# Nonparametric Bayesian Methods: Models, Algorithms, and Applications

Tamara Broderick
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

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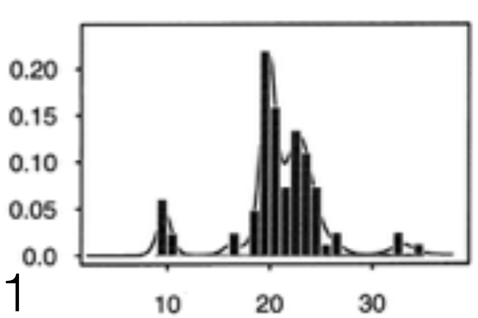


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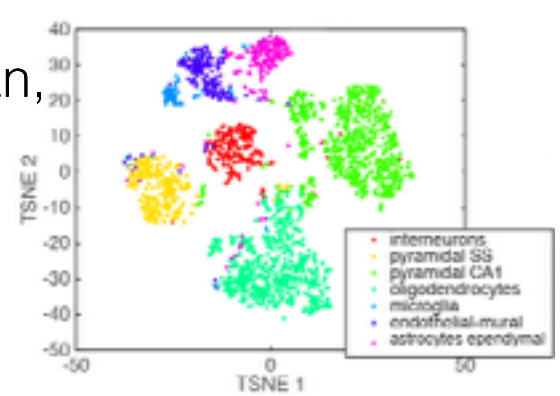


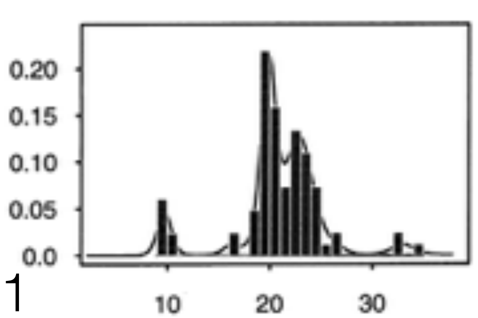


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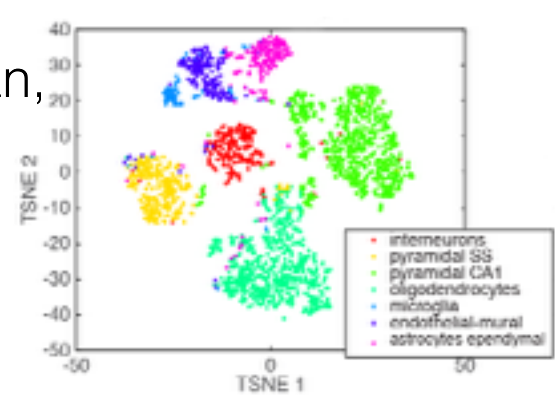


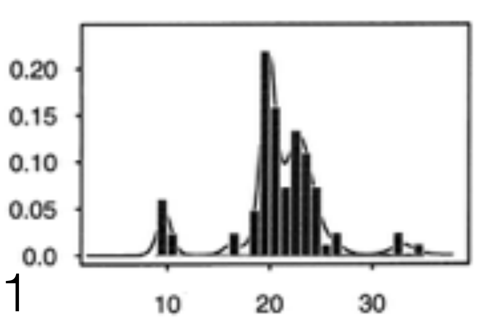


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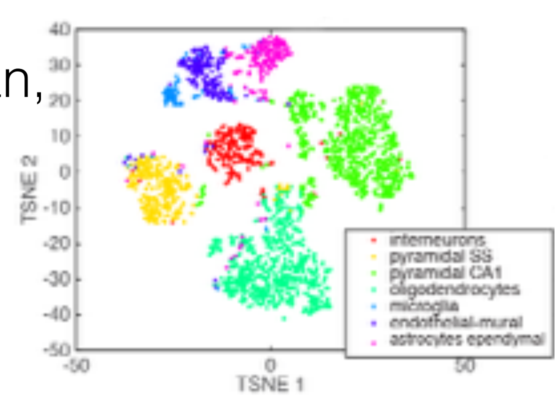


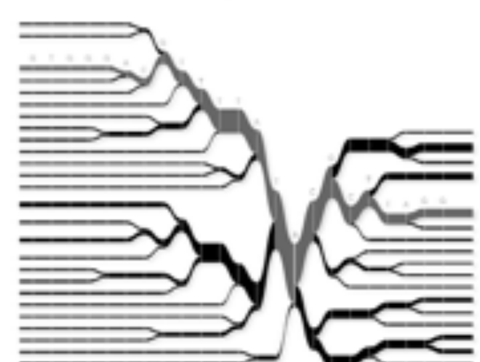

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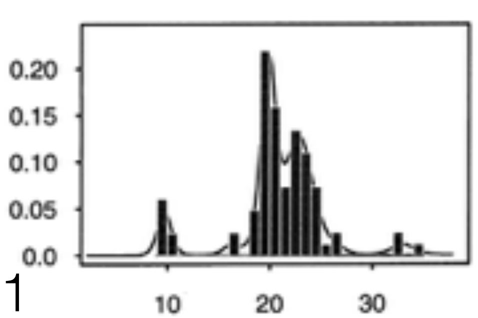


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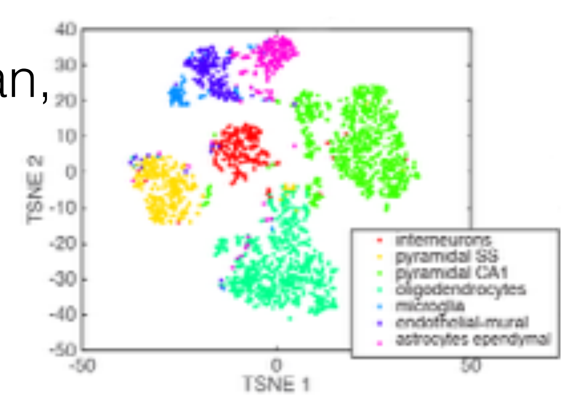

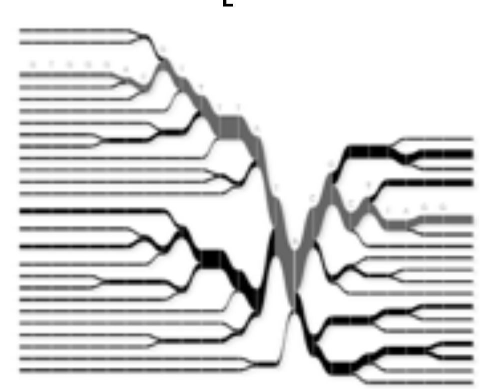

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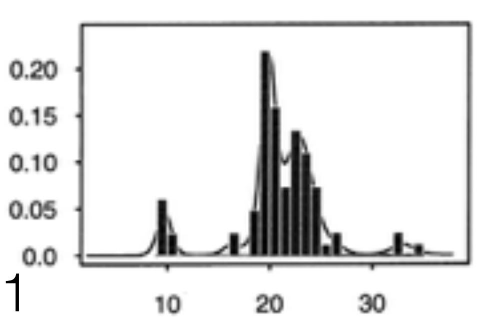

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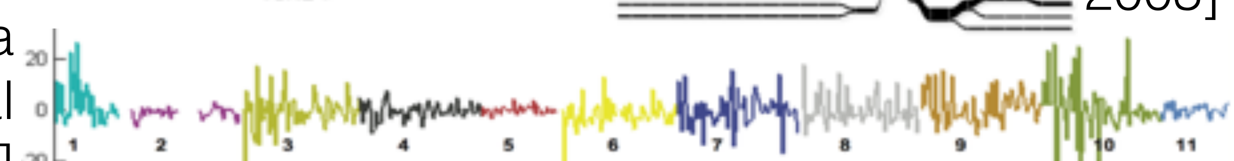

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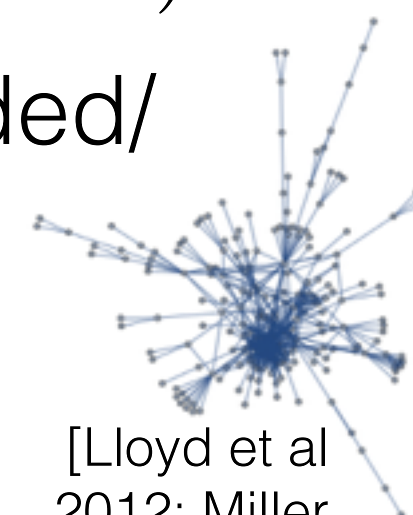

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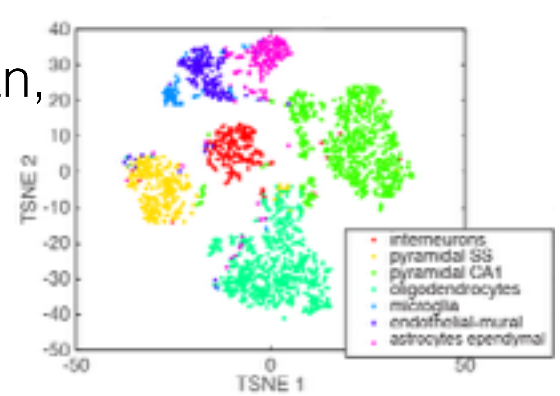


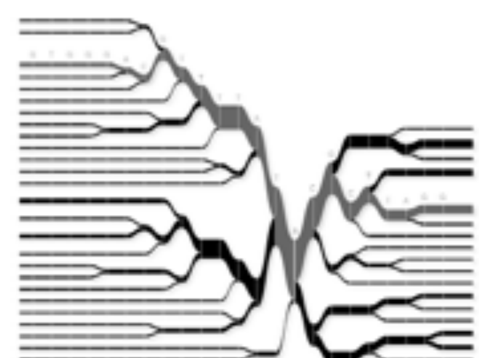

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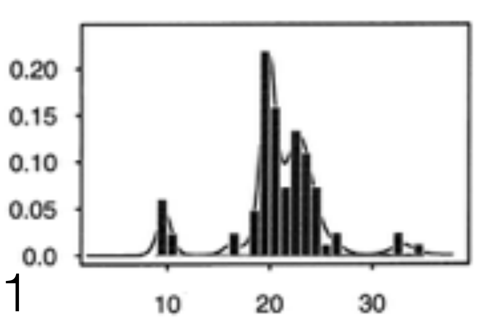

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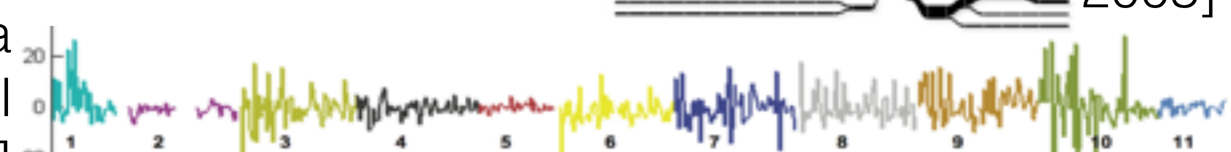

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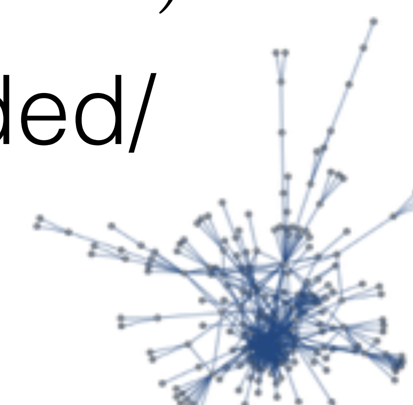

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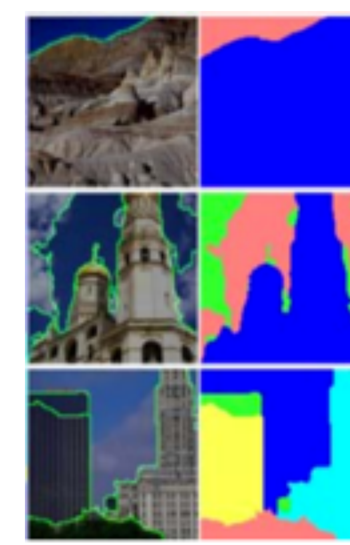

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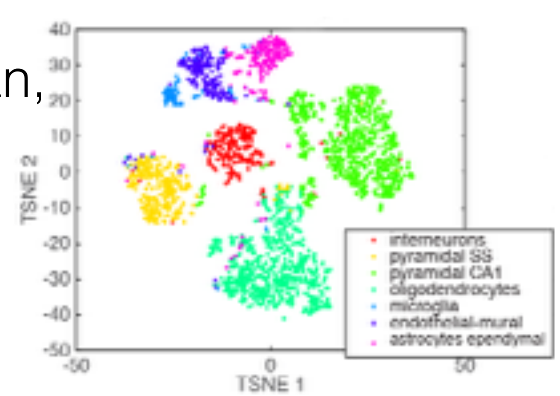


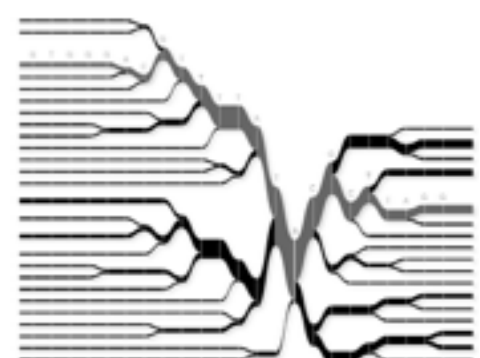

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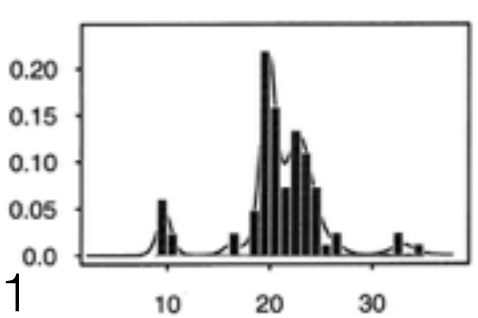

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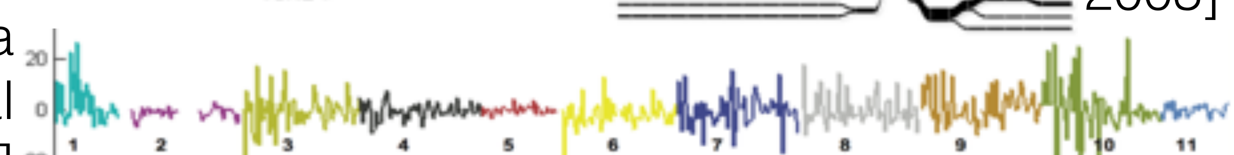

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  - Why is NPBayes challenging but practical?

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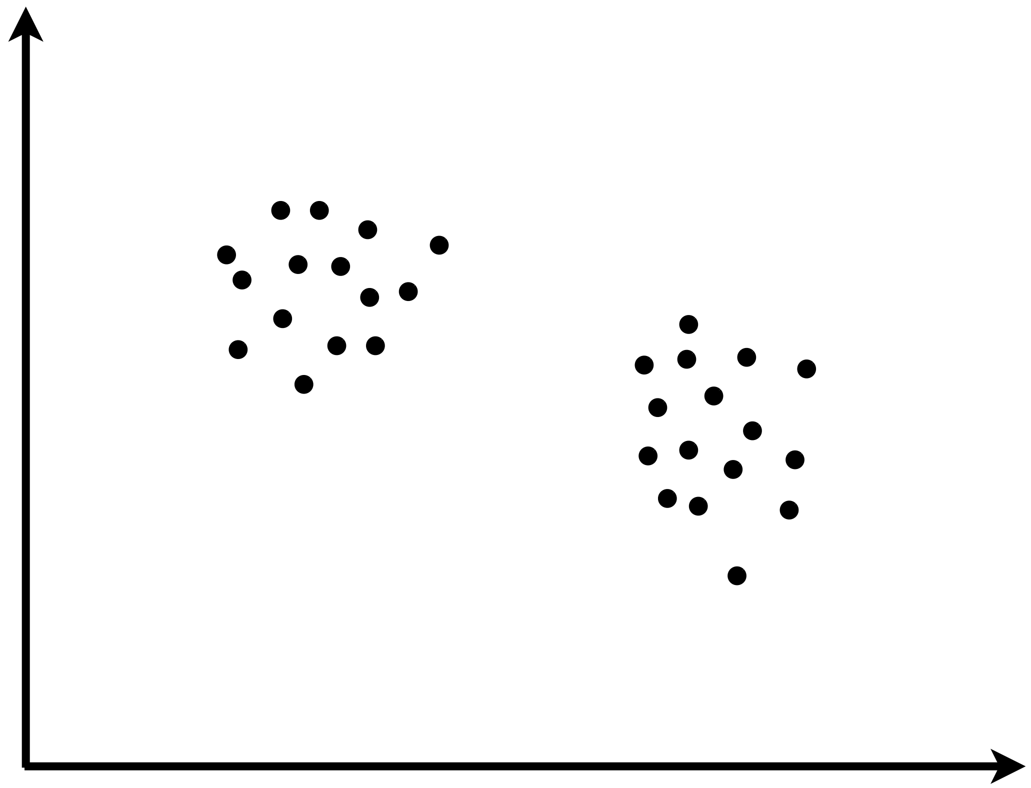

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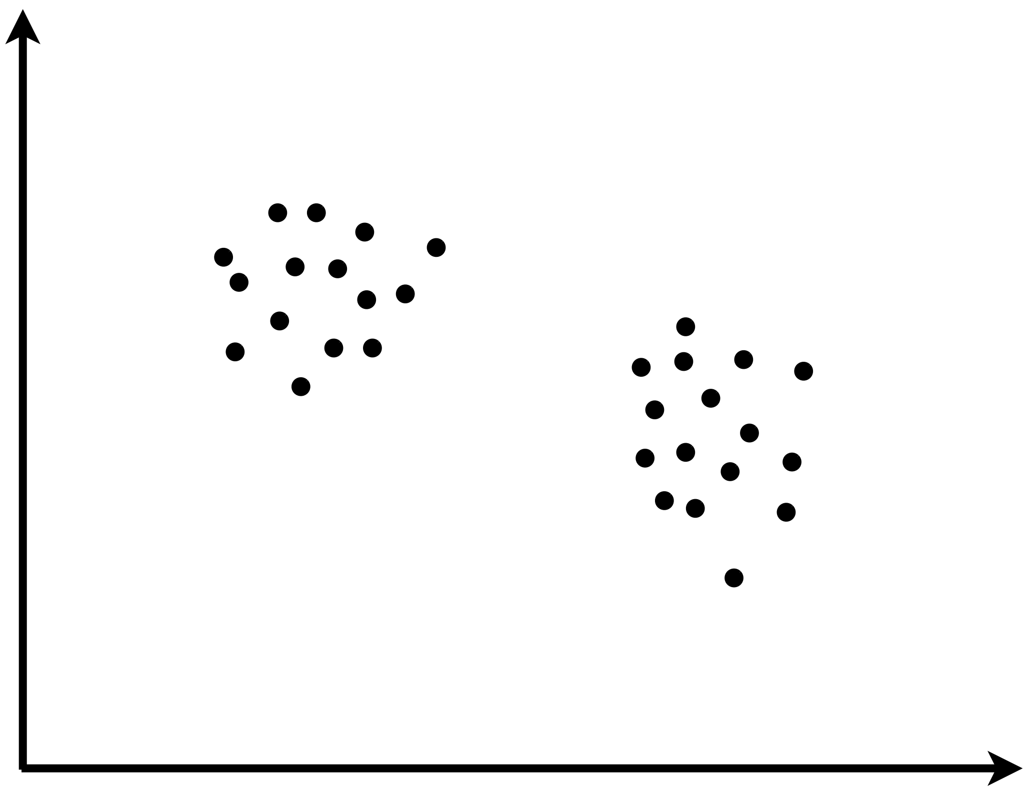

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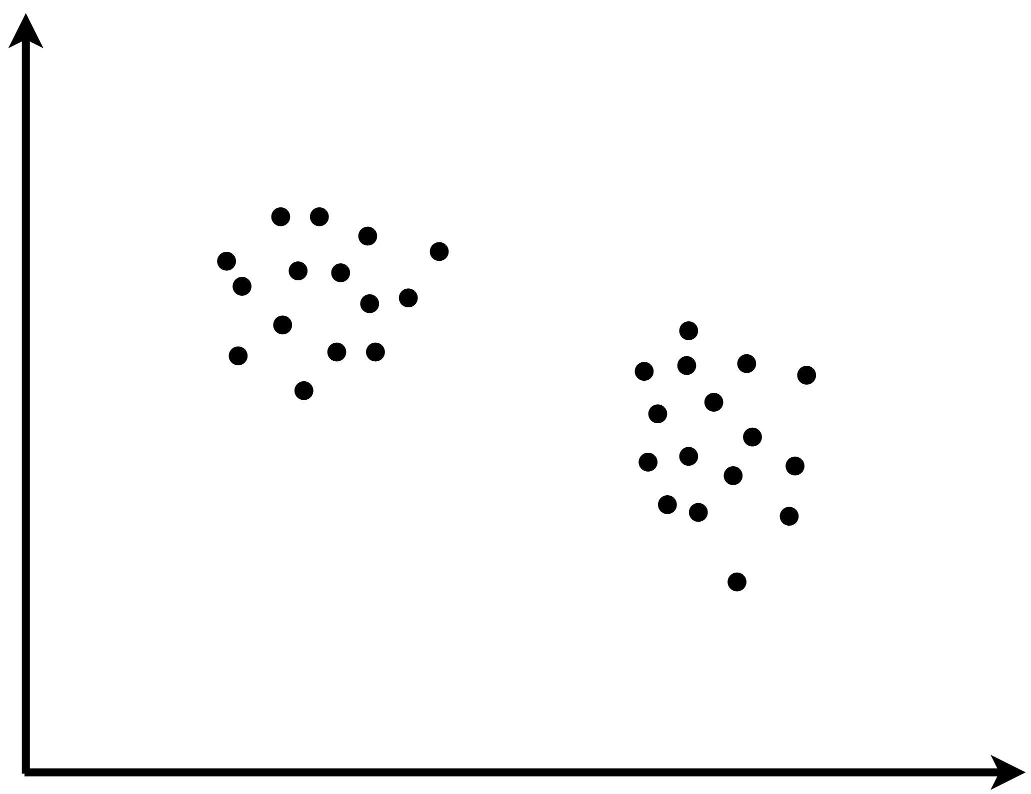

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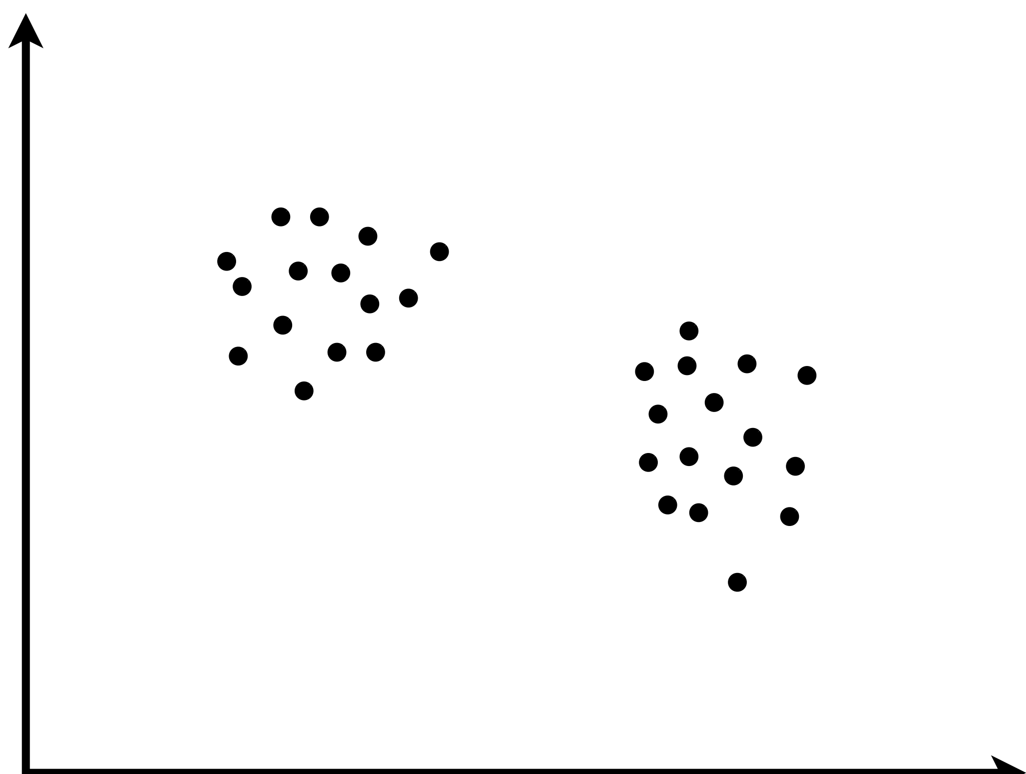

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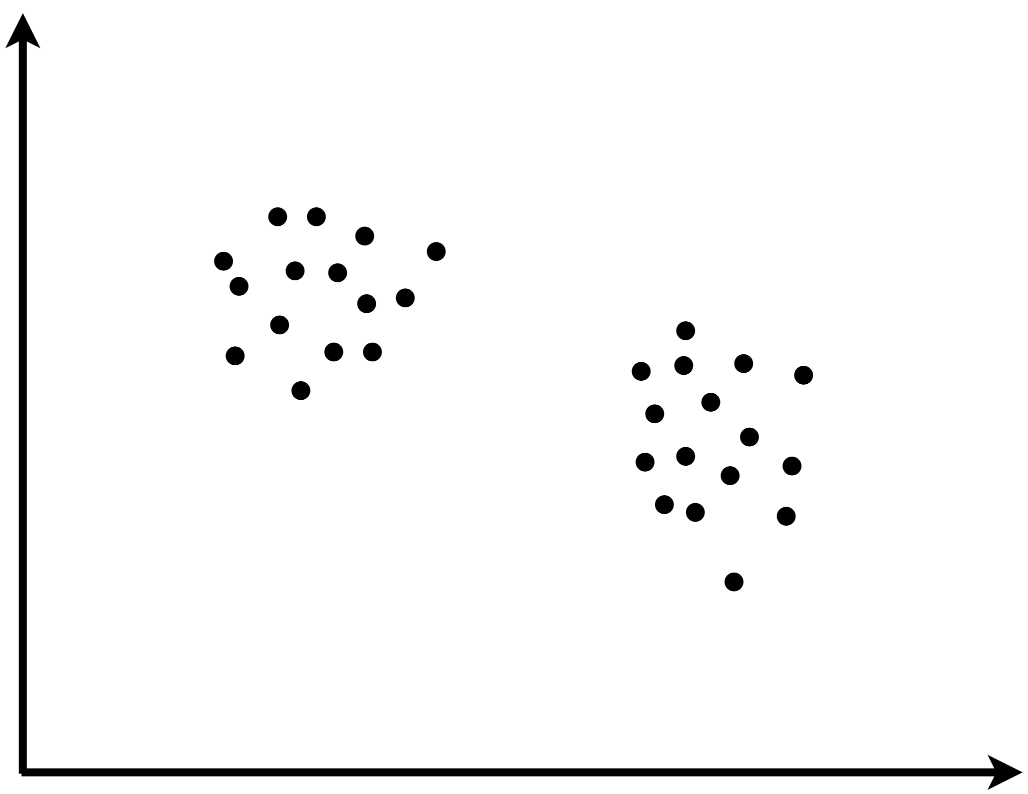

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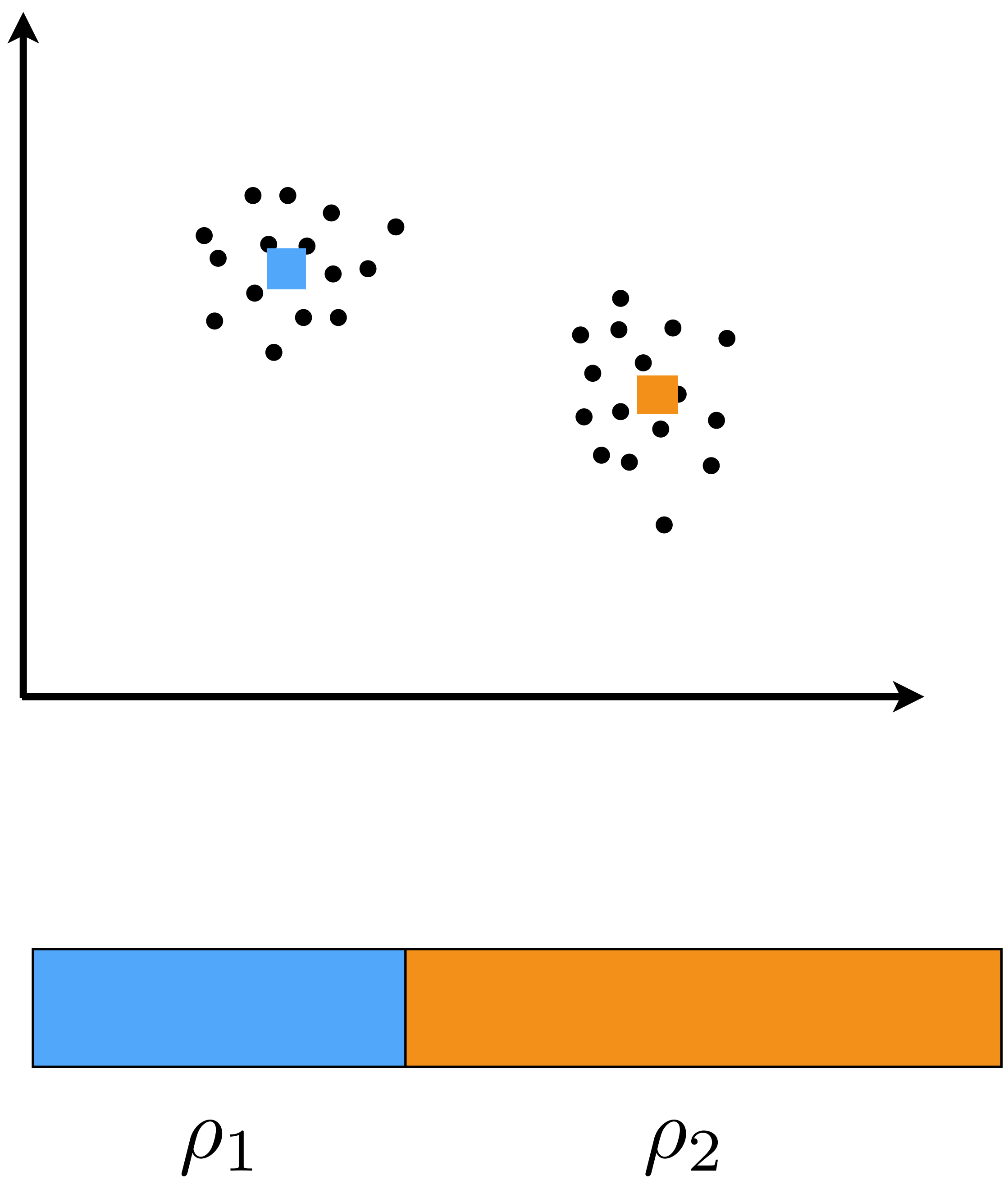

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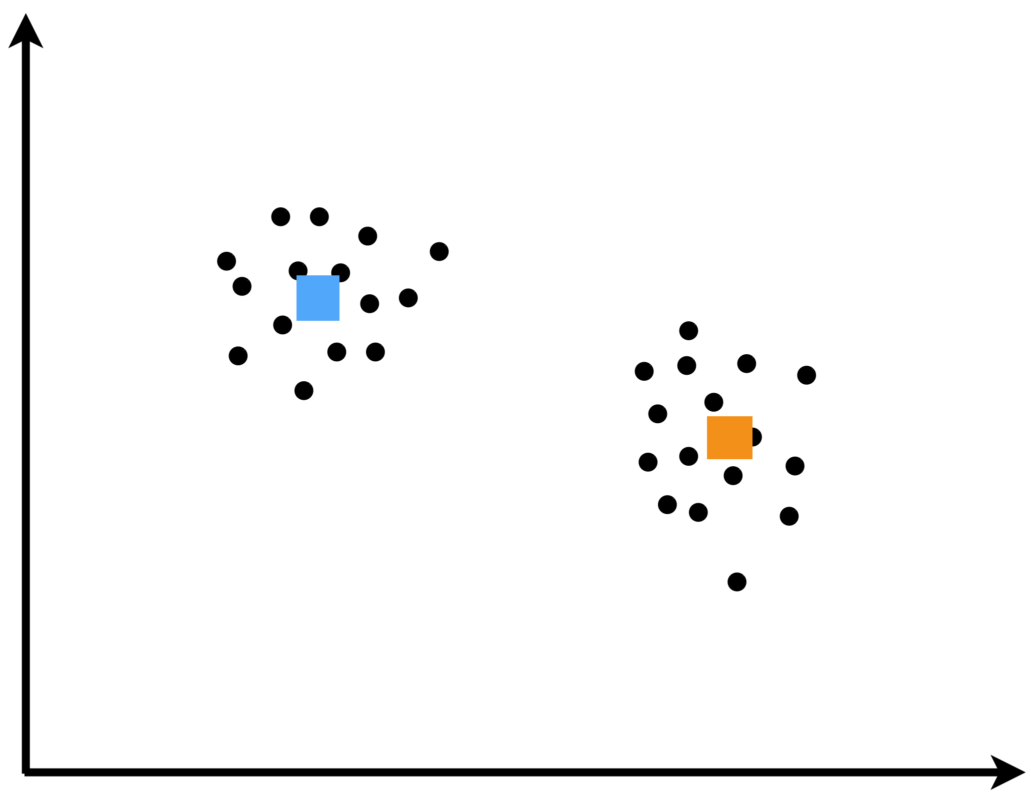

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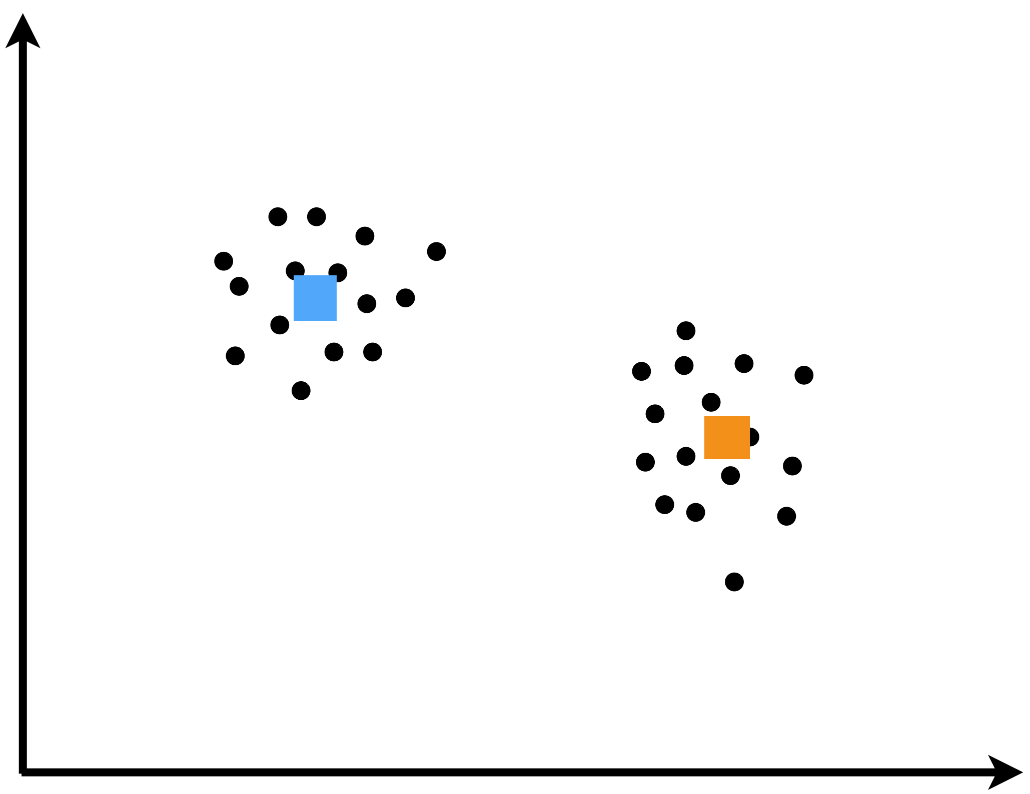

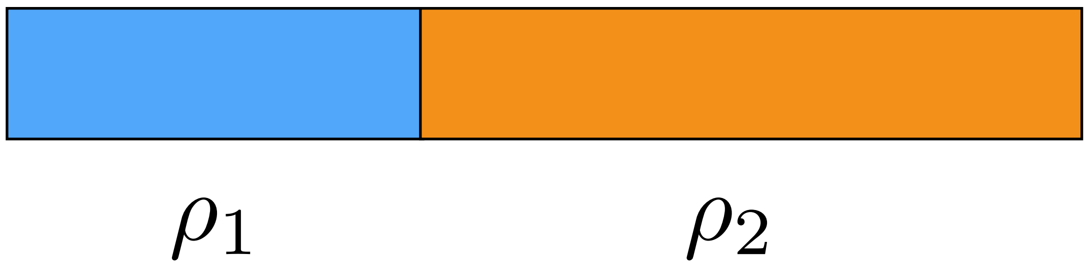


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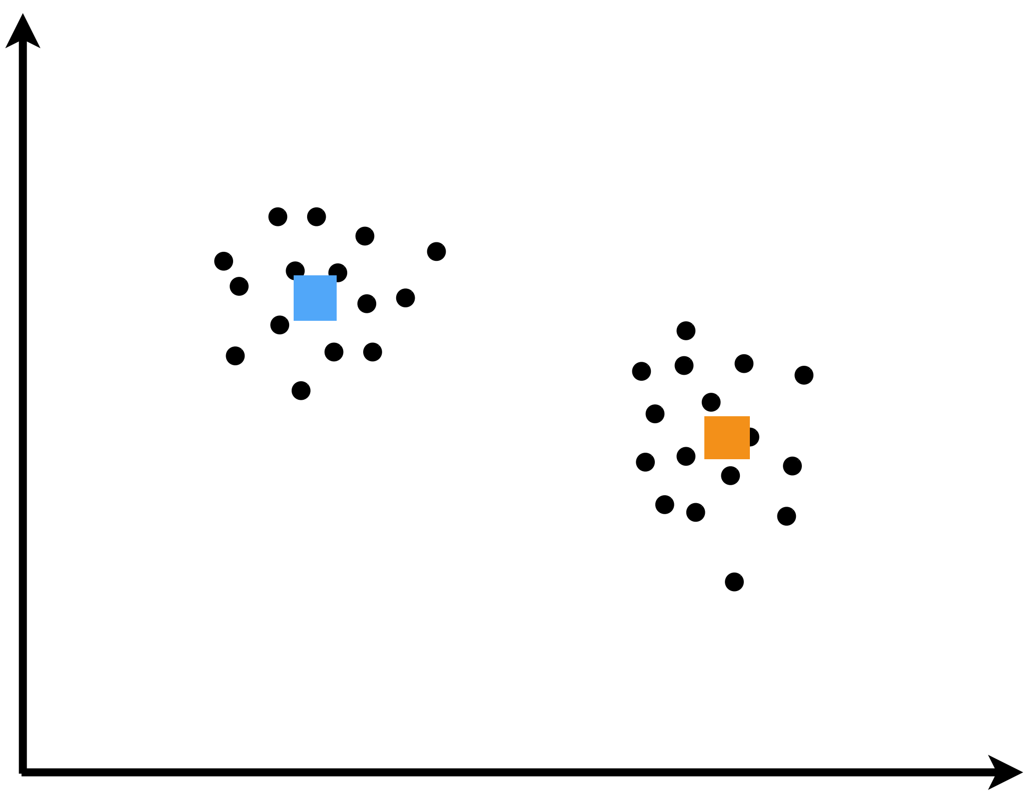

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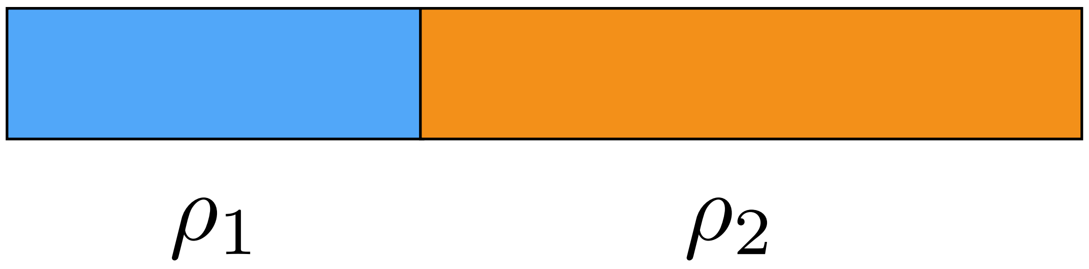

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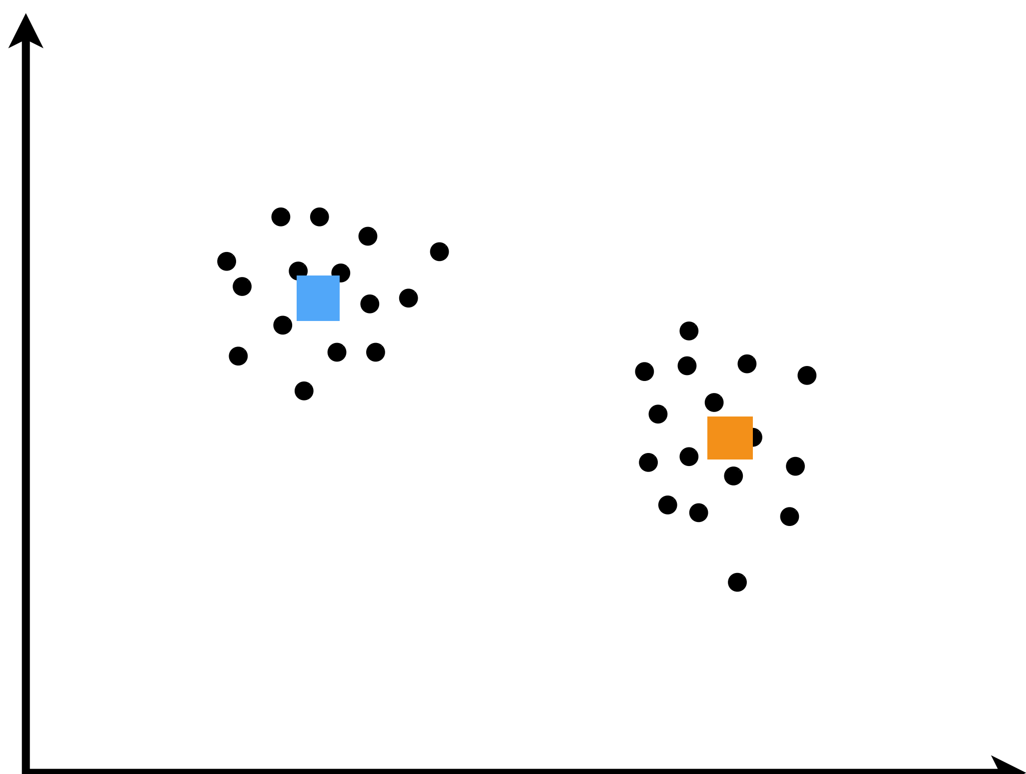

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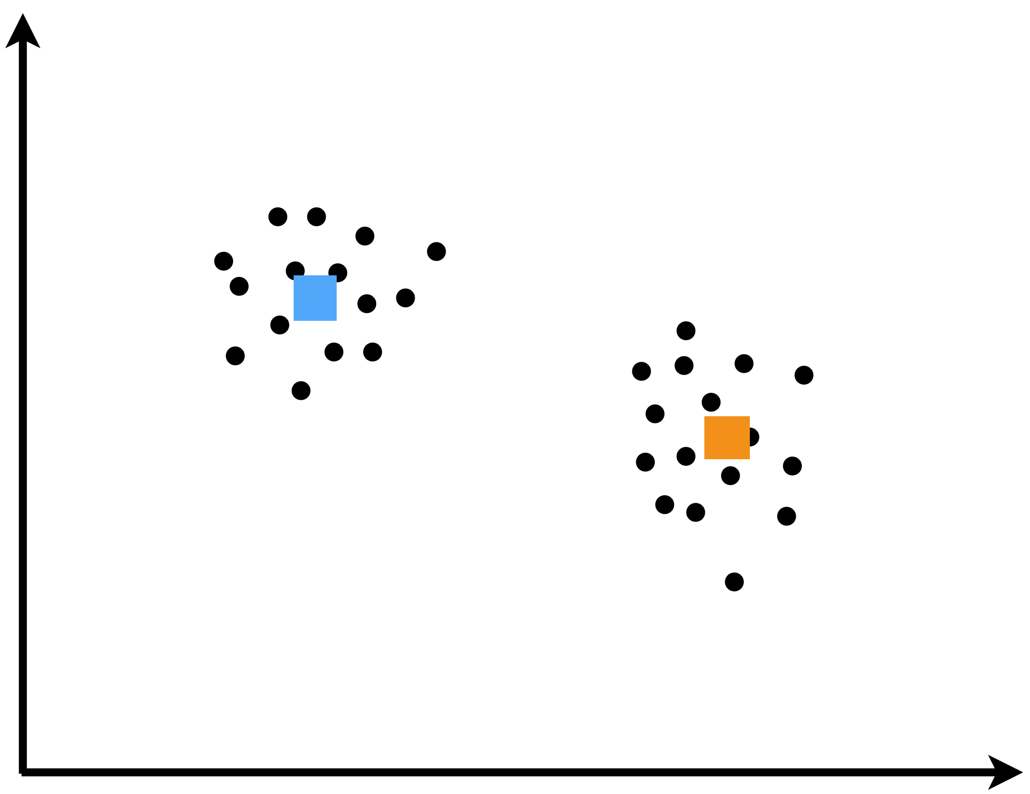


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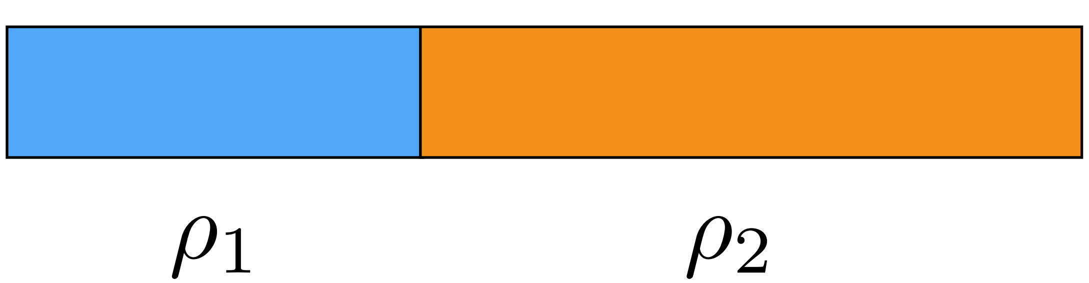


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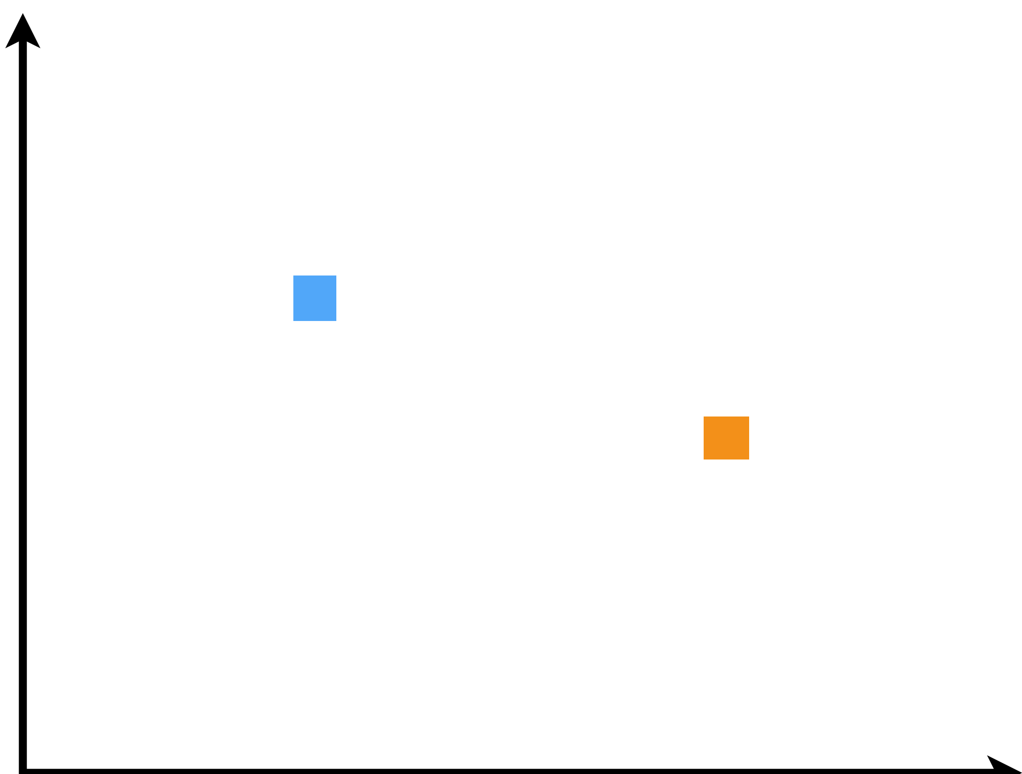

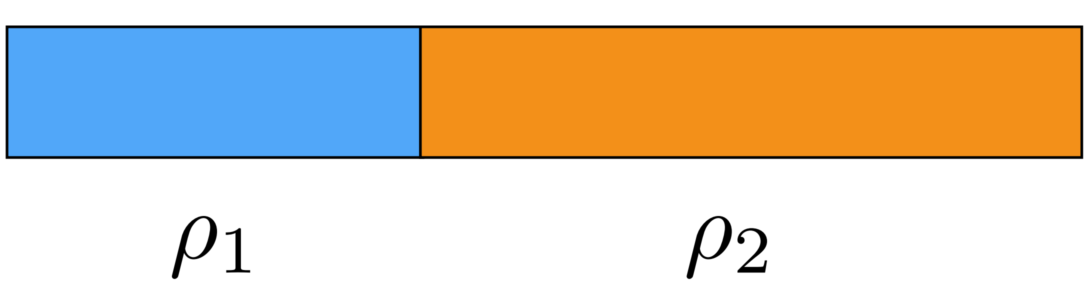


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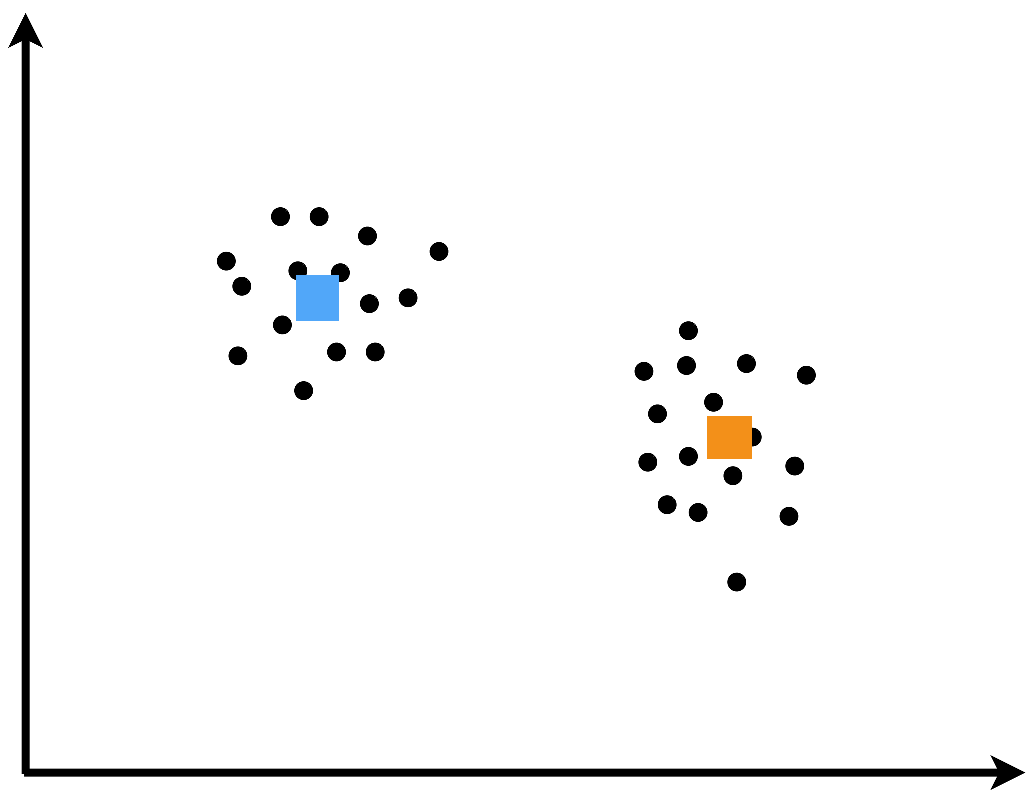


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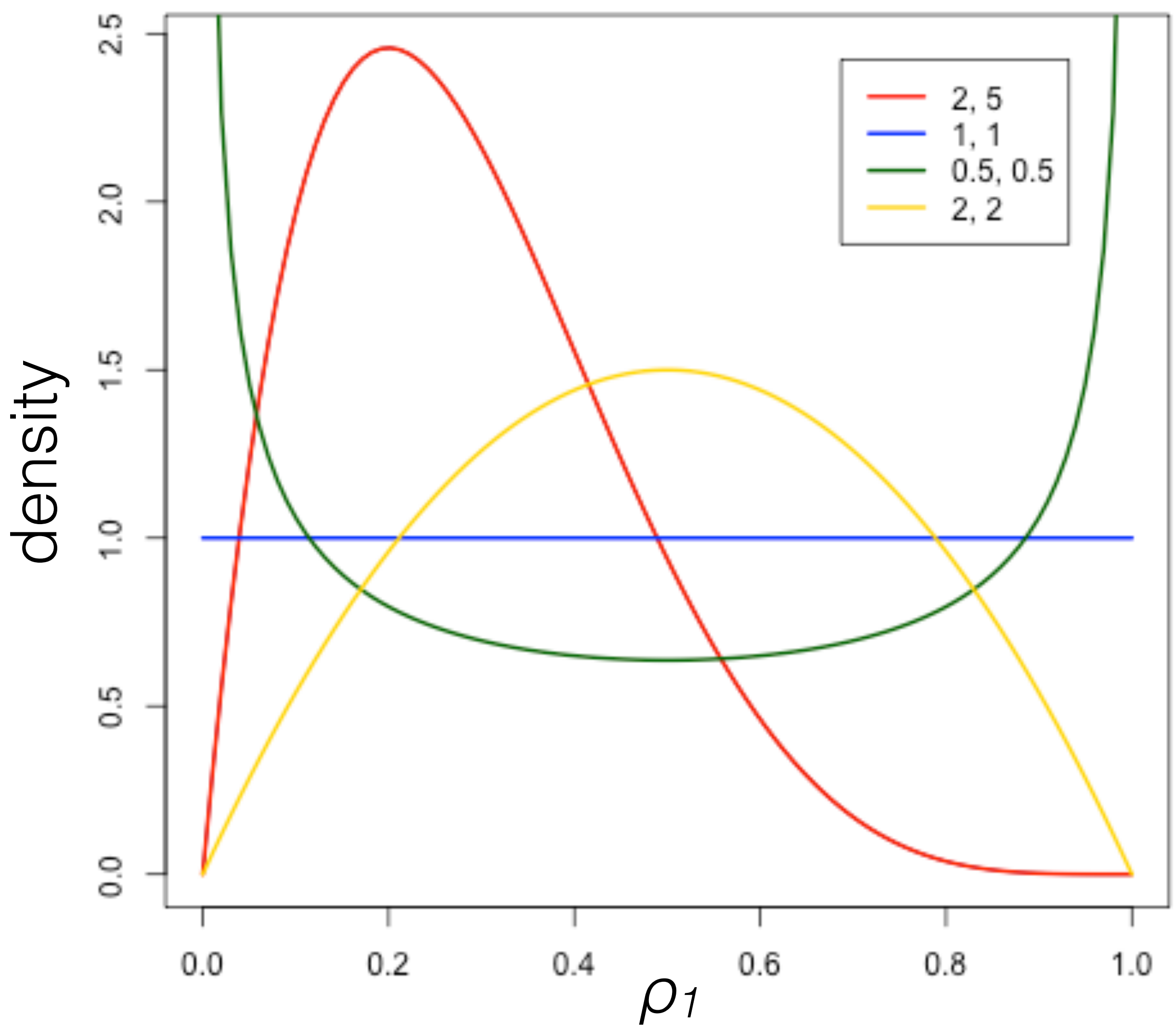


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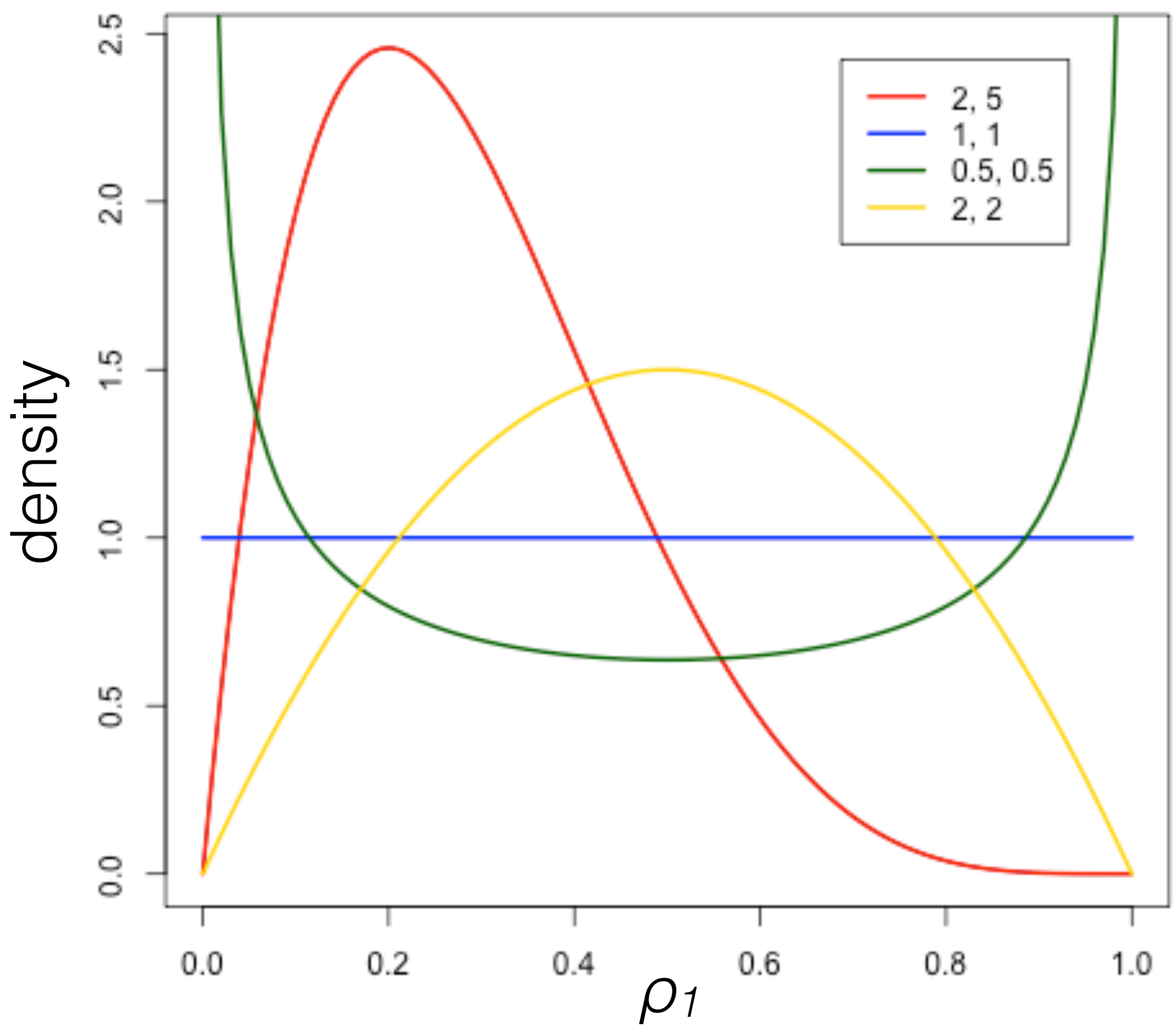


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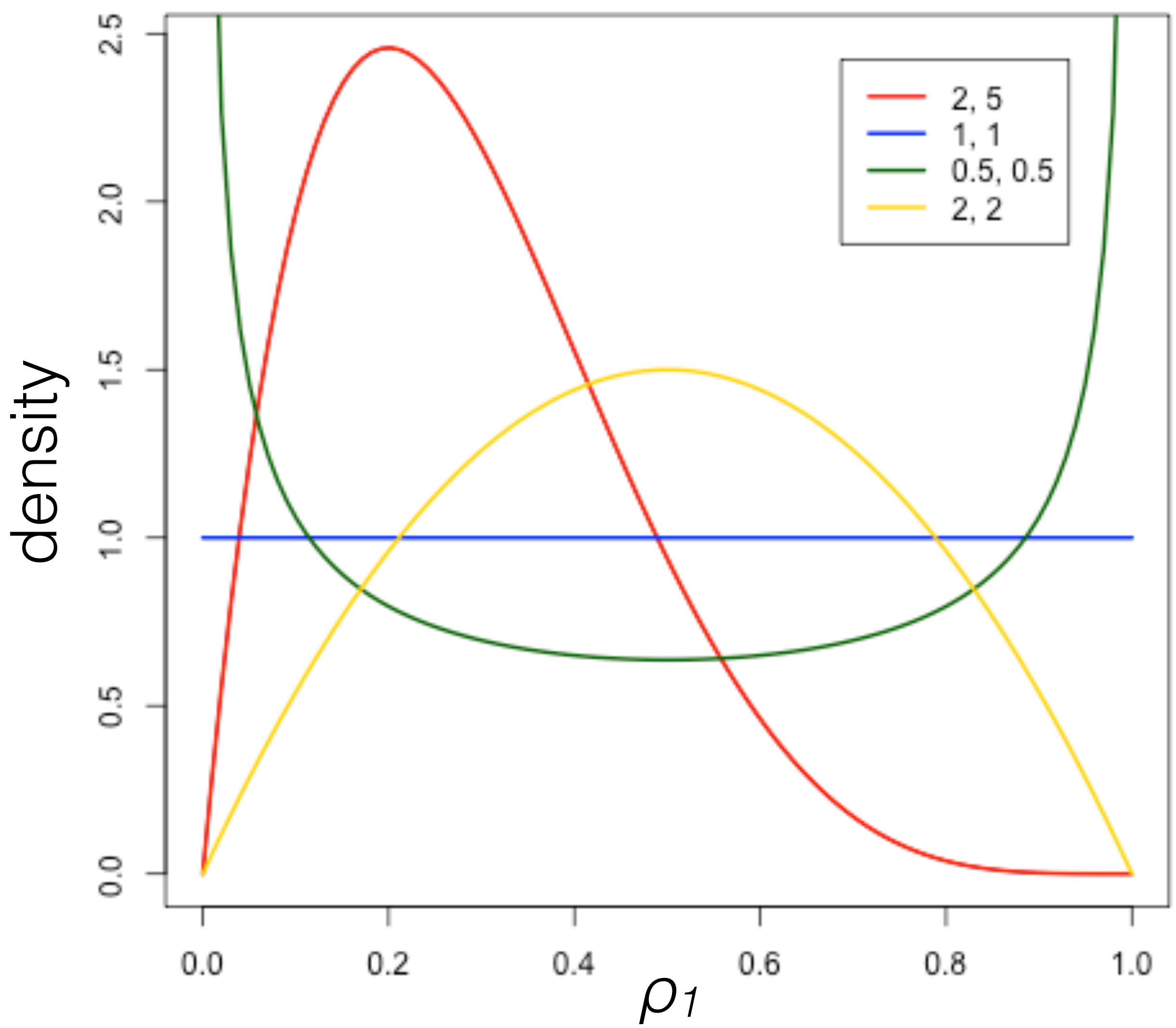


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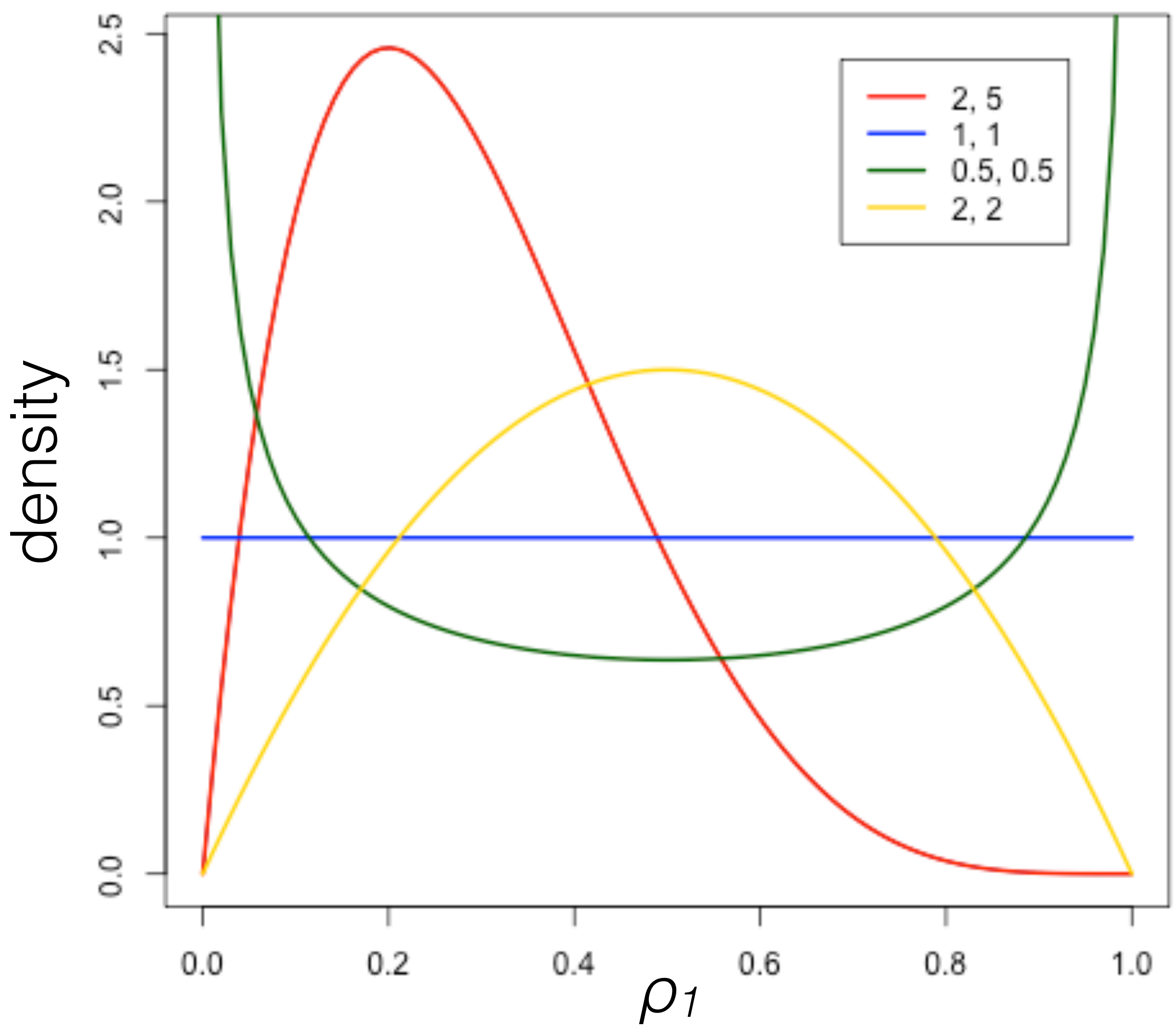


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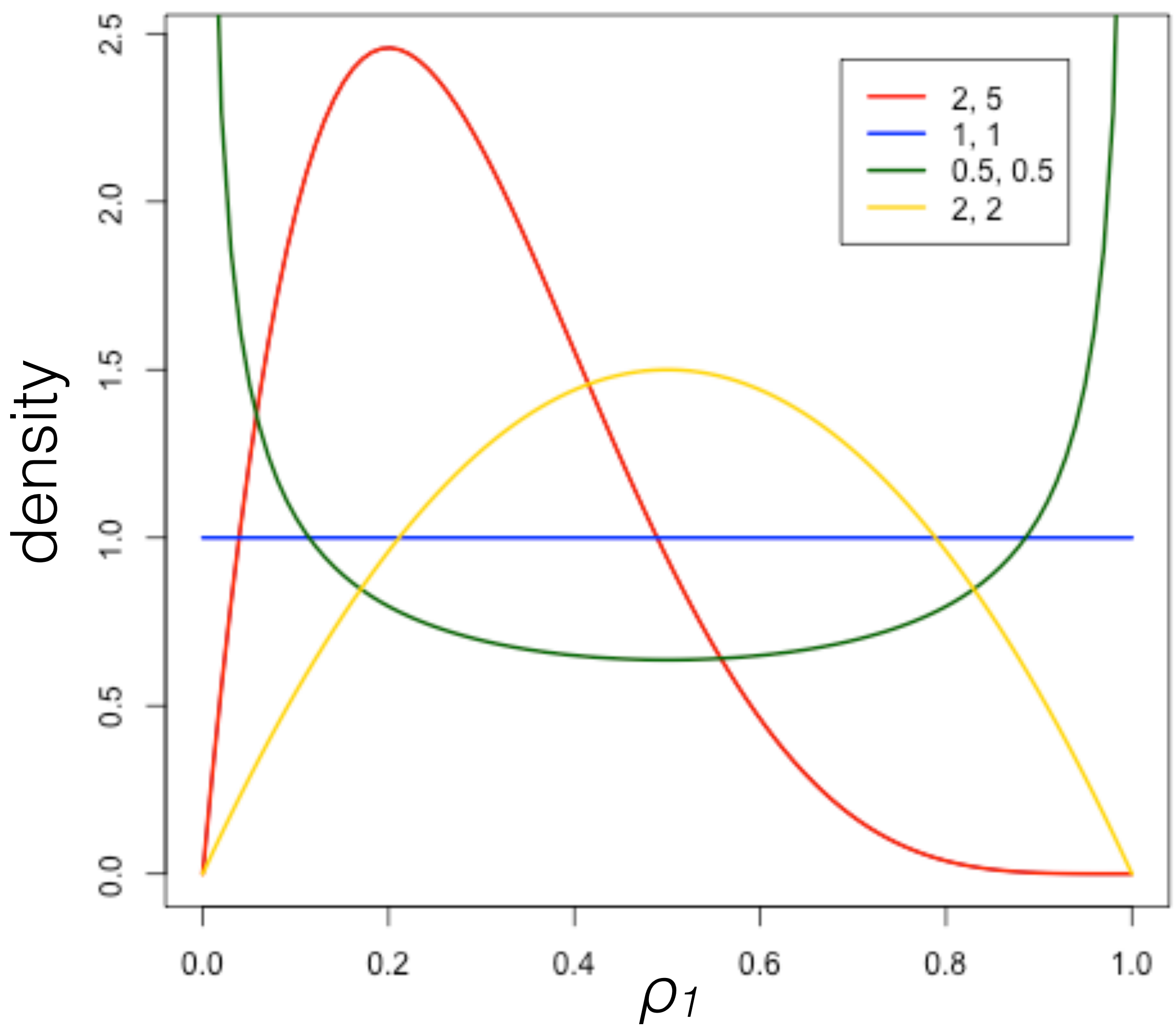


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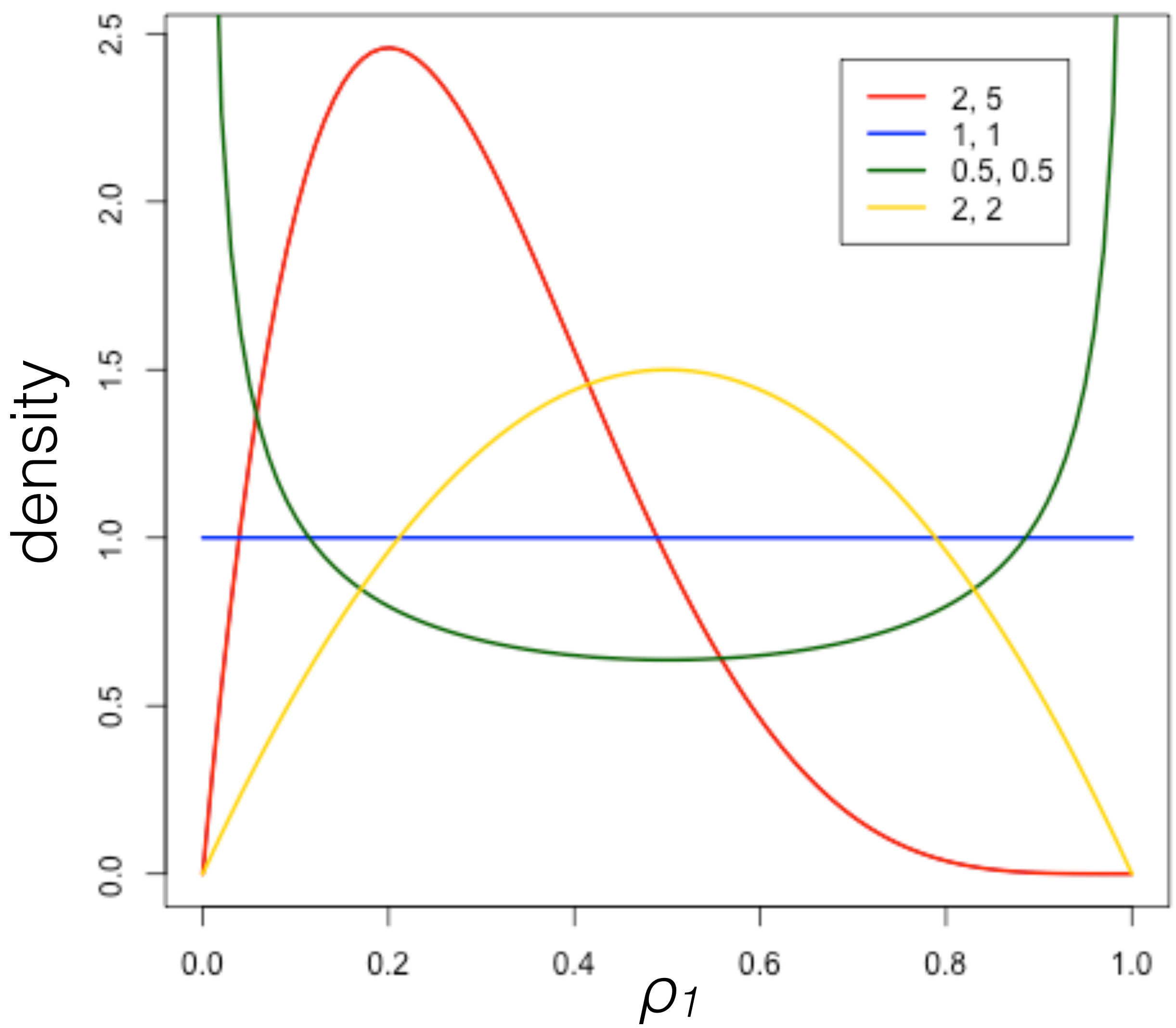


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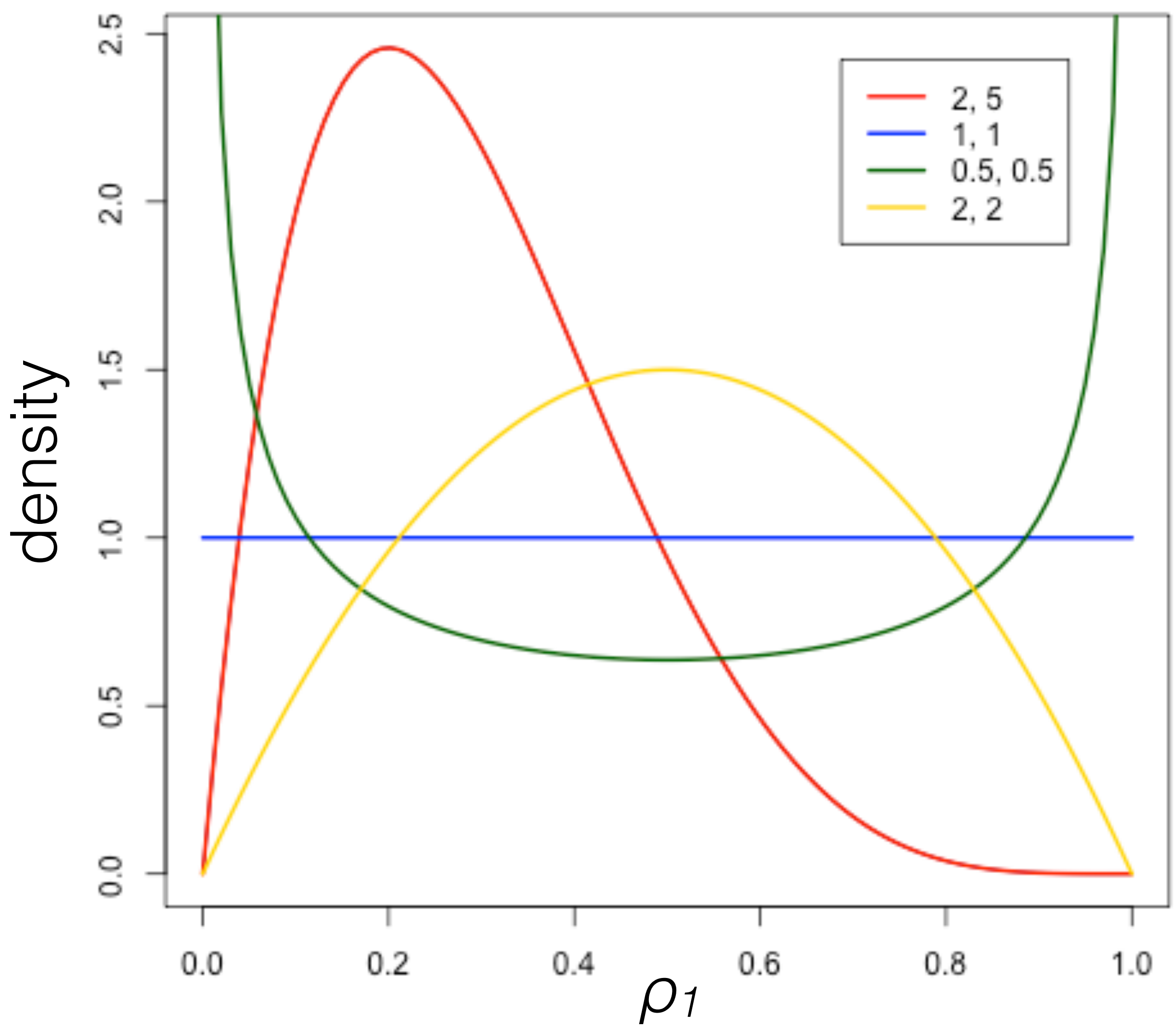


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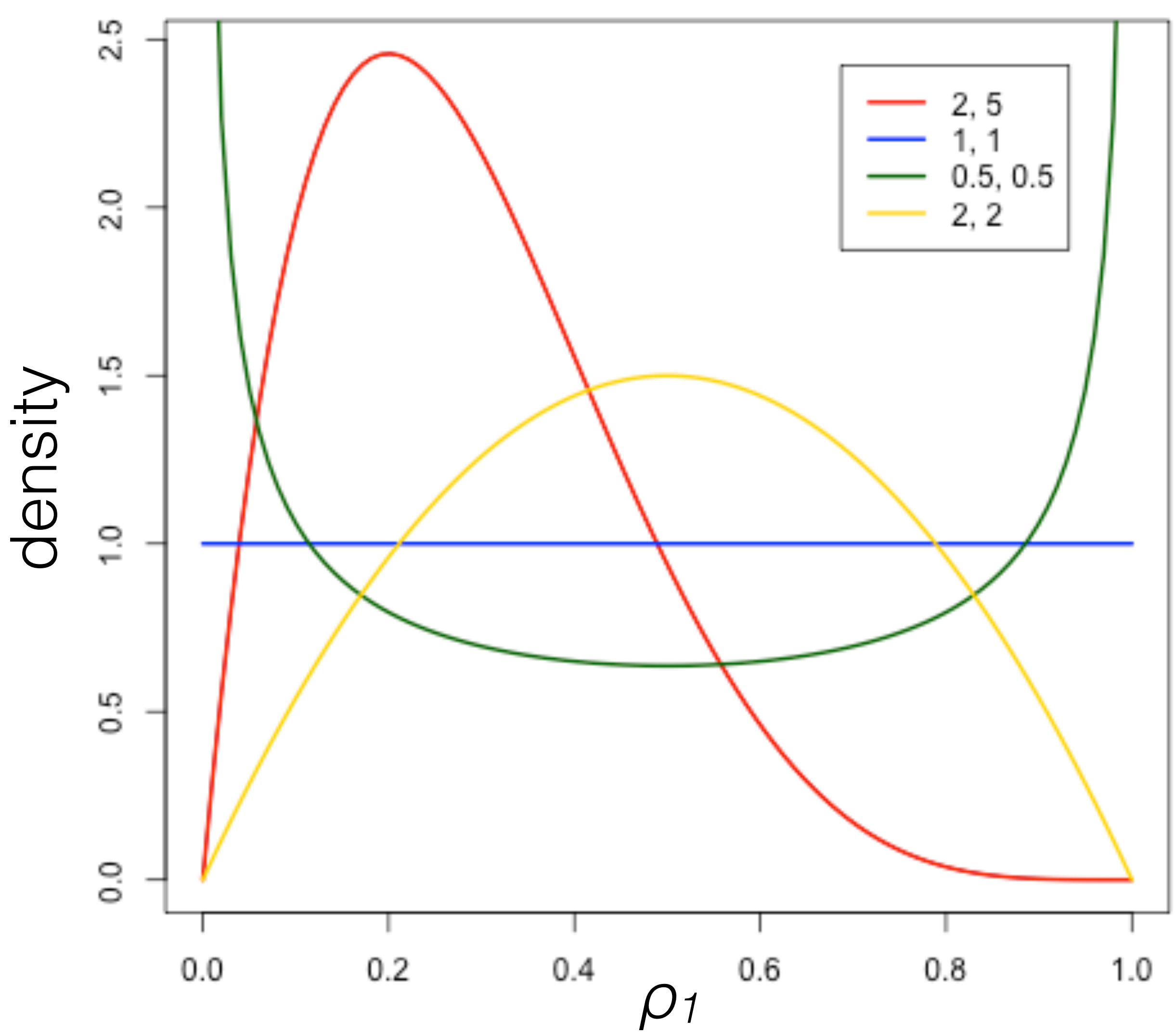


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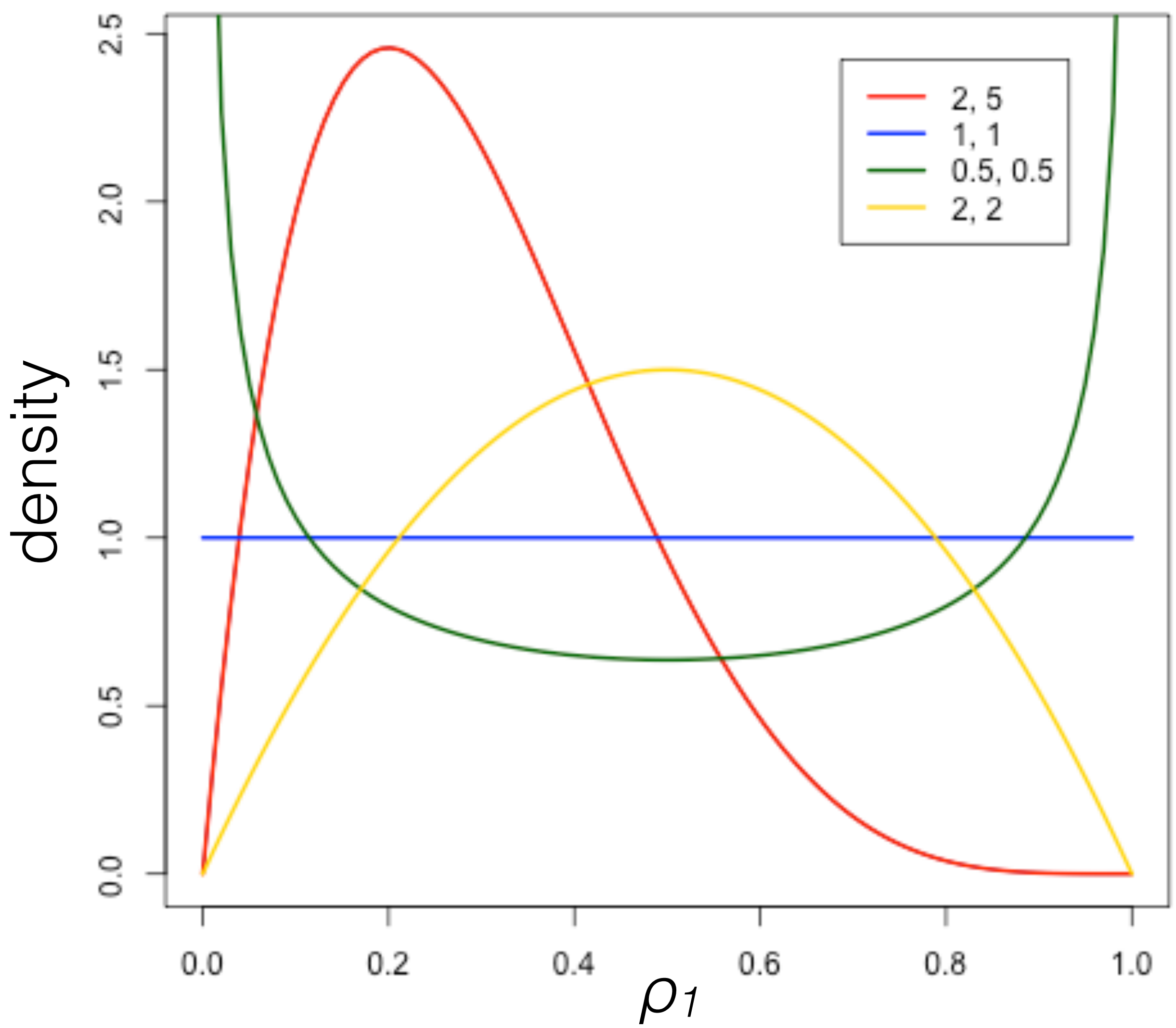


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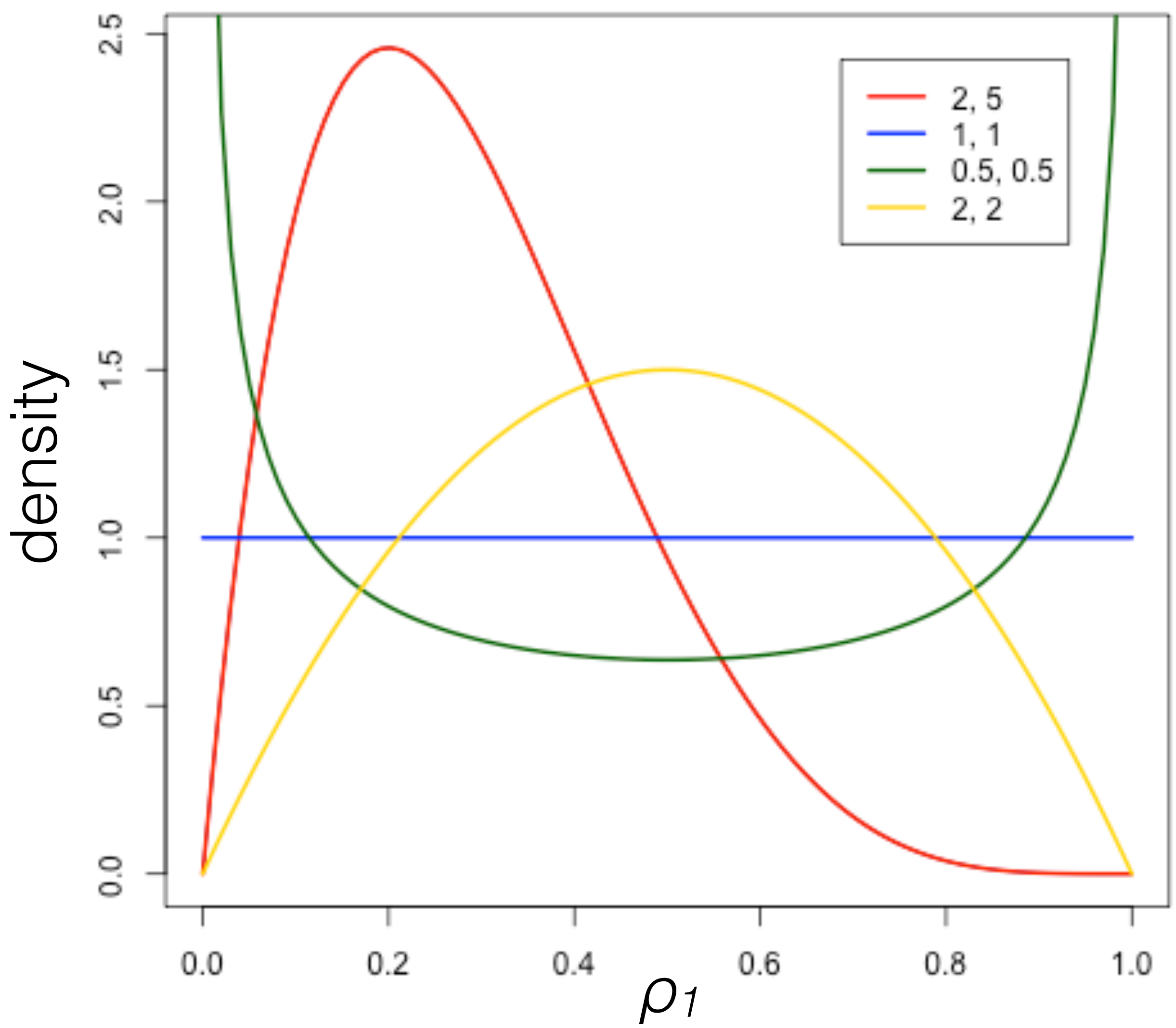


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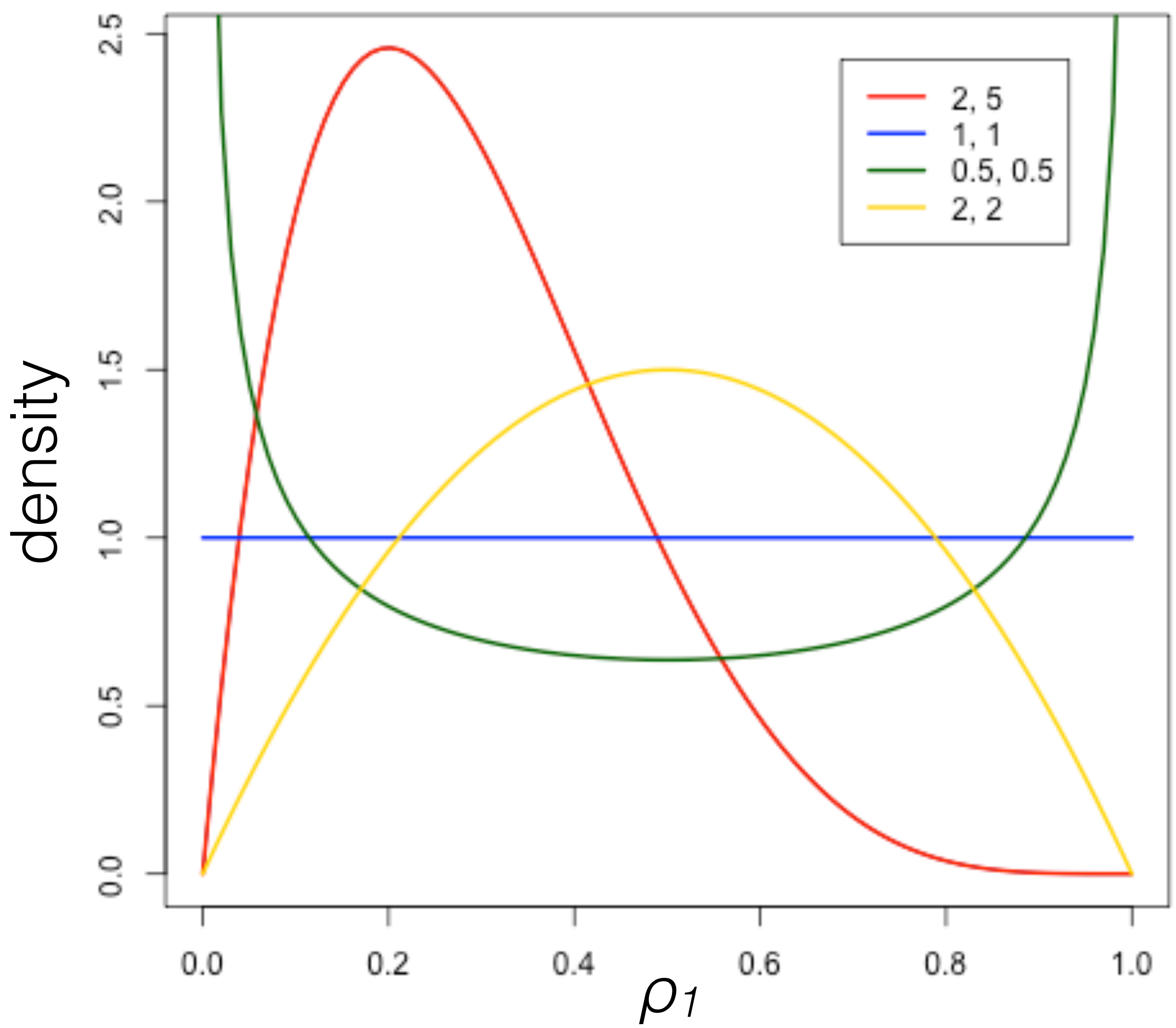


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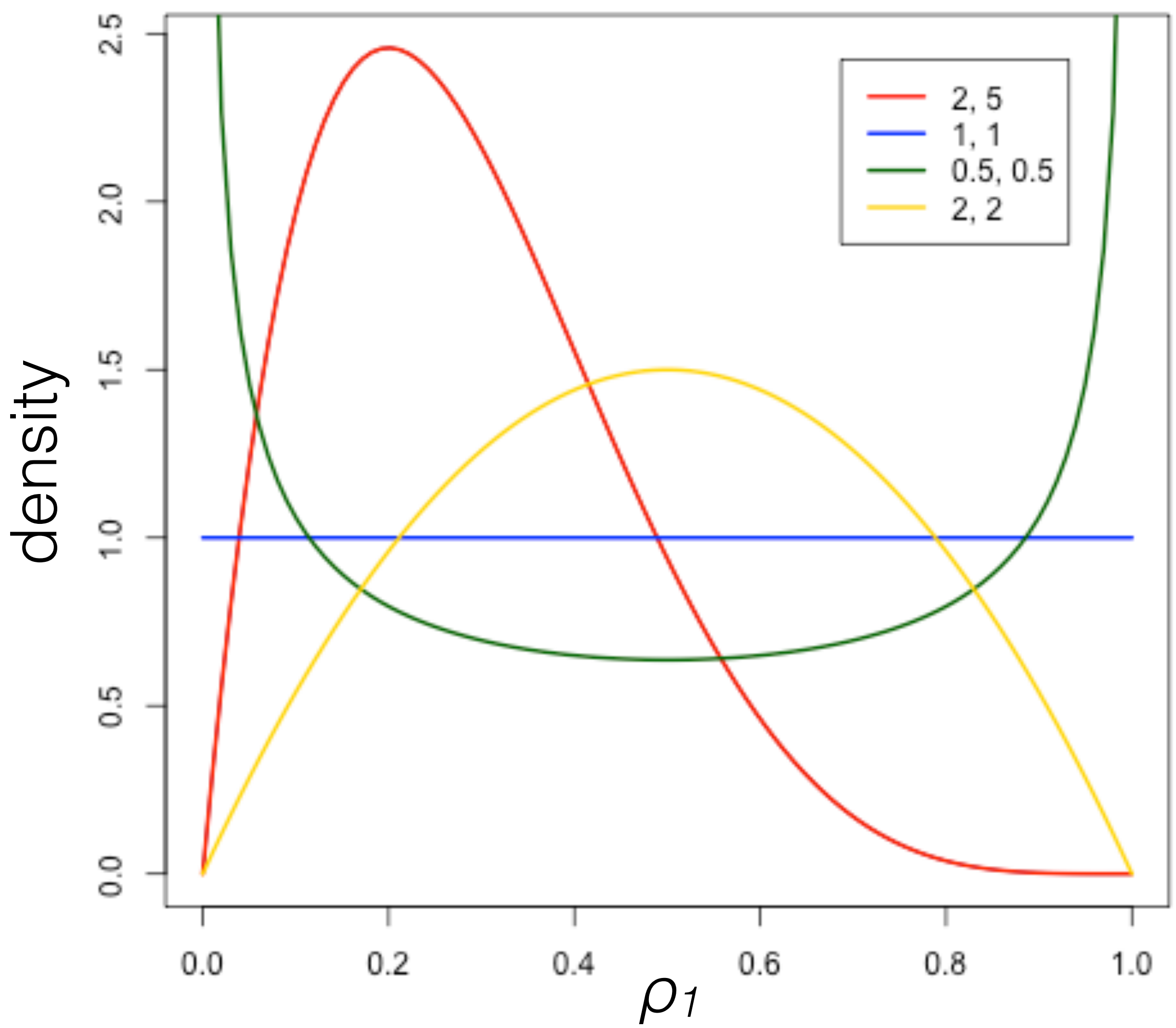


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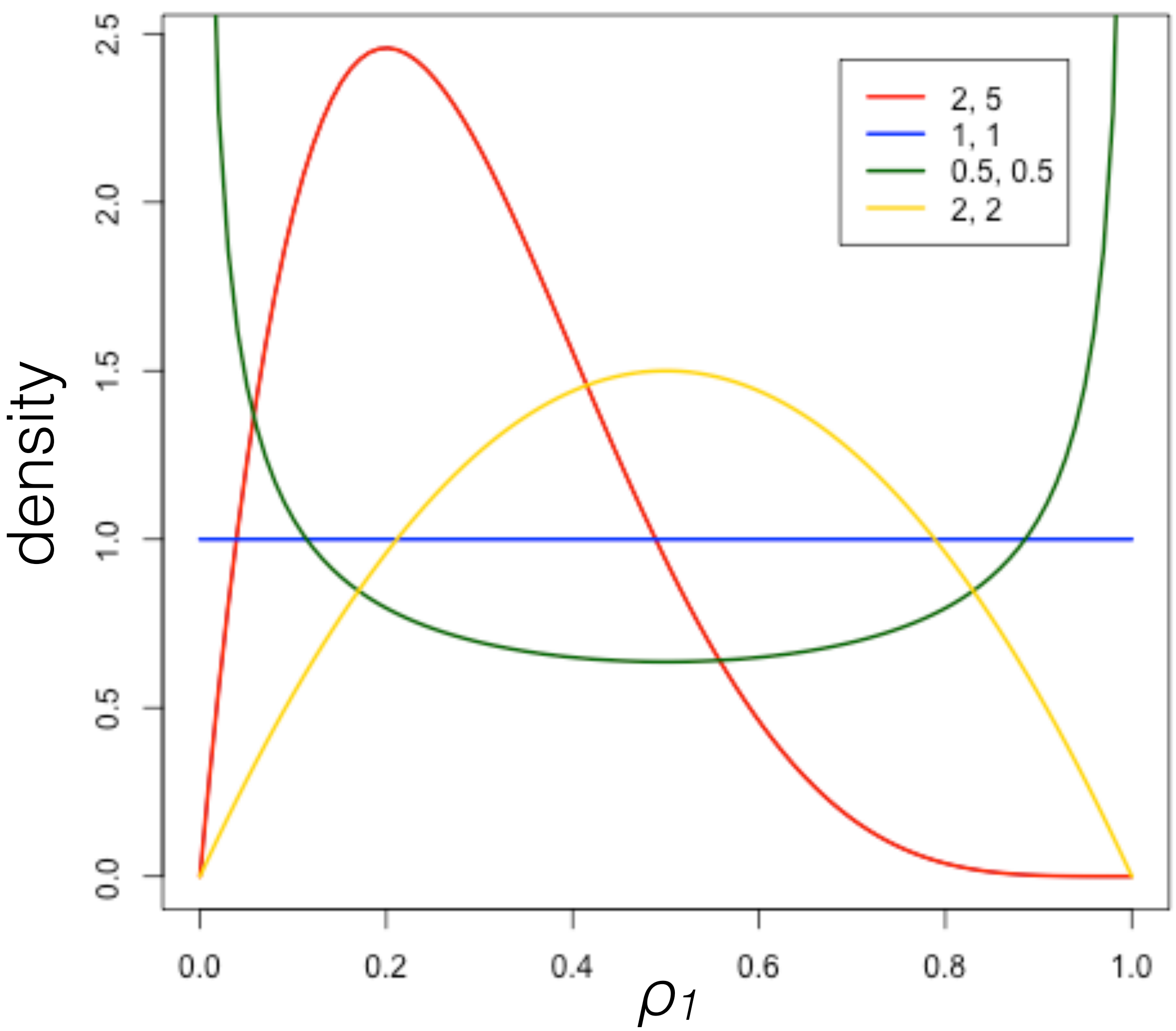


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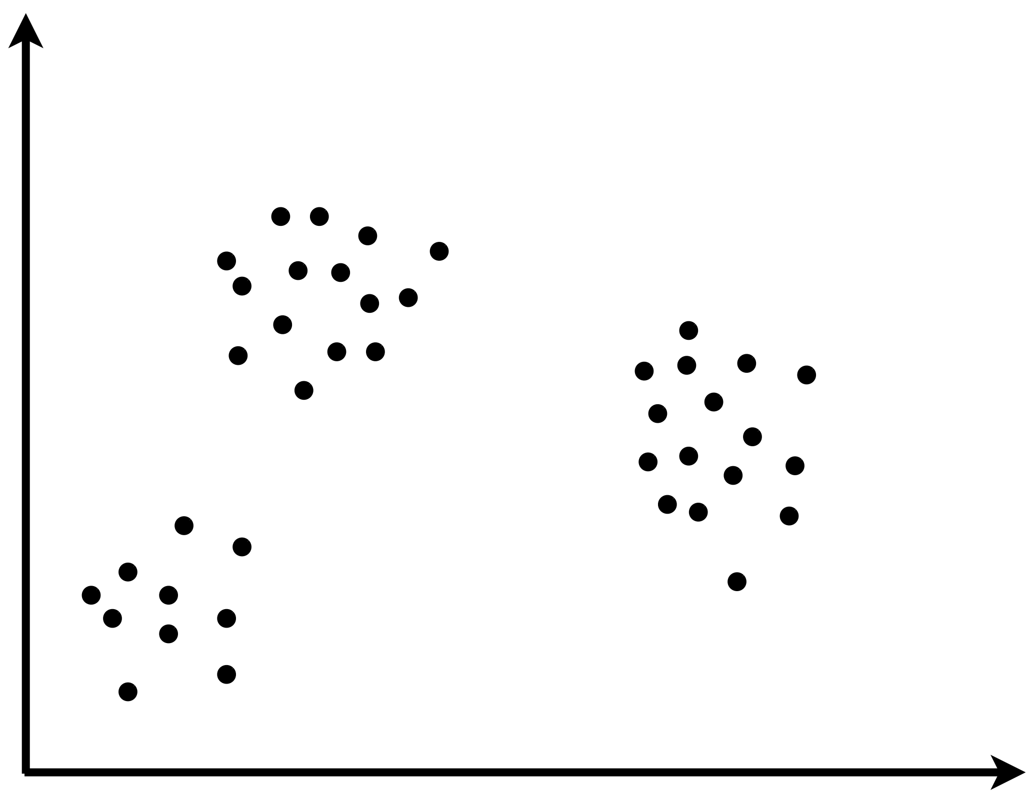

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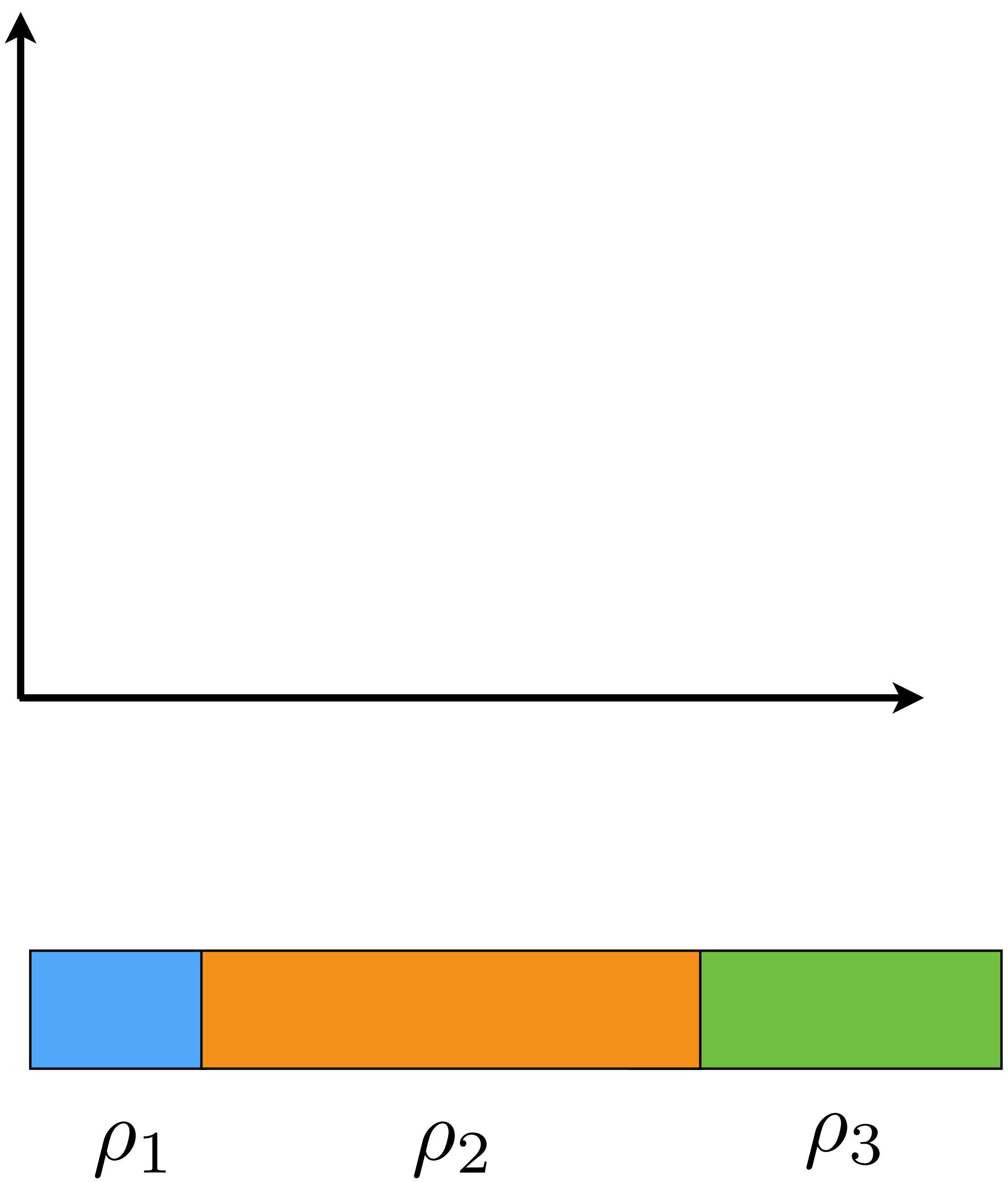


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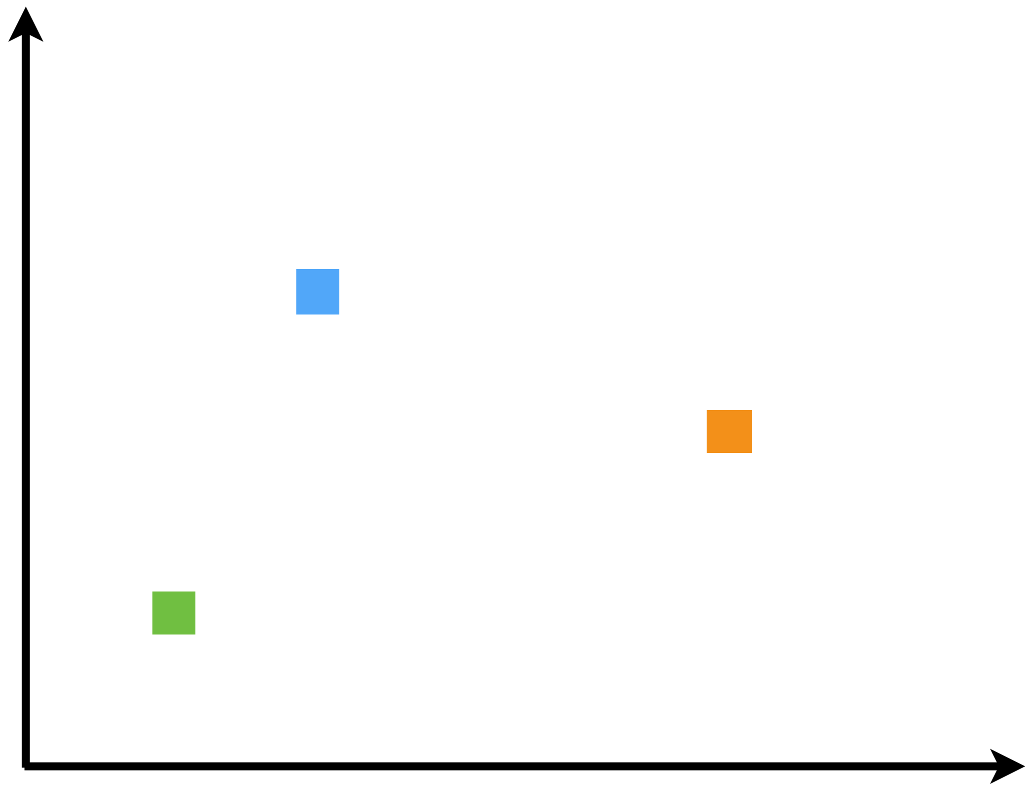

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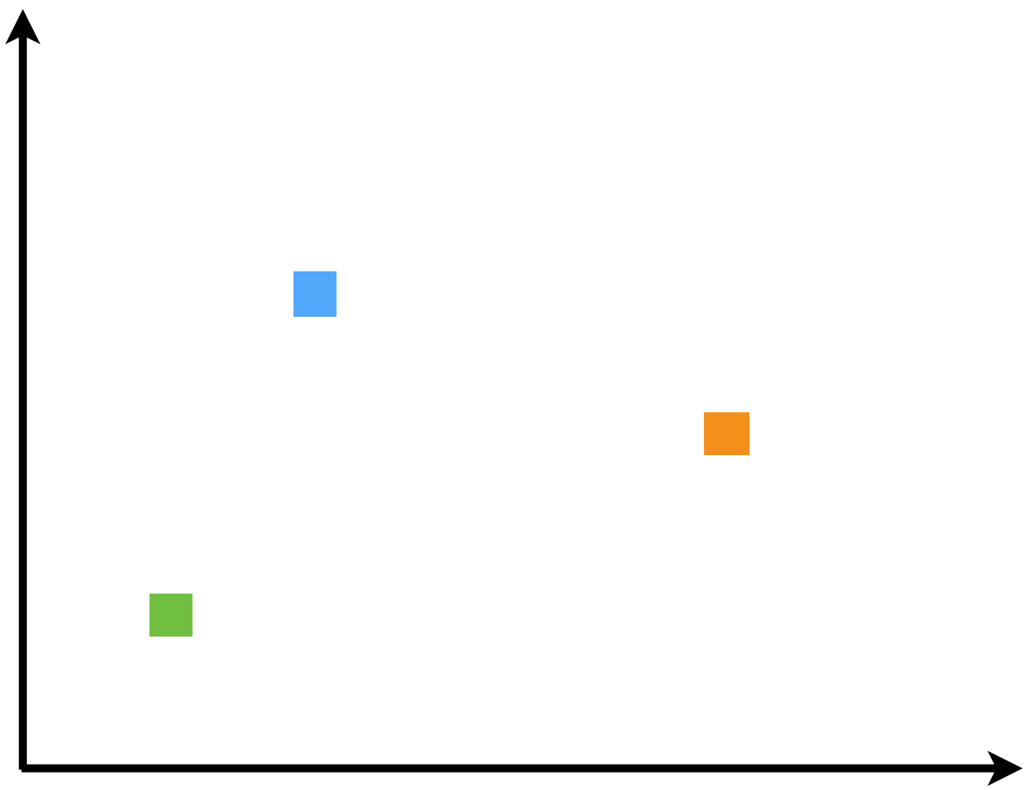

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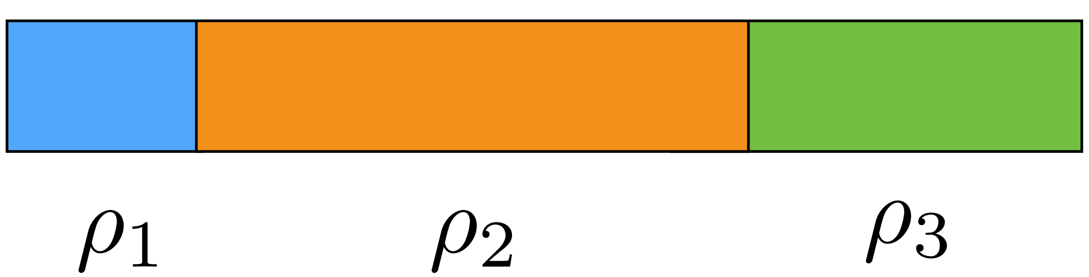

# Generative model

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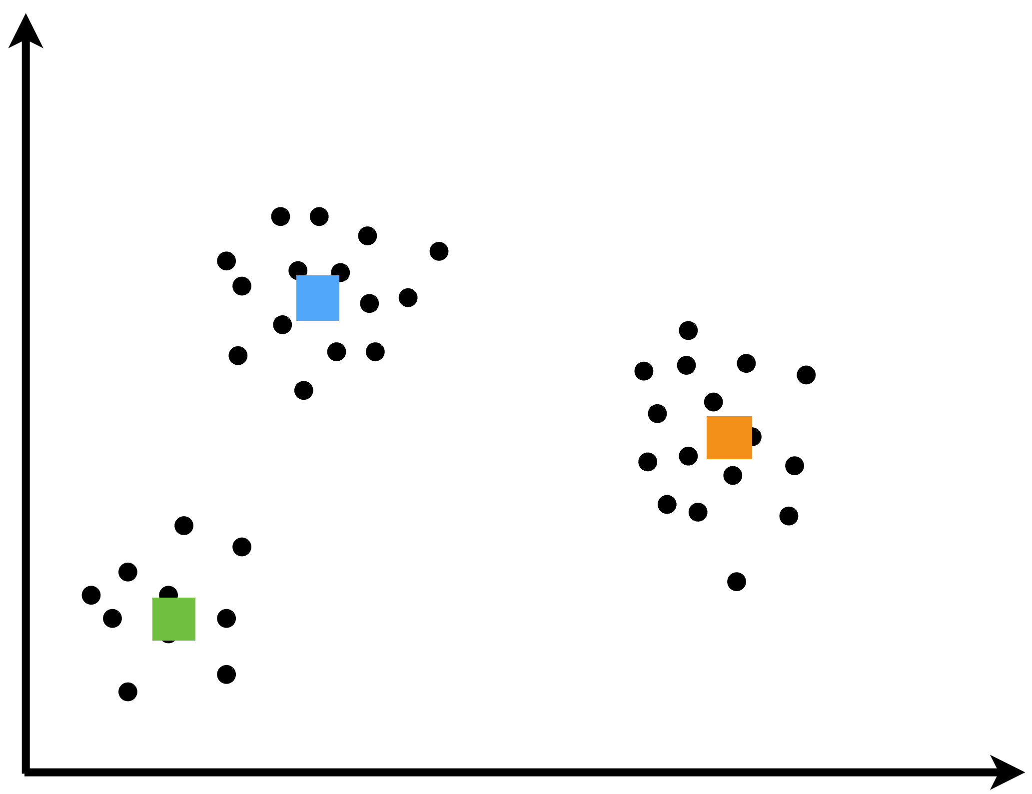


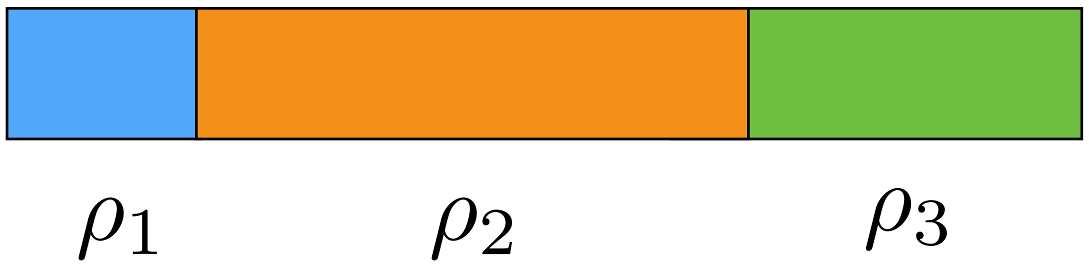
- Finite Gaussian mixture model (*K* clusters)

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$$

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_{1:K})$$

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$



# Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

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$$a_k > 0$$

$$\rho_k \in (0,1)$$

$$\sum_k \rho_k = 1$$

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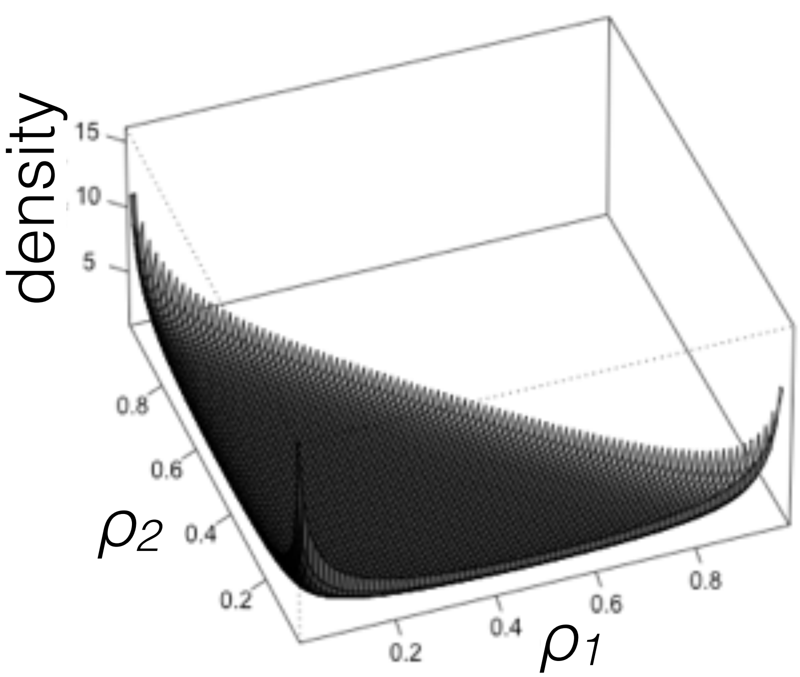


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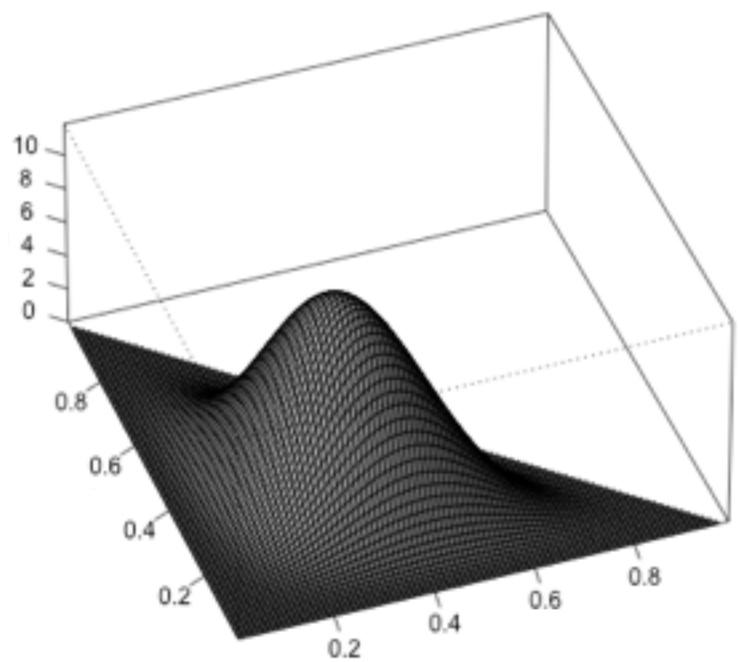
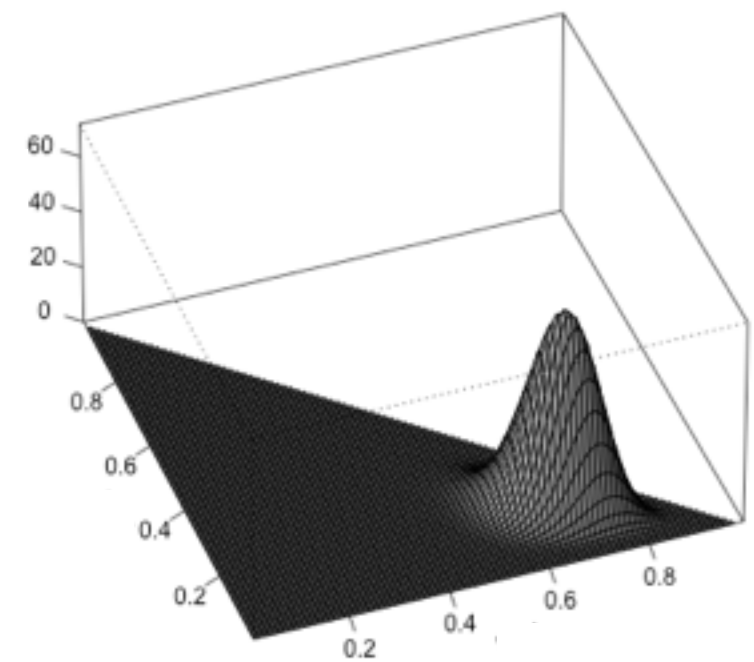



a = (40,10,10)

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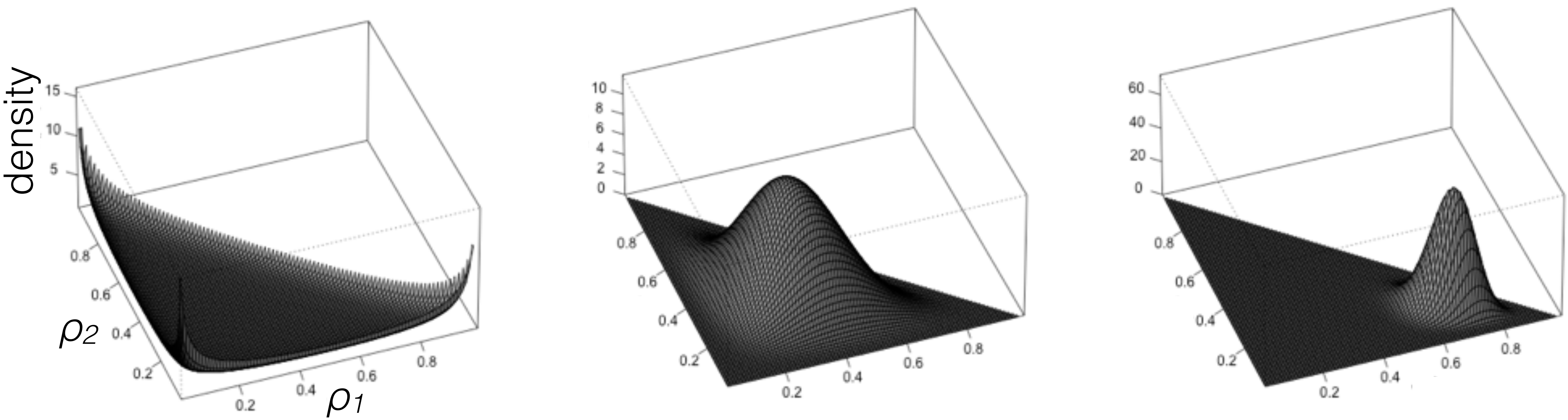


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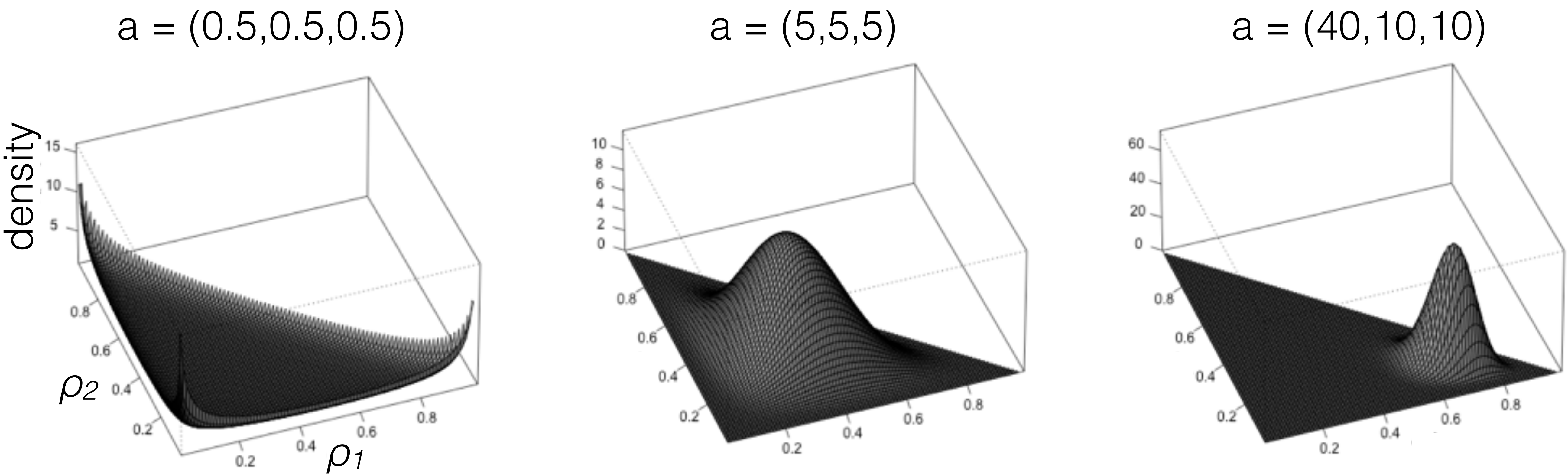


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[demo]

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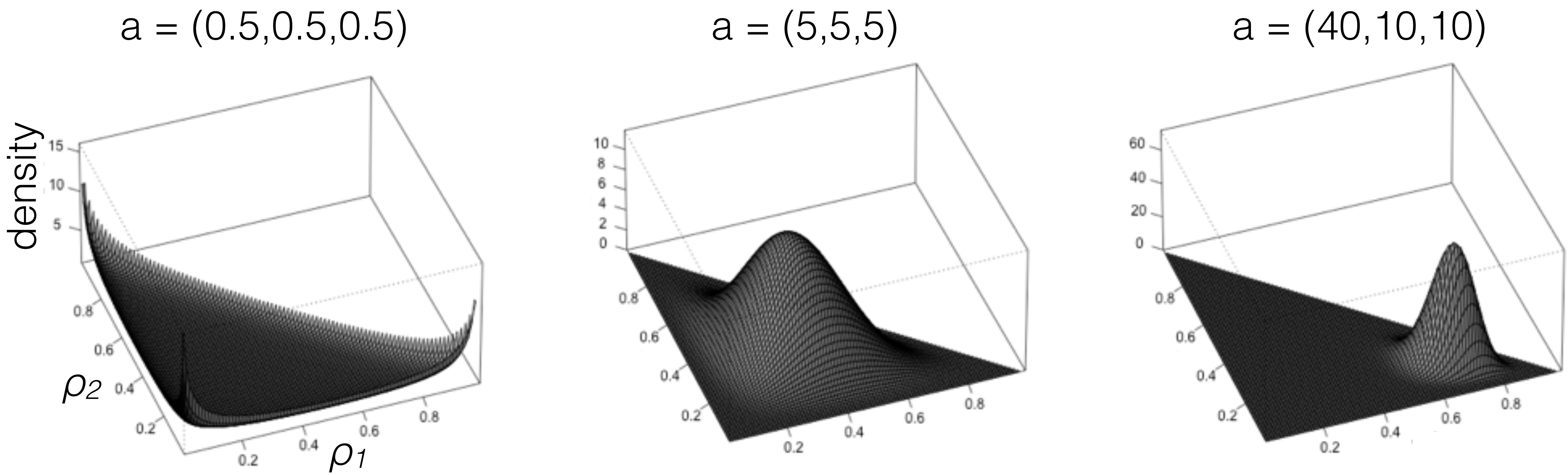


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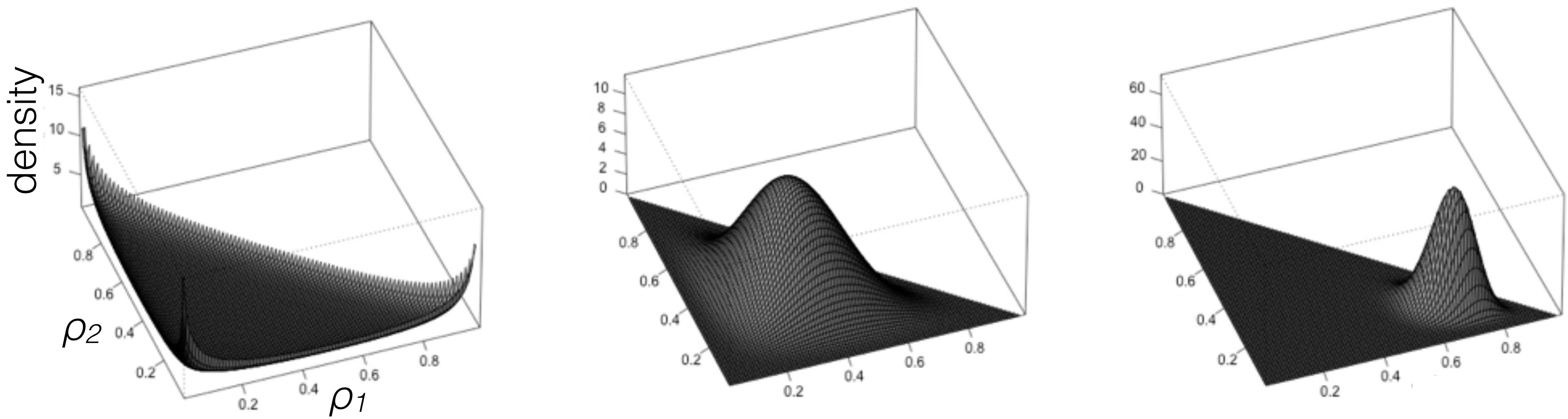


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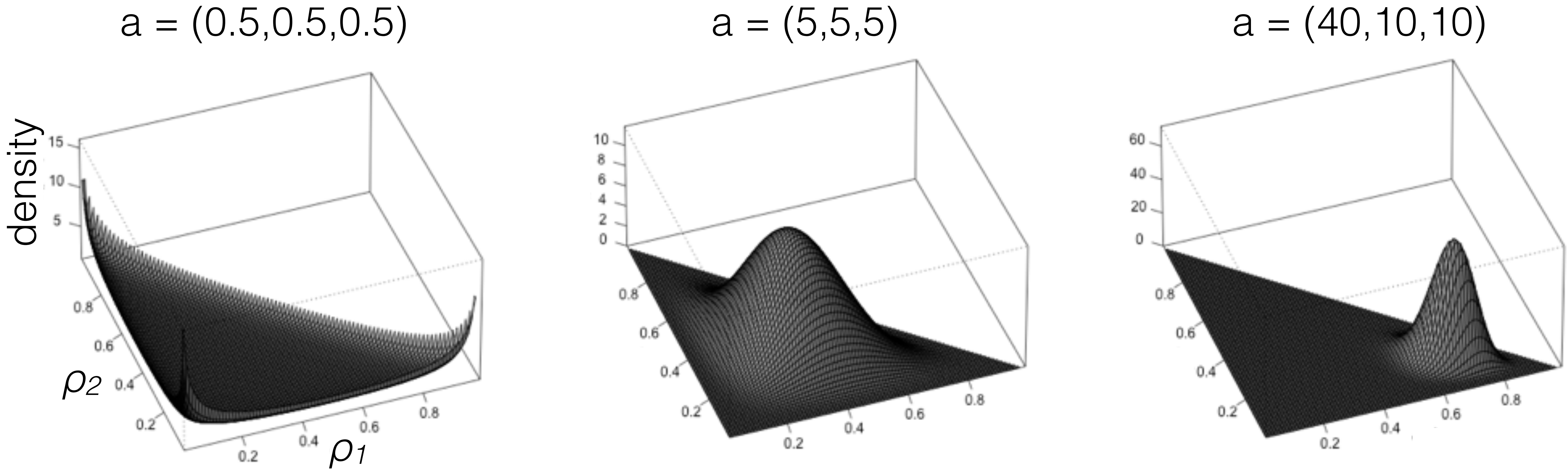


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[demo]

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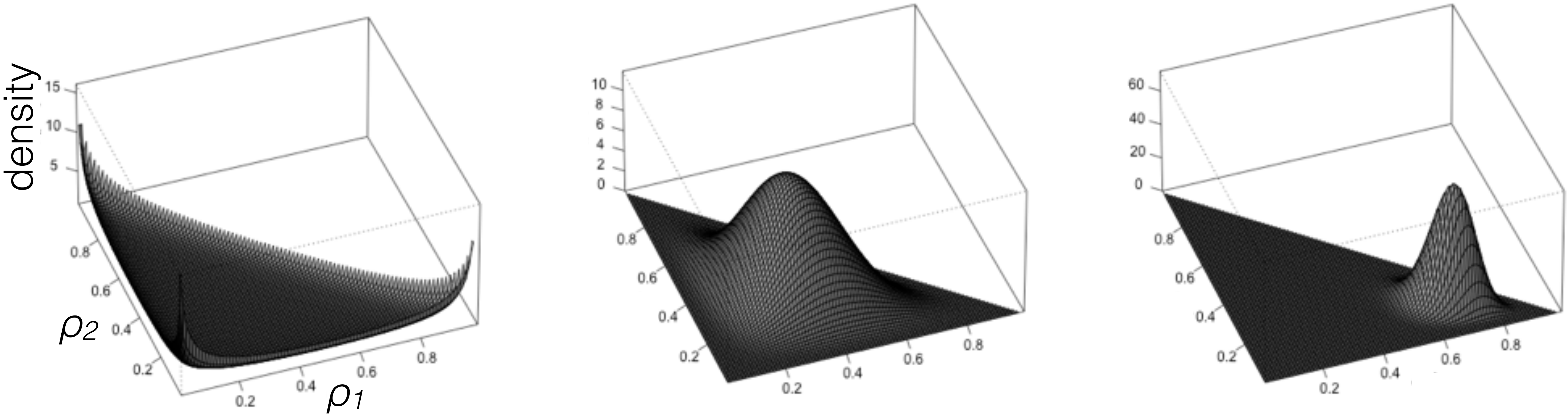


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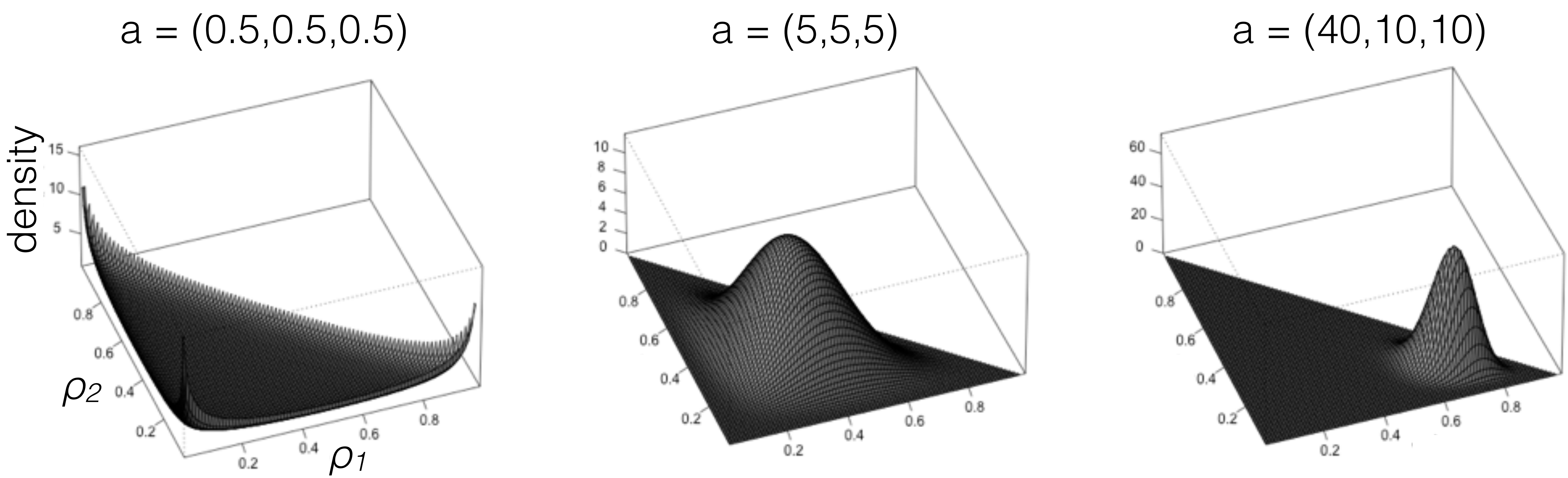


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[demo]

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$$\rho_{1:K} | z \stackrel{d}{=} \mathrm{Dirichlet}(a'_{1:K}), a'_k = a_k + \mathbf{1}\{z = k\}$$

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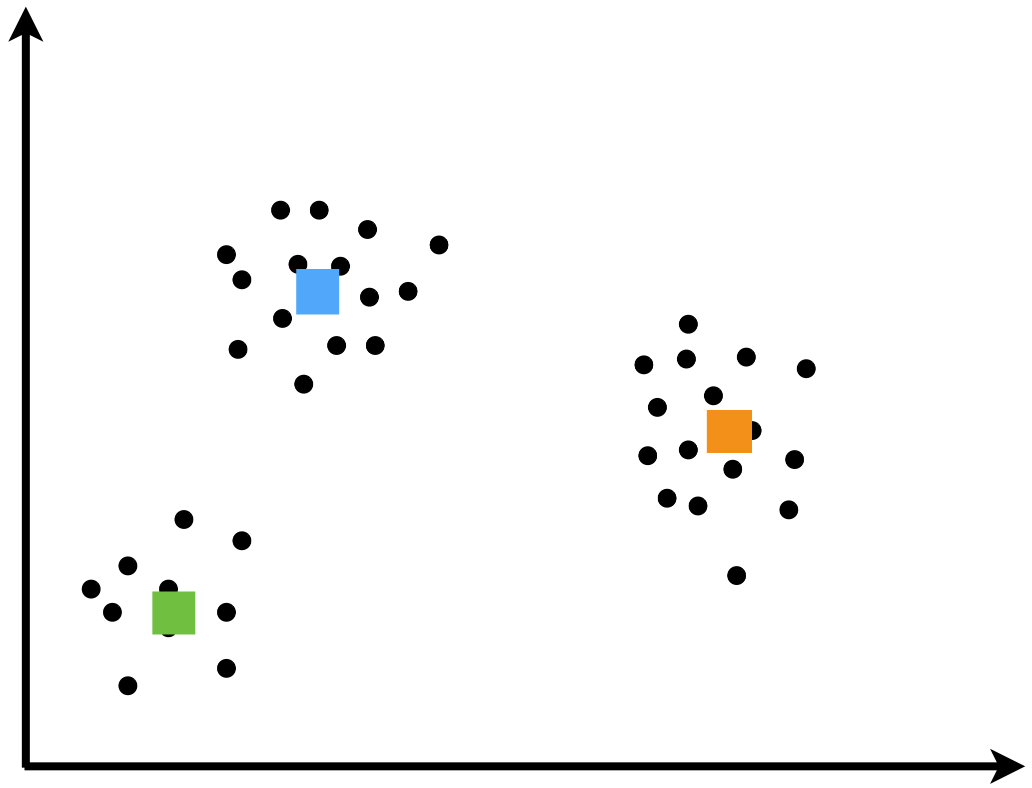

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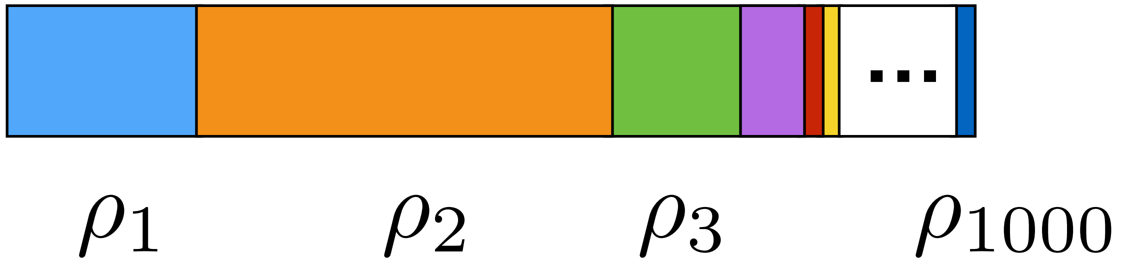

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- e.g. species sampling, topic modeling, groups on a social network, etc.

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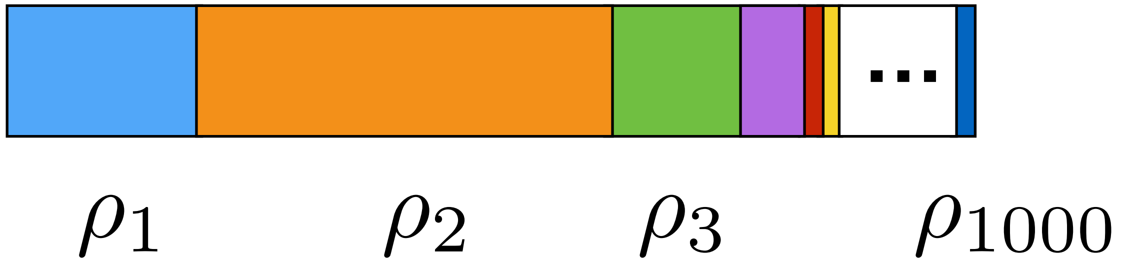


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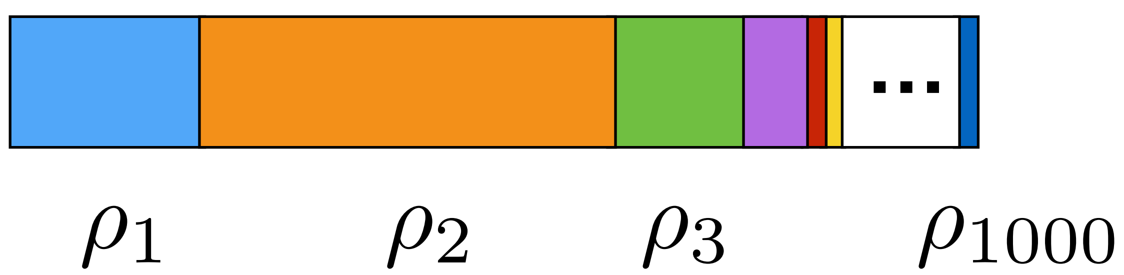

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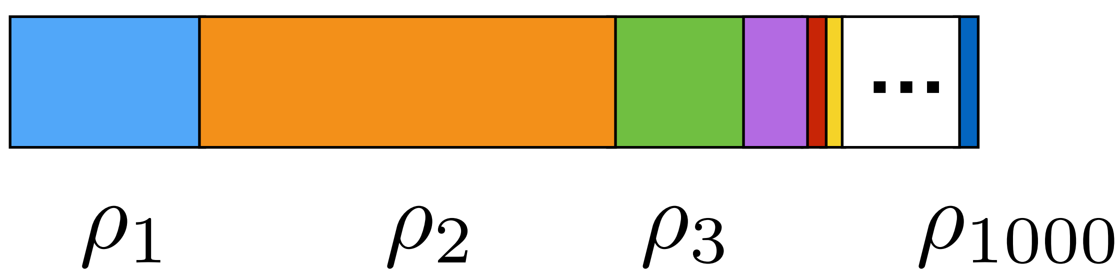


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- [demo 1, demo 2]

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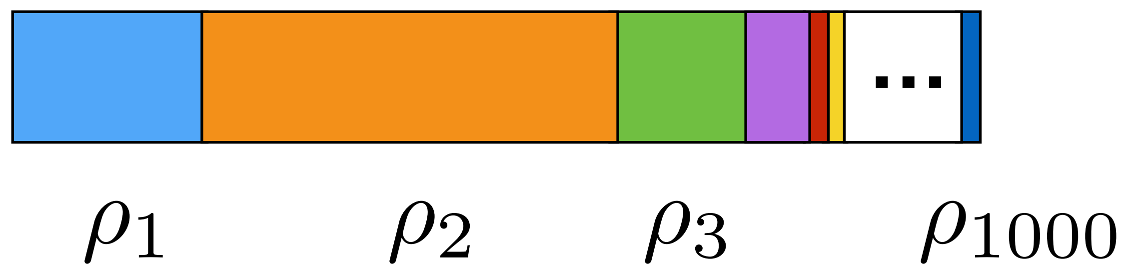


- Components: number of latent groups
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- [demo 1, demo 2]
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- Number of clusters grows with $N$

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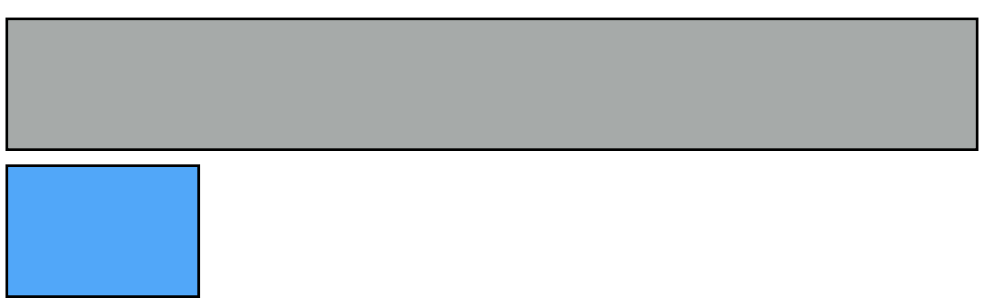

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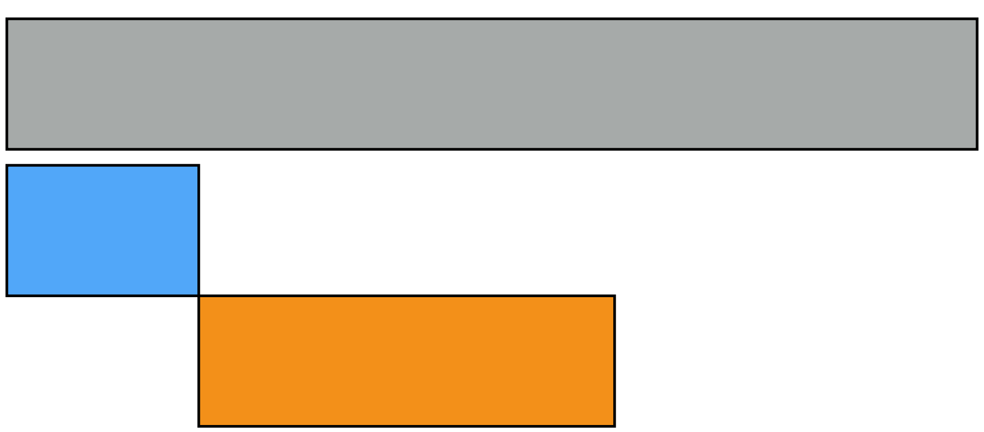

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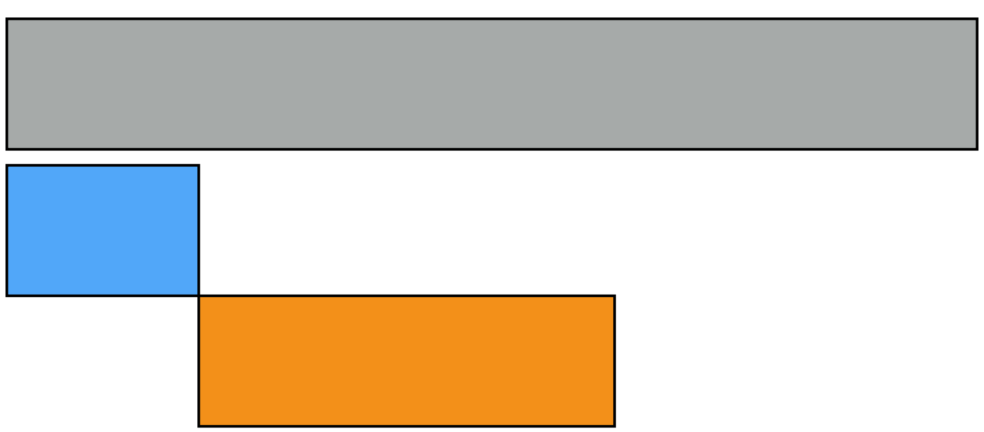

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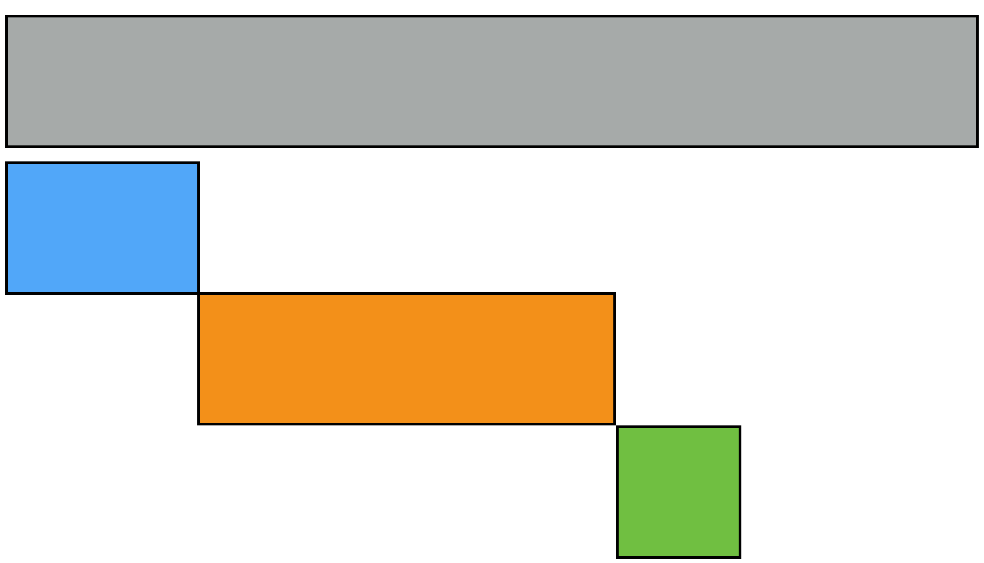

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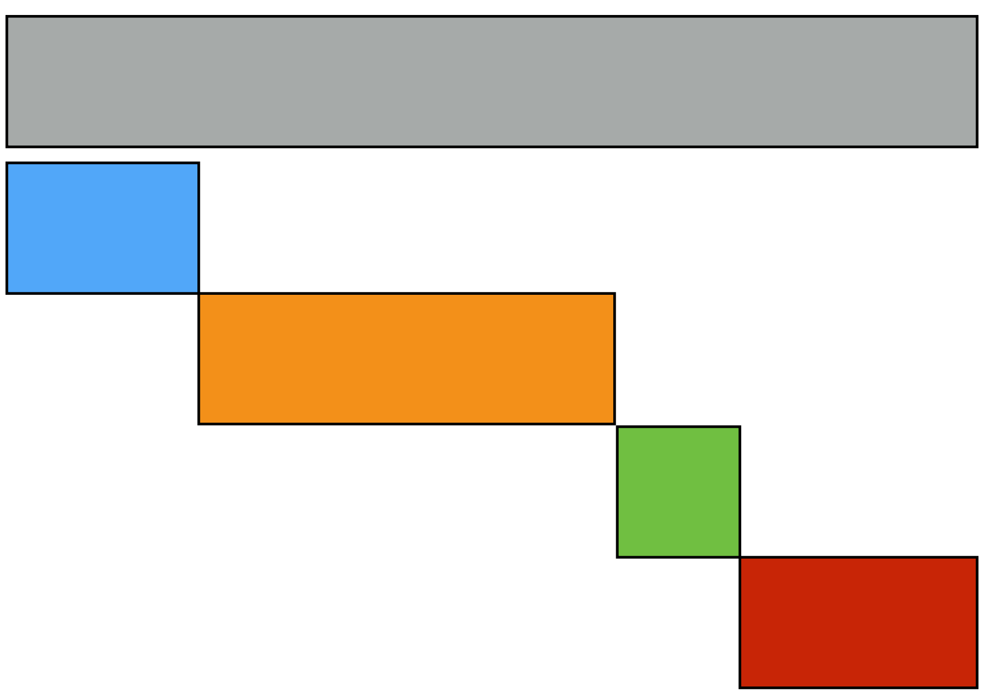

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$$\rho_4 = 1 - \sum_{k=1}^{3} \rho_k$$

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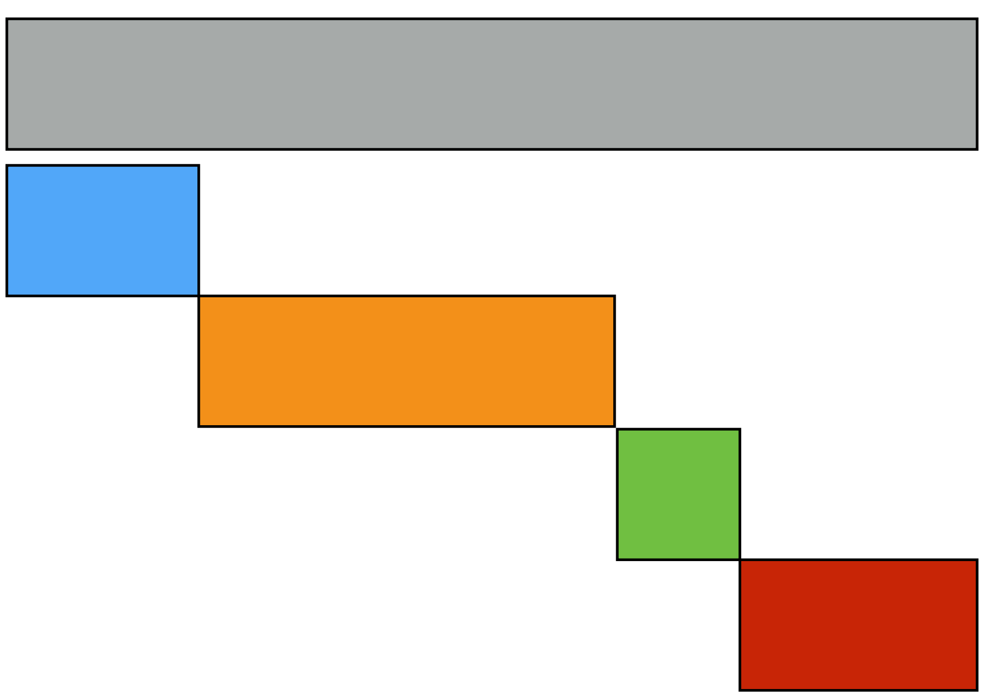

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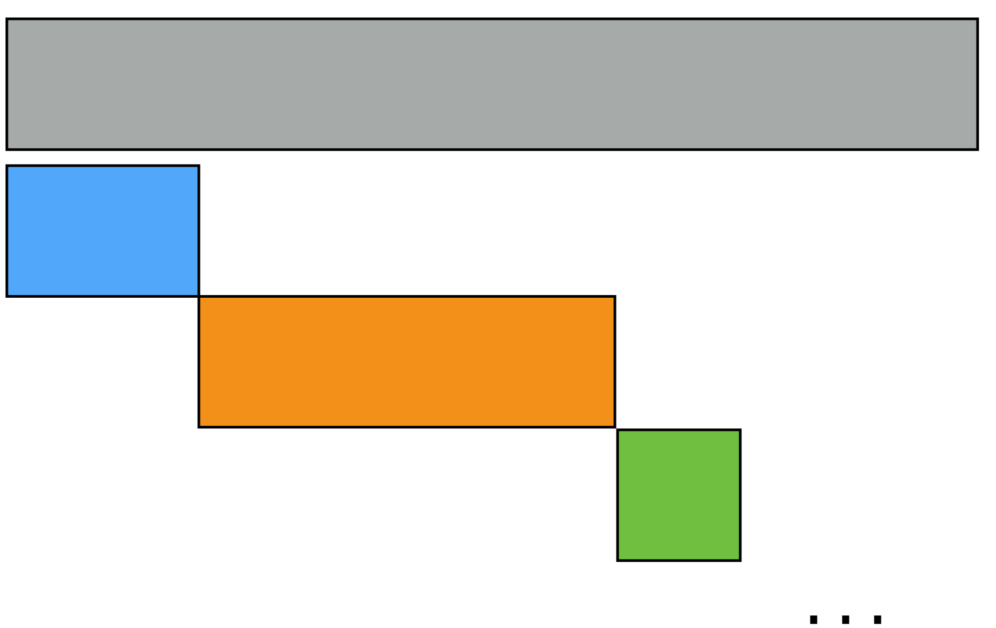

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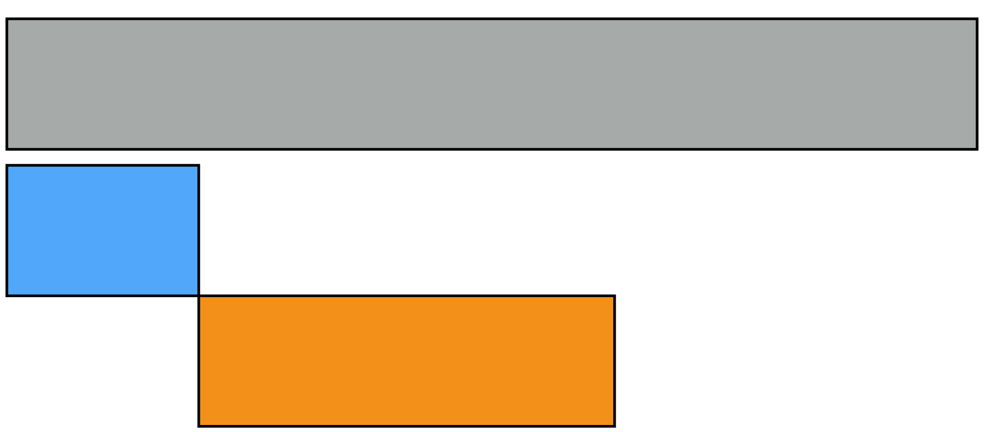

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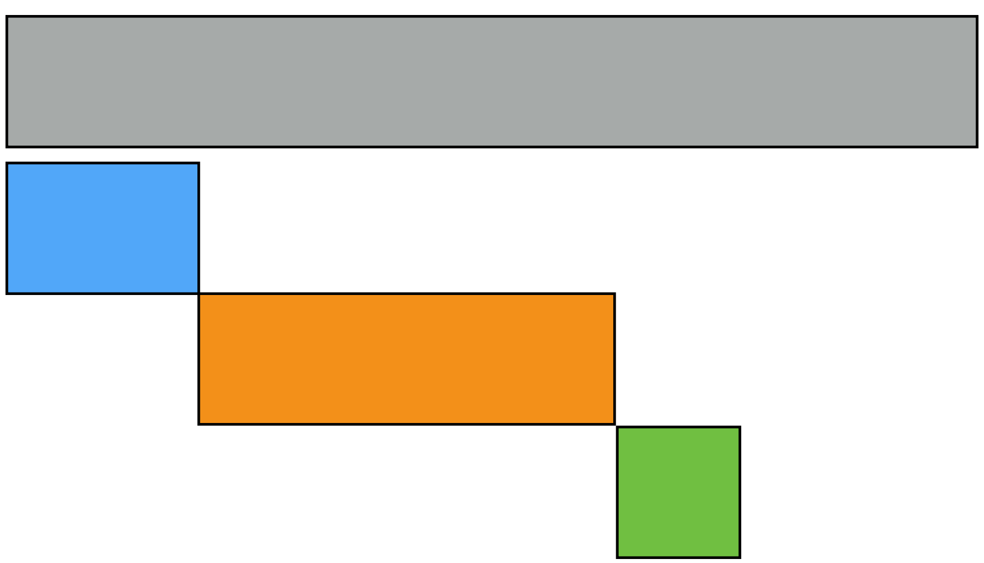

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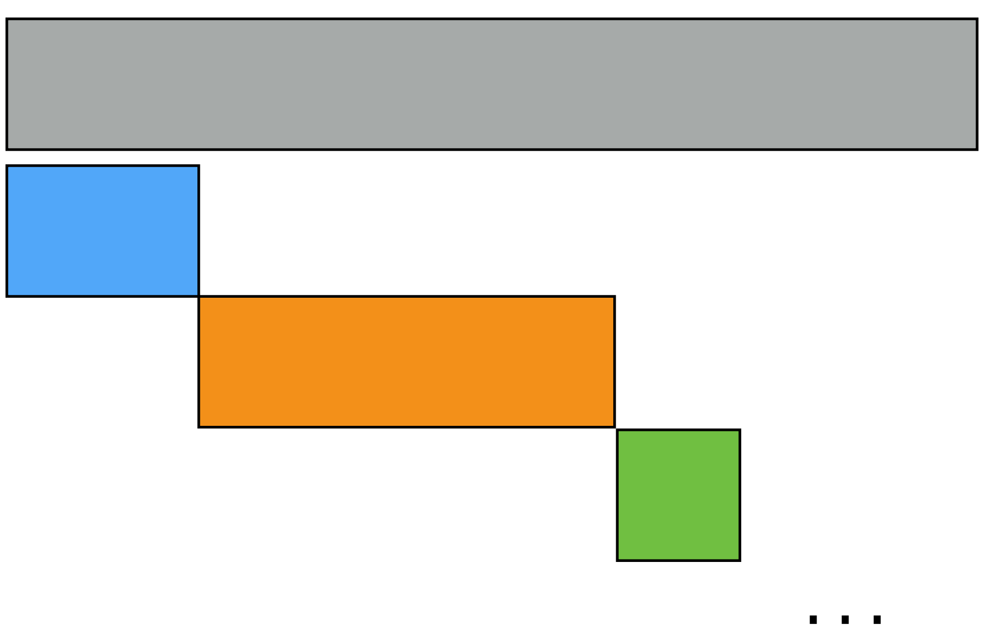

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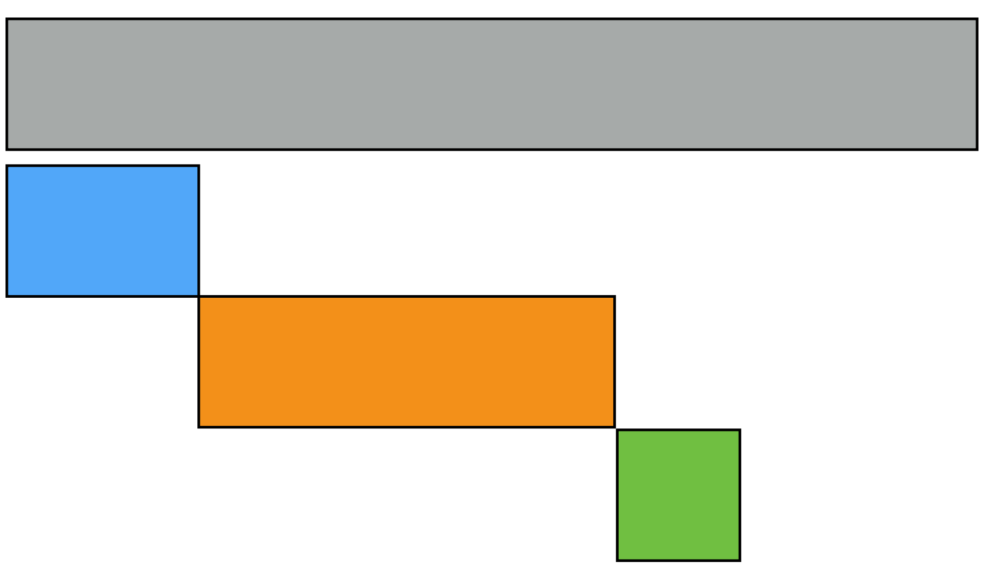

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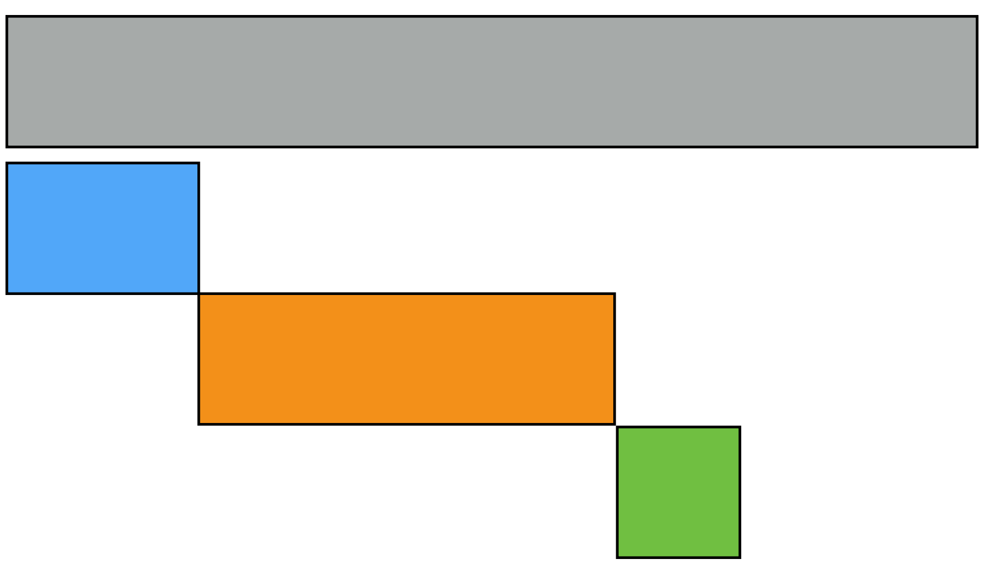

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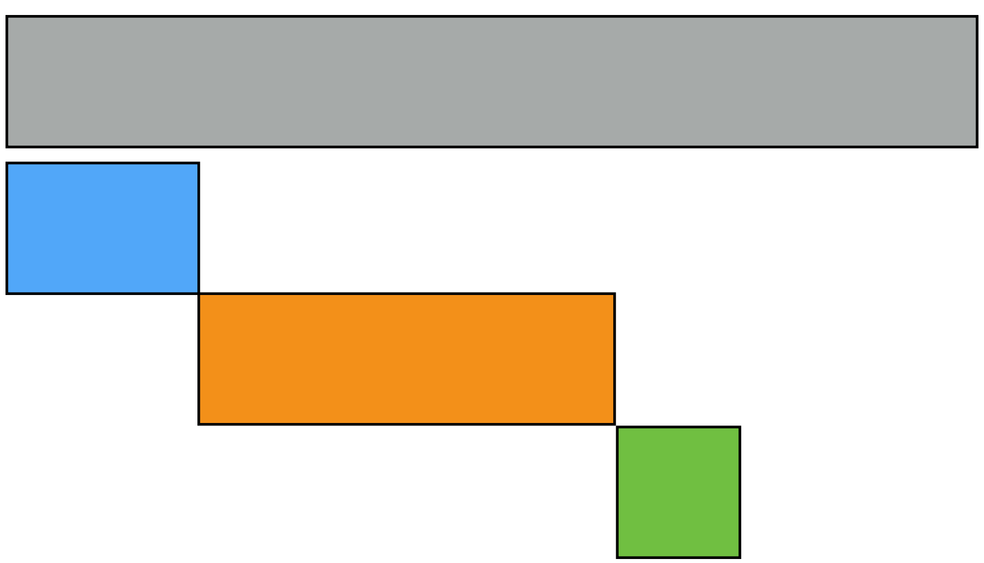

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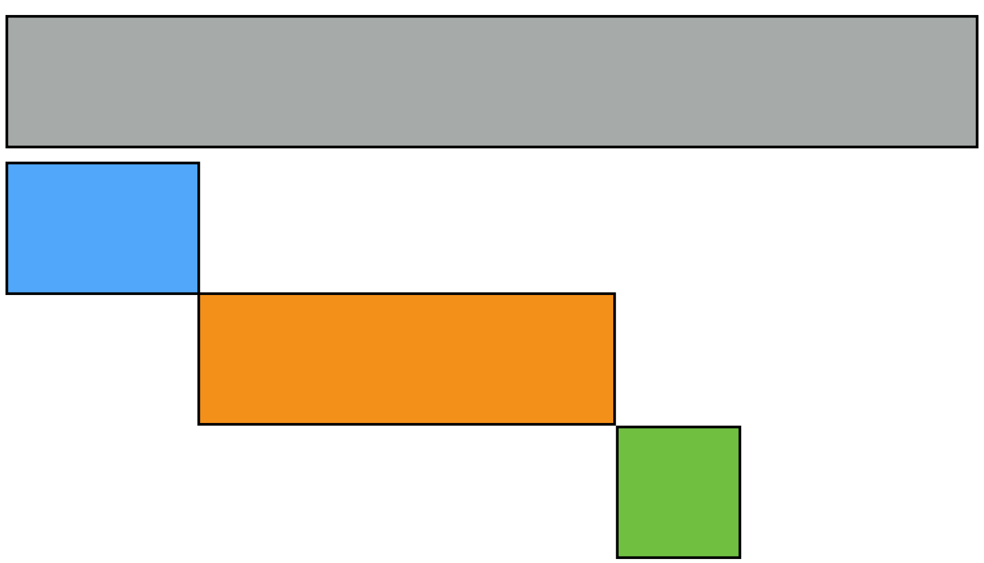

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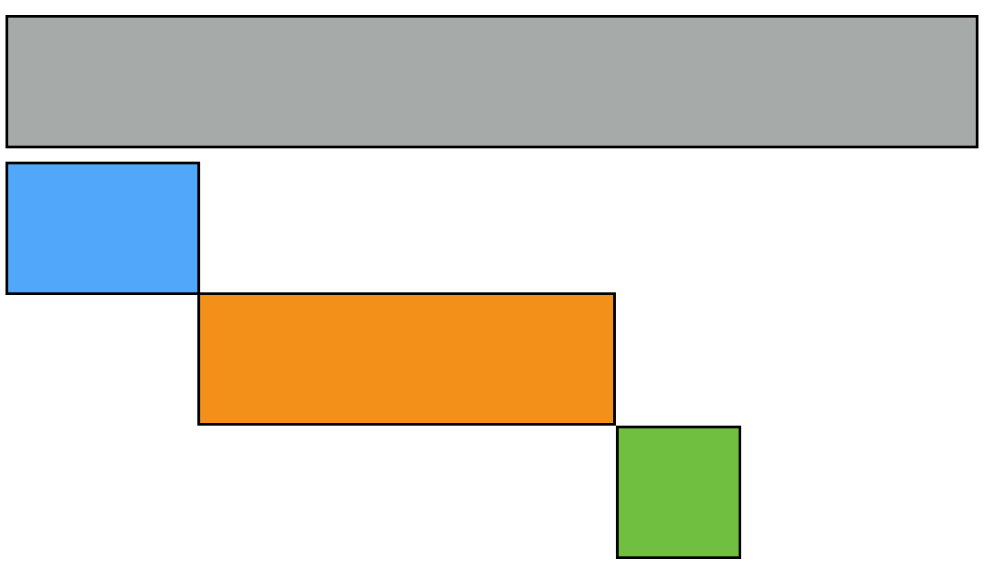

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[McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; Ishwaran, James 2001]

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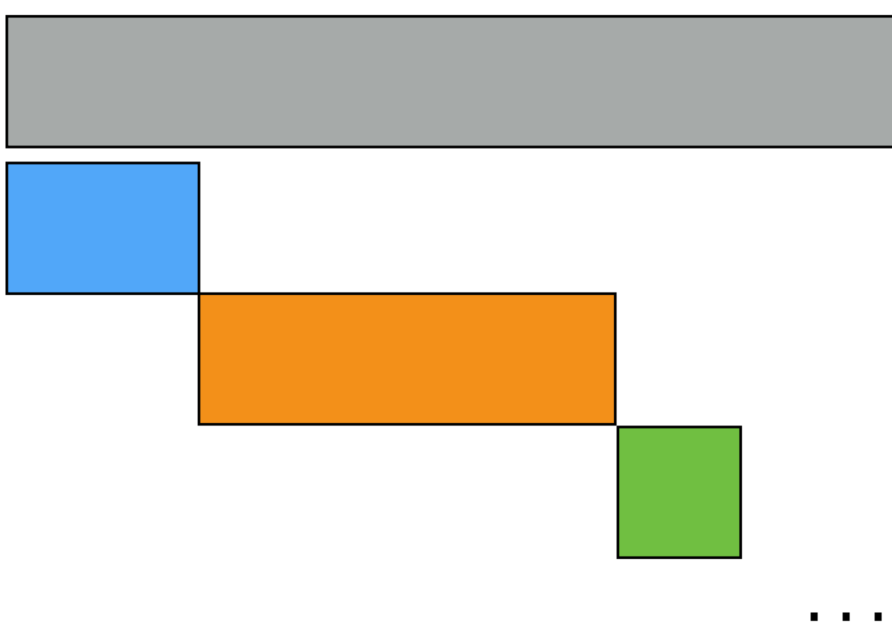

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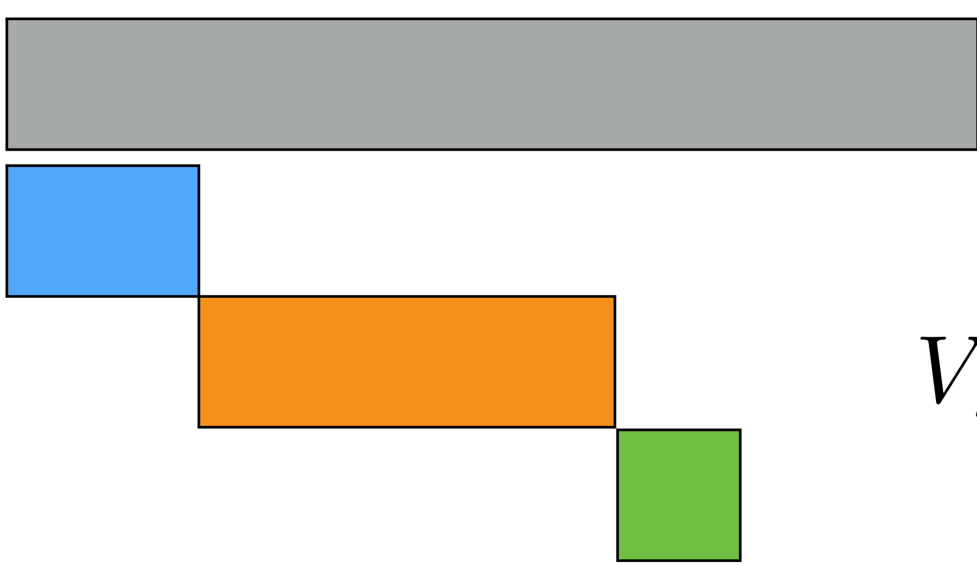

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[demo]

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# Distributions

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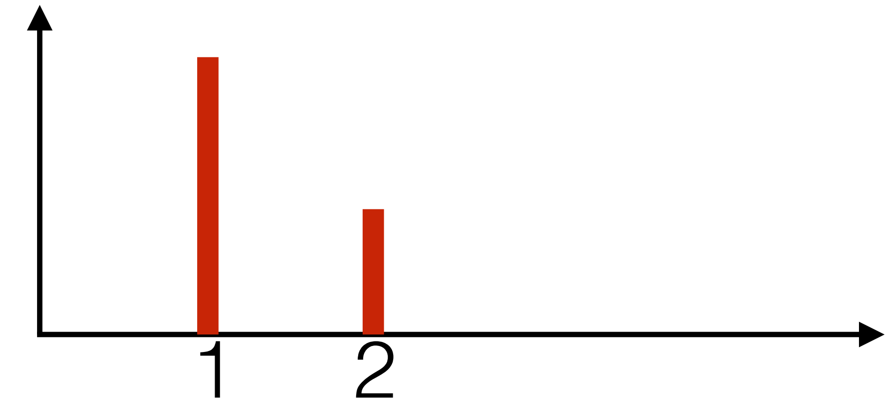

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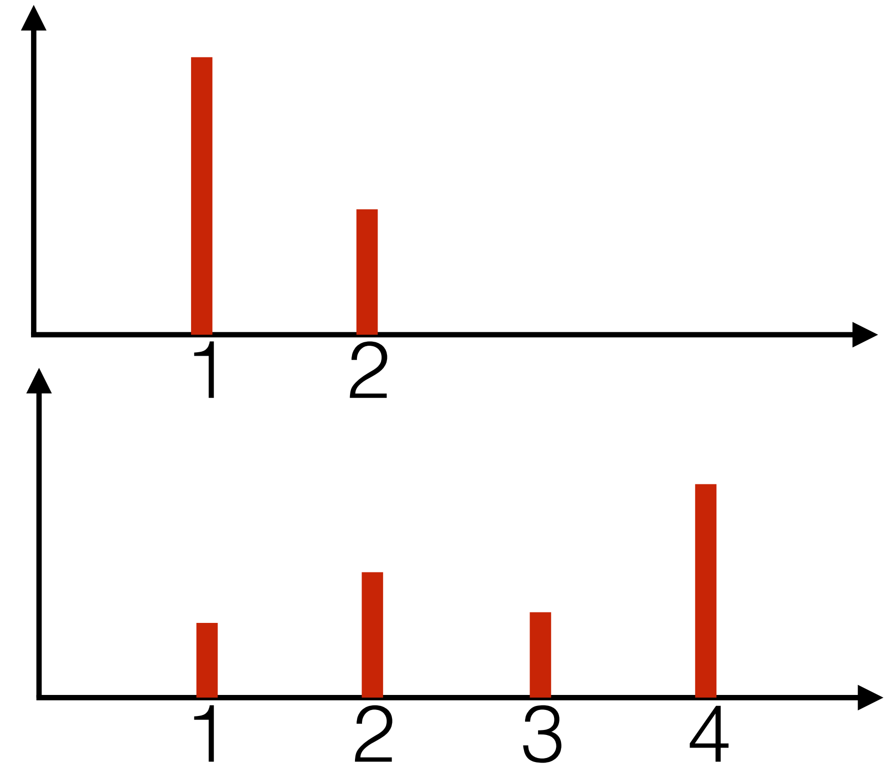

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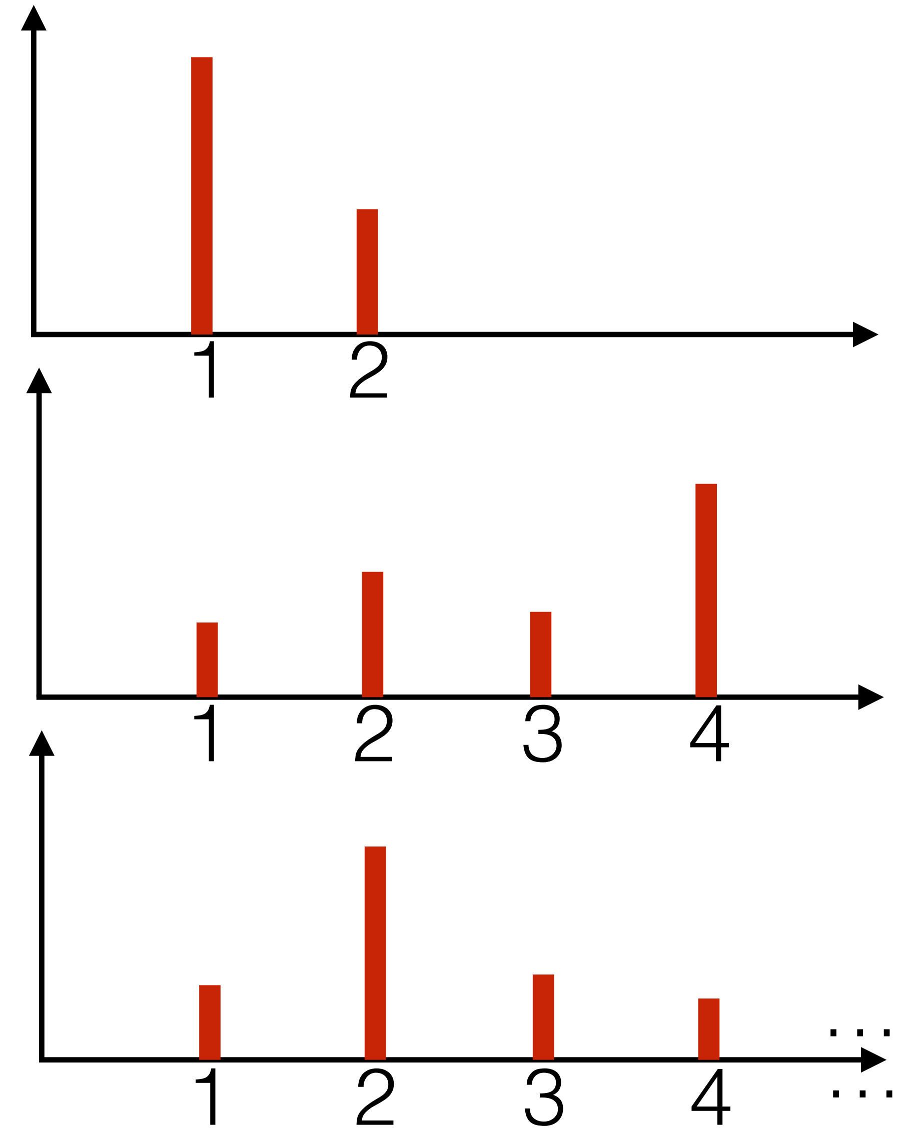

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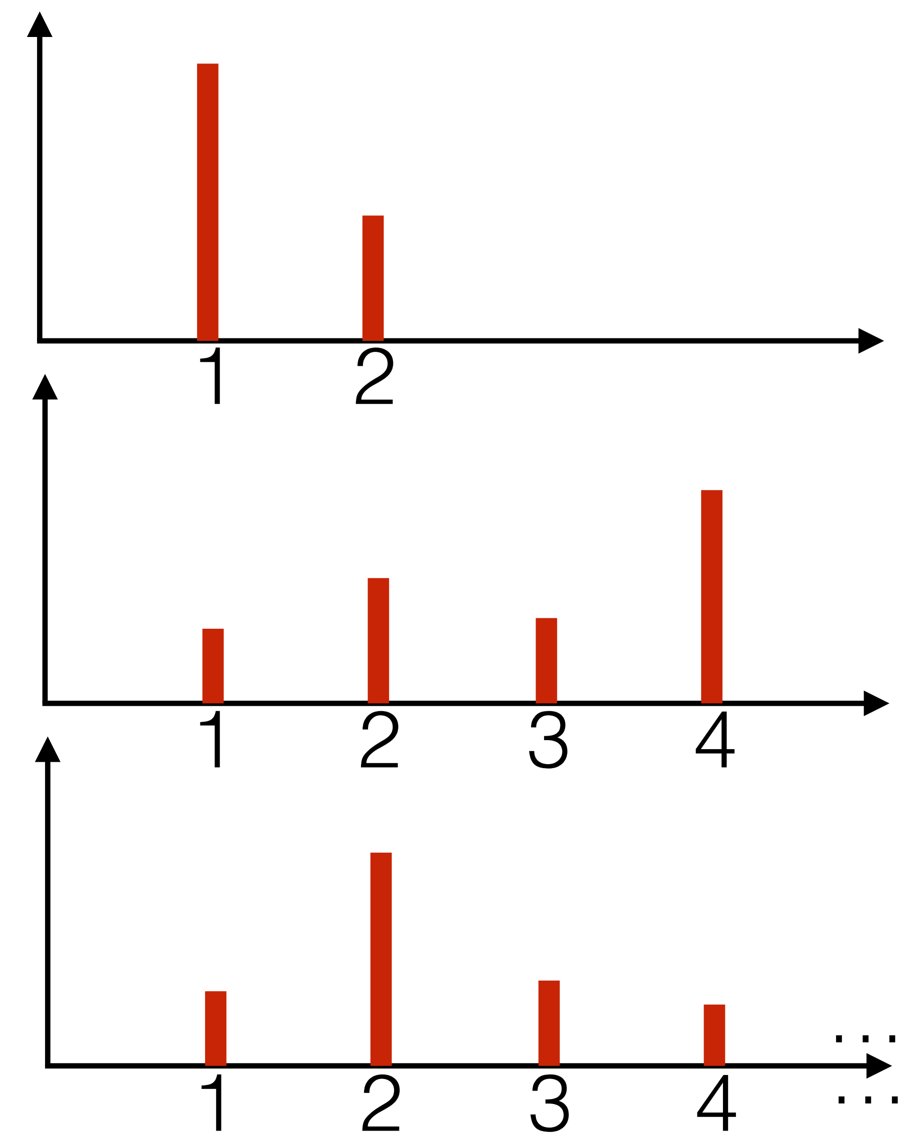


- Infinity of parameters: components
- Growing number of parameters: clusters

# Roadmap

- Example problem: clustering
- Example NPBayes model: Dirichlet process
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayes

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  - Why NPBayes?
  - What does an infinite/growing number of parameters really mean (in NPBayes)?
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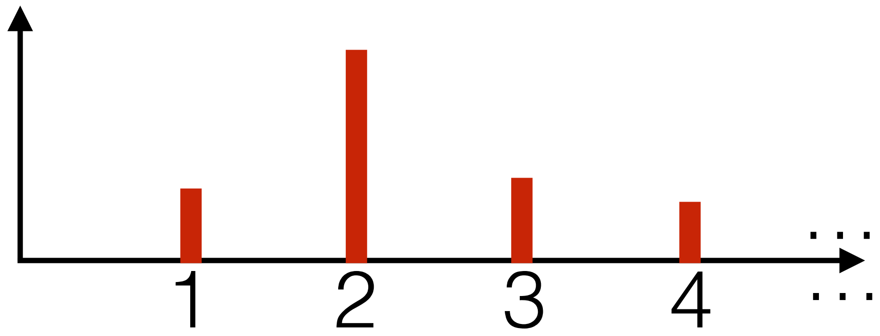
# Exercises



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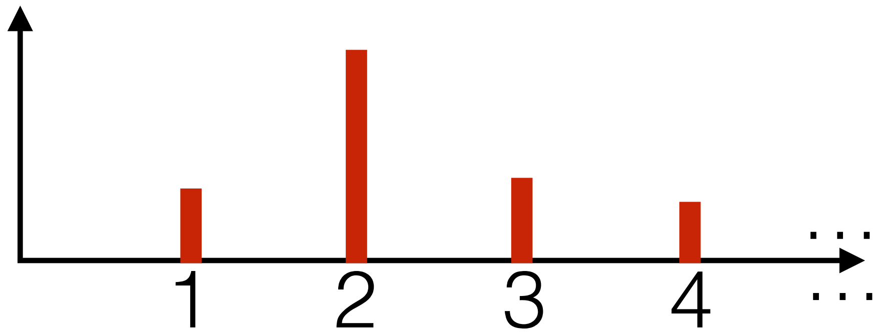
[slides, code: www.tamarabroderick.com/tutorials.html]



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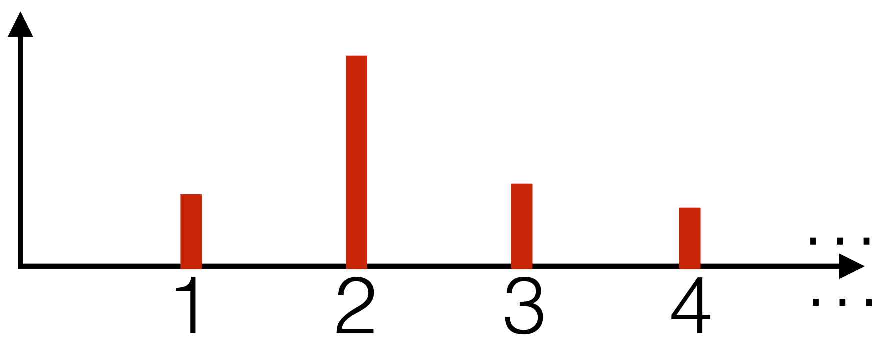

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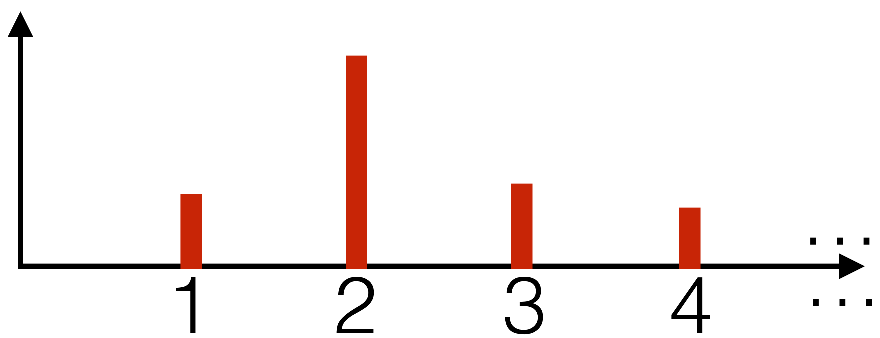

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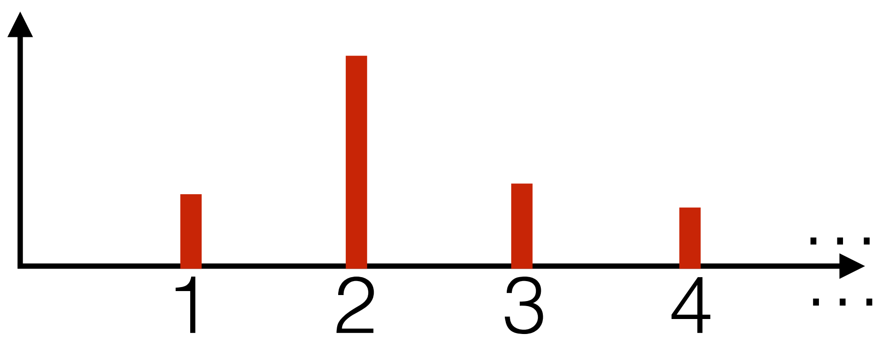

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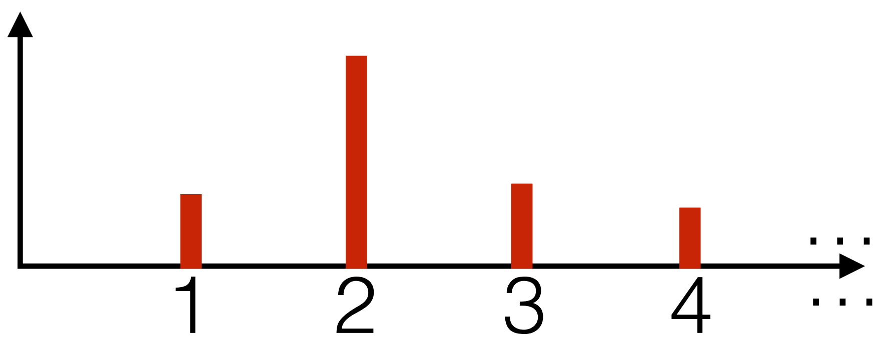

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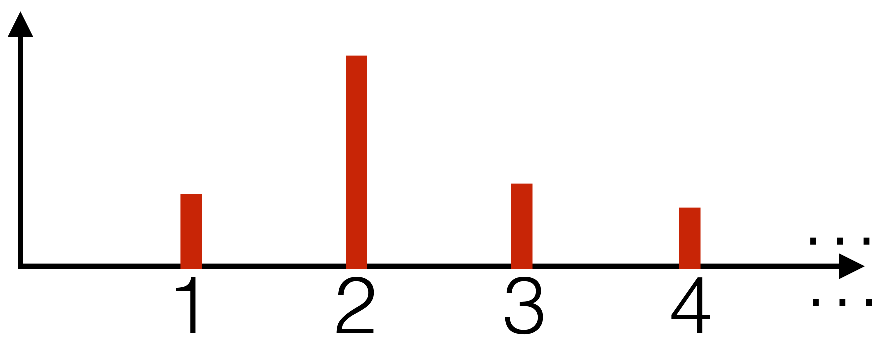


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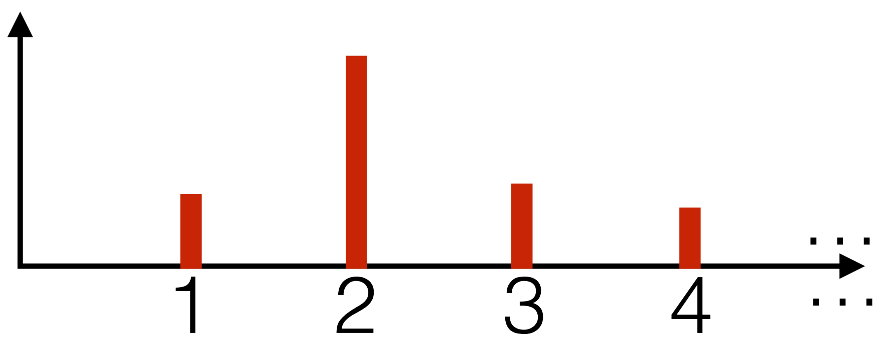


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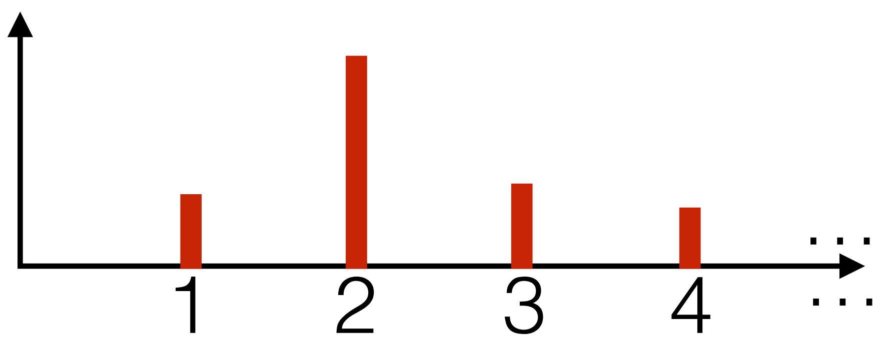


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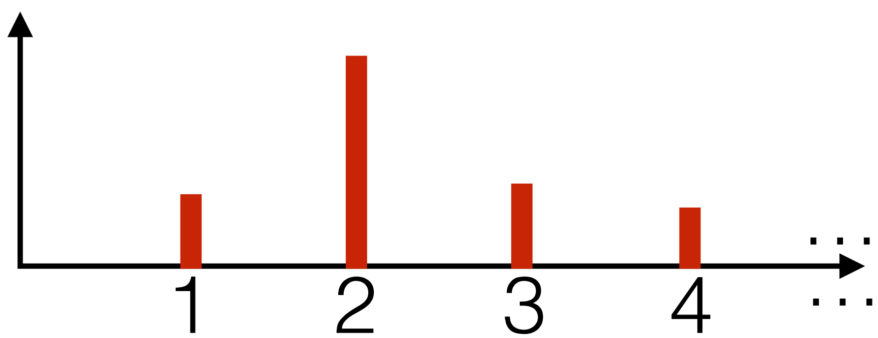


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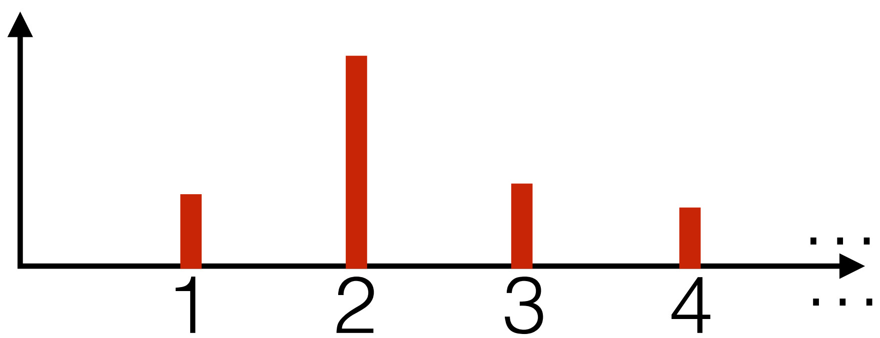


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- For which stick-breaking $(a_k, b_k)$ can you prove $\sum_{k=1}^{\infty} \rho_k = 1$ ?

# References

A full reference list is provided at the end of the “Part III” slides.