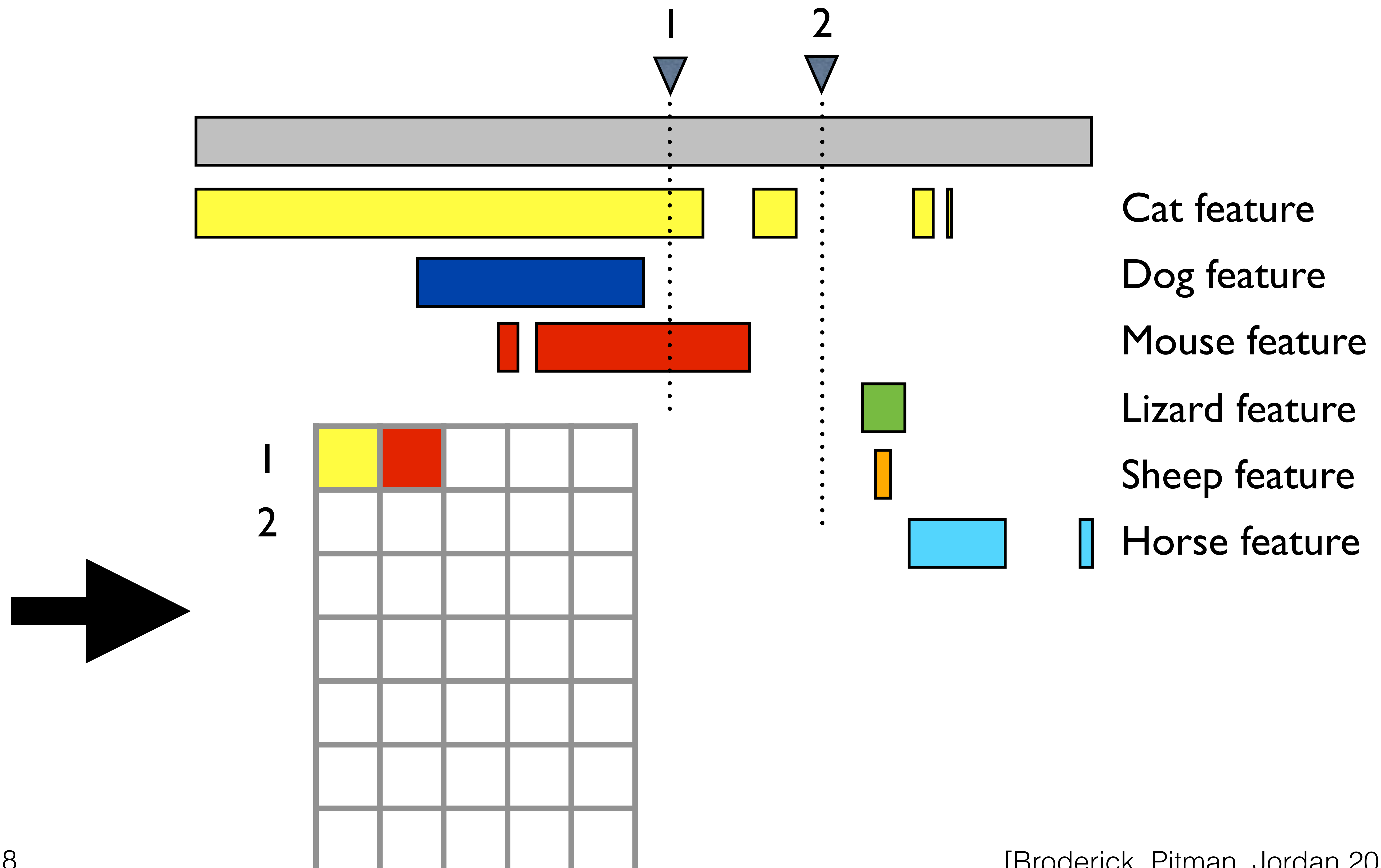




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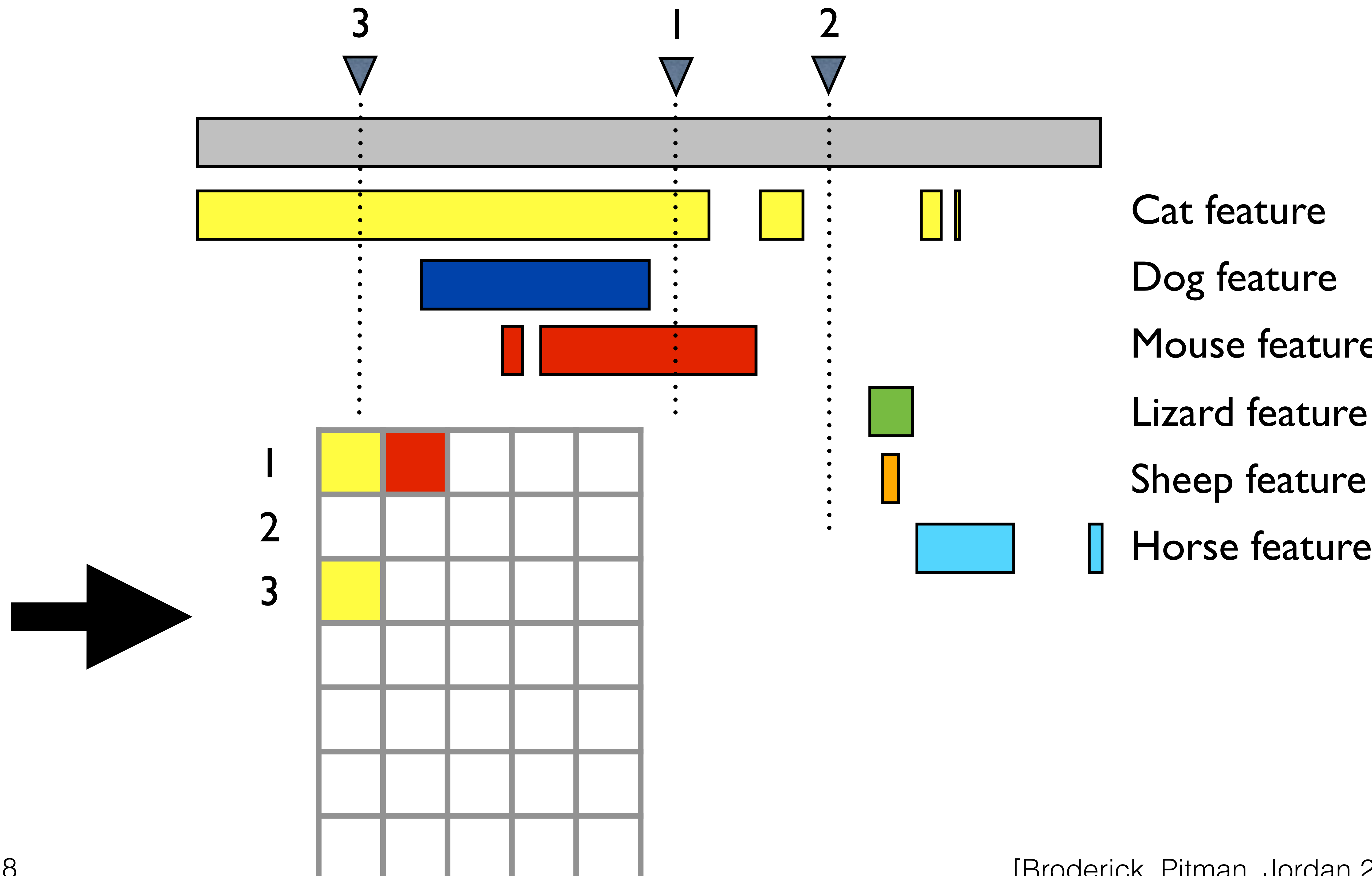


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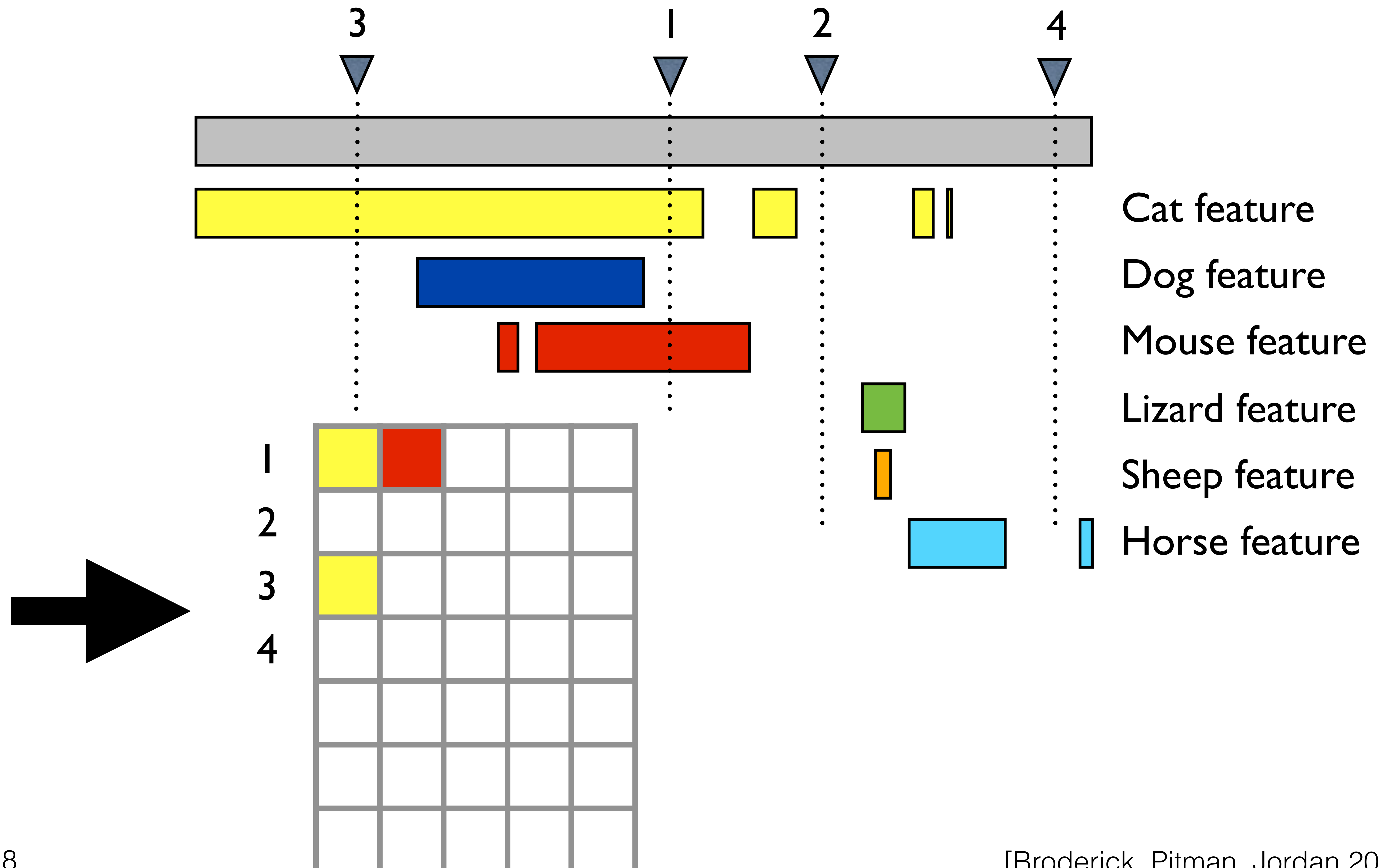




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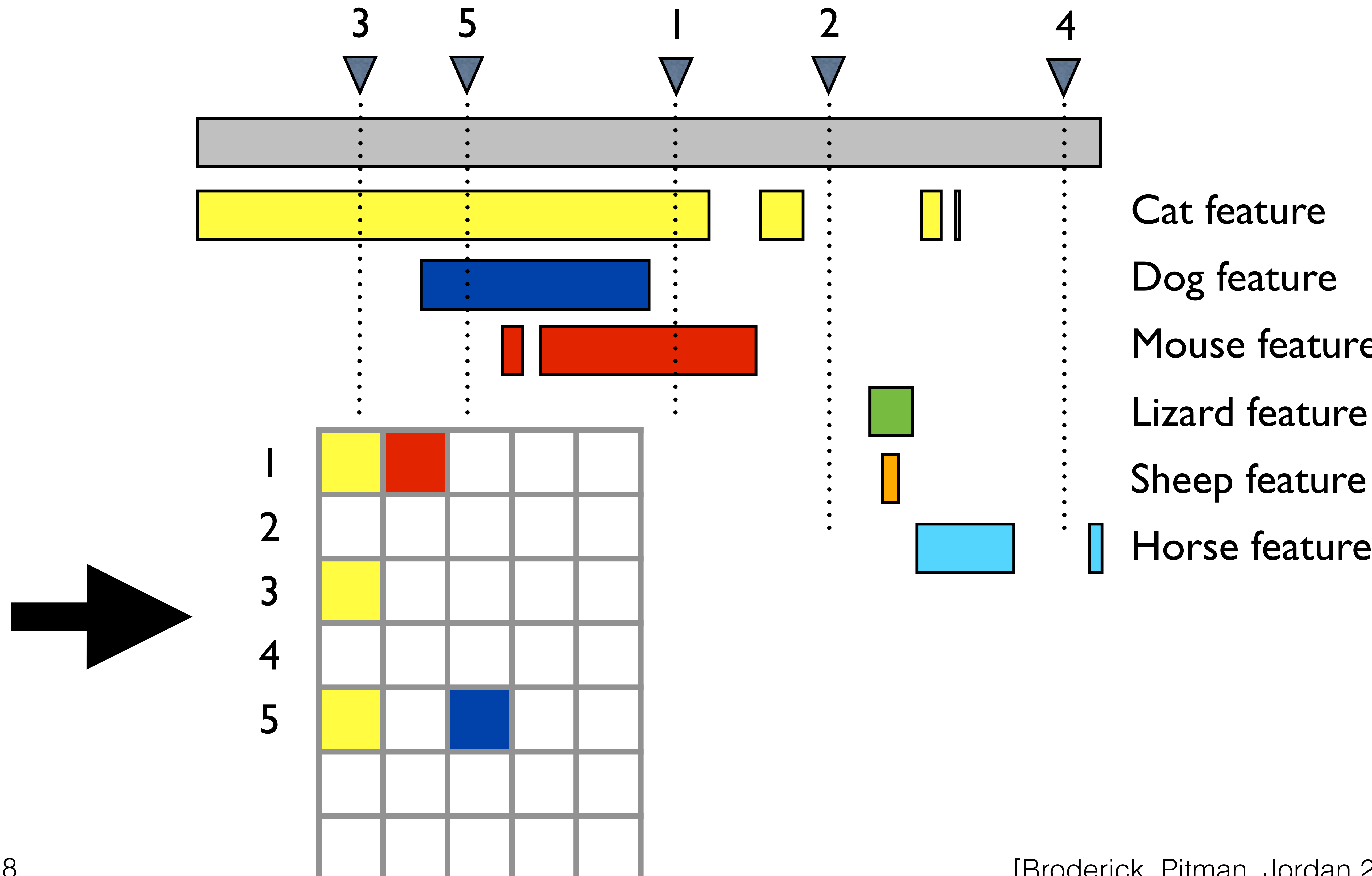


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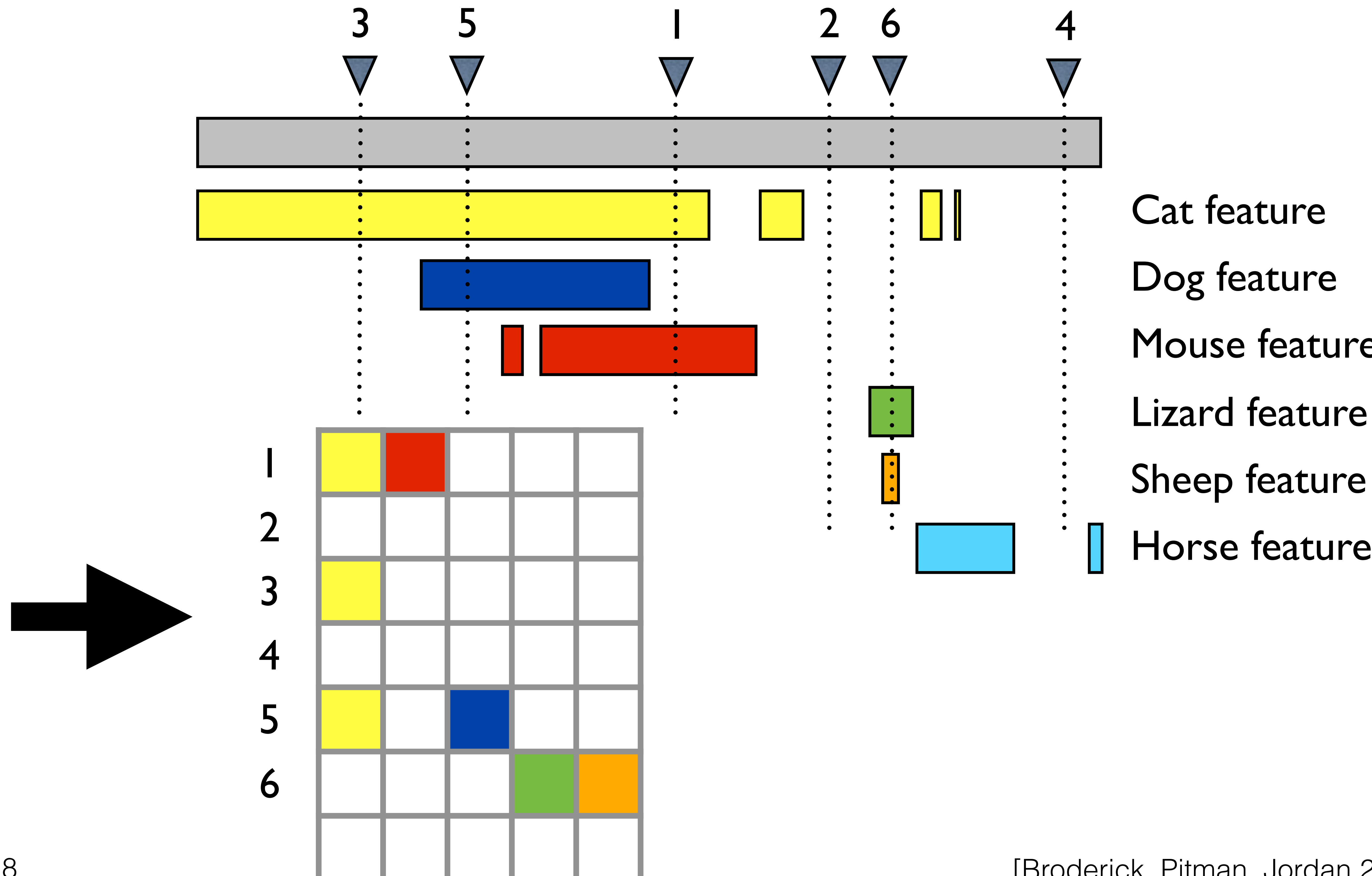




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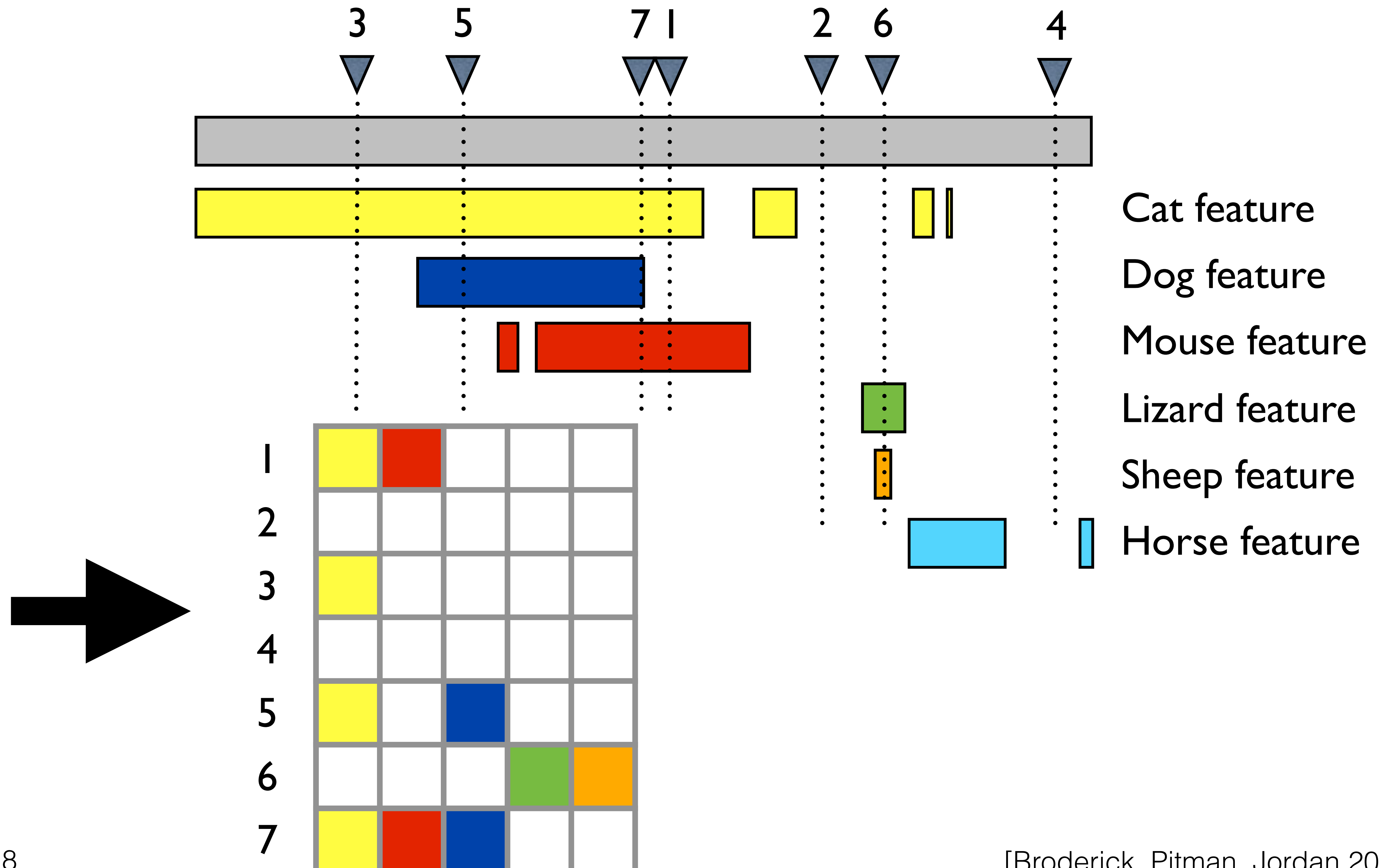




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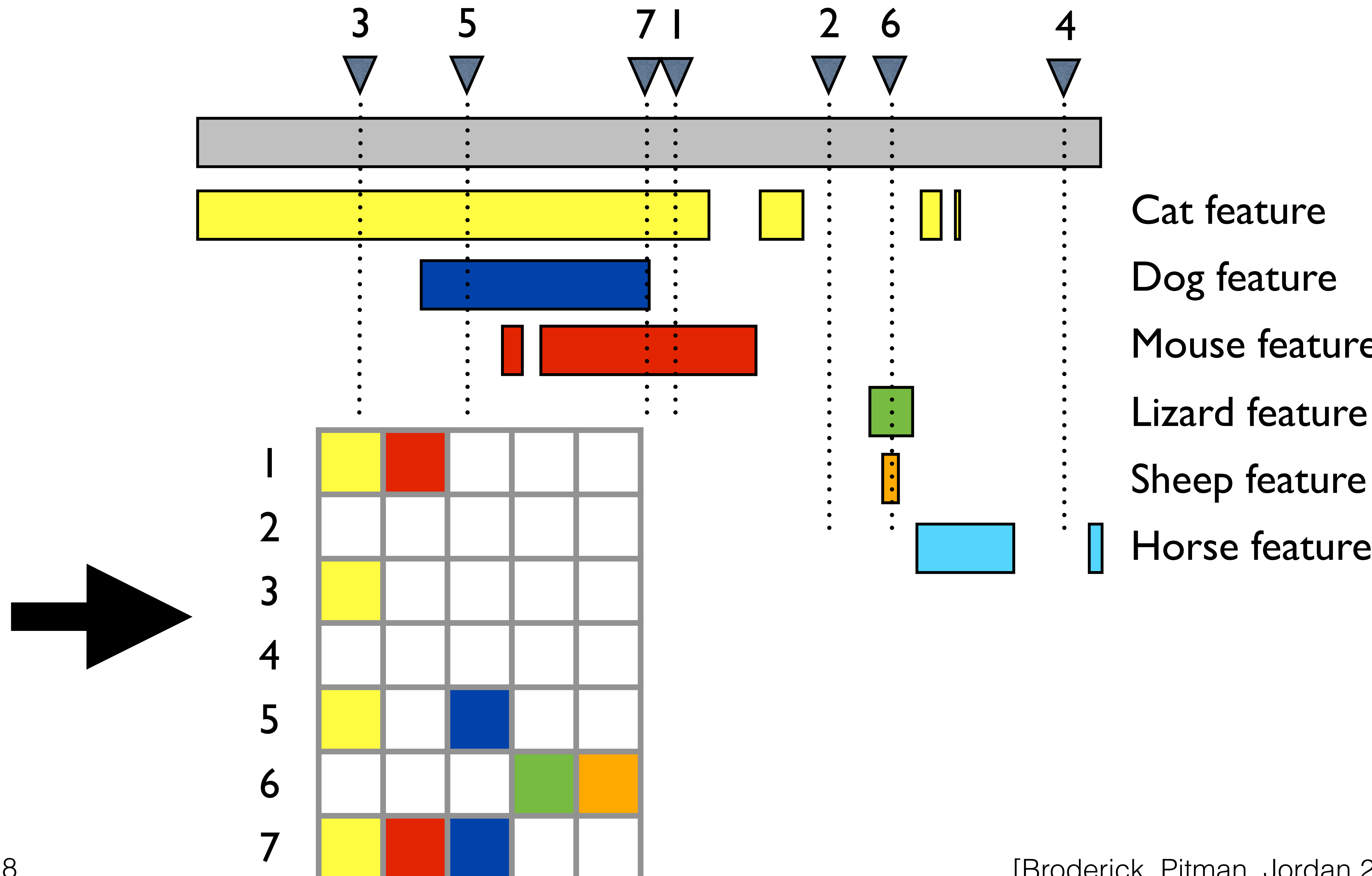
Thm (BPJ). A feature allocation is exchangeable iff it has a feature paintbox representation



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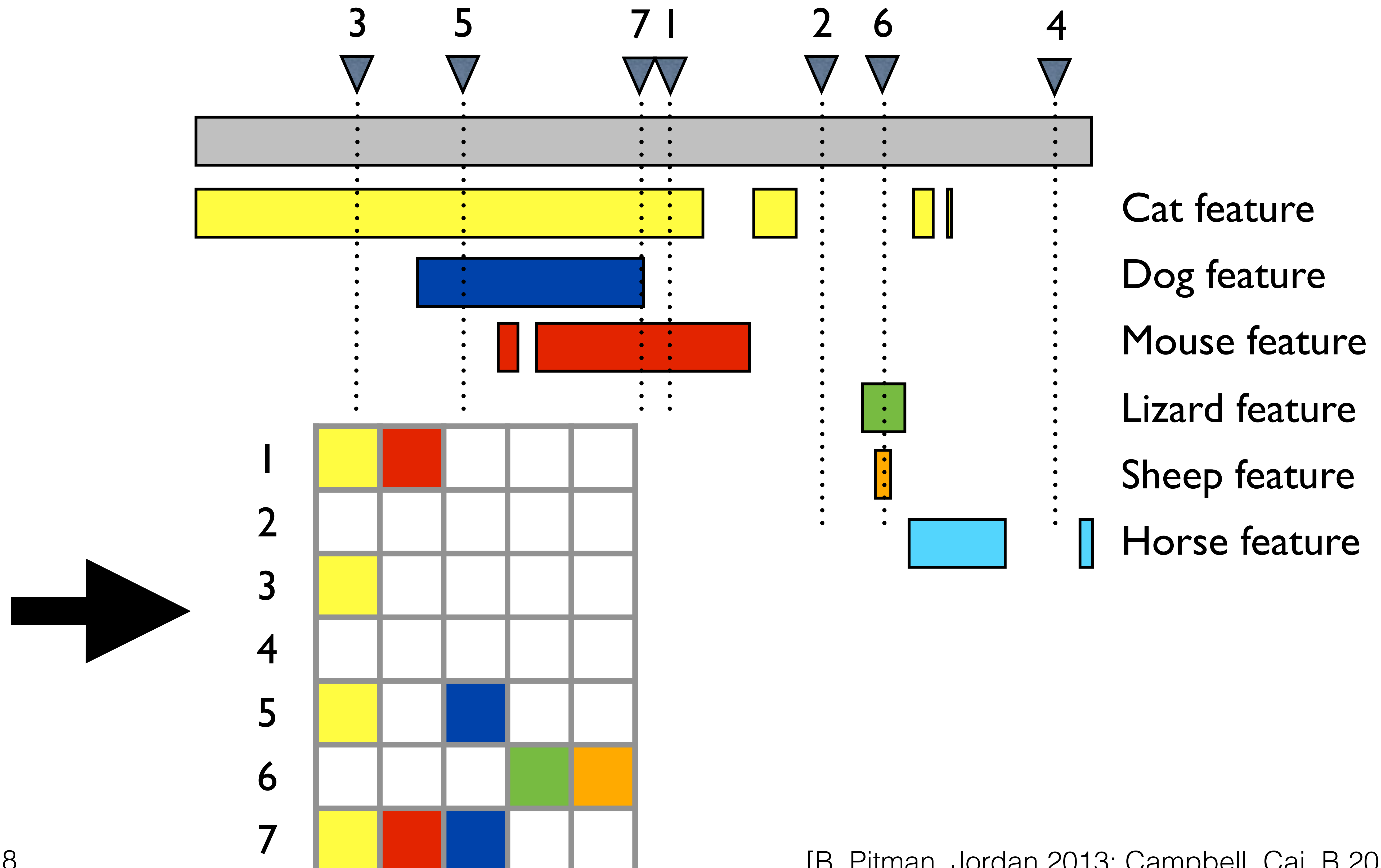
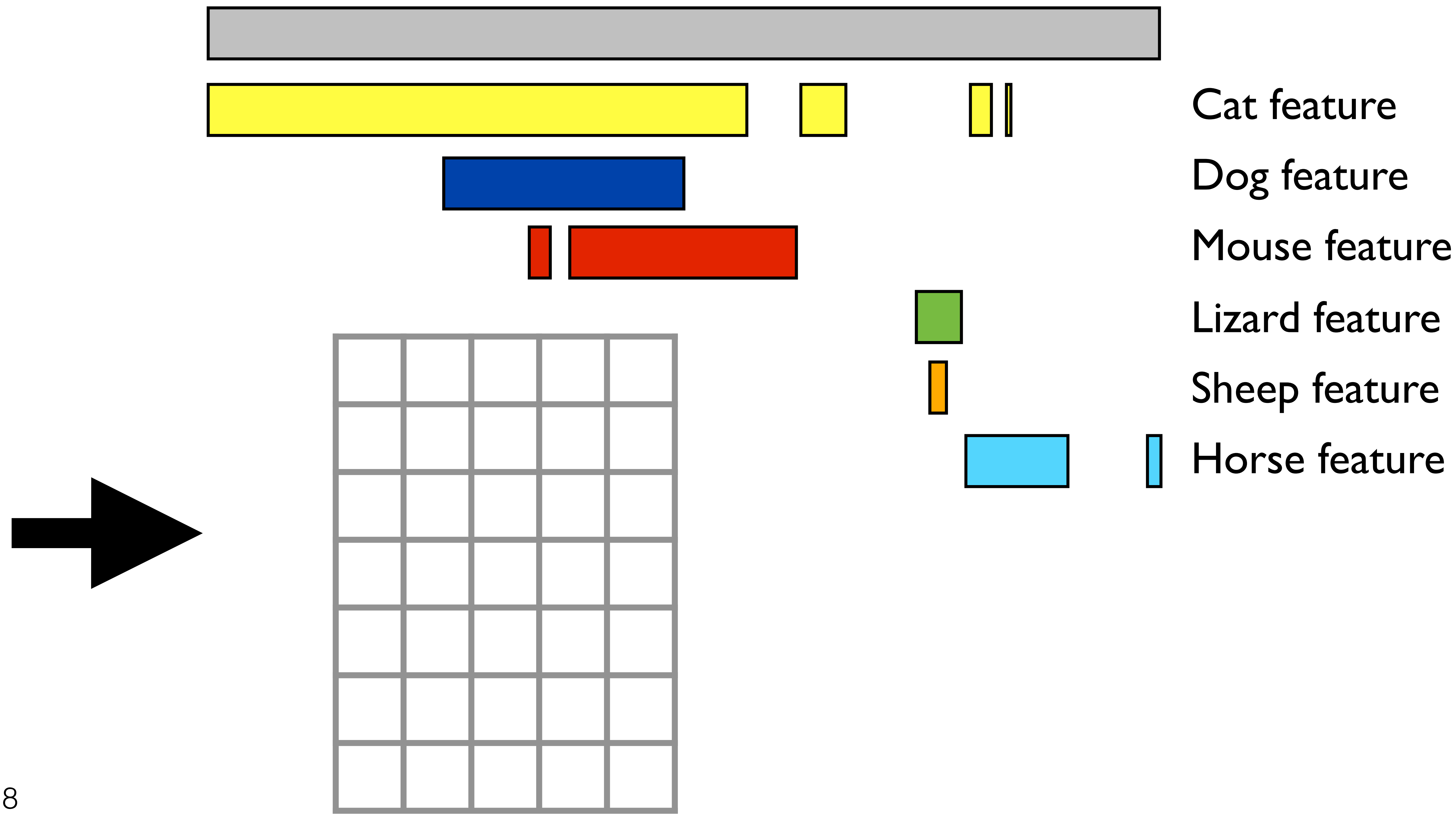
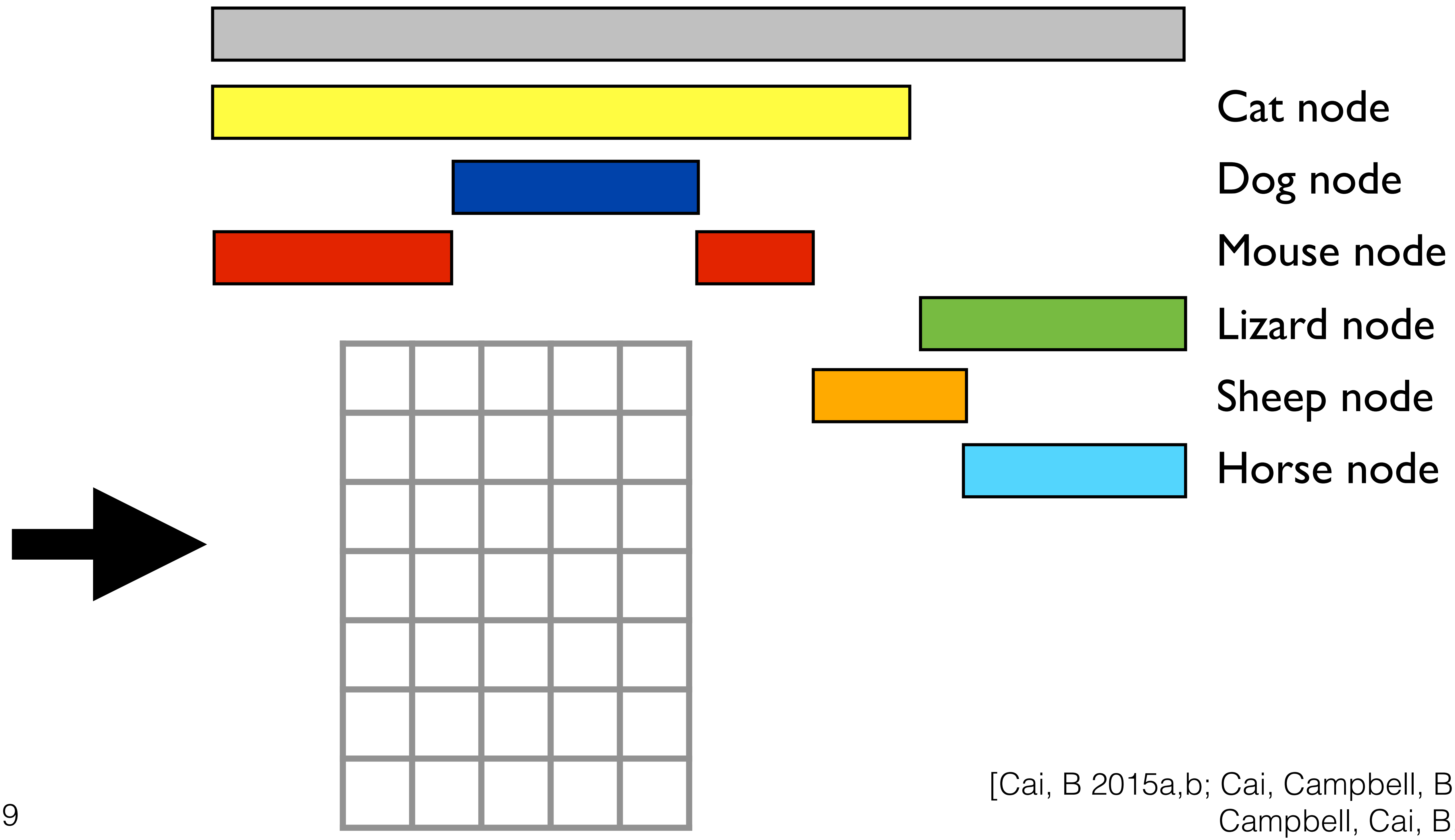





[B, Pitman, Jordan 2013; Campbell, Cai, B 2016]


Cat feature
Dog feature
Mouse feature
Lizard feature
Sheep feature
Horse feature




[Cai, B 2015a,b; Cai, Campbell, B, 2016;
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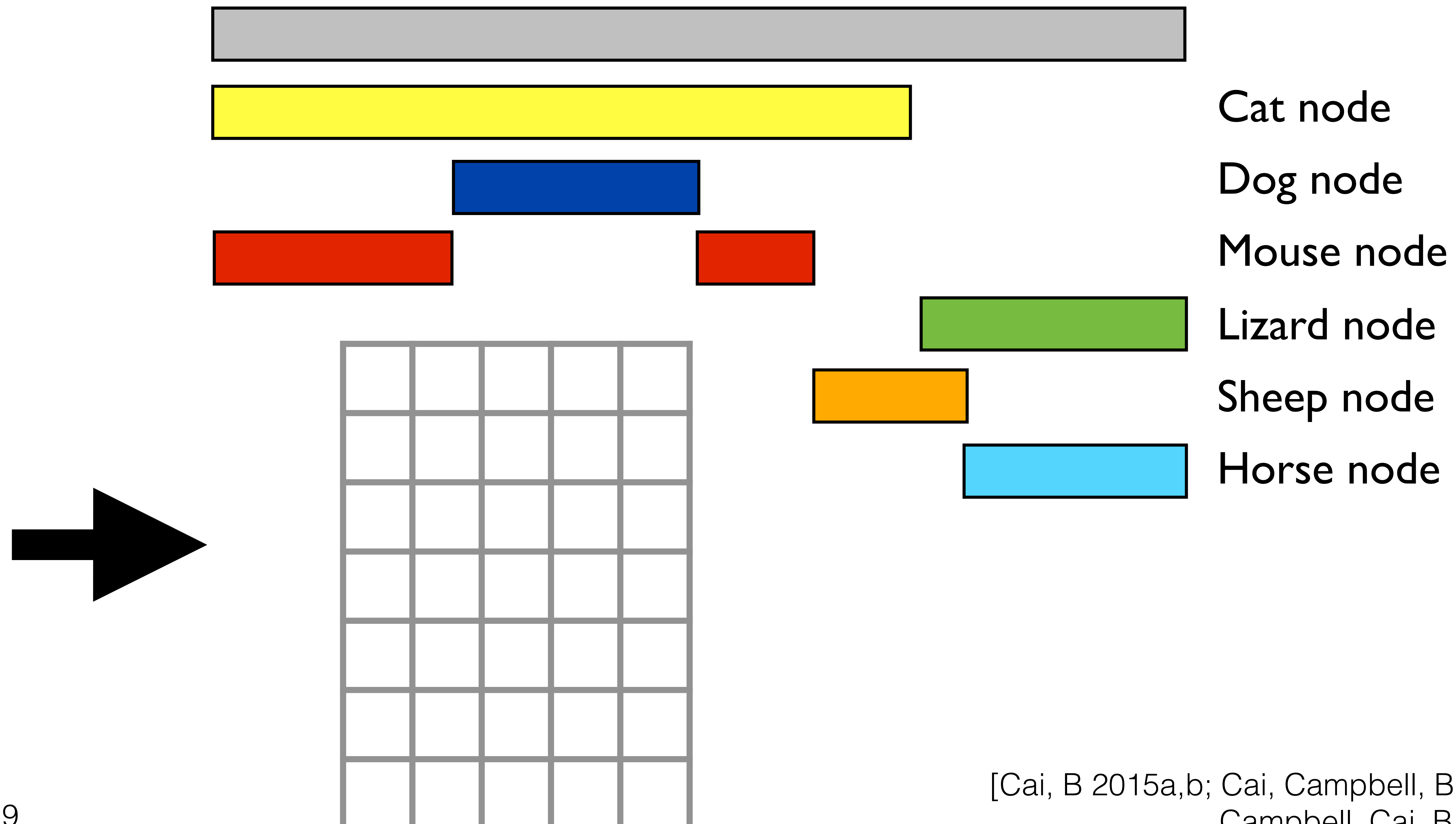
# Edge-exchangeable graph

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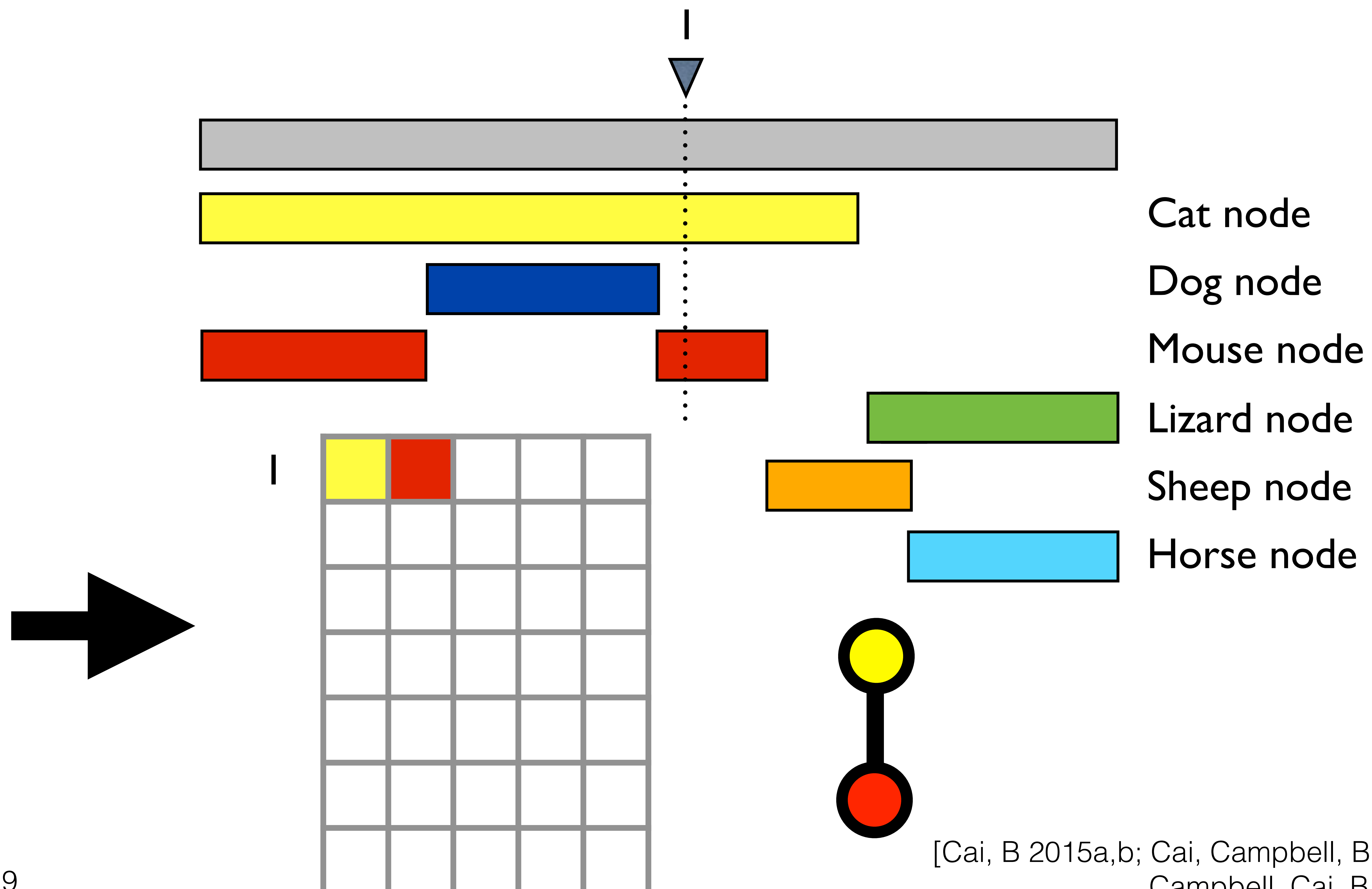


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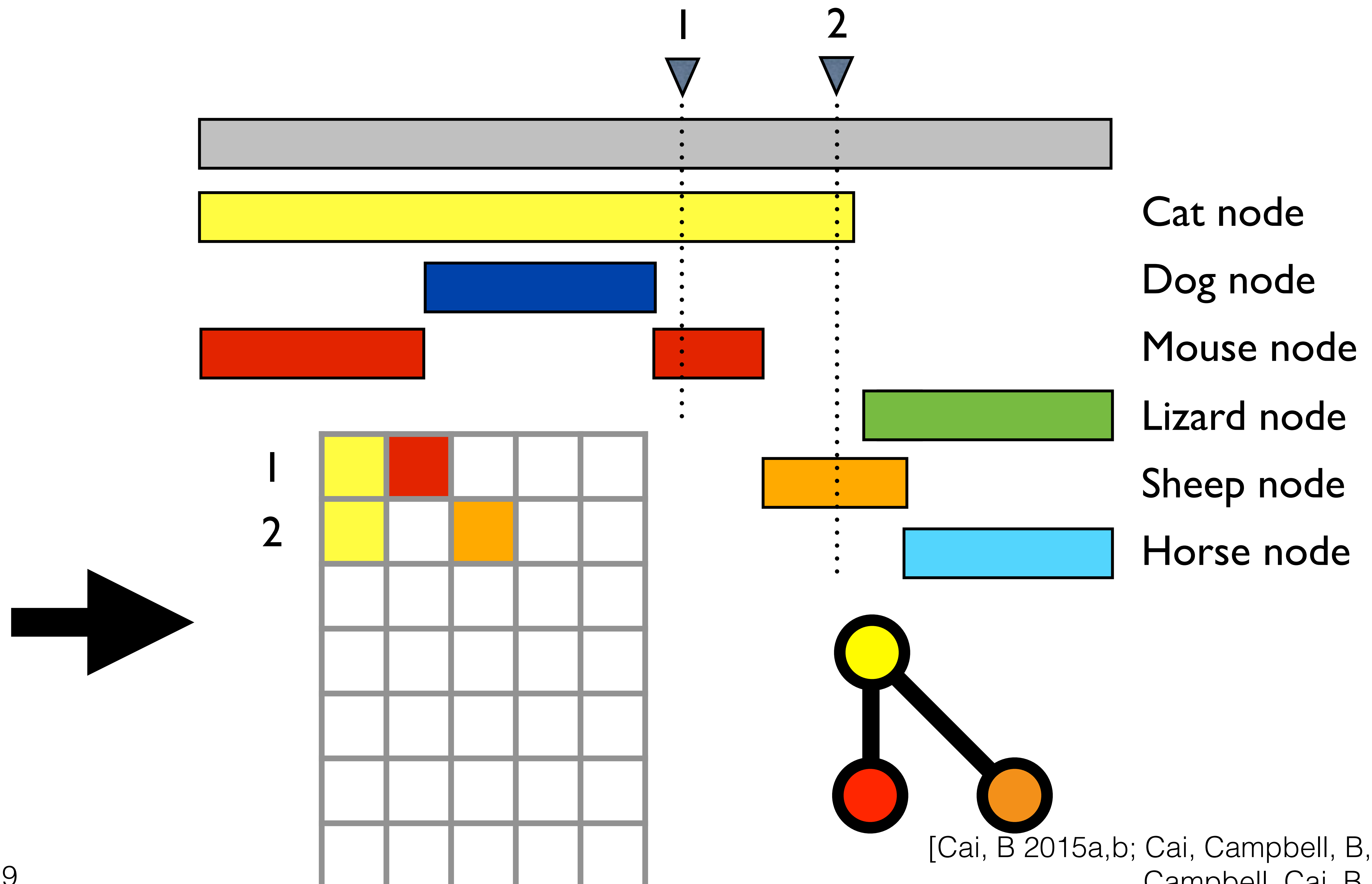


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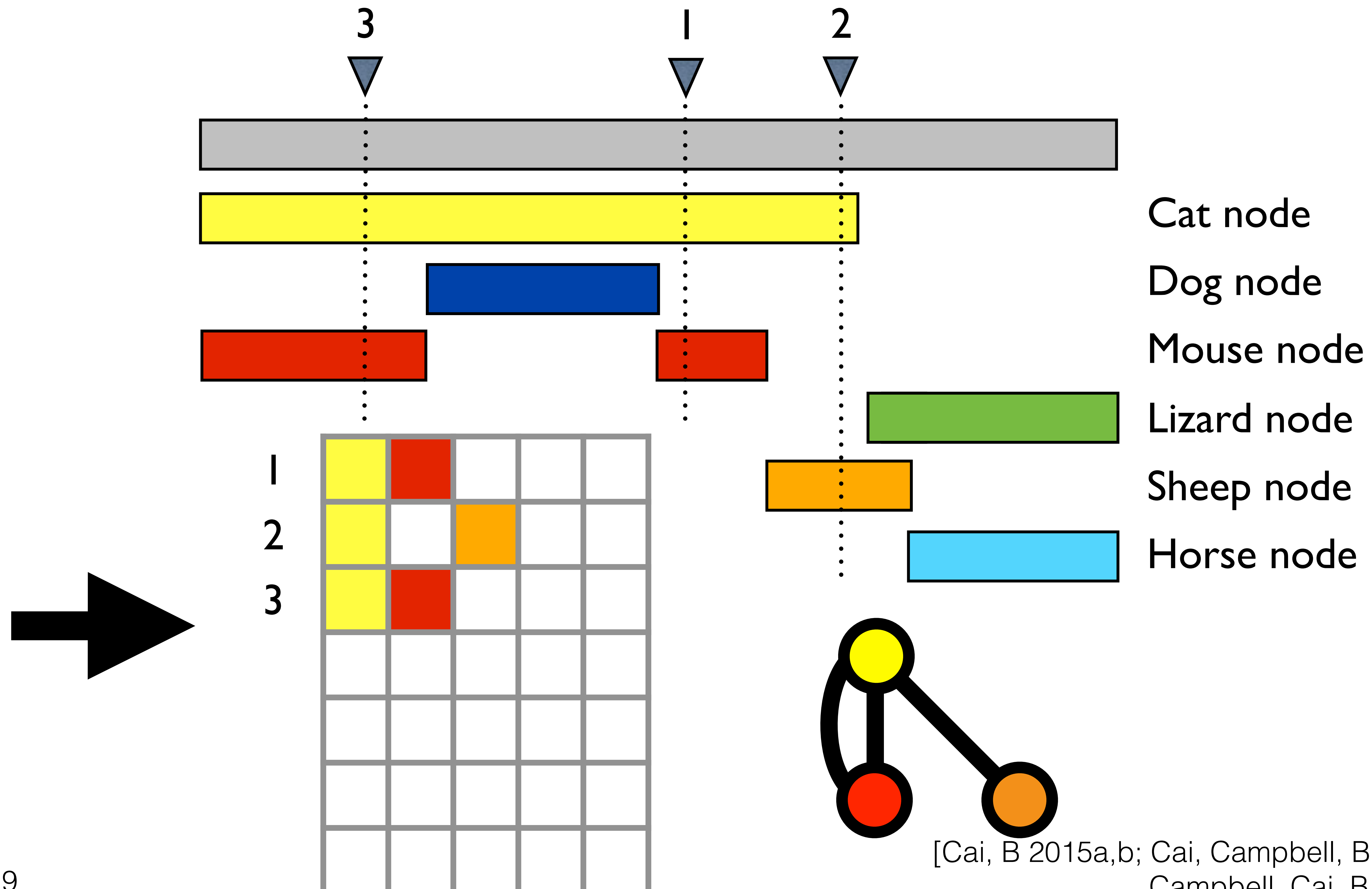


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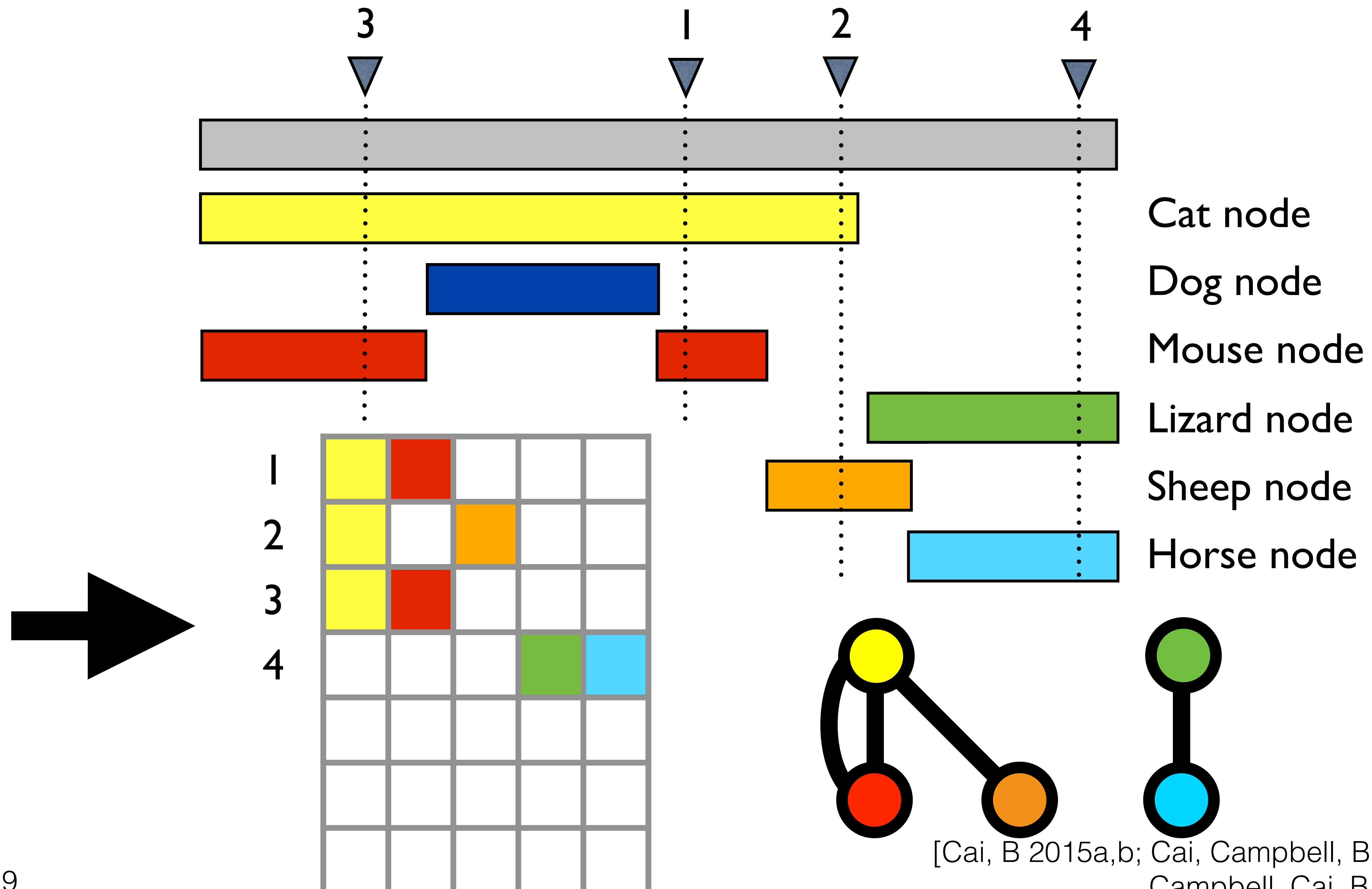
# Edge-exchangeable graph

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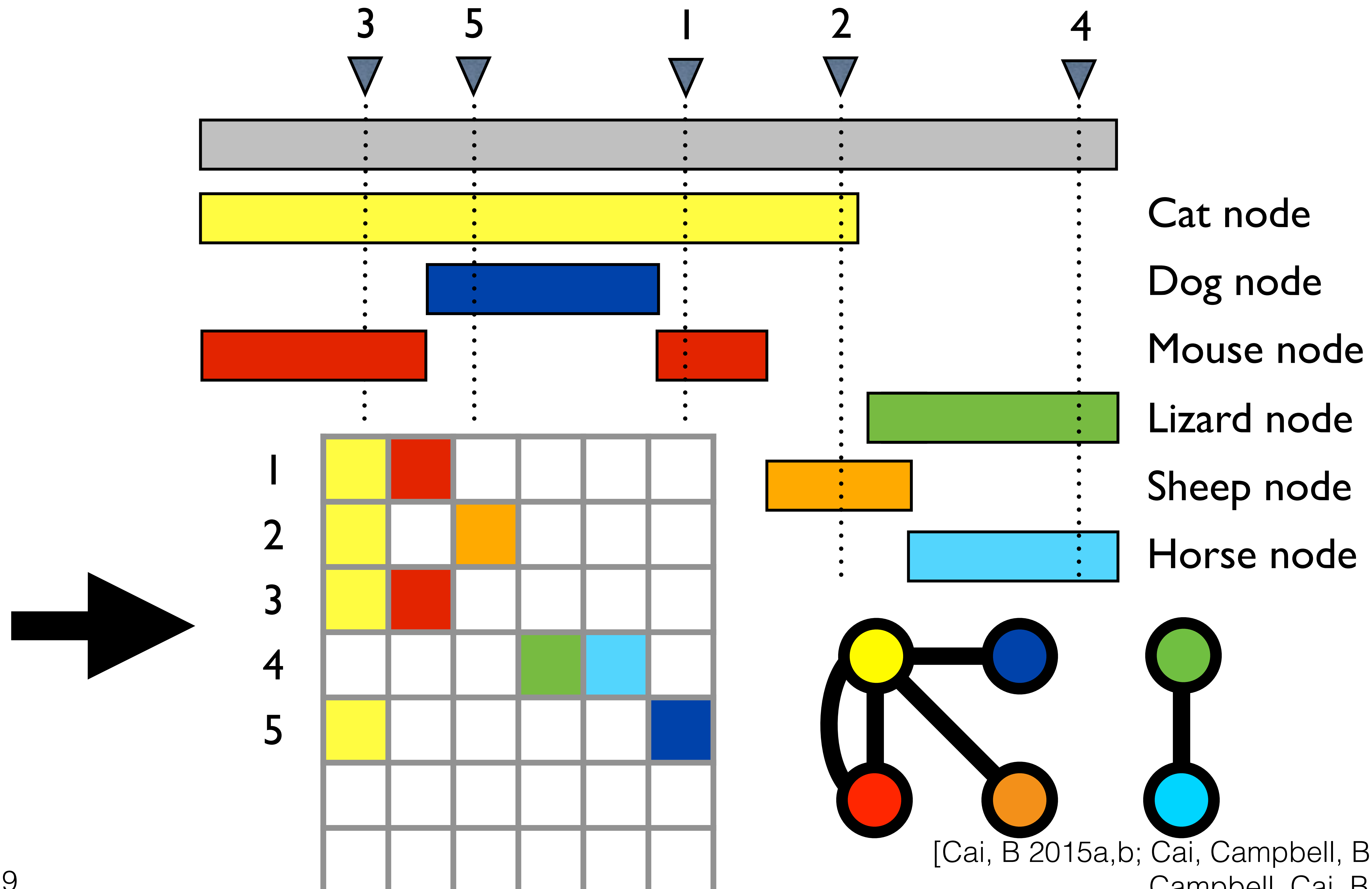


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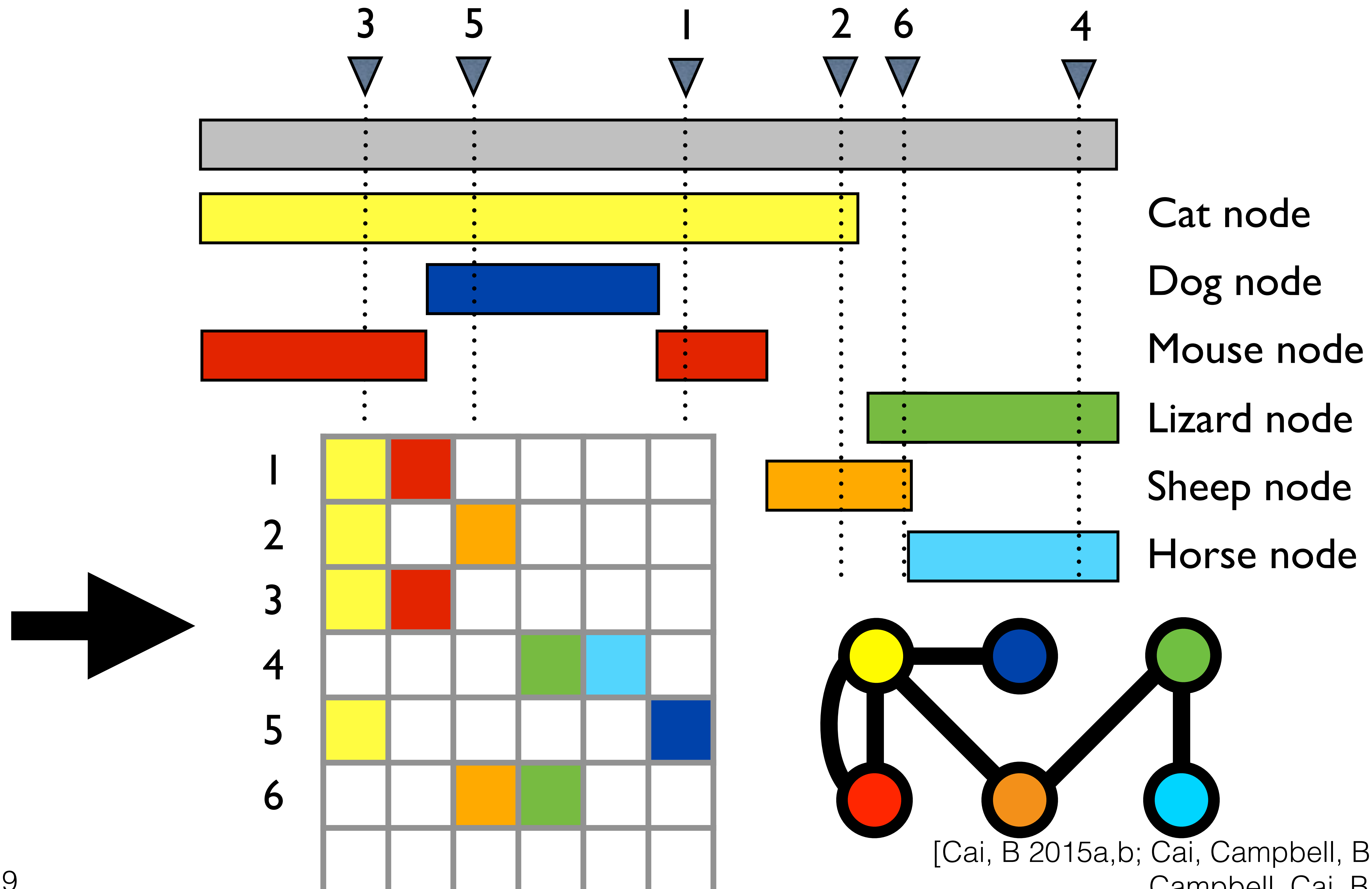


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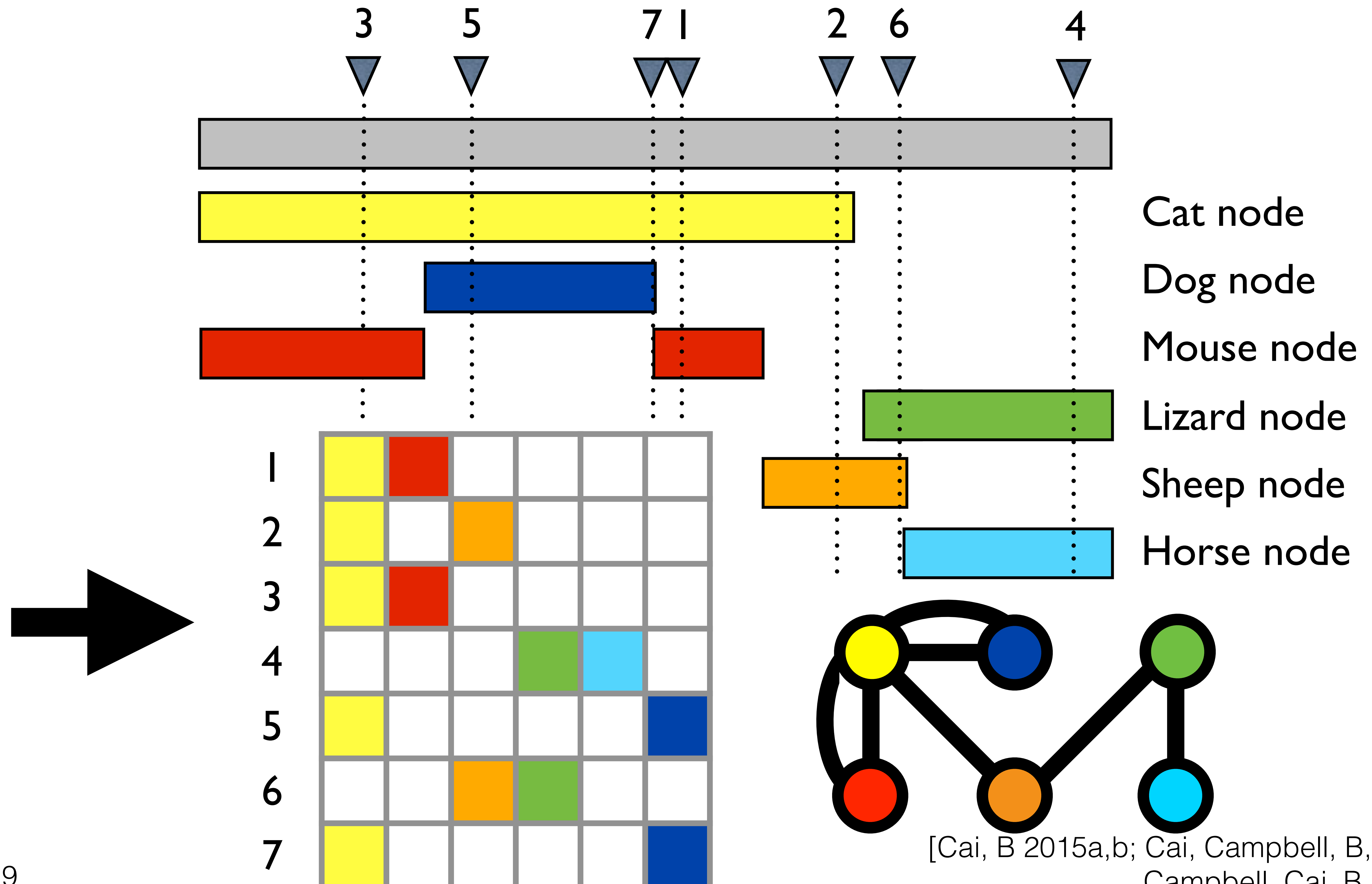


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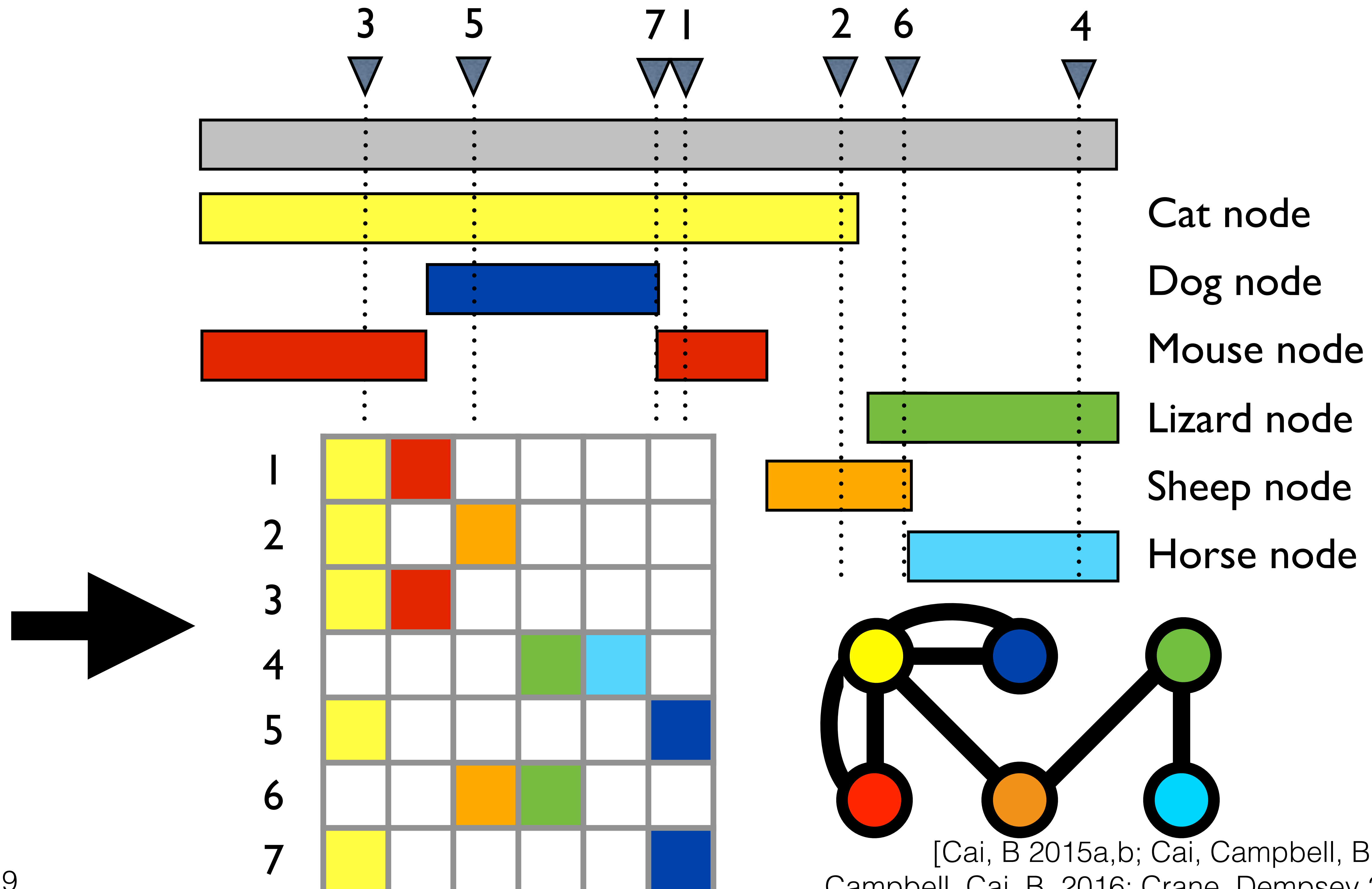
# Cor (CCB). A graph sequence is edge-exchangeable iff it has a graph paintbox

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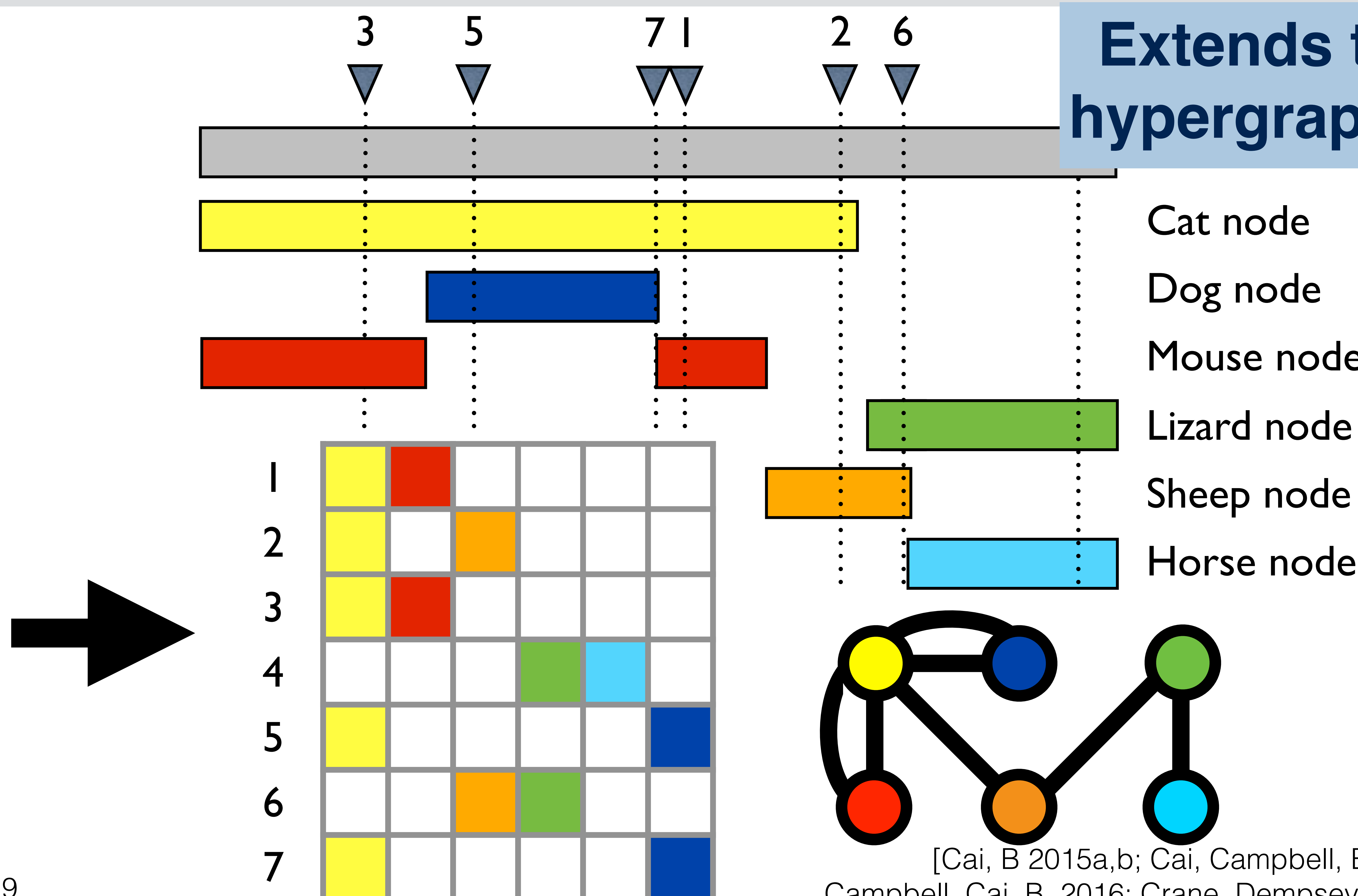
**Extends to hypergraphs**

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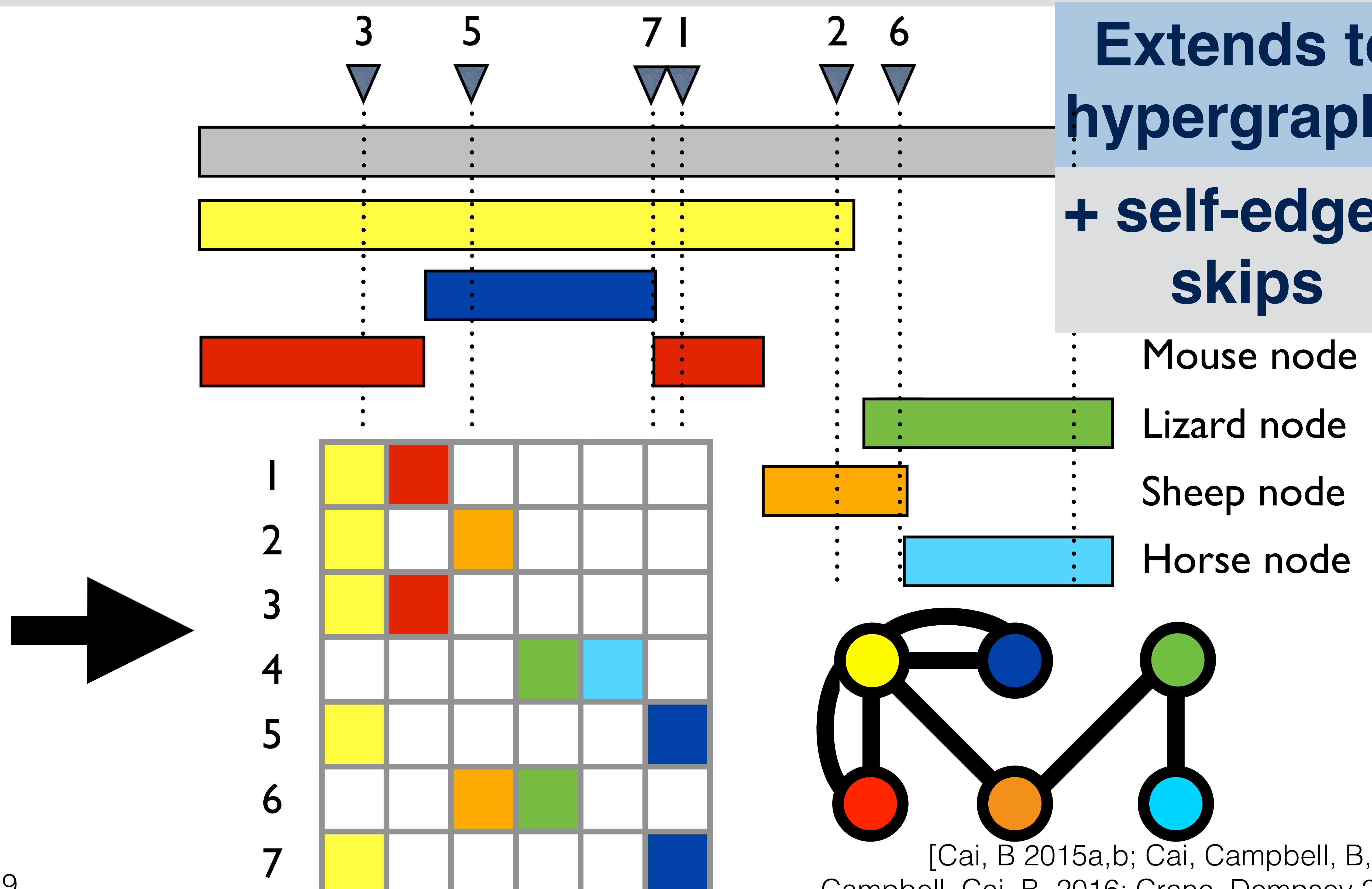
**Extends to hypergraphs**

**+ self-edges, skips**

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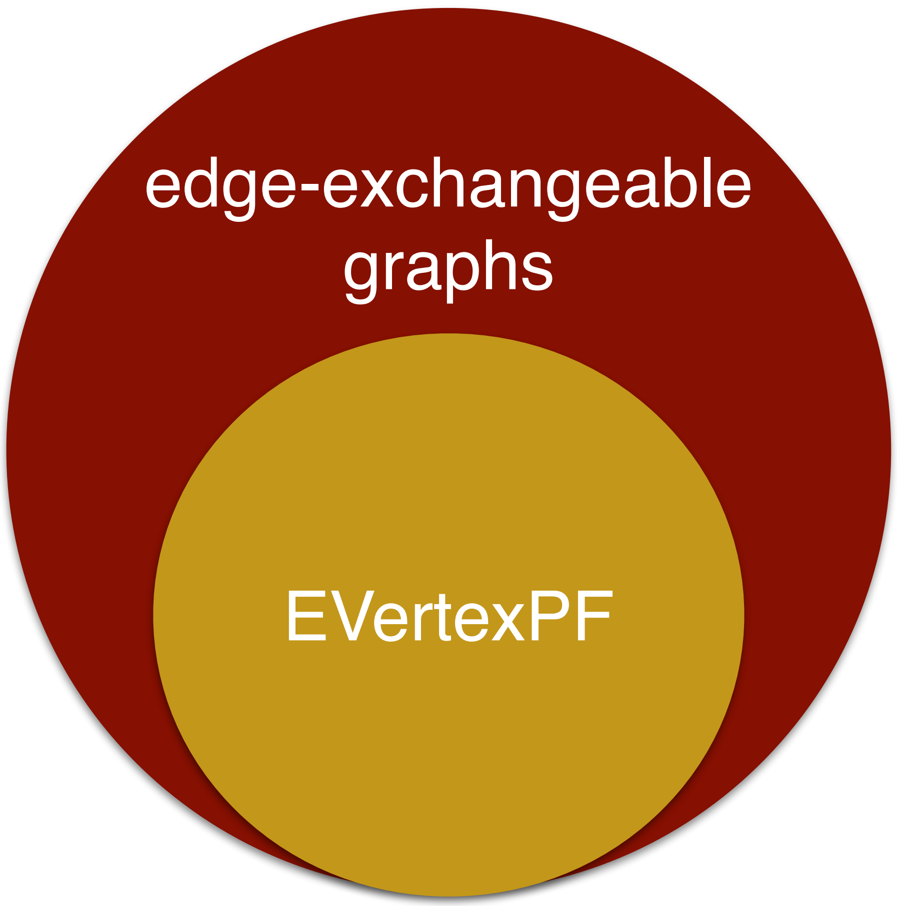


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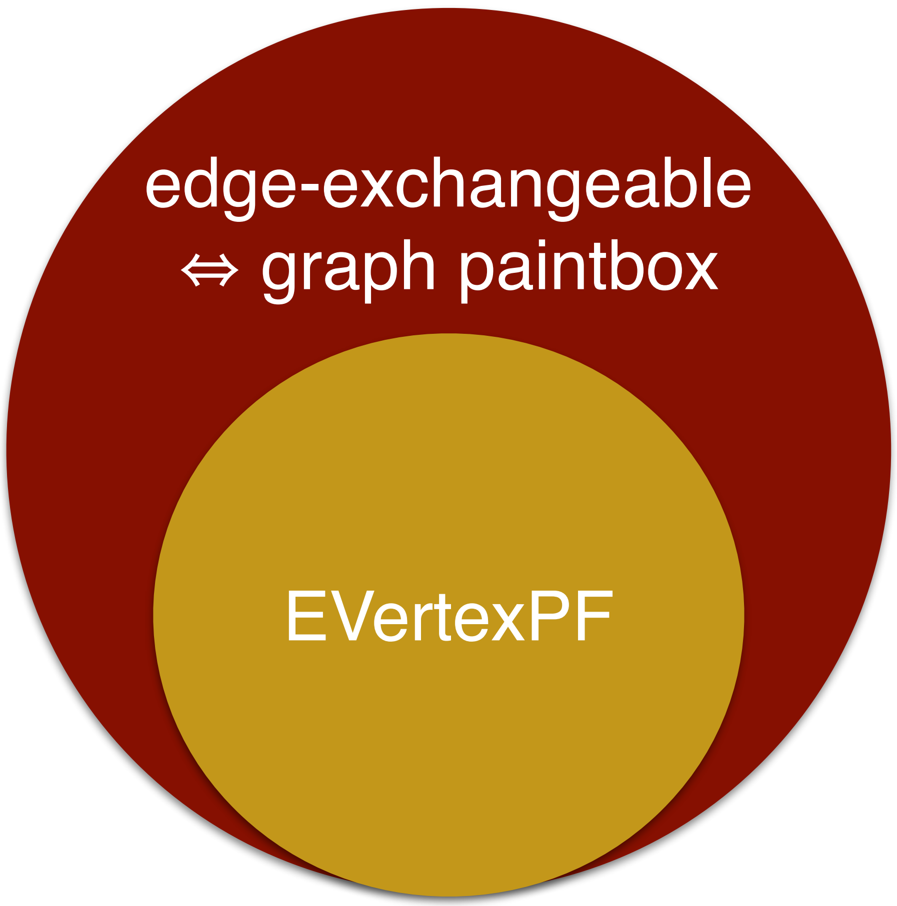


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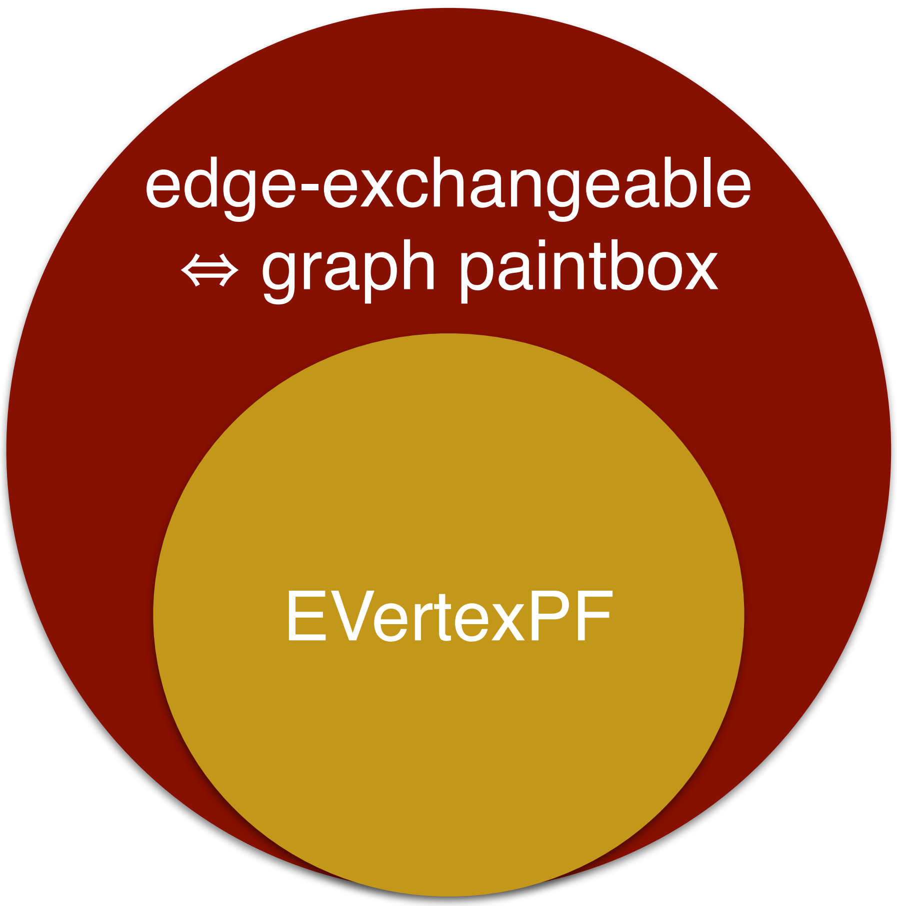


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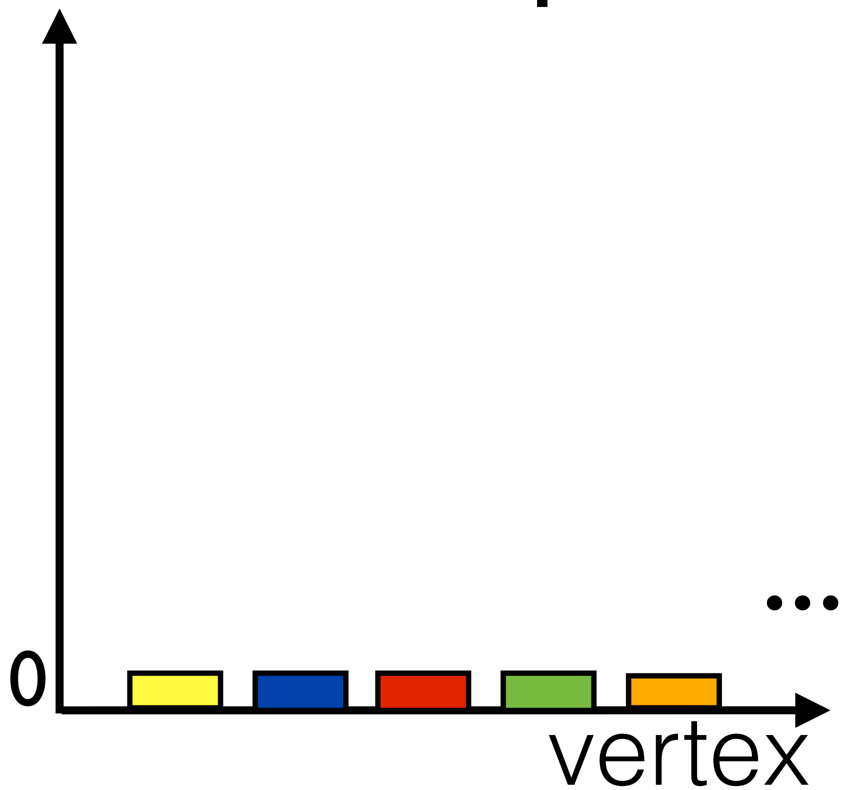


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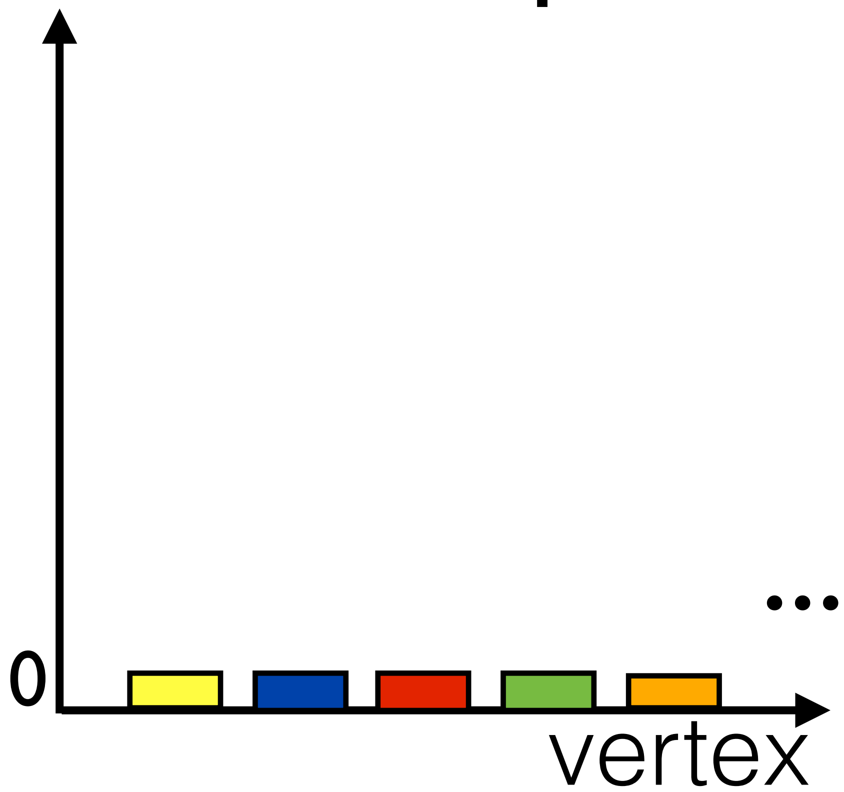


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[Caron, Fox 2014; Cai, Campbell, B 2016; Crane, Dempsey 2016a; Palla, Caron, Teh 2016; Herlau, Schmidt 2016]

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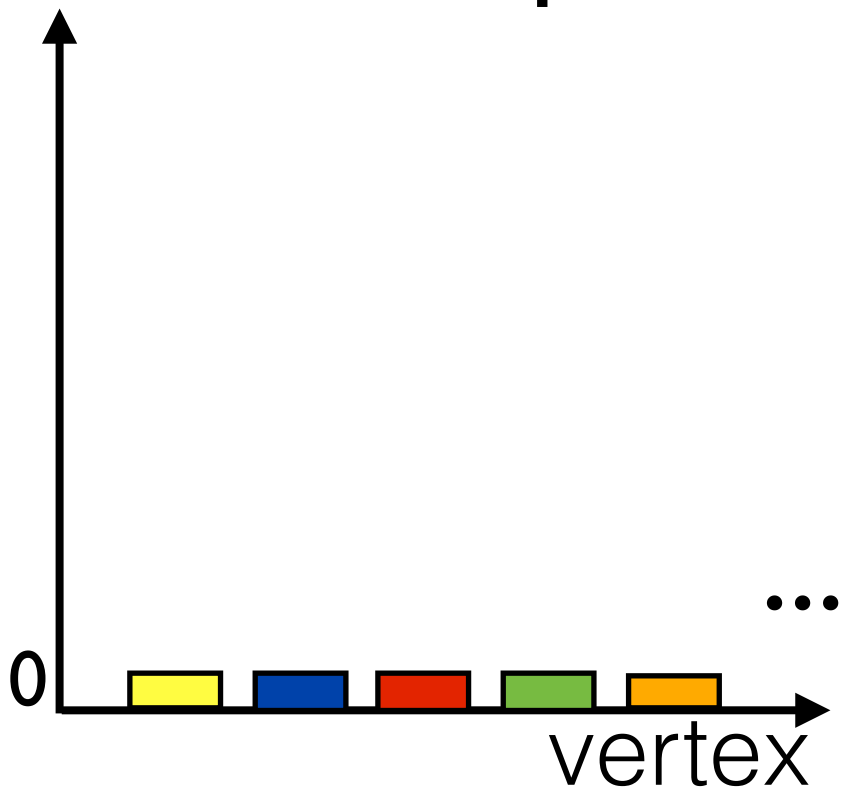


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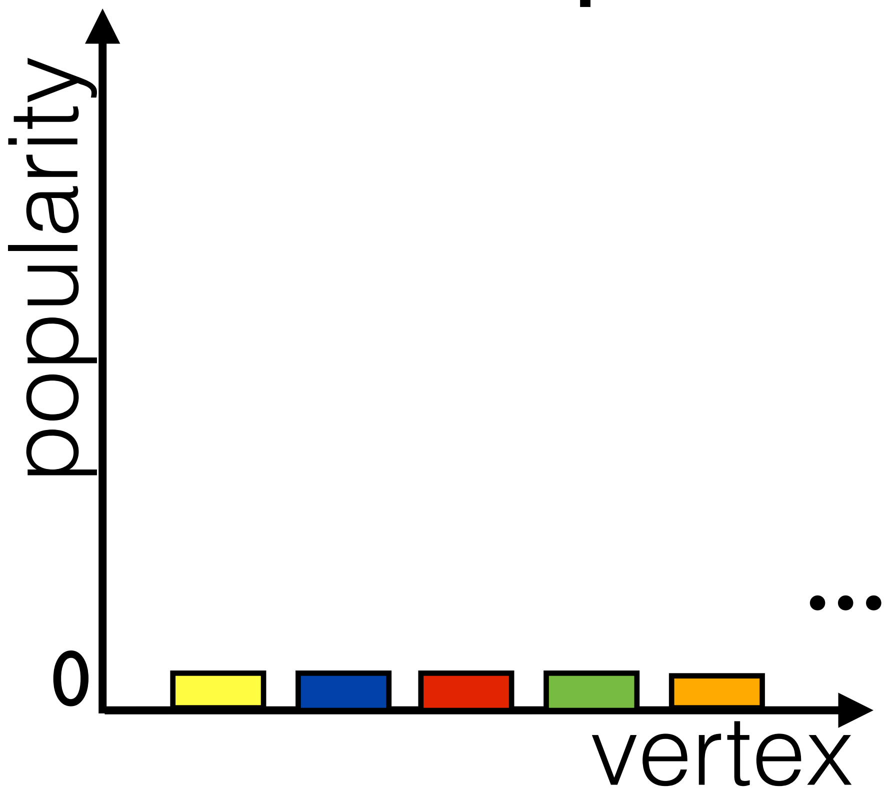


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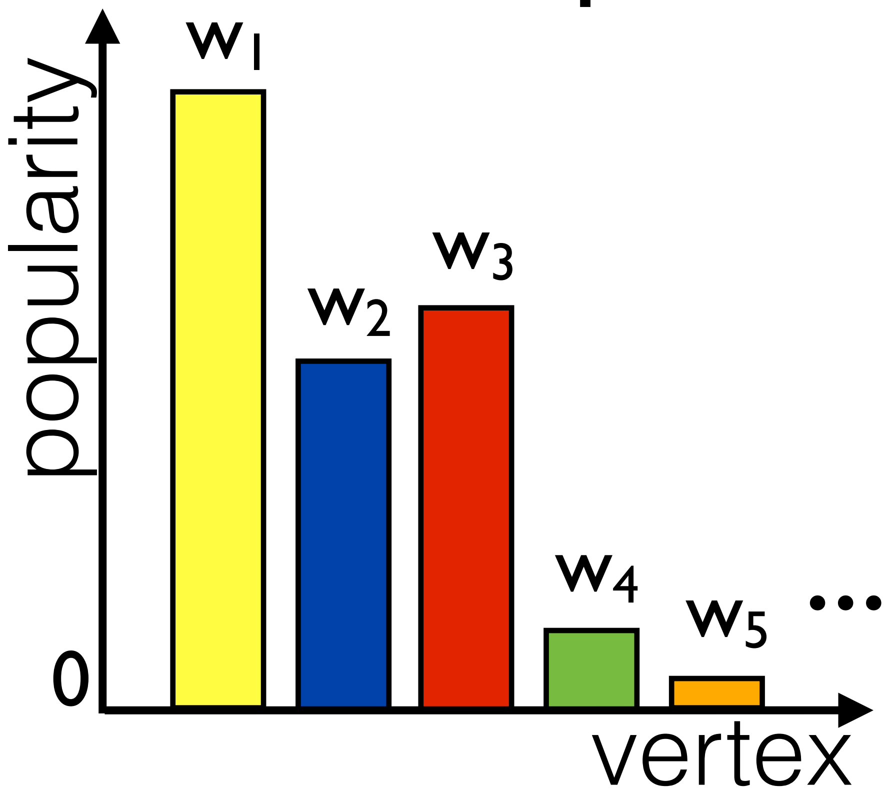


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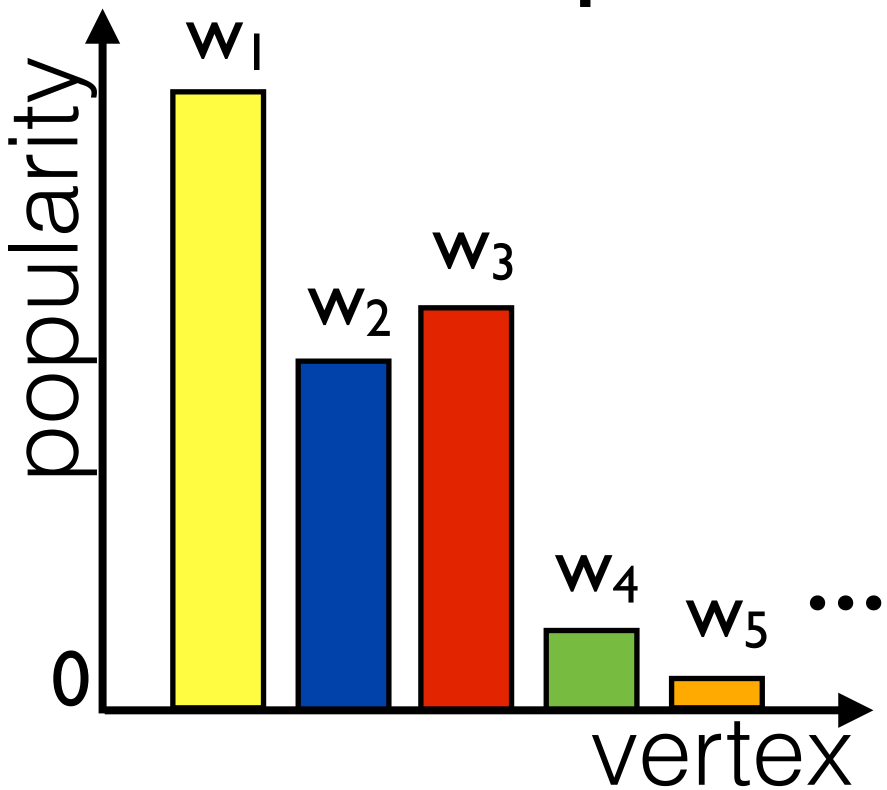


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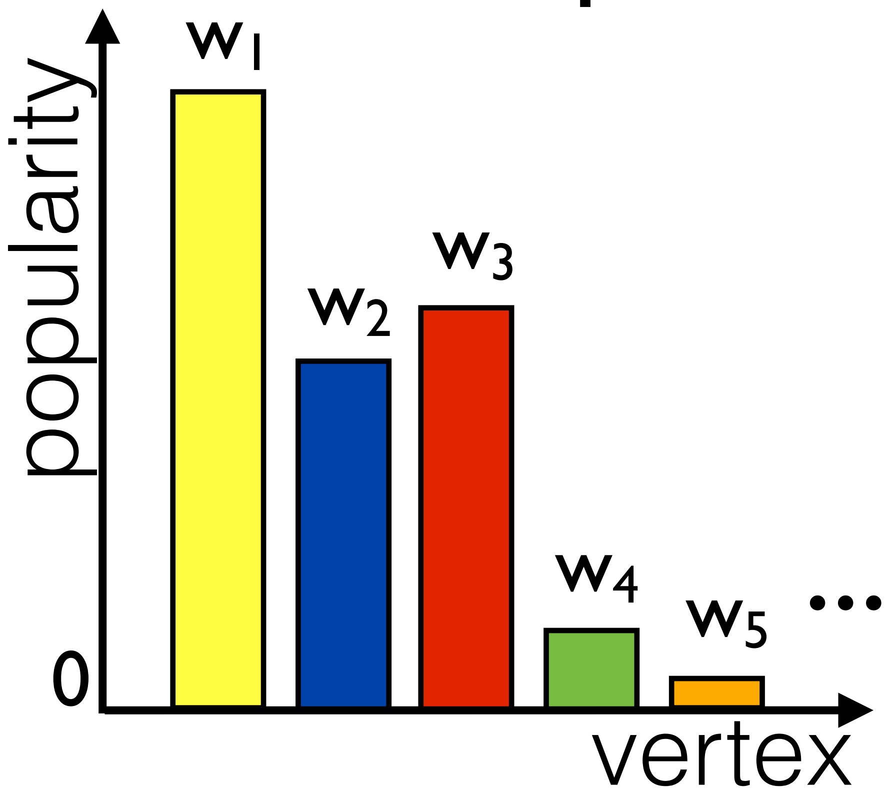


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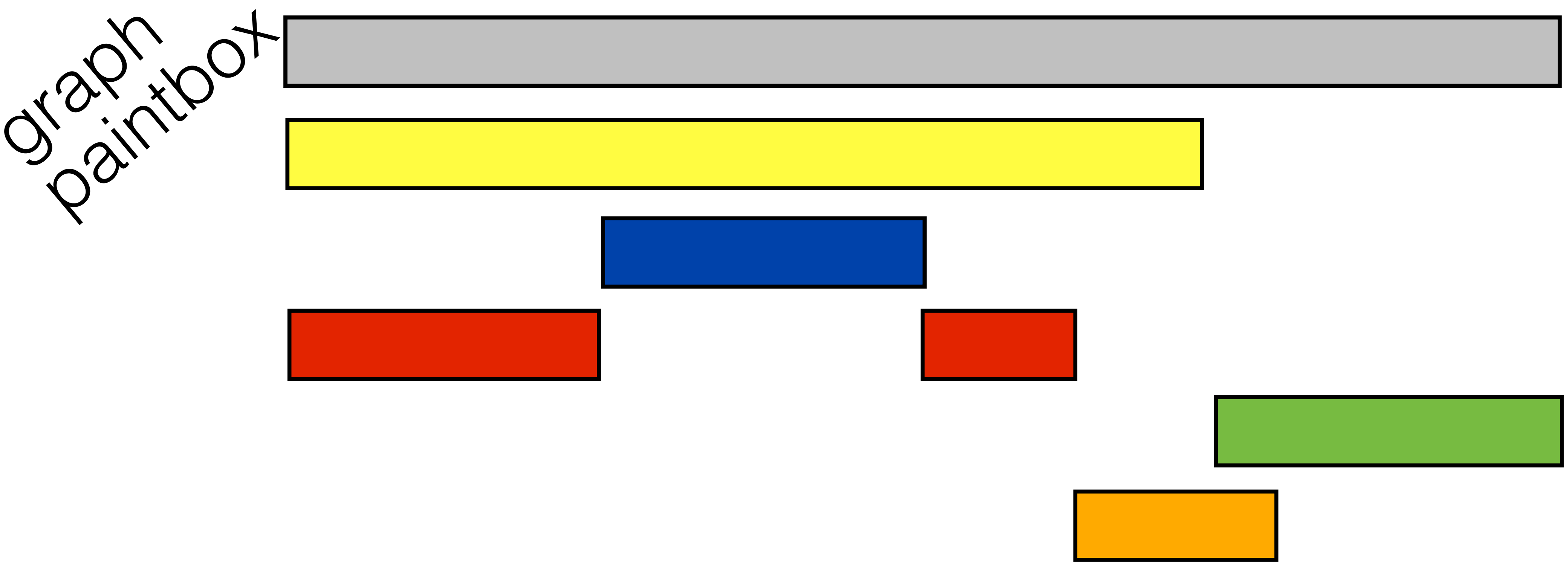




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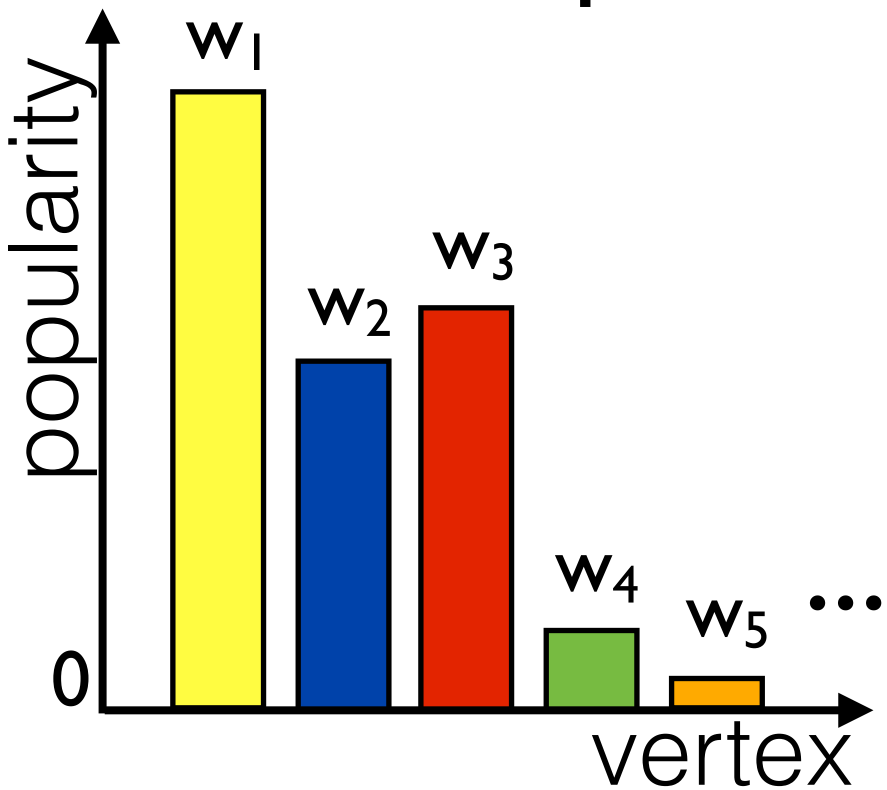


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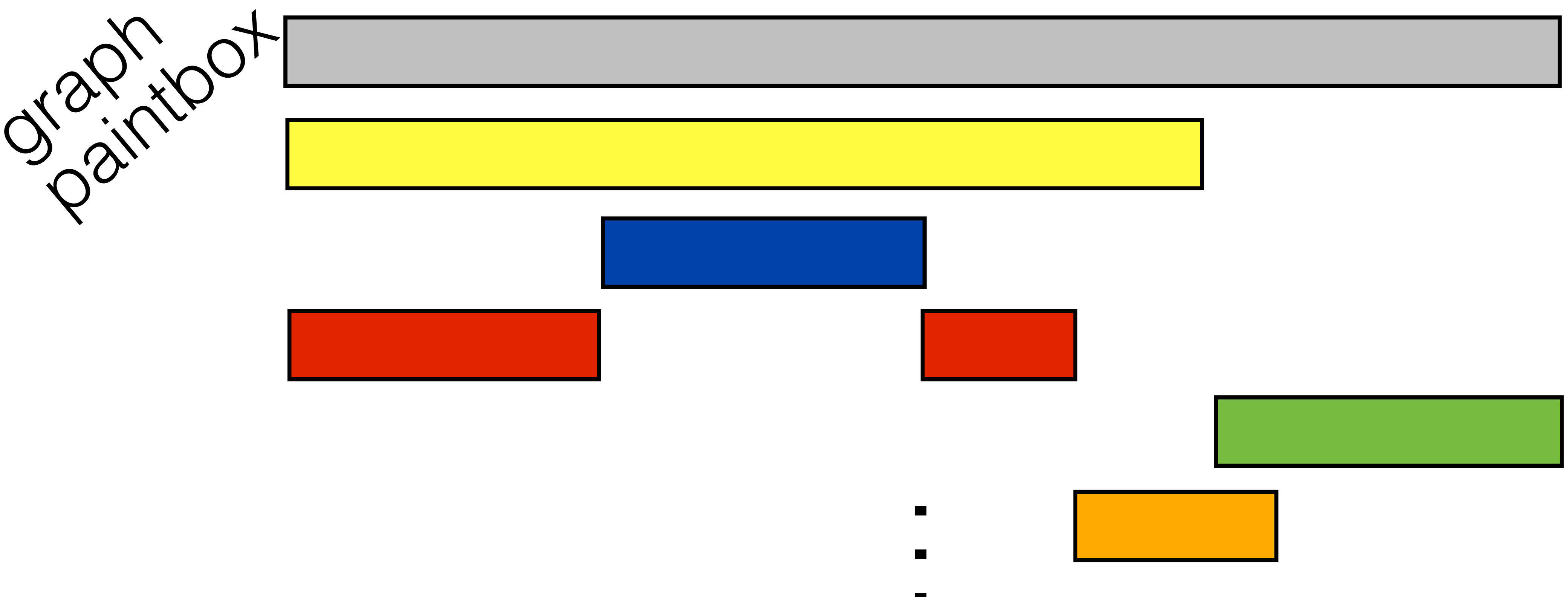




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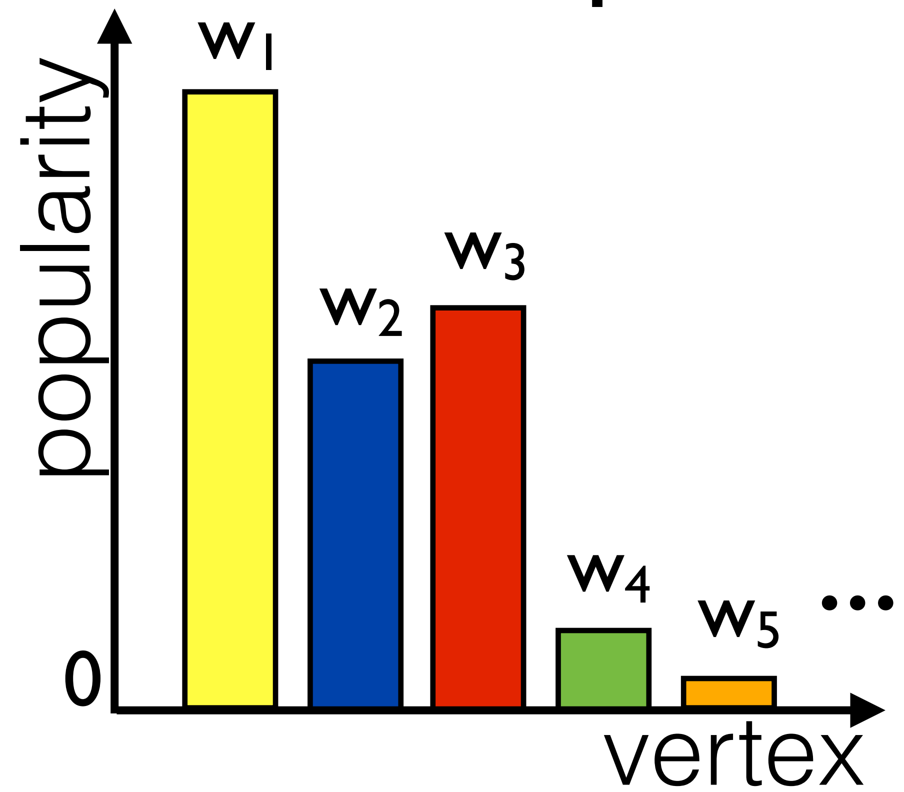


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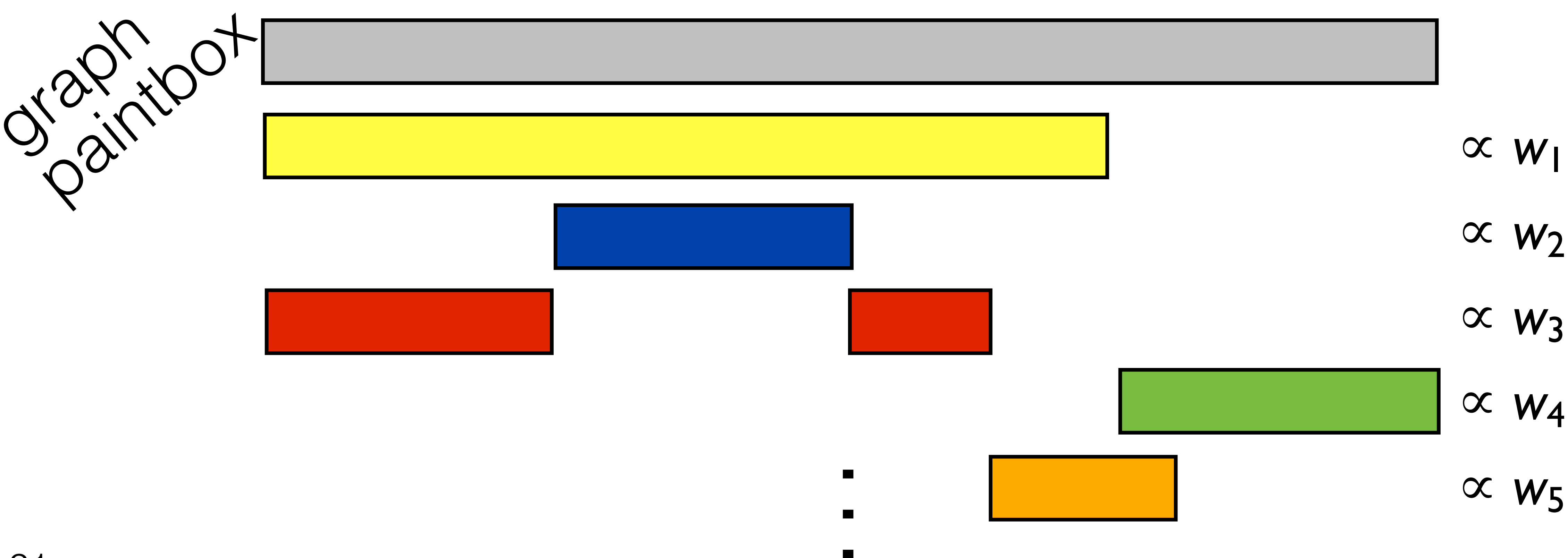




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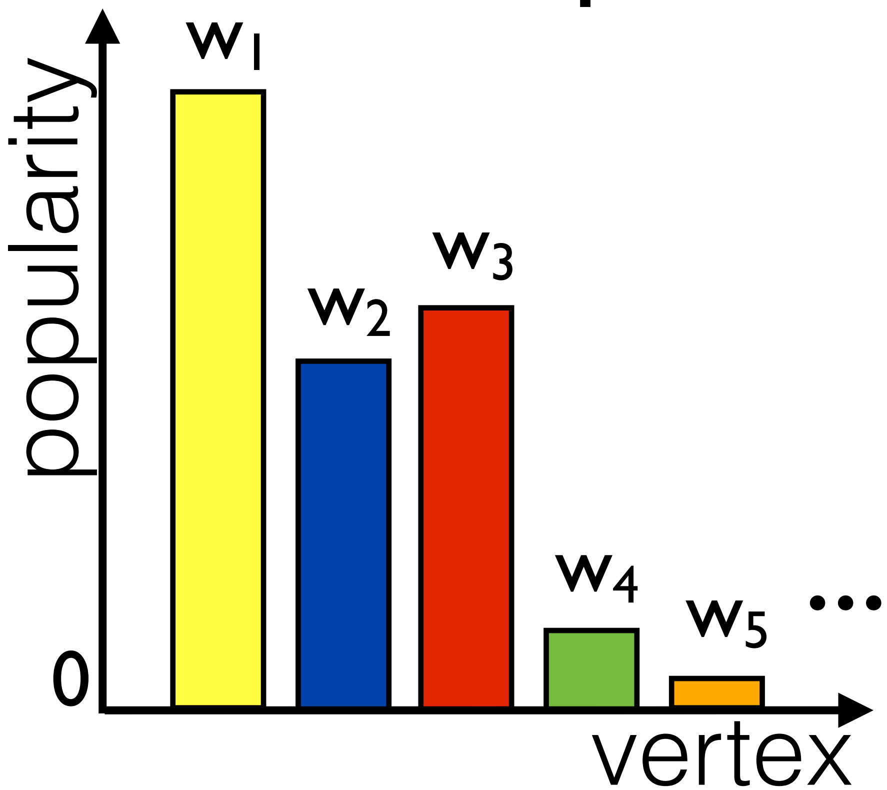


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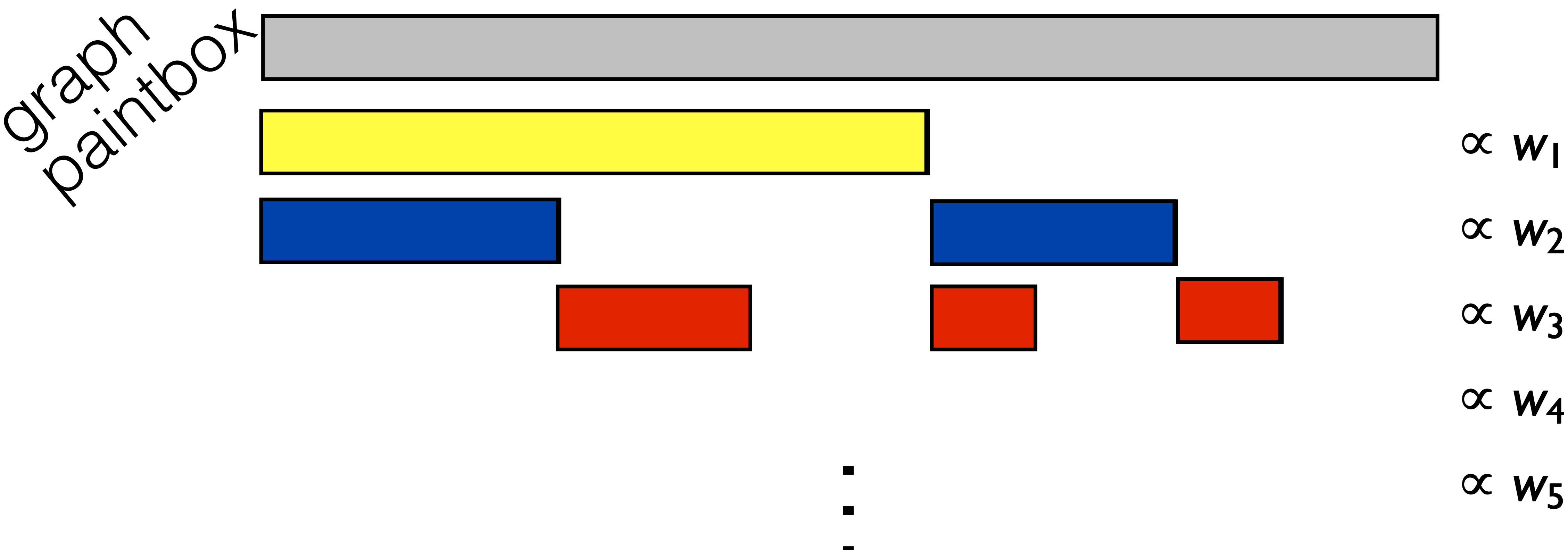




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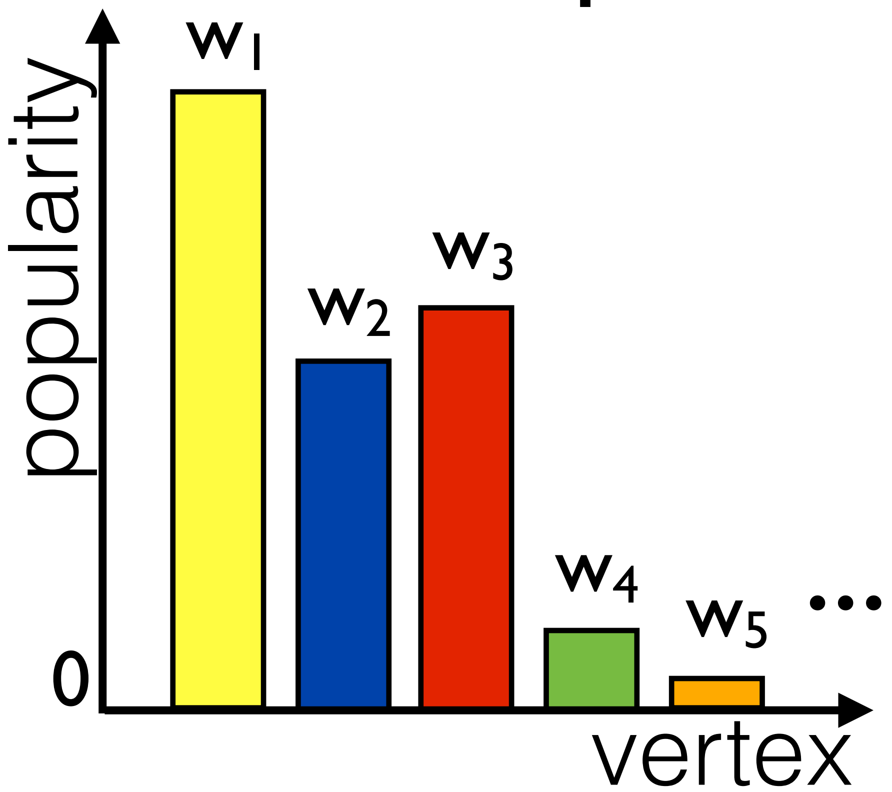


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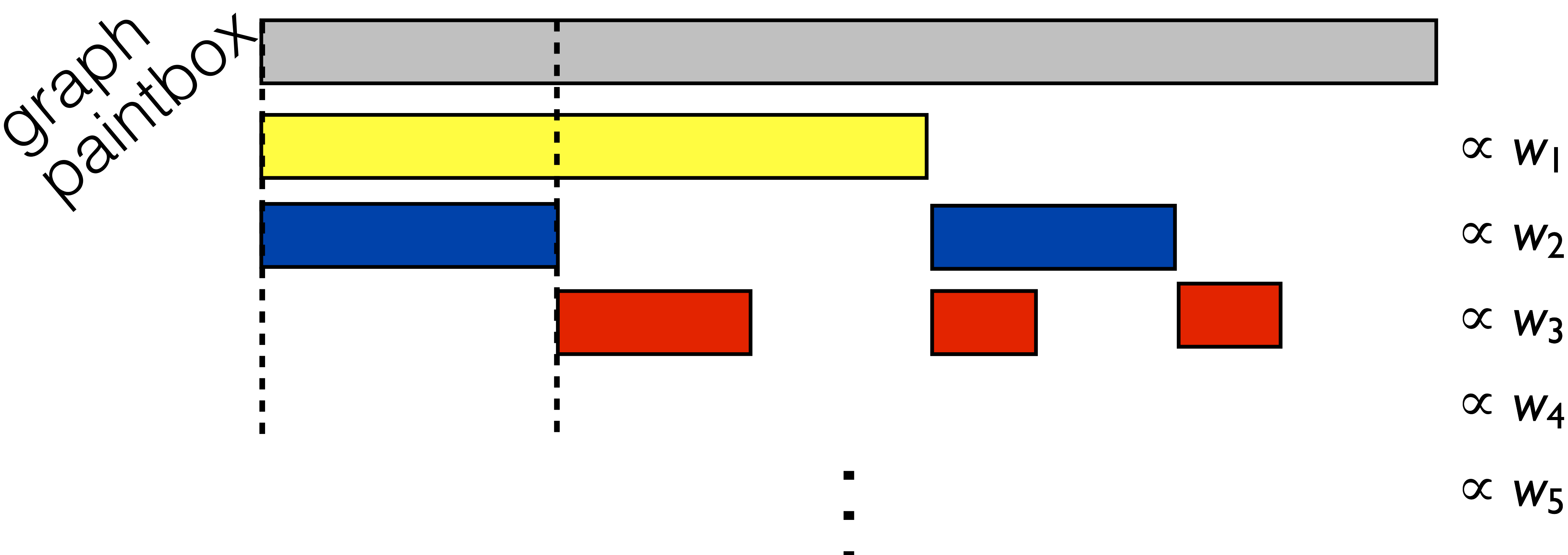




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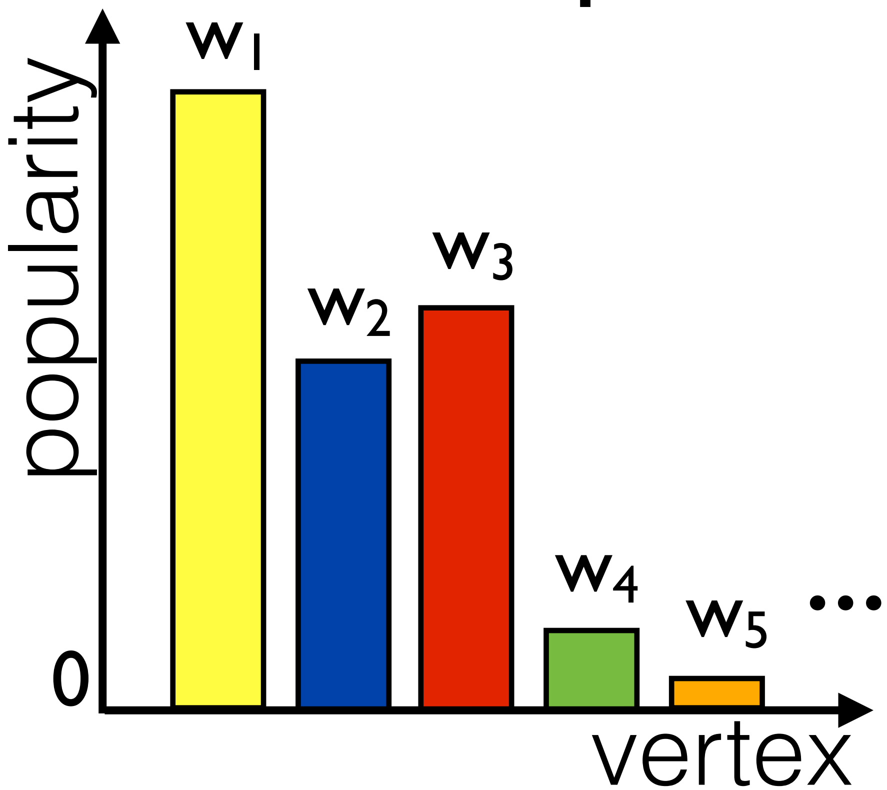


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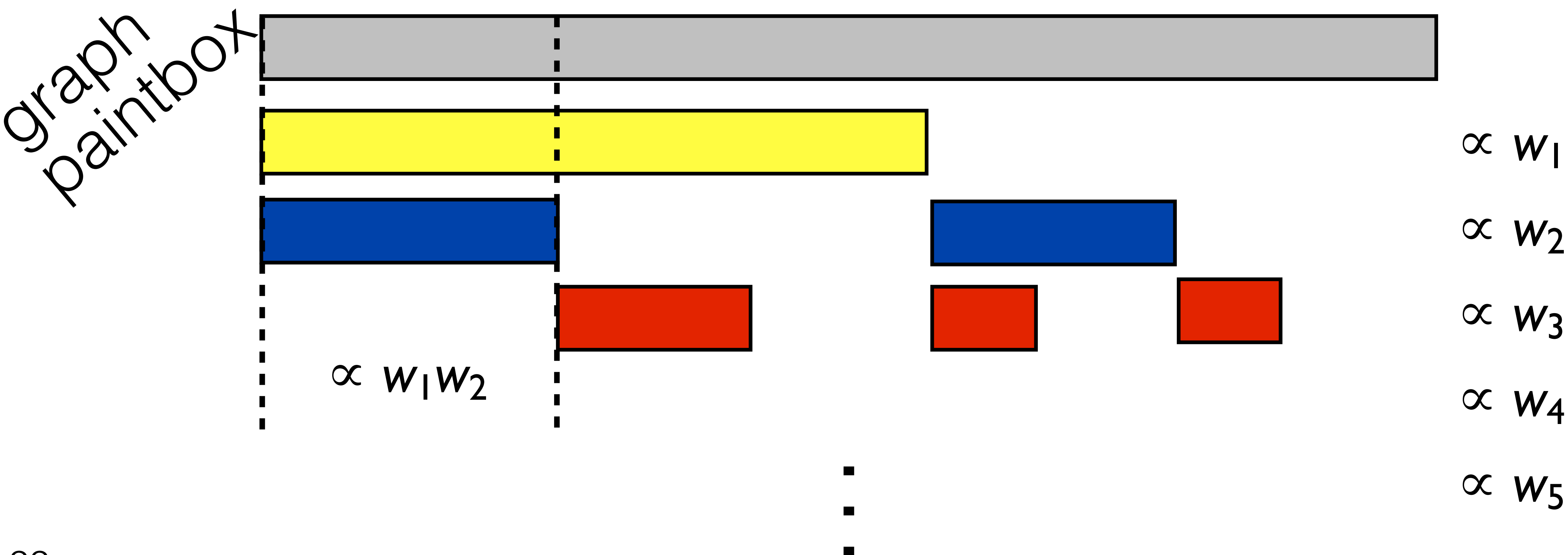



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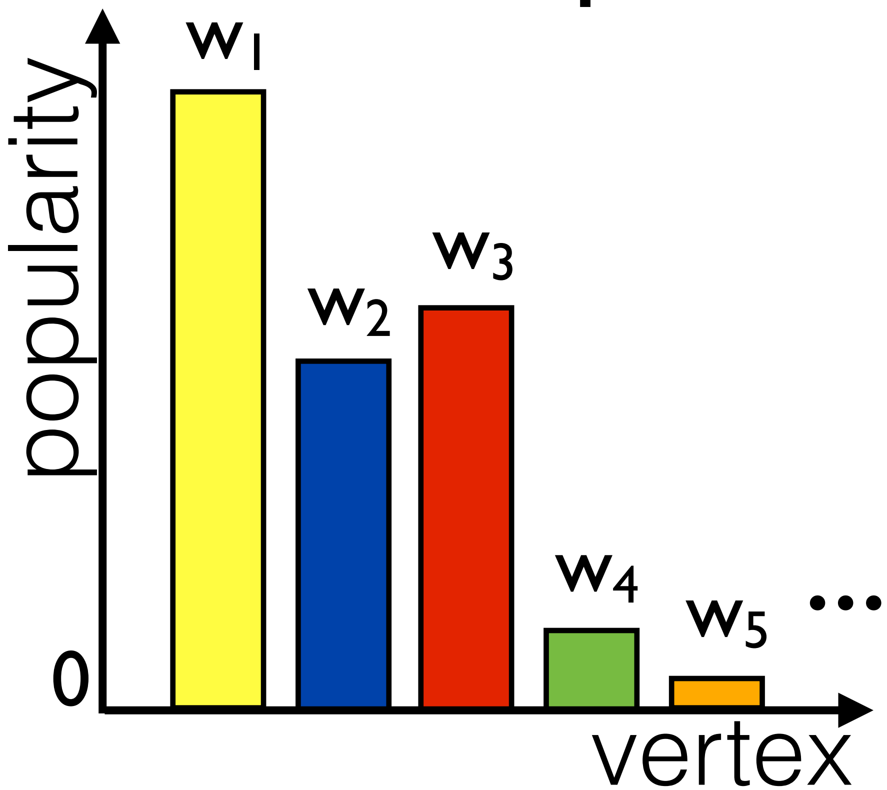


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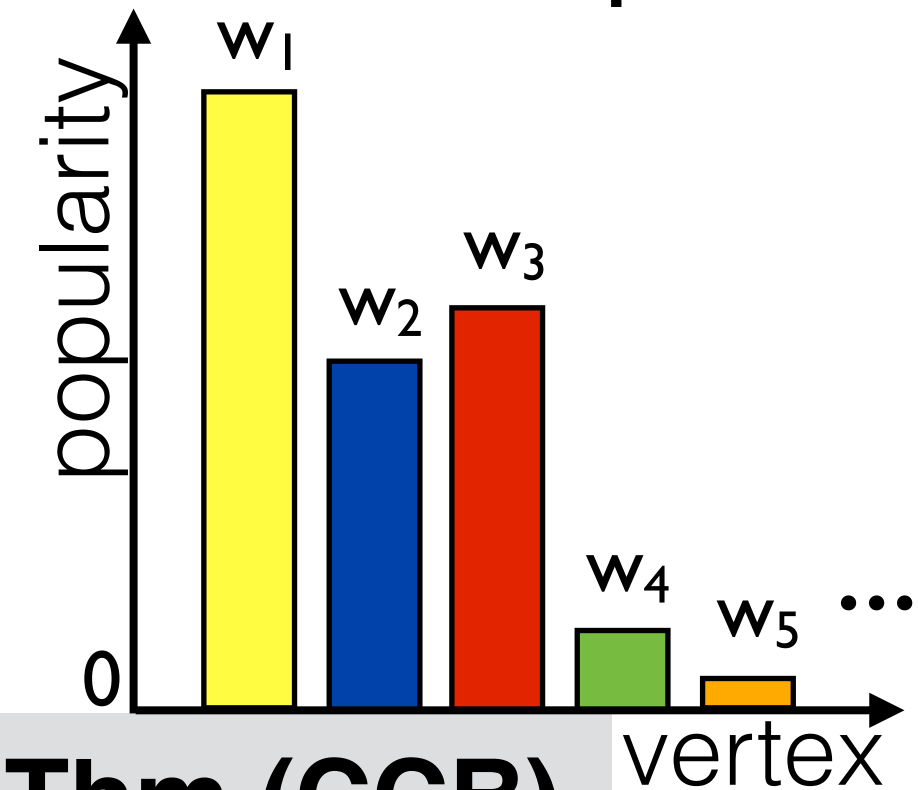


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[Cai, Campbell, Broderick 2016]

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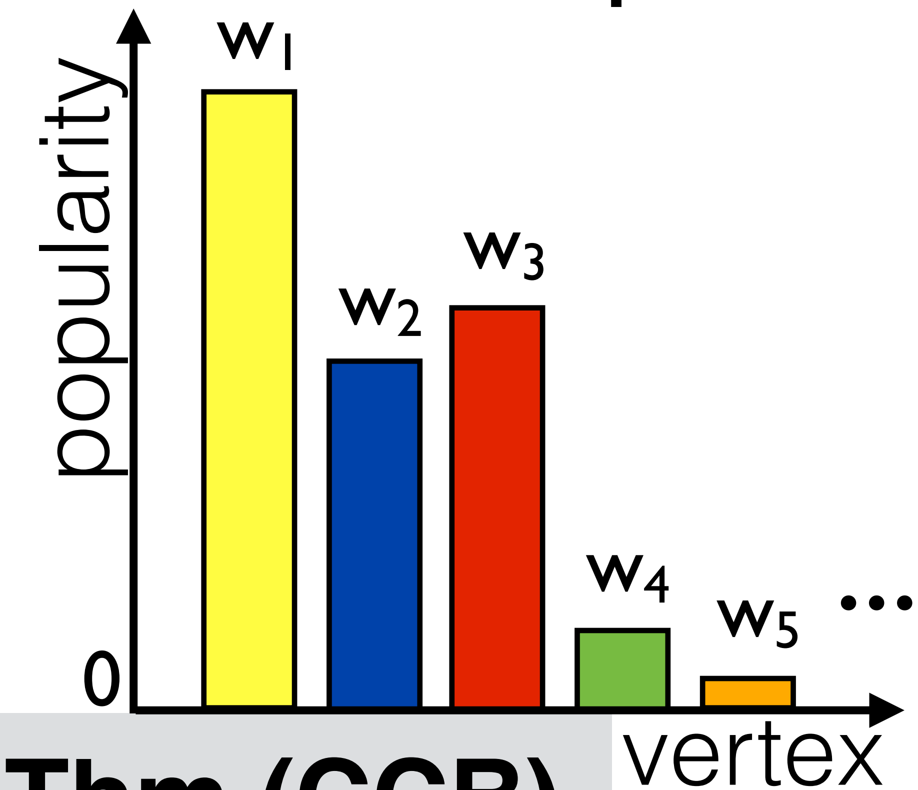


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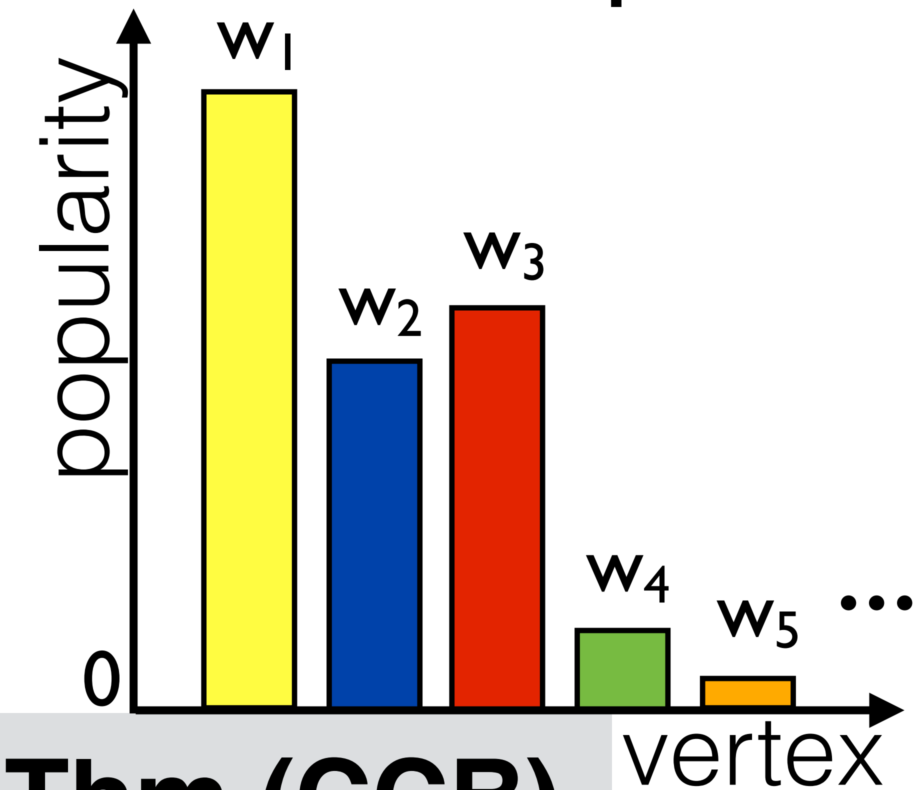


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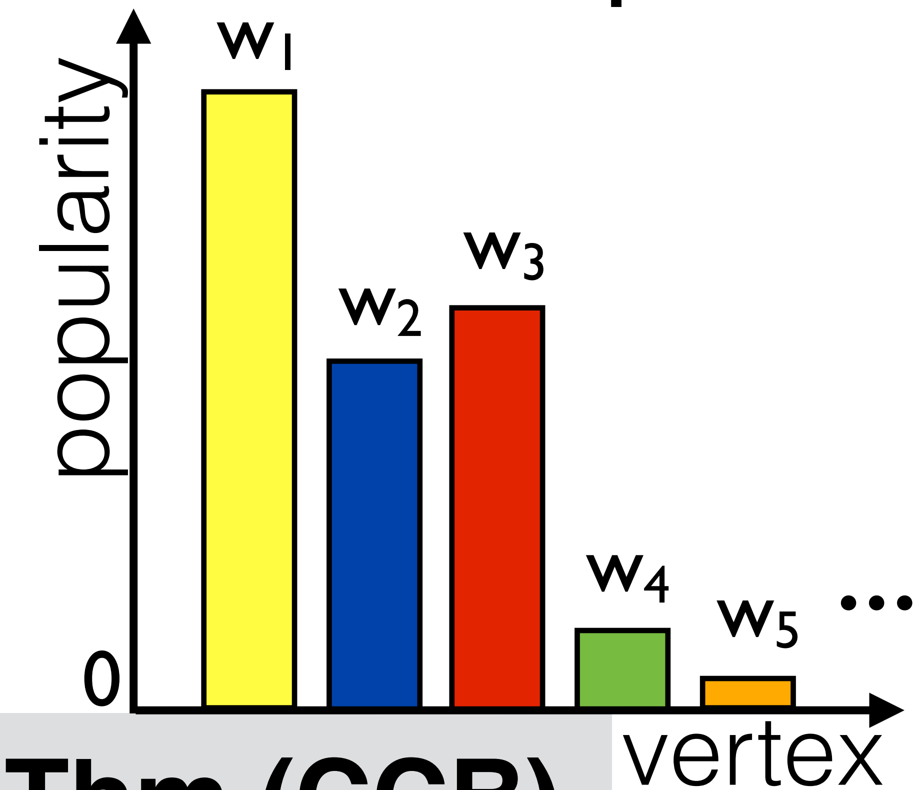


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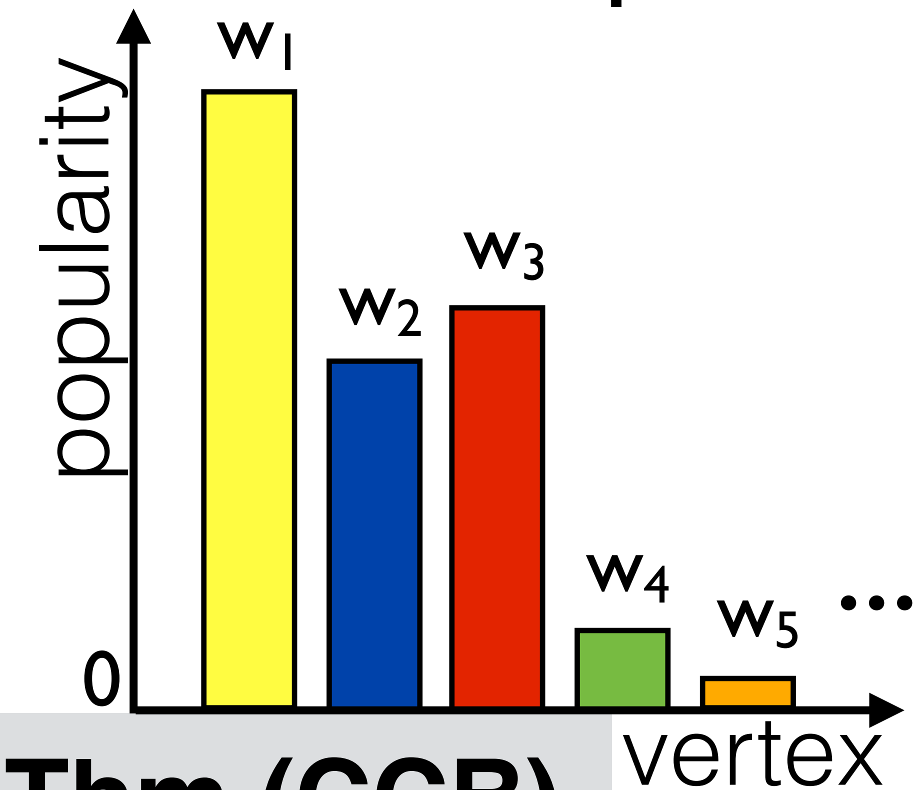


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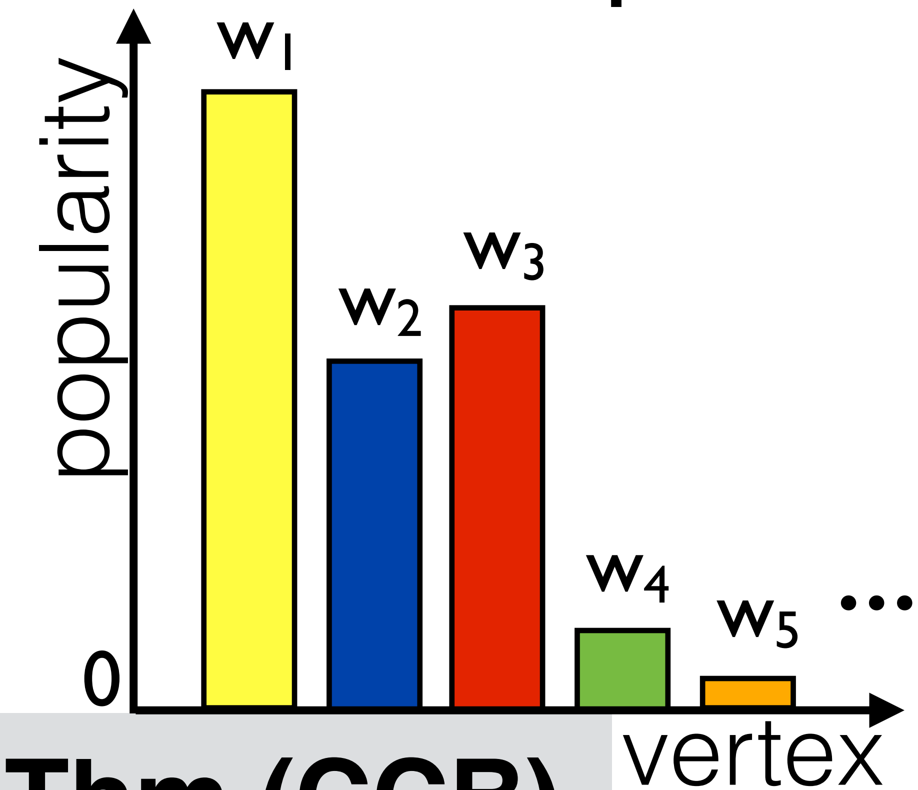


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**Cor (CCB). A wide class of edge-exchangeable graph models yields sparse graph sequences.**



[Cai, Campbell, Broderick 2016]

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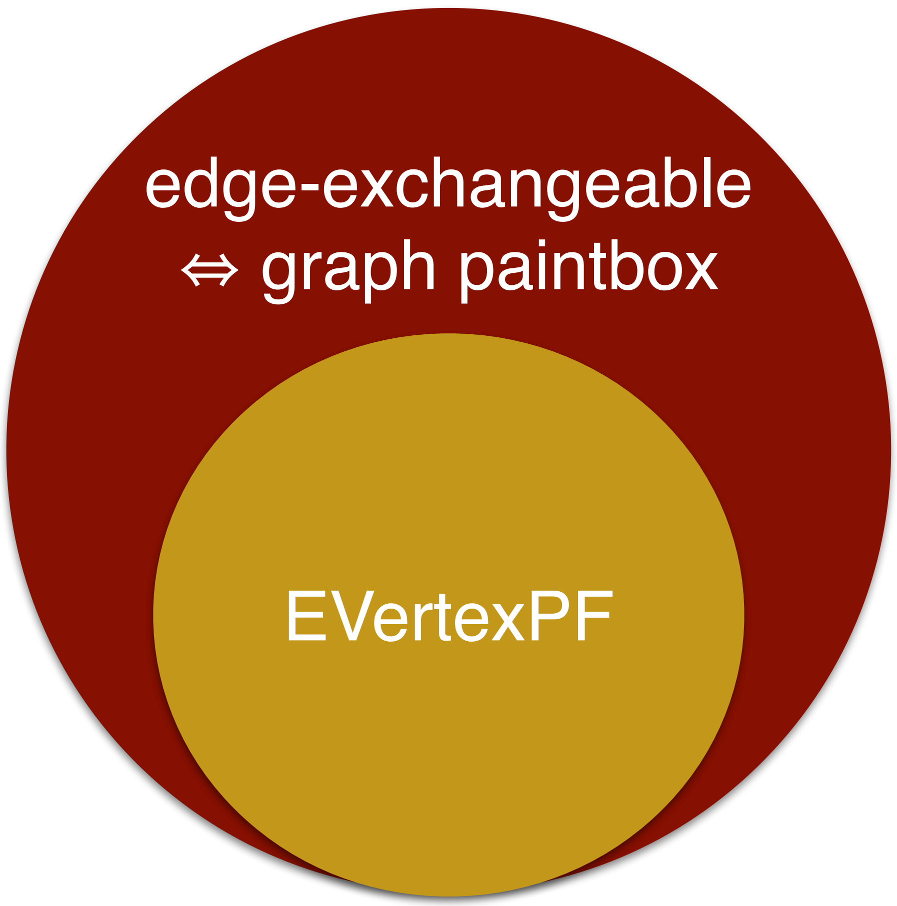


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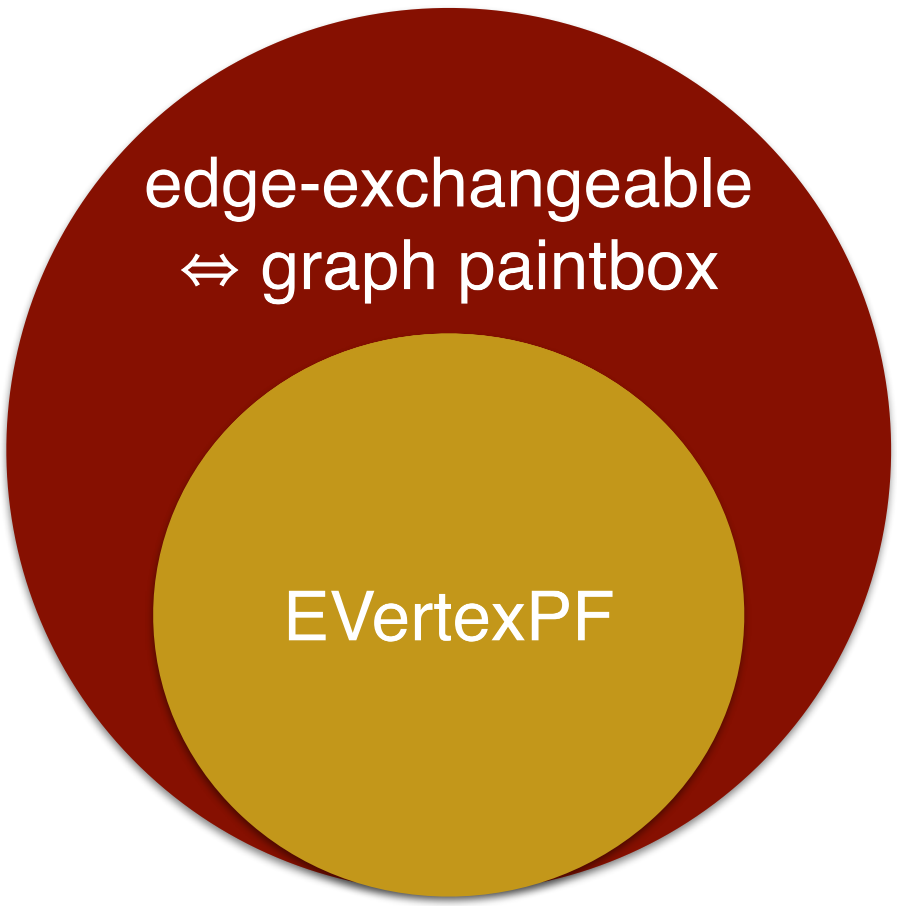


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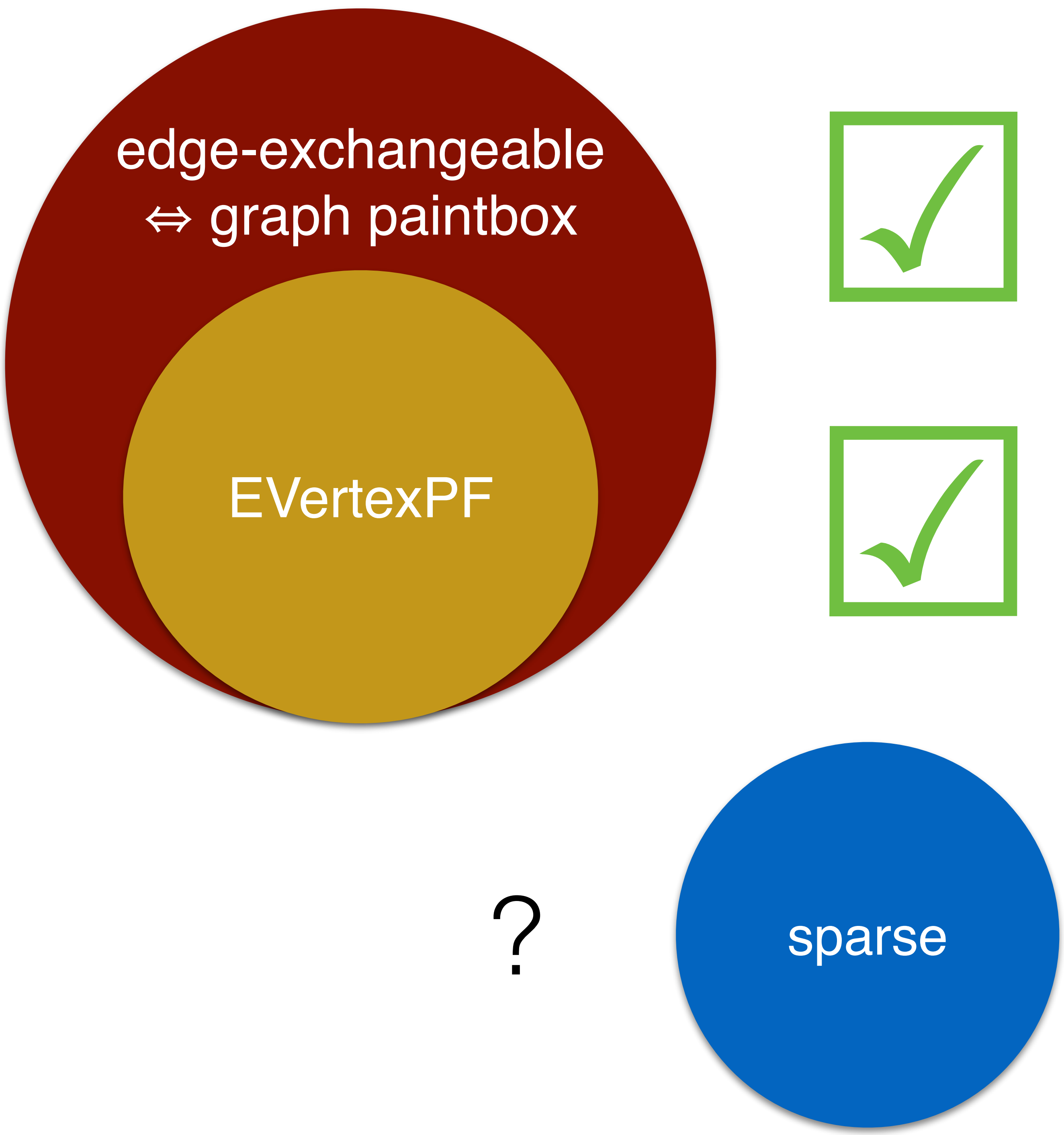


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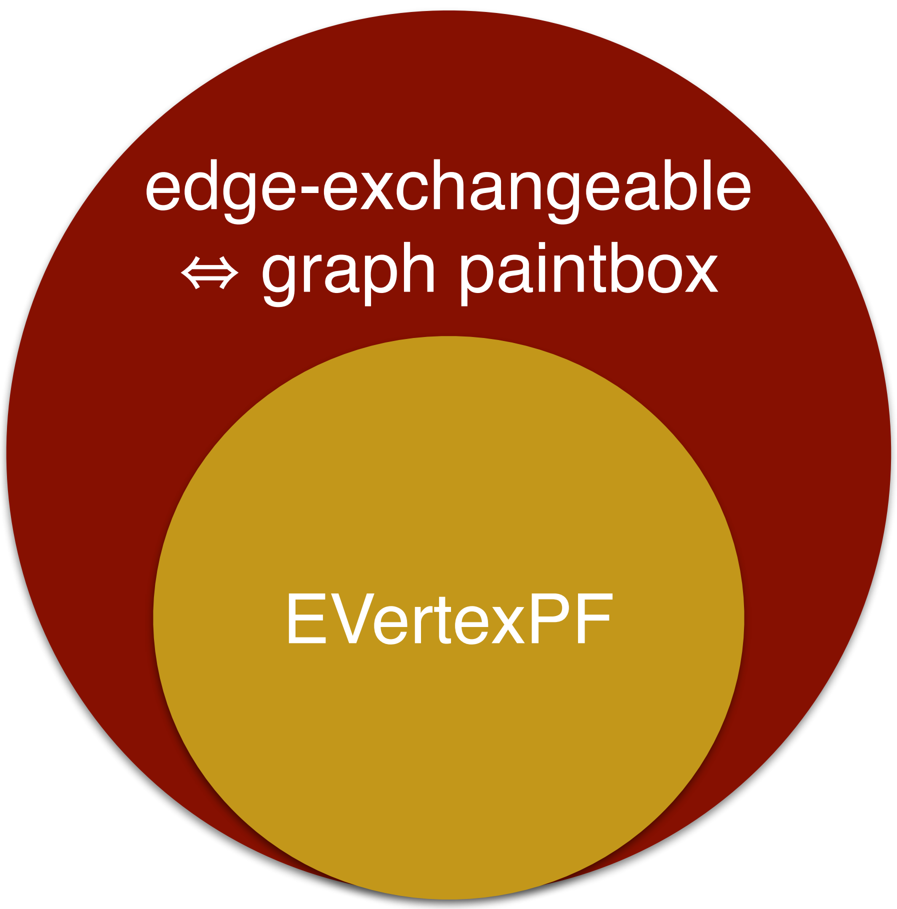


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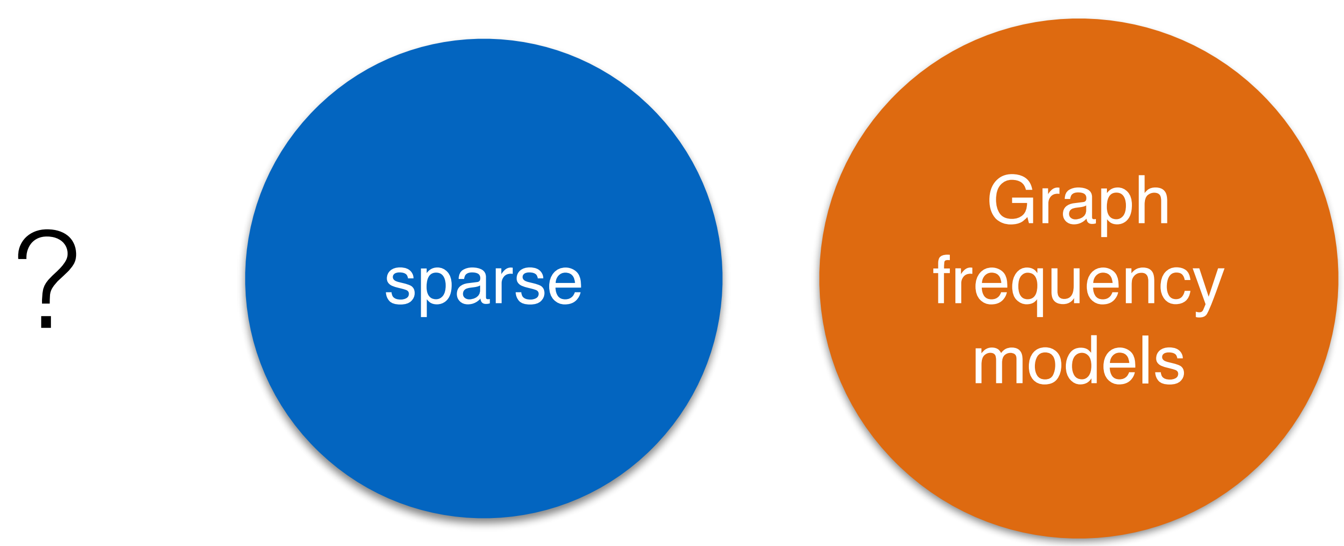

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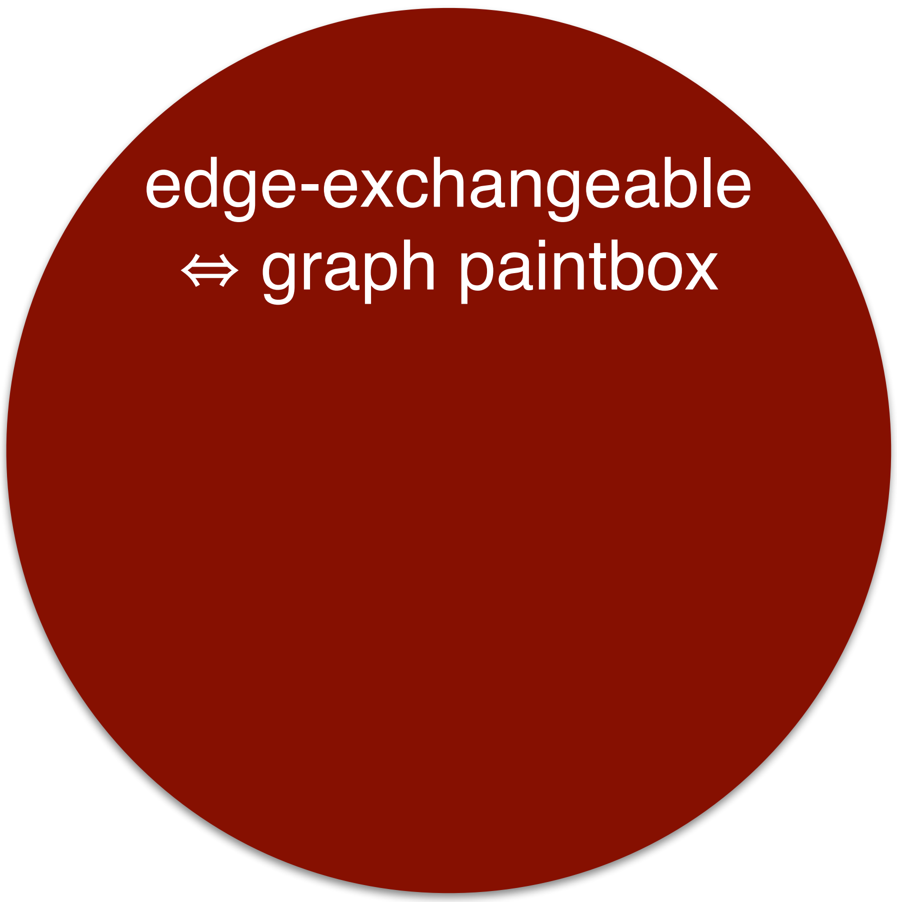


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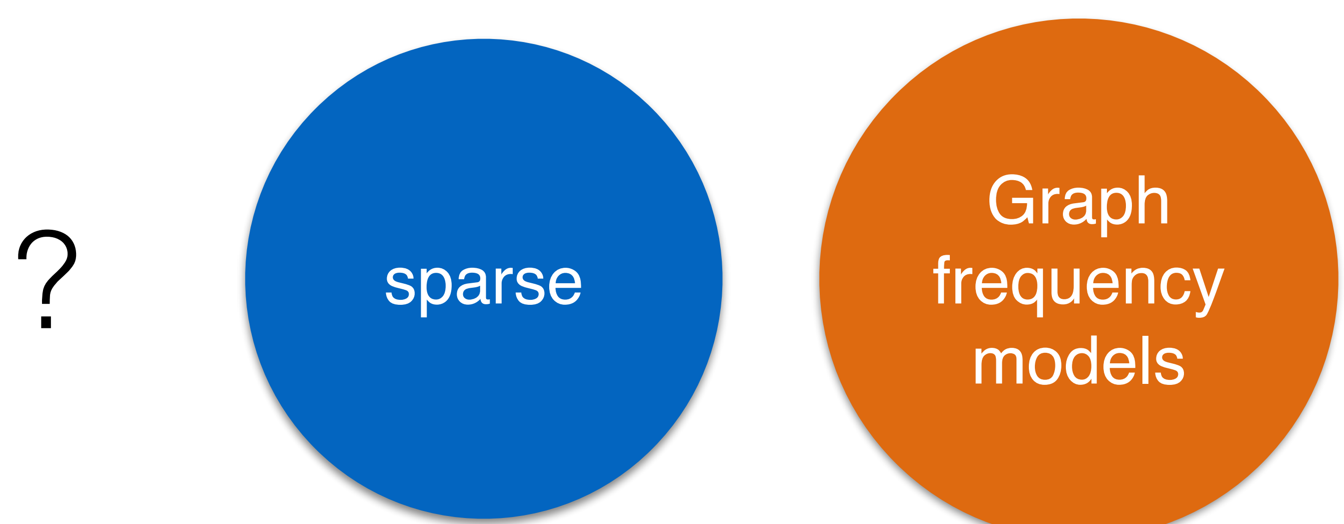

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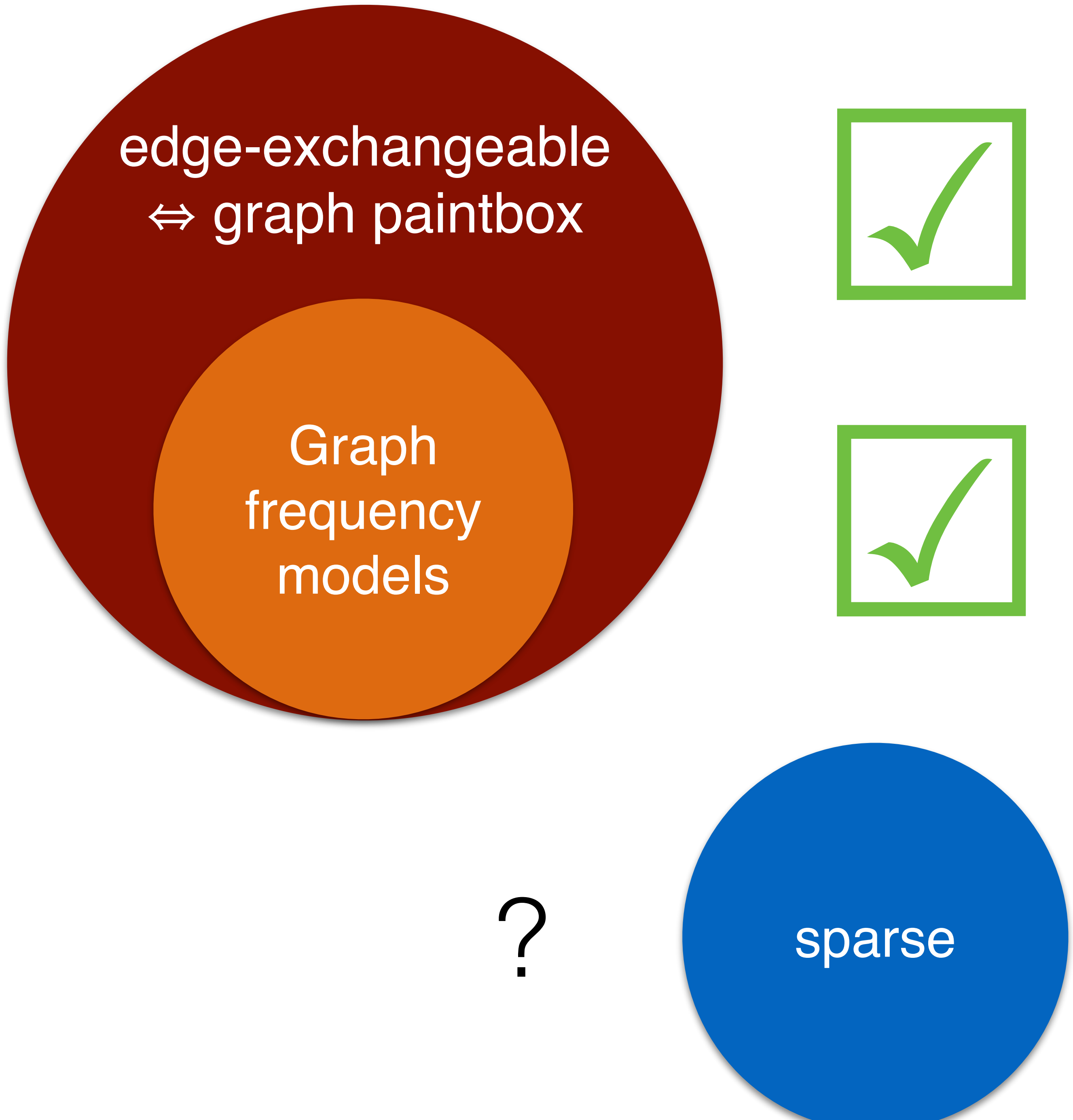


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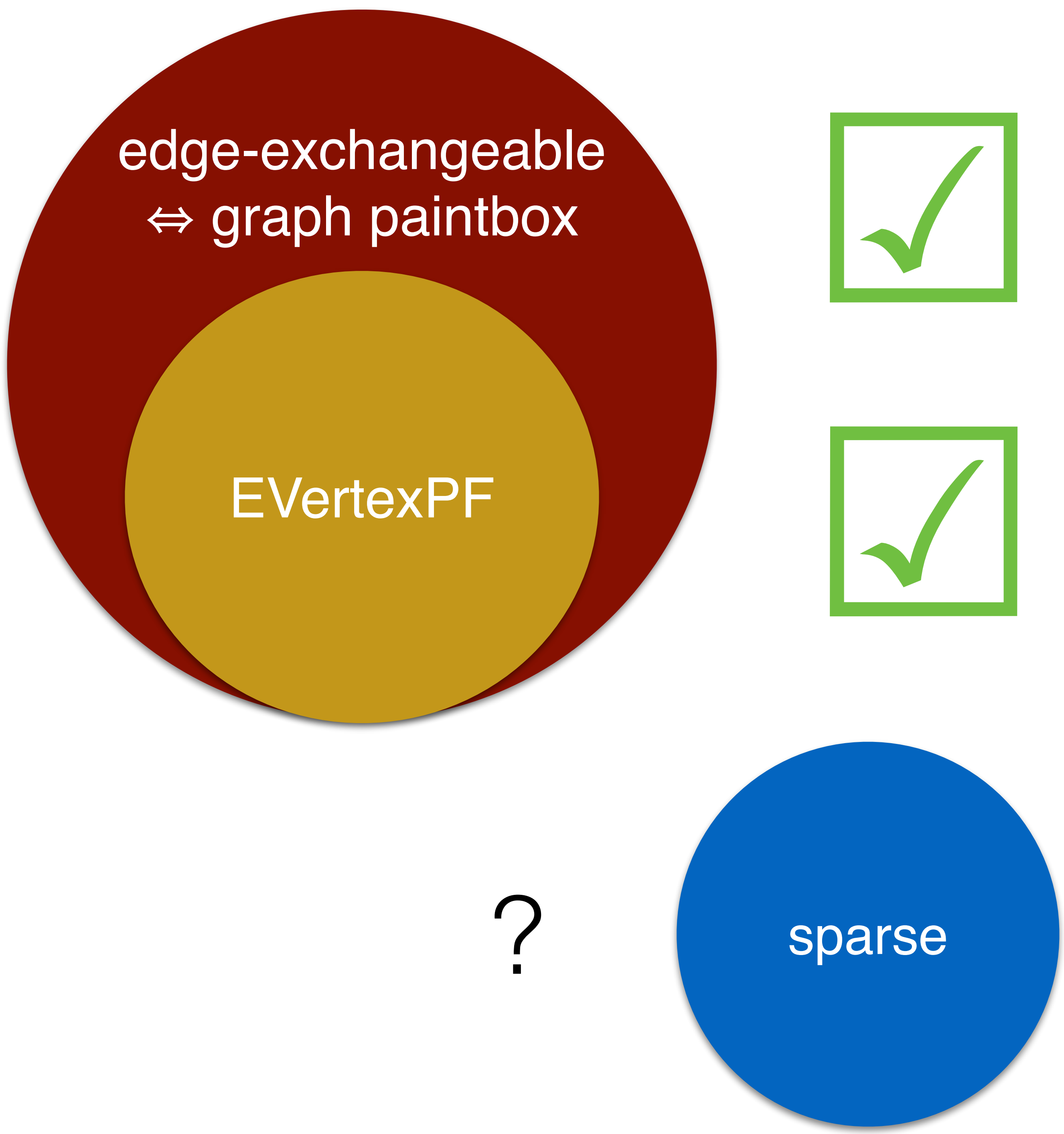


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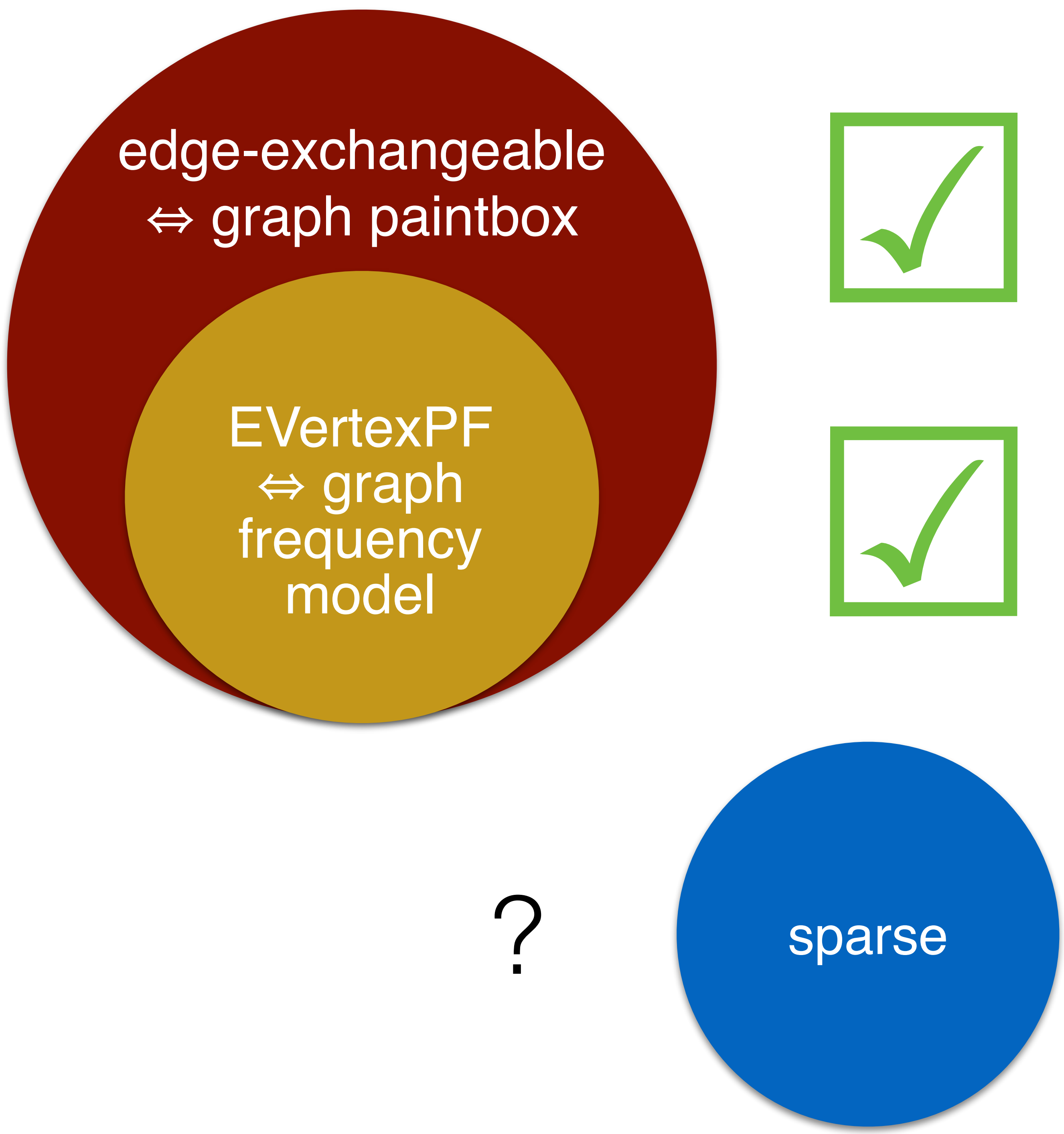
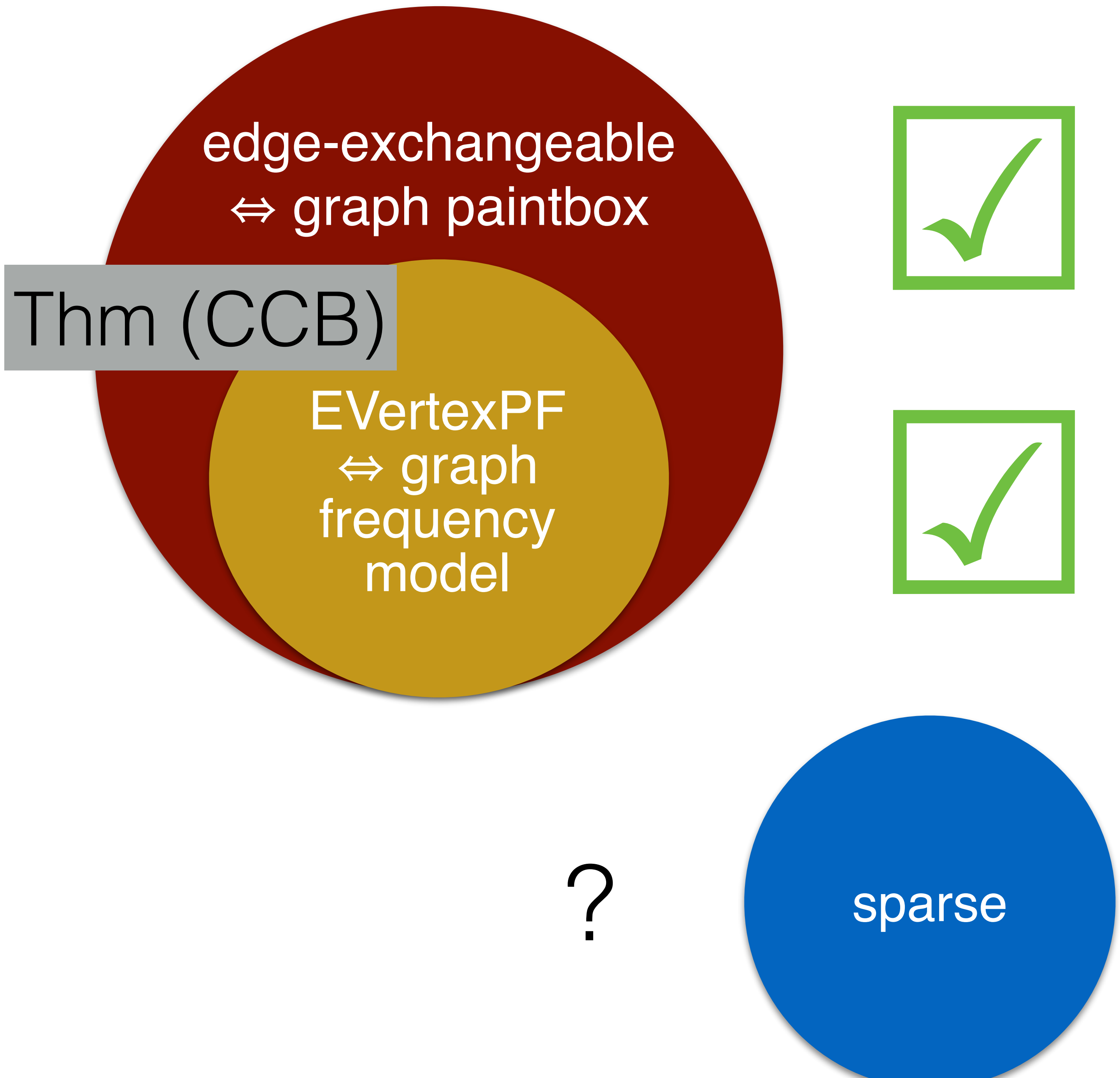



- Thm 1: characterization theorem for edge-exchangeable graphs
- Thm 2: sparsity exists in edge-exchangeable graphs

# What we know so far

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[Campbell, Cai, Broderick 2016]

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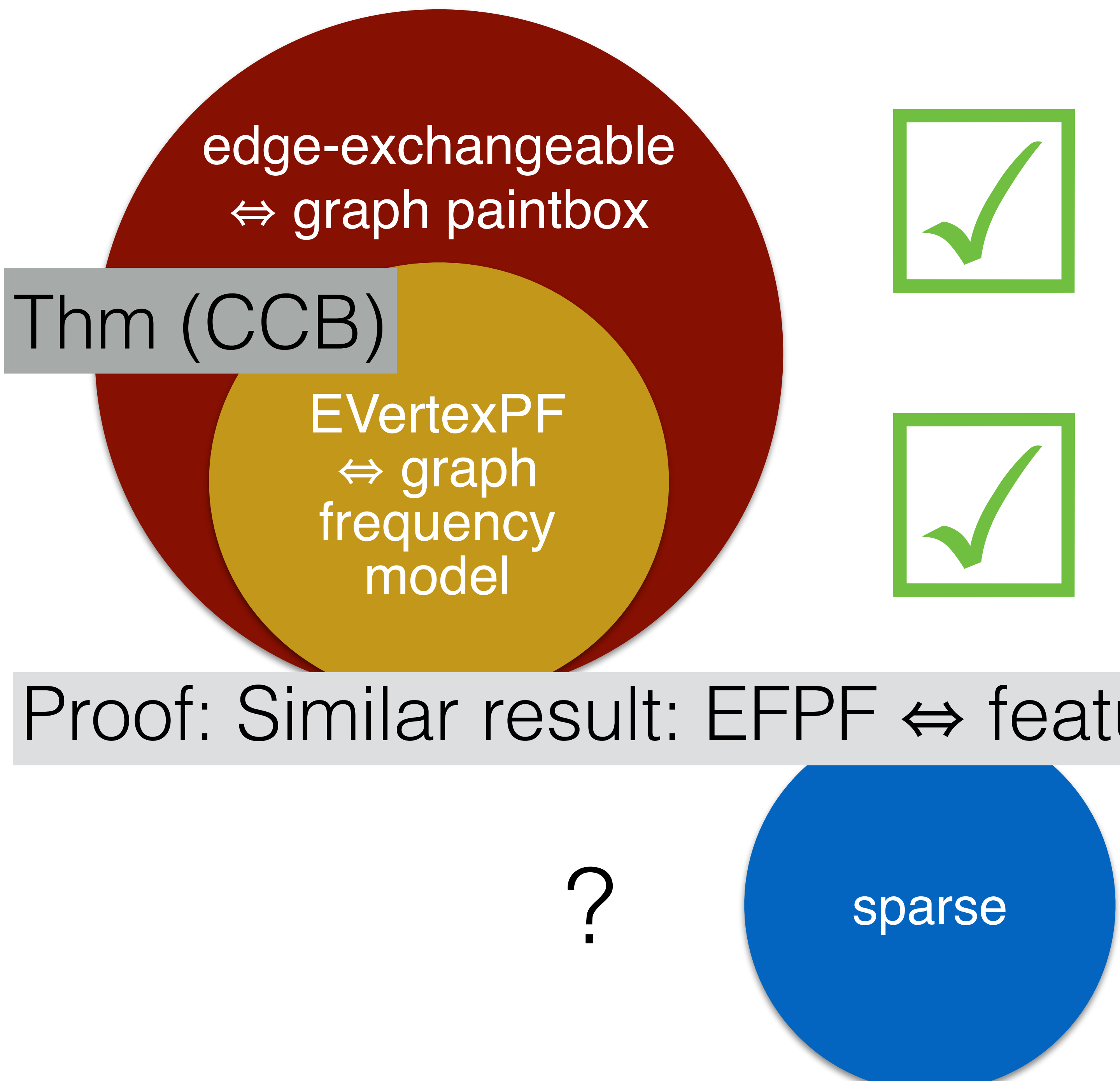


- Thm 1: characterization theorem for edge-exchangeable graphs
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Proof: Similar result: EFPF ⇔ feature frequency model

[Campbell, Cai, Broderick 2016; Broderick, Jordan, Pitman 2013]

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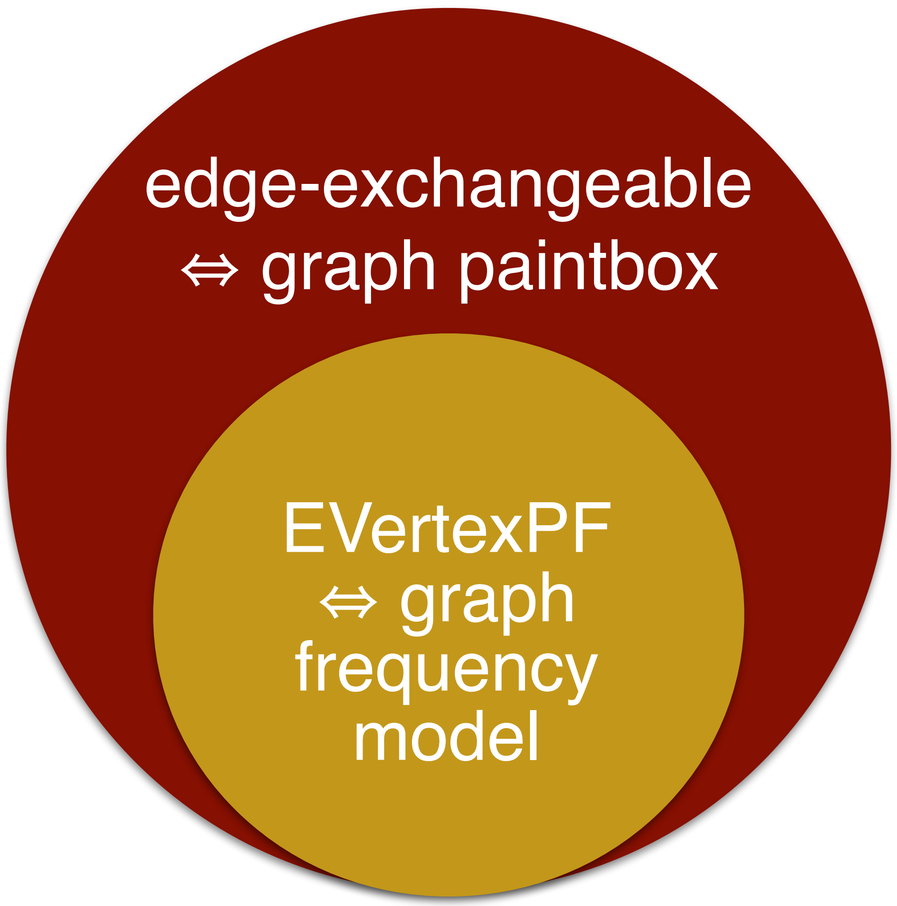


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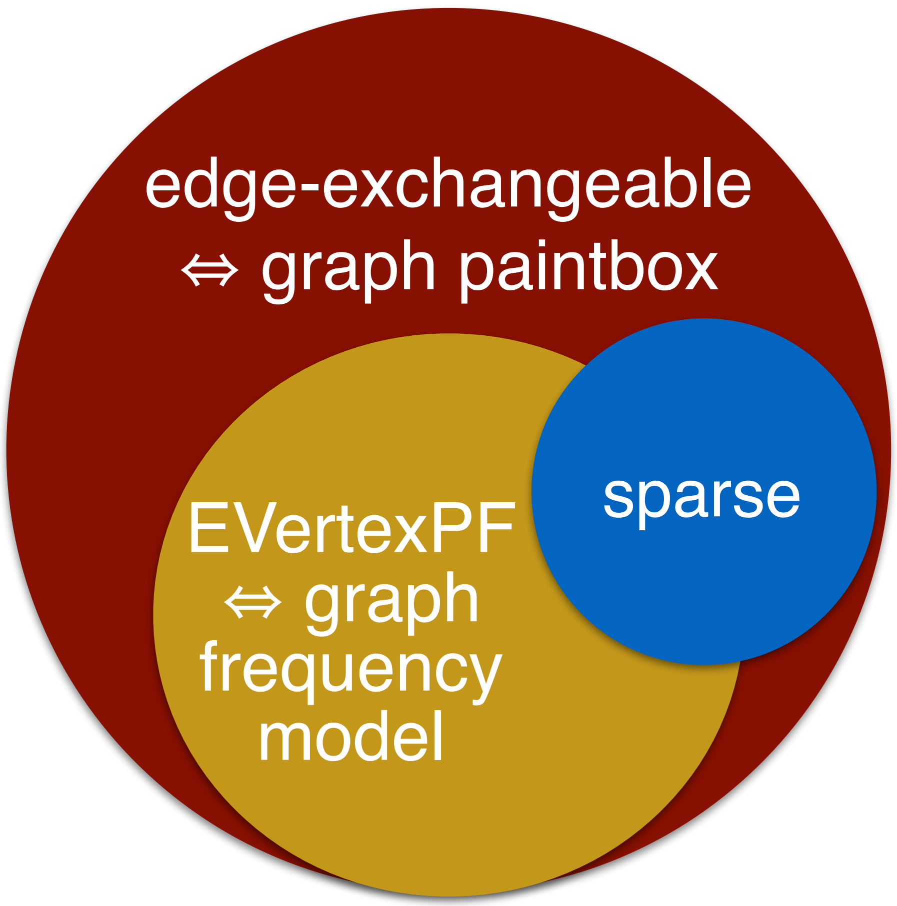


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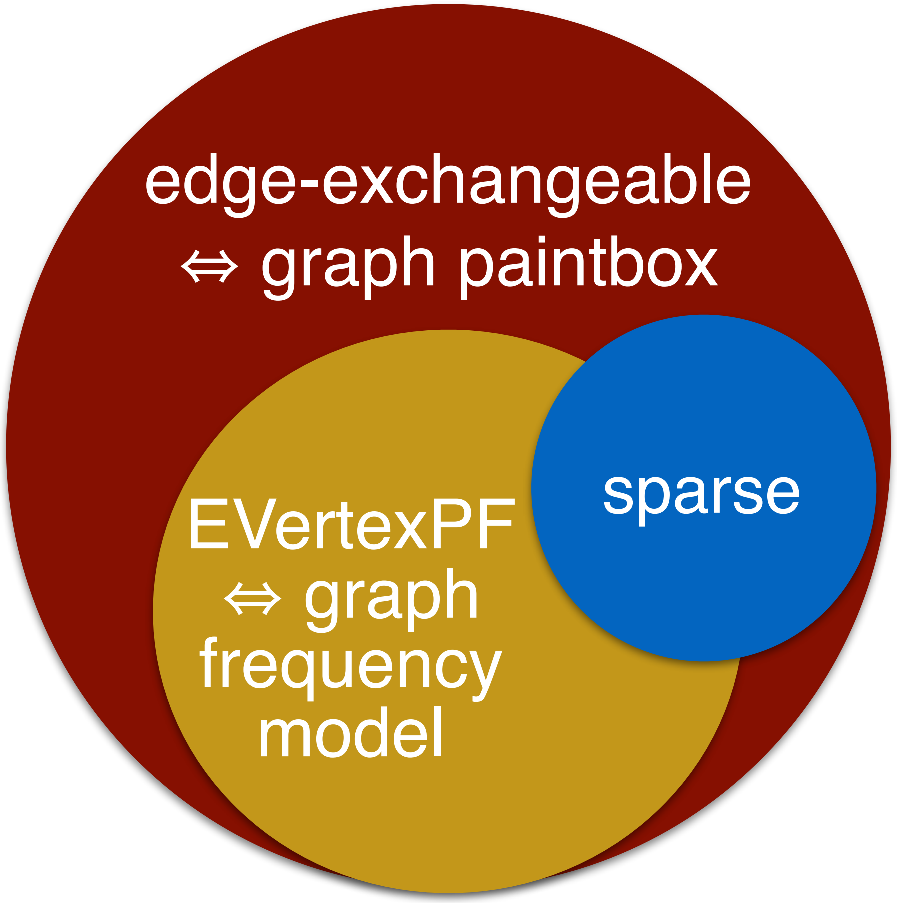


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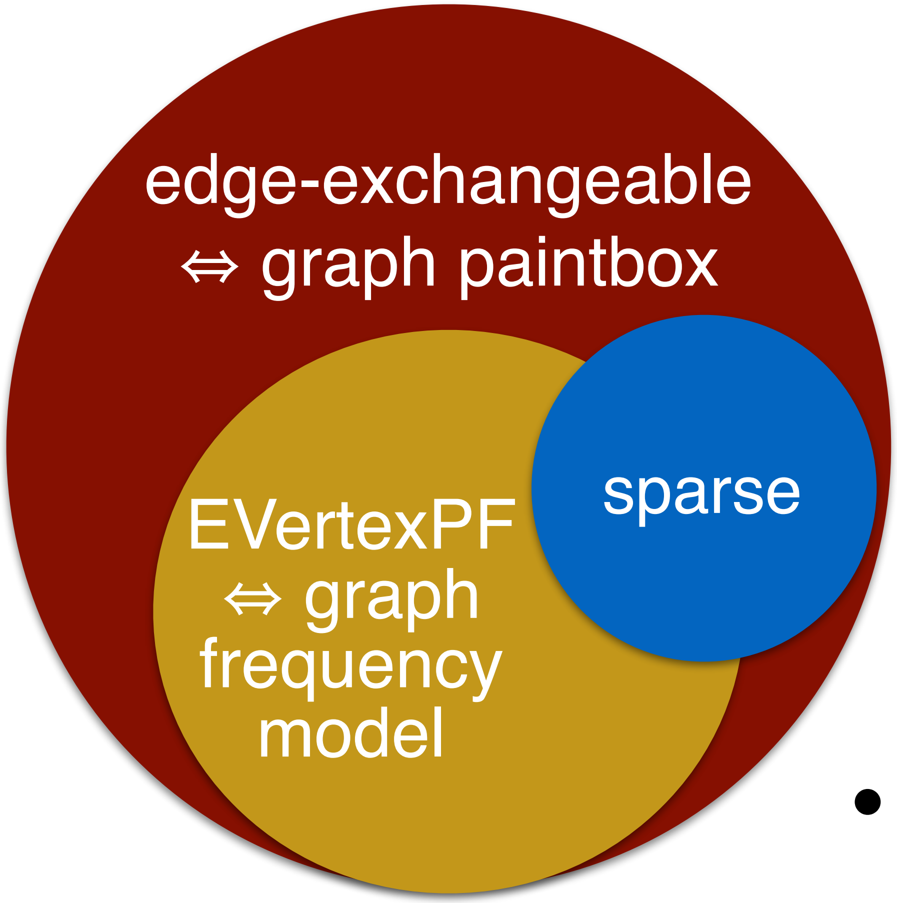
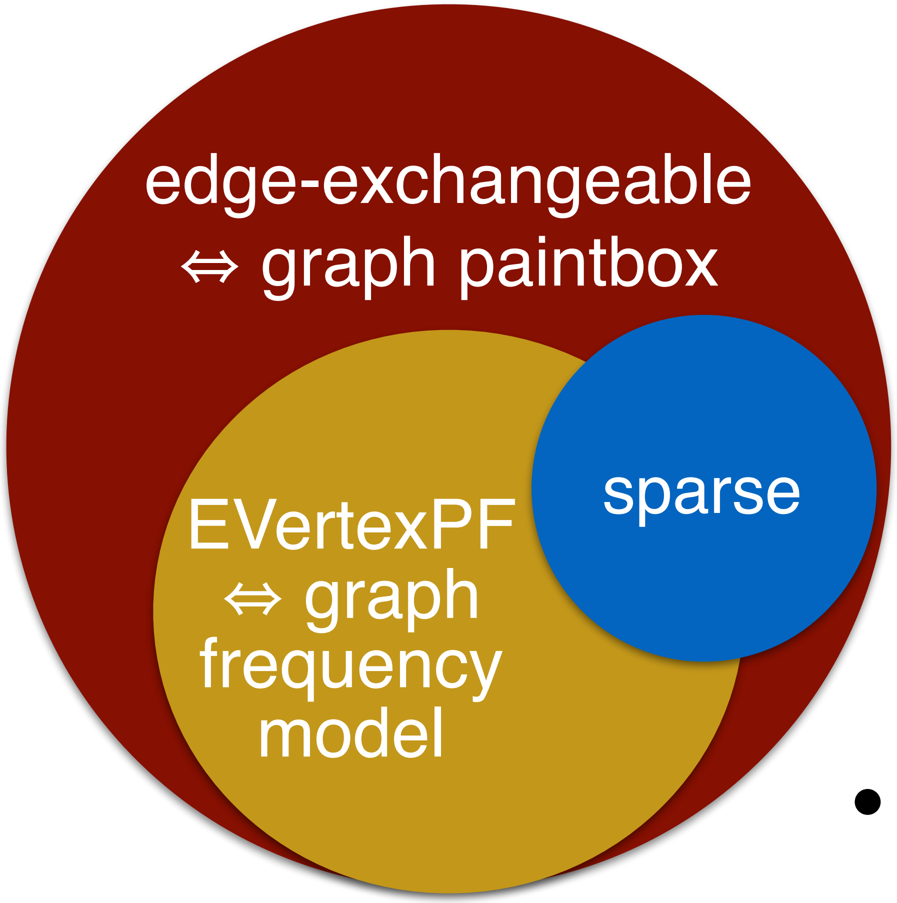



- Thm 1: characterization theorem for edge-exchangeable graphs
- Thm 2: sparsity exists in edge-exchangeable graphs
- Also: Characterization of “dust” (new for features, traits, and graphs)



[Campbell, Cai, Broderick 2016]

# What we know so far

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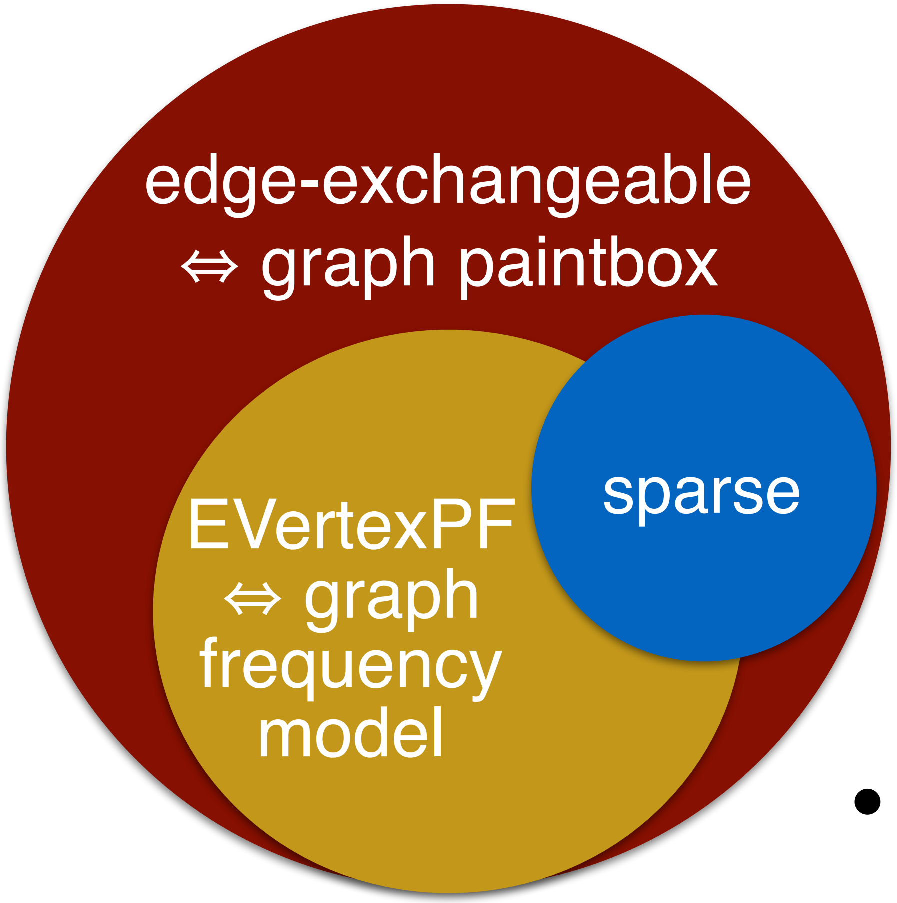
# Frontier



[Campbell, Cai, Broderick 2016]

# What we know so far




- Thm 1: characterization theorem for edge-exchangeable graphs
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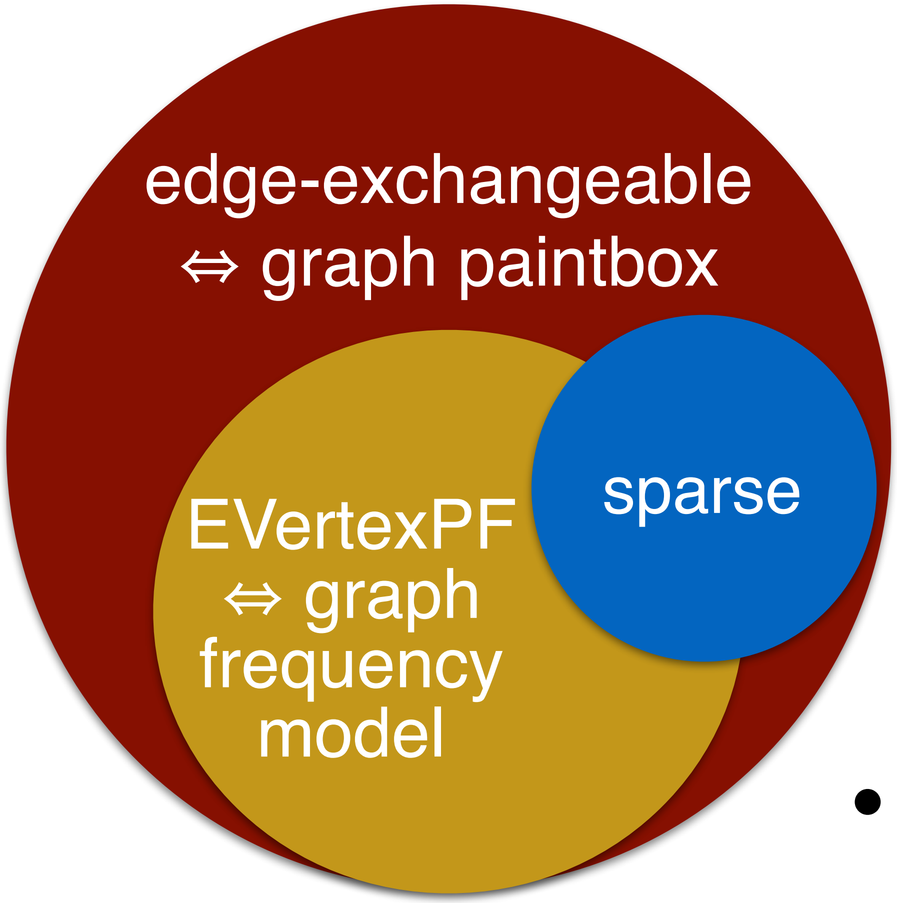
# Frontier

- Characterize all sparse, edge-exchangeable graphs



[Campbell, Cai, Broderick 2016]

# What we know so far




- Thm 1: characterization theorem for edge-exchangeable graphs
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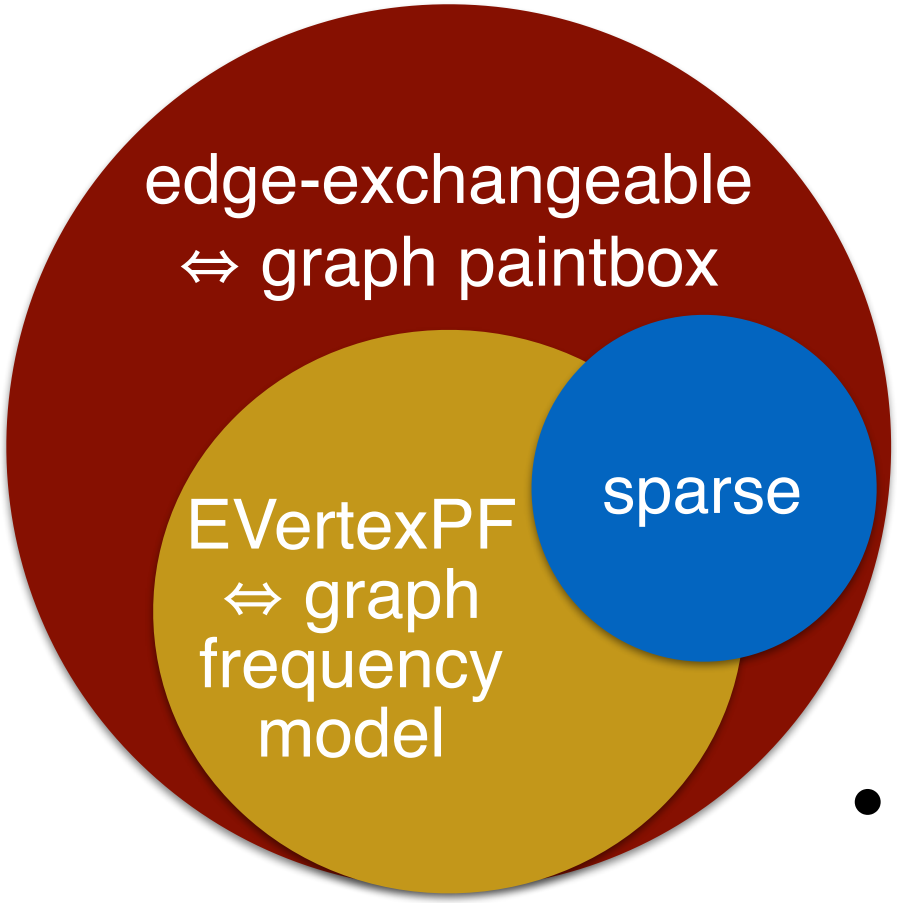
# Frontier

- Characterize all sparse, edge-exchangeable graphs
- Models and inference



[Campbell, Cai, Broderick 2016; Broderick, Jordan, Pitman 2012]

# What we know so far




- ✓ Thm 1: characterization theorem for edge-exchangeable graphs
- ✓ Thm 2: sparsity exists in edge-exchangeable graphs
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# Frontier

- Characterize all sparse, edge-exchangeable graphs
- Models and inference
- Truncation approximations

 [Campbell, Cai, Broderick 2016; Broderick, Jordan, Pitman 2012; Campbell*, Huggins*, How, Broderick 2016]

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