User-friendly conjugacy for completely random measures

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Conjugacy

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(Nonparametric) Bayesian concerns:

- Calculating the posterior can be hard

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- How to choose the prior

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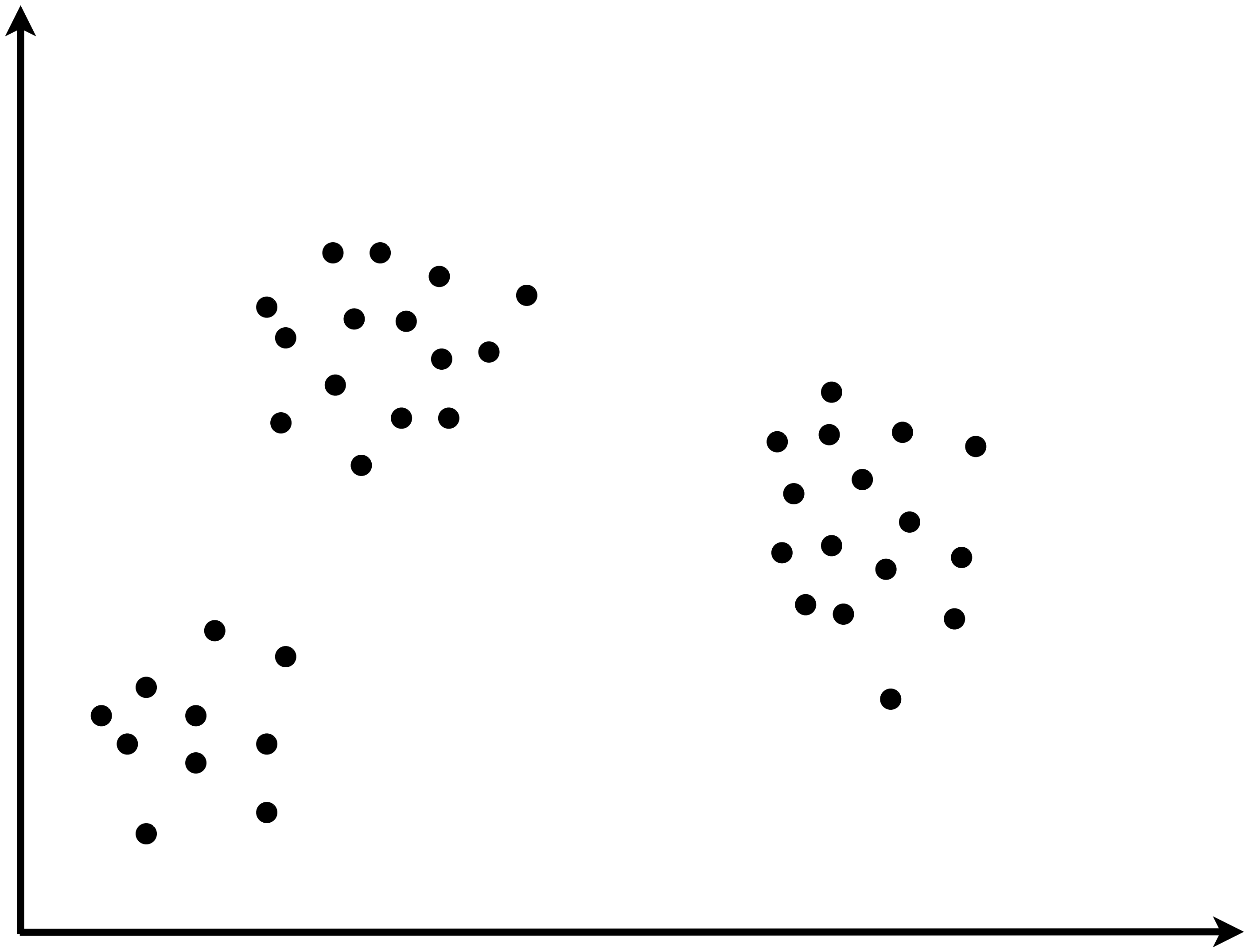
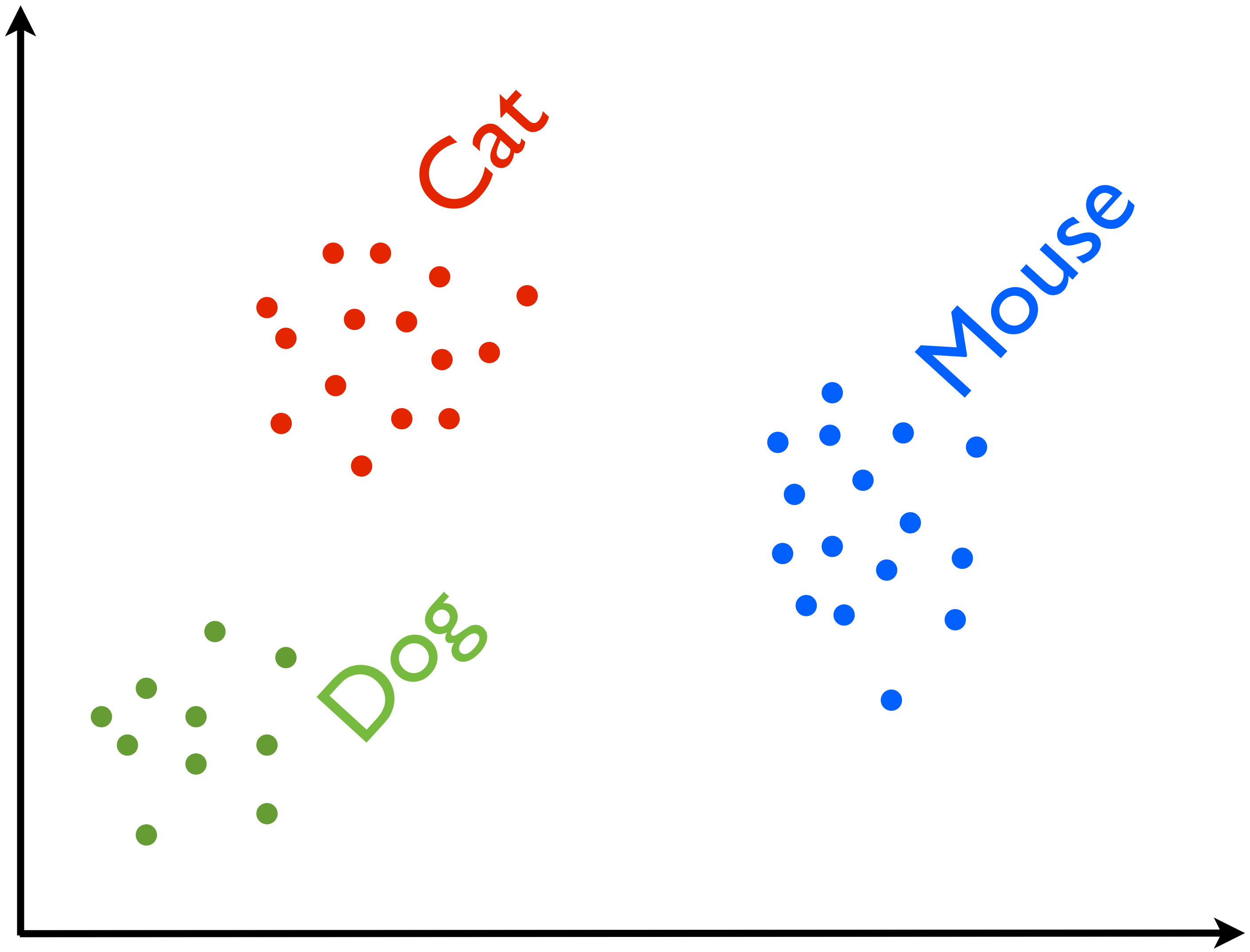
- Computational imperative in nonparametric setting

We want conjugacy to be user-friendly in Bayesian nonparametrics

Clustering

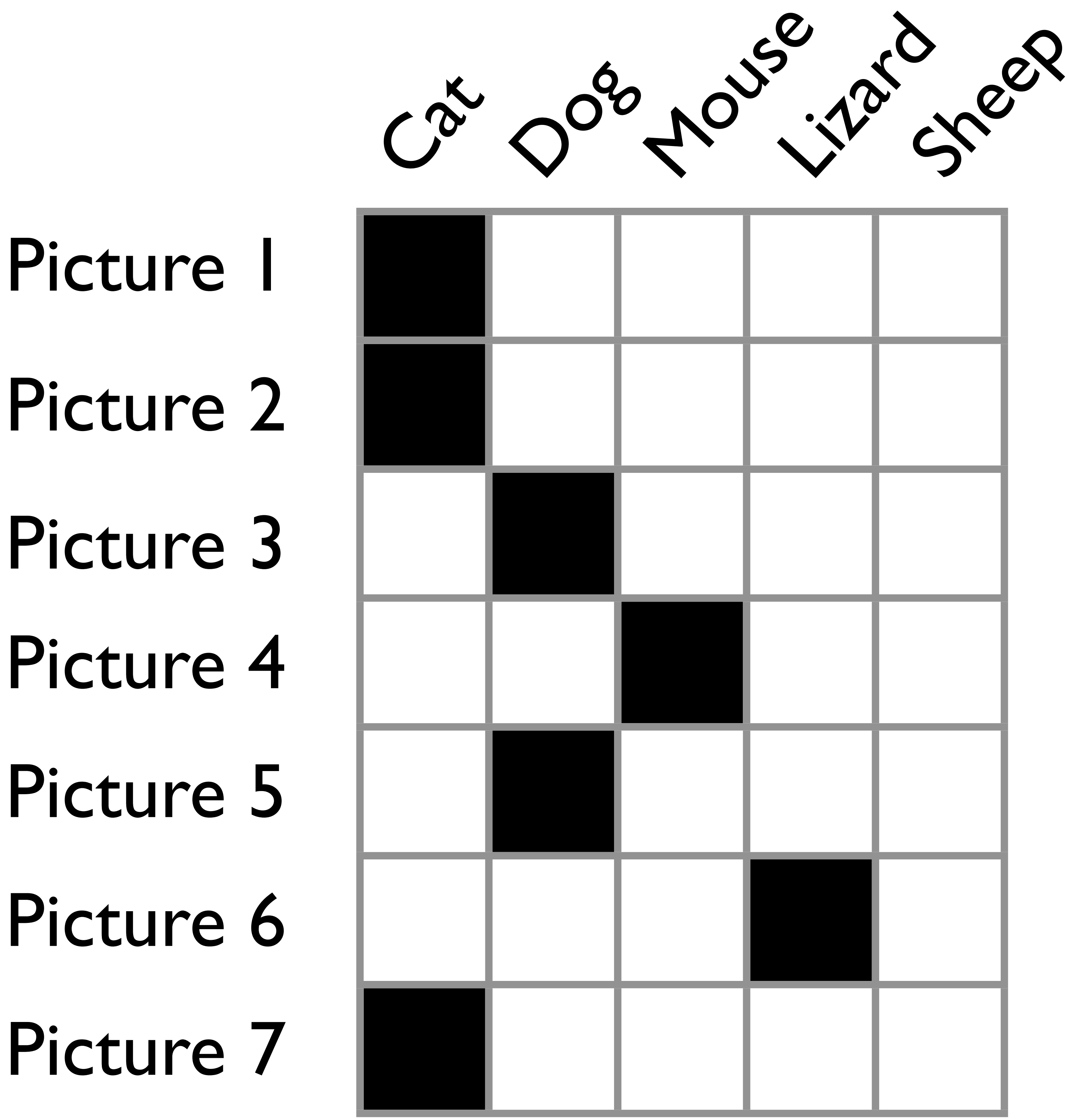


Clustering

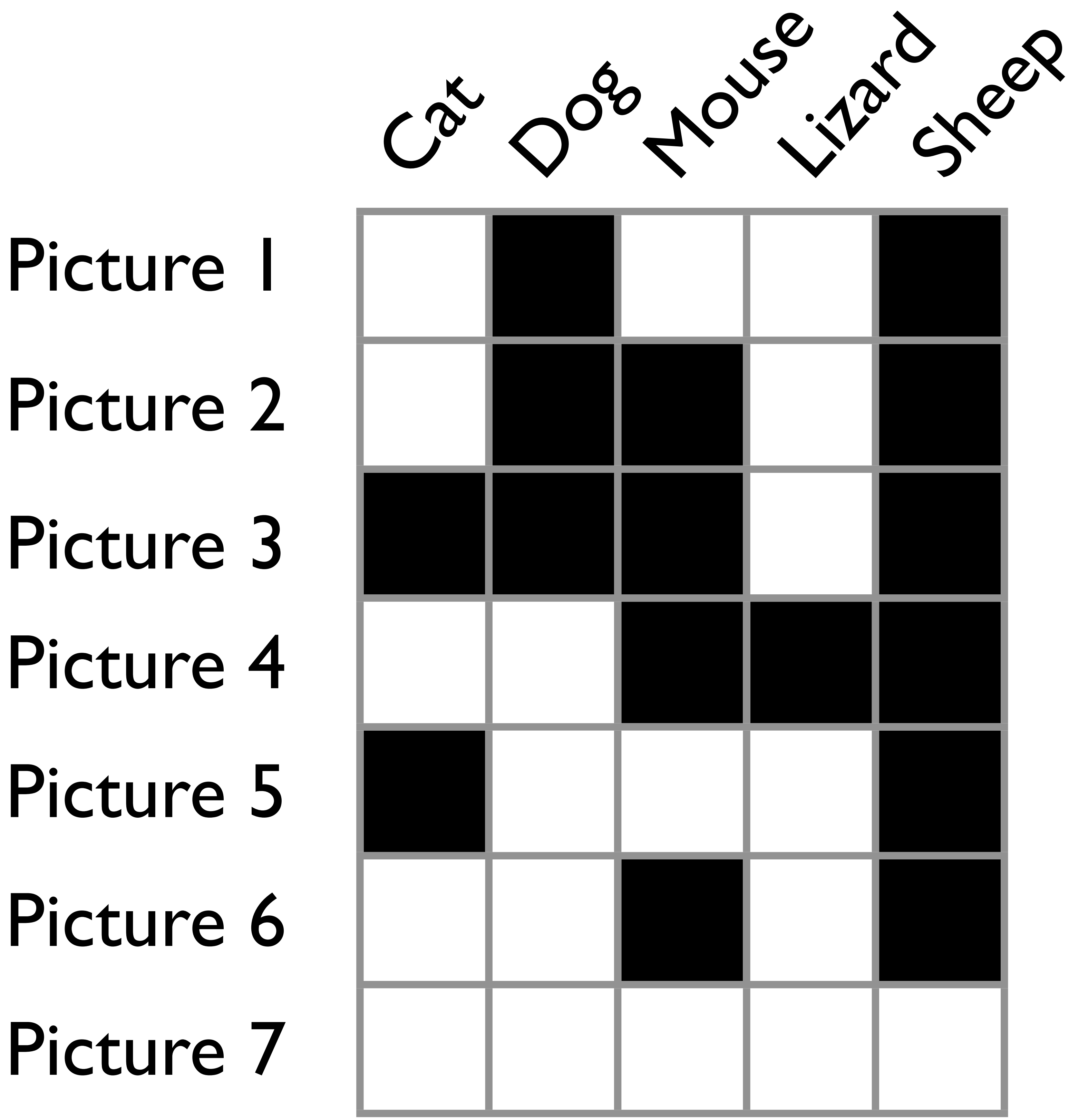


“clusters”

Clustering

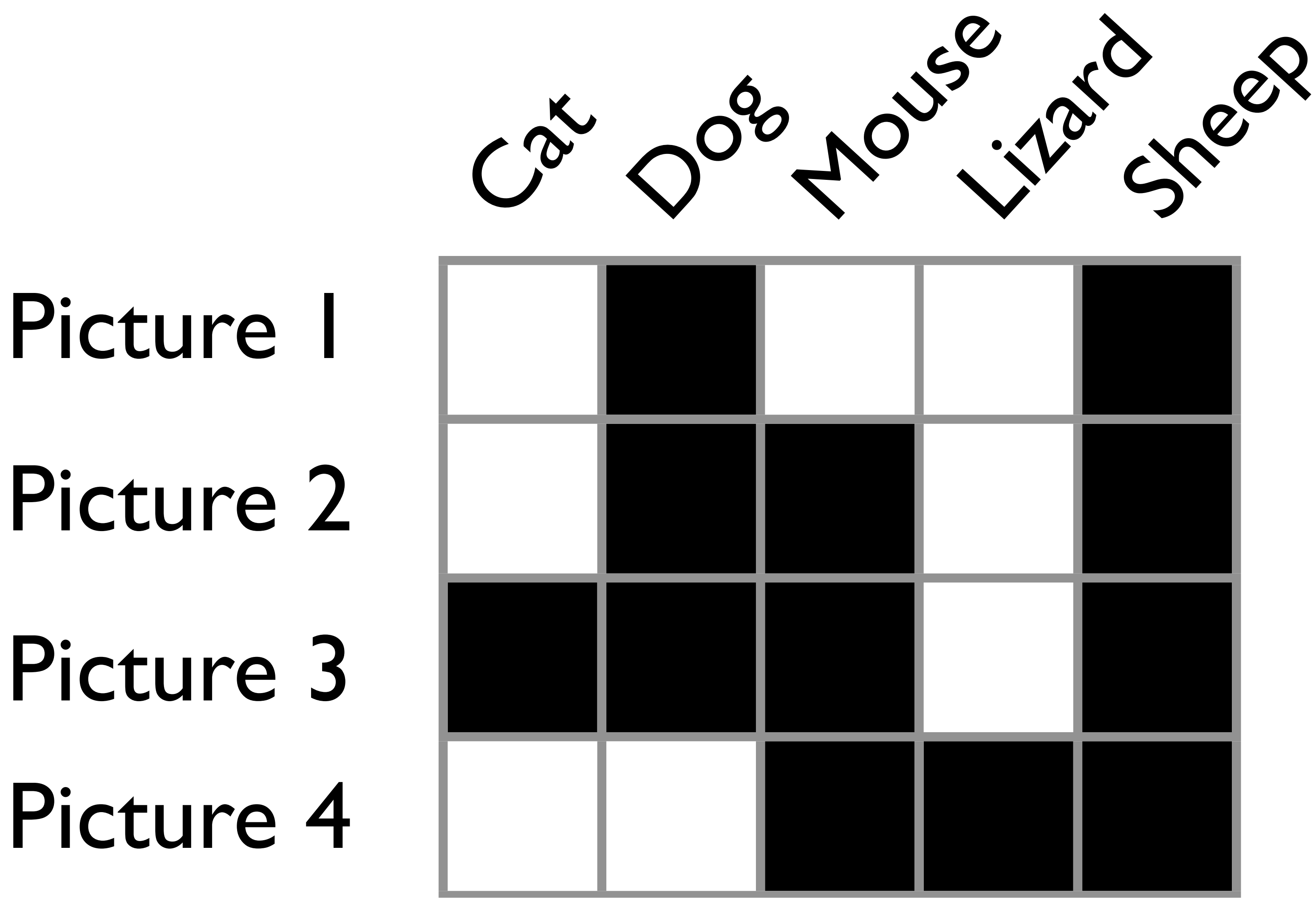

Cat
Dog
Mouse
Lizard
Sheep
Picture 1
Picture 2
Picture 3
Picture 4
Picture 5
Picture 6
Picture 7

Feature allocation



“features”

Feature allocation

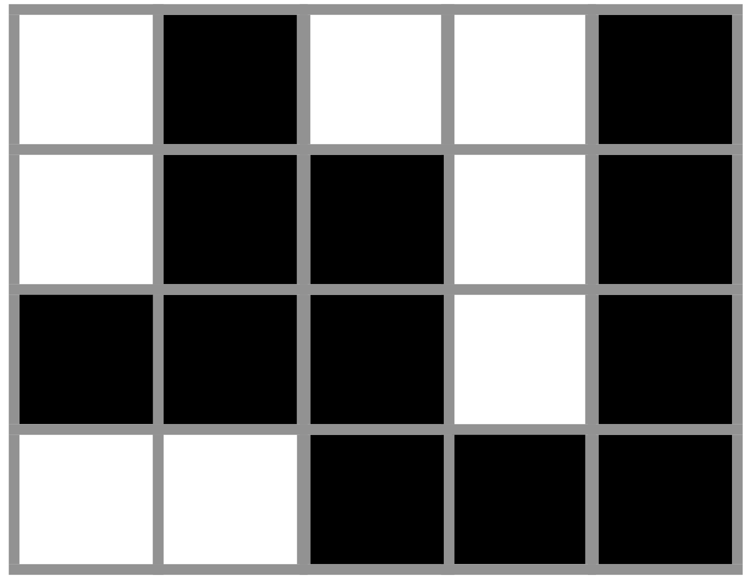


“features”

Indian buffet process [Griffiths, Ghahramani 2006]

Feature allocation

Picture 1
Picture 2
Picture 3
Picture 4



[Thibaux, Jordan 2007]

Feature allocation

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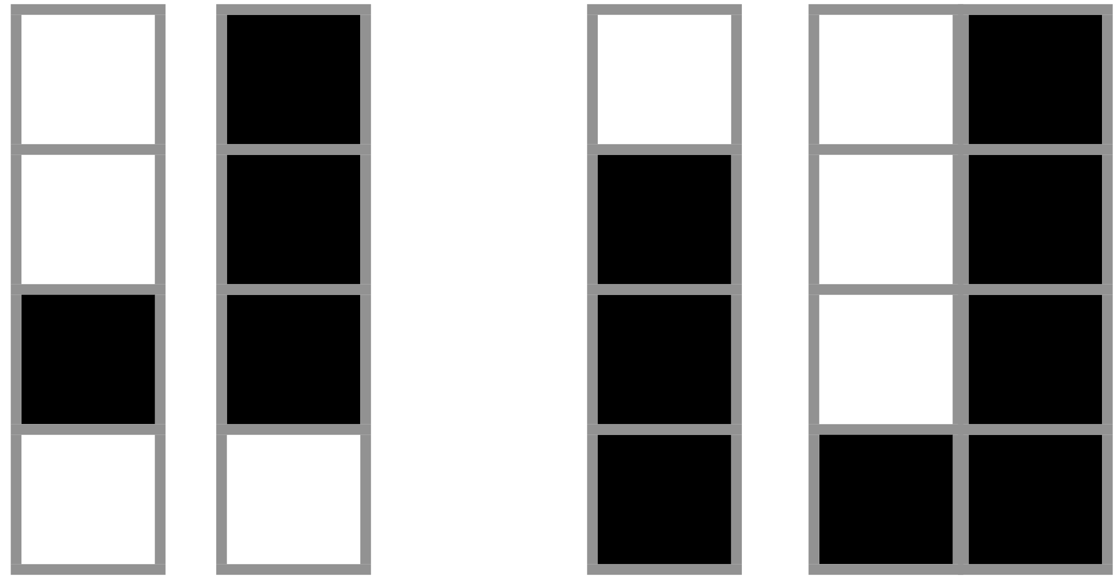

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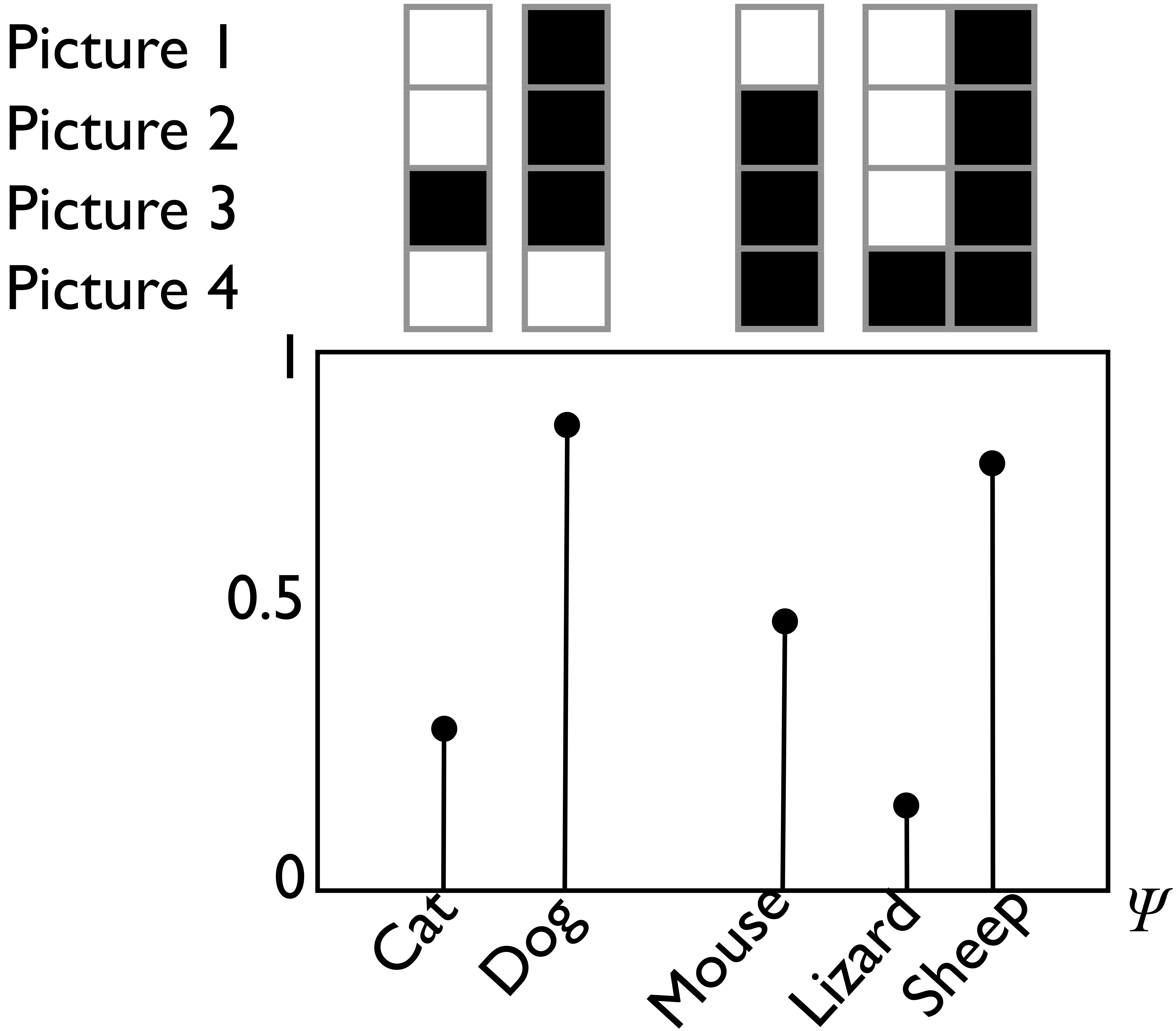

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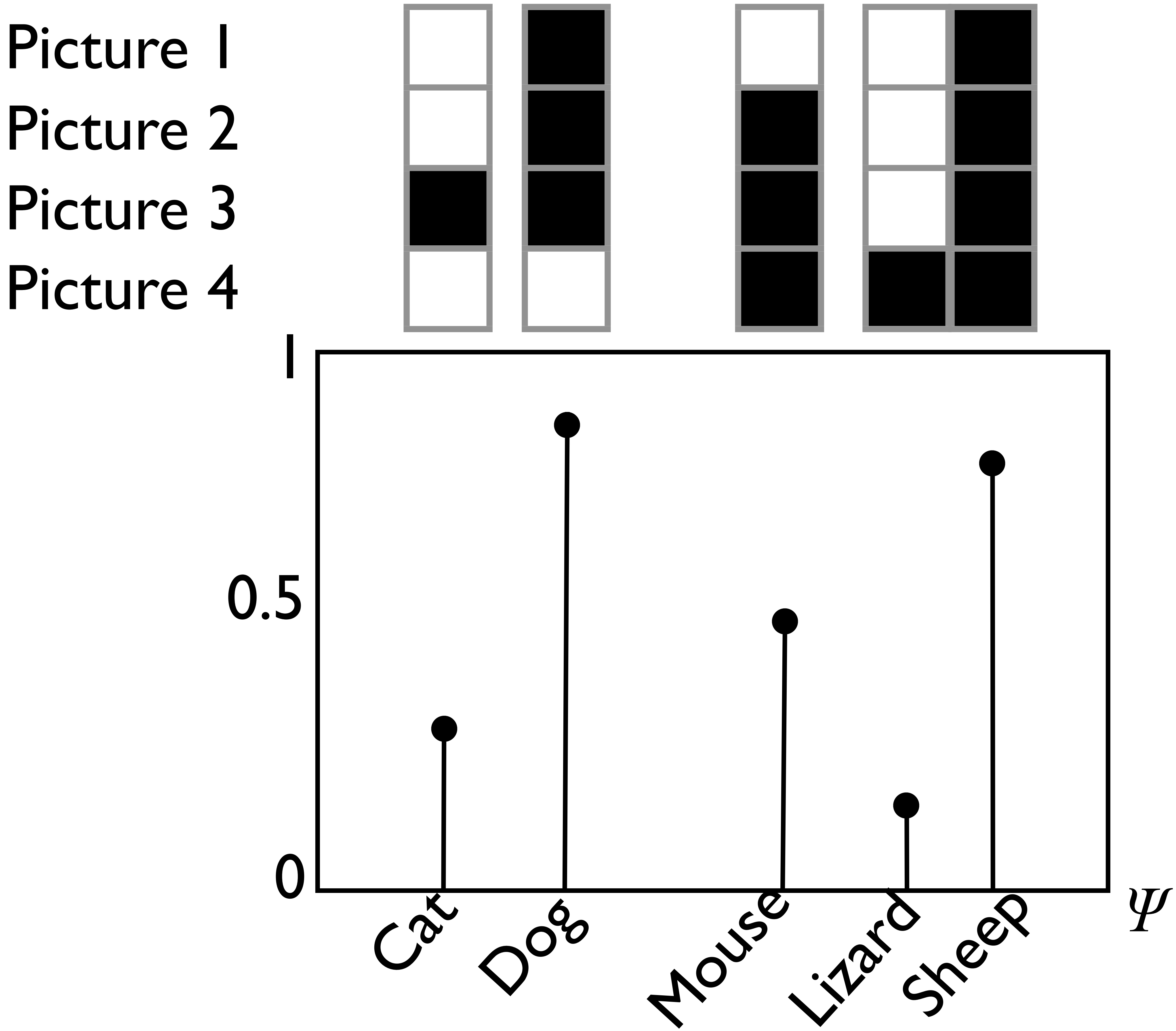
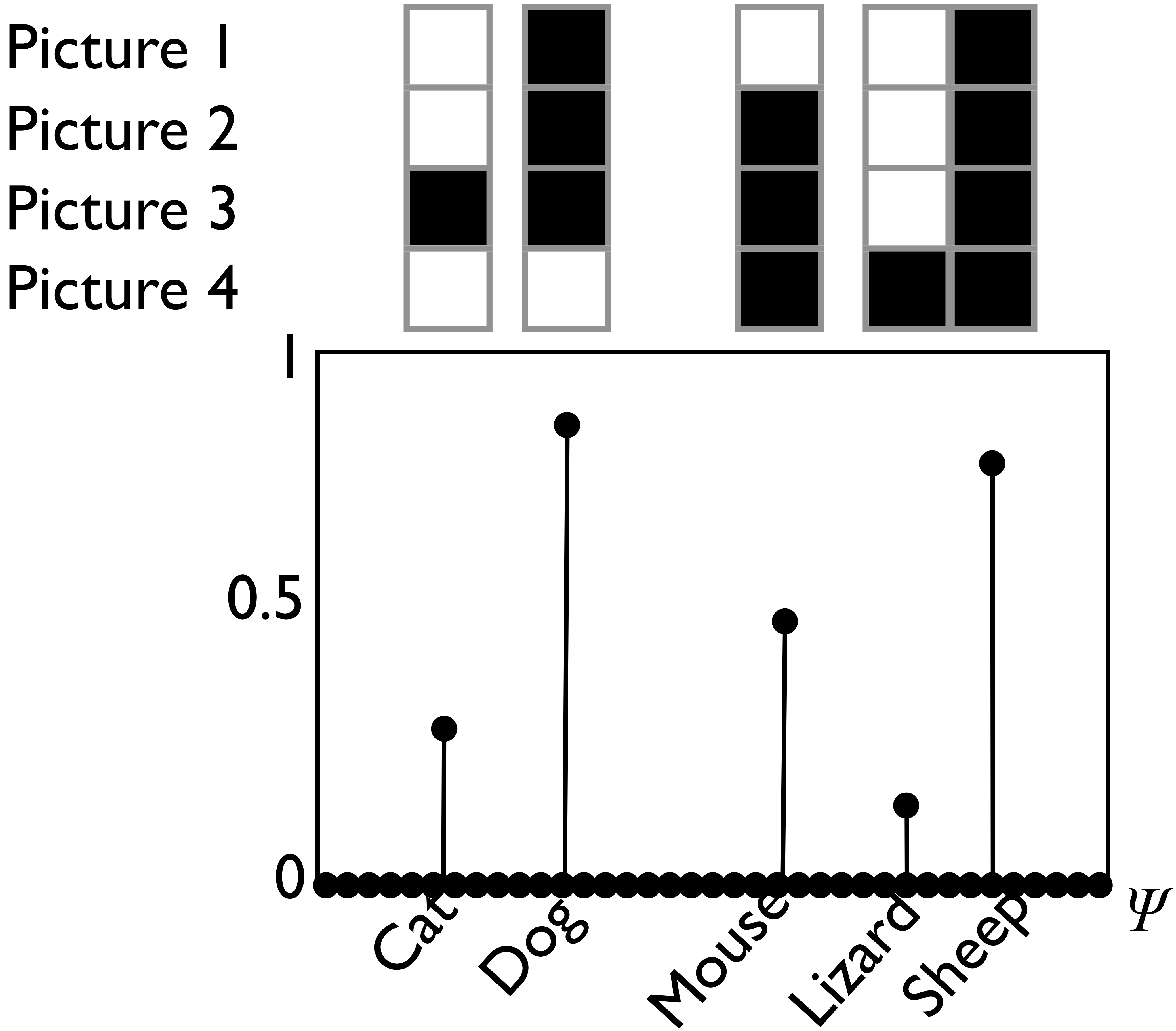


"beta process"

Feature allocation



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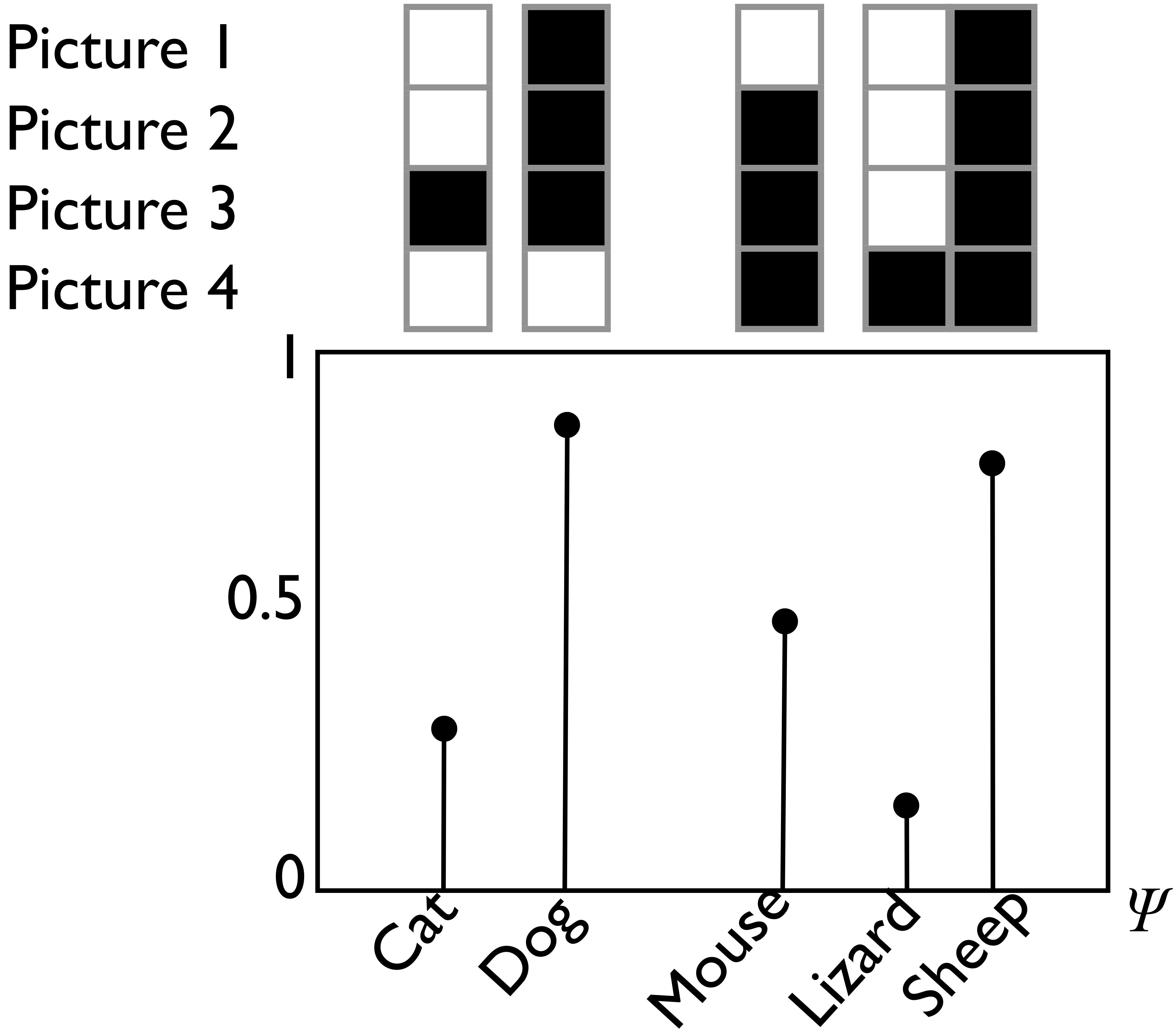

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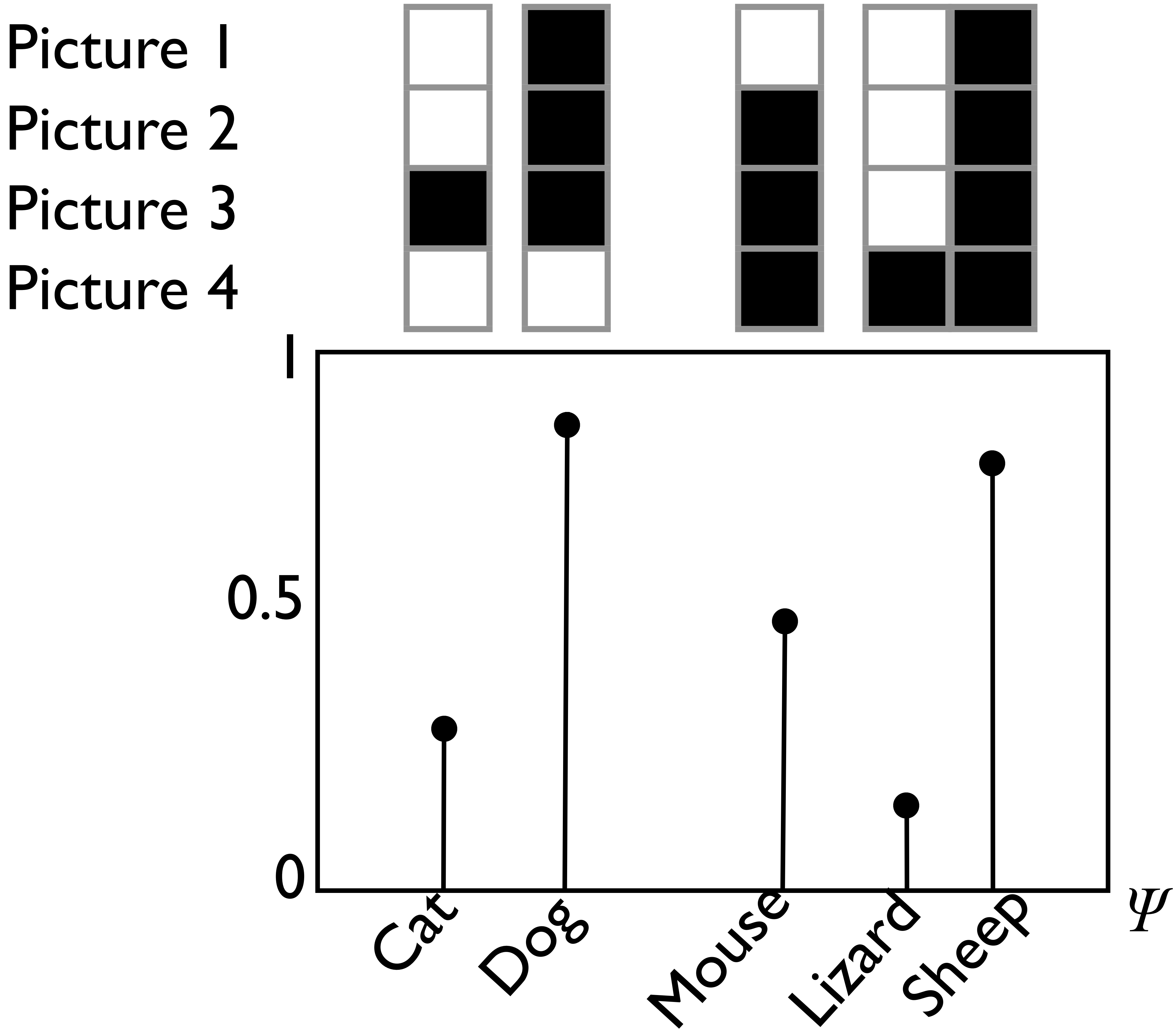
[Hjort 1990; Kim 1999; Thibaux, Jordan 2007]



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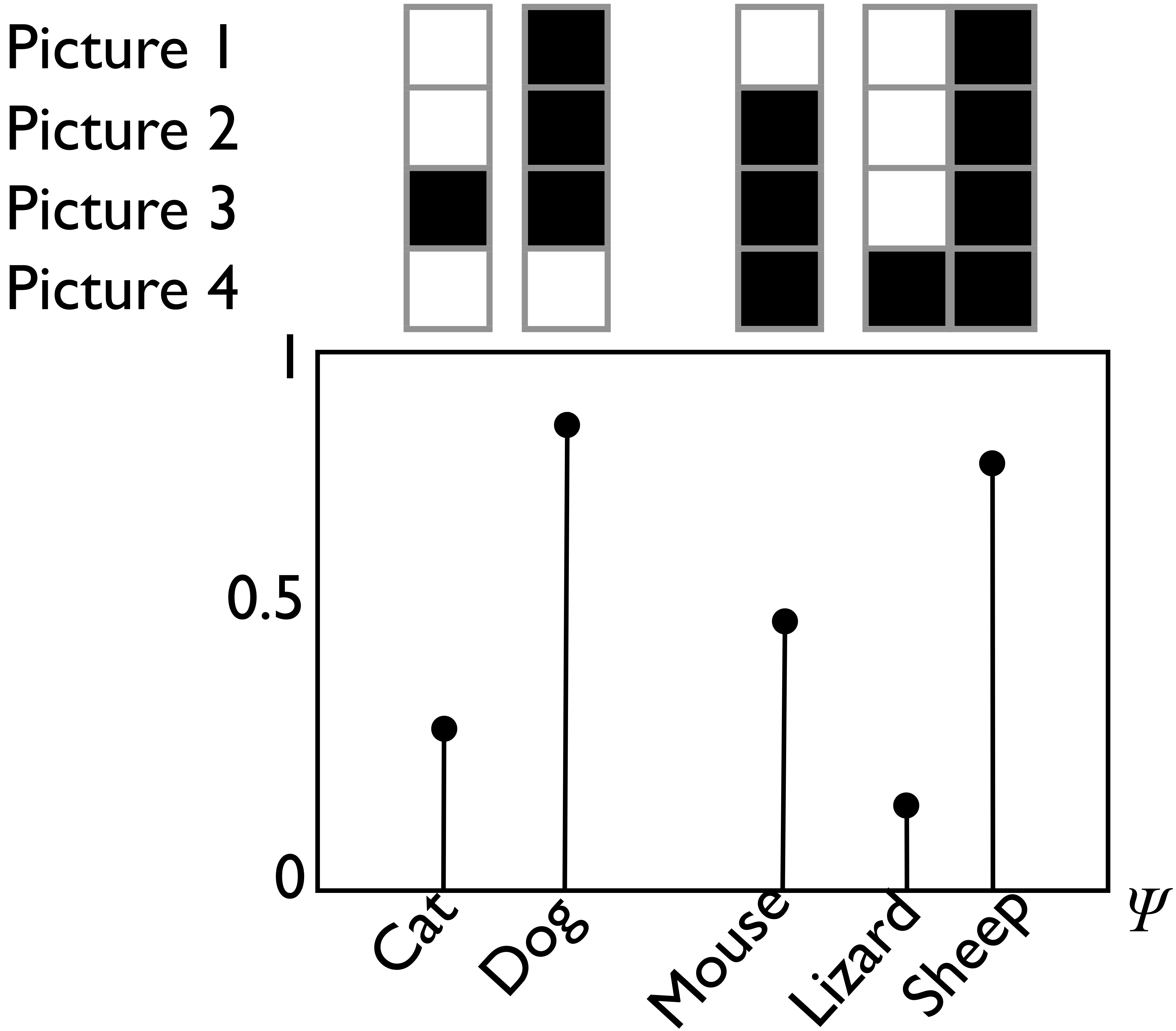

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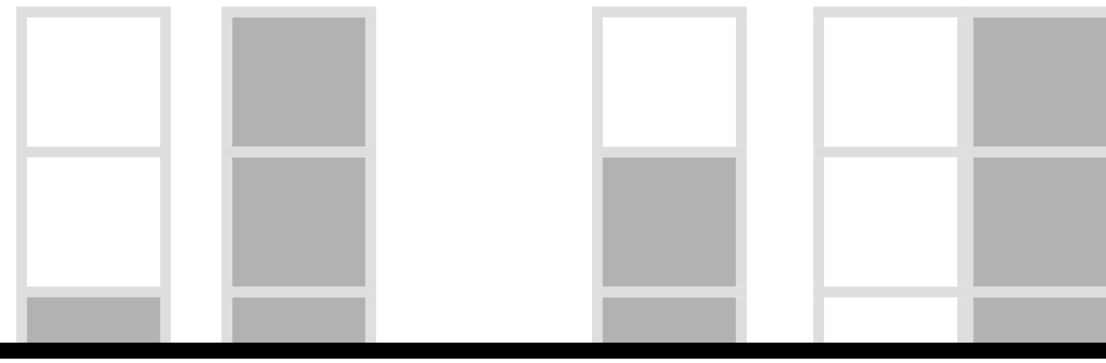
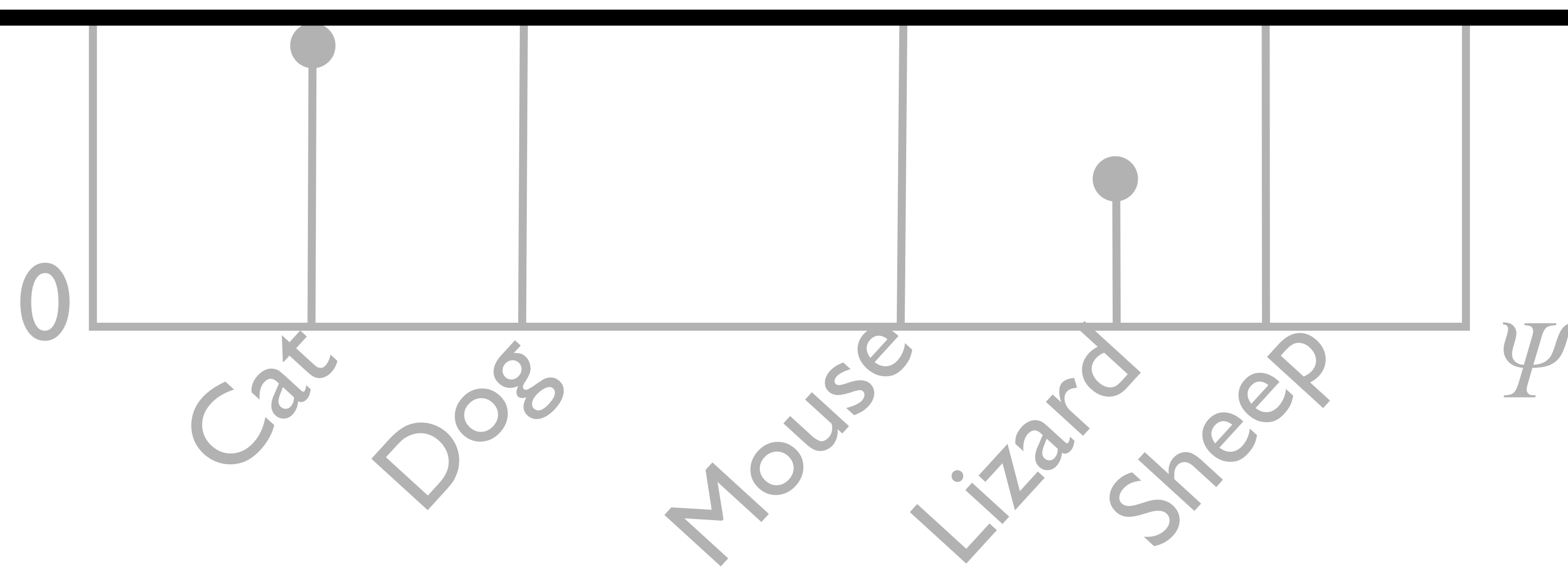


Want to show:

Posterior is a beta process

- No taking limits to infinite case
- Understandable
- Generalizable (other priors, likelihoods)



Poisson process

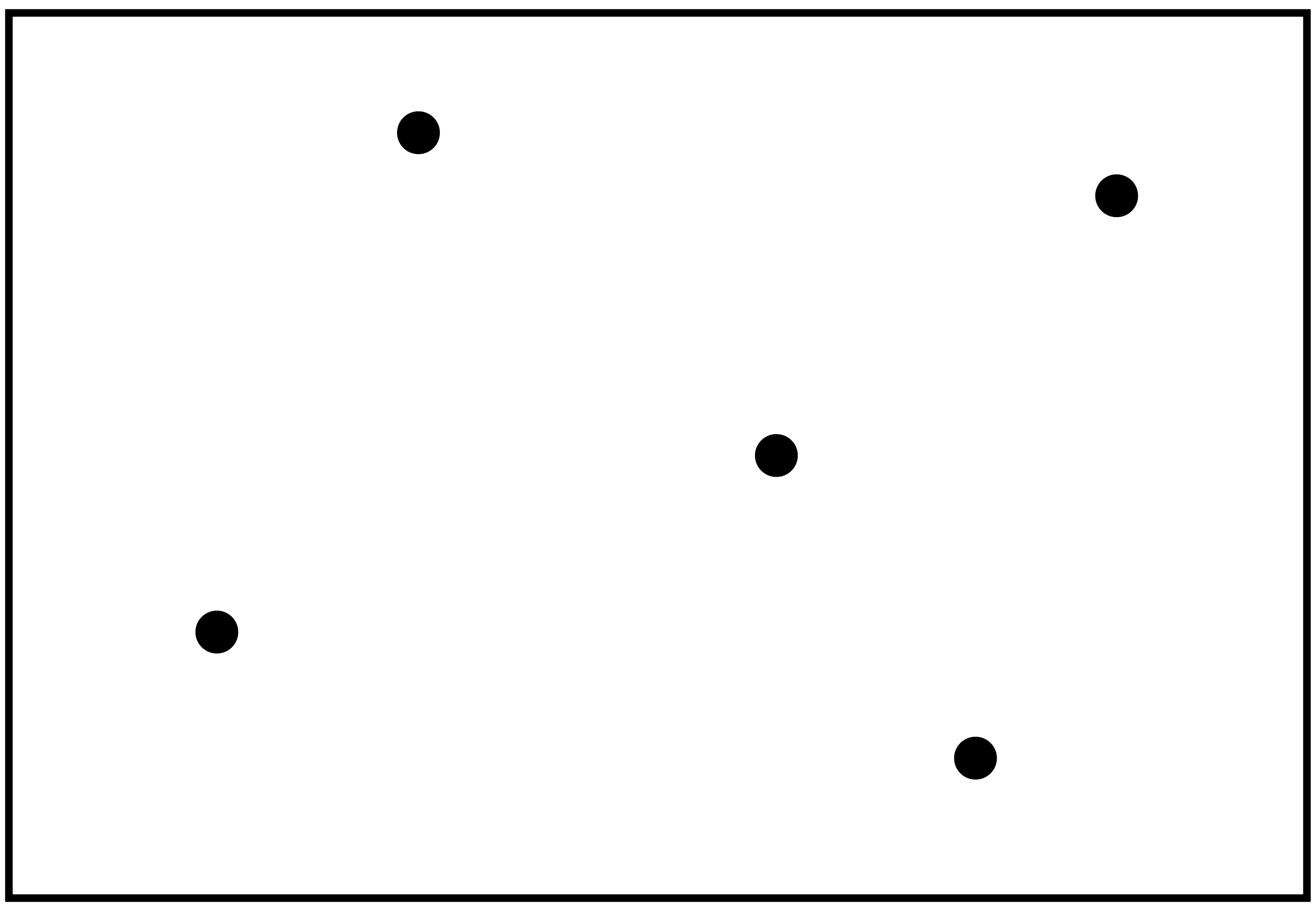
with mean measure ν

[Kingman 1993]

Poisson process

with mean measure ν

Random, countable set of points with:



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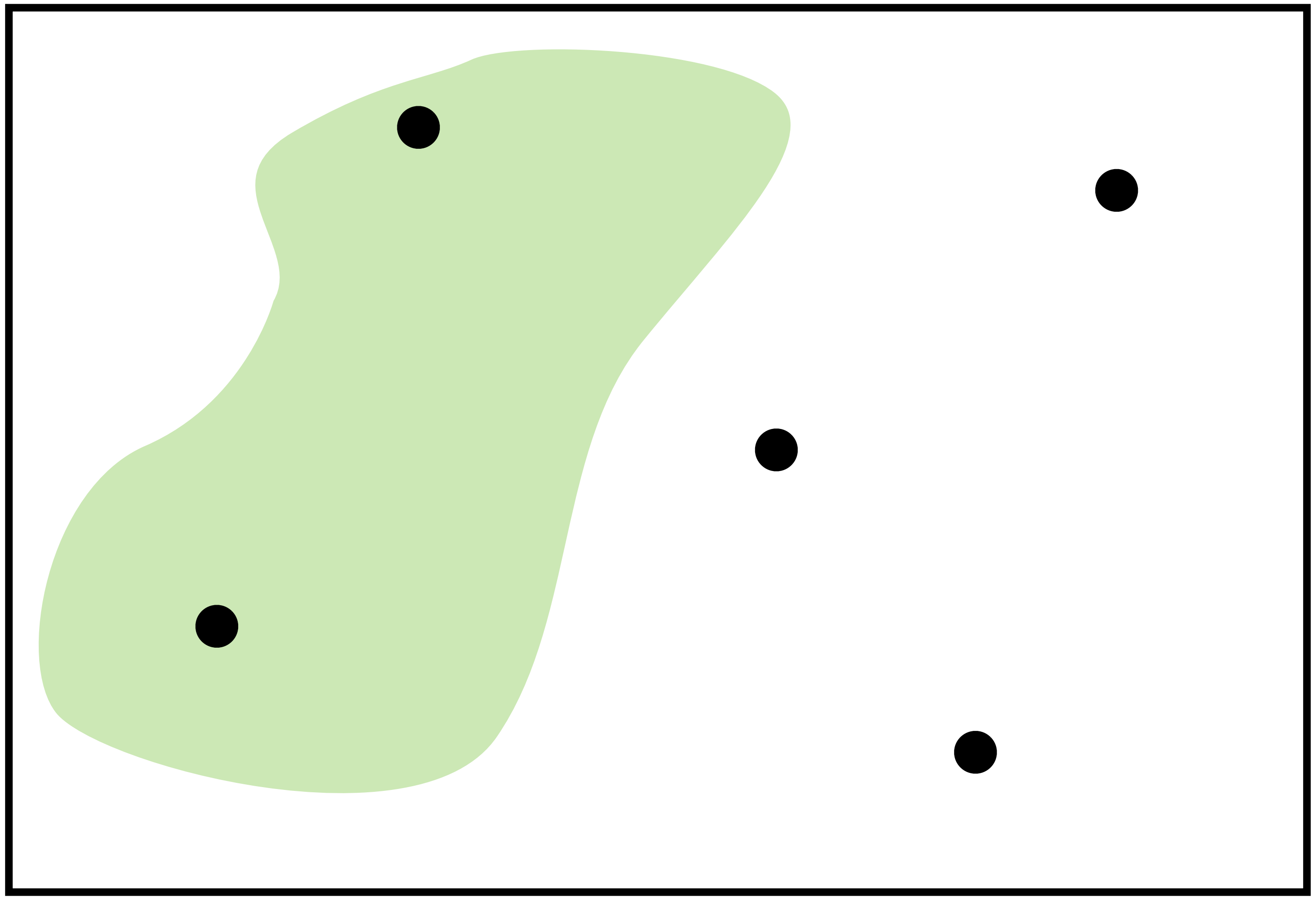
Poisson process
with mean measure ν

Random, countable set of points with:

- # points in set *A* is $\sim \mathrm{Poisson}[\nu(A)]$



[Kingman 1993]

Poisson process

with mean measure ν

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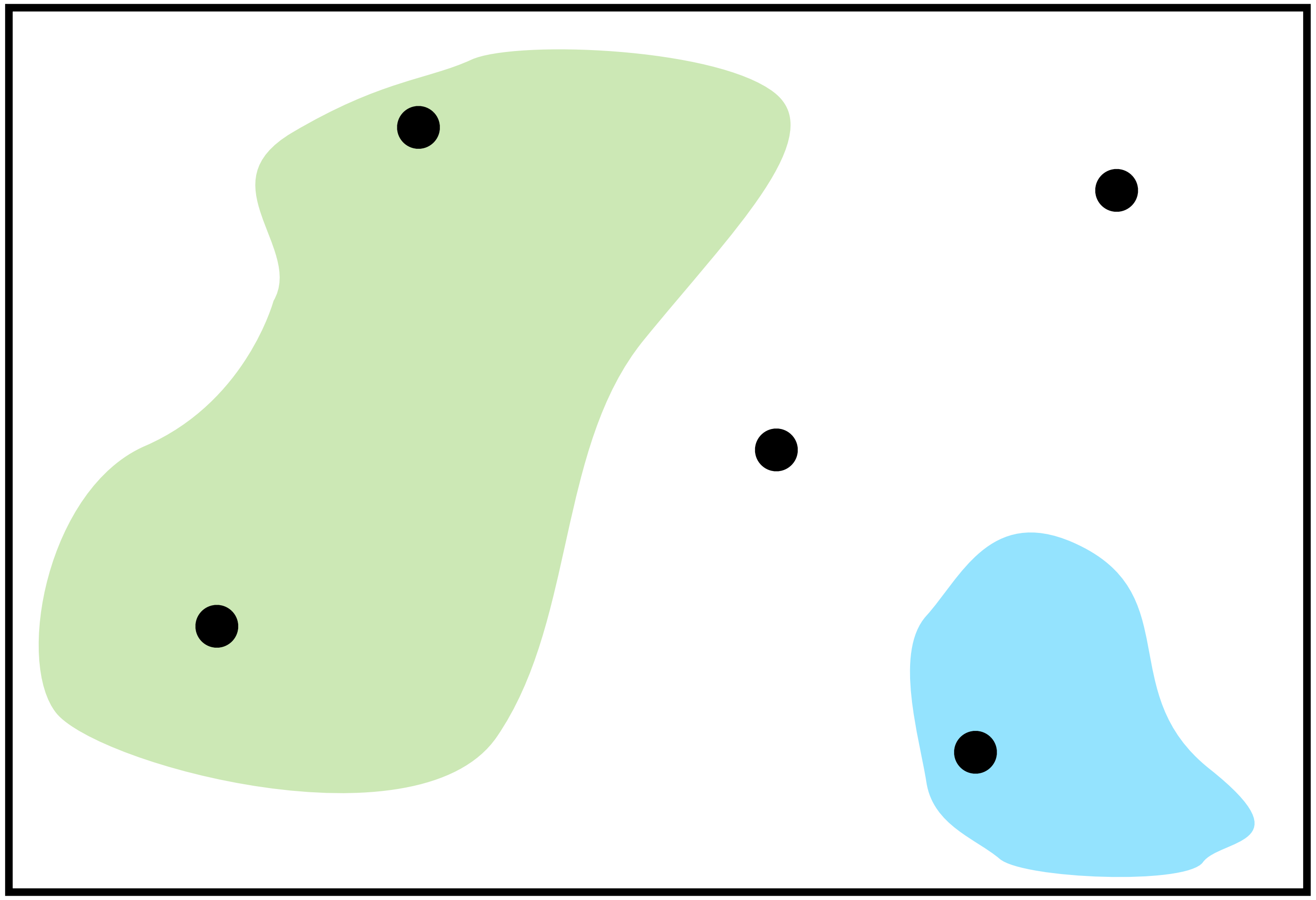
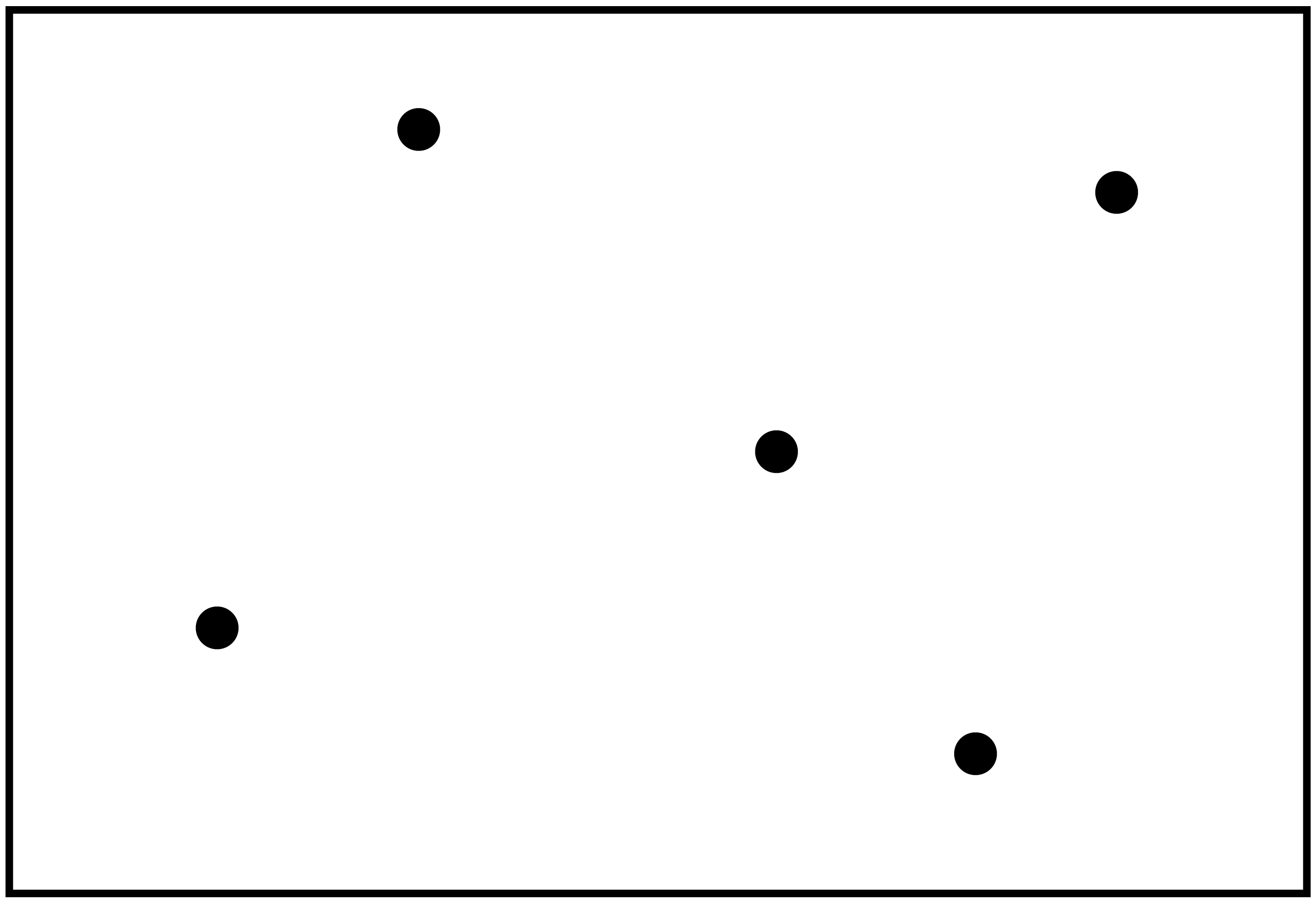
- Independent #s of points in disjoint sets



[Kingman 1993]

Poisson process fact: Thinning



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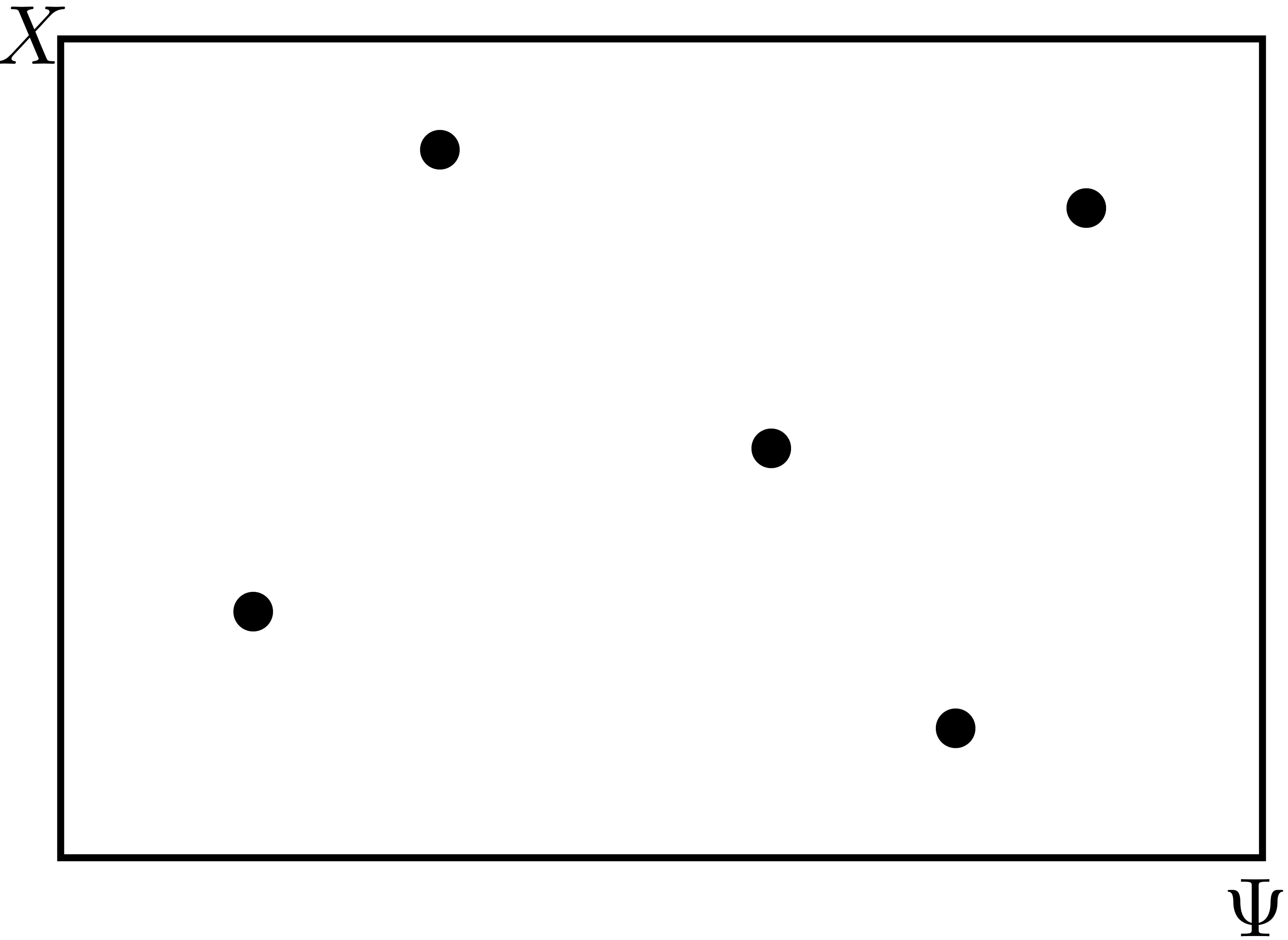

- Initial process has mean measure $\nu(dx, d\psi)$

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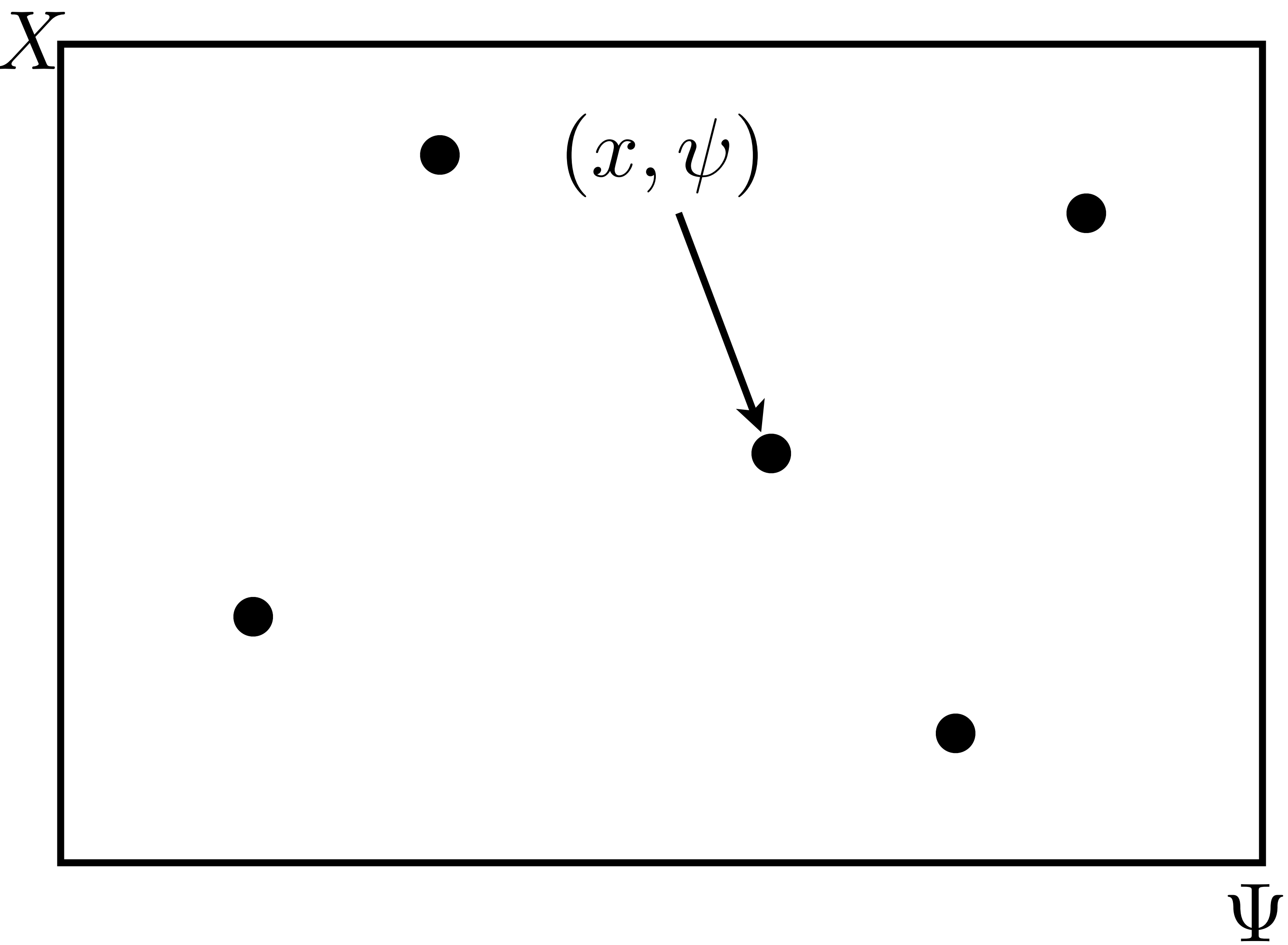

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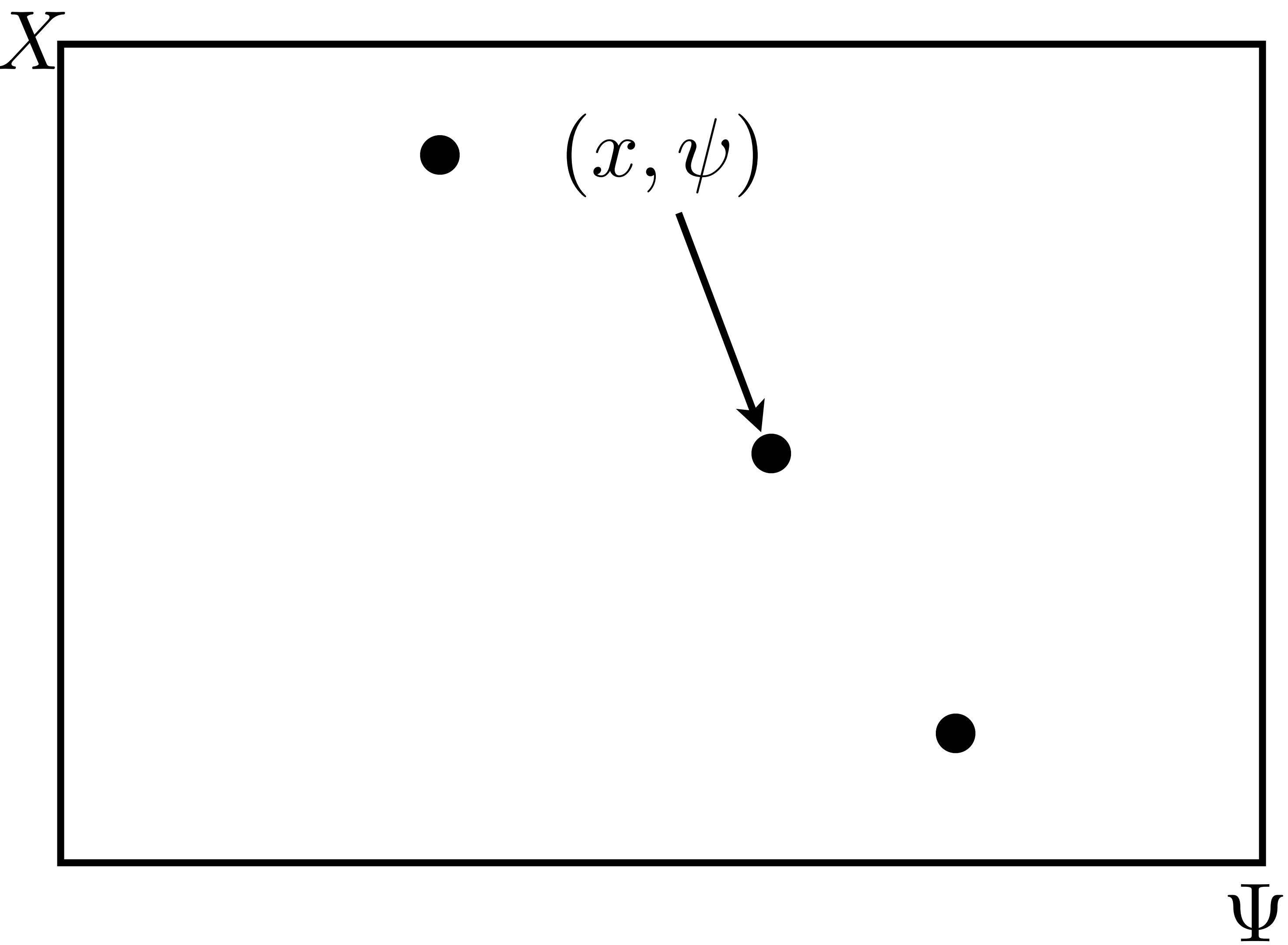

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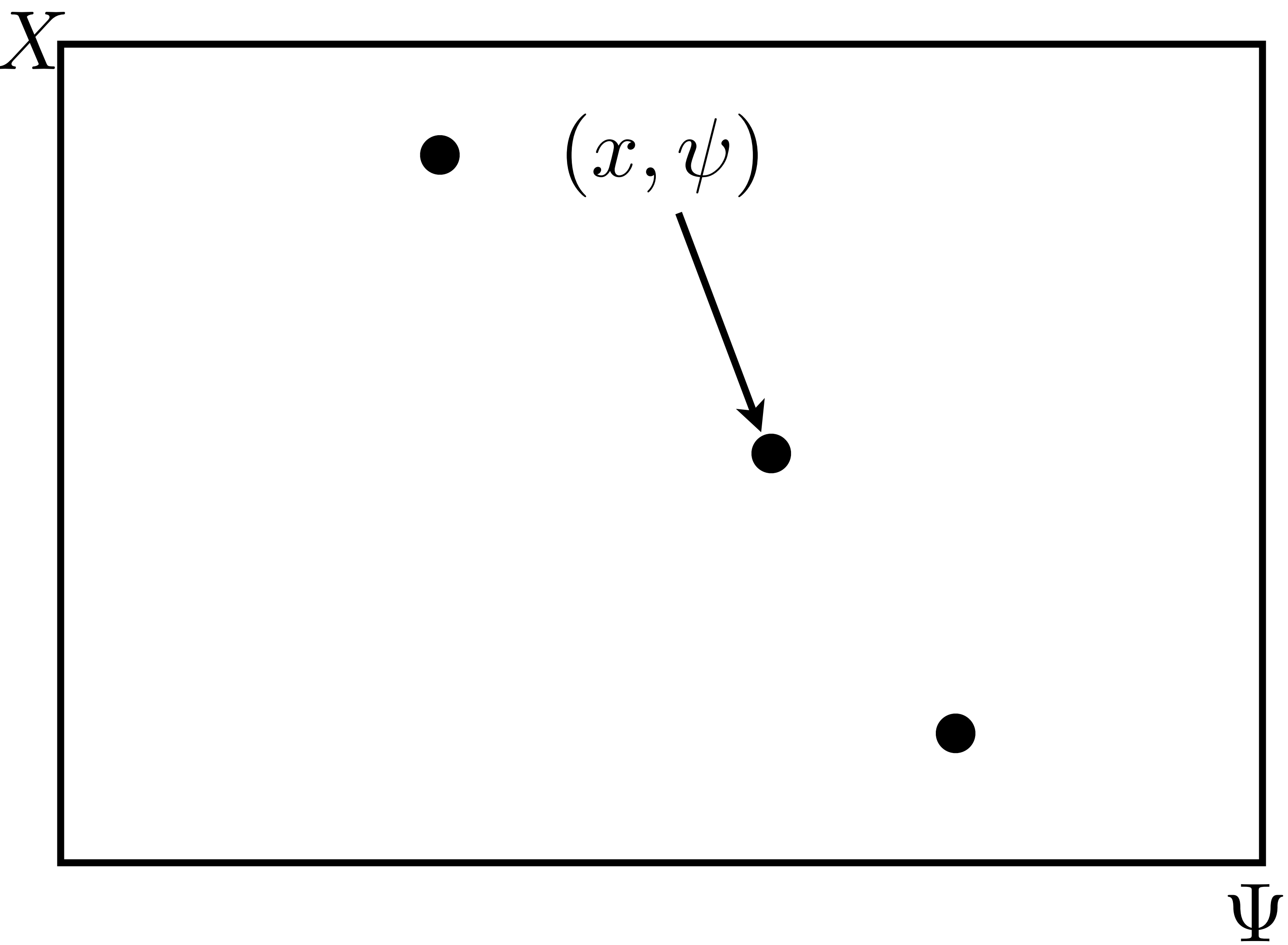

- Initial process has mean measure $\nu(dx, d\psi)$
- Keep point at (x, ψ) with probability $h(x, \psi)$
- Thinned process has mean measure

$$\nu_{thin}(A) = \int_{(x,\psi)\in A} \nu(dx, d\psi)\; h(x, \psi)$$

Beta process

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Two parts:

Beta process

Two parts:

- Poisson process component

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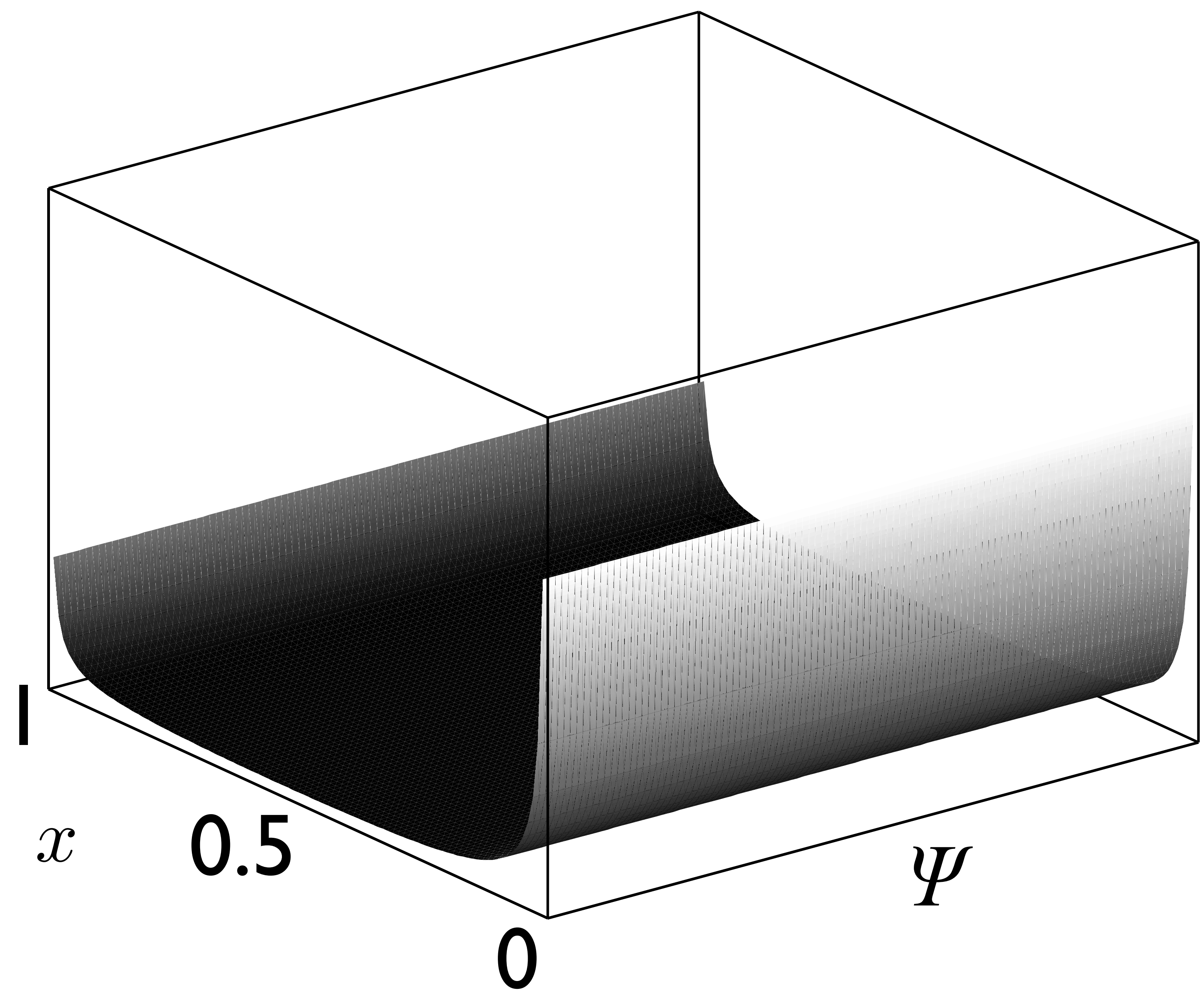
$$\nu(dx, d\psi) = \gamma\theta x^{-1}(1-x)^{\theta-1}\, dx\, d\psi$$



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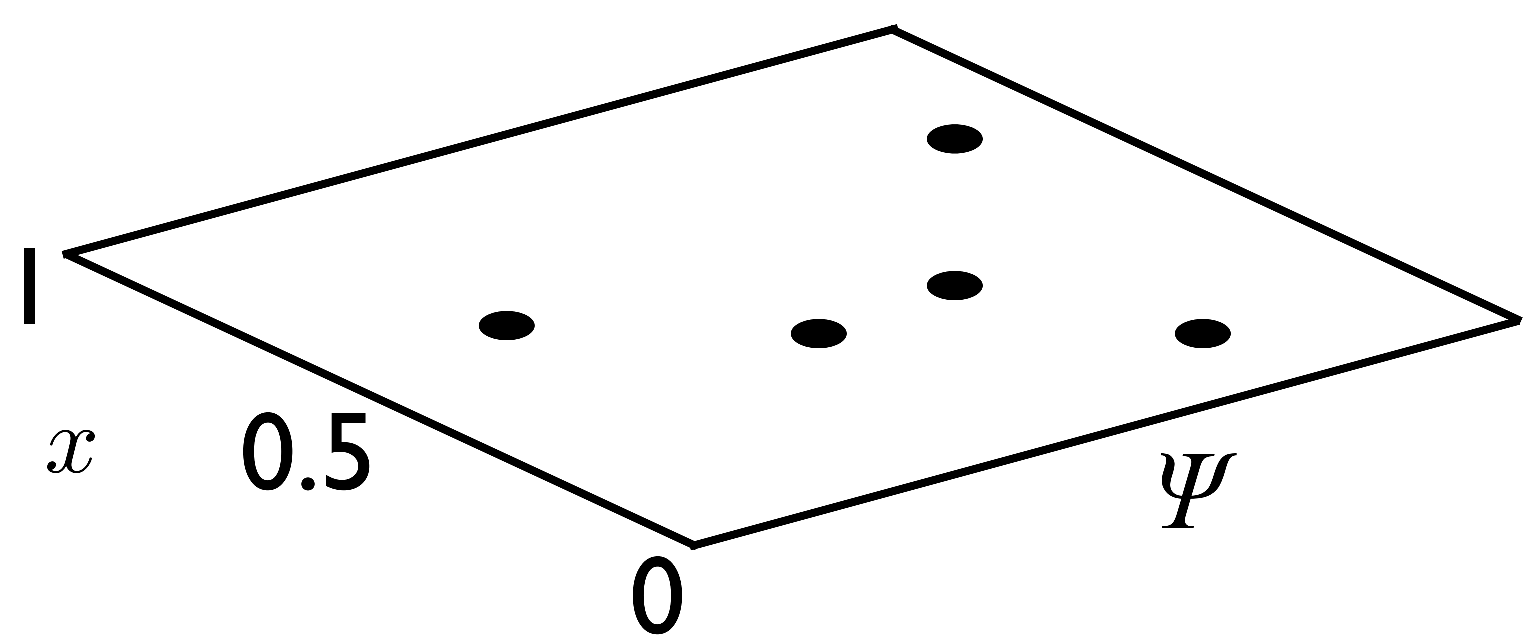
$$\nu(dx, d\psi) = \gamma\theta x^{-1}(1-x)^{\theta-1}\, dx\, d\psi$$



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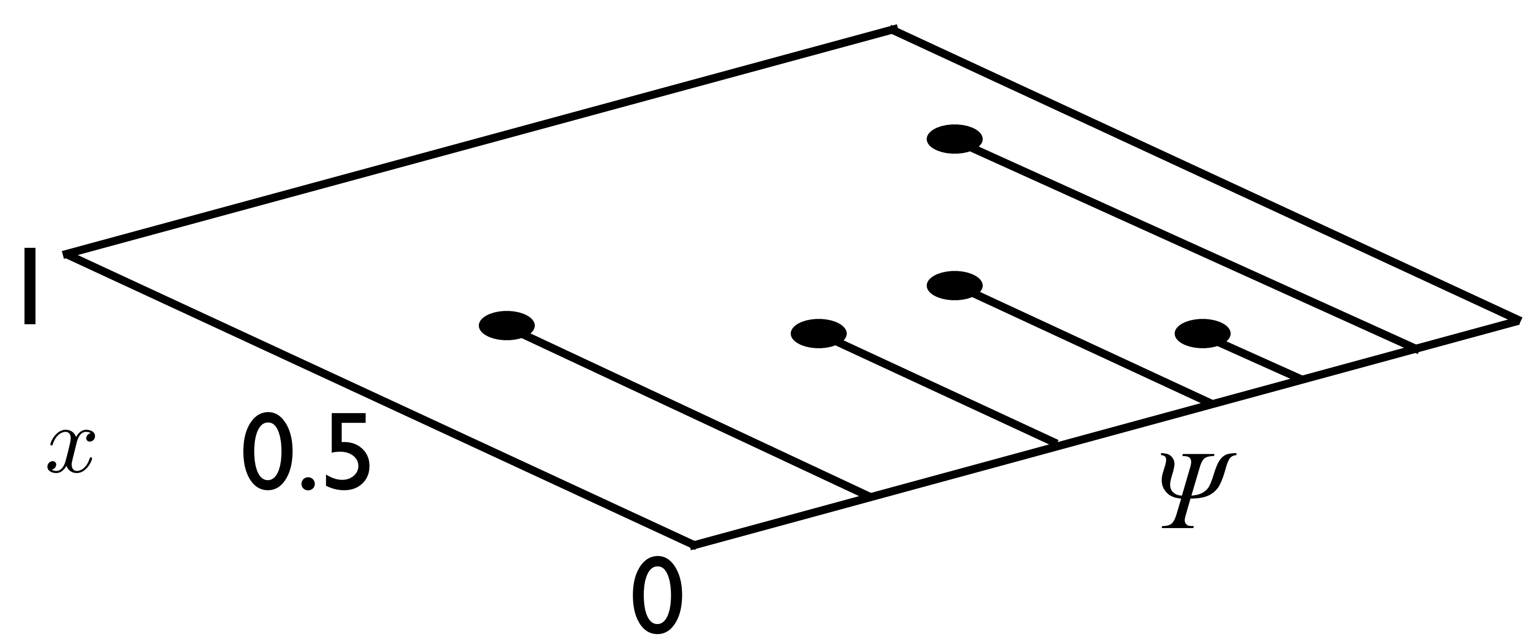

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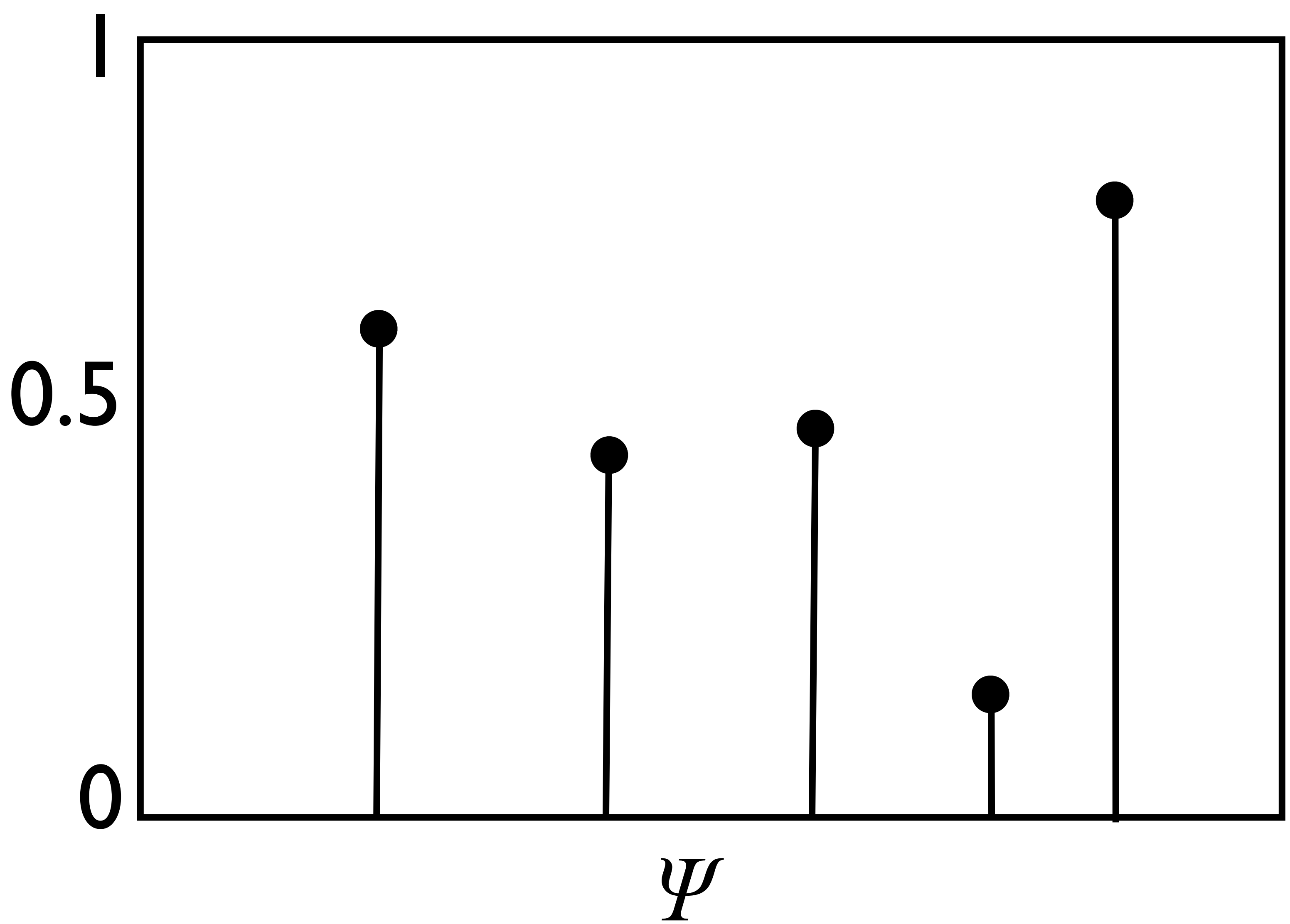

Beta process

Two parts:

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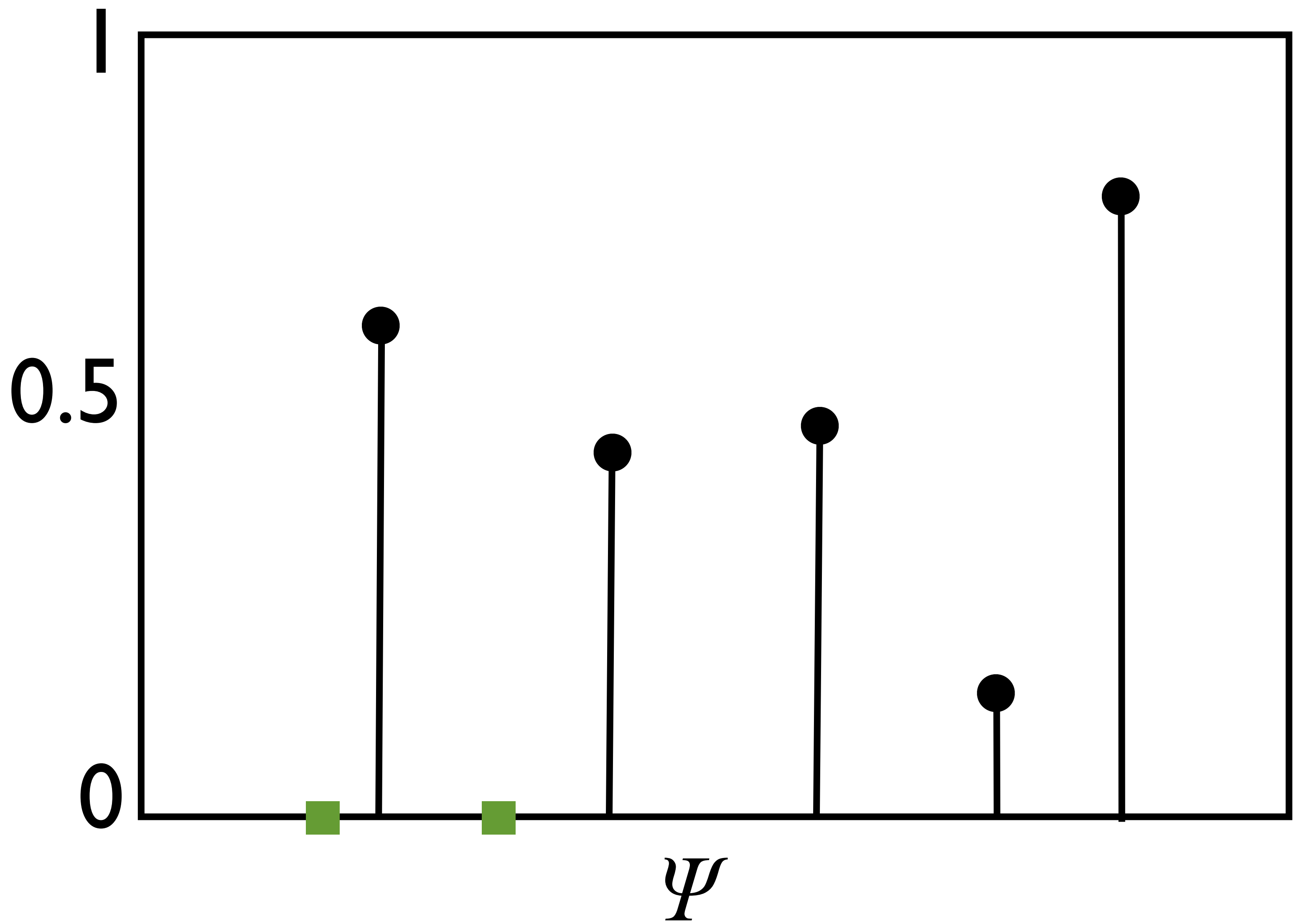
$$\nu(dx, d\psi) = \gamma\theta x^{-1}(1-x)^{\theta-1}\, dx\, d\psi$$

- Fixed atoms



Beta process

Two parts:

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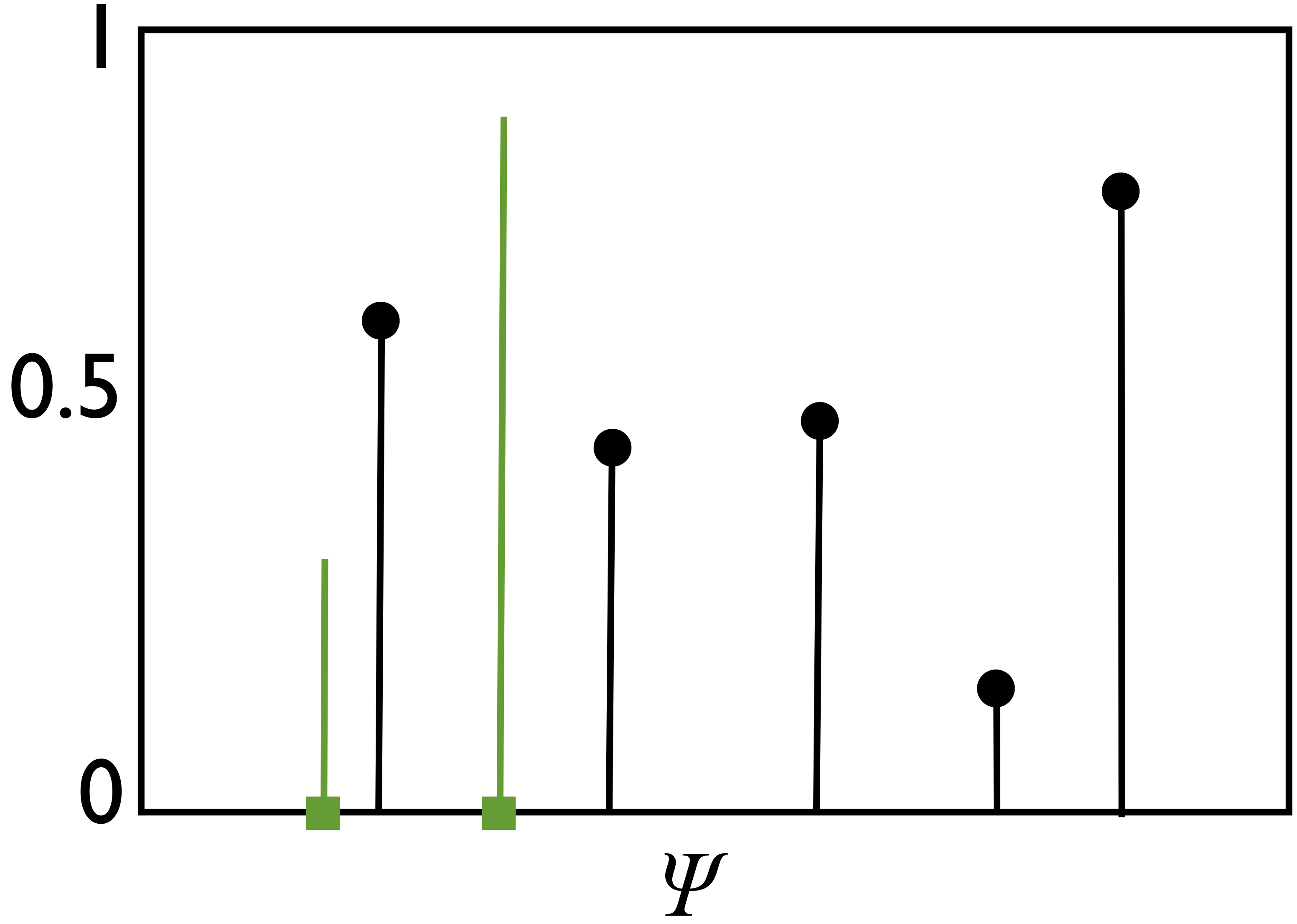
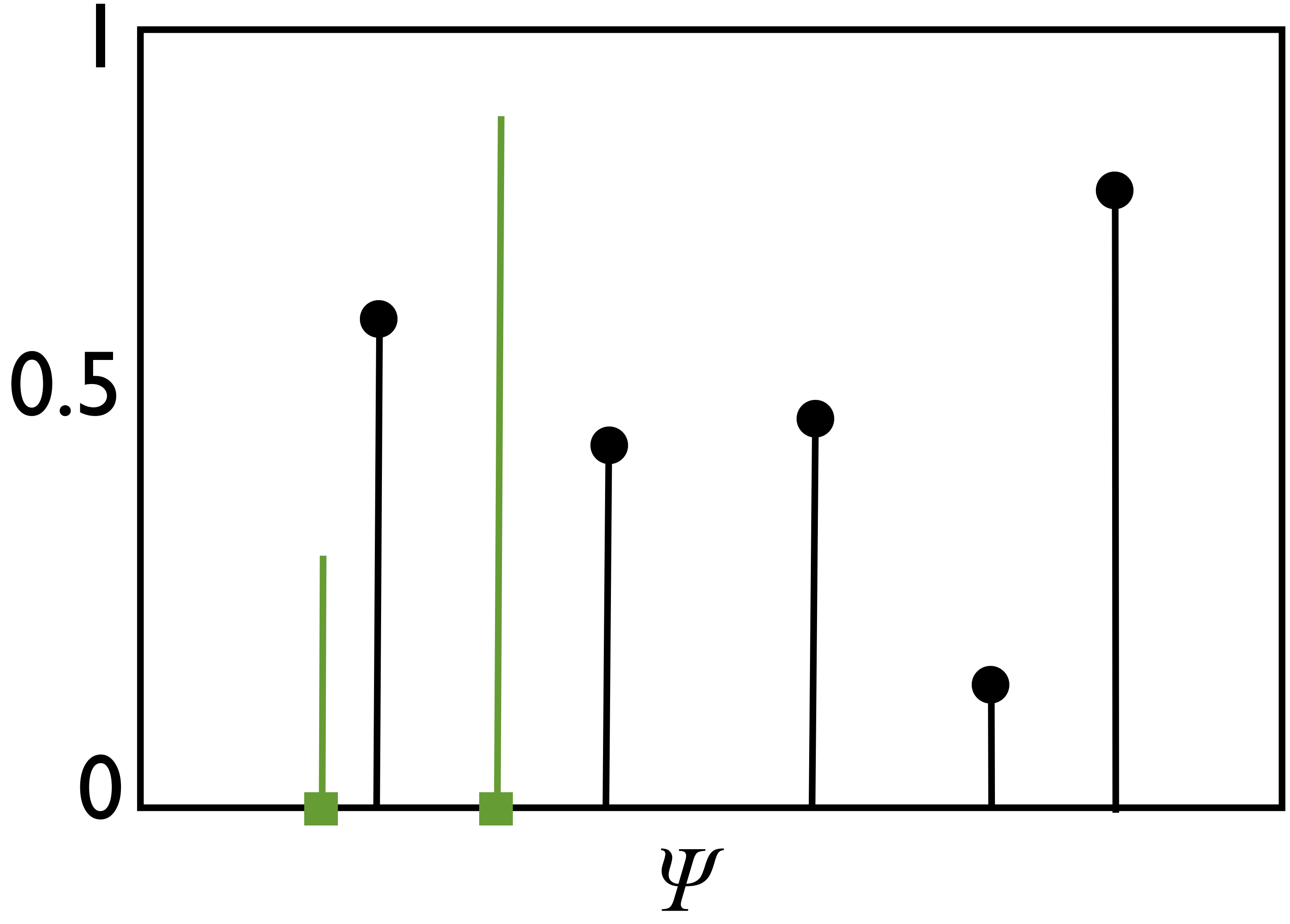
$$\nu(dx, d\psi) = \gamma\theta x^{-1}(1-x)^{\theta-1}\, dx\, d\psi$$

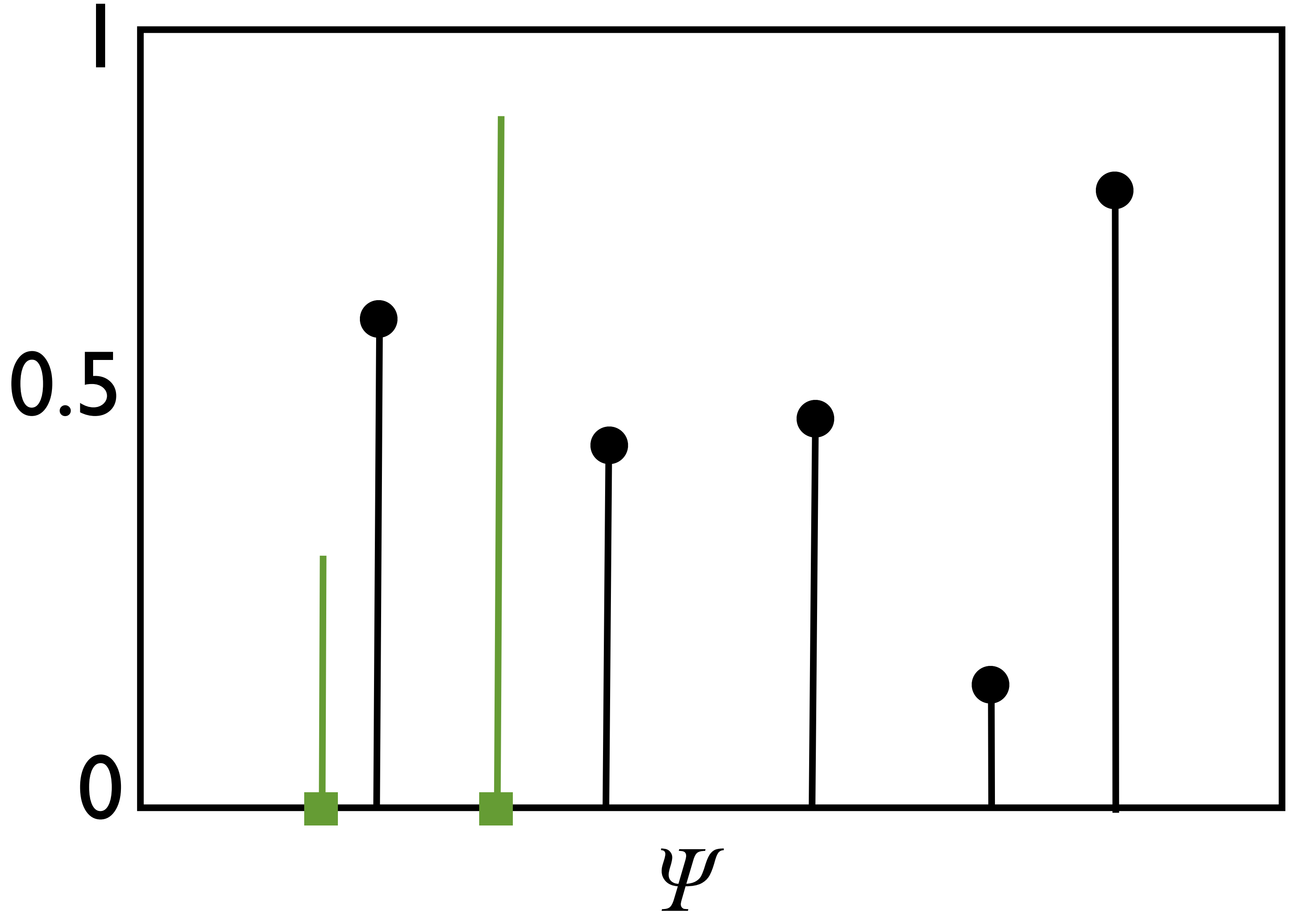
- Fixed atoms $x'_j \sim \mathrm{Beta}(a_j, b_j)$ at ψ'_j



- Prior: beta process



- Prior: beta process
- Likelihood: Bernoulli process



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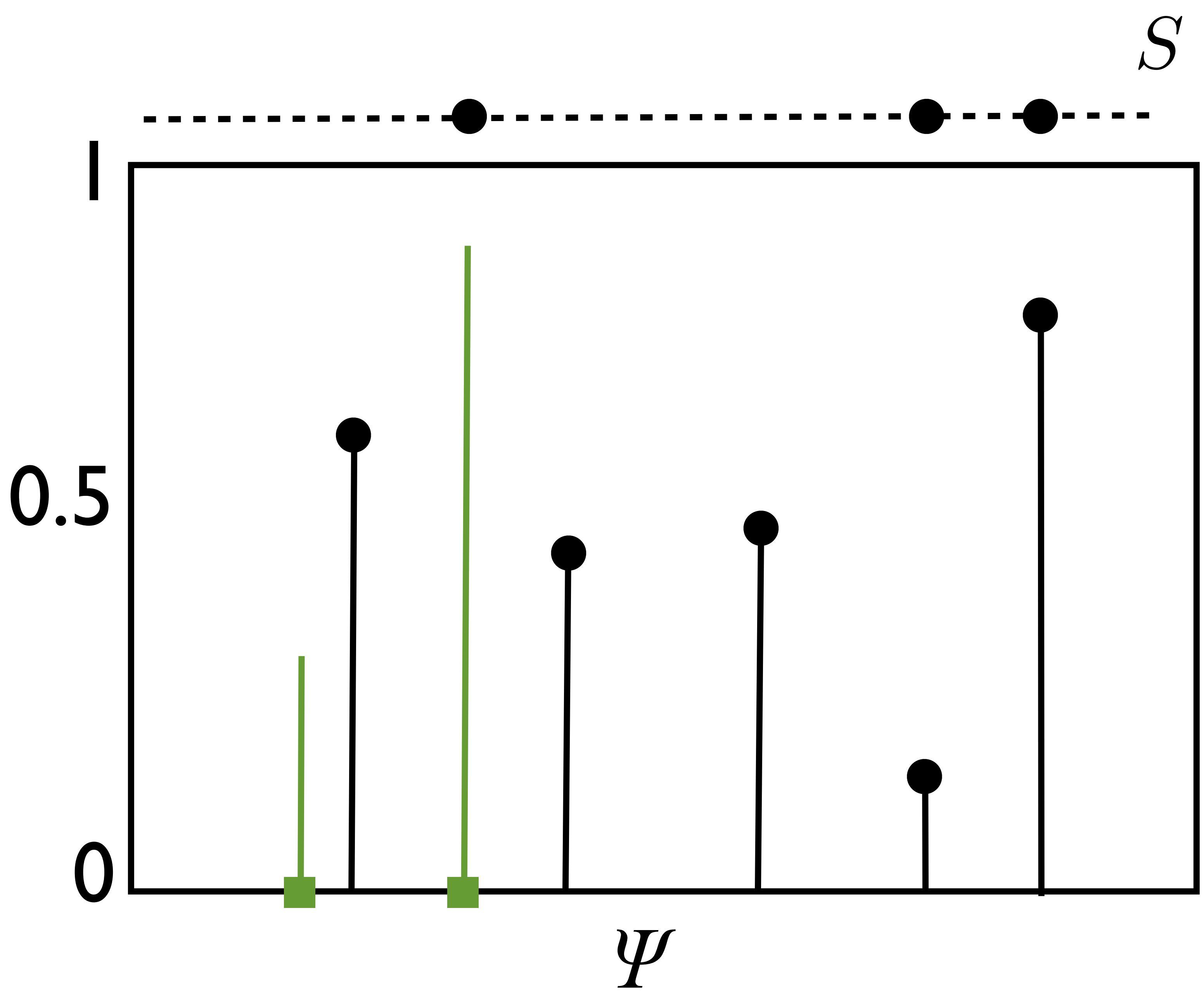

Posterior

- Prior: beta process
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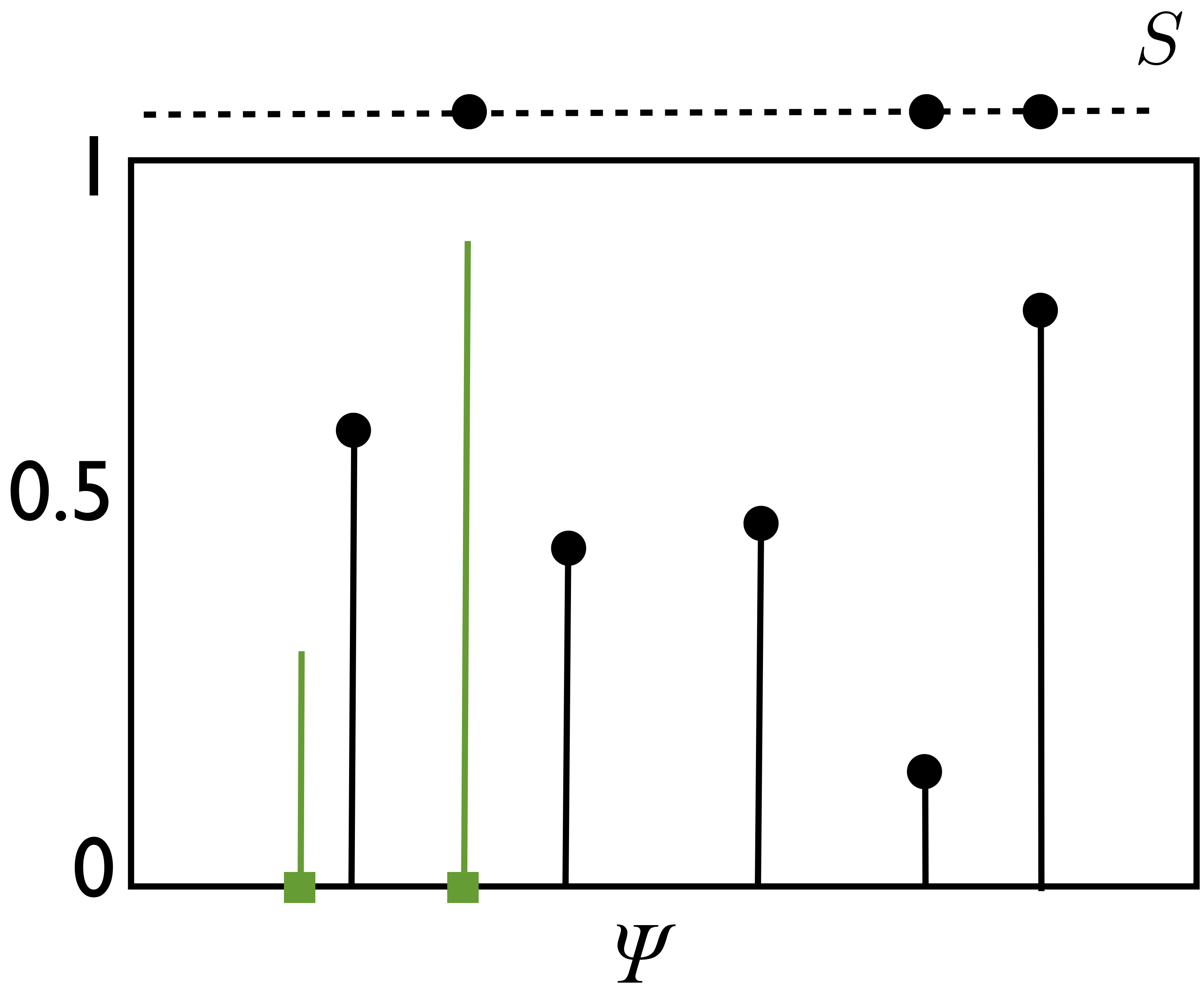

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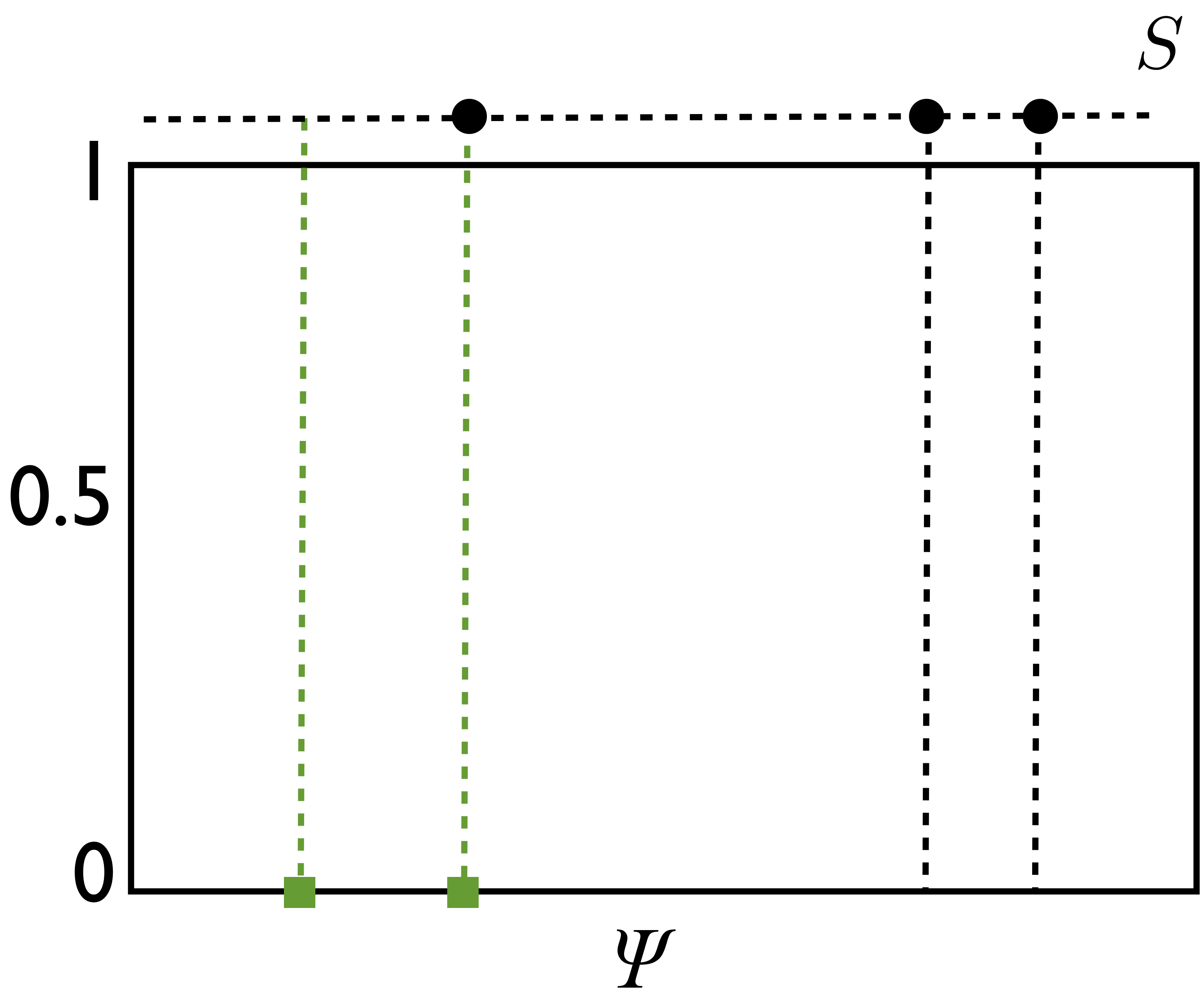

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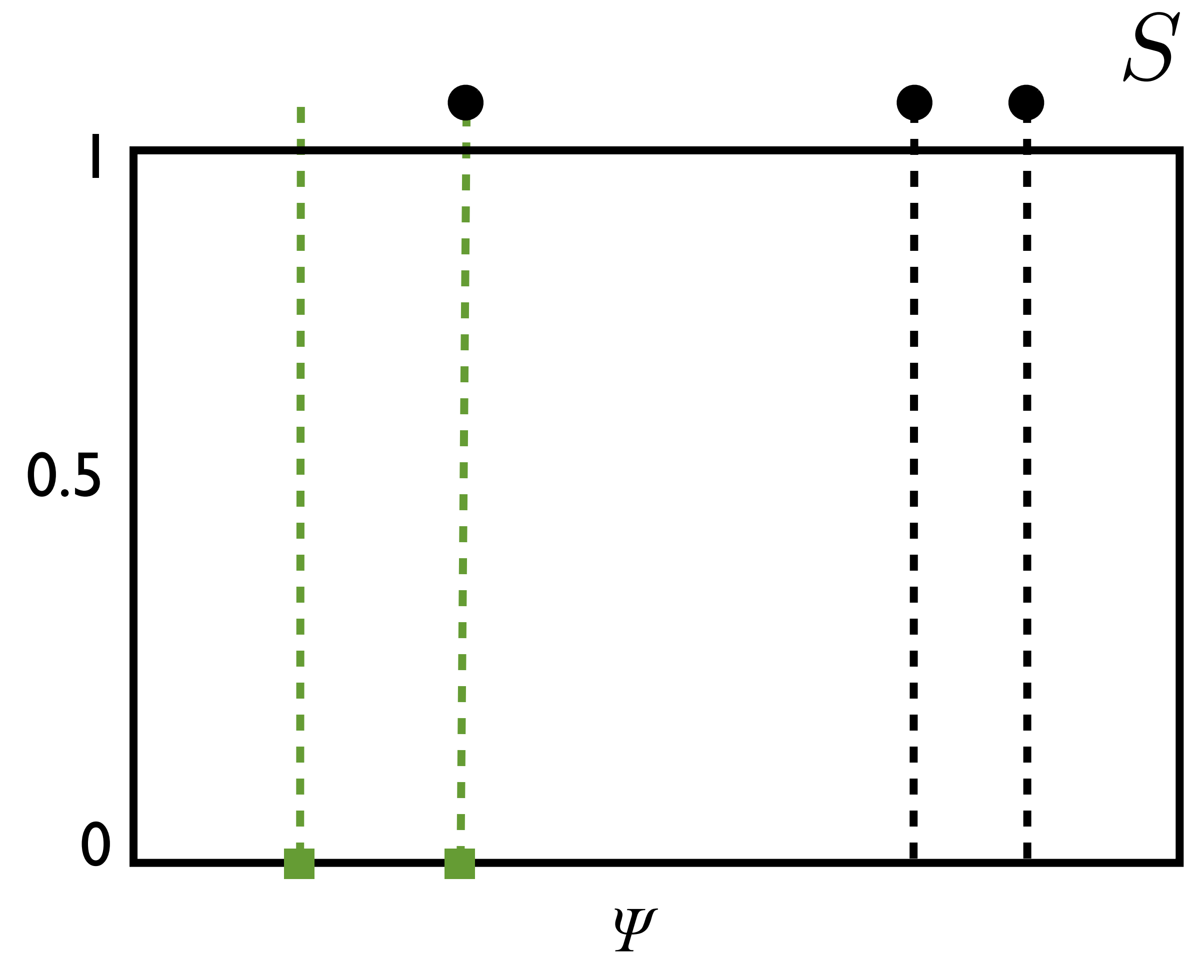

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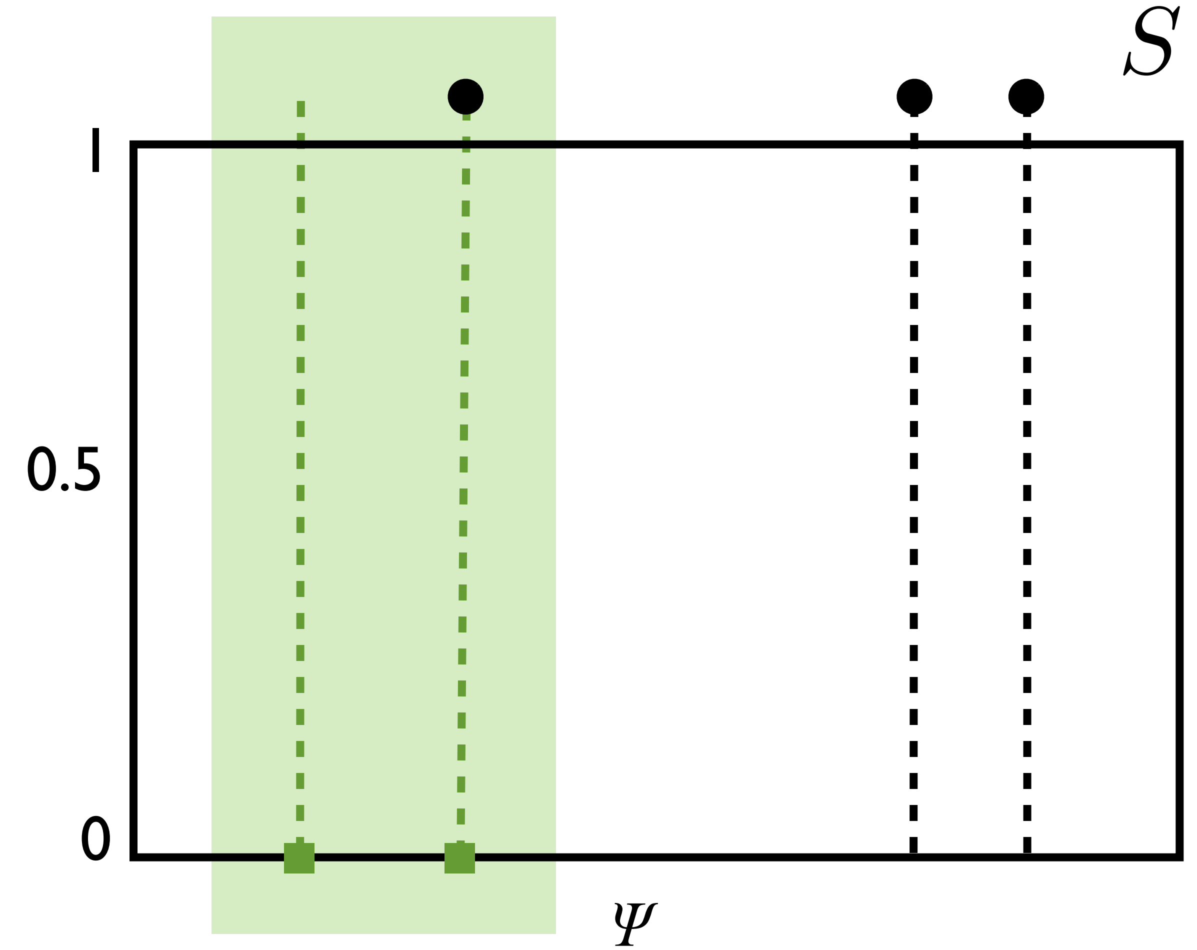

- Old fixed atom at ψ_j

Posterior

- Prior: beta process
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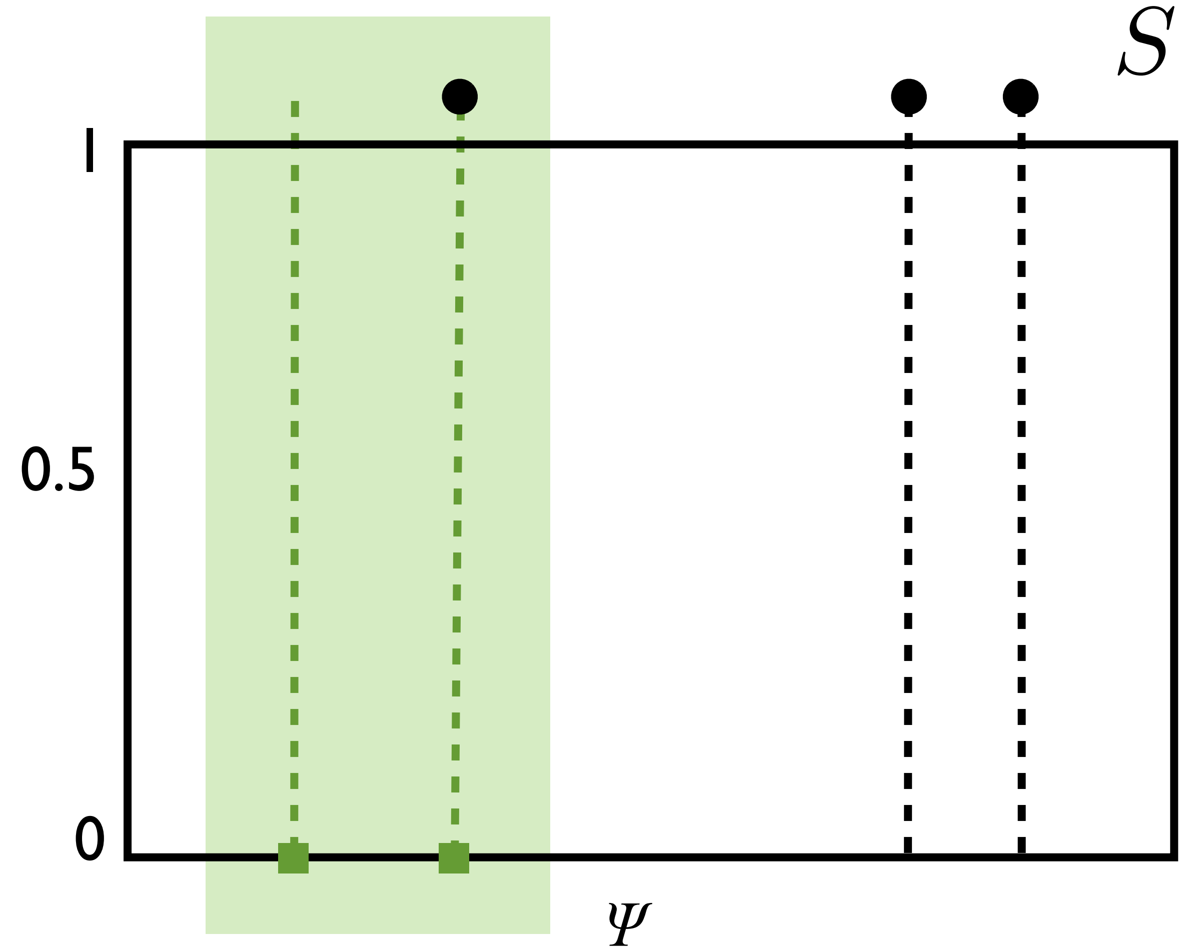

- Old fixed atom at ψ_j

$\mathrm{Beta}(x_j | a_j, b_j)$

Posterior

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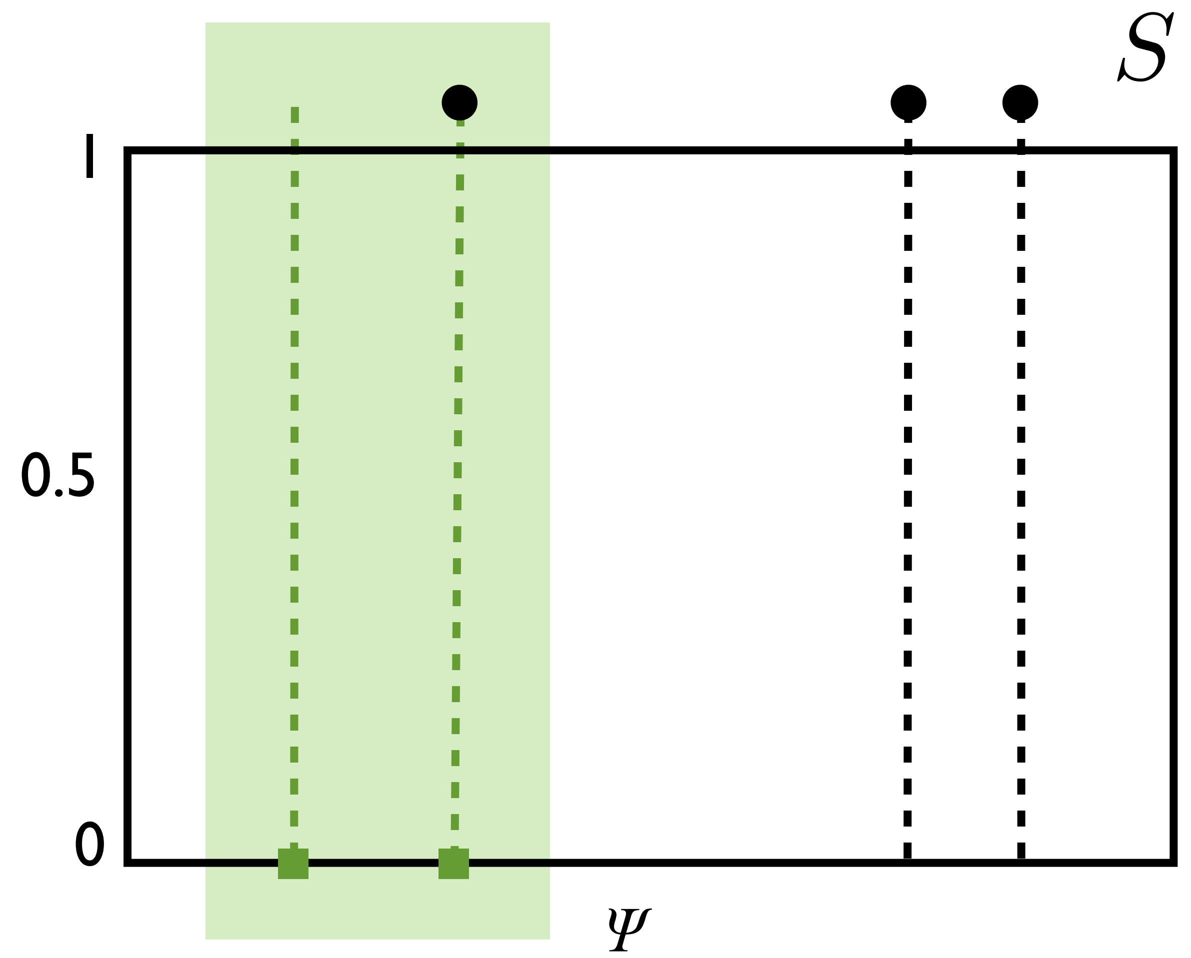

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$$\mathrm{Beta}(x_j|a_j, b_j) \Rightarrow \mathrm{Beta}\left(x_j|a_j + \mathbf{1}\{\psi_j \in S\}, b_j + \mathbf{1}\{\psi_j \notin S\}\right)$$

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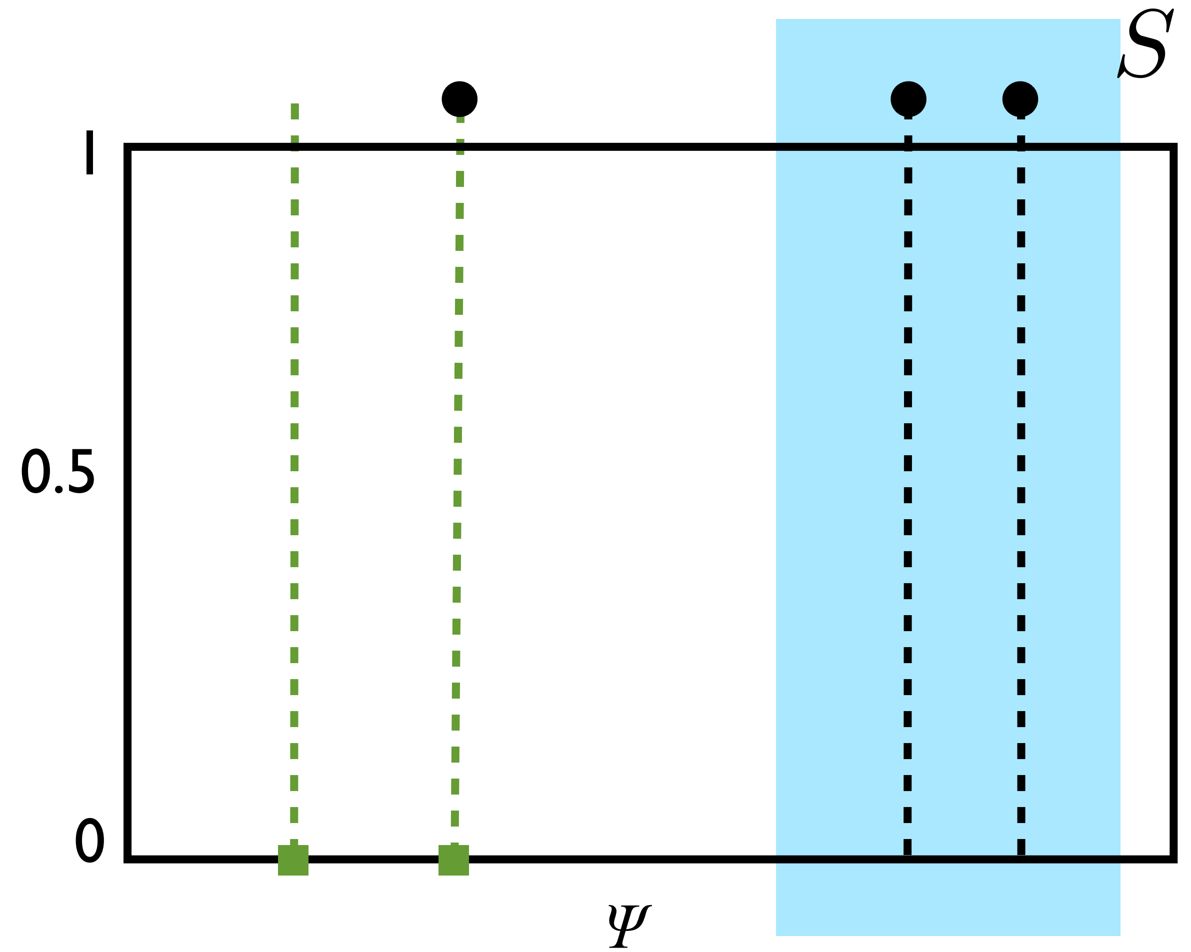

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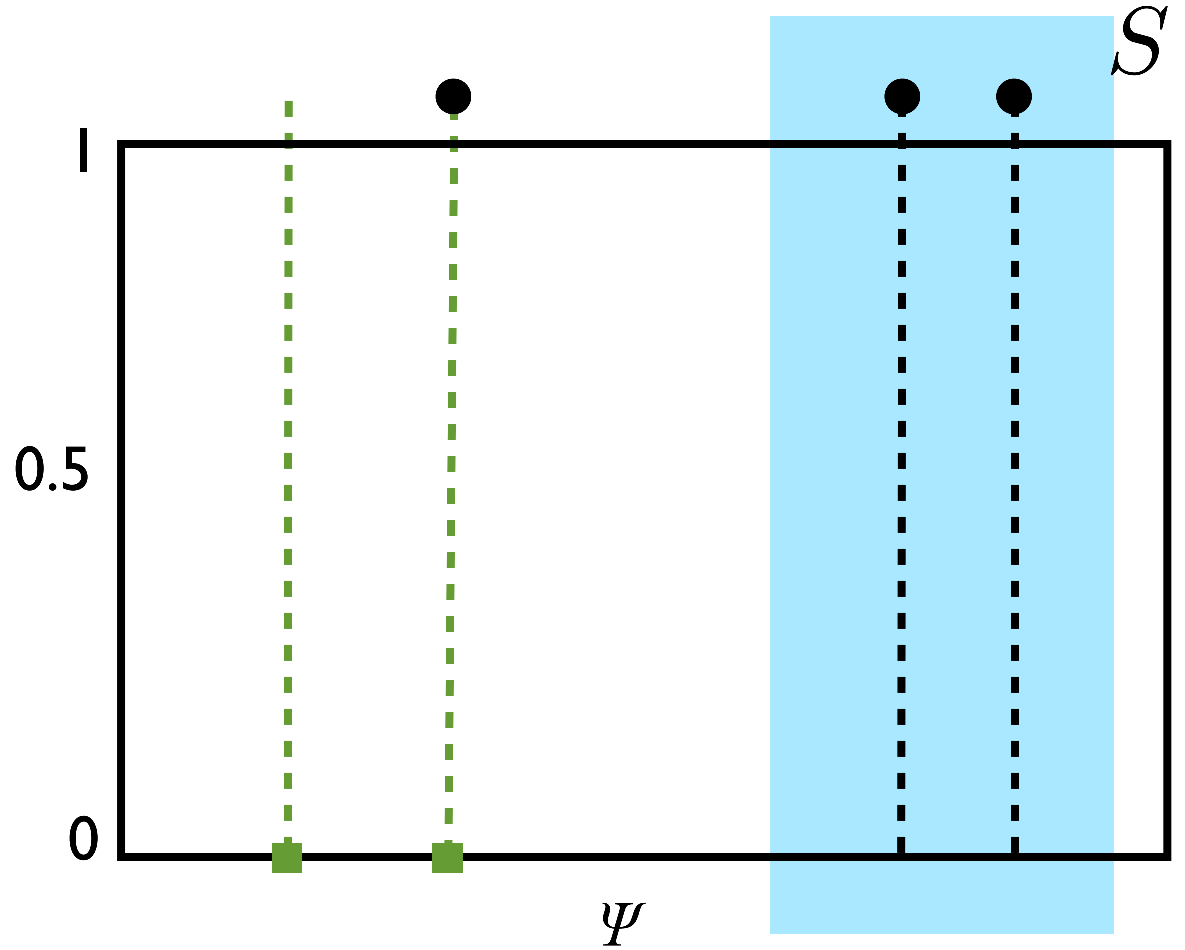

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- New fixed atom at ψ

$$\gamma \theta x^{-1} (1-x)^{\theta - 1} \, dx$$

Posterior

- Prior: beta process
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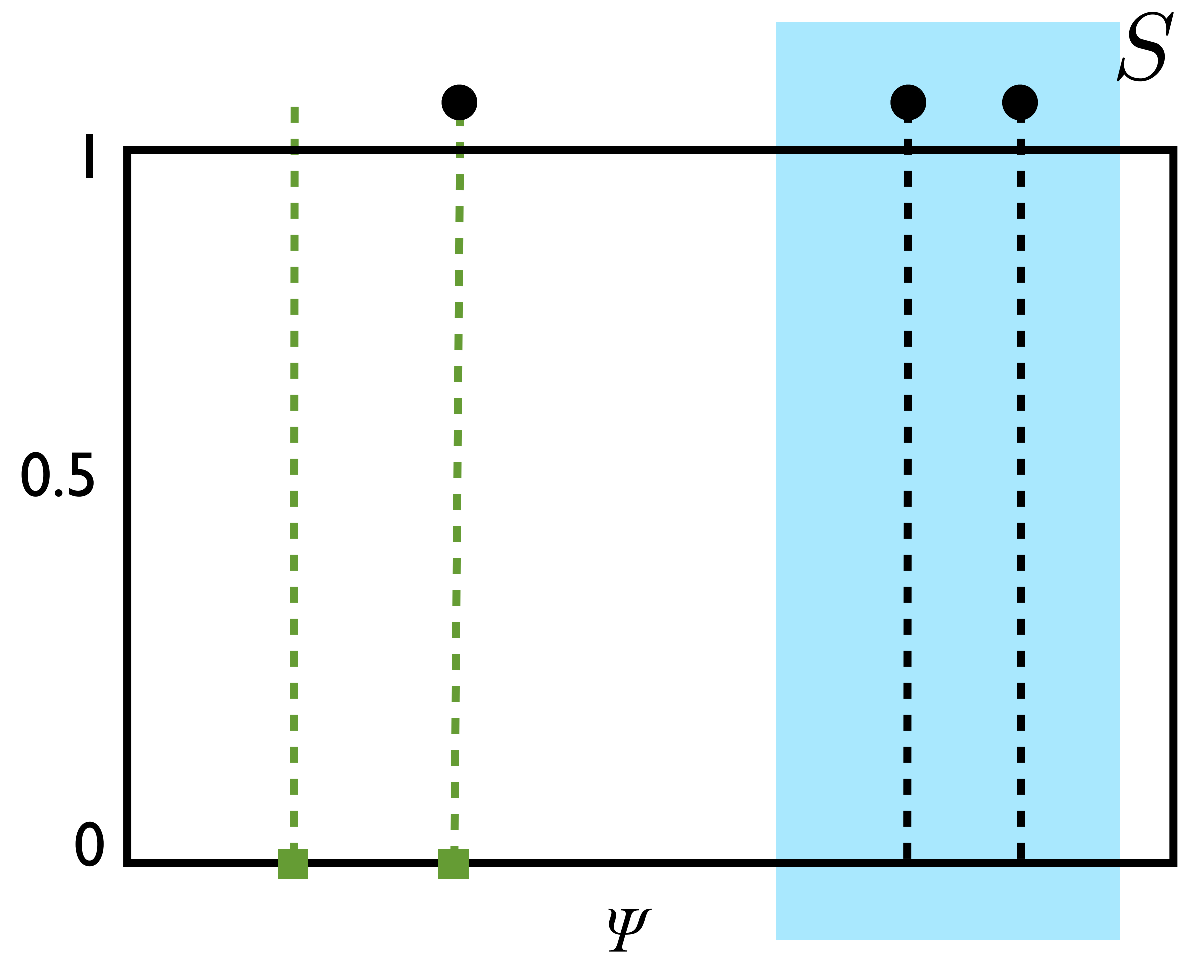

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$$\gamma\theta x^{-1}(1-x)^{\theta-1}\,dx \Rightarrow \qquad \propto x \cdot \gamma\theta x^{-1}(1-x)^{\theta-1}dx$$

Posterior

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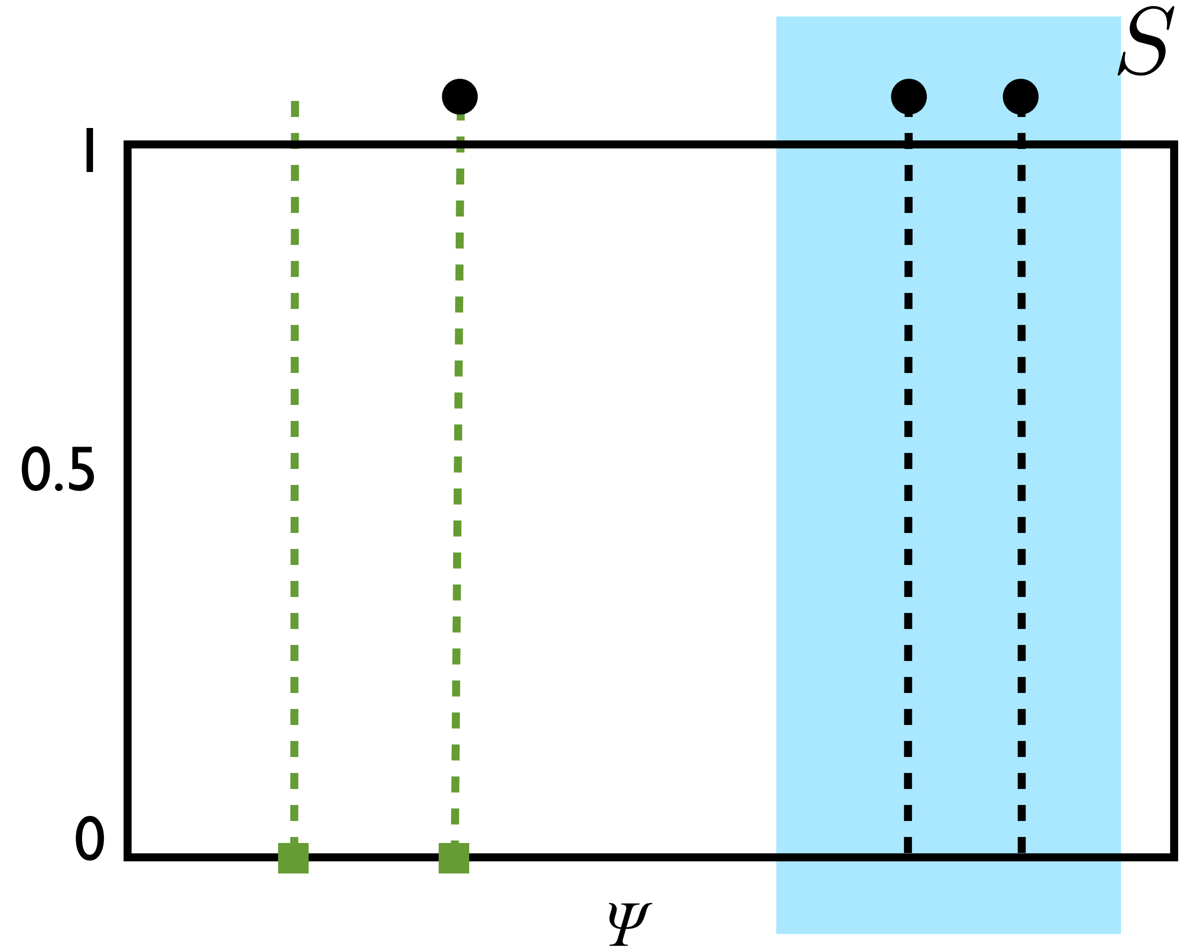

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- New fixed atom at ψ

$$\gamma\theta x^{-1}(1-x)^{\theta-1}\,dx \Rightarrow \mathrm{Beta}(x|1,\theta) \propto x \cdot \gamma\theta x^{-1}(1-x)^{\theta-1}dx$$

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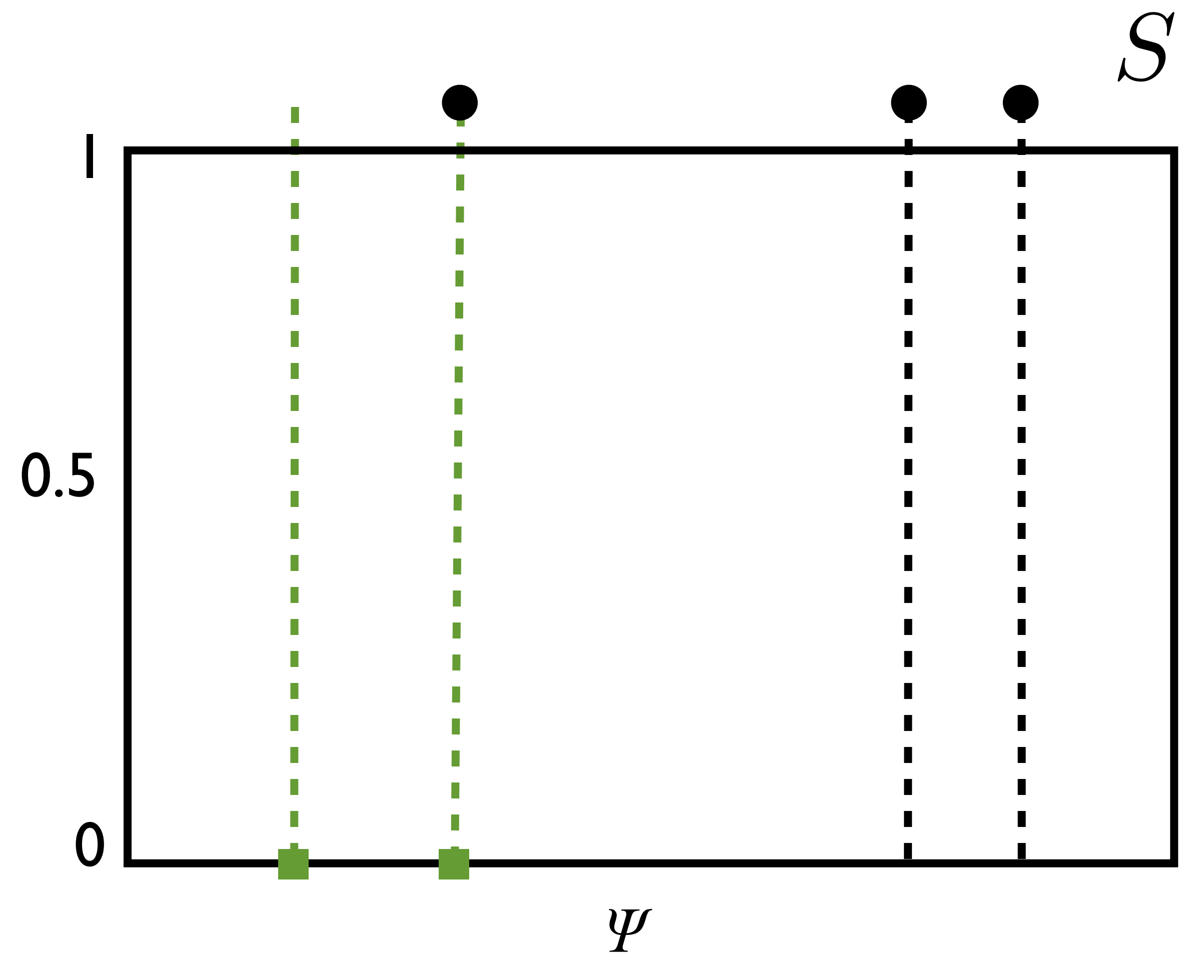

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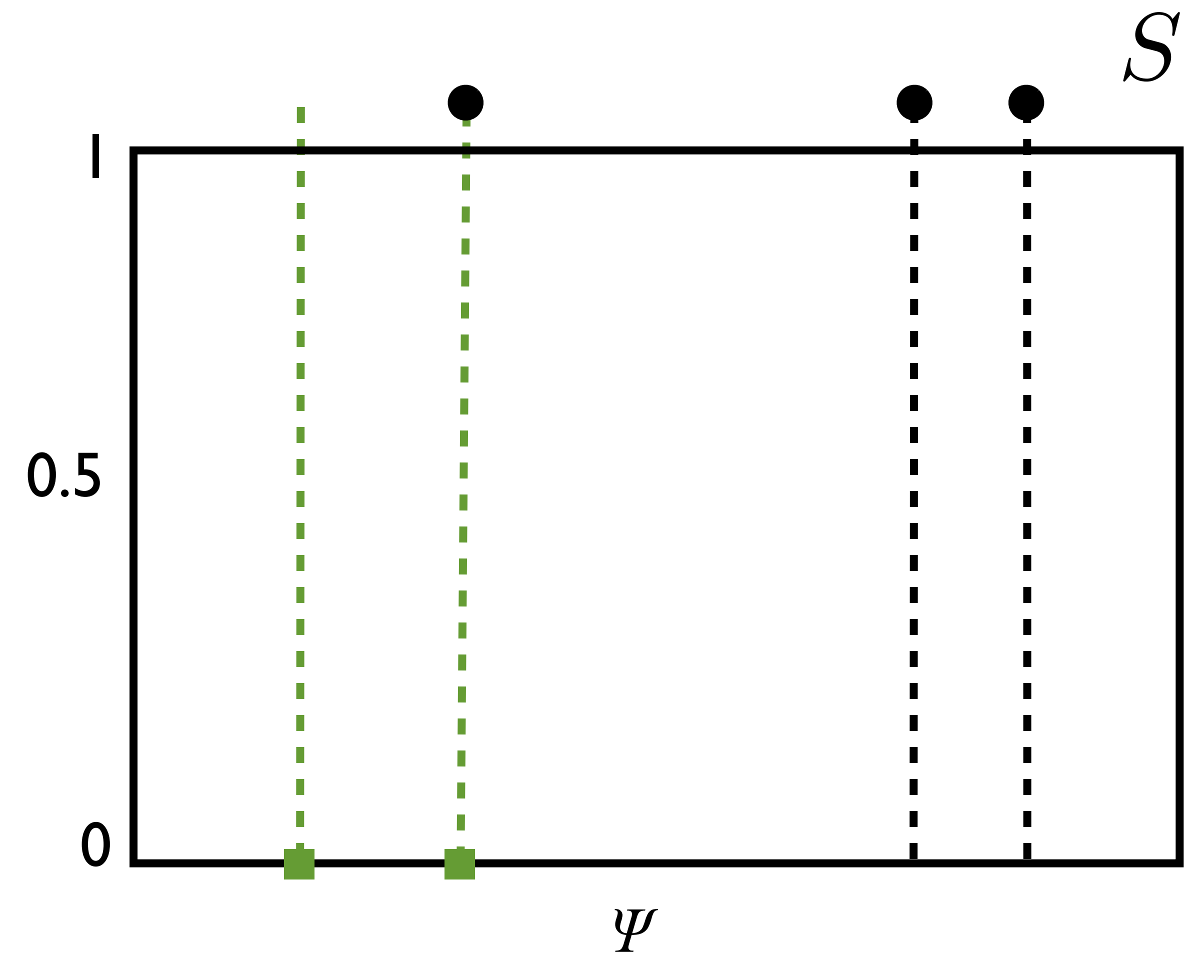
- New fixed atom at ψ

$$\gamma\theta x^{-1}(1-x)^{\theta-1}\,dx \Rightarrow \mathrm{Beta}(x|1,\theta) \propto x \cdot \gamma\theta x^{-1}(1-x)^{\theta-1} dx$$

- Unobserved Poisson process

Posterior

- Prior: beta process
- Likelihood: Bernoulli process



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$$\gamma\theta x^{-1}(1-x)^{\theta-1}\,dx$$

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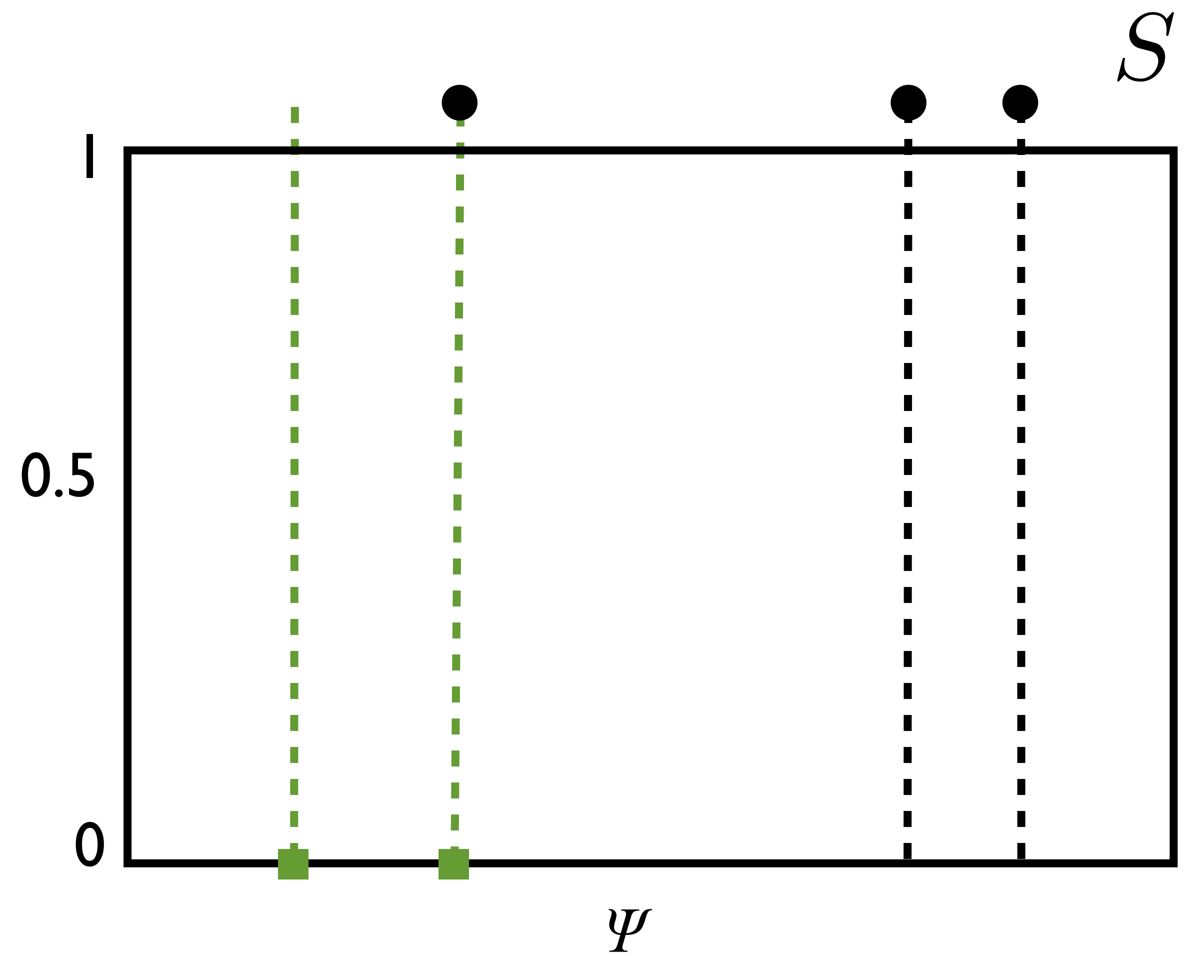

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$$\gamma\theta x^{-1}(1-x)^{\theta-1}\, dx \Rightarrow \mathrm{Beta}(x|1,\theta) \propto x \cdot \gamma\theta x^{-1}(1-x)^{\theta-1} dx$$

- Unobserved Poisson process

$$\gamma\theta x^{-1}(1-x)^{\theta-1}\, dx \Rightarrow (1-x) \cdot \gamma\theta x^{-1}(1-x)^{\theta-1} dx$$

Posterior

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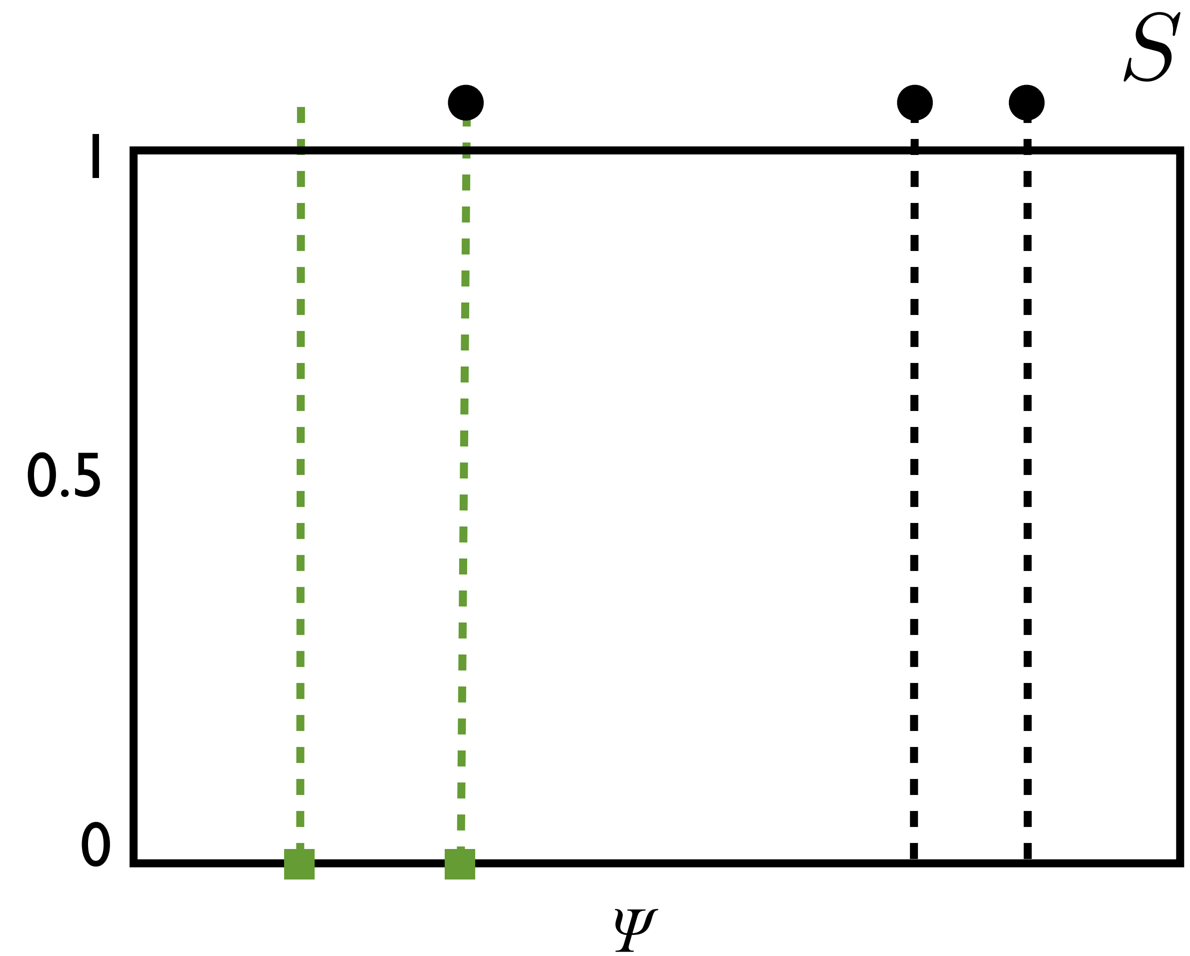

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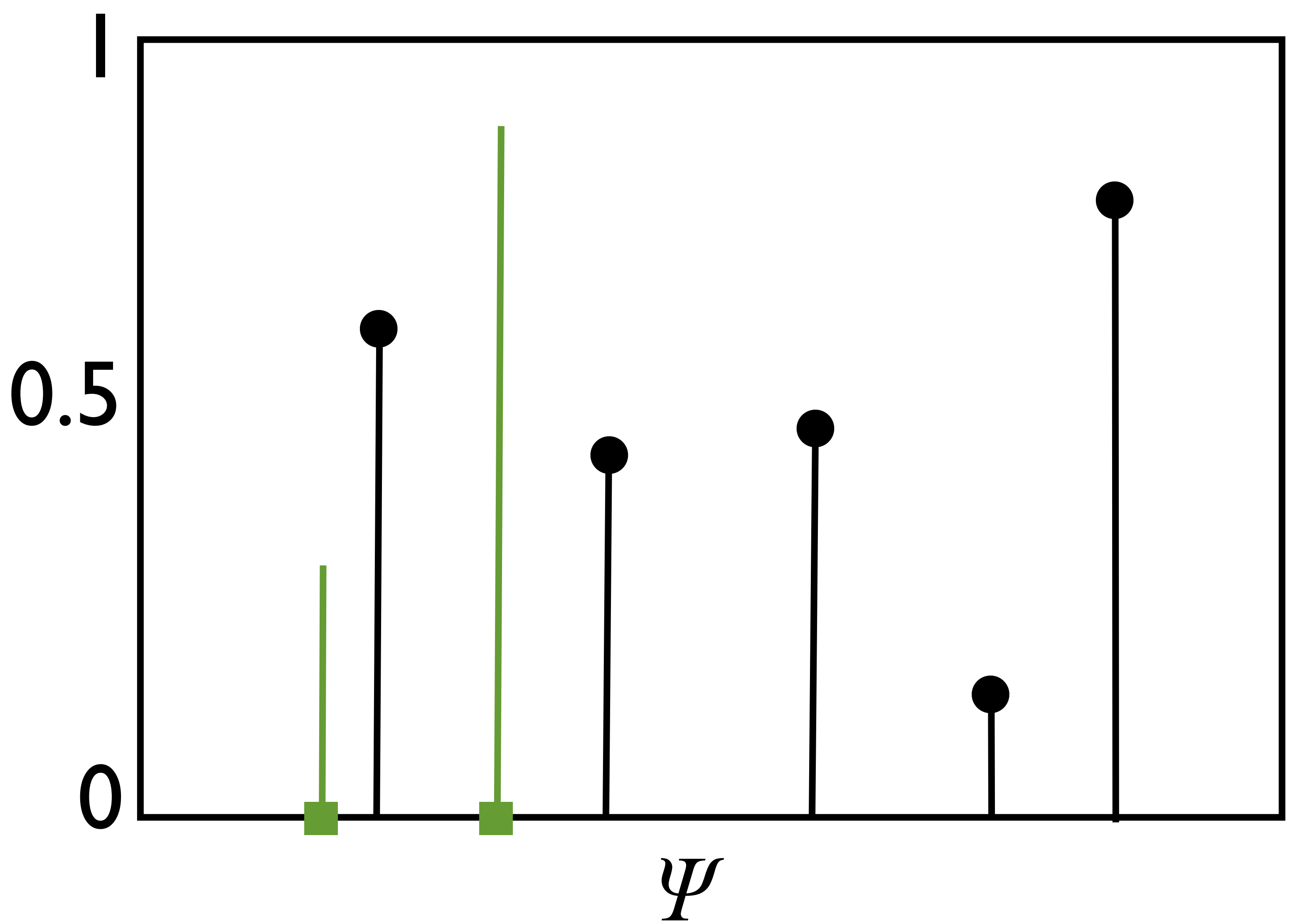
- Unobserved Poisson process

$$\gamma\theta x^{-1}(1-x)^{\theta-1}\,dx \Rightarrow \frac{\gamma\theta}{\theta+1}(\theta+1)x^{-1}(1-x)^{(\theta+1)-1}dx$$

Completely random measure

Two parts (for our purposes):

- Poisson process component
- Fixed atoms



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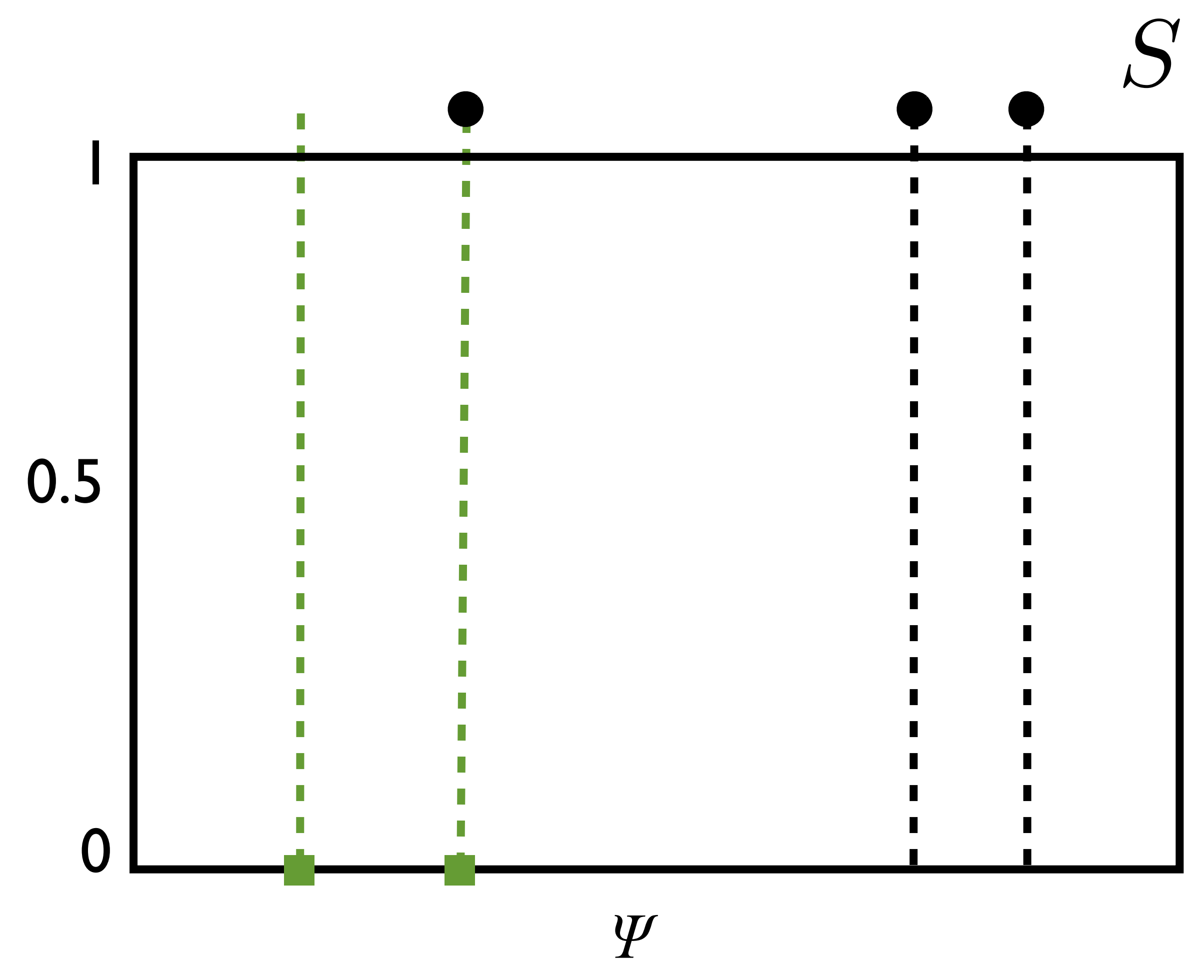
- Prior: beta process
- Likelihood: Bernoulli process



- Old fixed atom at ψ_j

$$\mathrm{Beta}(x_j | a_j, b_j) \Rightarrow \mathrm{Beta}\left(x_j | a_j + \mathbf{1}\{\psi_j \in S\}, b_j + \mathbf{1}\{\psi_j \notin S\}\right)$$

- New fixed atom at ψ

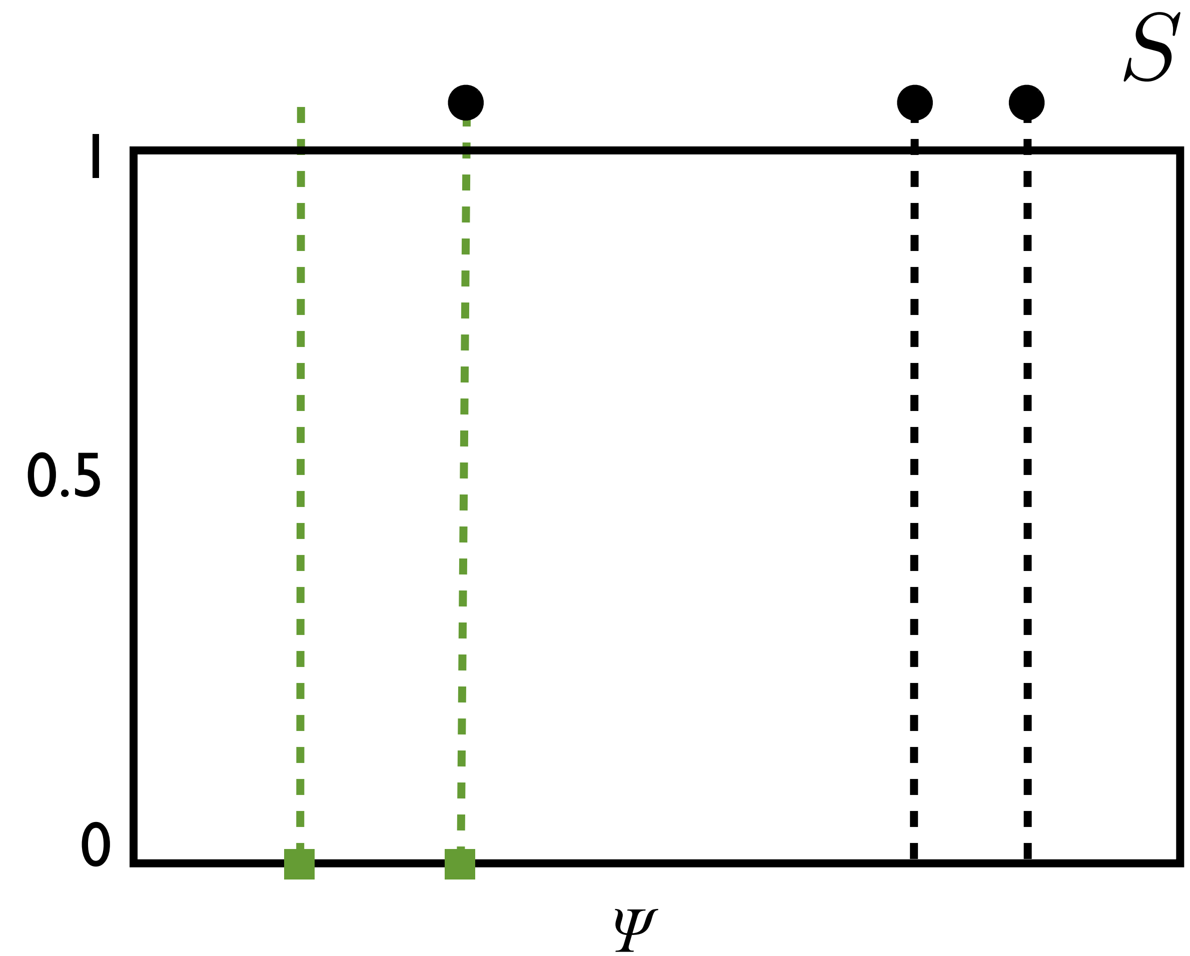
$$\gamma\theta x^{-1}(1-x)^{\theta-1}\, dx \Rightarrow \mathrm{Beta}(x|1,\theta) \propto x \cdot \gamma\theta x^{-1}(1-x)^{\theta-1}$$

- Unobserved Poisson process

$$\gamma\theta x^{-1}(1-x)^{\theta-1}\, dx \Rightarrow \frac{\gamma\theta}{\theta+1}(\theta+1)x^{-1}(1-x)^{(\theta+1)-1}$$

Posterior

- Prior: Completely random measure
- Likelihood: Bernoulli process



- Old fixed atom at ψ_j

$$\text{Usual parametric posterior}$$

- New fixed atom at ψ

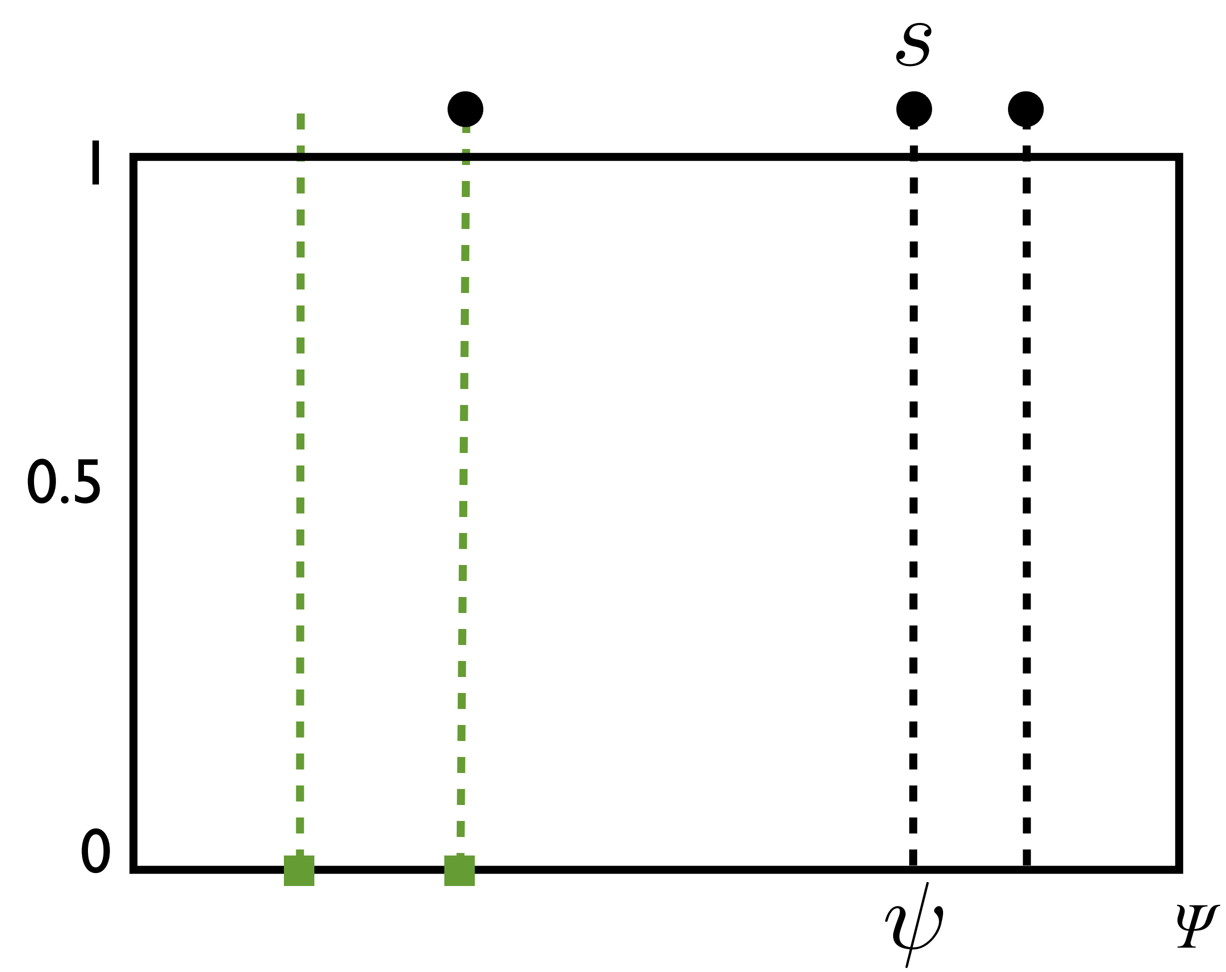
$$\nu(dx) \Rightarrow \text{density} \propto x \cdot \nu(dx)$$

- Unobserved Poisson process

$$\nu(dx) \Rightarrow (1 - x) \cdot \nu(dx)$$

Posterior

- Prior: Completely random measure
- Likelihood: Discrete process



General Recipe

- Old fixed atom at ψ_j

$$\text{Usual parametric posterior}$$

- New fixed atom at ψ

$$\nu(dx) \Rightarrow \text{density} \propto p(s|x) \cdot \nu(dx)$$

- Unobserved Poisson process

$$\nu(dx) \Rightarrow p(0|x) \cdot \nu(dx)$$

Conclusions

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  - And identify conjugate priors
- Extend other ideas from parametric conjugacy

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