Covariances, Robustness, and Variational Bayes

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Jonathan H. Huggins, Michael I. Jordan

Microcredit Experiment

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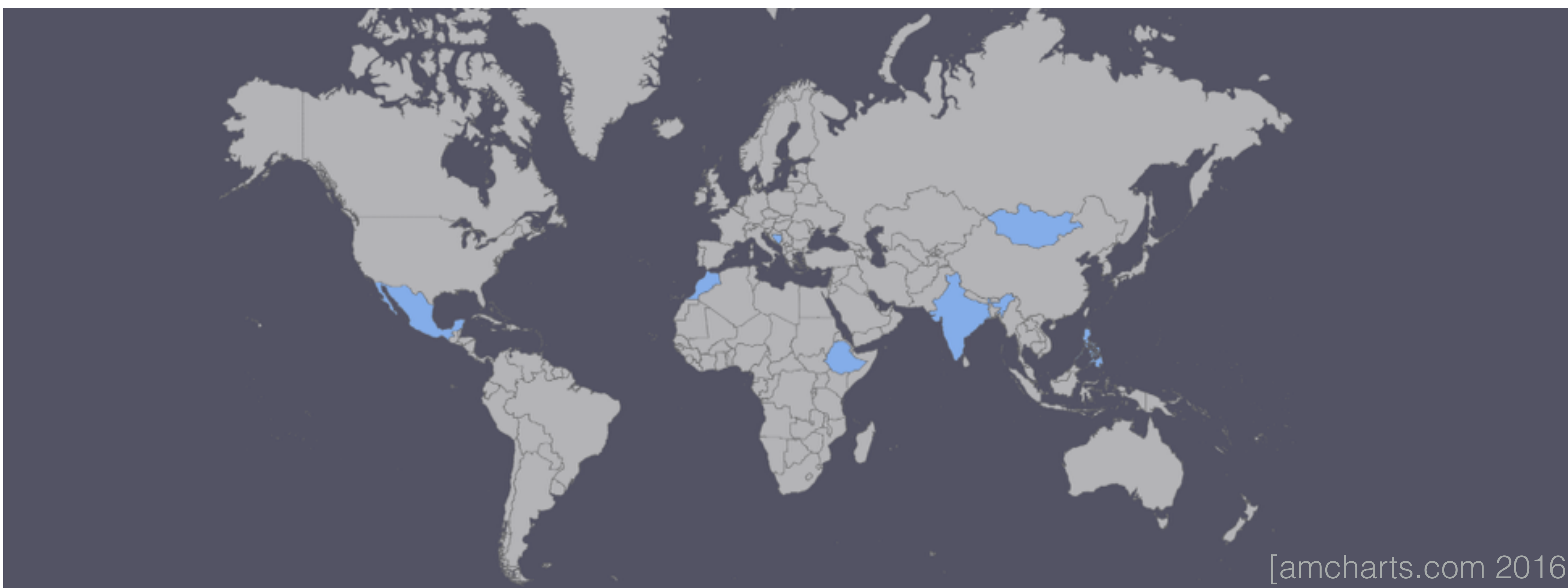
- K = 7 sites with microcredit trials (Mexico, Mongolia, Bosnia, India, Morocco, Philippines, Ethiopia)



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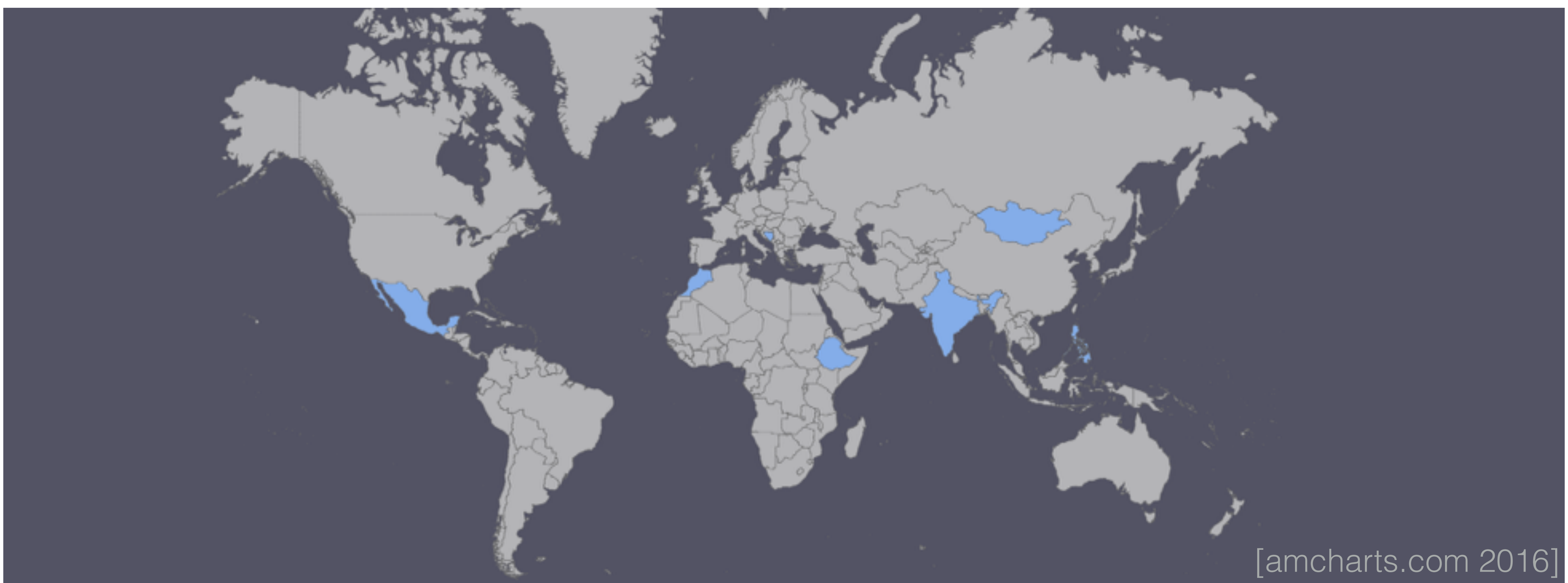

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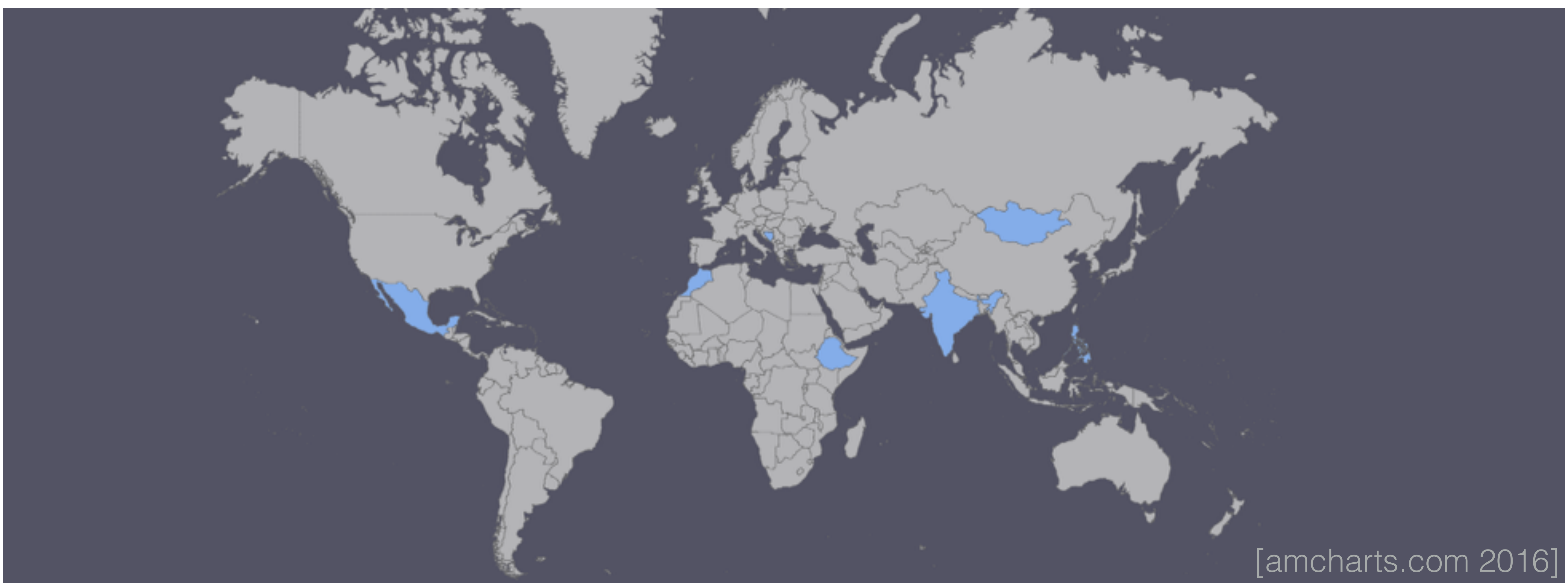

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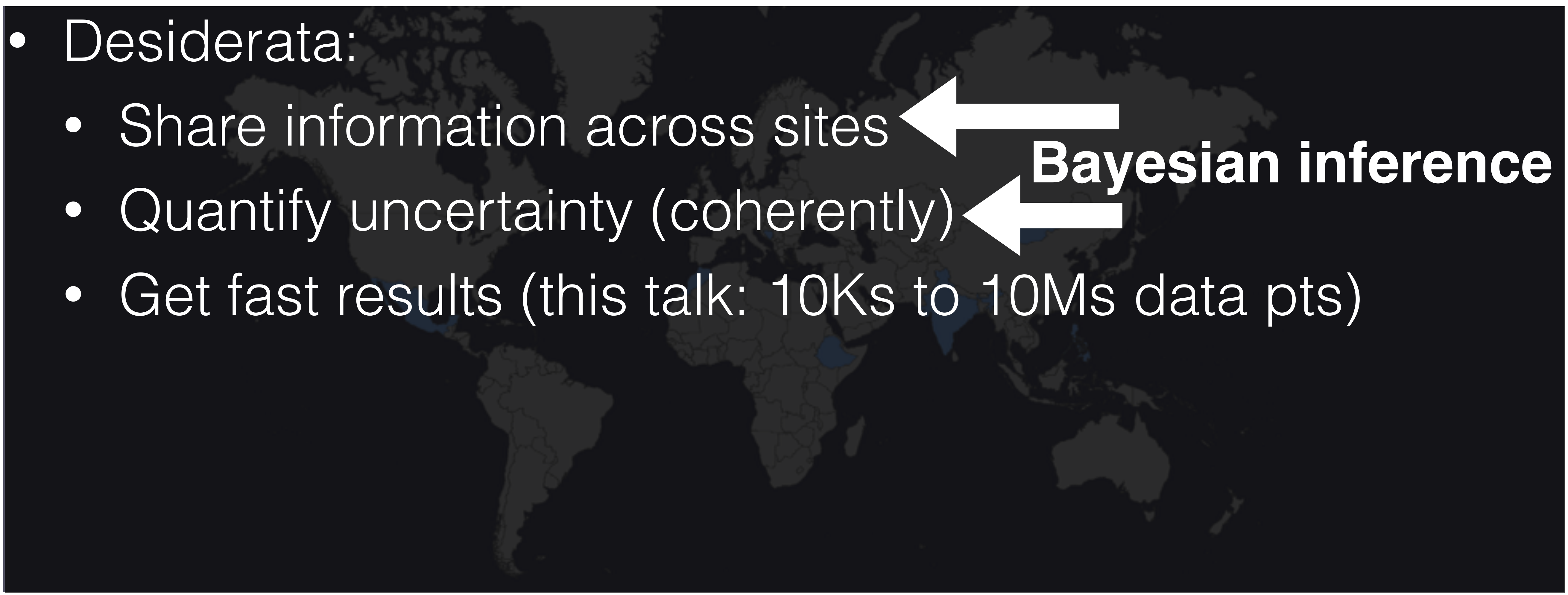

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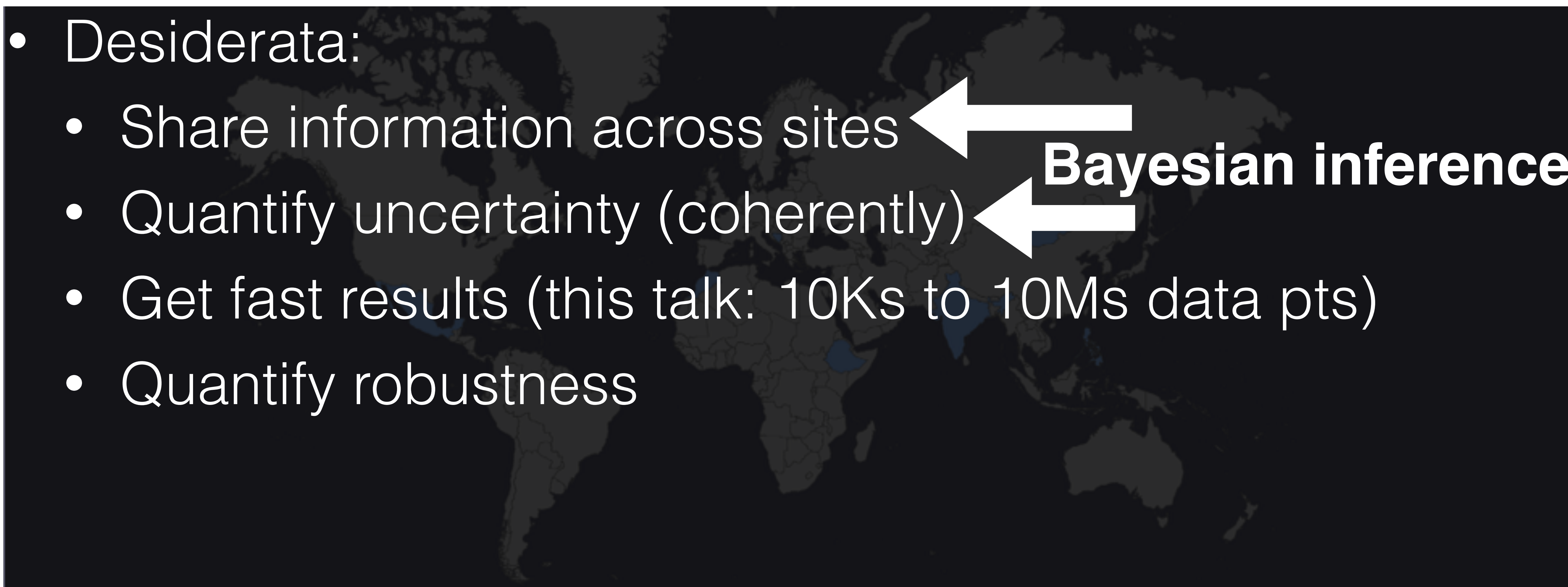

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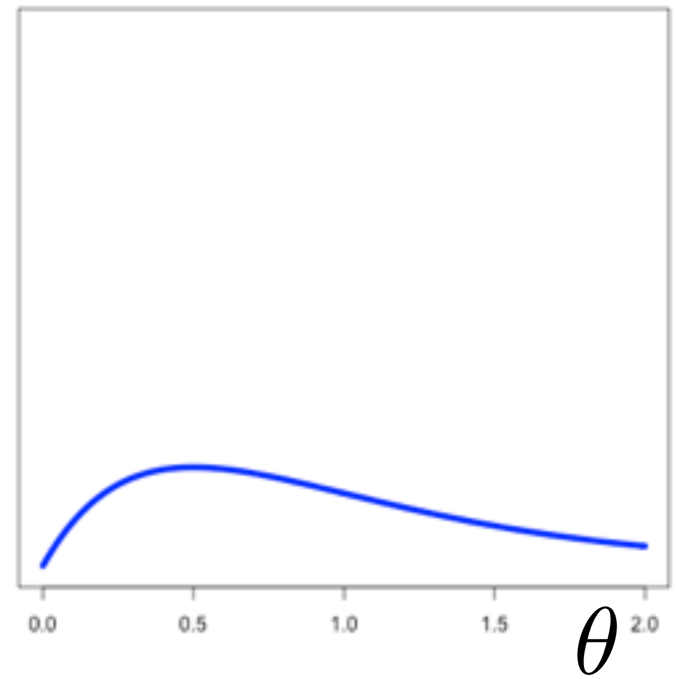

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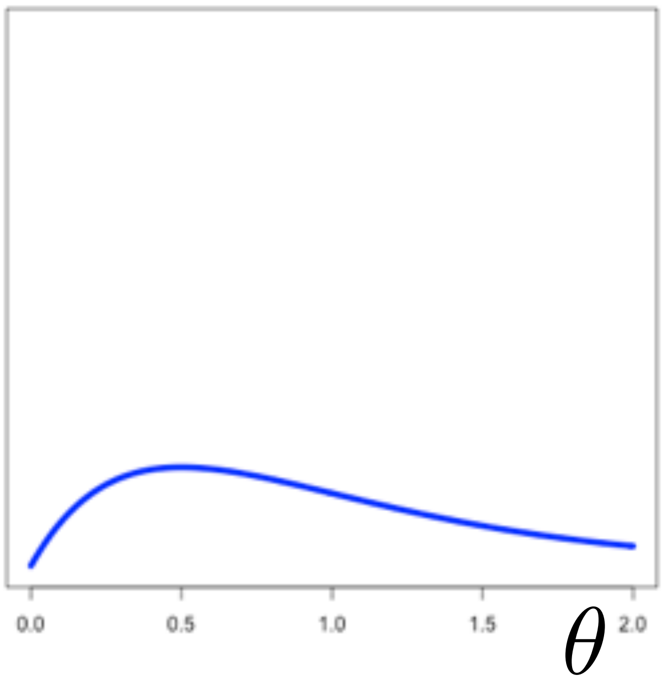

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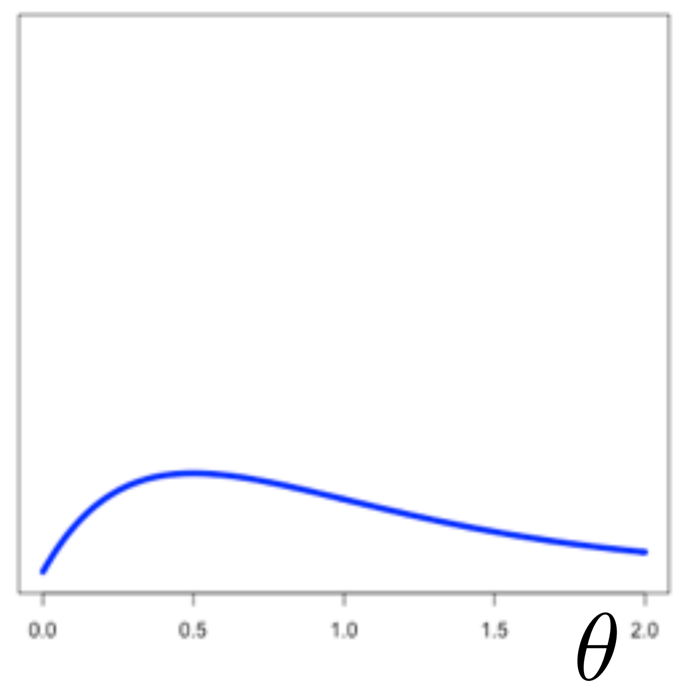

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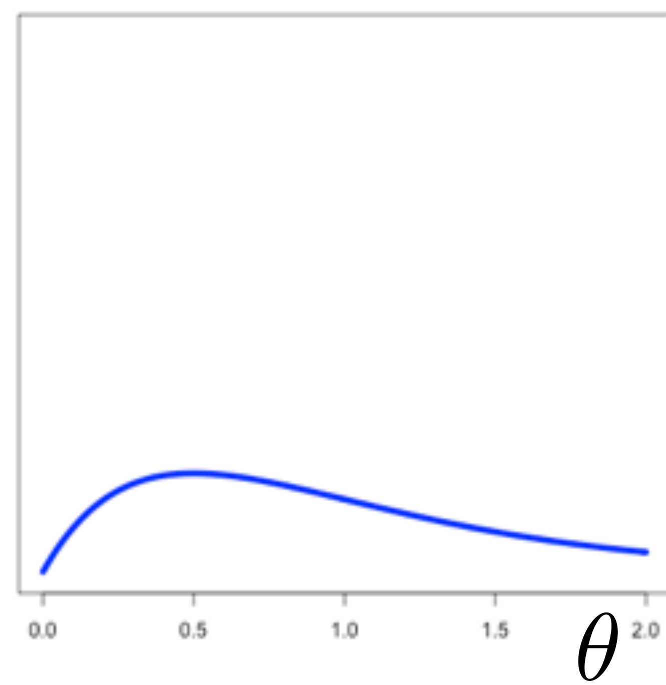

Bayes Theorem →

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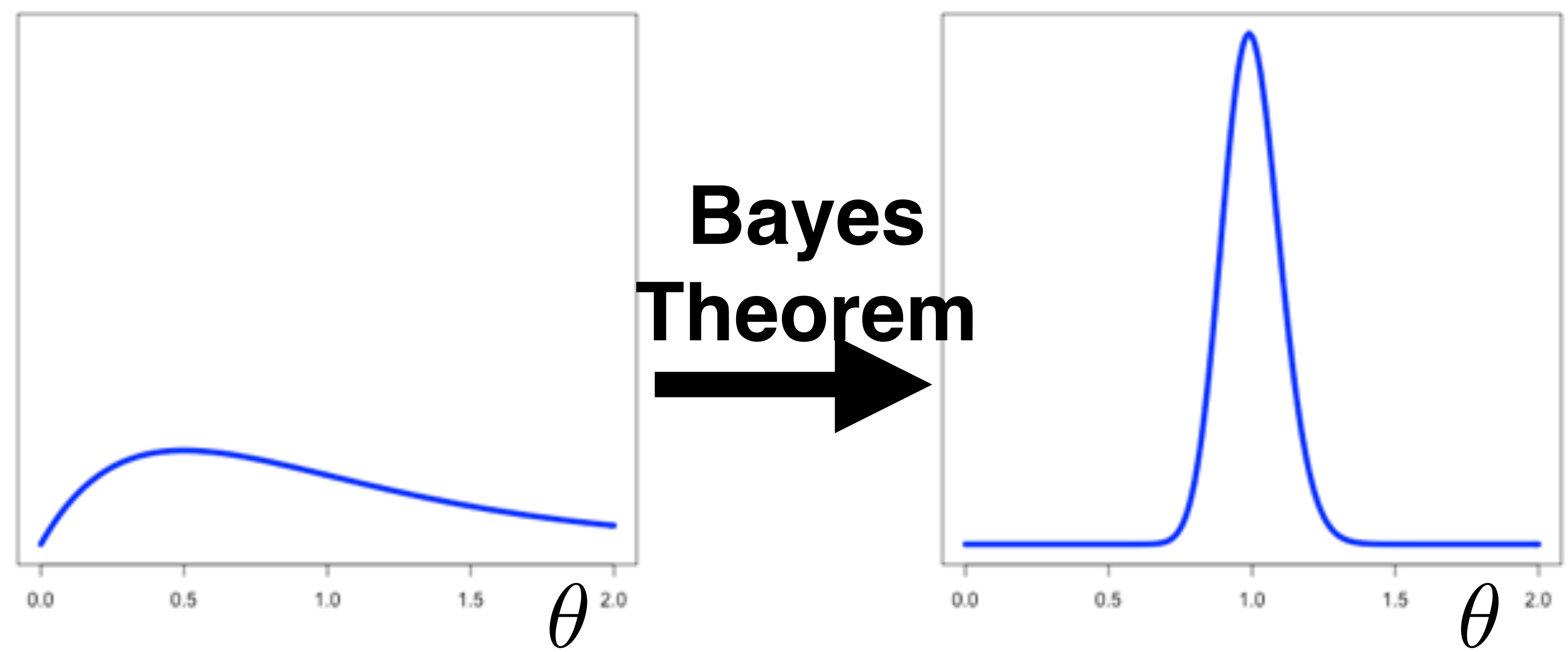

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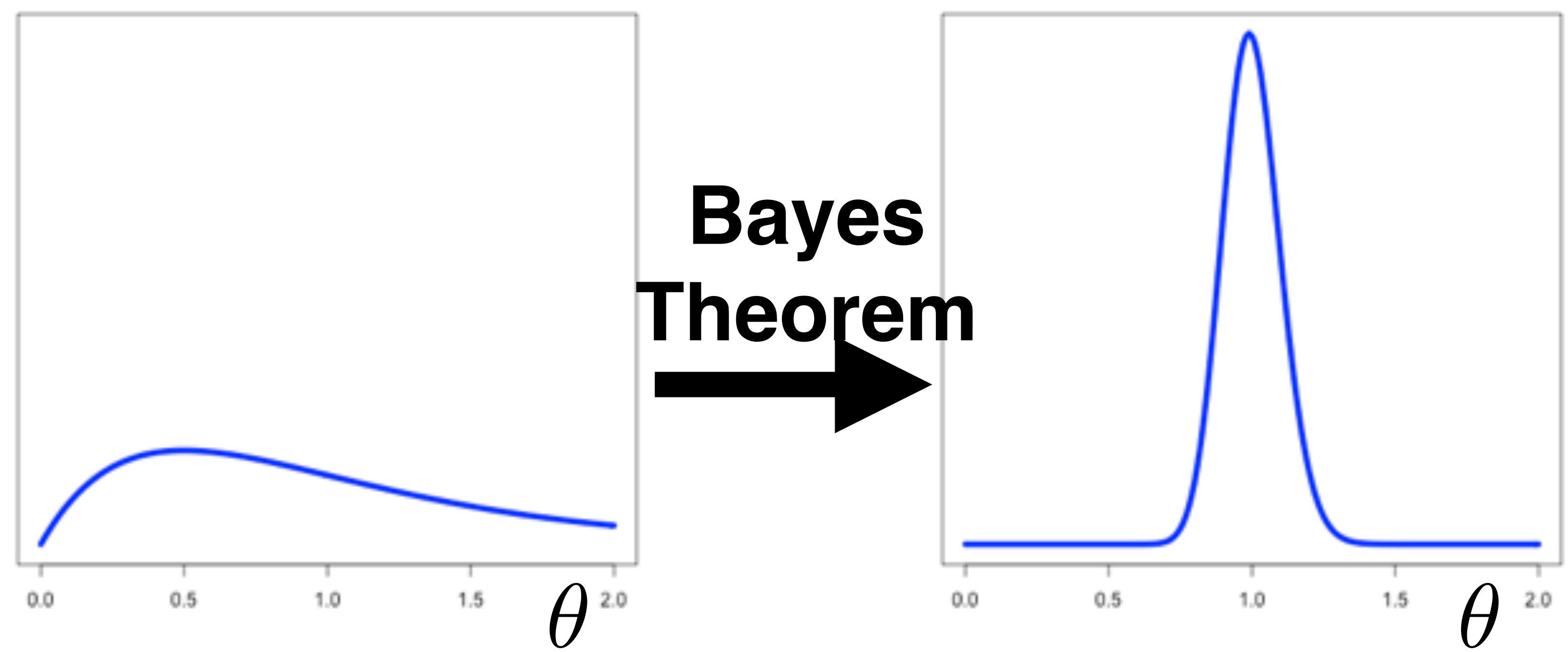

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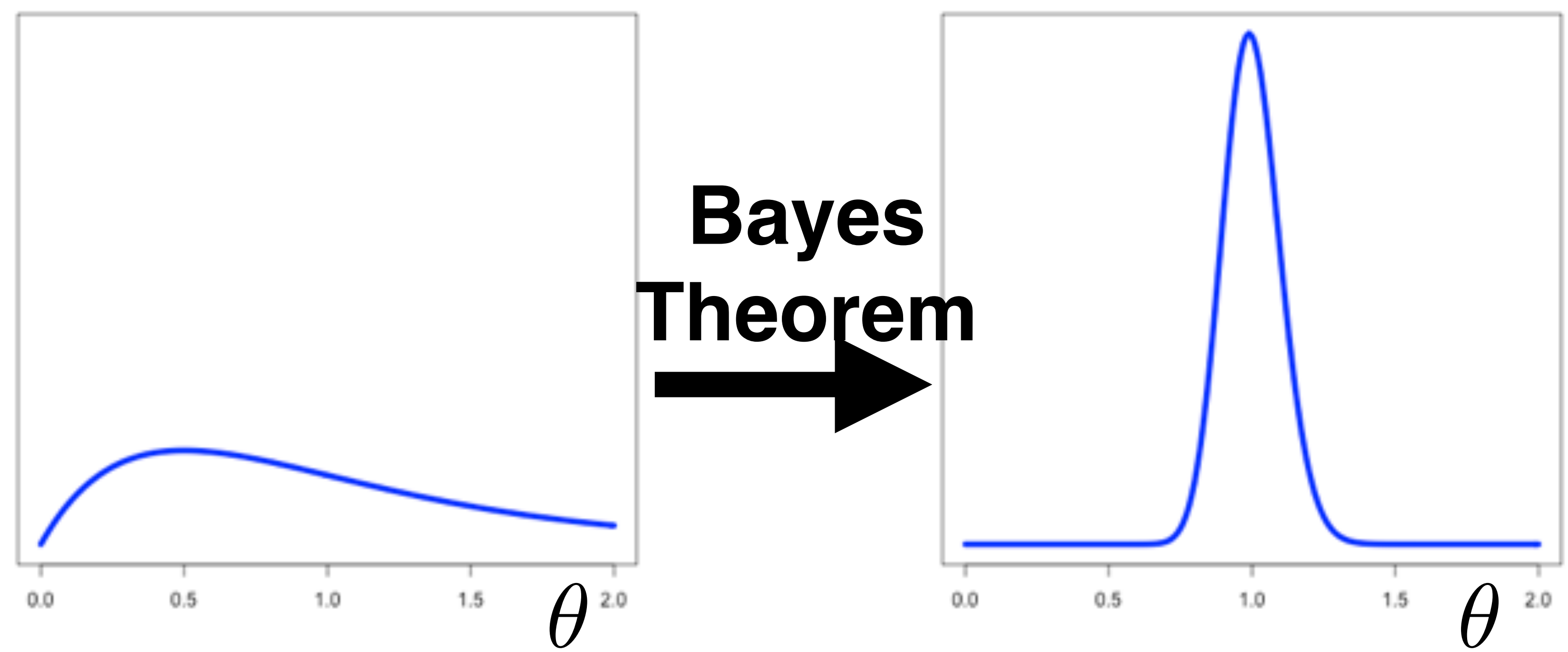

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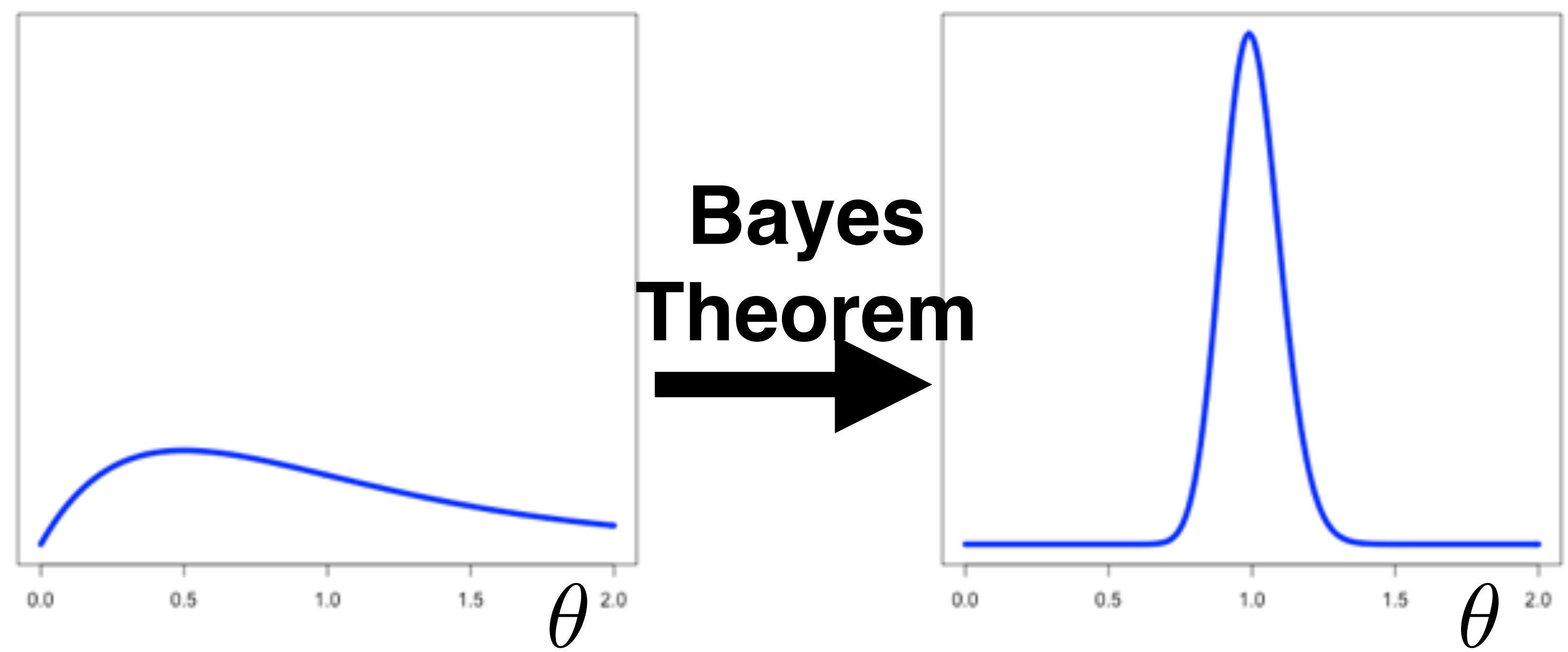

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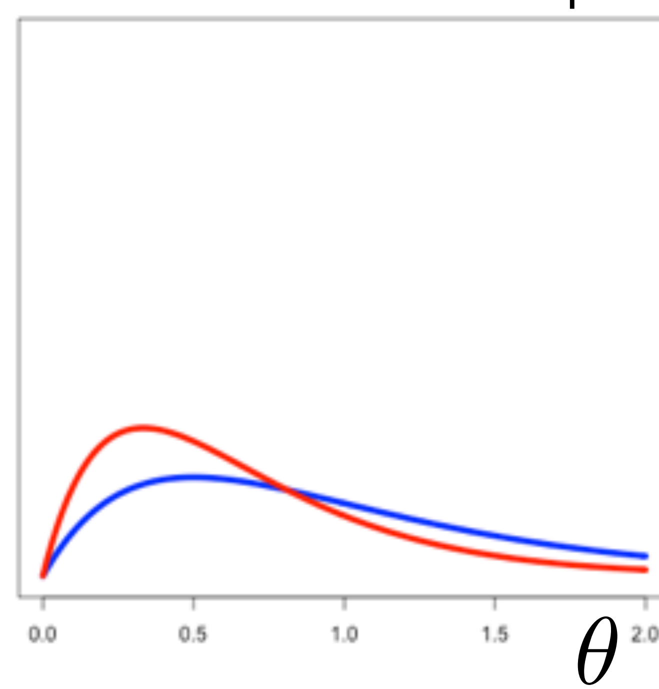

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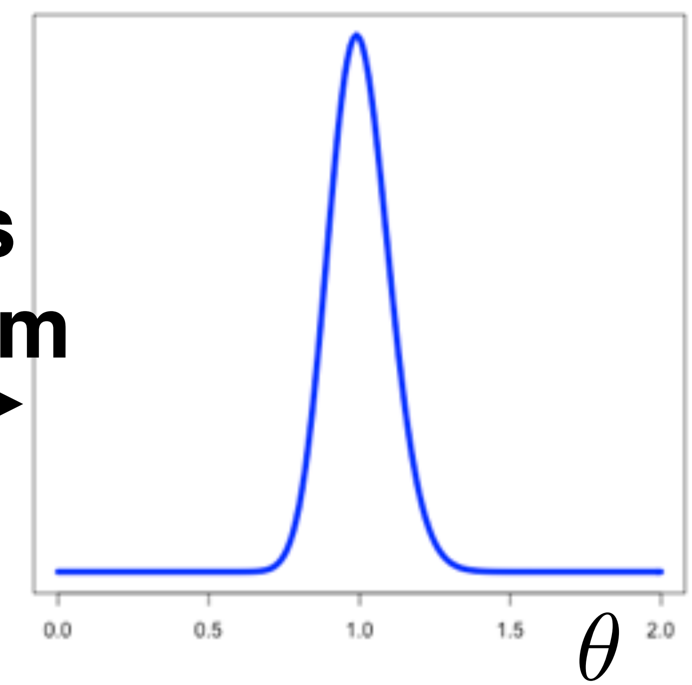

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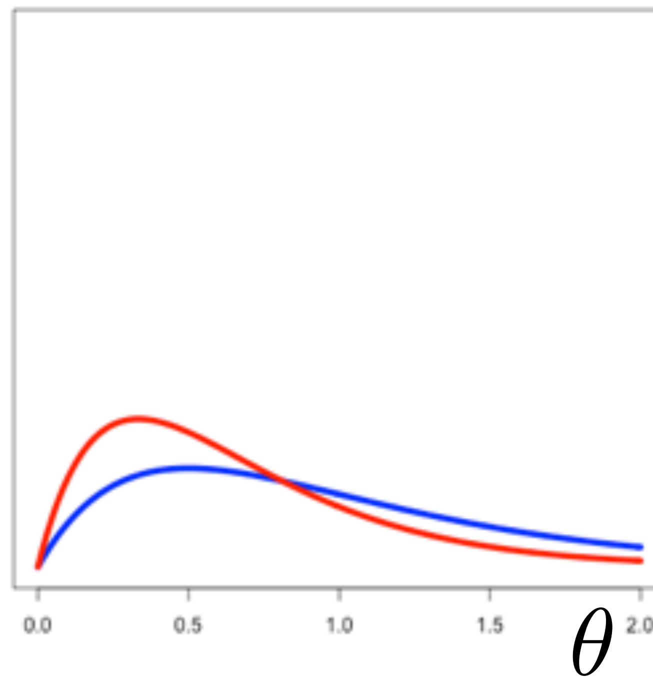

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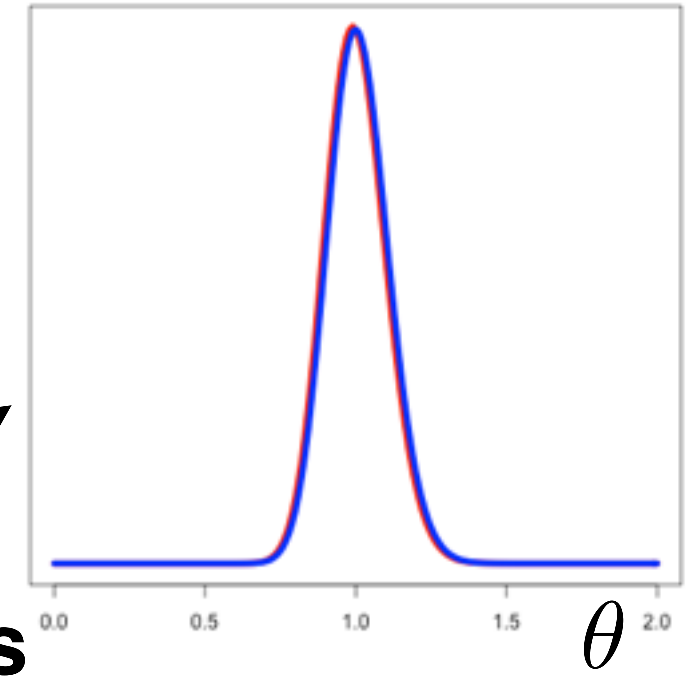

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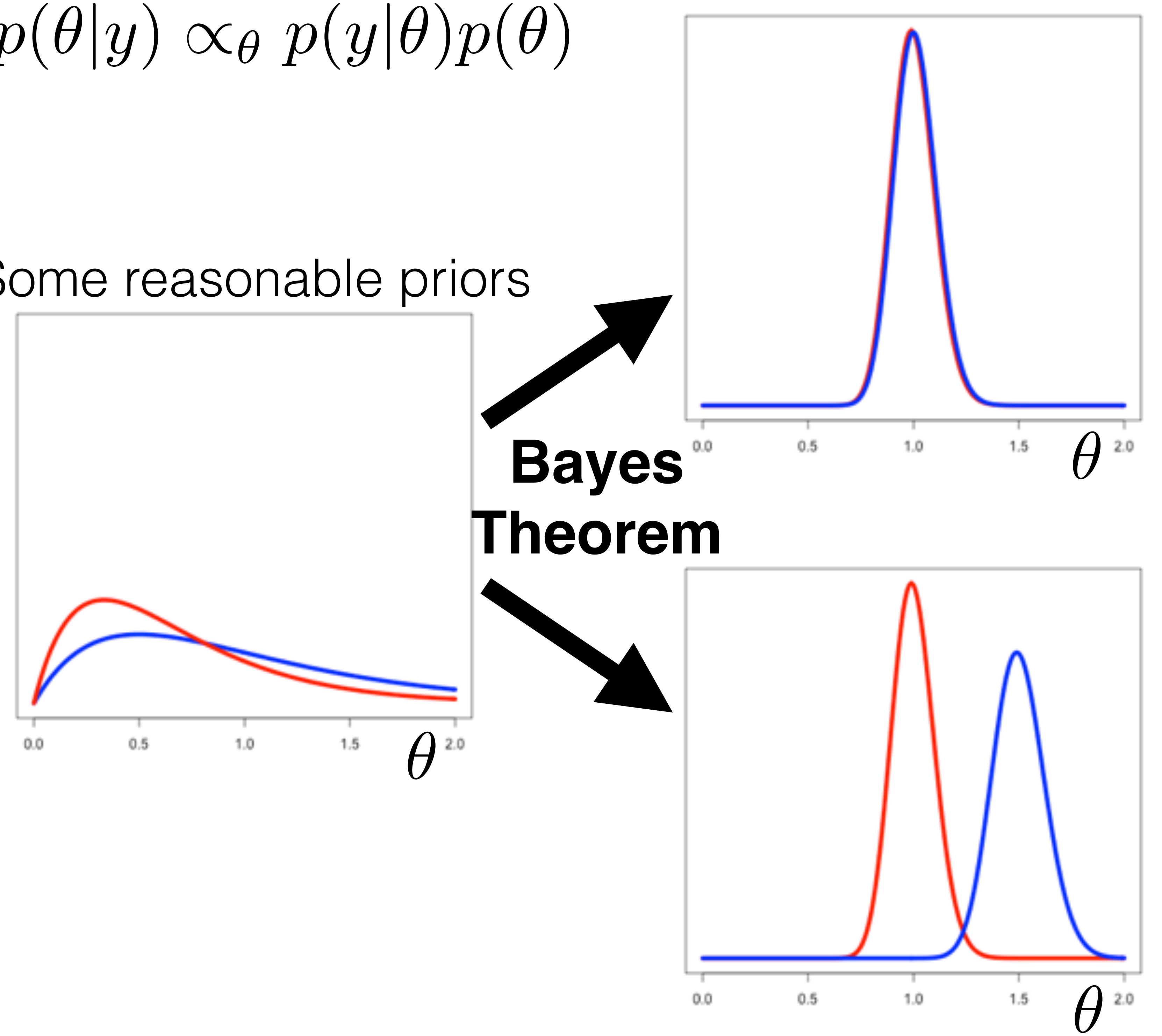

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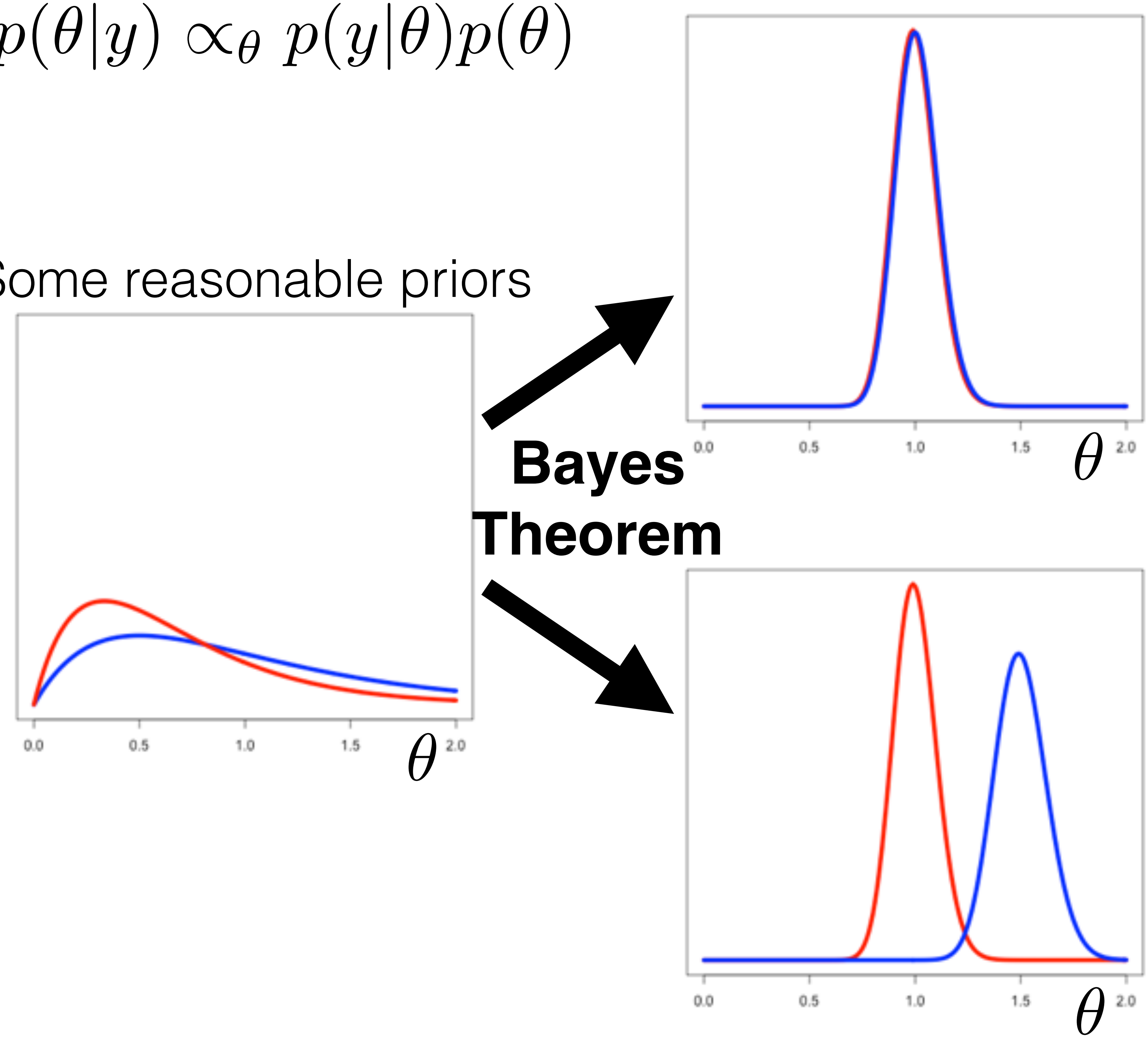

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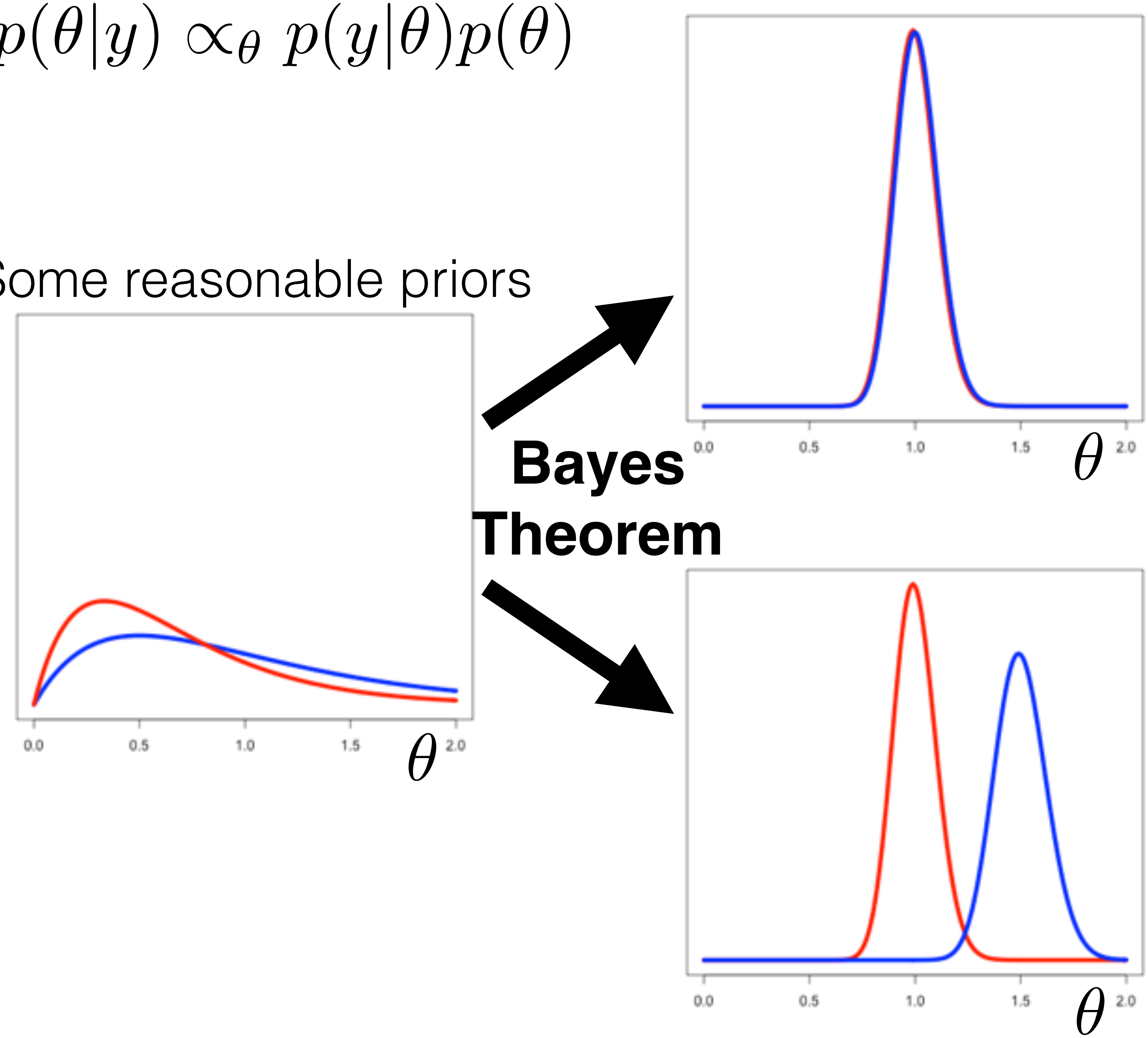

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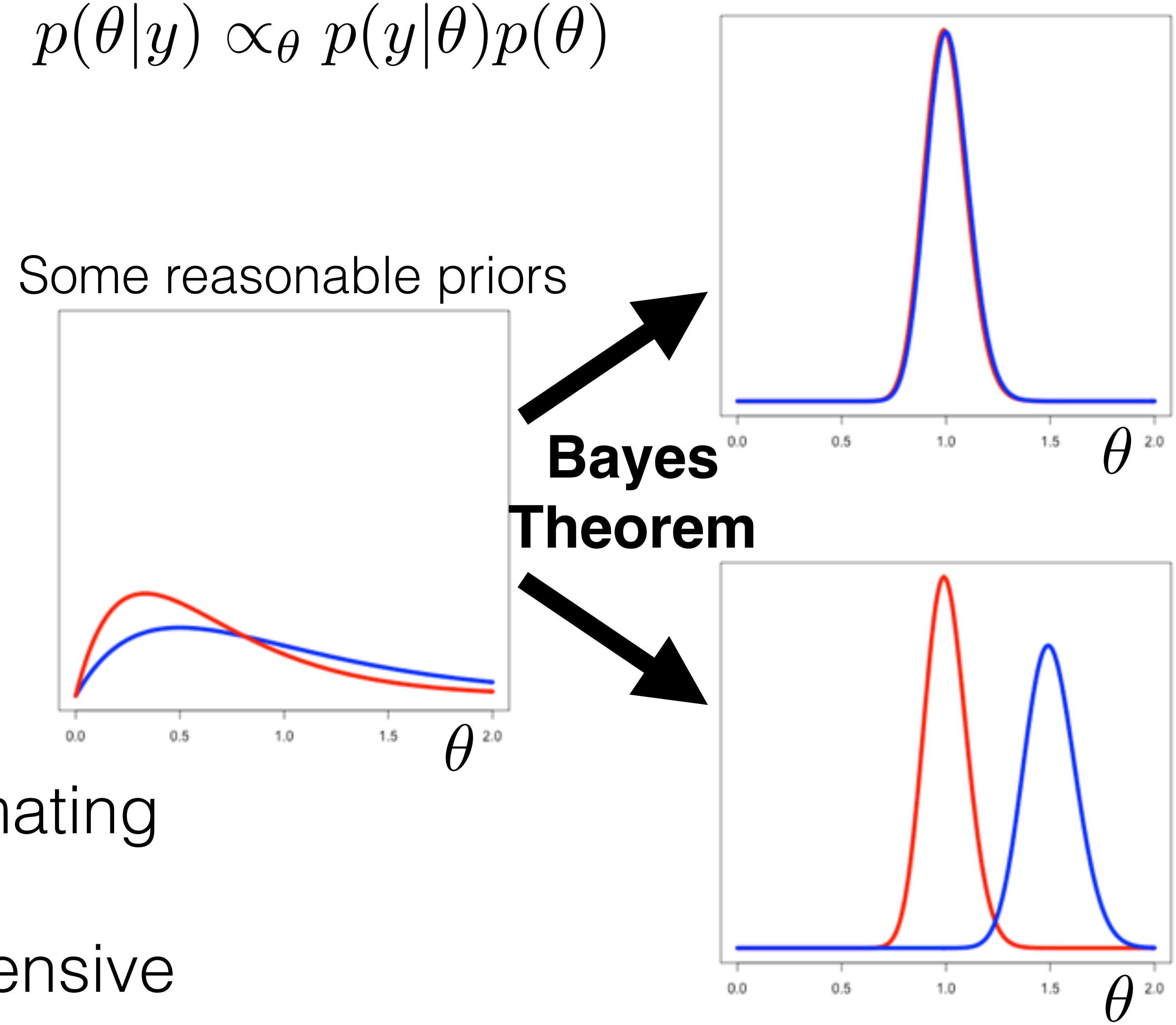

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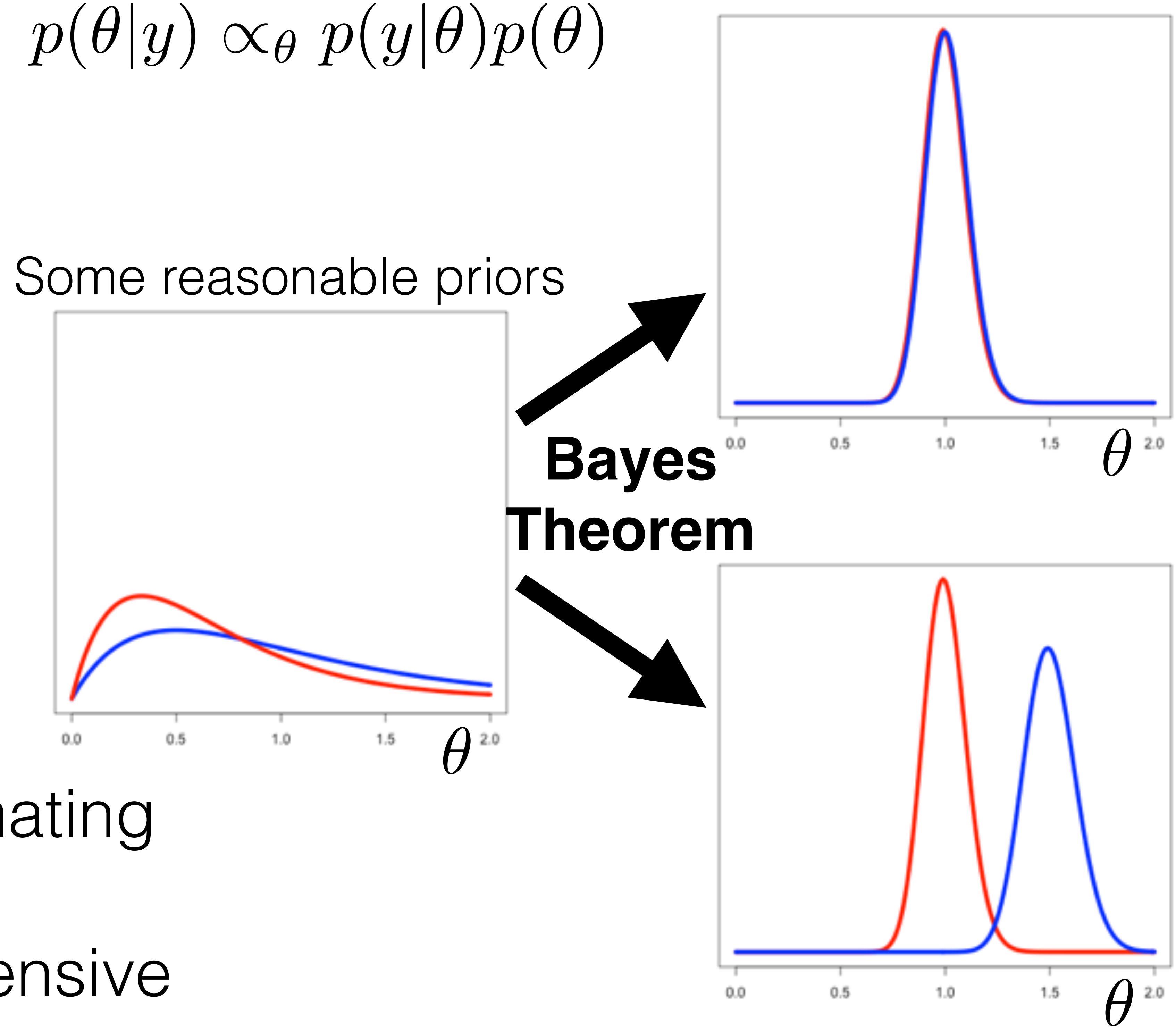

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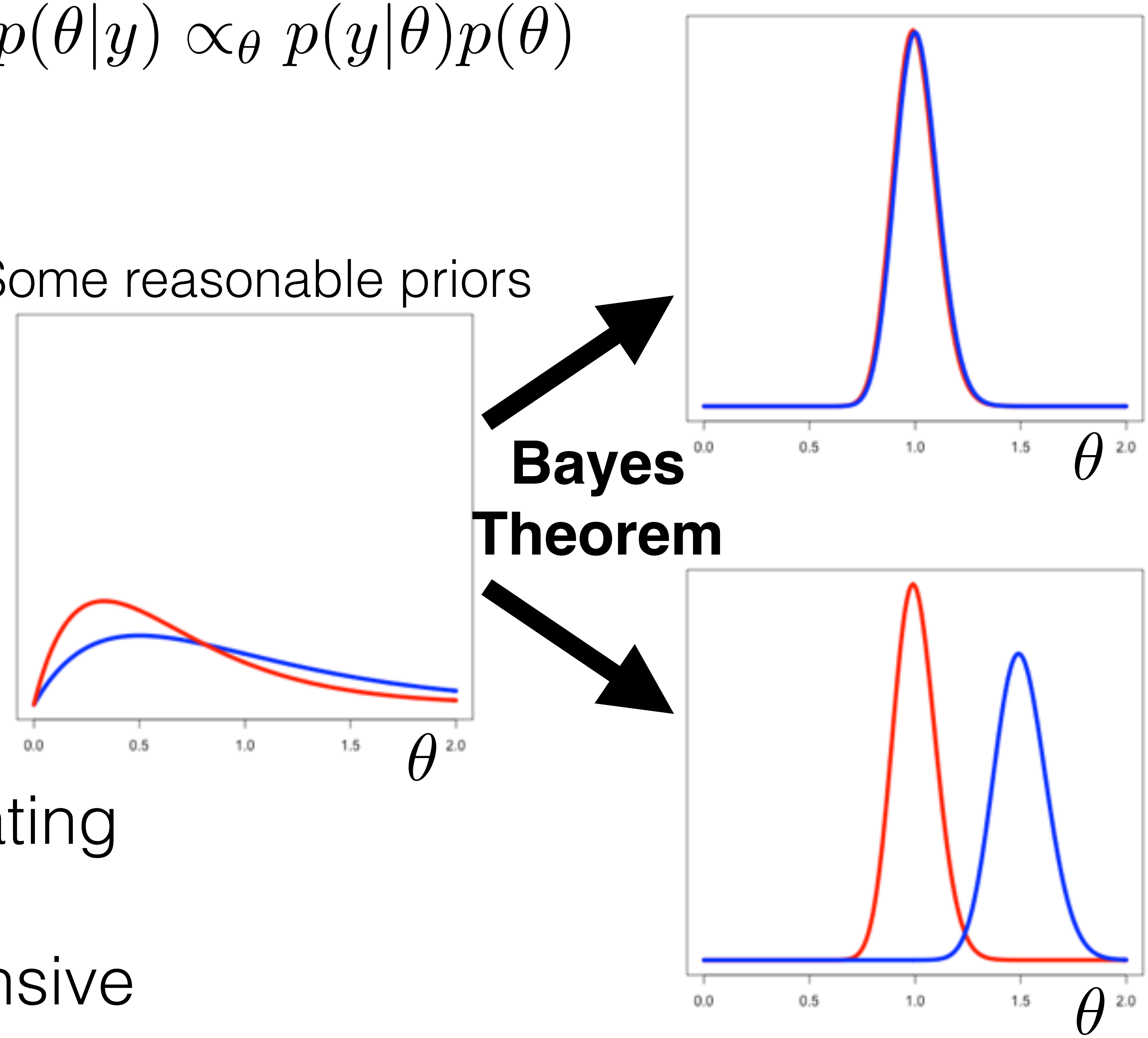

variational Bayes

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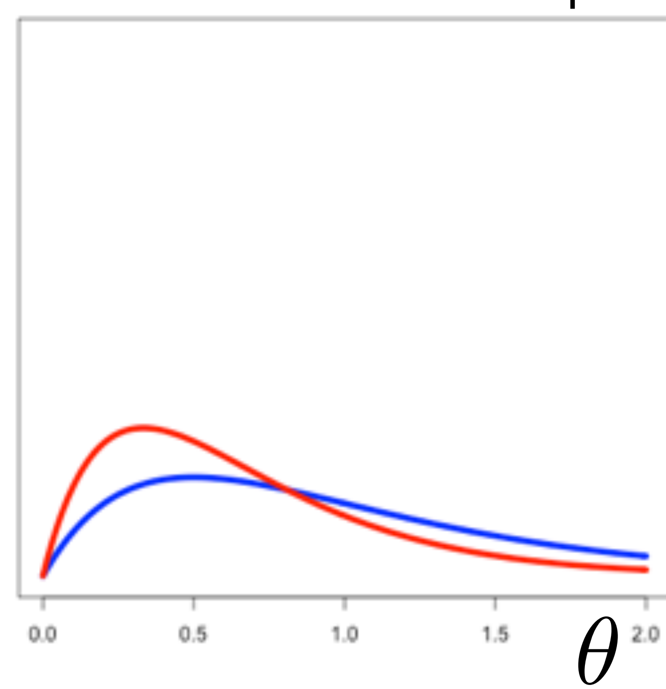

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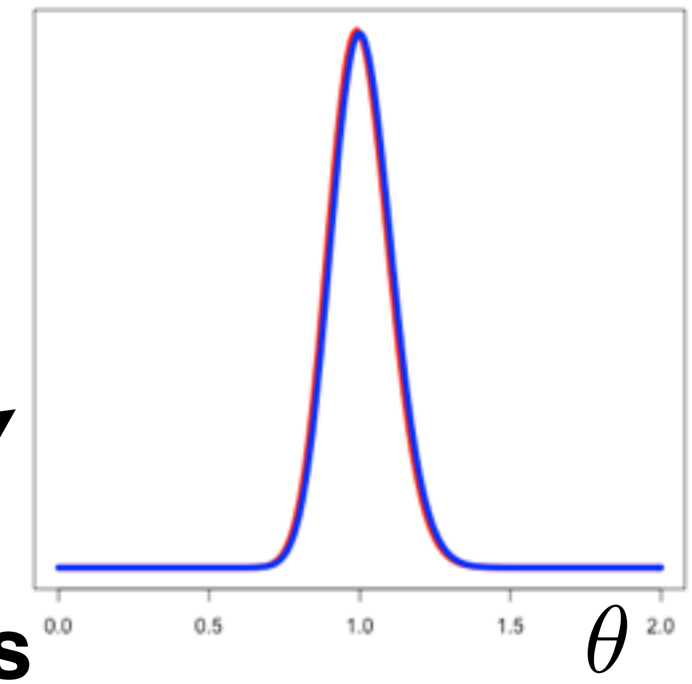

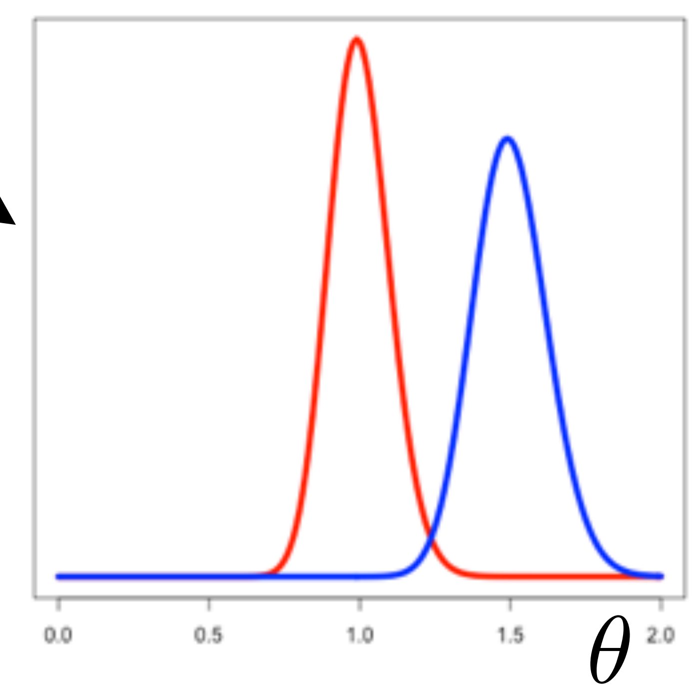

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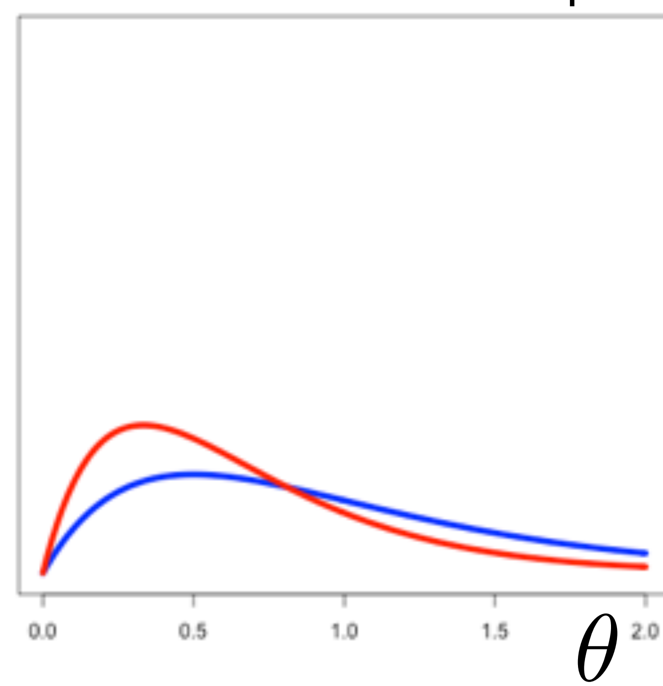

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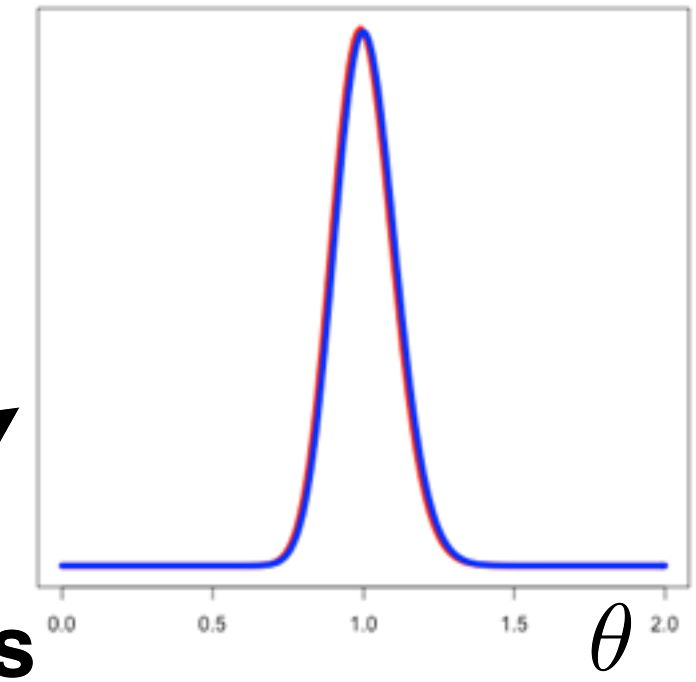

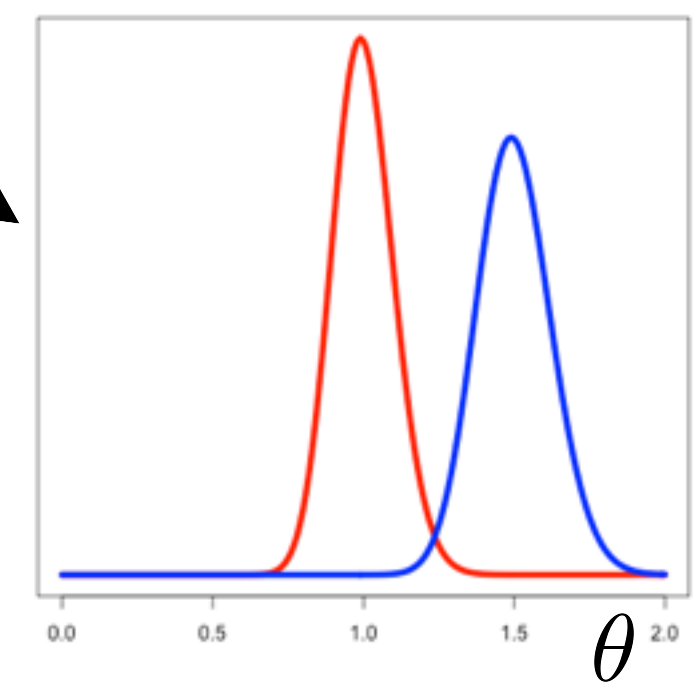

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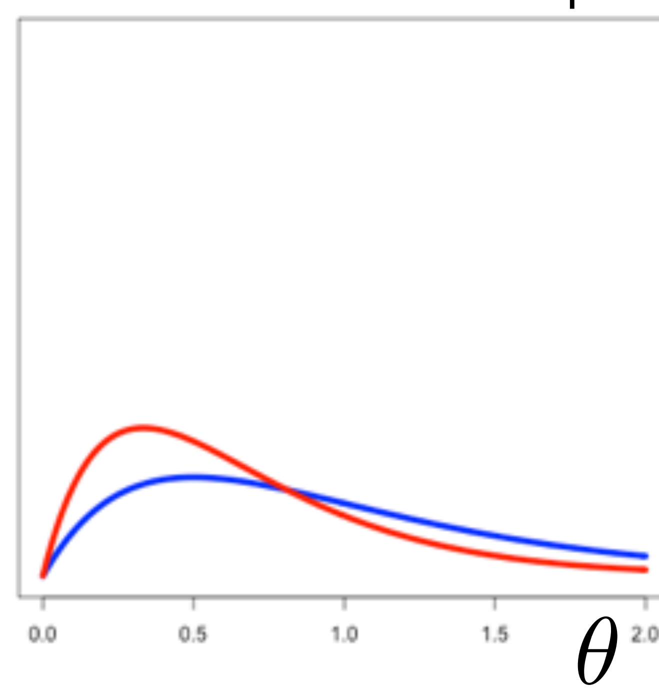

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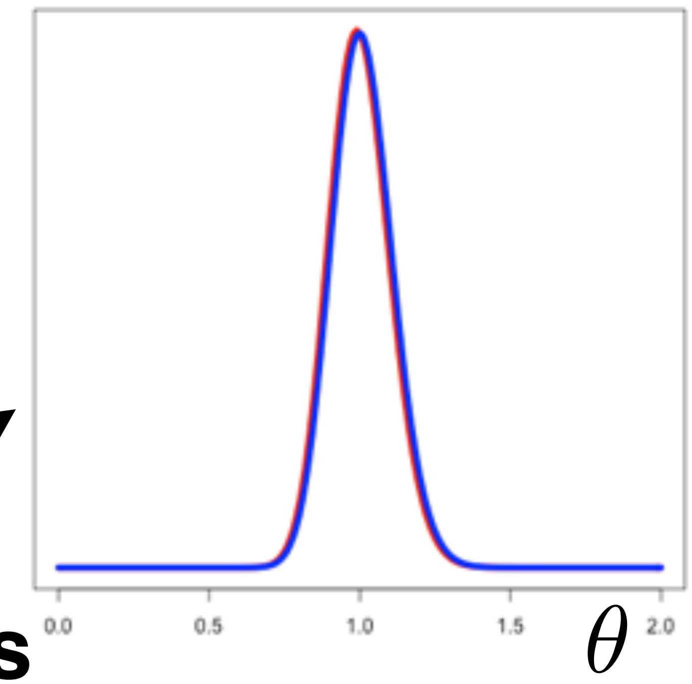

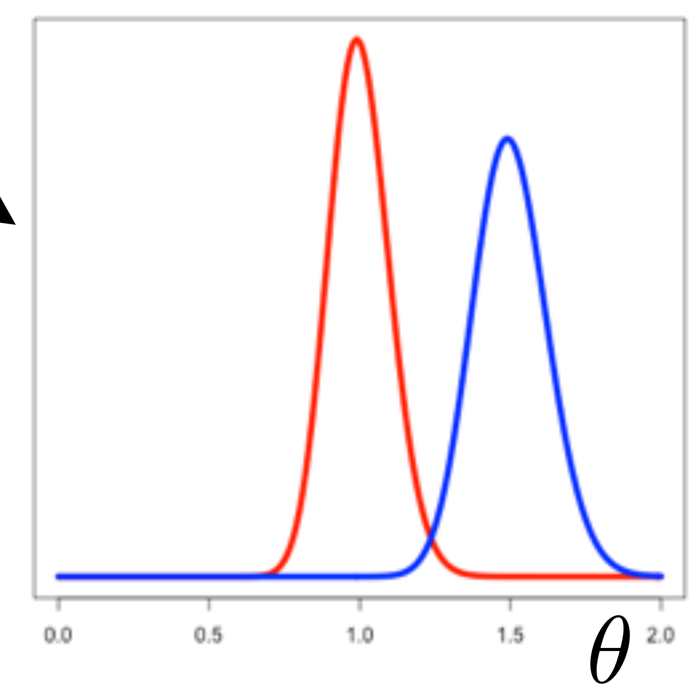

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[see also Opper, Winther 2003]

Roadmap

- Challenges of VB
- Accurate uncertainties from VB
- Accurate robustness quantification from VB
  - Big idea: derivatives/perturbations are relatively easy in VB

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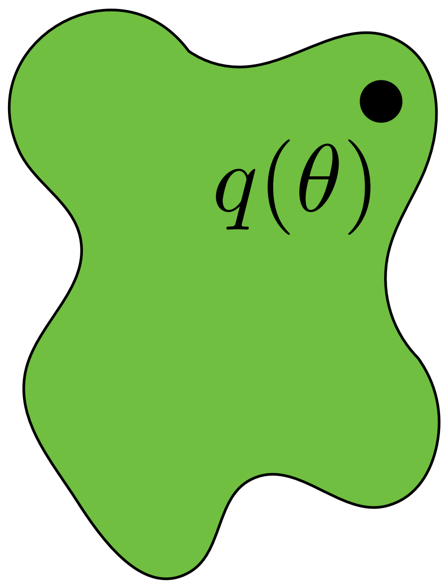

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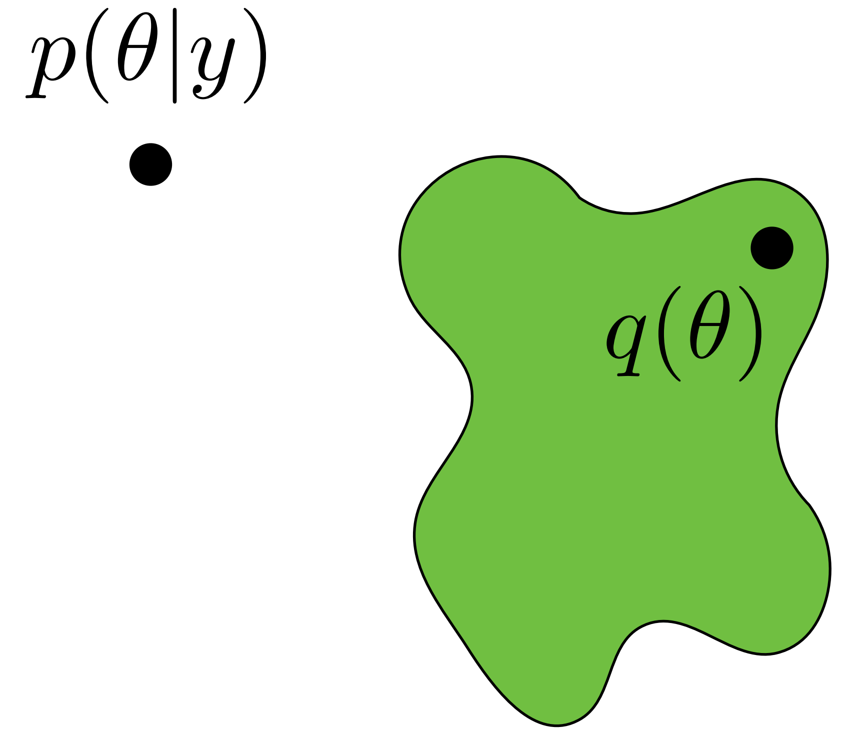

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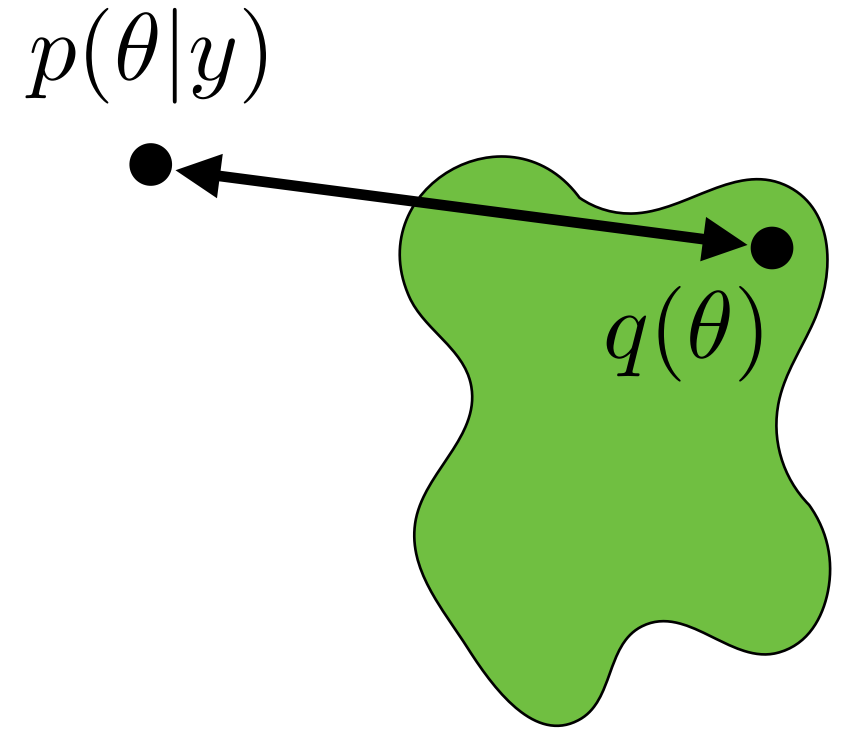

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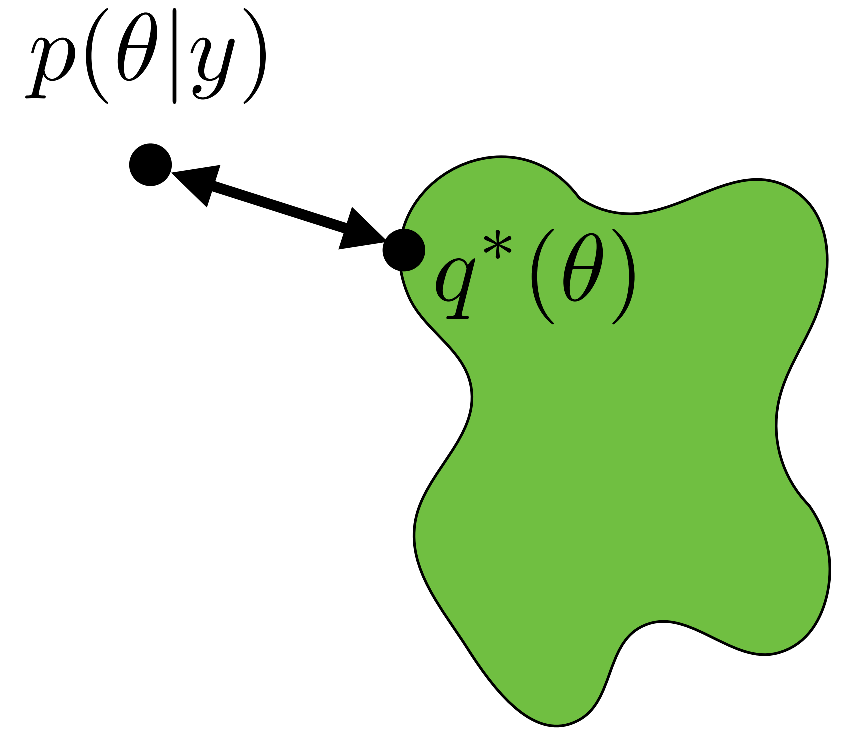

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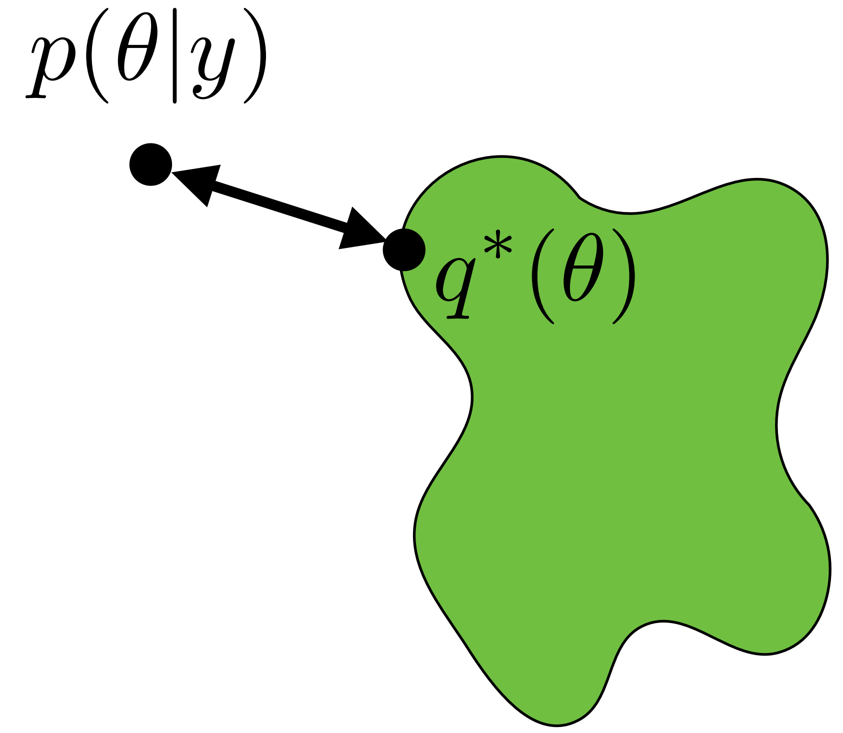

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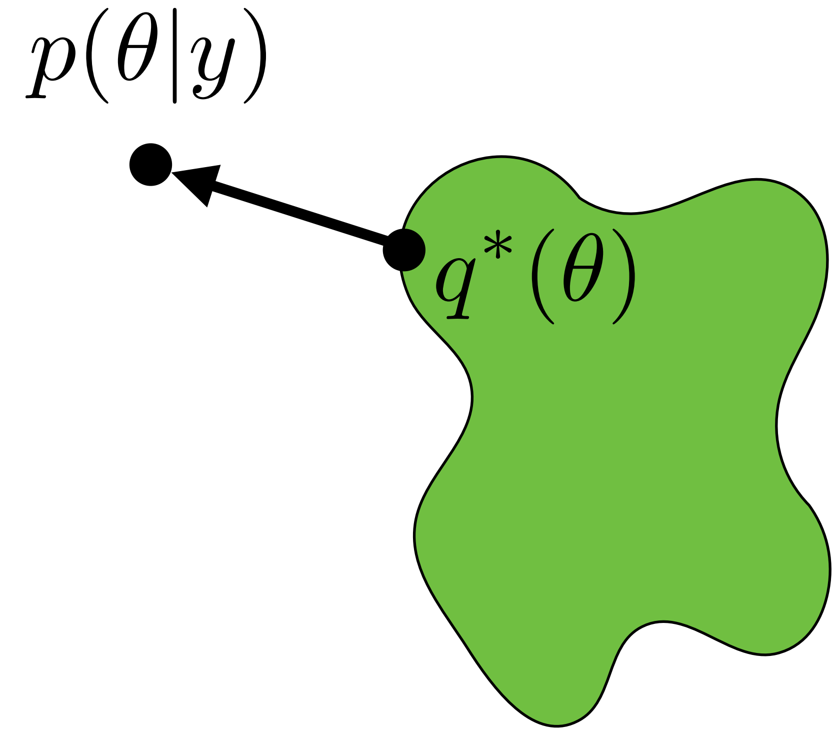

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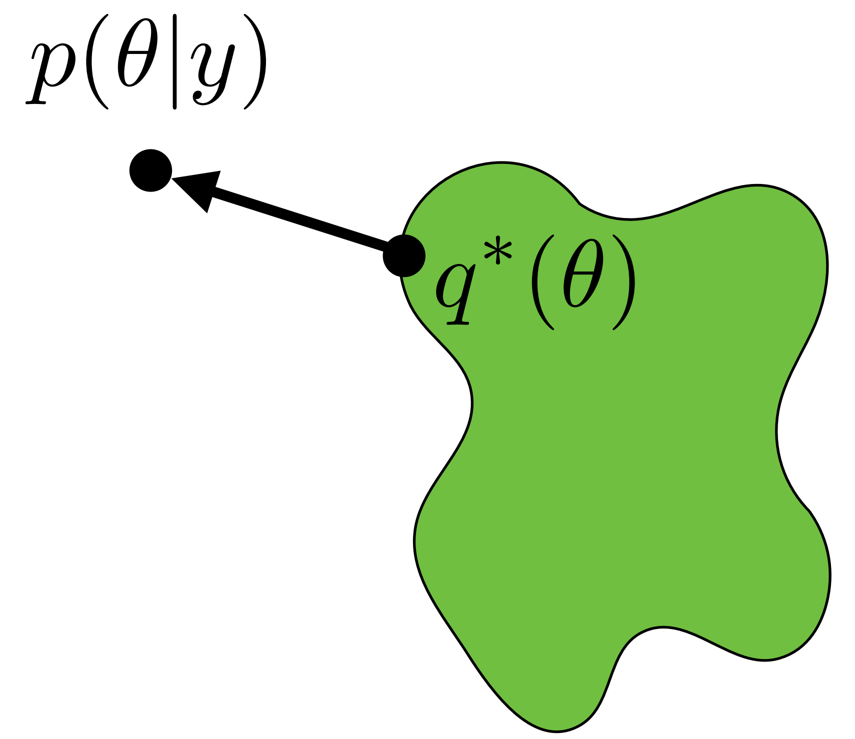

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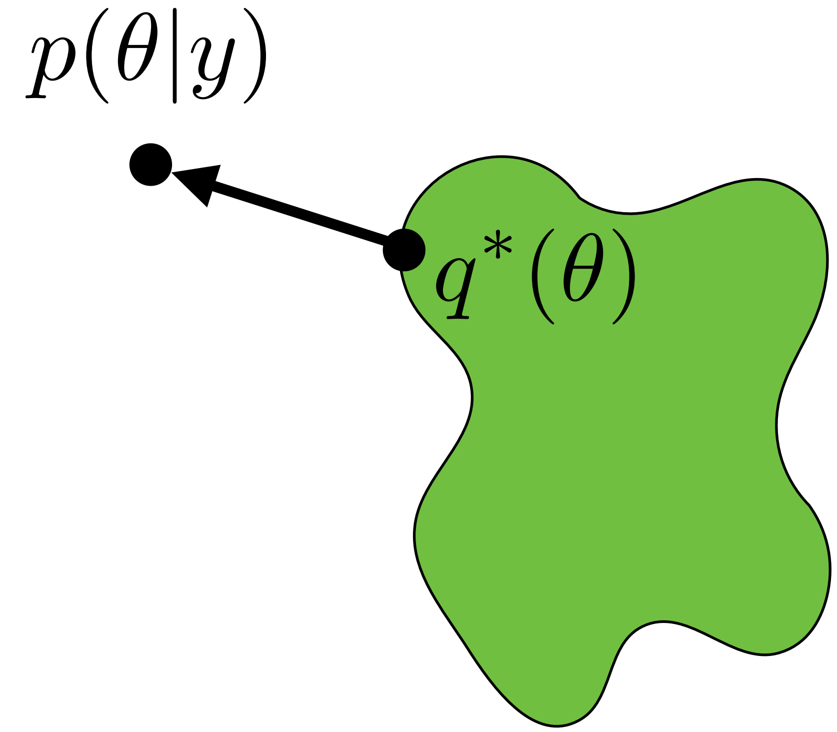

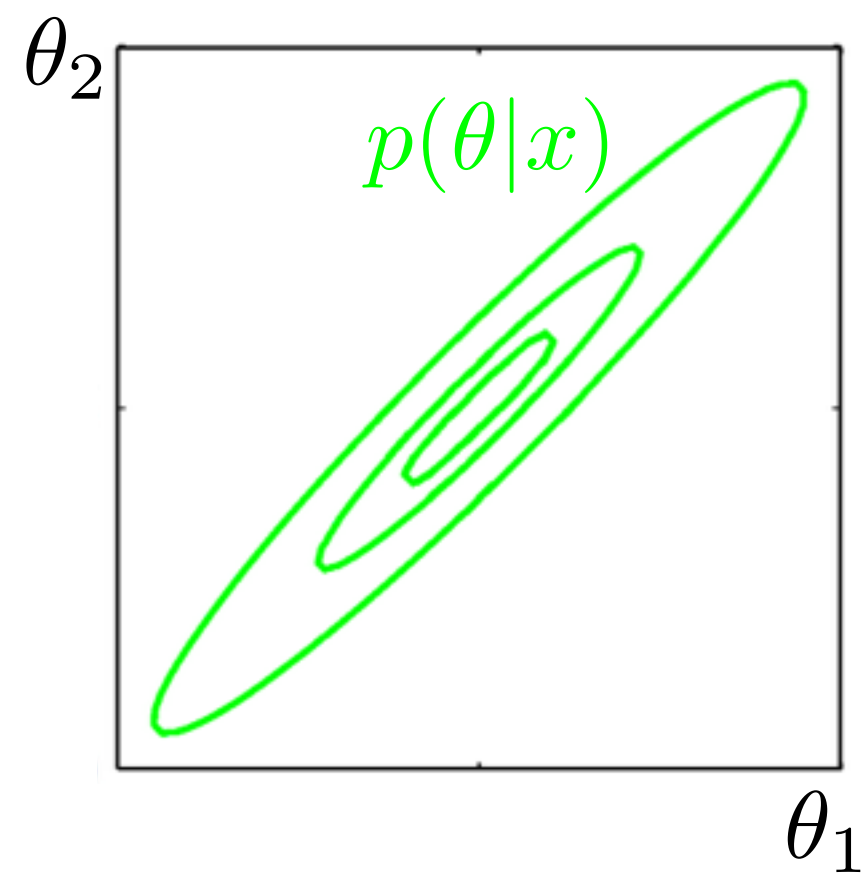

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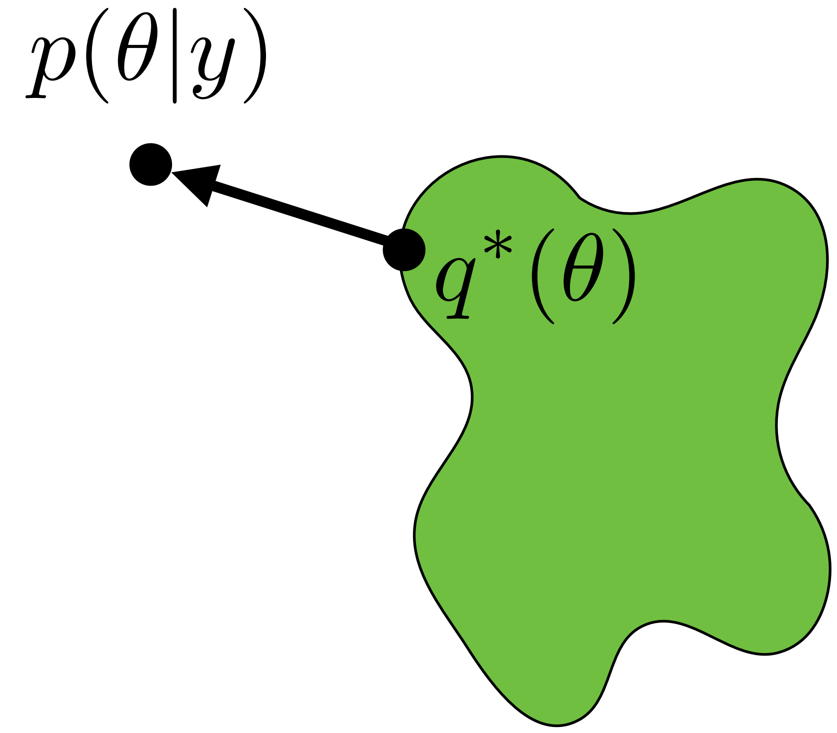

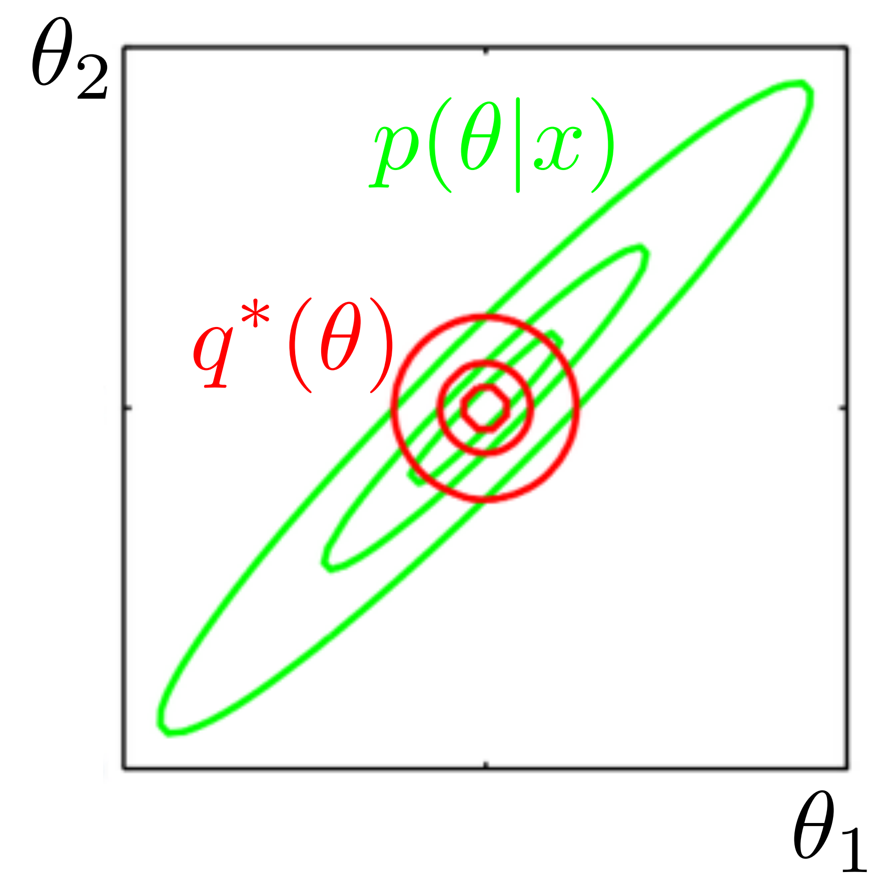

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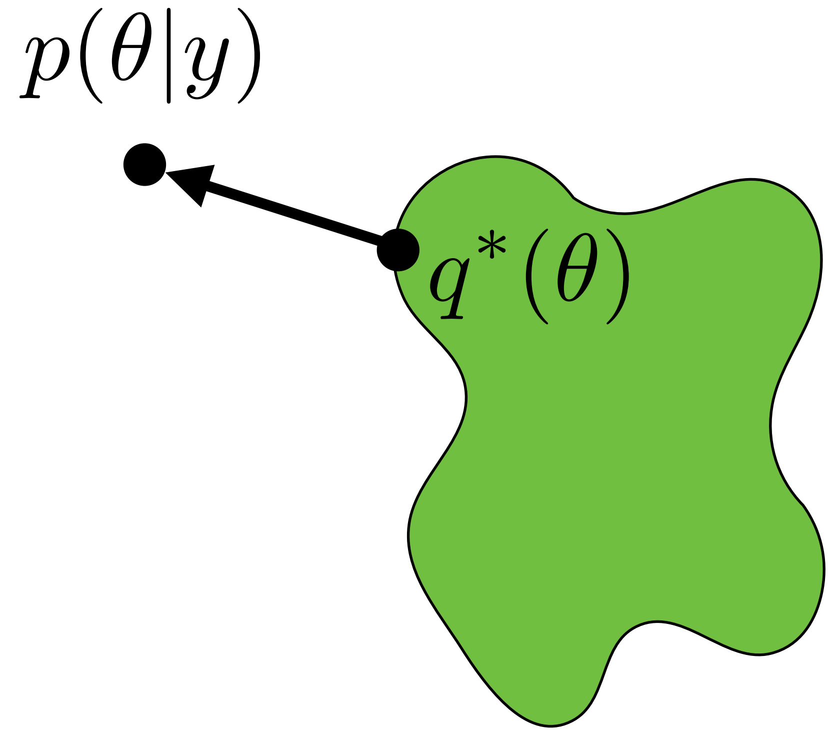

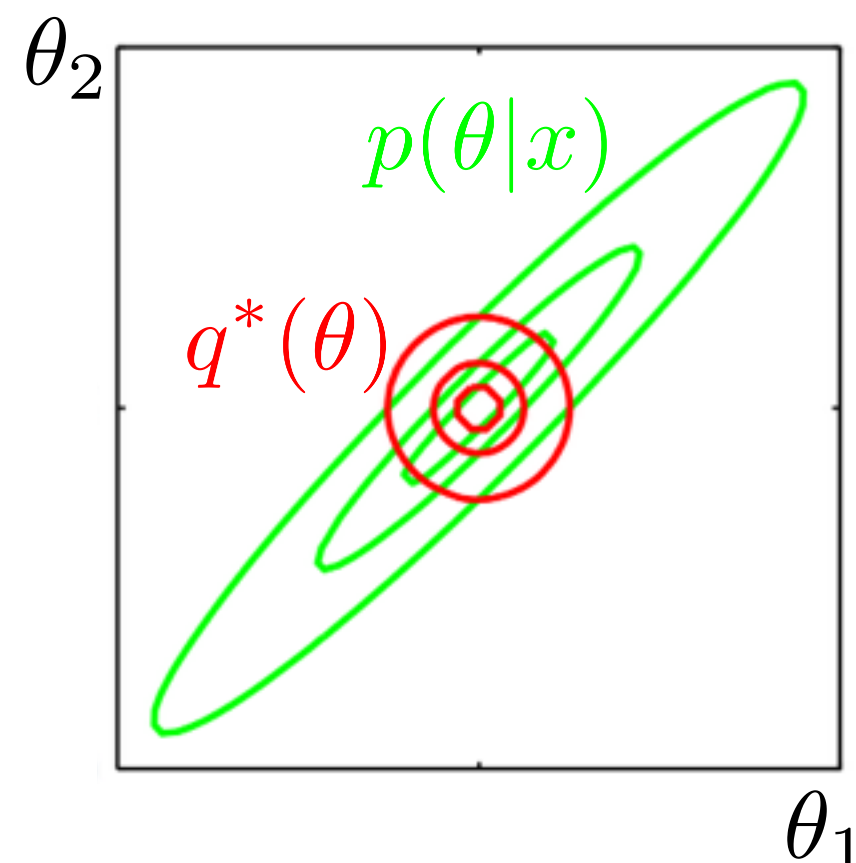

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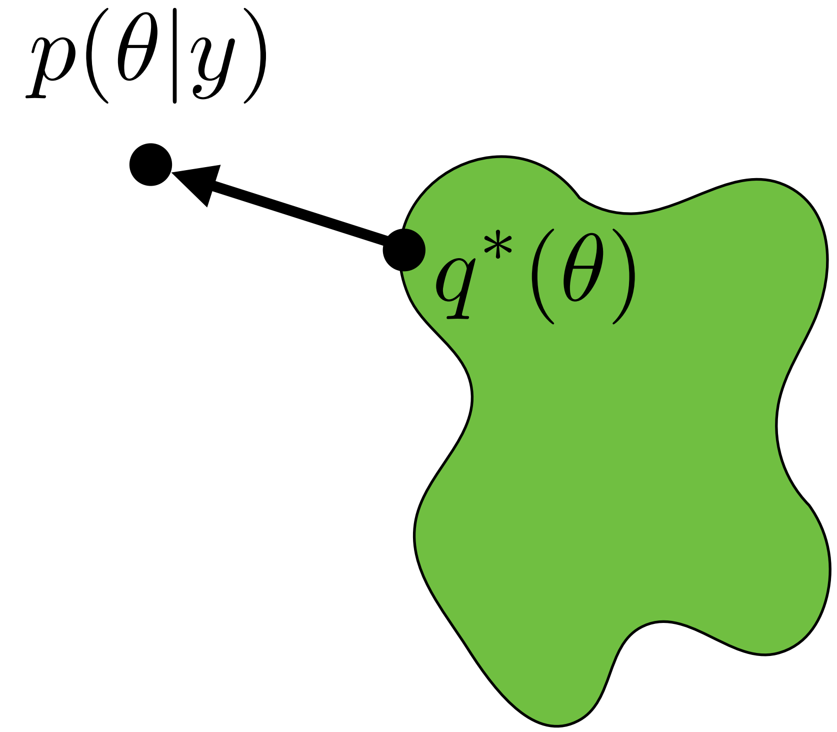

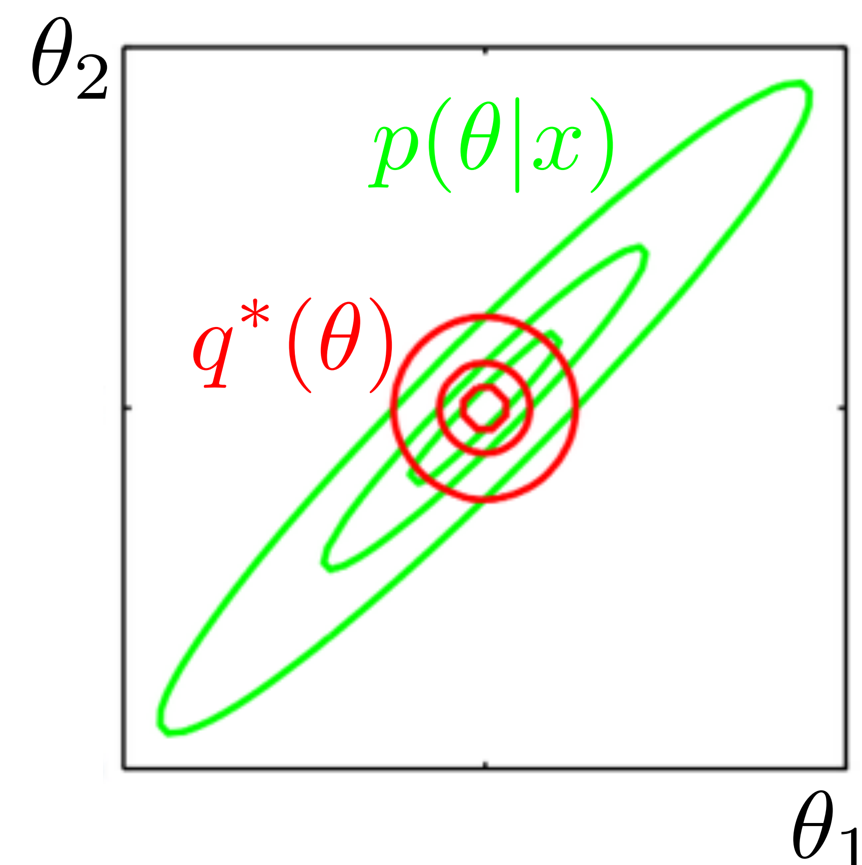

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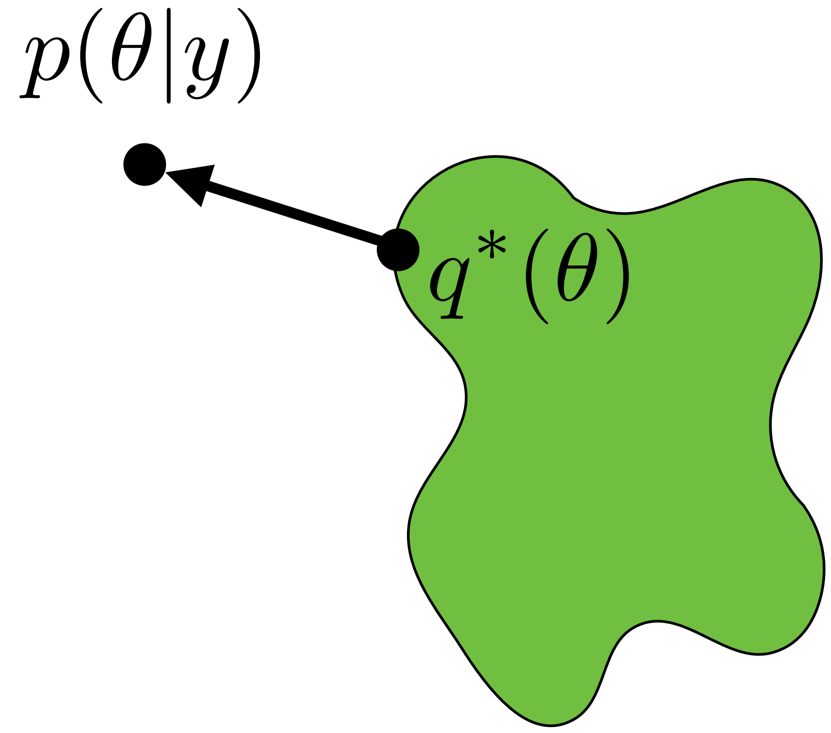
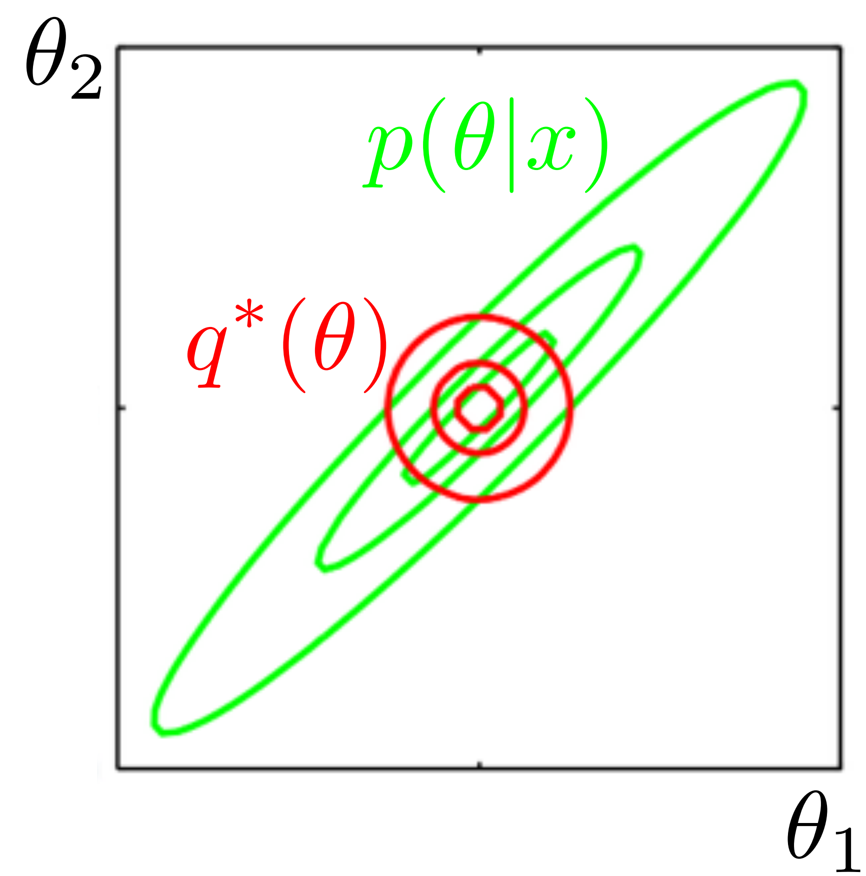




[MacKay 2003; Bishop 2006; Wang, Titterington 2004; Turner, Sahani 2011]

What about uncertainty?

- Variational Bayes

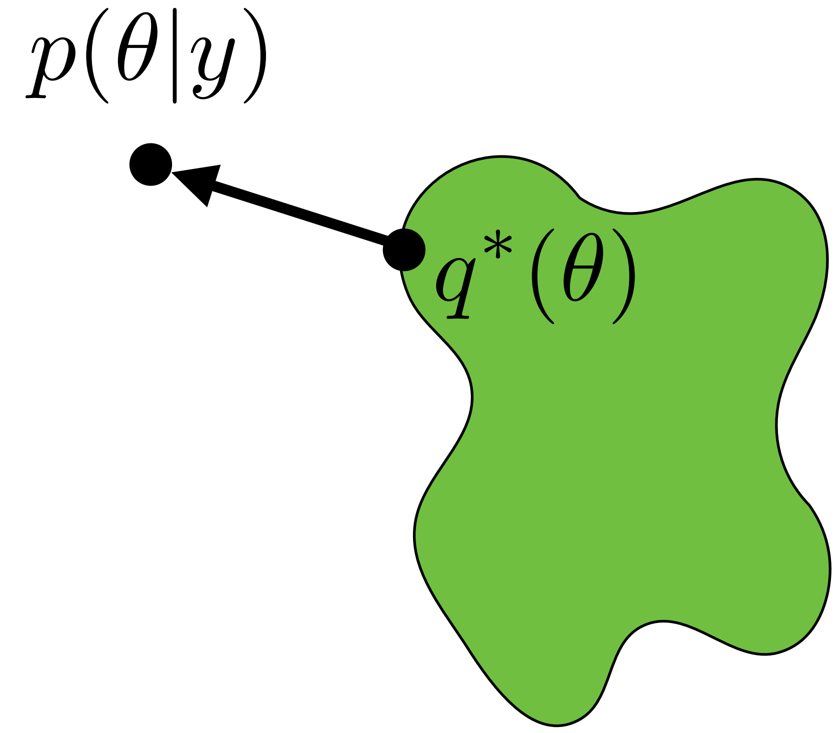
$$KL(q||p(\cdot|x)) = \int_\theta q(\theta) \log \frac{q(\theta)}{p(\theta|x)} d\theta$$

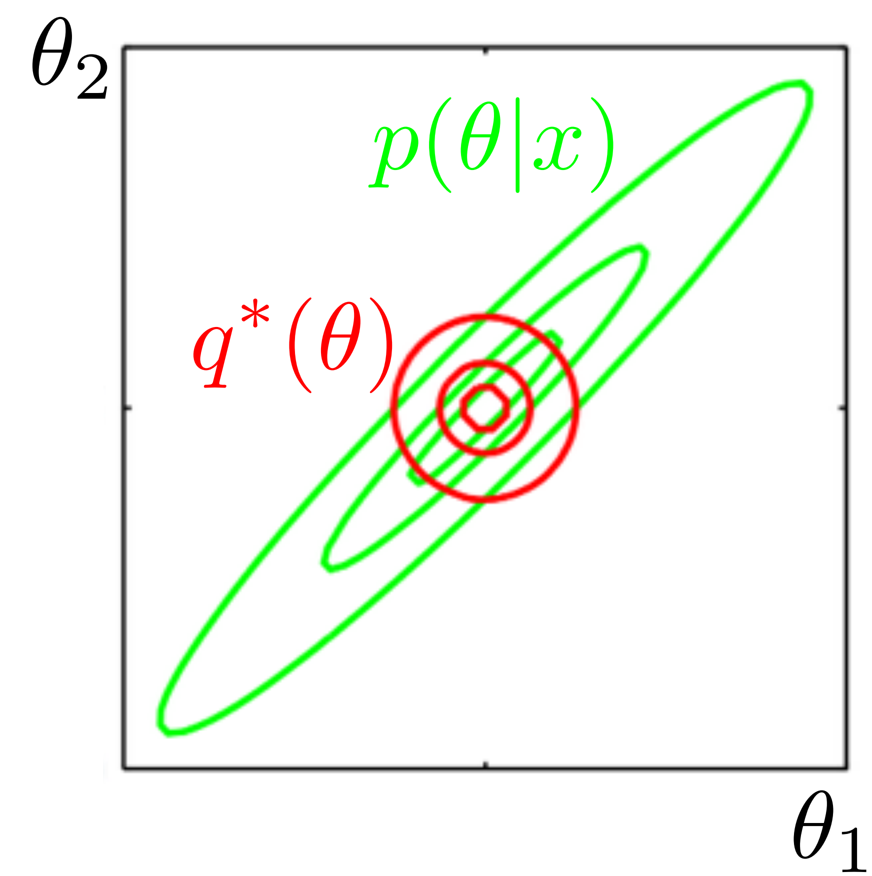
- Mean-field variational Bayes (MFVB)

$$q(\theta) = \prod_{j=1}^{J} q(\theta_j)$$

- Underestimates variance (sometimes severely)

- No covariance estimates





[MacKay 2003; Bishop 2006; Wang, Titterington 2004; Turner, Sahani 2011]
[Fosdick 2013; Dunson 2014; Bardenet, Doucet, Holmes 2015]

Linear response

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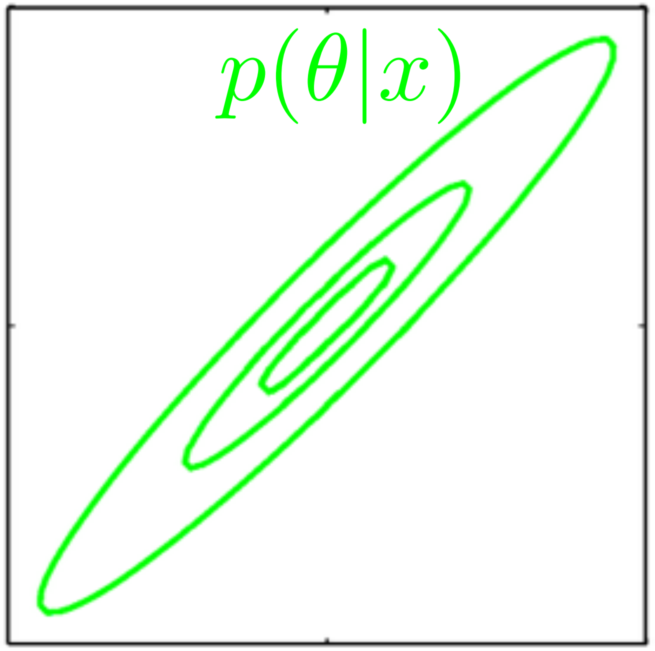

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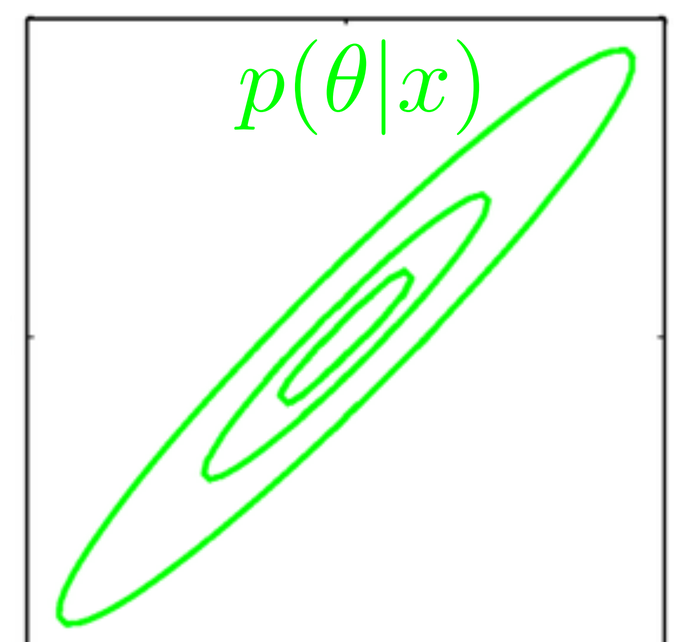

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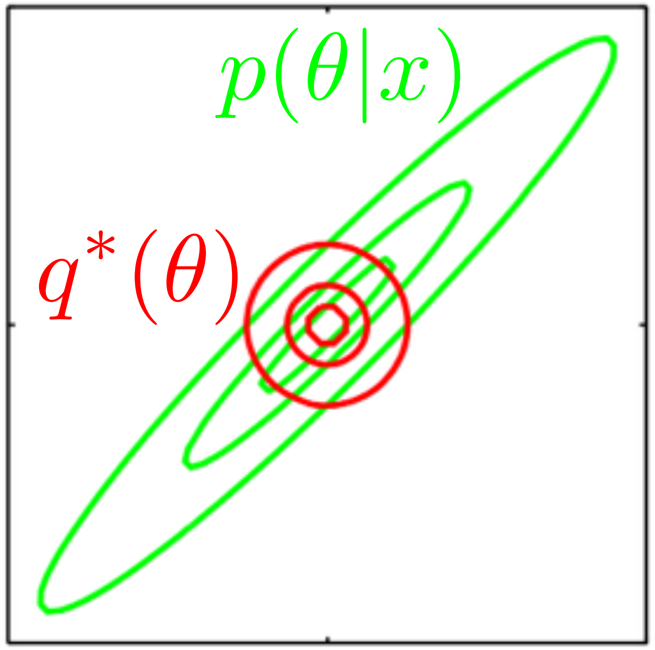

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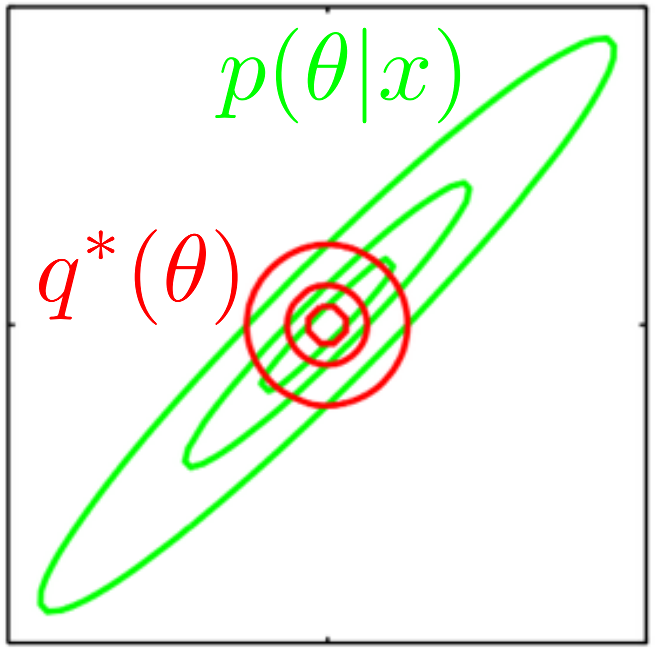

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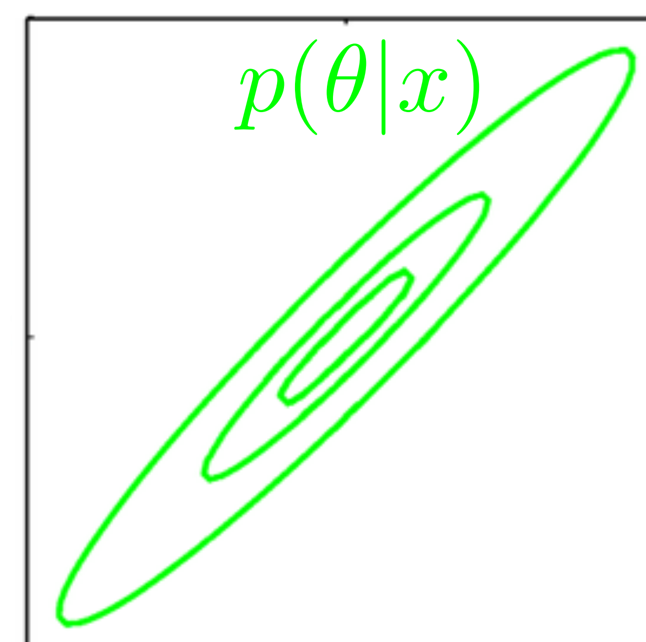

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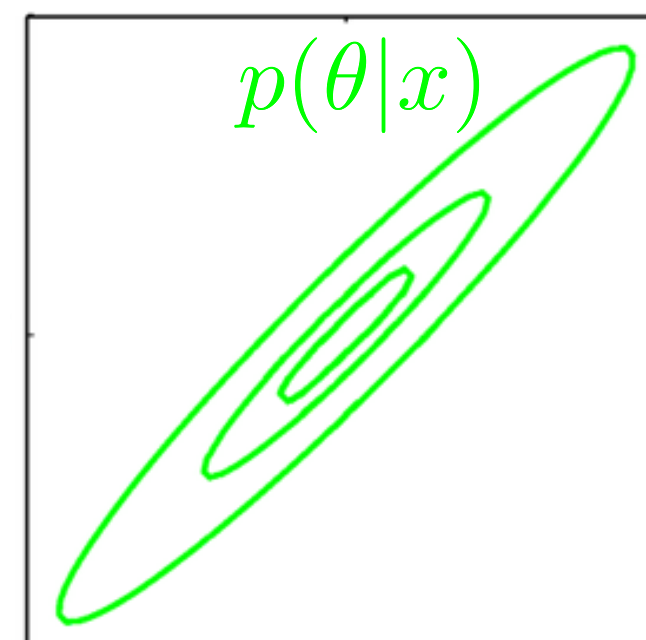

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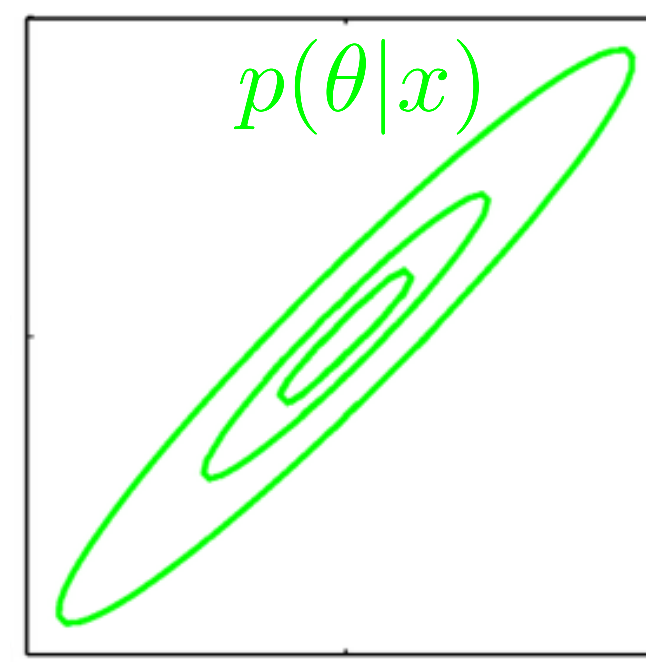

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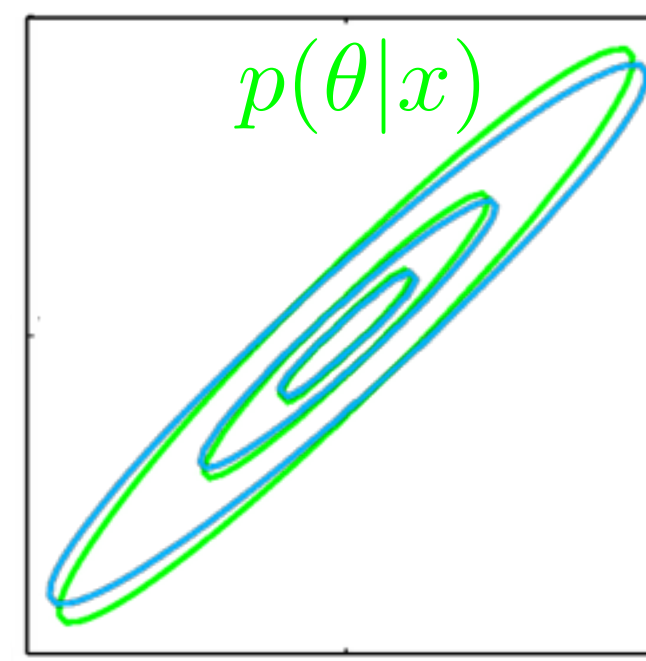

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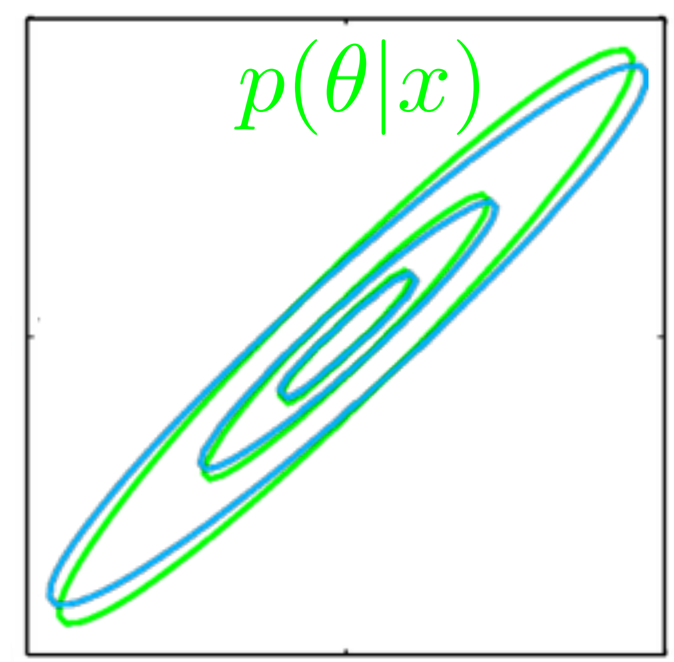

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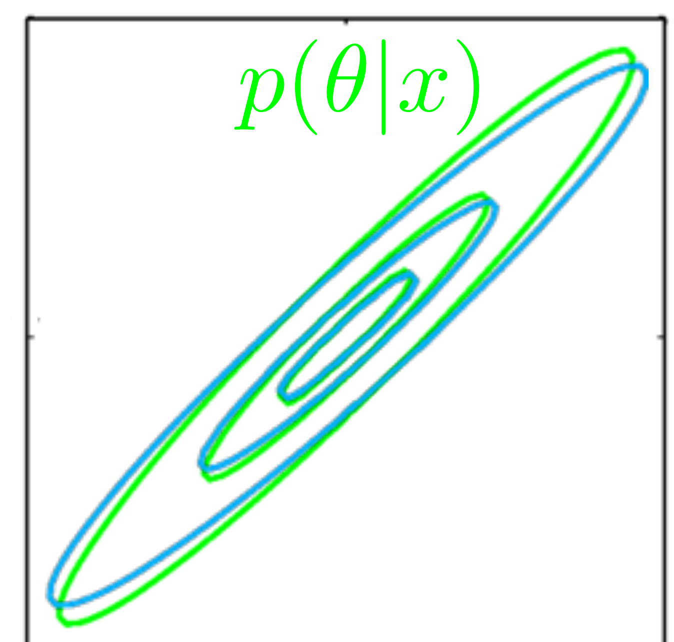

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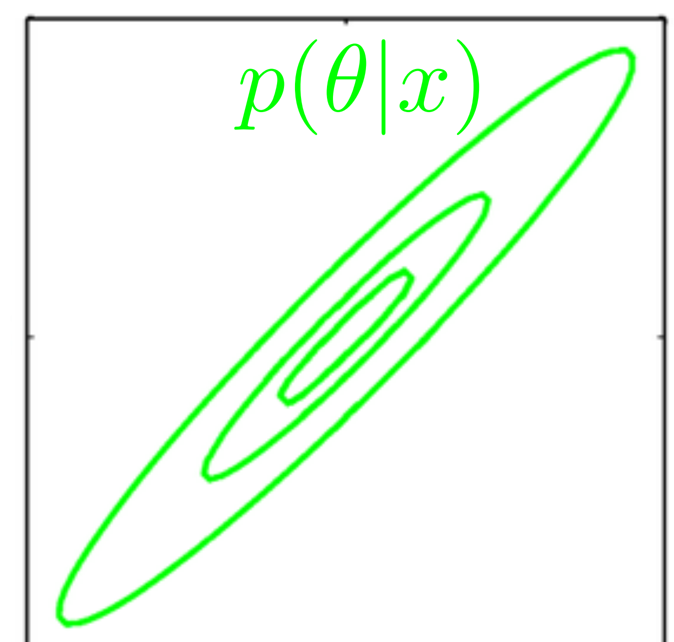

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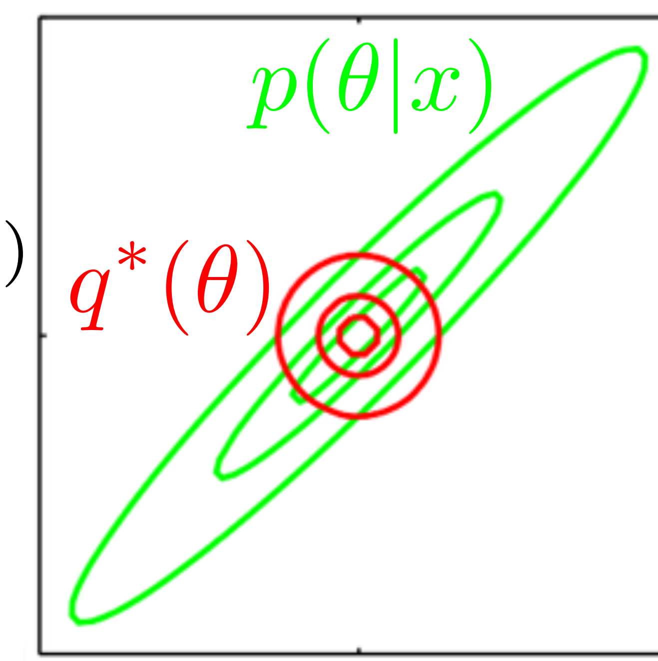

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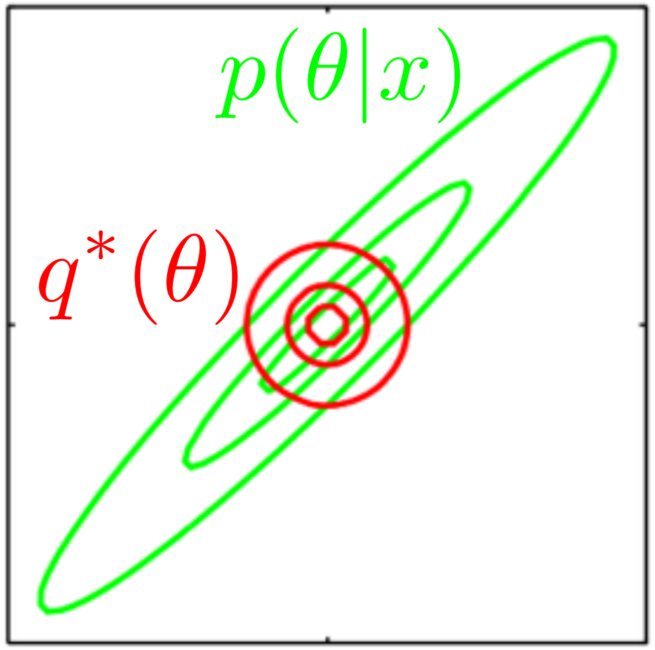

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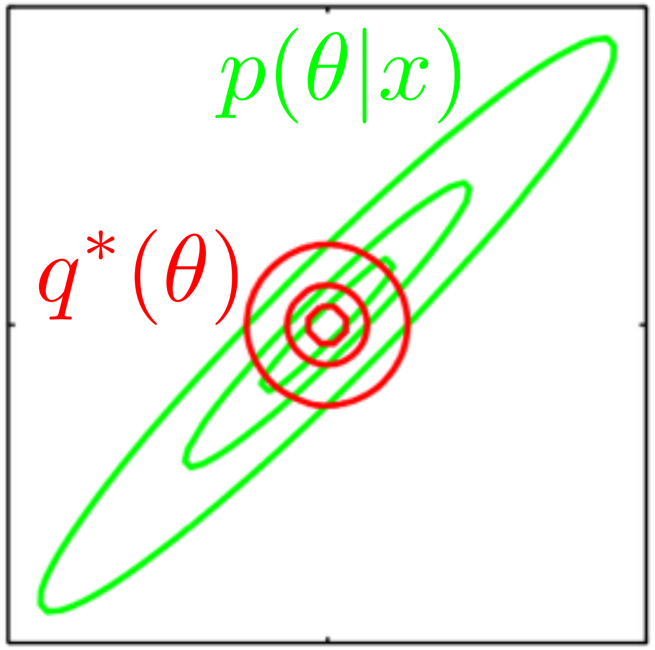

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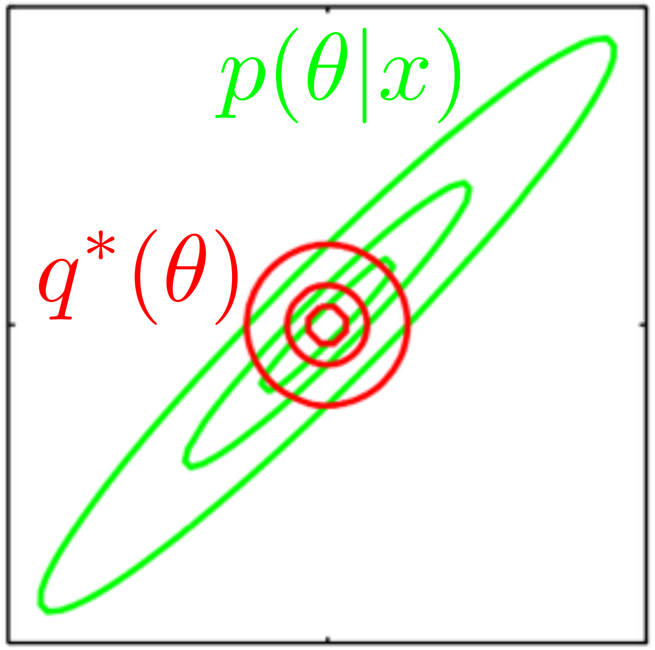

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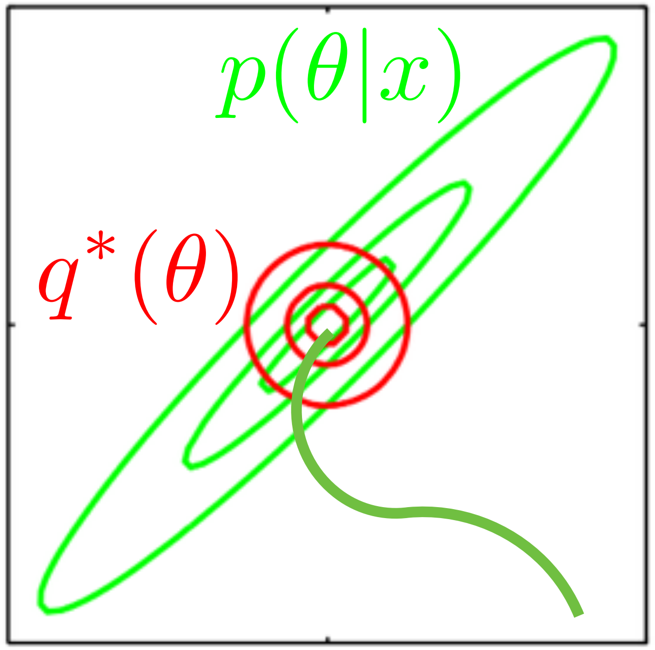

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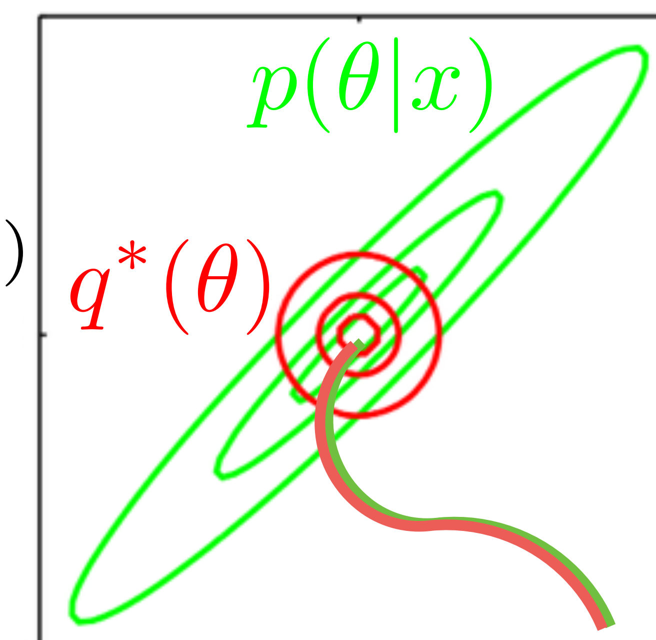

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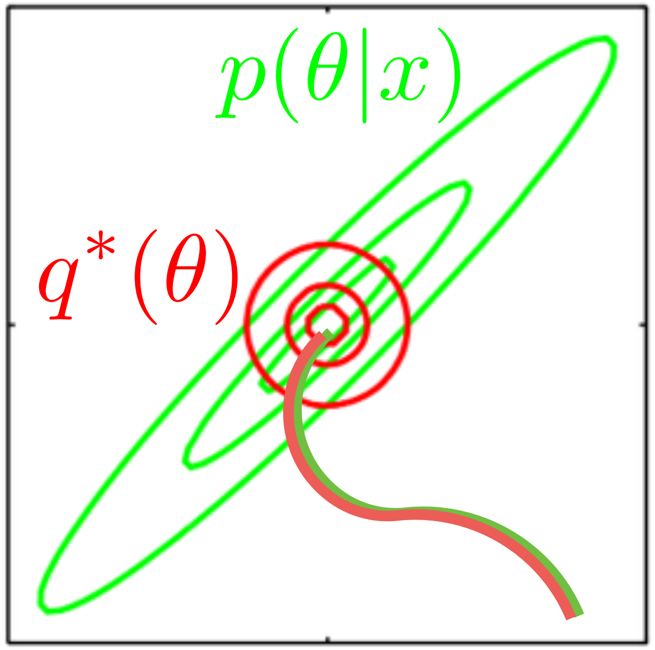

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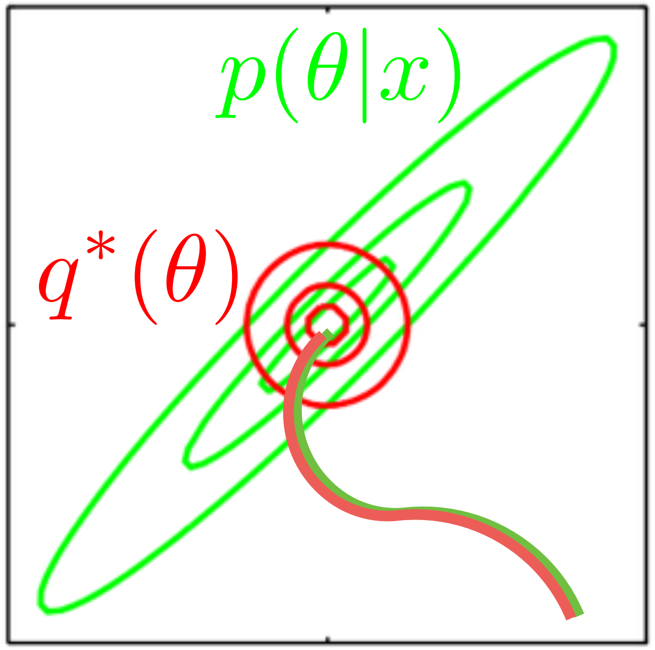

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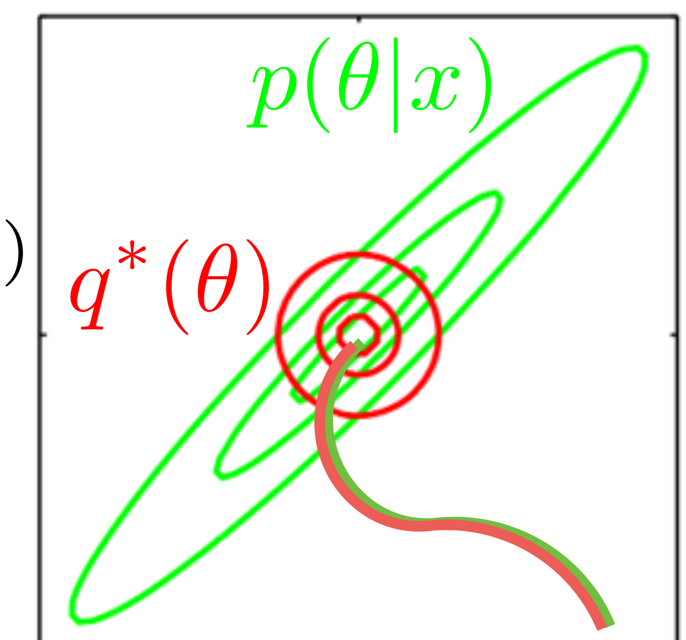

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- Goal: $\hat{\Sigma} := \left. \frac{d}{dt^T} \mathbb{E}_{q_{\eta^*(t)}} \theta \right|_{t=0}$

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- More general: $\log p_t(\theta) = \log p(\theta | y) + f(\theta, t) - \mathrm{Const}(t)$
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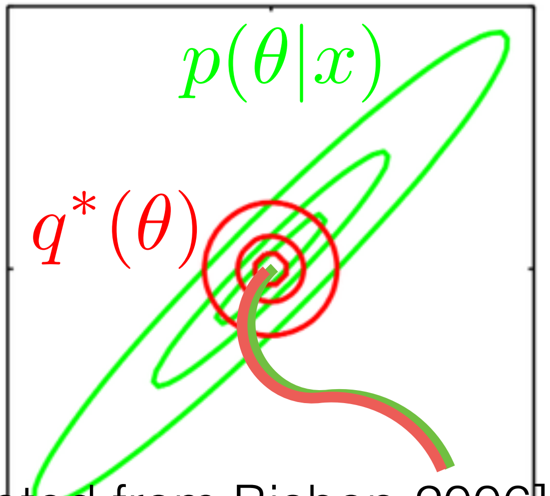

[adapted from Bishop 2006]

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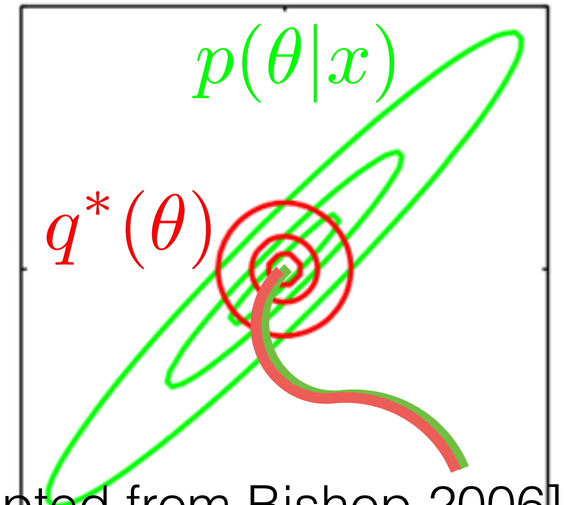

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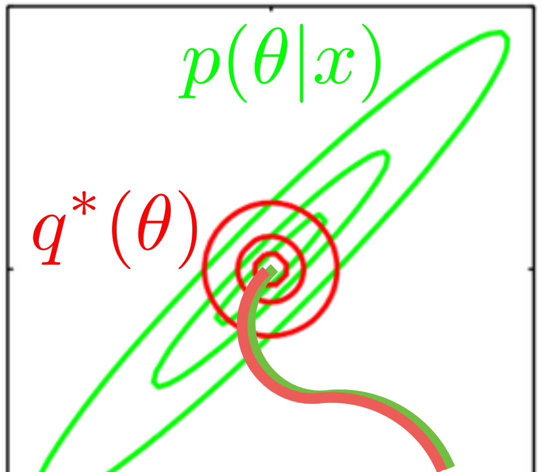

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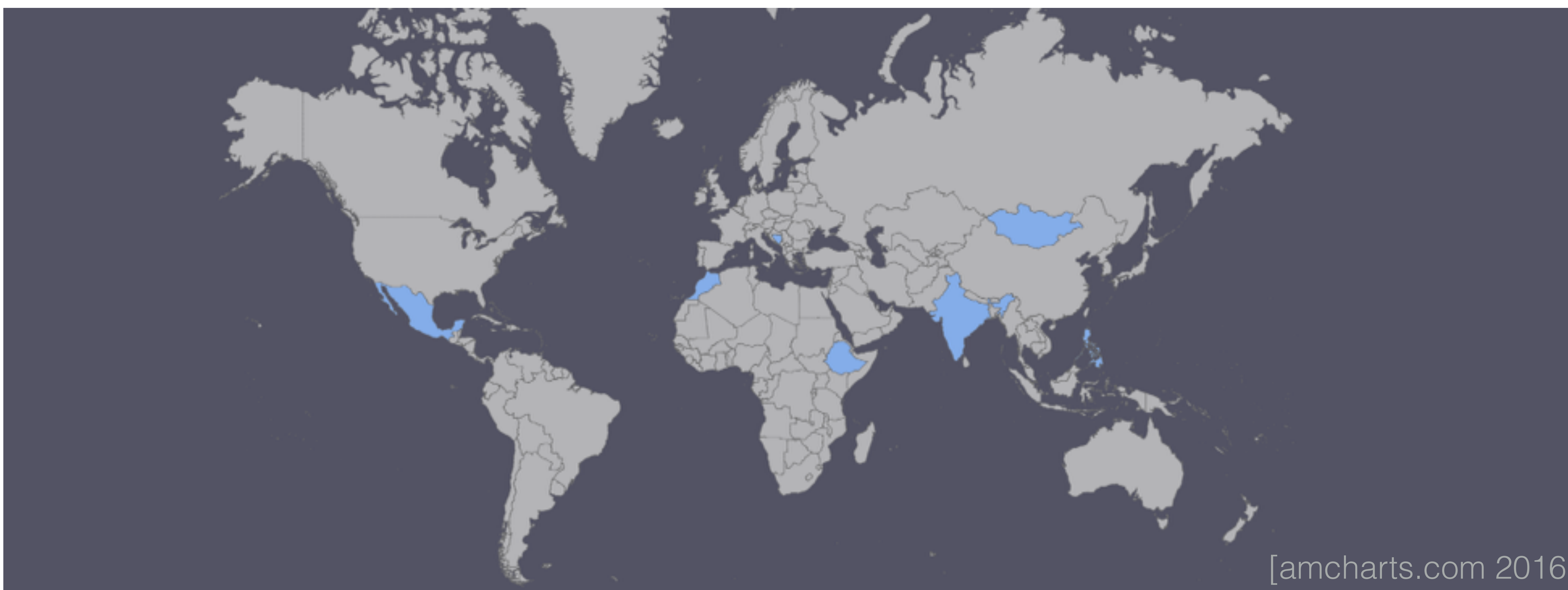
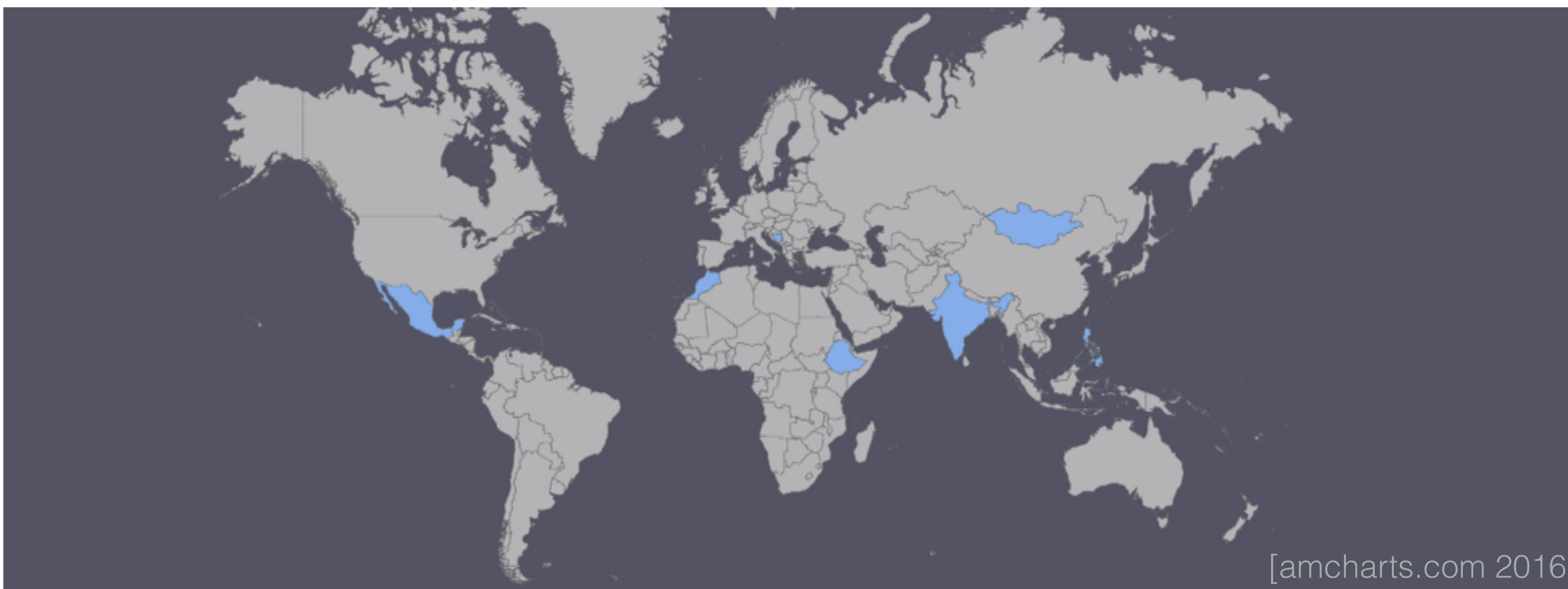
Microcredit Experiment



Microcredit Experiment

- Simplified from Meager (2018a)
- K microcredit trials (Mexico, Mongolia, Bosnia, India, Morocco, Philippines, Ethiopia)
- N_k businesses in kth site (~900 to ~17K)



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$$\underbrace{y_{kn}}_{\text{profit}}$$

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Microcredit Experiment

- Simplified from Meager (2018a)
- K microcredit trials (Mexico, Mongolia, Bosnia, India, Morocco, Philippines, Ethiopia)
- N_k businesses in kth site (~900 to ~17K)
- Profit of nth business at kth site:

1 if microcredit

profit

$$y_{kn} \overset{indep}{\sim} \mathcal{N}(\mu_k + T_{kn}\tau_k, \sigma_k^2)$$

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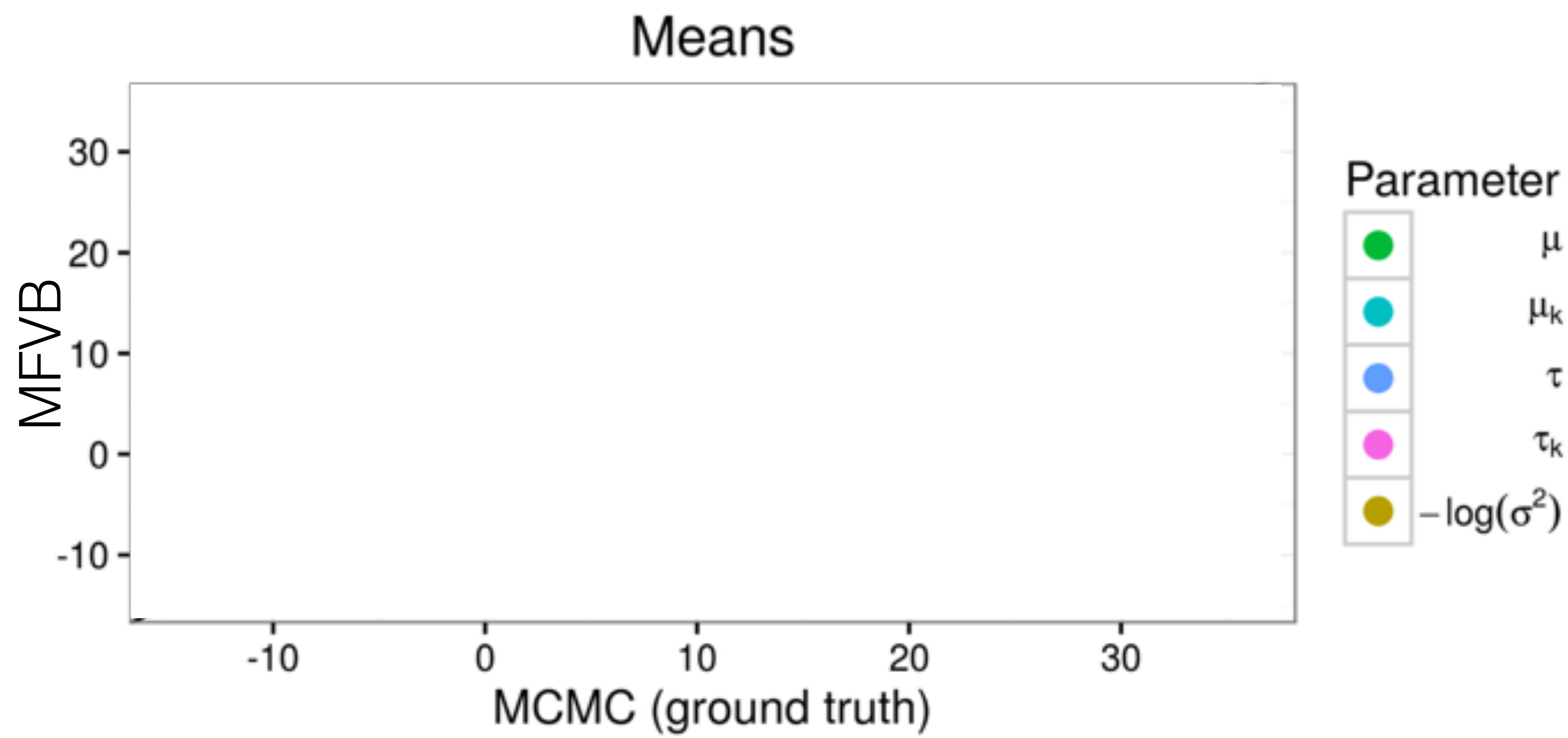
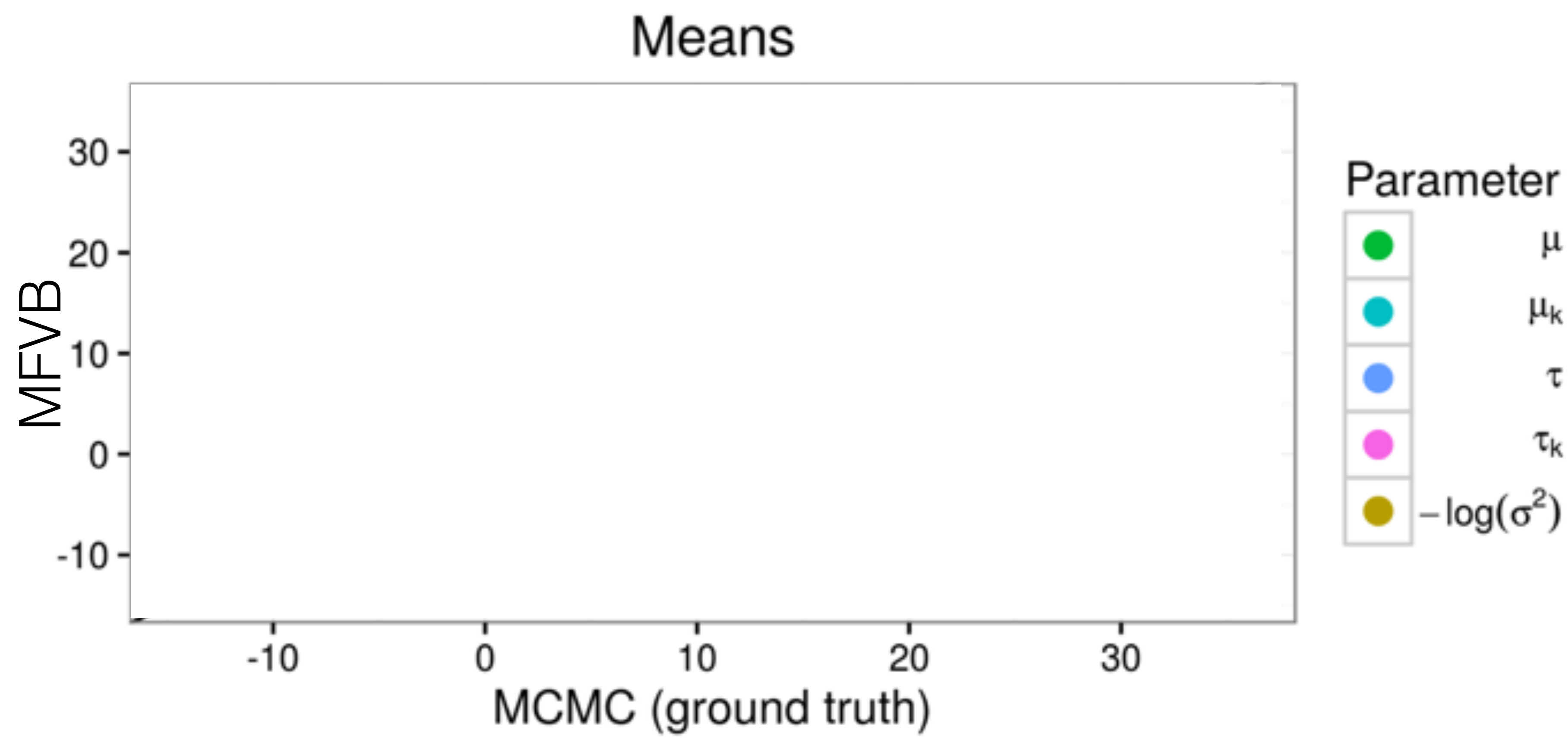
$$\sigma_k^{-2} \overset{iid}{\sim} \Gamma(a, b) \qquad C \sim \mathrm{Sep\&LKJ}(\eta, c, d)$$

Microcredit Experiment



Microcredit Experiment

- *One set* of 2500 MCMC draws: **45 minutes**



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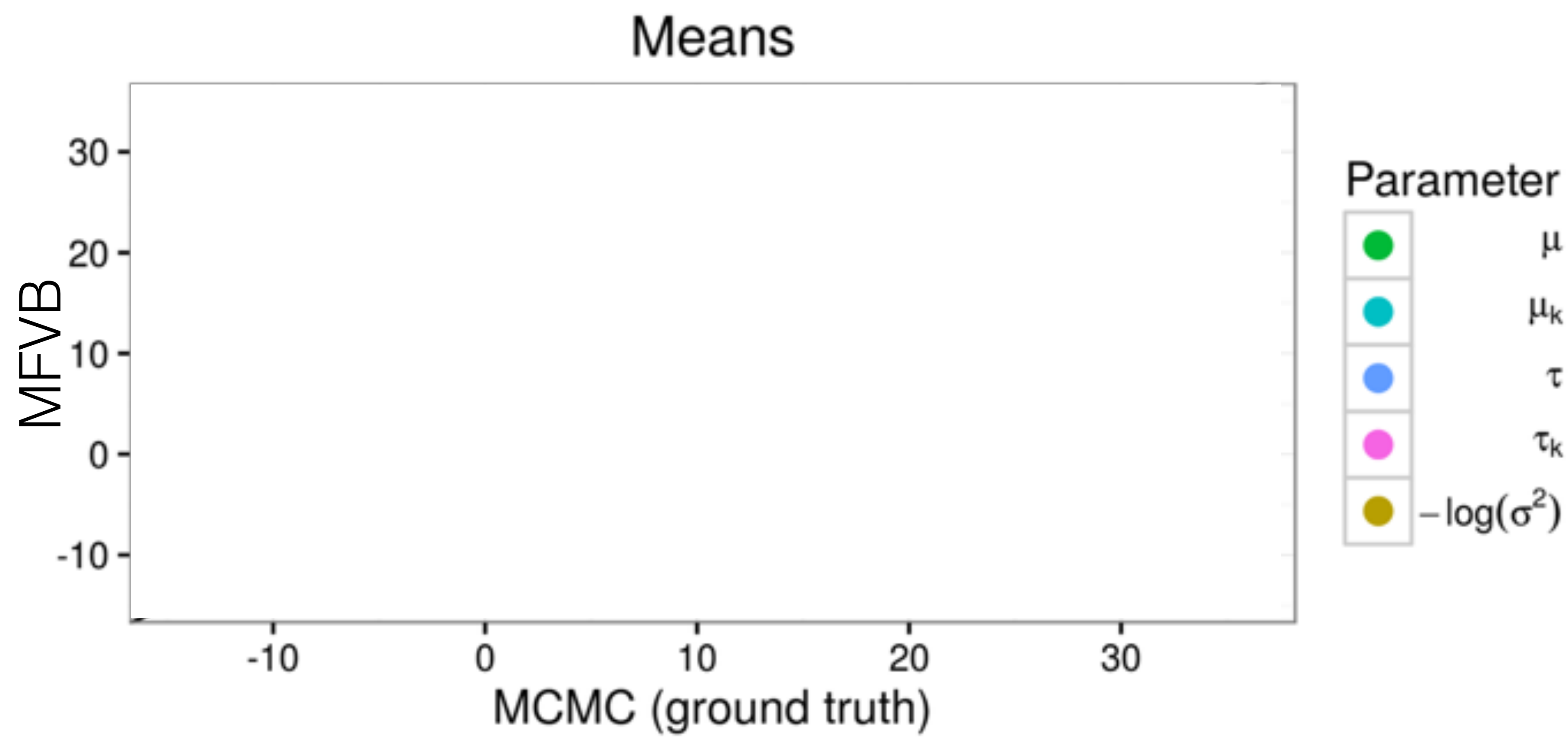

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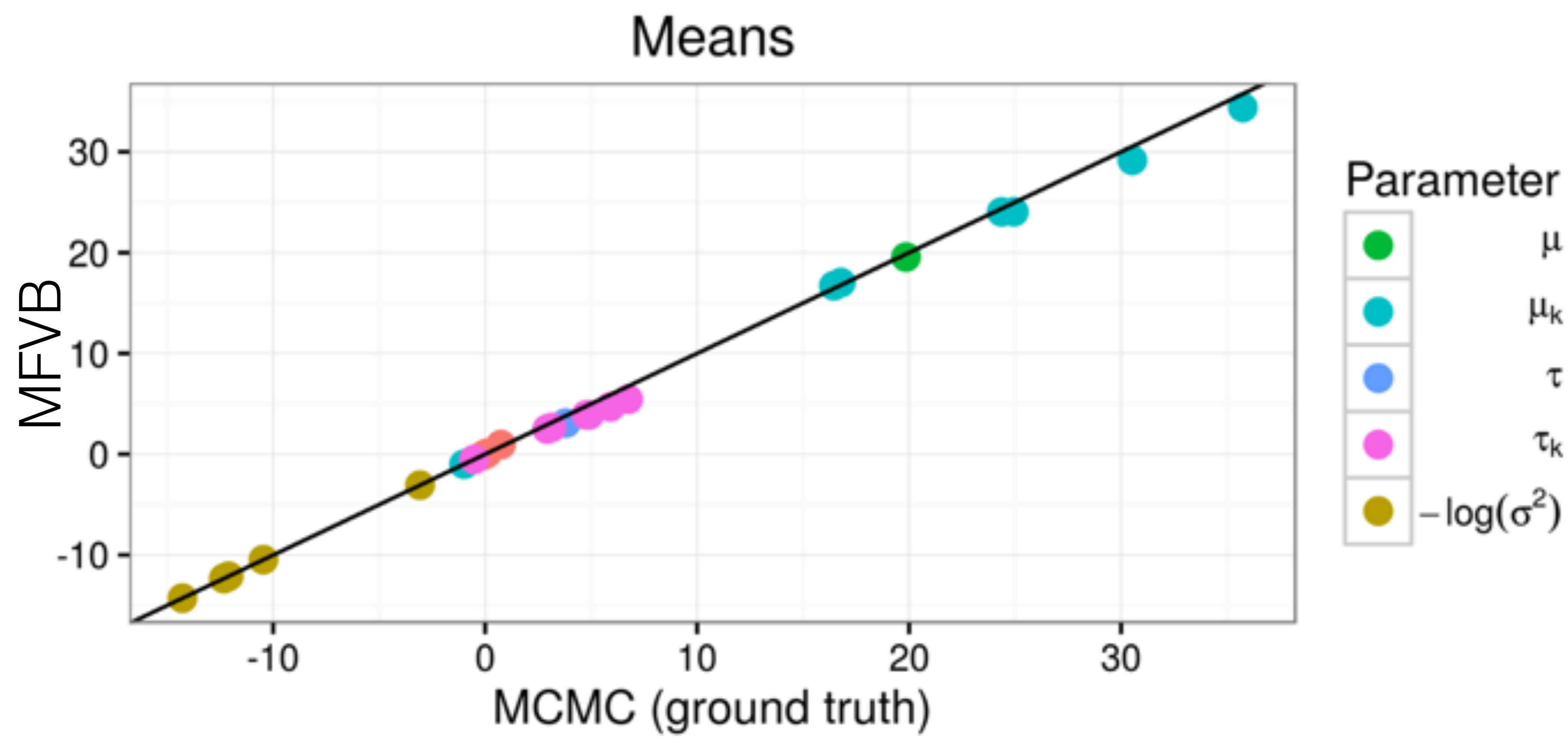

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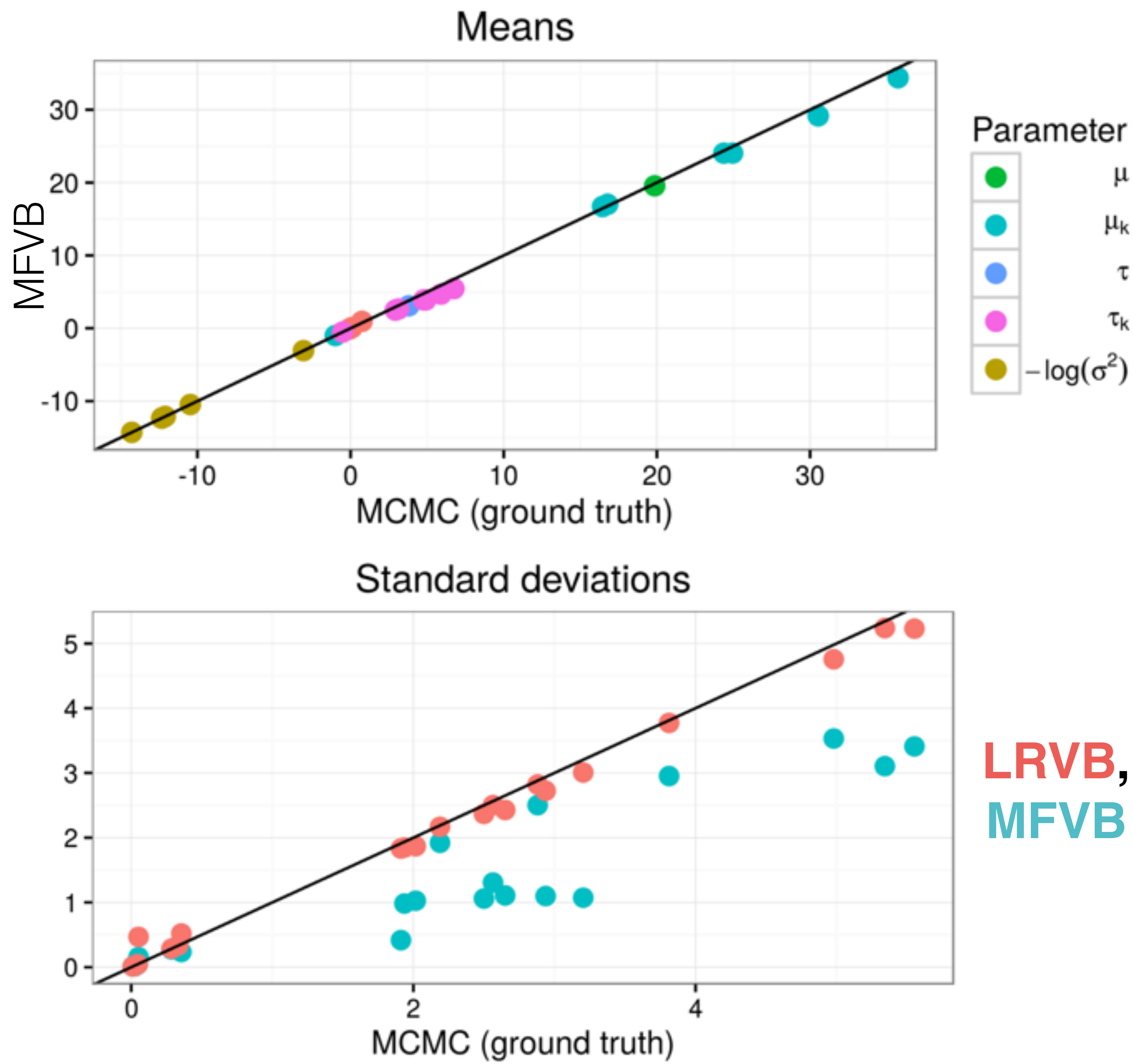

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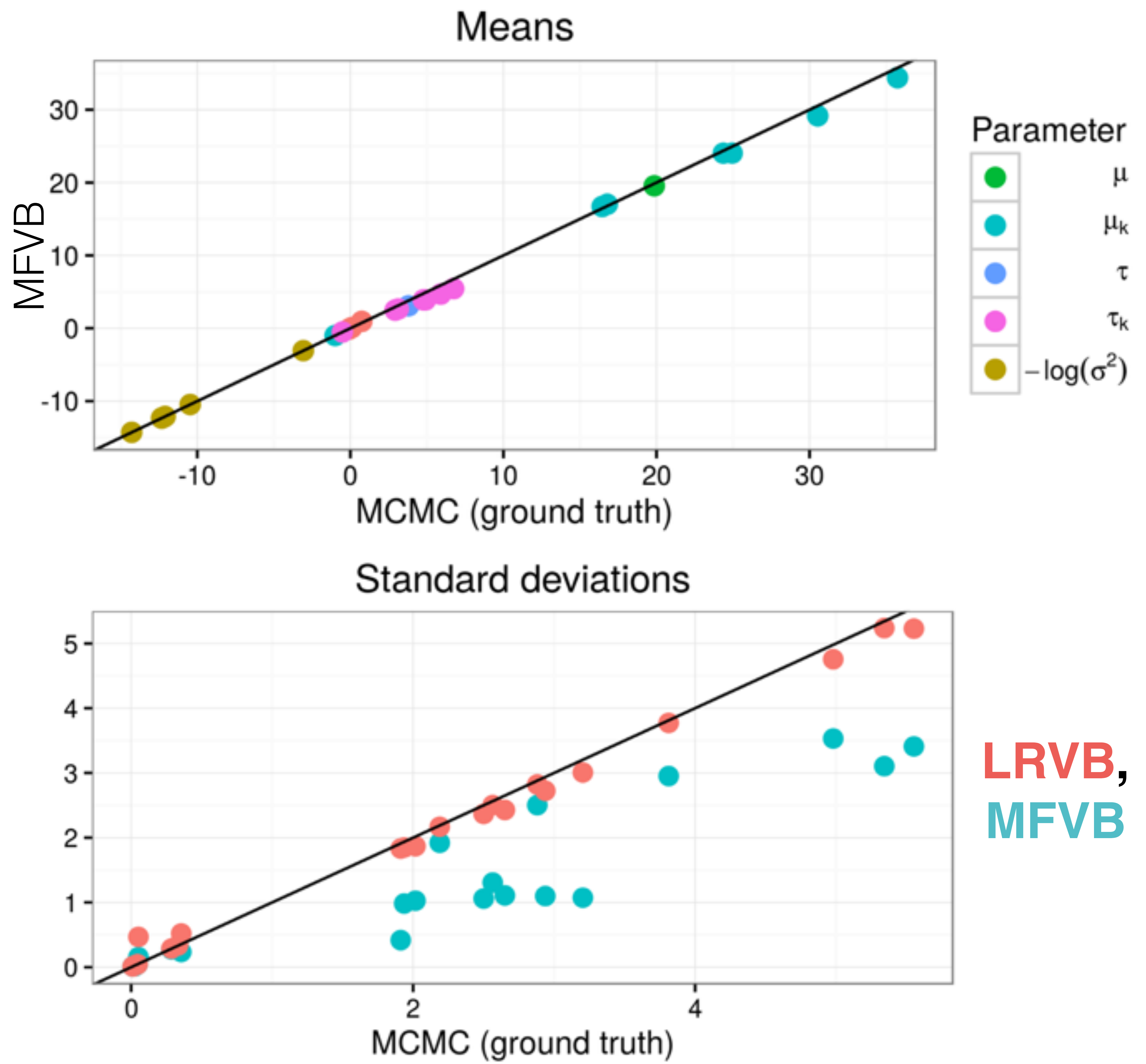

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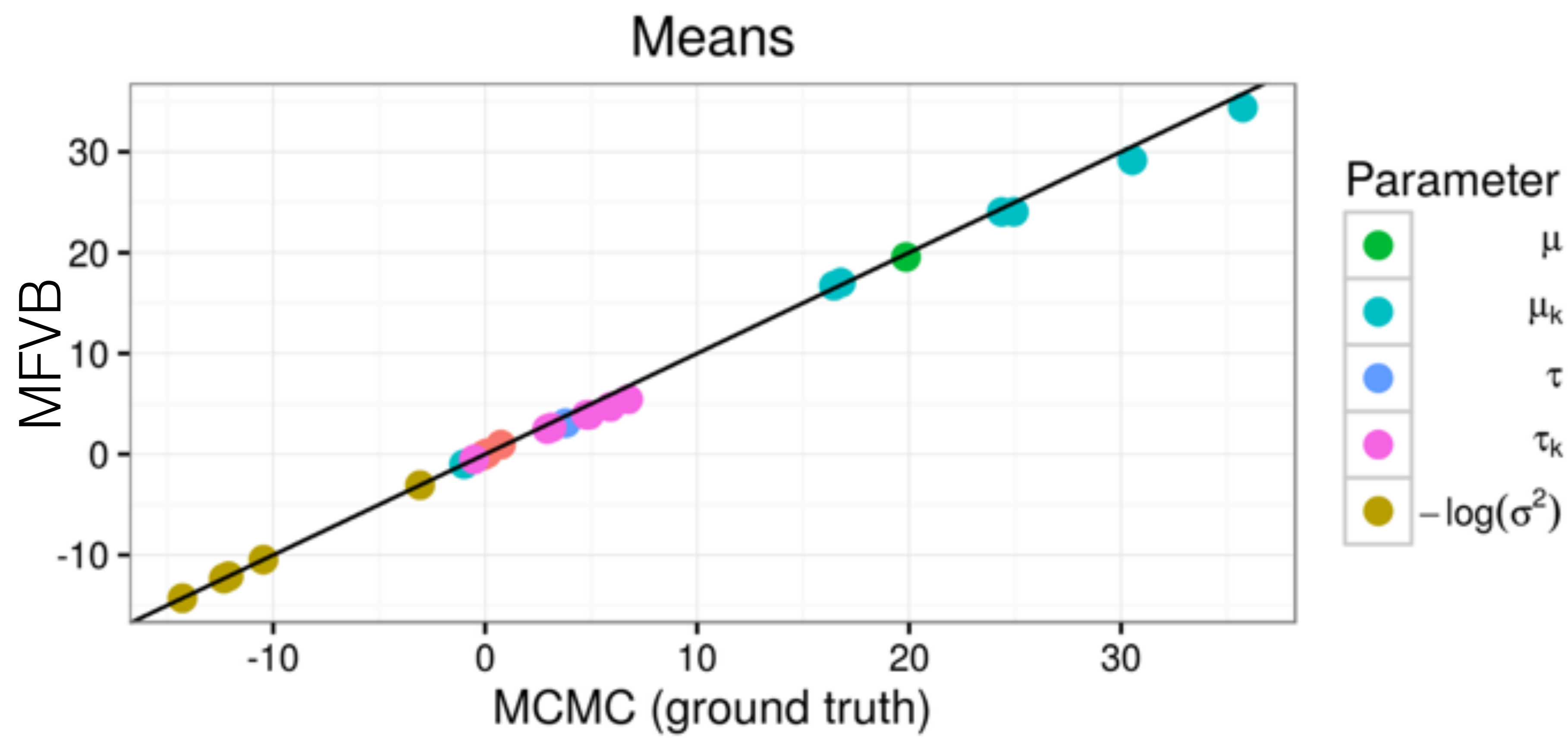
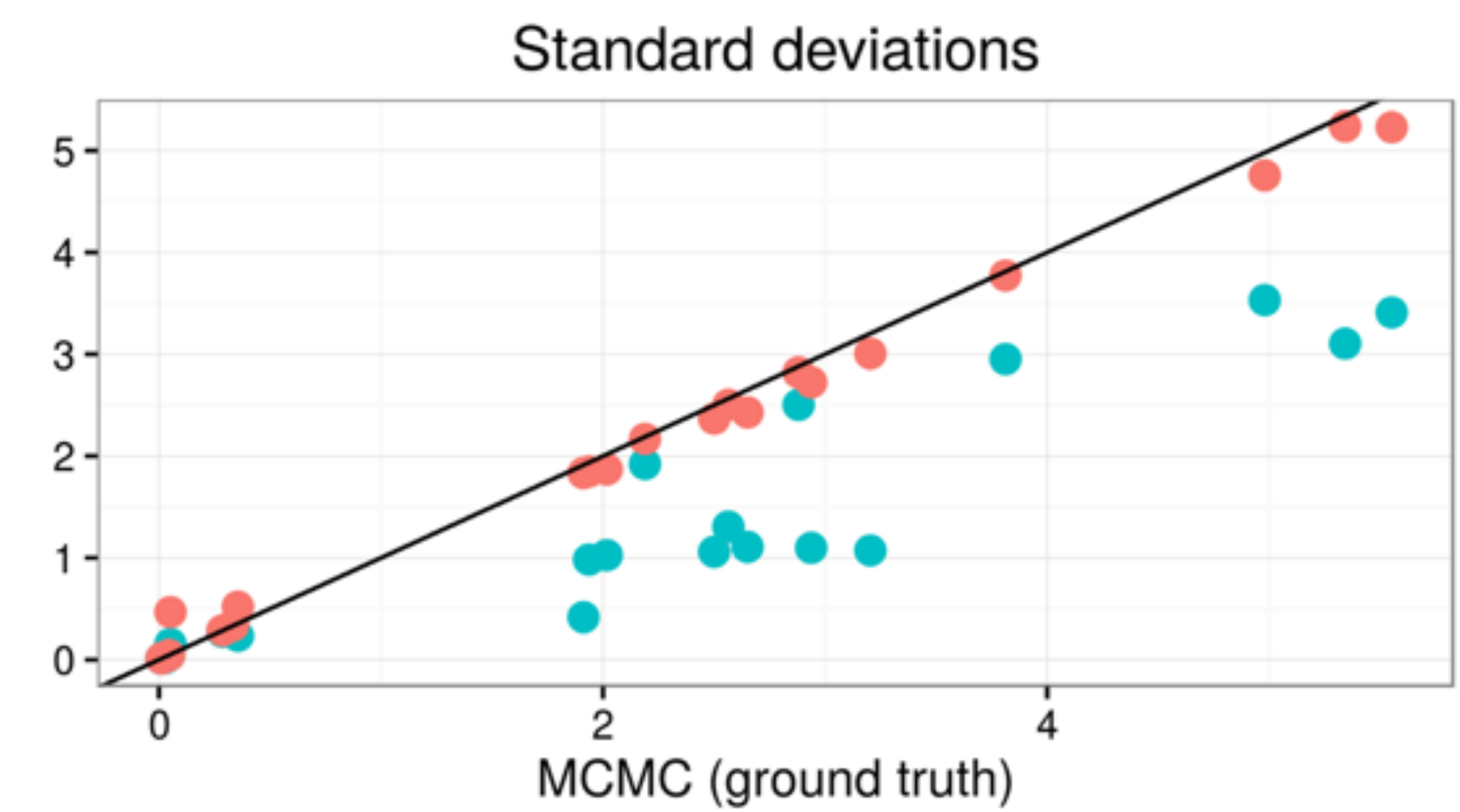
- τ mean (MFVB): 3.08 USD PPP
- τ std dev (LRVB): 1.83 USD PPP

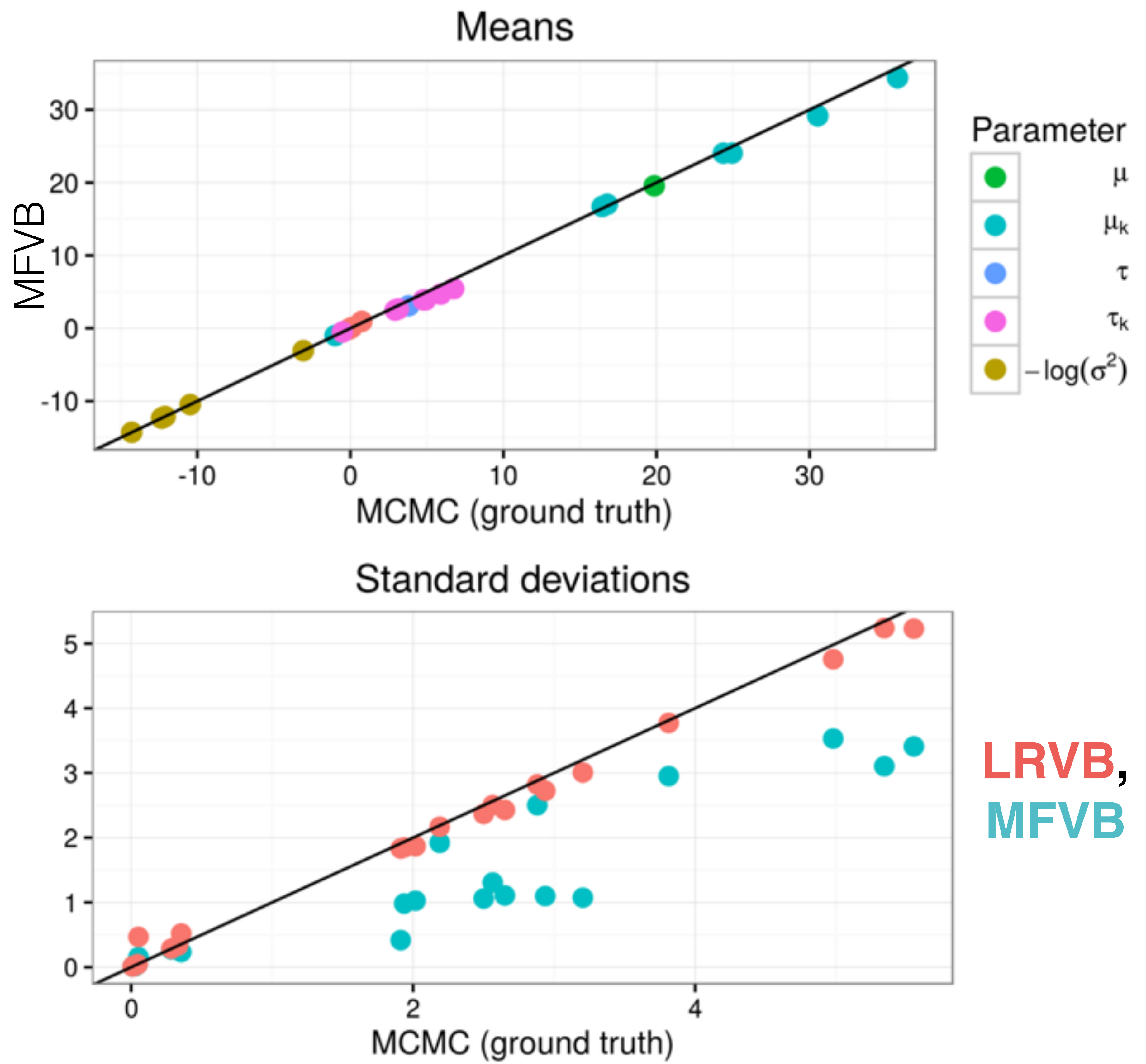




LRVB, **MFVB**

Microcredit Experiment

- *One set* of 2500 MCMC draws: **45 minutes**
- All of MFVB optimization, LRVB uncertainties, all sensitivity measures: **58 seconds**
- τ mean (MFVB): 3.08 USD PPP
- τ std dev (LRVB): 1.83 USD PPP
- Mean is 1.68 std dev from 0



Scaling the matrix inverse

$$\hat{\Sigma} = - \left. \frac{\partial \mathbb{E}_{q_\eta} \theta}{\partial \eta} \right|_{\eta=\eta^*} \left[\left. \frac{\partial^2 \mathrm{ELBO}(q_\eta, p_0)}{\partial \eta \partial \eta^T} \right|_{\eta=\eta^*} \right]^{-1} \left[\left. \frac{\partial \mathbb{E}_{q_\eta} \theta}{\partial \eta} \right|_{\eta=\eta^*} \right]^T$$

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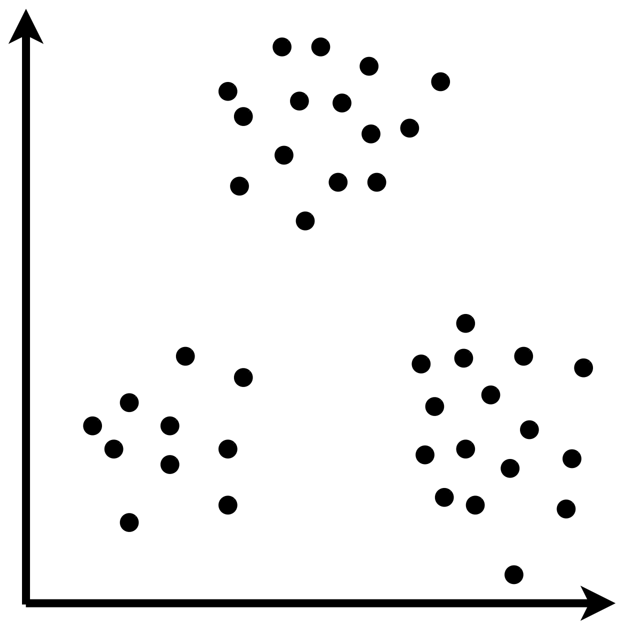

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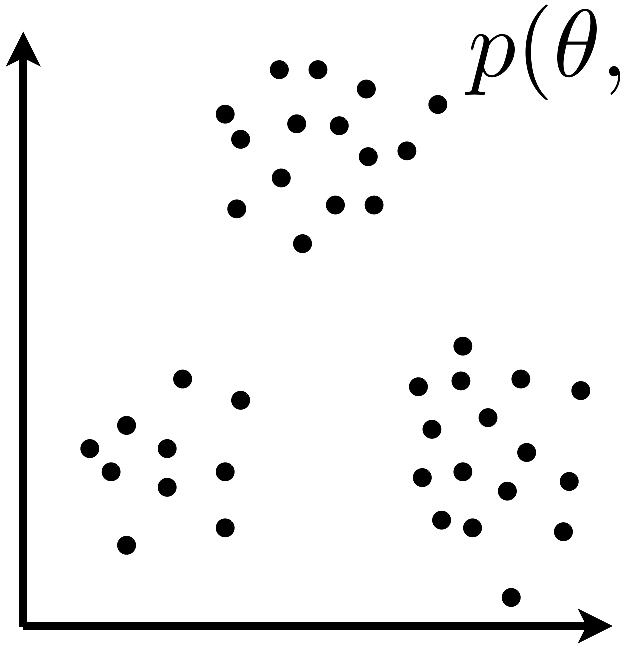
$$p(\theta, y) = \prod_{n=1}^{N} \mathcal{N}(y_n | \mu_{z_n}, \Sigma_{z_n})$$



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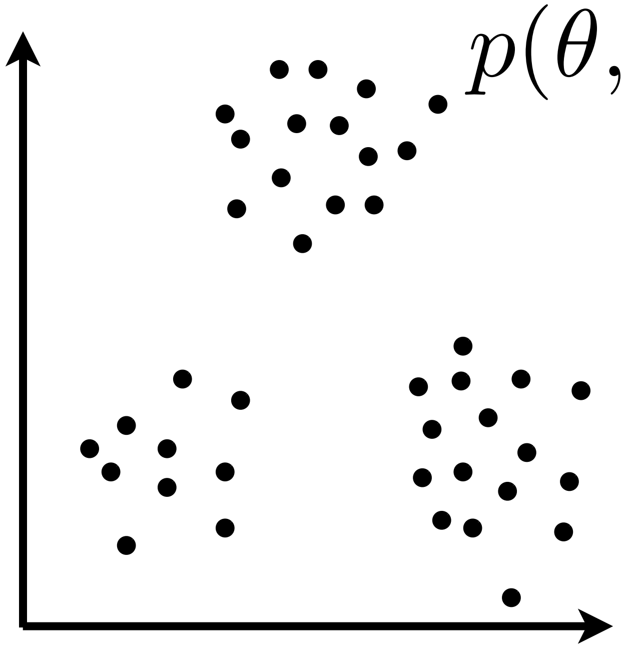

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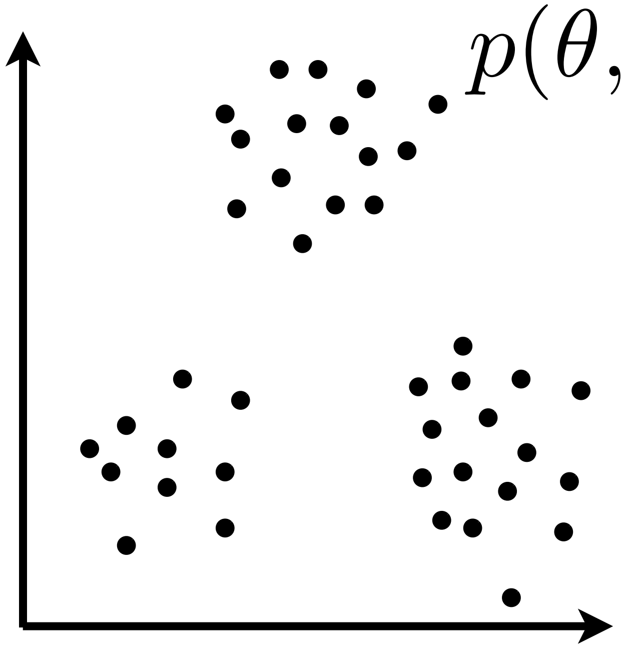

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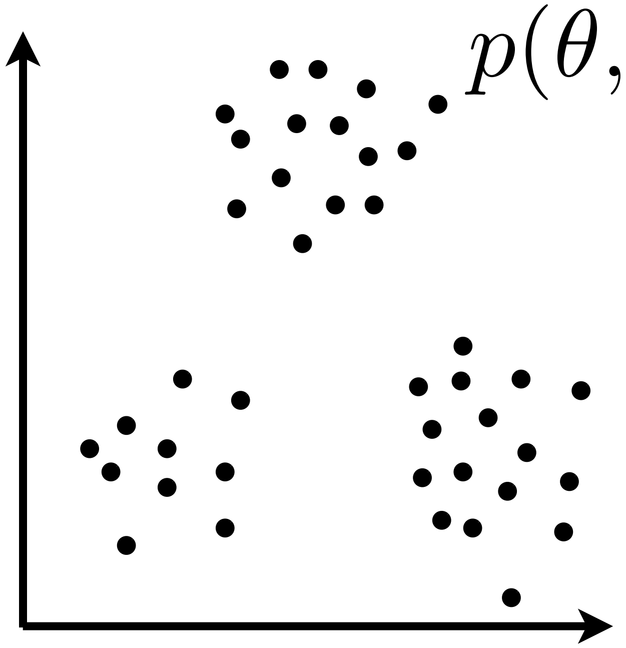

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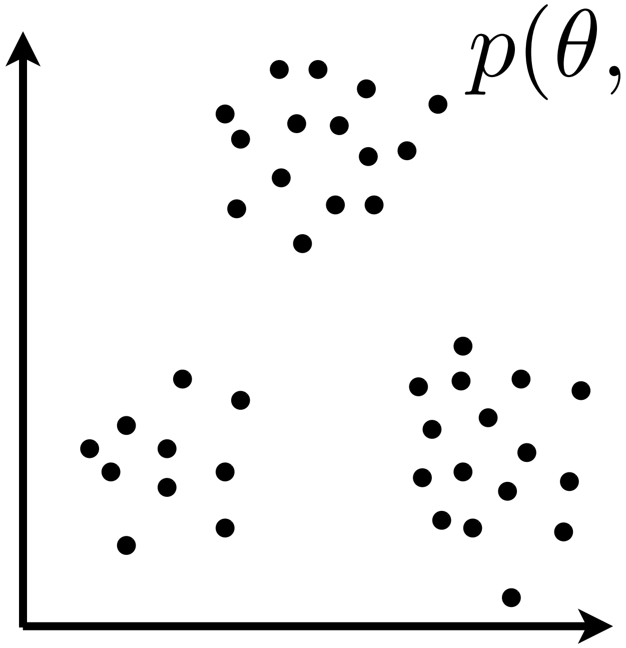

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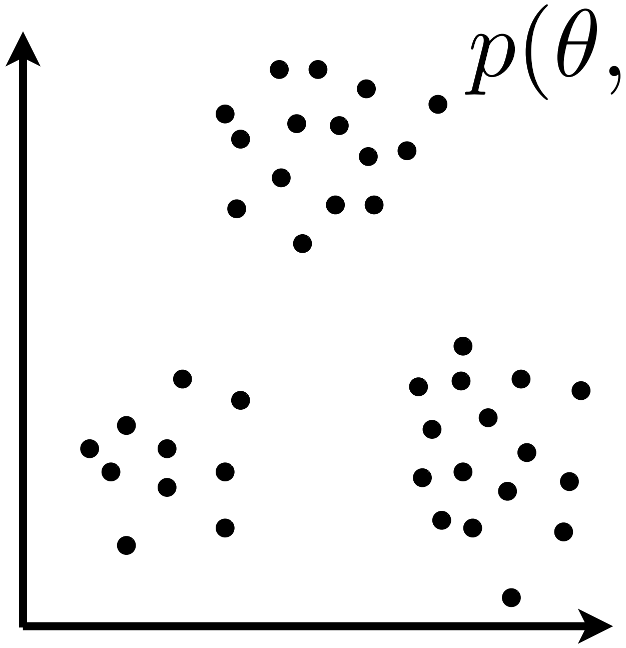

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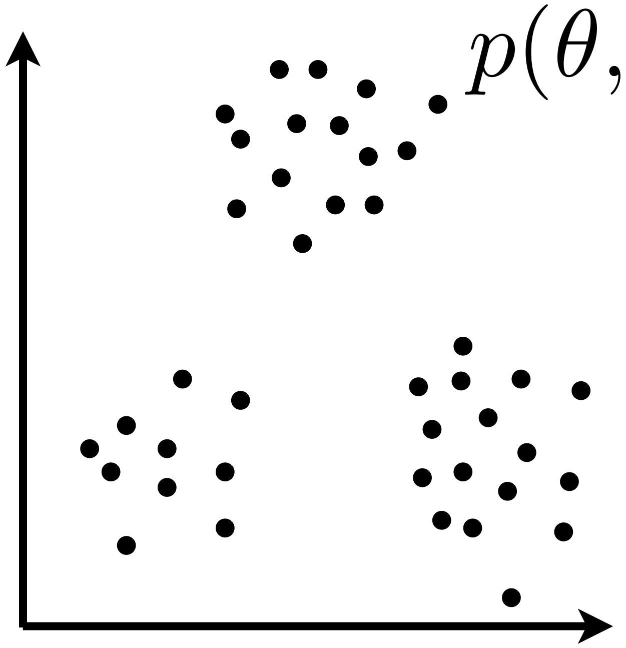

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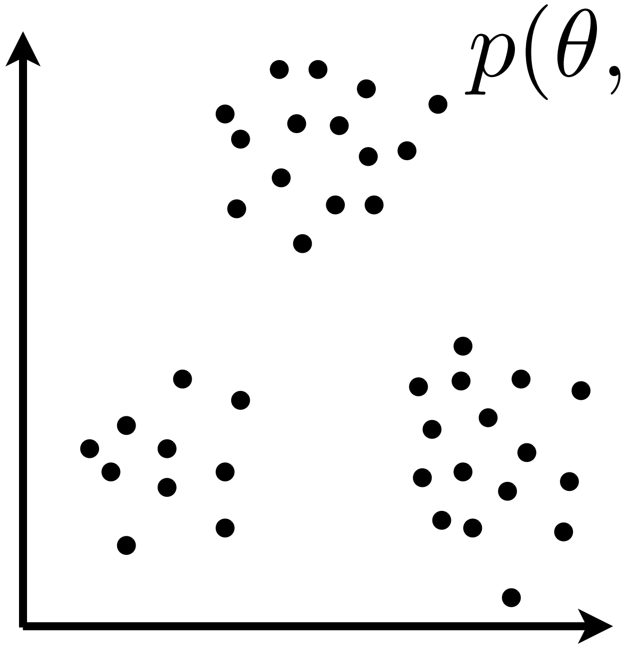

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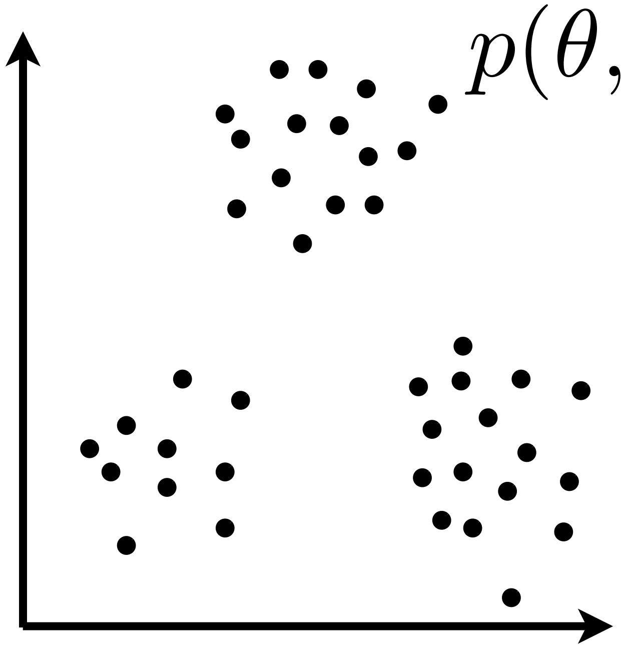

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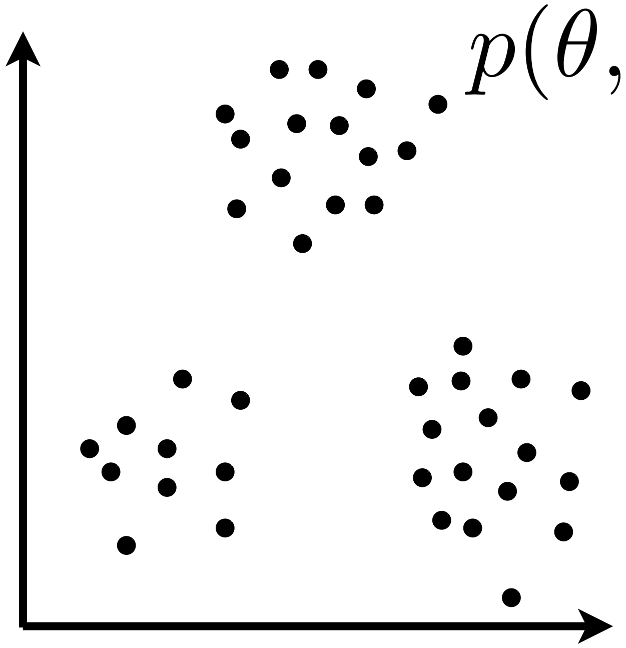

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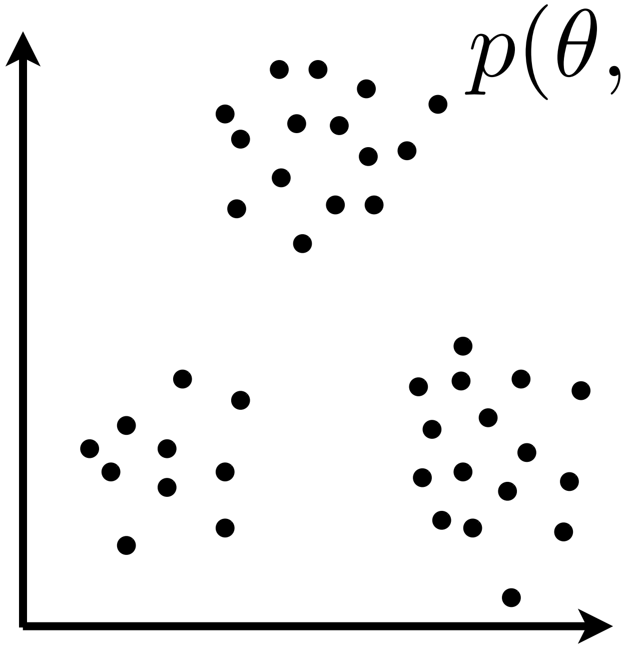

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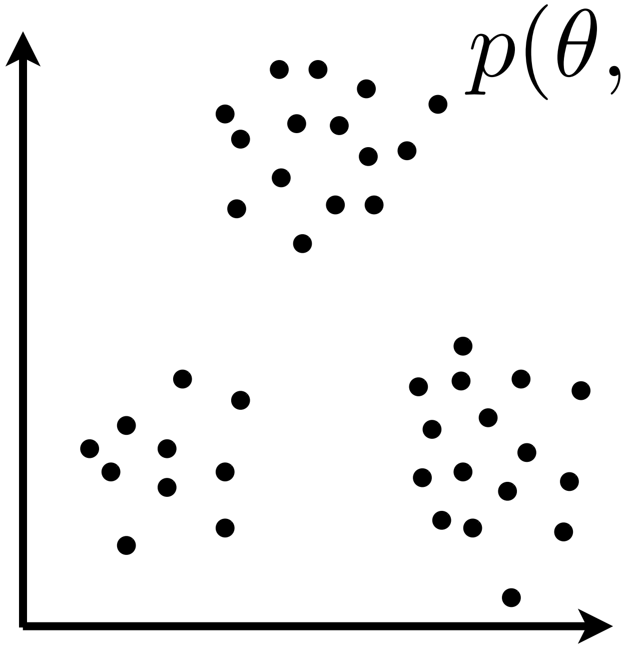

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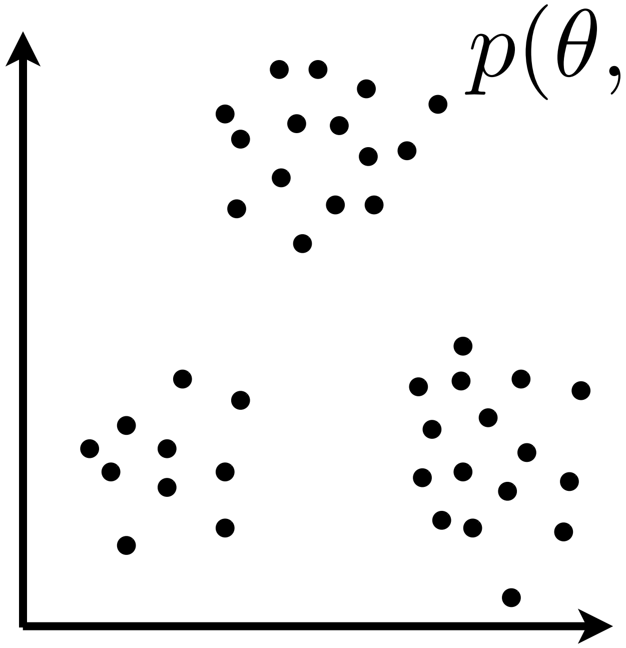

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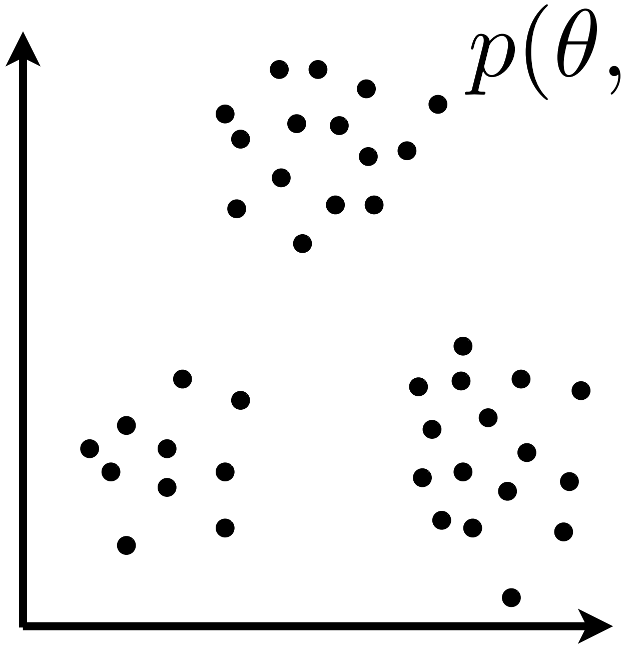

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$$\log p(\theta, y)$$
$$= \left[\sum_{n=1}^{N} \log p(y_n|\theta_{loc,n}, \theta_{glob}) + \log p(\theta_{loc,n}|\theta_{glob})\right]$$
$$+ \log p(\theta_{glob})$$

Scaling the matrix inverse

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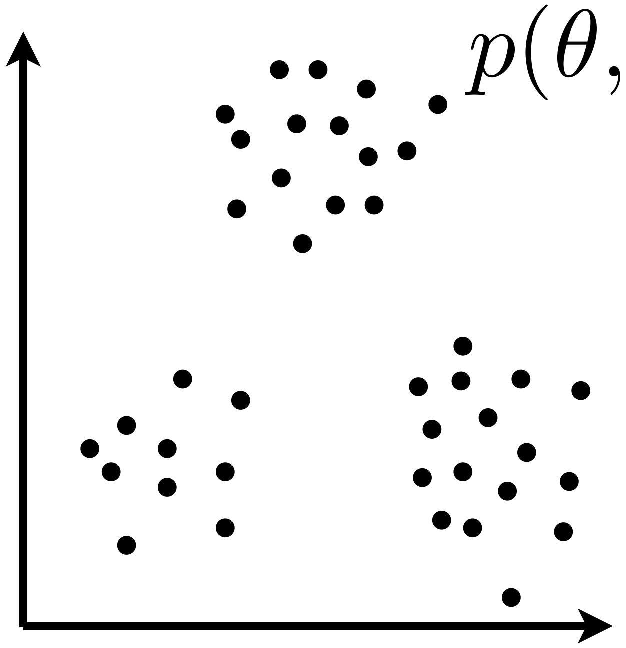

$$p(\theta, y) = \left[\prod_{n=1}^{N} \mathcal{N}(y_n | \mu_{z_n}, \Sigma_{z_n}) \mathrm{Cat}(z_n | \pi_{1:K}) \right]$$

$$\cdot \left[\prod_{k=1}^{K} \mathcal{N}(\mu_k | \mu_0, \Sigma_0) \mathrm{InvWish}(\Sigma_k | A, \nu) \right] \mathrm{Dir}(\pi | a)$$

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$$= \left[\sum_{n=1}^{N} \log p(y_n | \theta_{loc,n}, \theta_{glob}) + \log p(\theta_{loc,n} | \theta_{glob}) \right]$$

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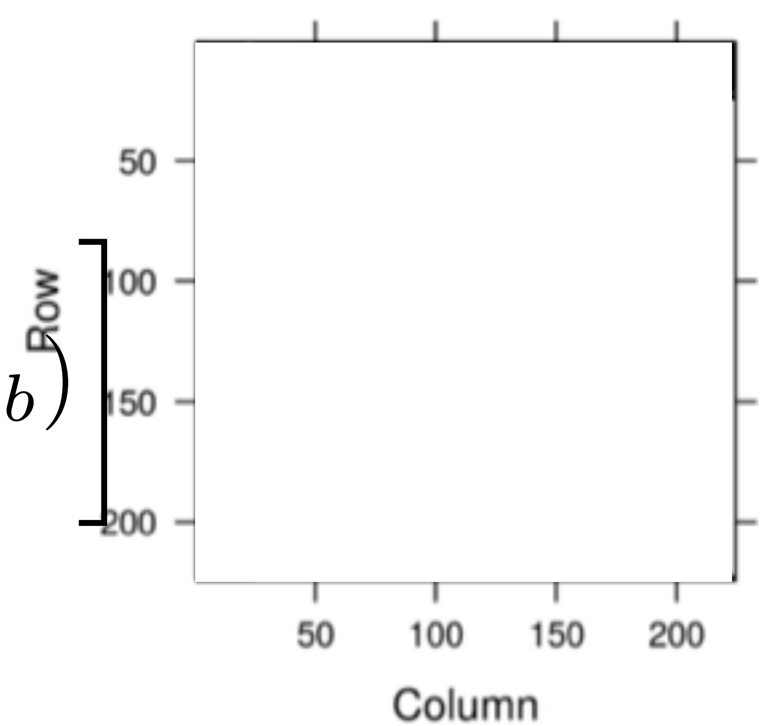

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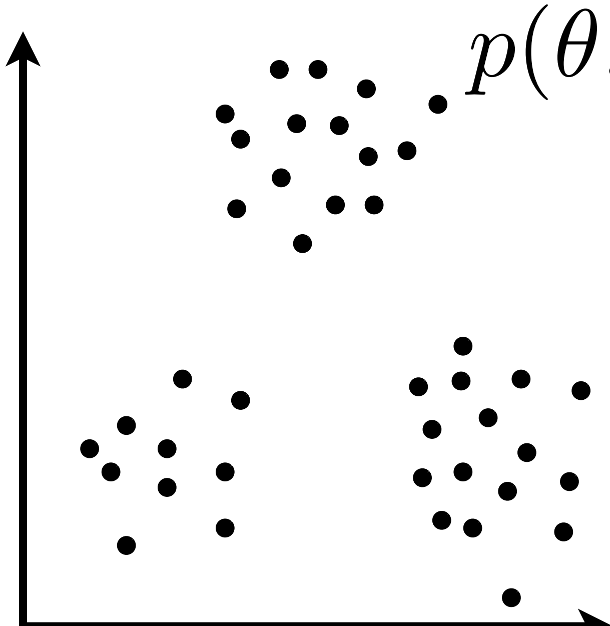

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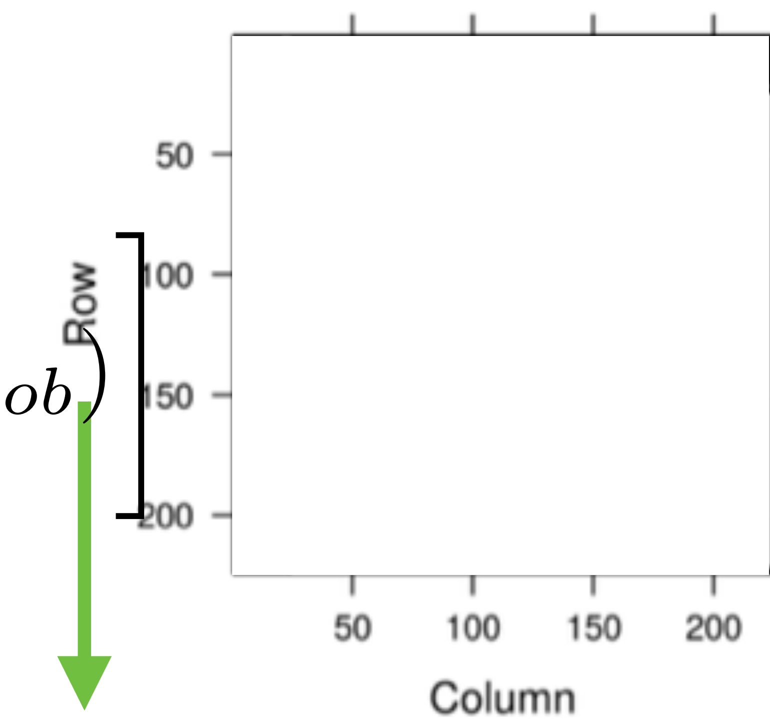
10,014 params



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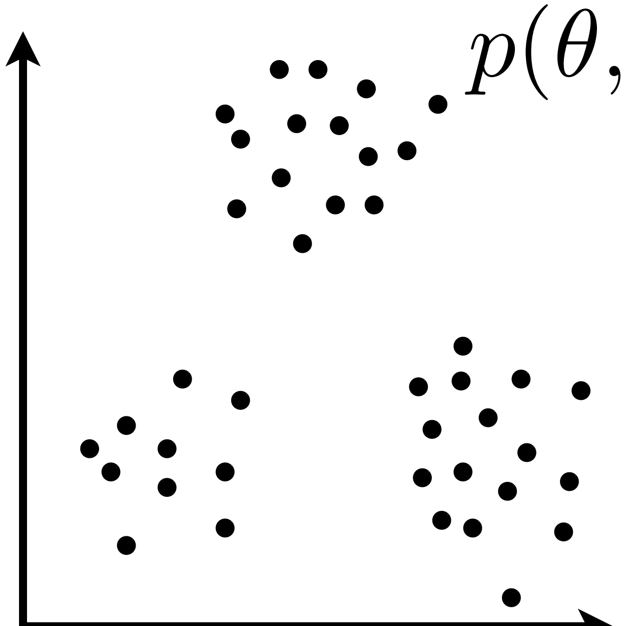

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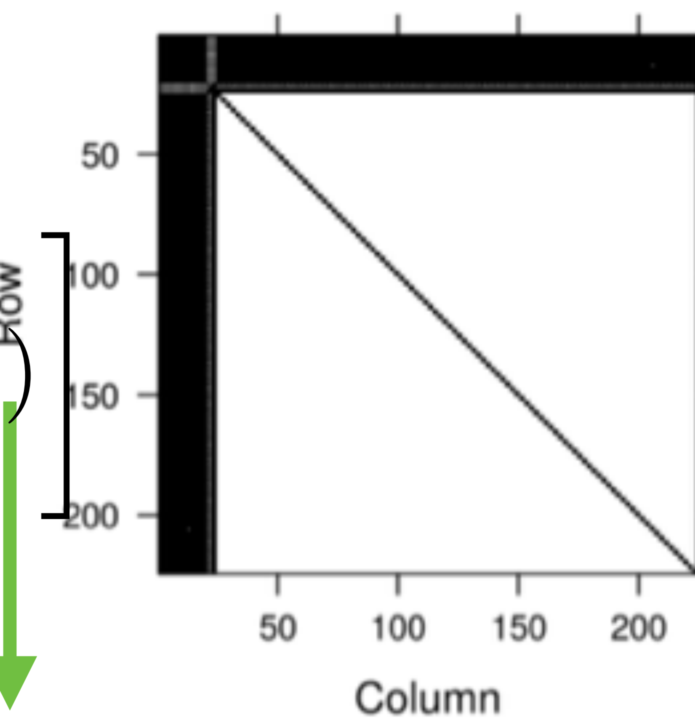

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Roadmap

- Challenges of VB
- Accurate uncertainties from VB
- Accurate robustness quantification from VB

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Robustness quantification

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$$p(\theta|y)$$

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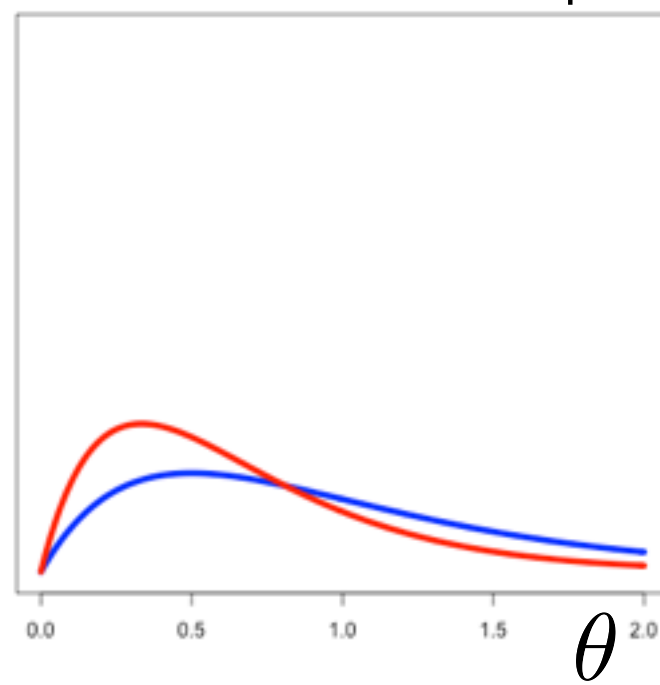

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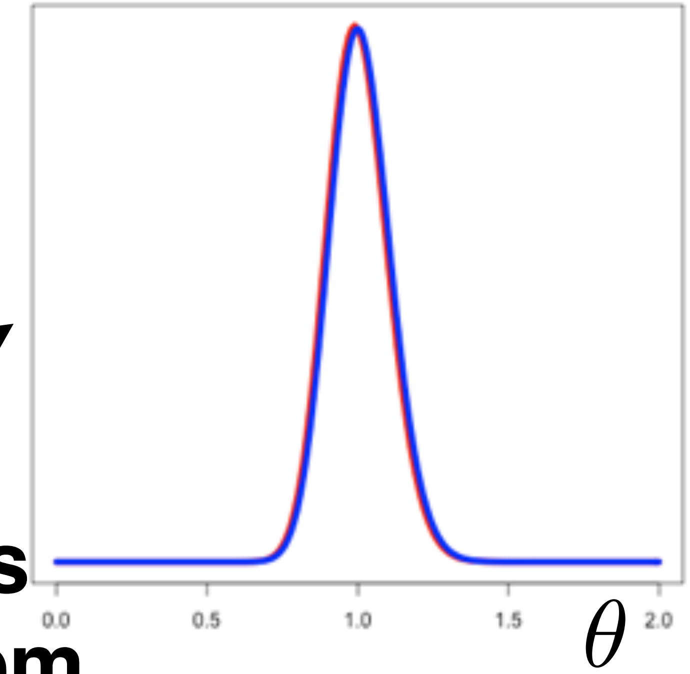

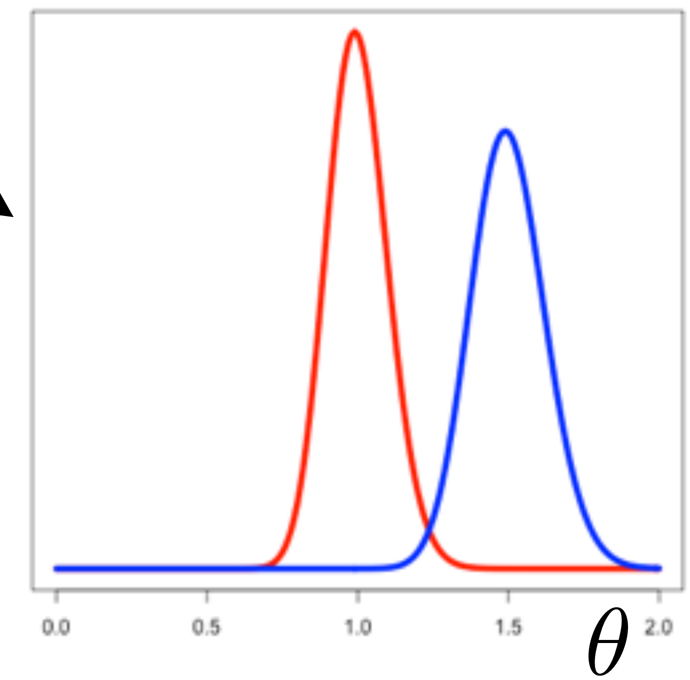

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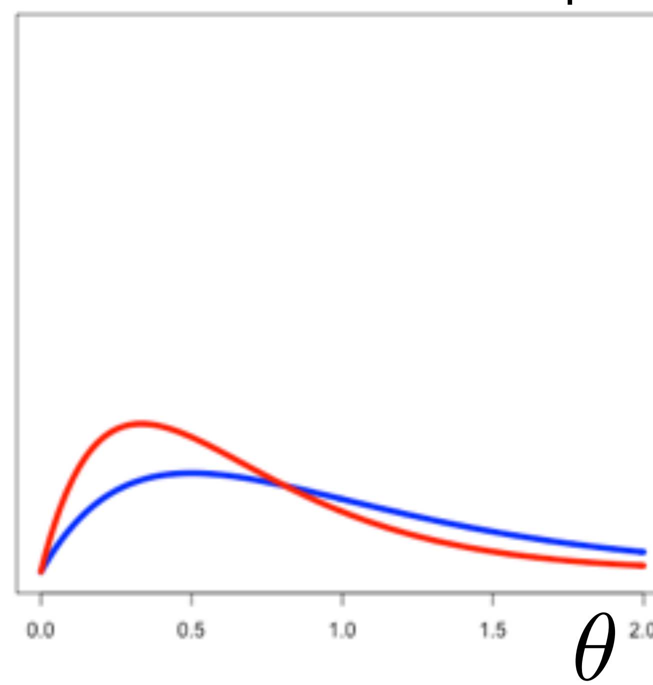

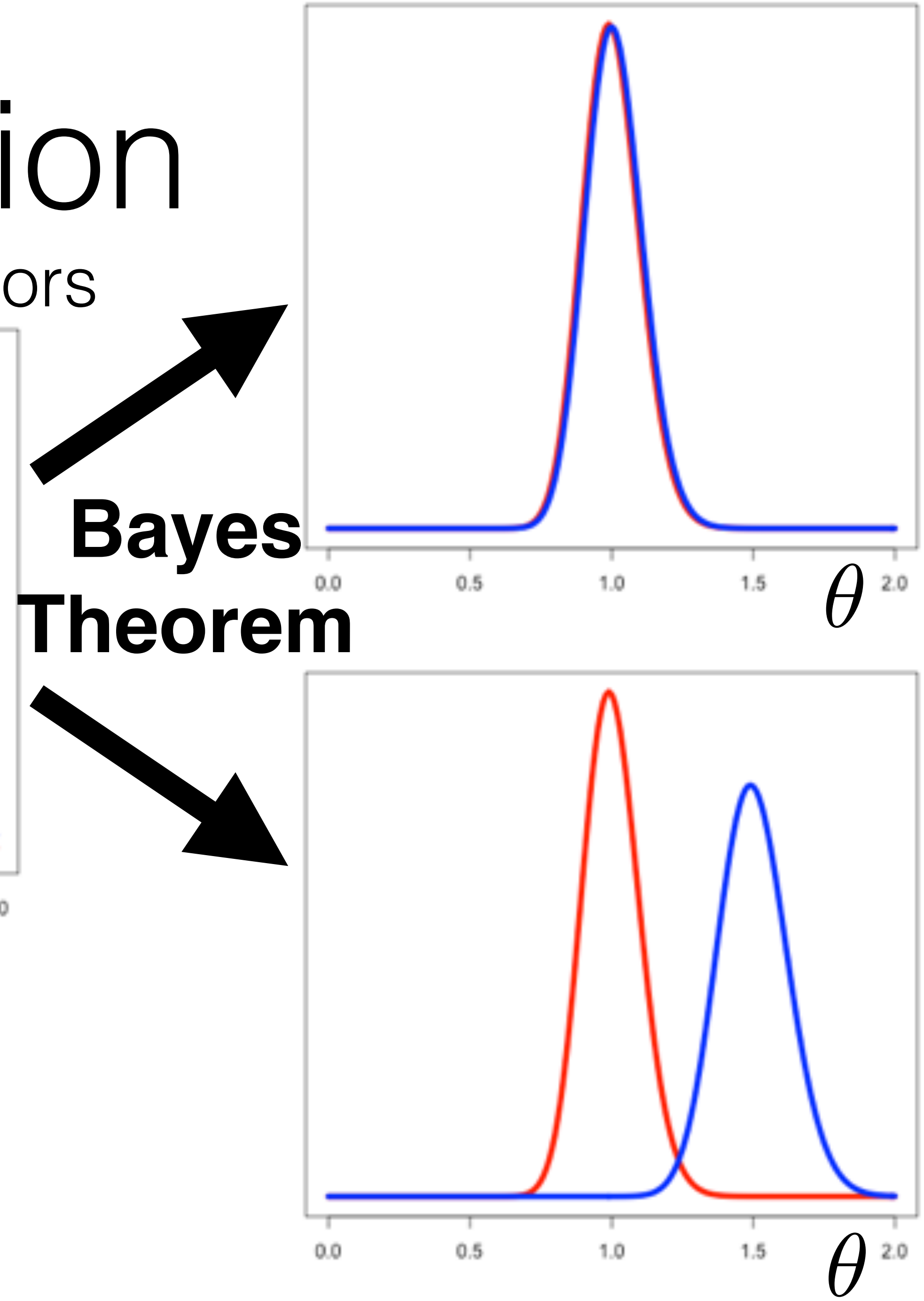

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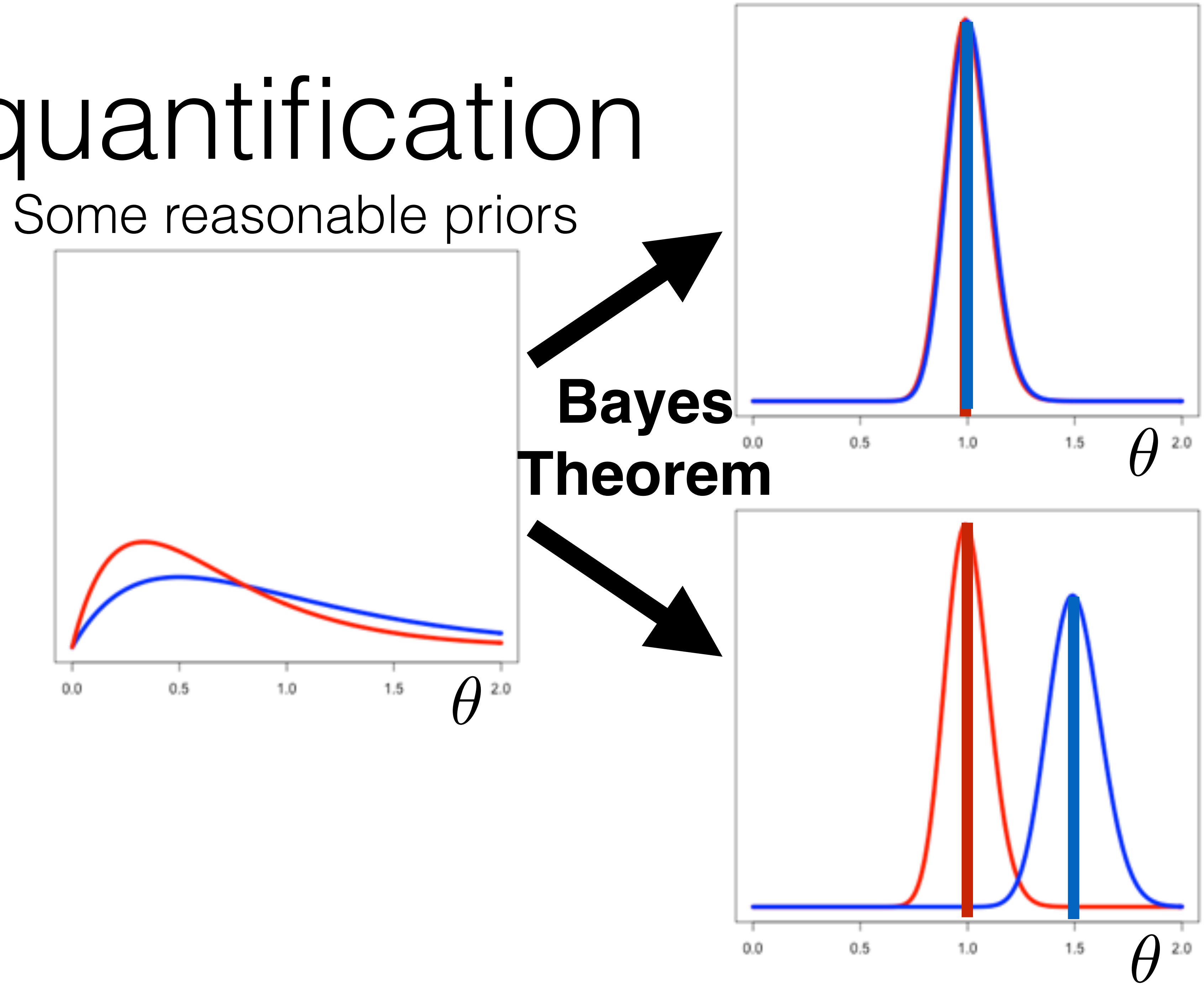

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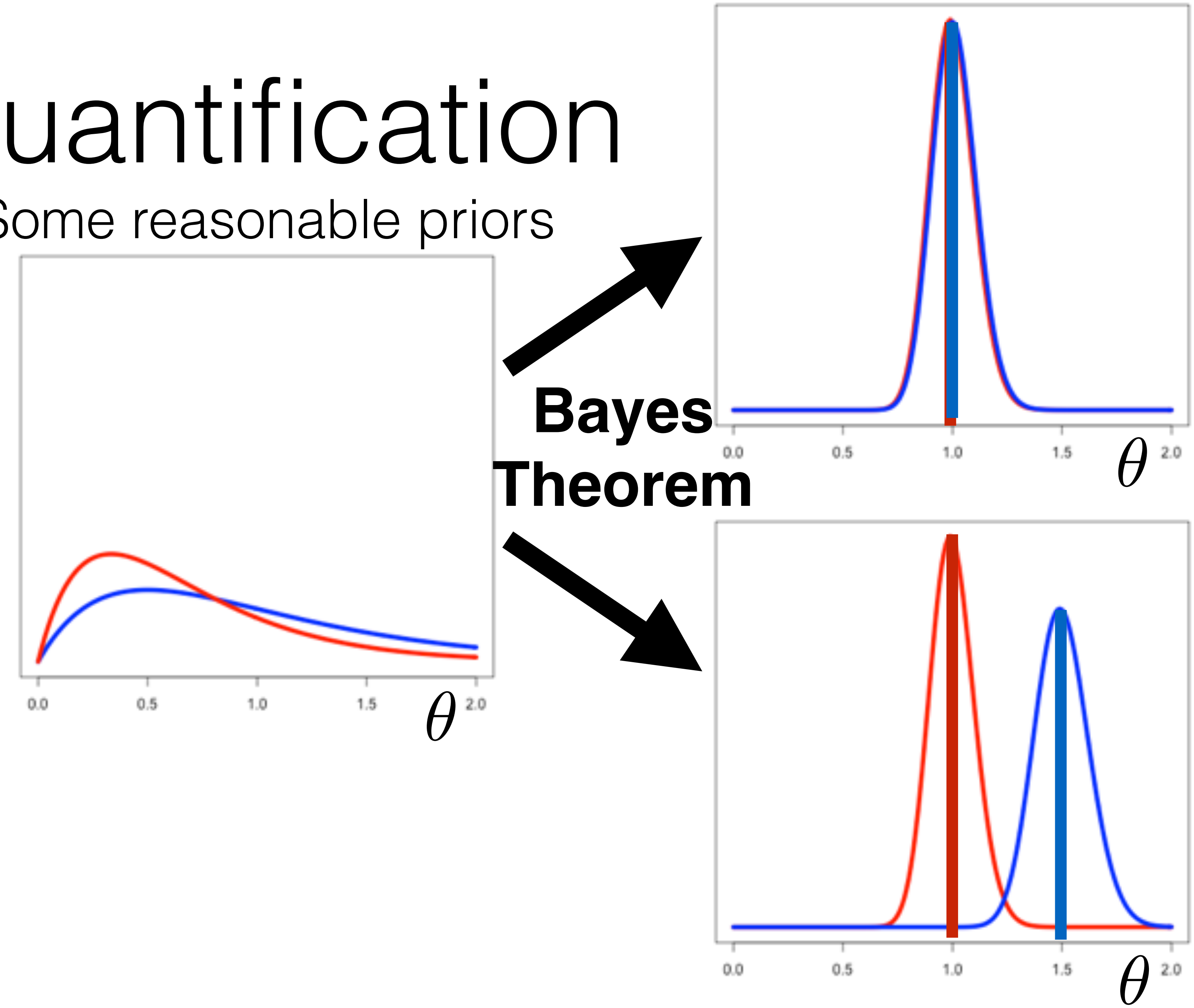

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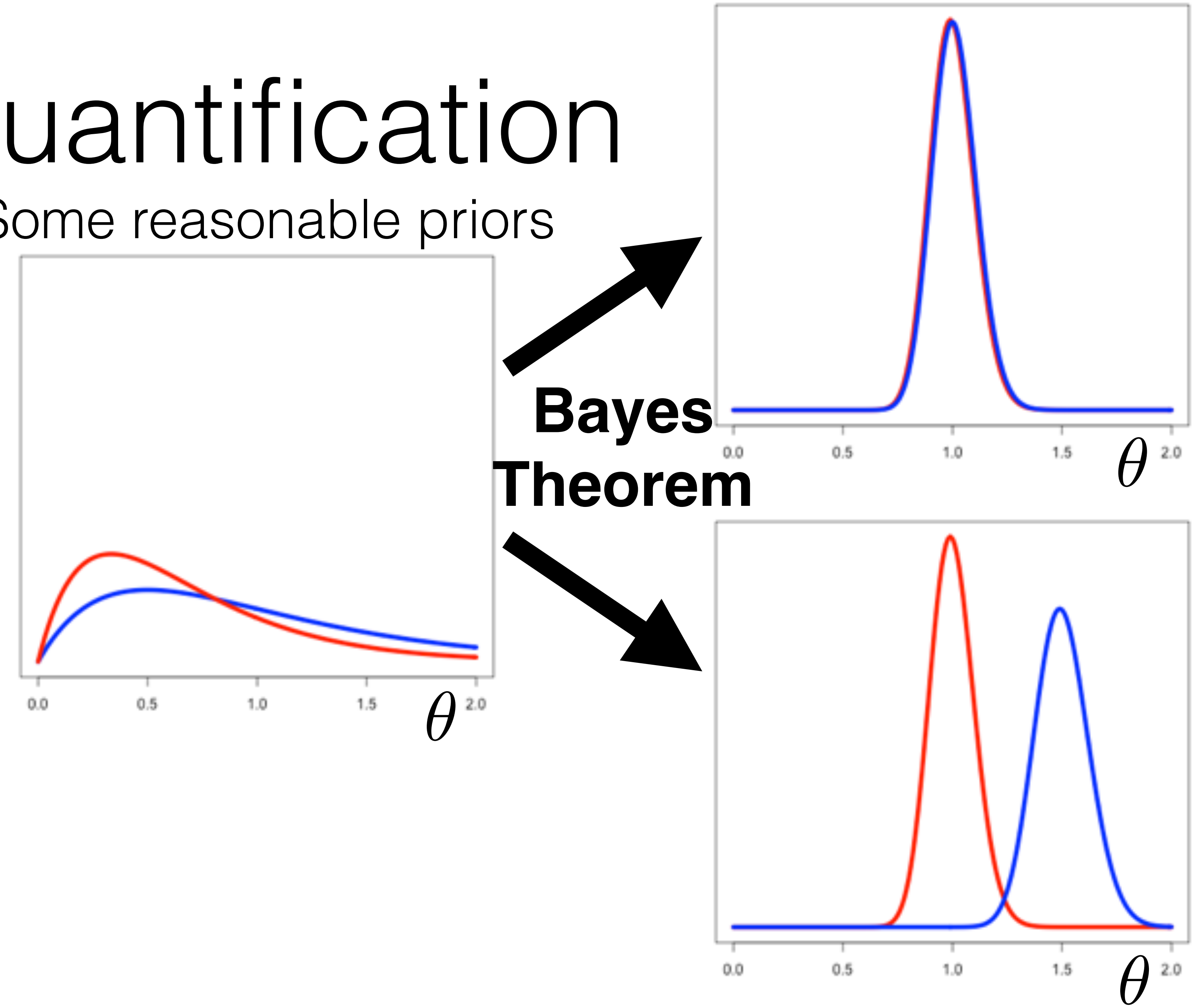

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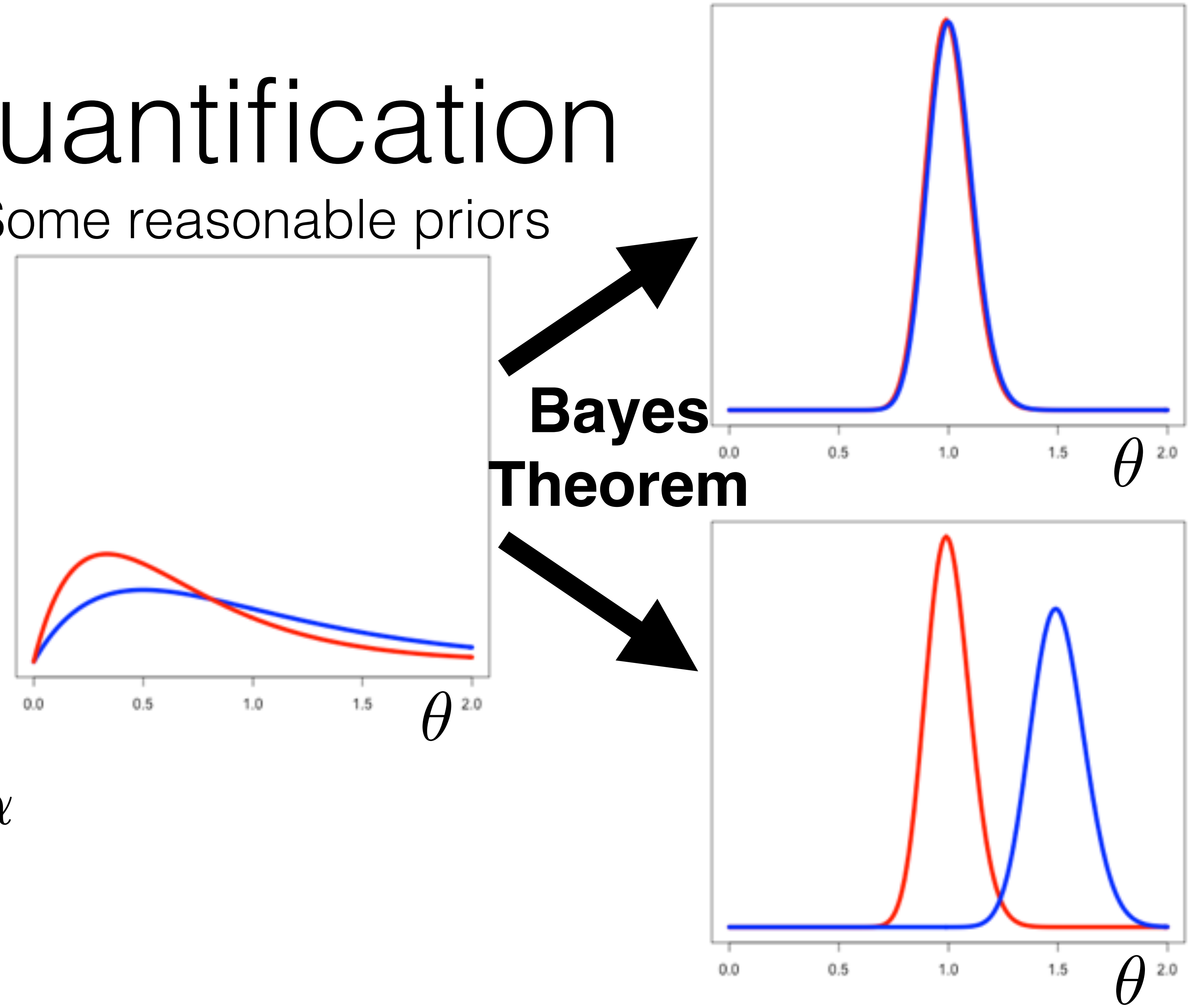

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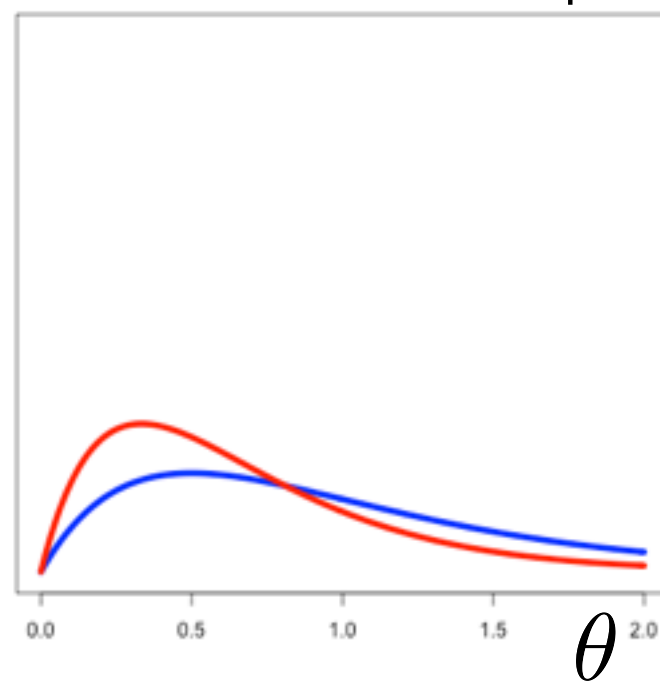

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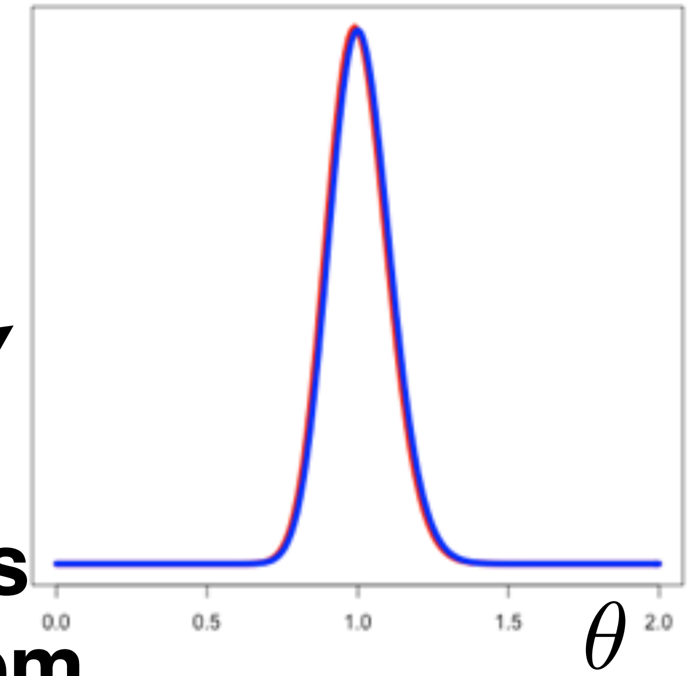

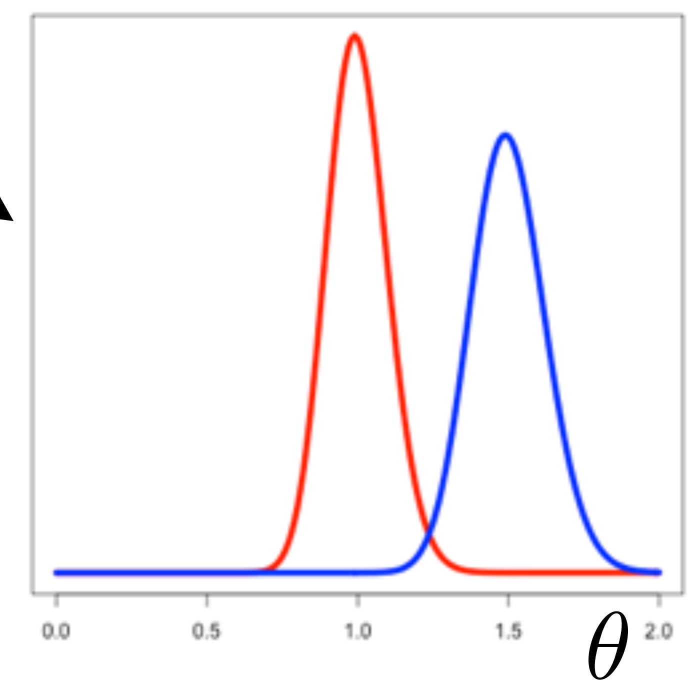

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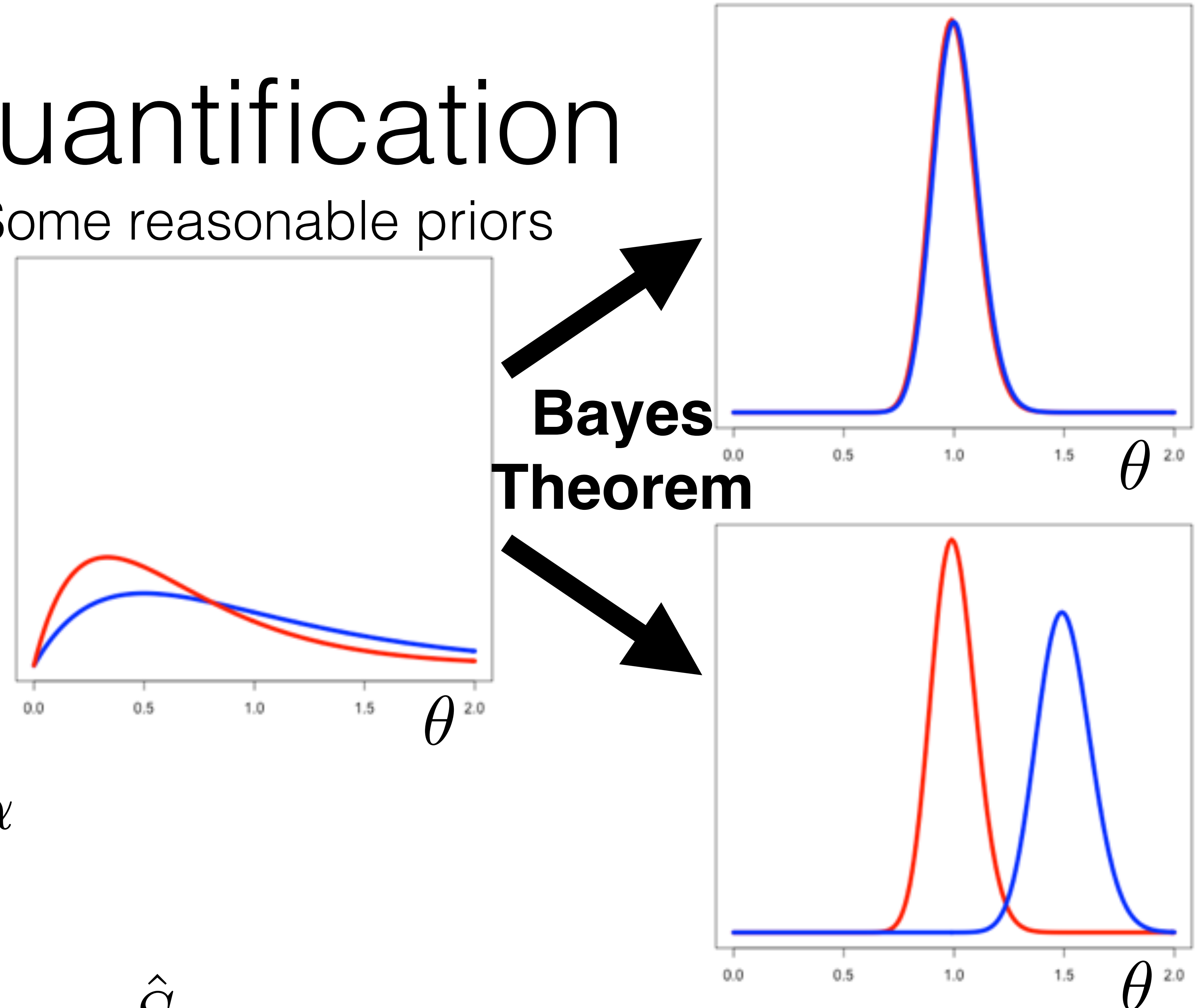

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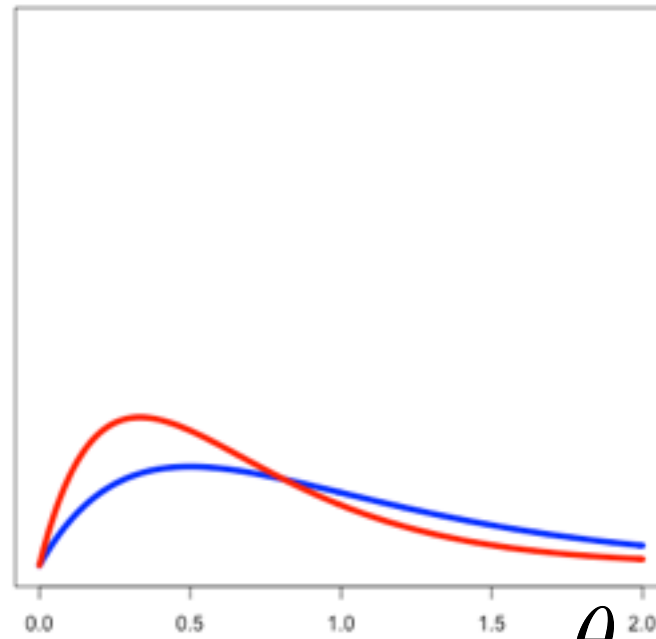

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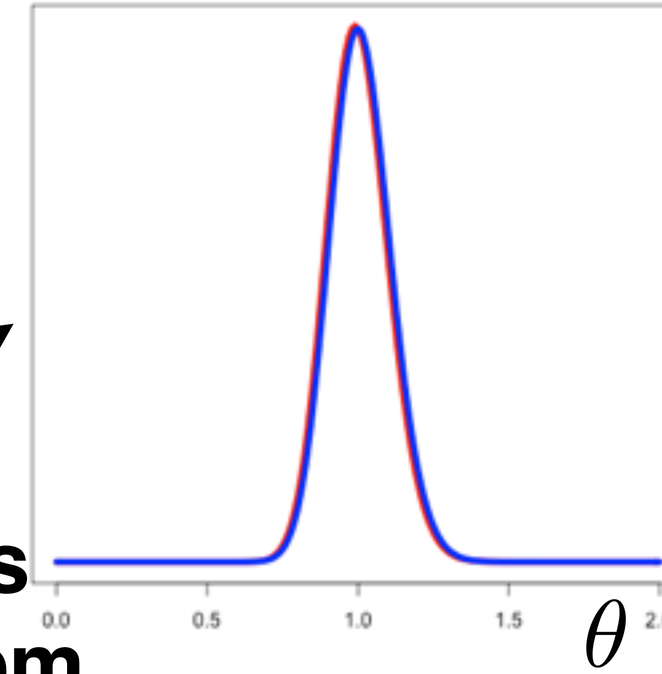

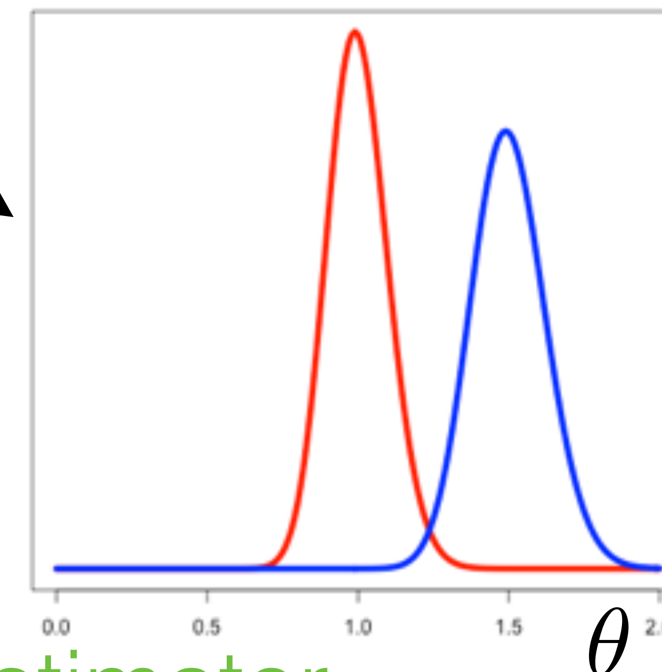

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LRVB estimator

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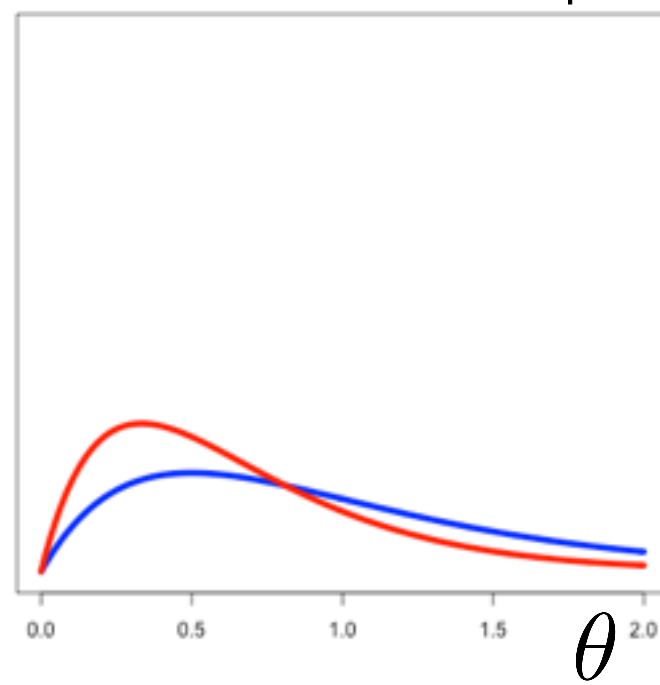

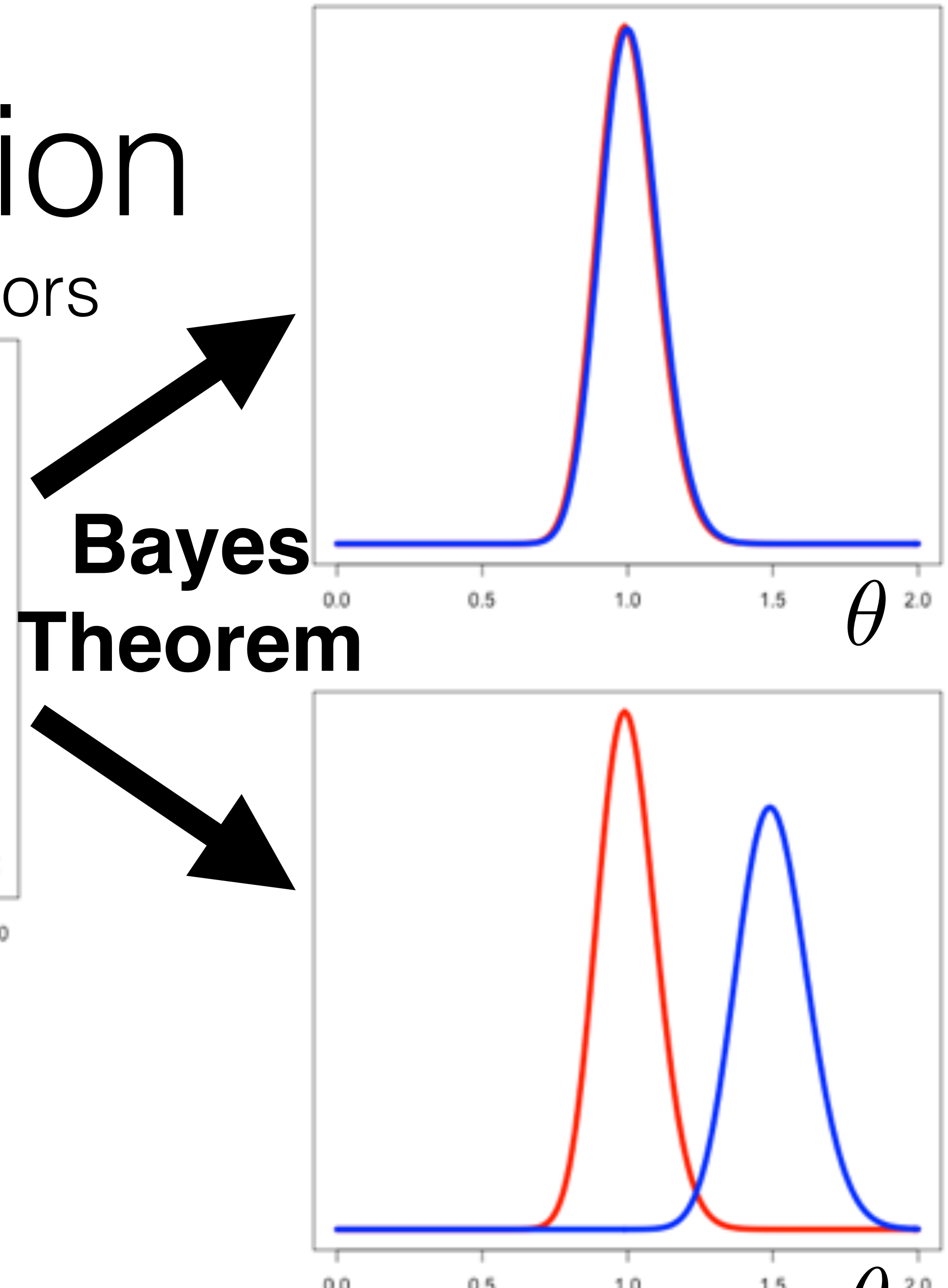

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- Recall: our general LRVB formula applies for:

$$\log p_t(\theta) = \log p(\theta|y) + f(\theta,t) - \mathrm{Const}(t)$$

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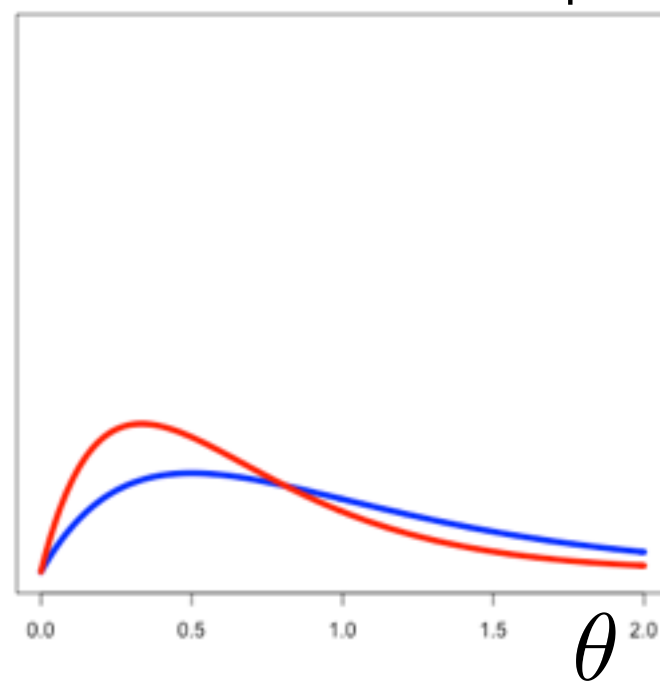

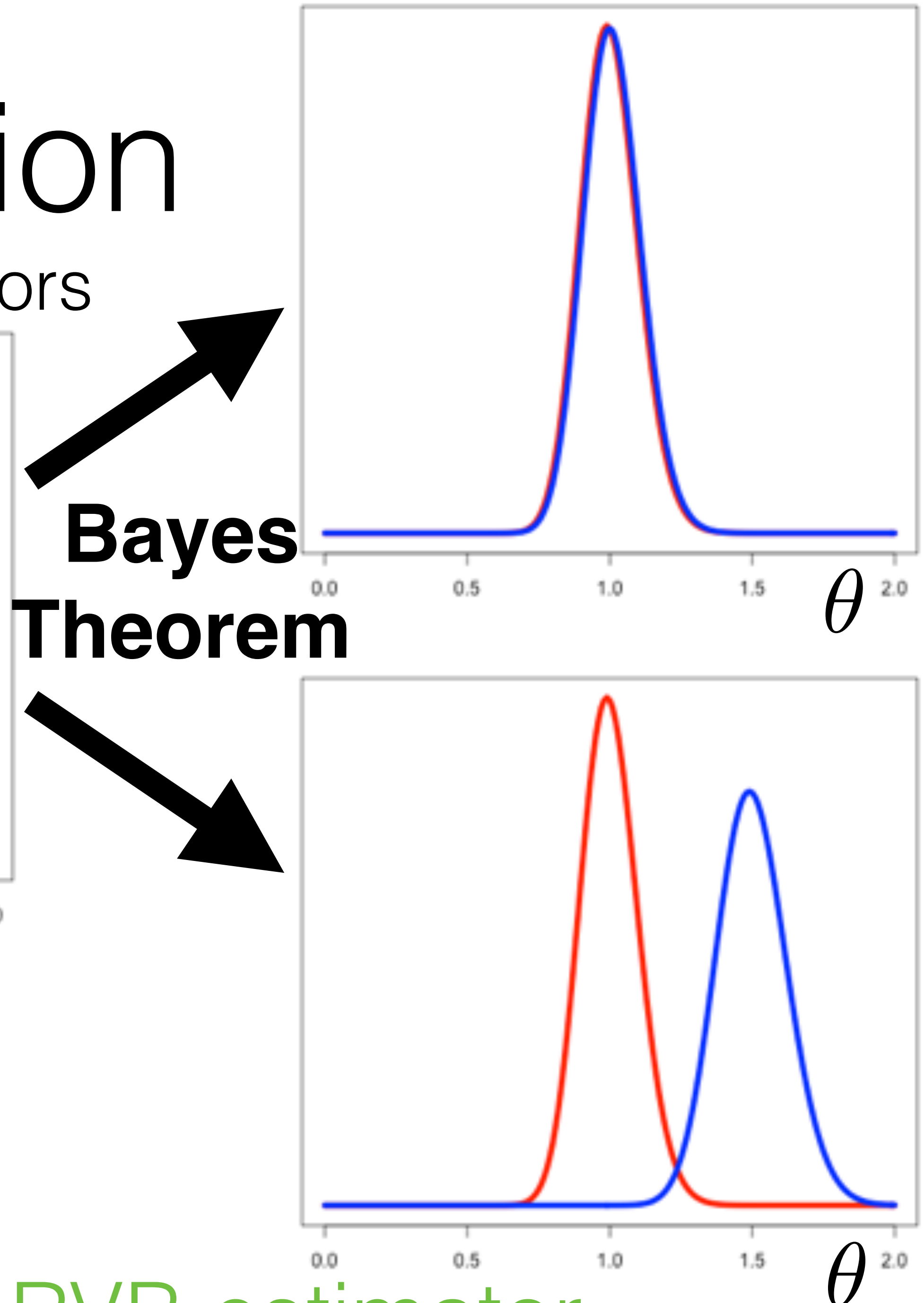

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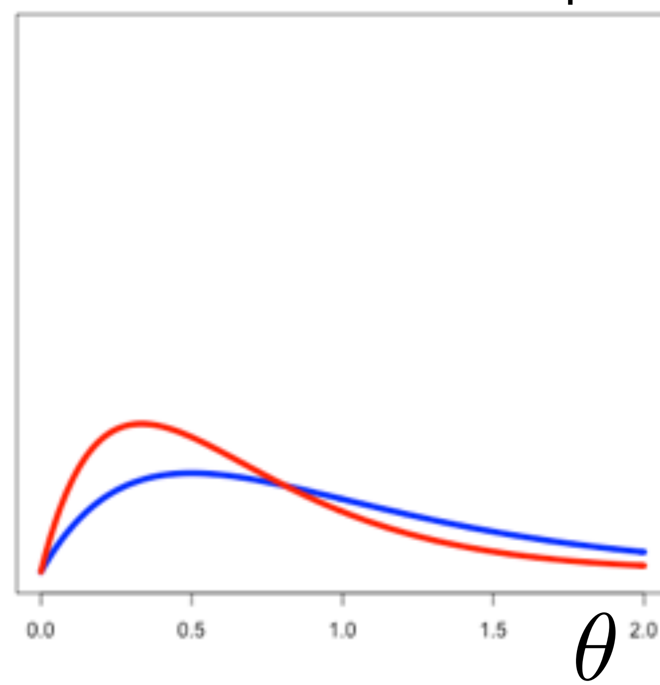

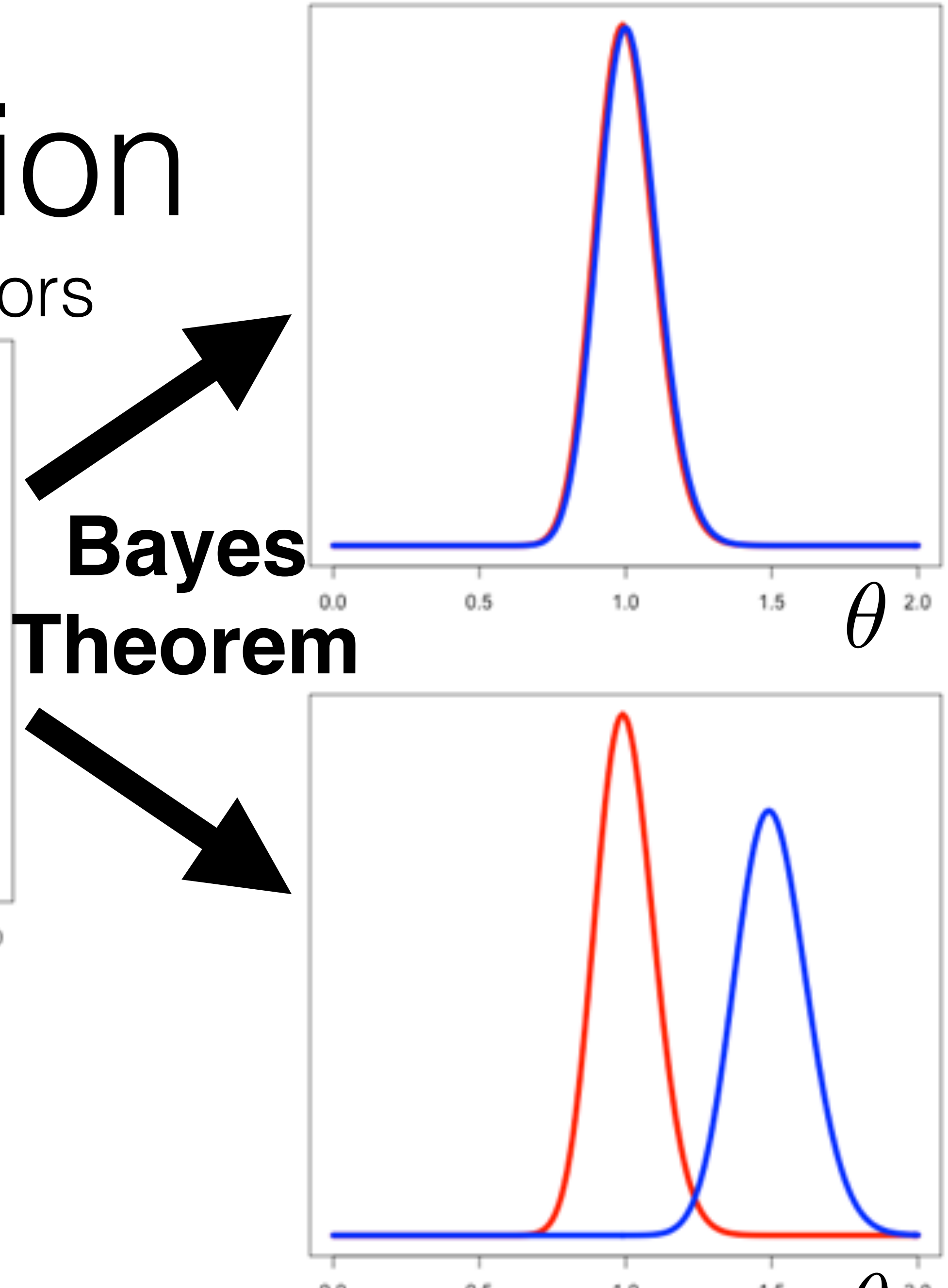

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Microcredit Experiment

- Simplified from Meager (2015)
- *K* microcredit trials (Mexico, Mongolia, Bosnia, India, Morocco, Philippines, Ethiopia)
- N_k businesses in *k*th site (~900 to ~17K)
- Profit of *n*th business at *k*th site:

1 if microcredit

profit

$$y_{kn} \overset{indep}{\sim} \mathcal{N}(\mu_k + T_{kn}\tau_k, \sigma_k^2)$$

- Priors and hyperpriors:

$$\begin{pmatrix} \mu_k \\ \tau_k \end{pmatrix} \overset{iid}{\sim} \mathcal{N}\left(\begin{pmatrix} \mu \\ \tau \end{pmatrix}, C\right) \qquad \begin{pmatrix} \mu \\ \tau \end{pmatrix} \overset{iid}{\sim} \mathcal{N}\left(\begin{pmatrix} \mu_0 \\ \tau_0 \end{pmatrix}, \Lambda^{-1}\right)$$

$$\sigma_k^{-2} \overset{iid}{\sim} \Gamma(a, b) \qquad C \sim \mathrm{Sep\&LKJ}(\eta, c, d)$$

Microcredit Experiment

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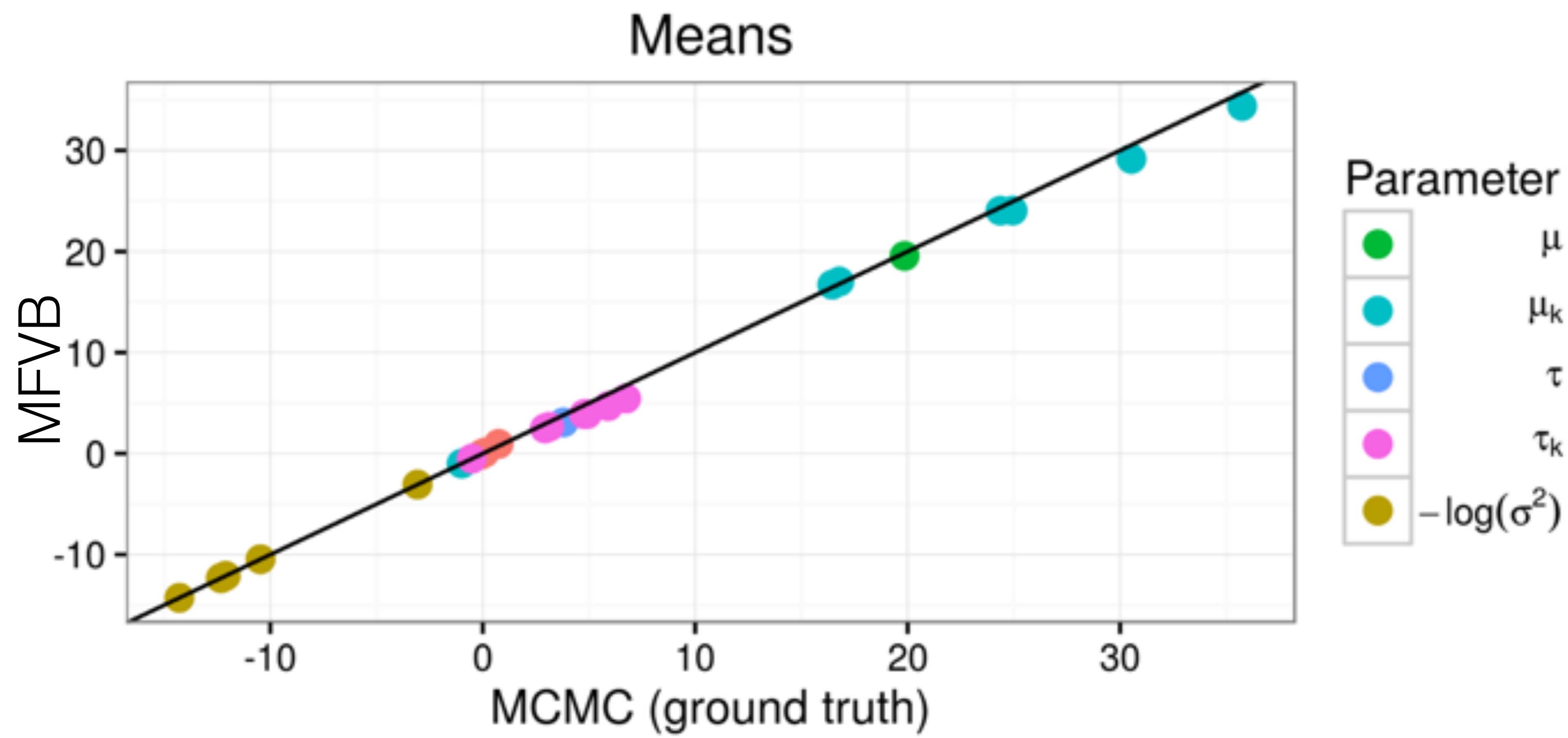
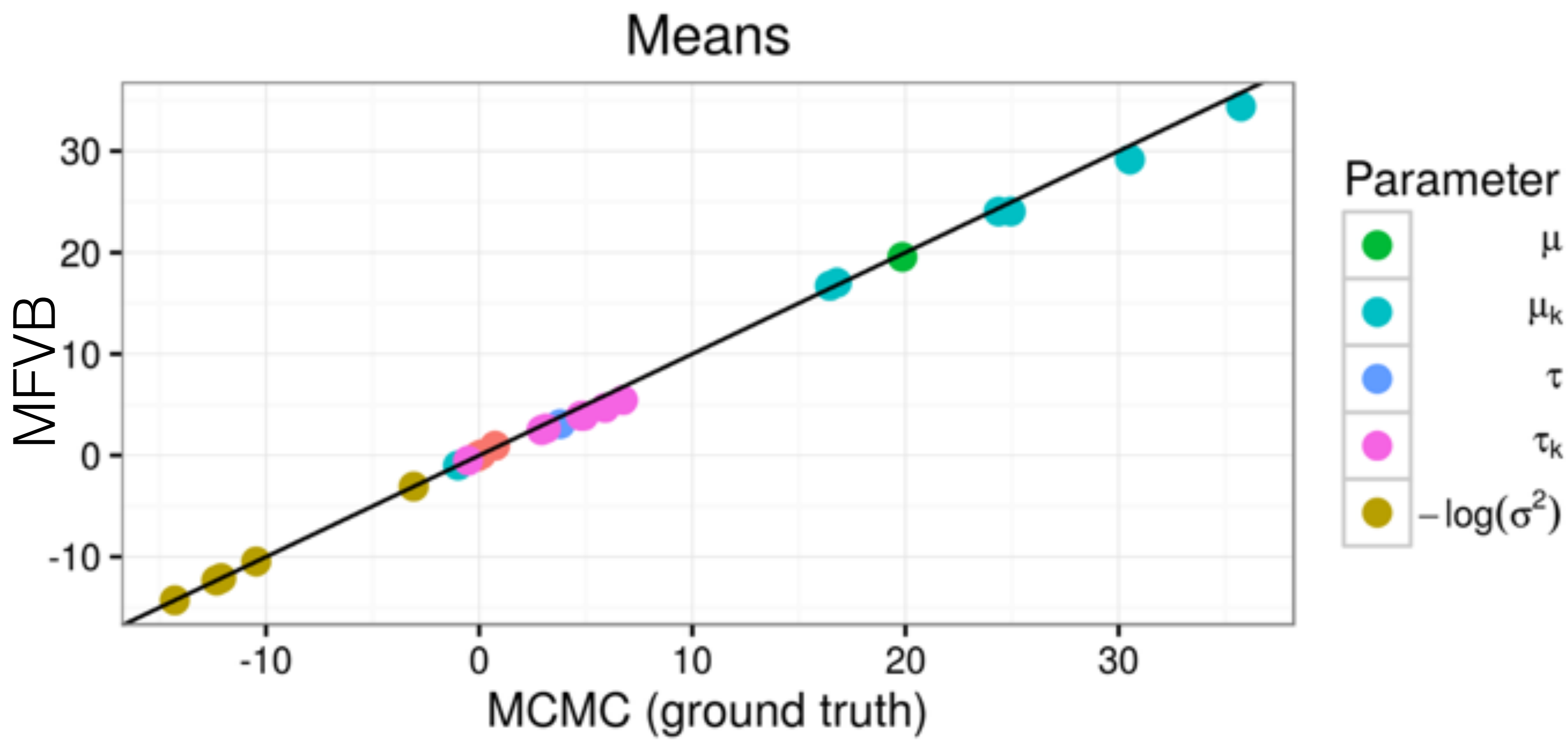


Microcredit Experiment

- Perturb Λ_{11}: 0.03 ➔ 0.04



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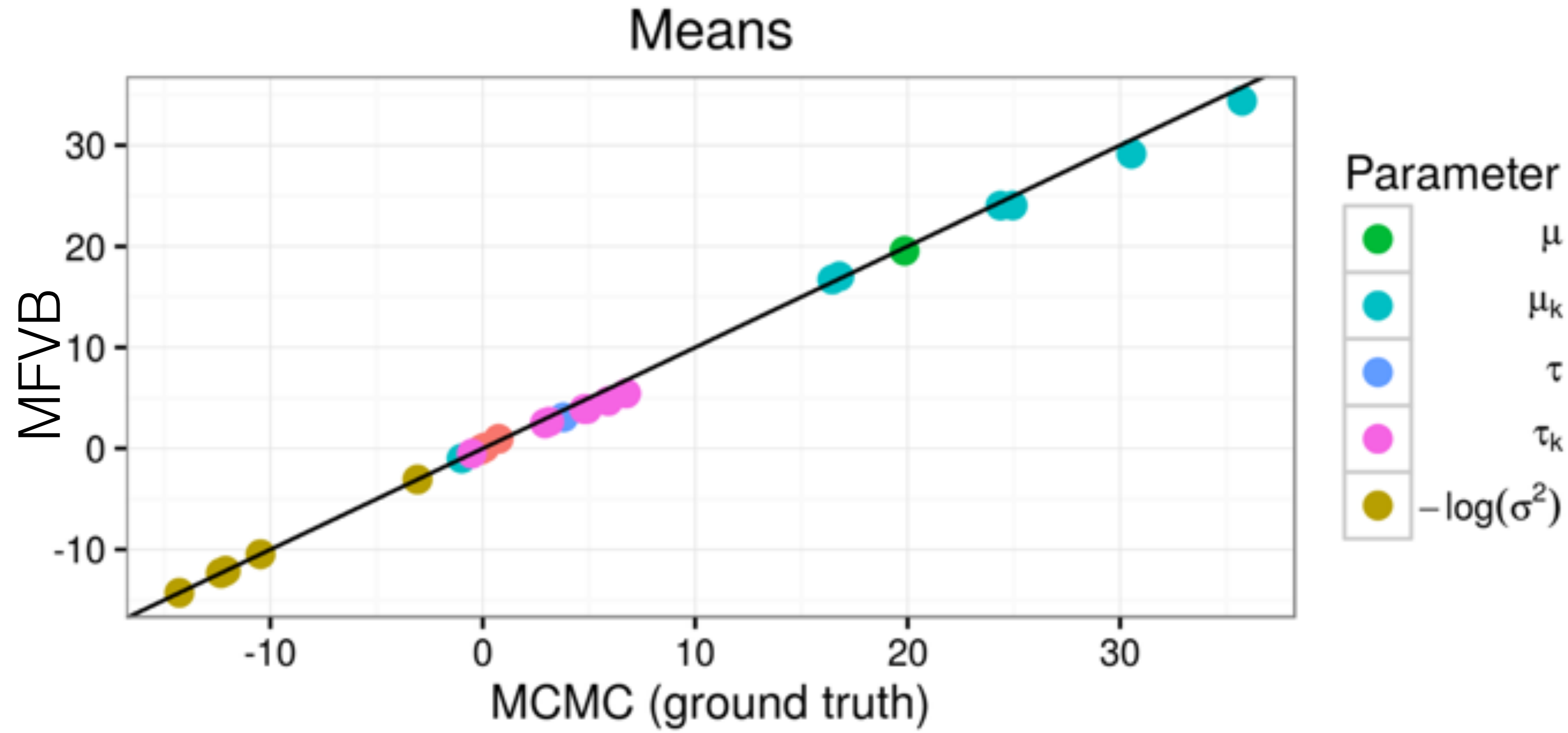

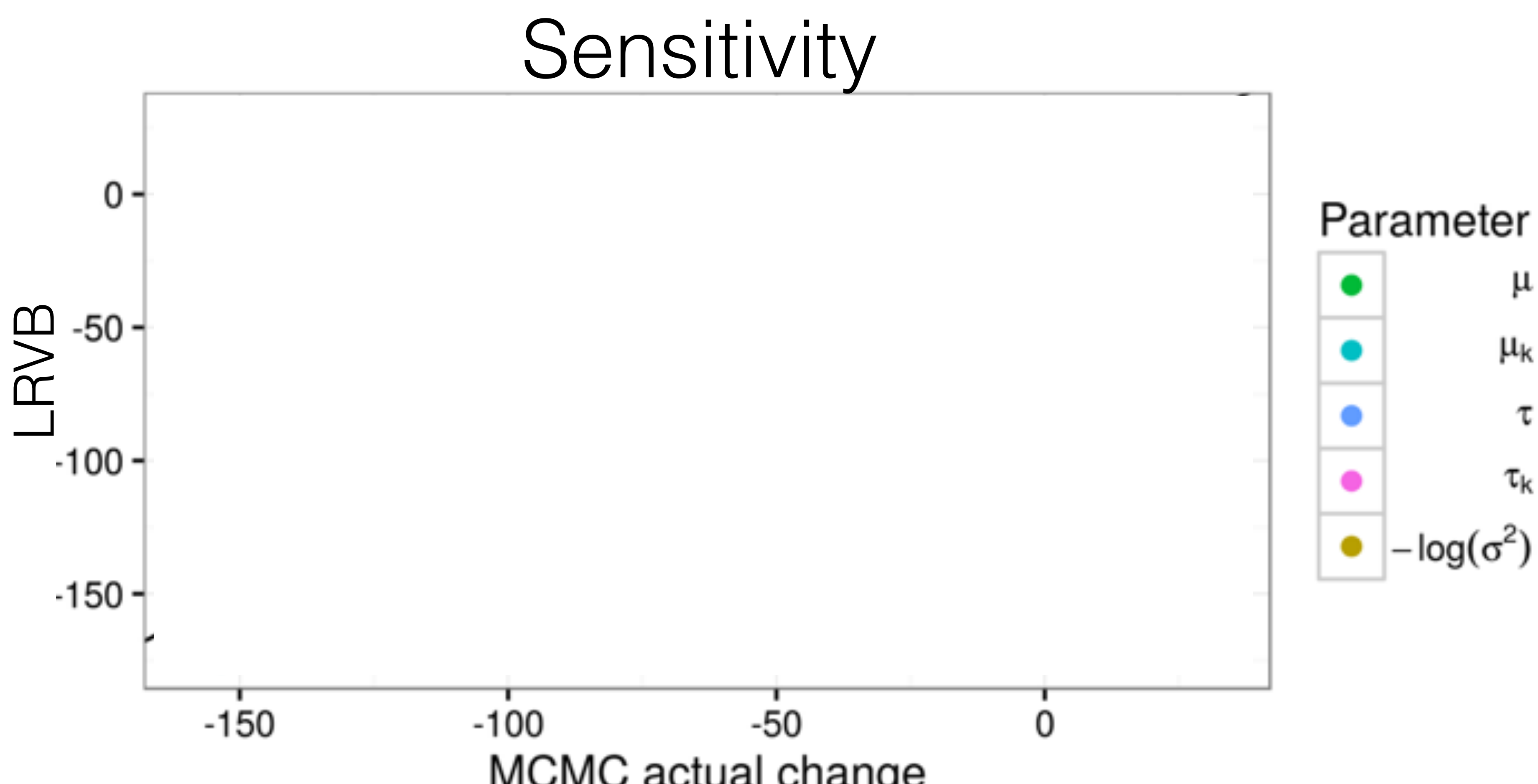

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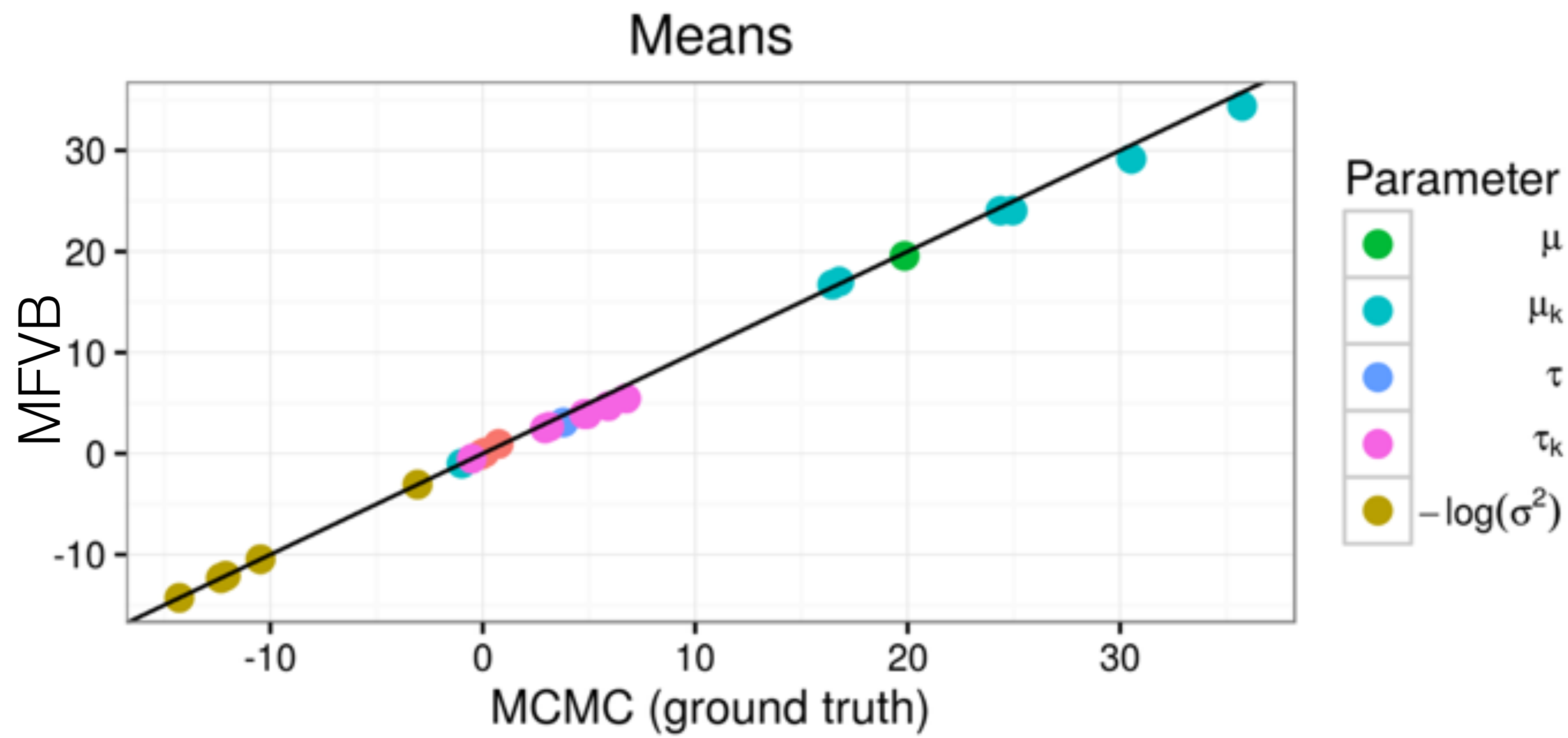

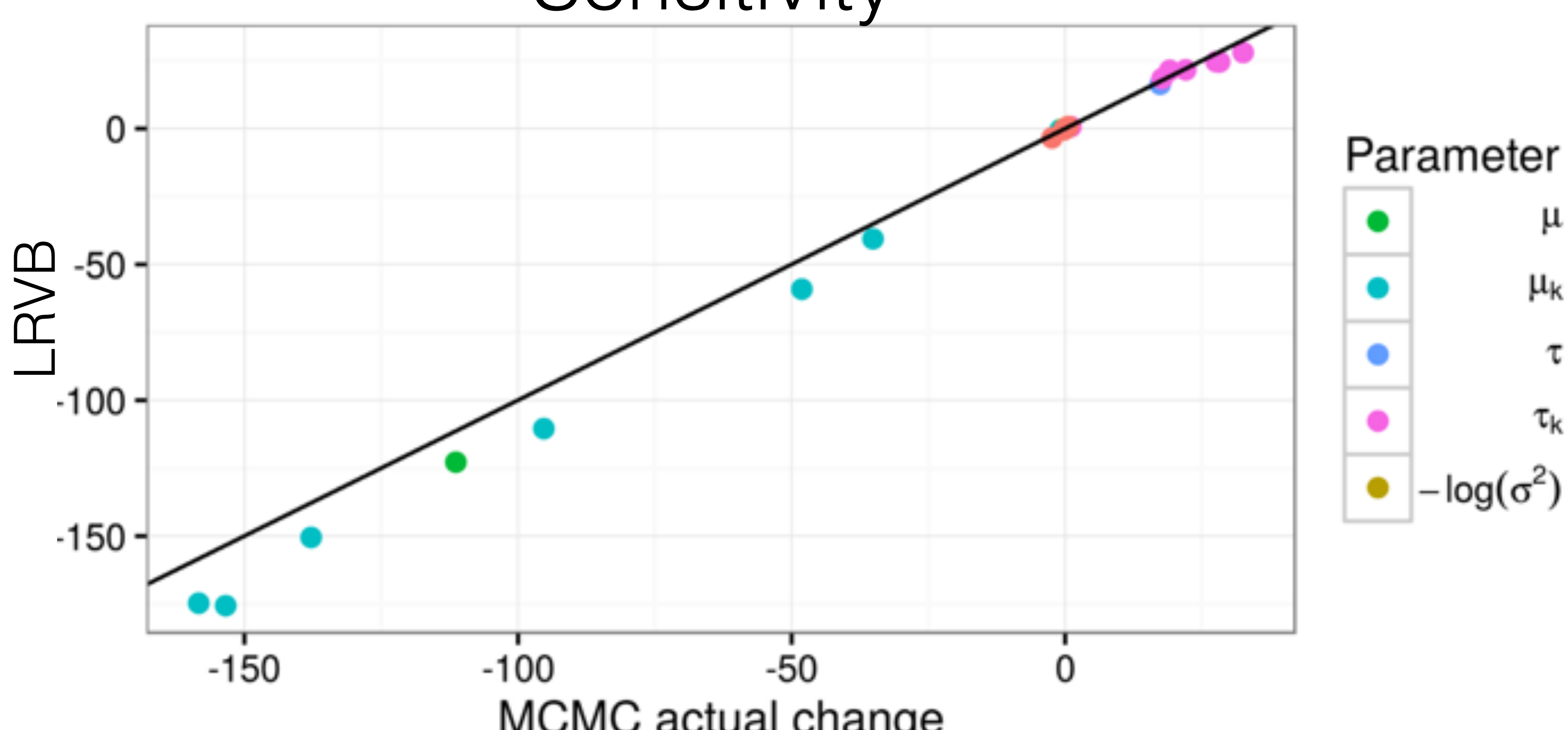

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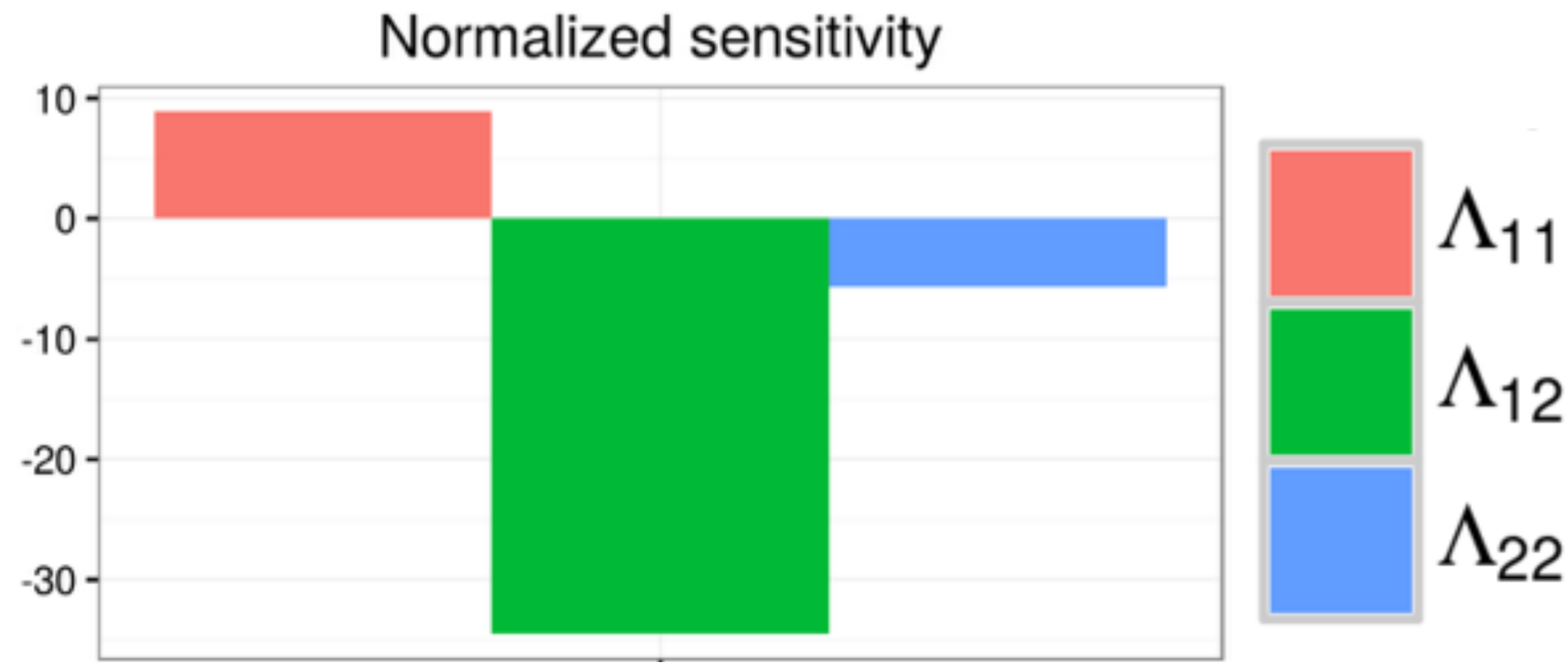

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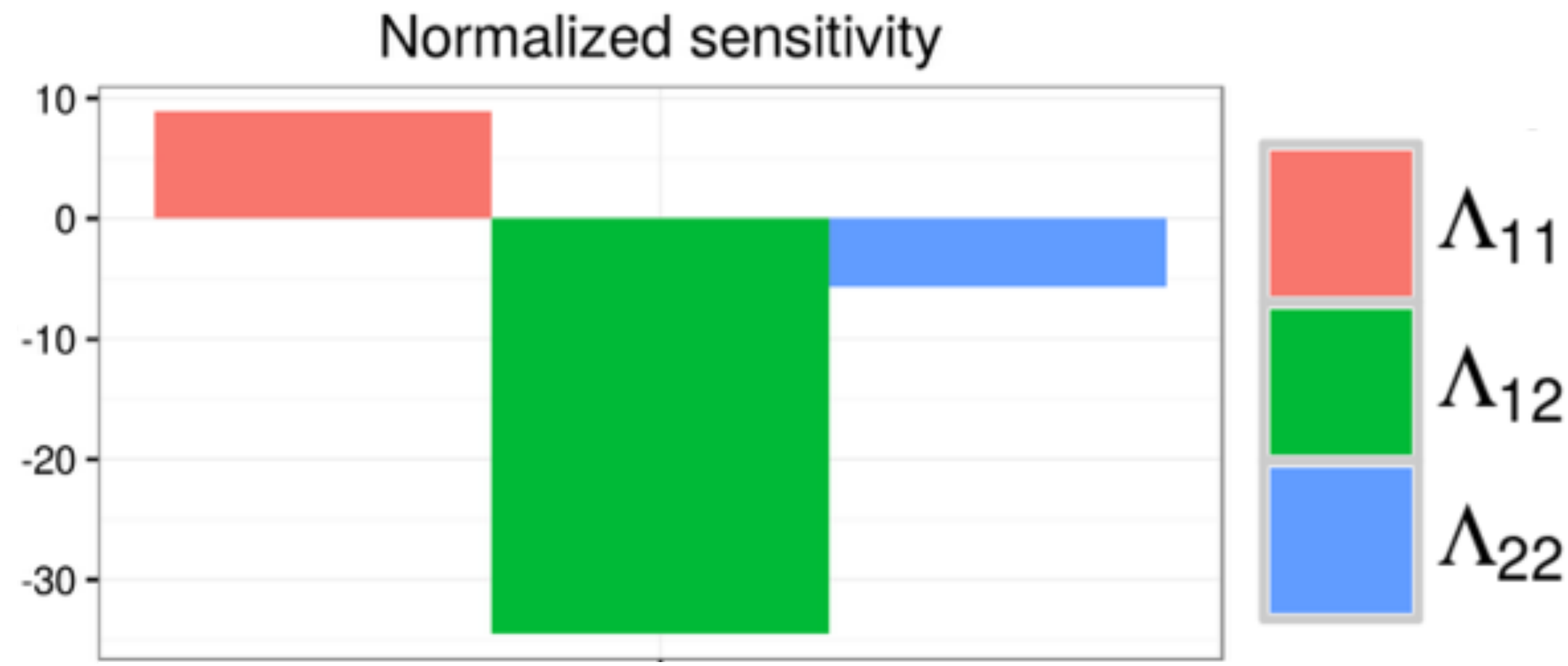

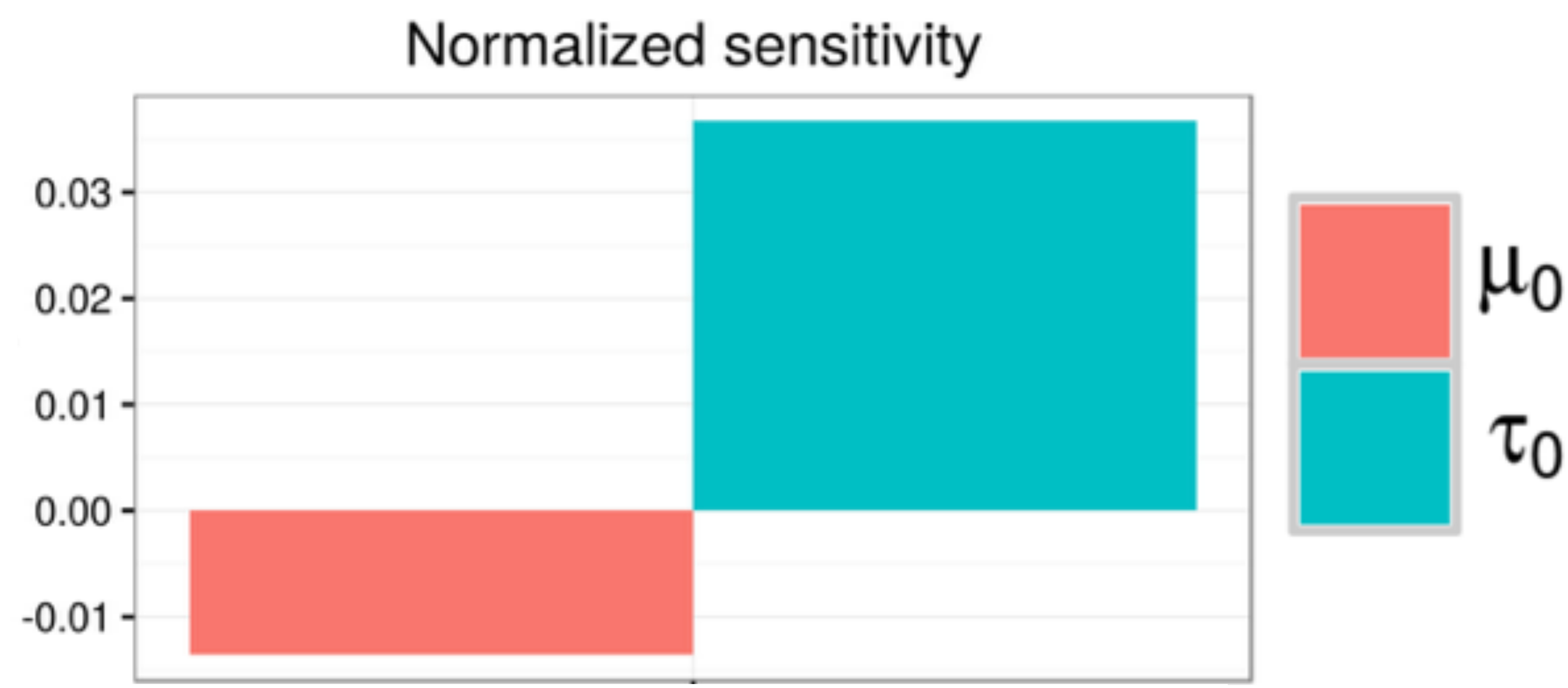

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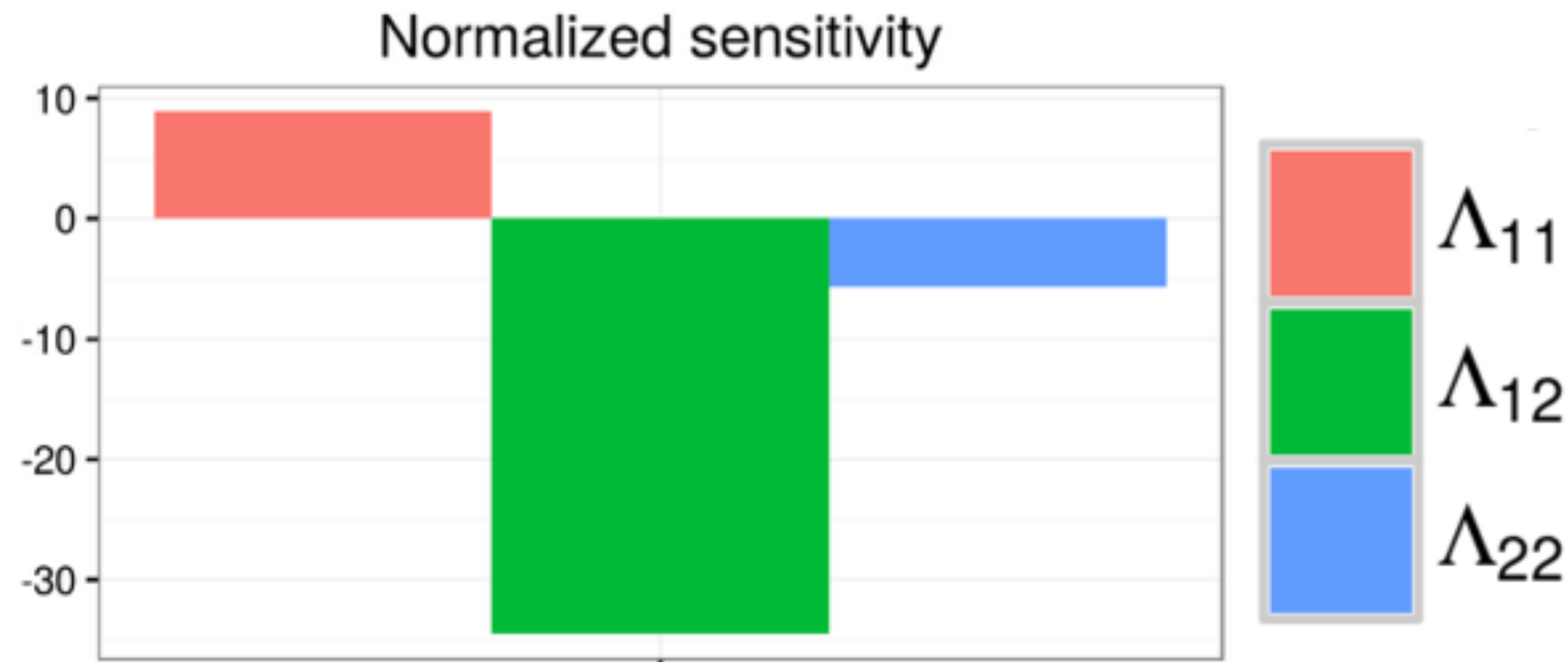

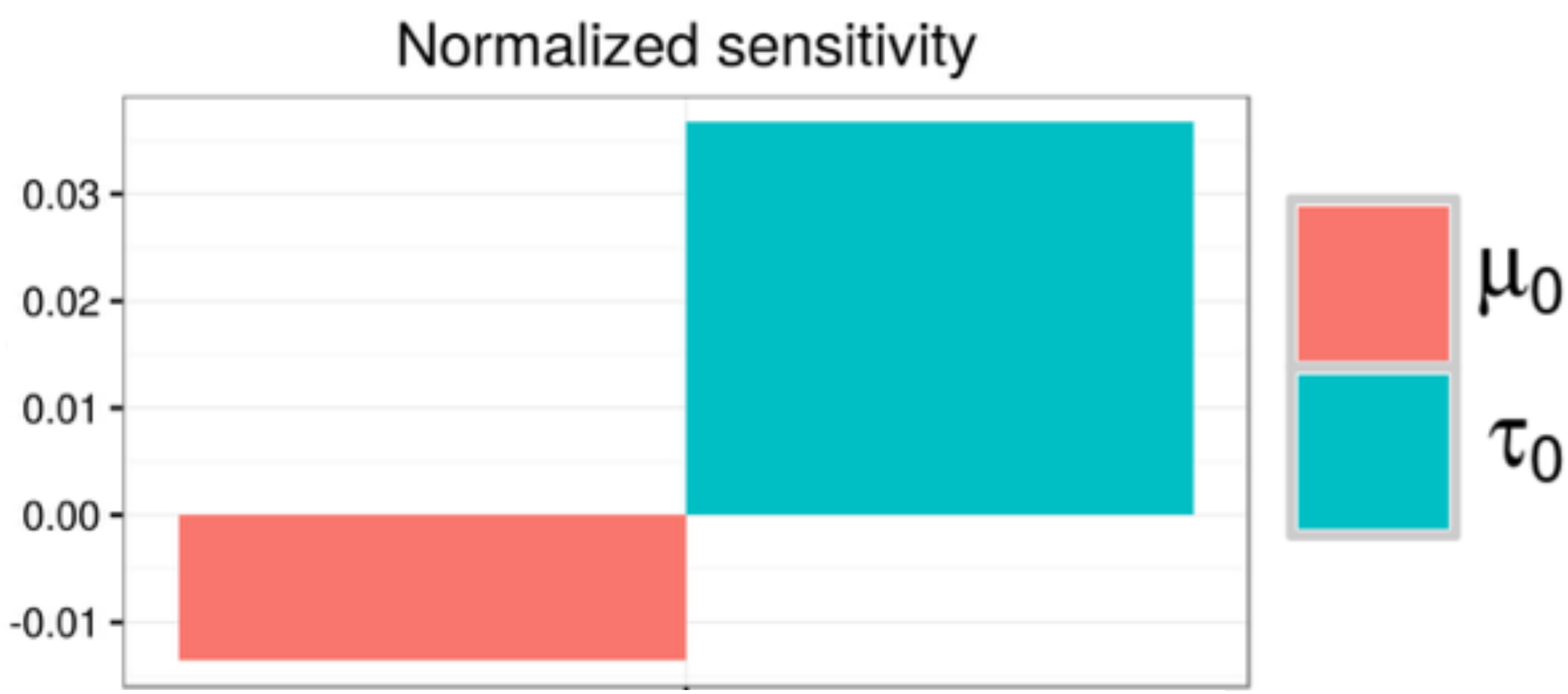

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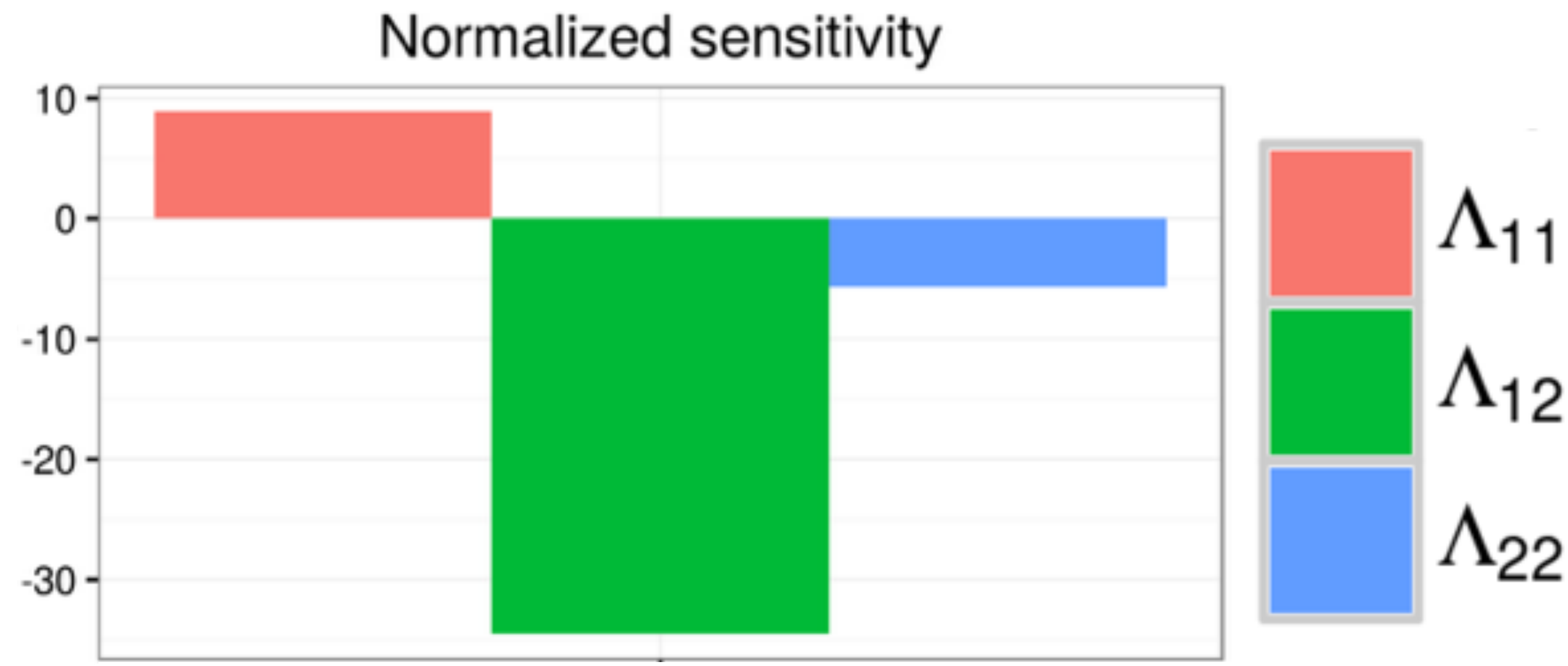

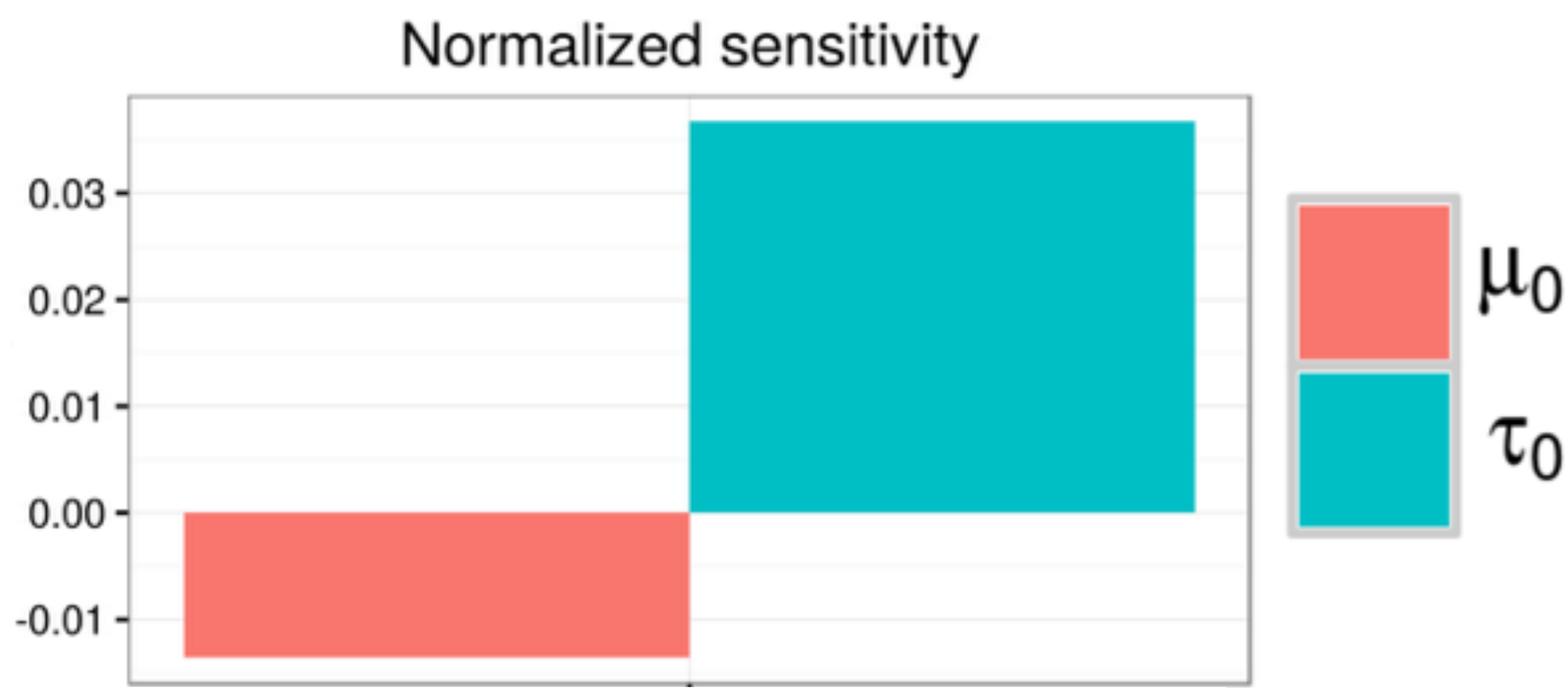

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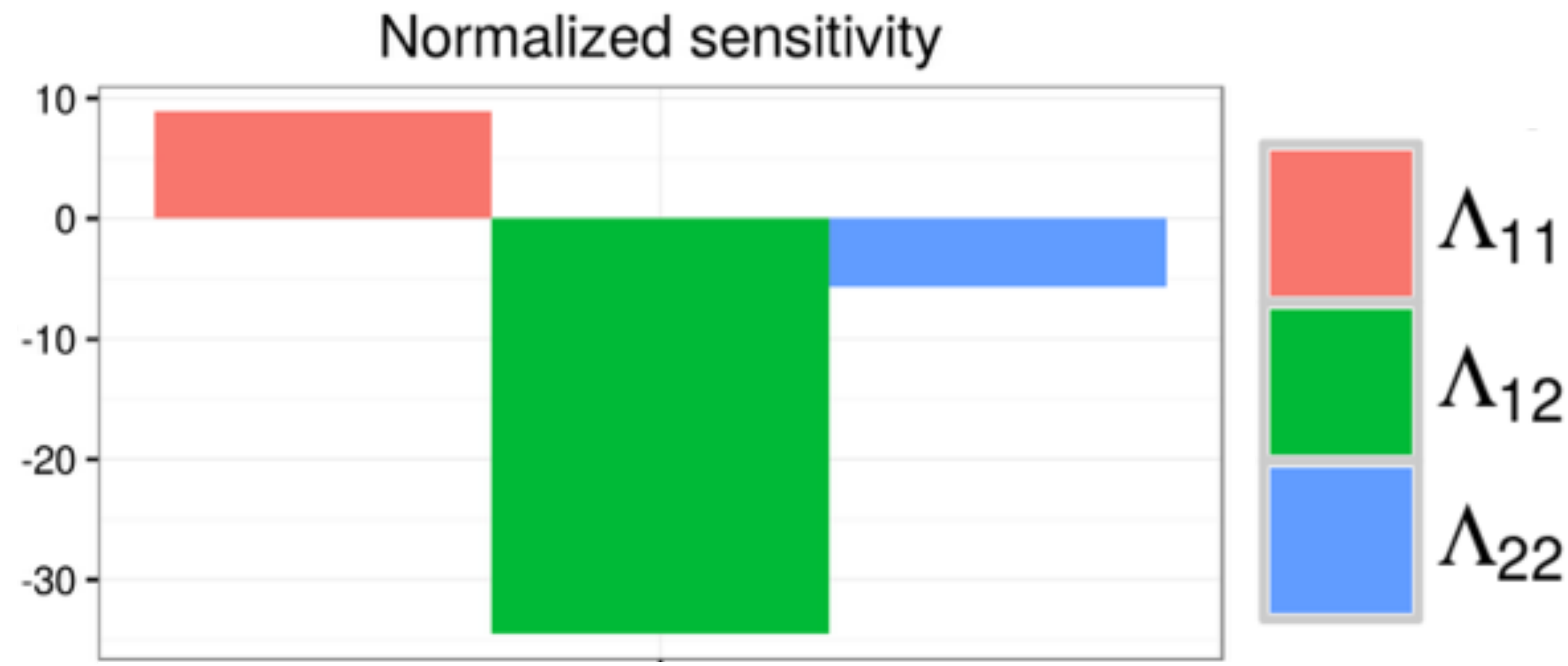
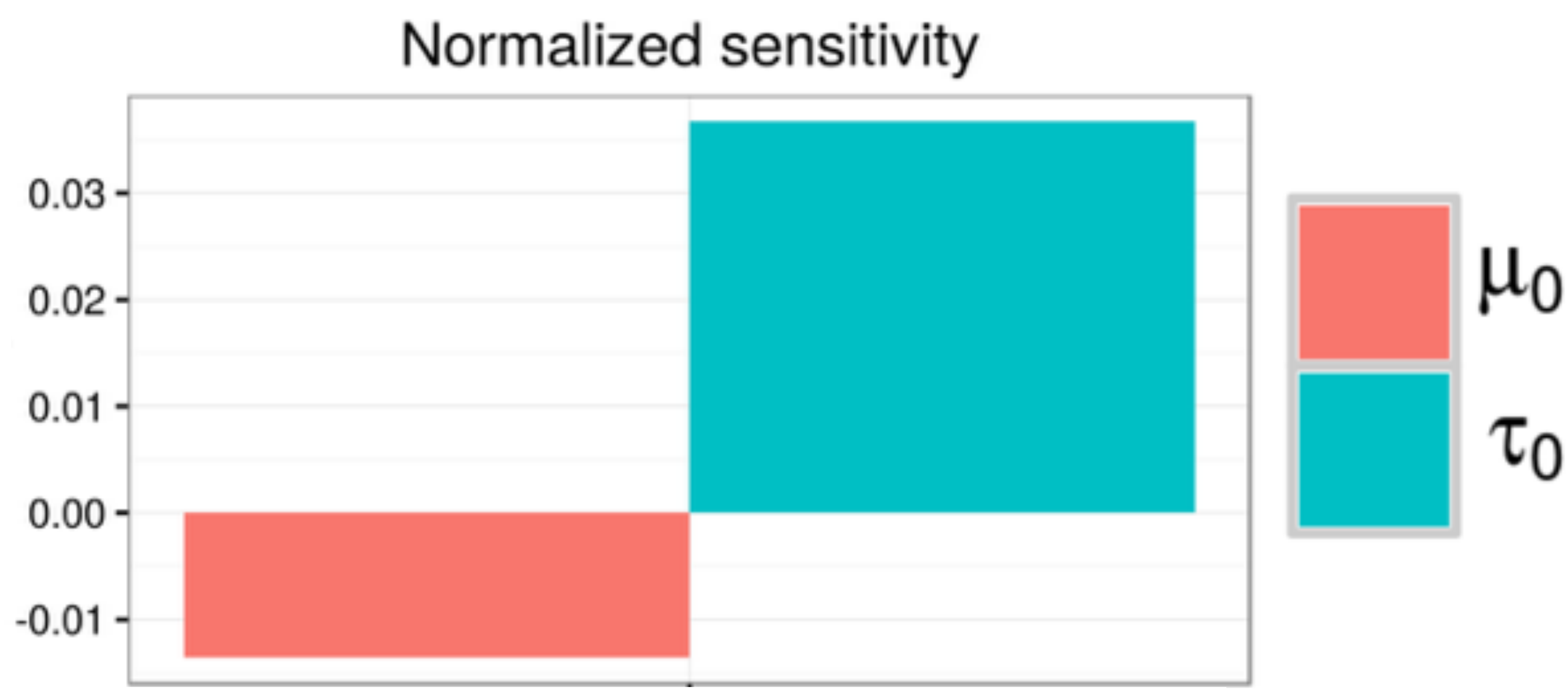
$\Rightarrow$ Mean > 2 std dev





Criteo Online Ads Experiment

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- Click-through conversion prediction

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- Q: Will a customer (e.g.) buy a product after clicking?
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Criteo Online Ads Experiment

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- Logistic GLMM; N = 61,895 subset to compare to MCMC

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Criteo Online Ads Experiment

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Criteo Online Ads Experiment

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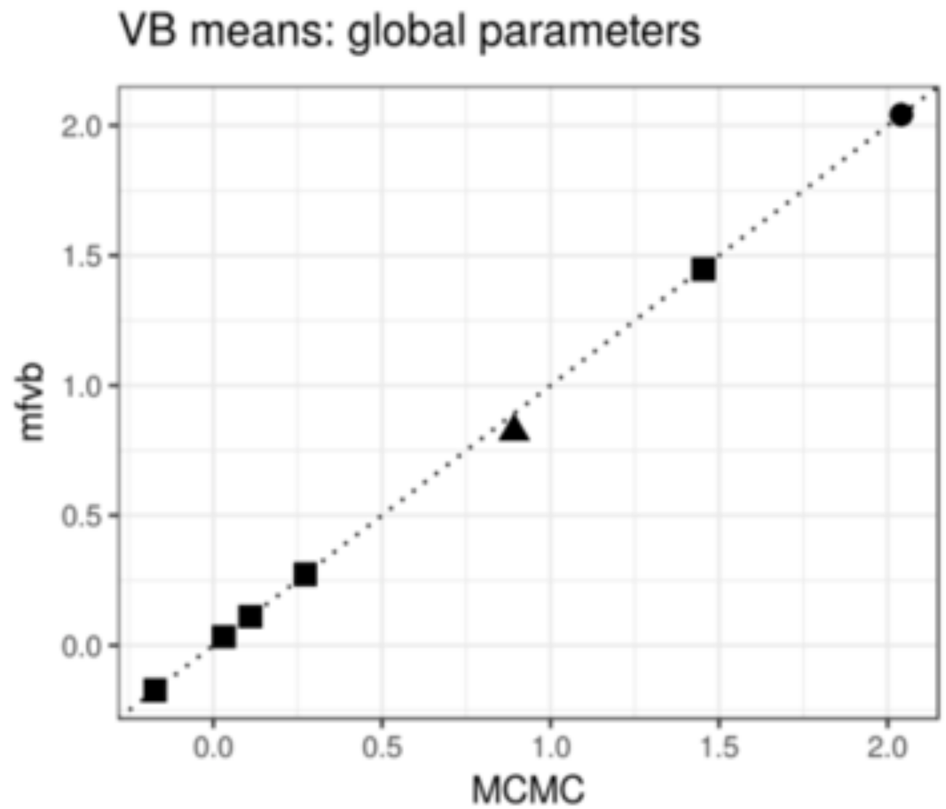

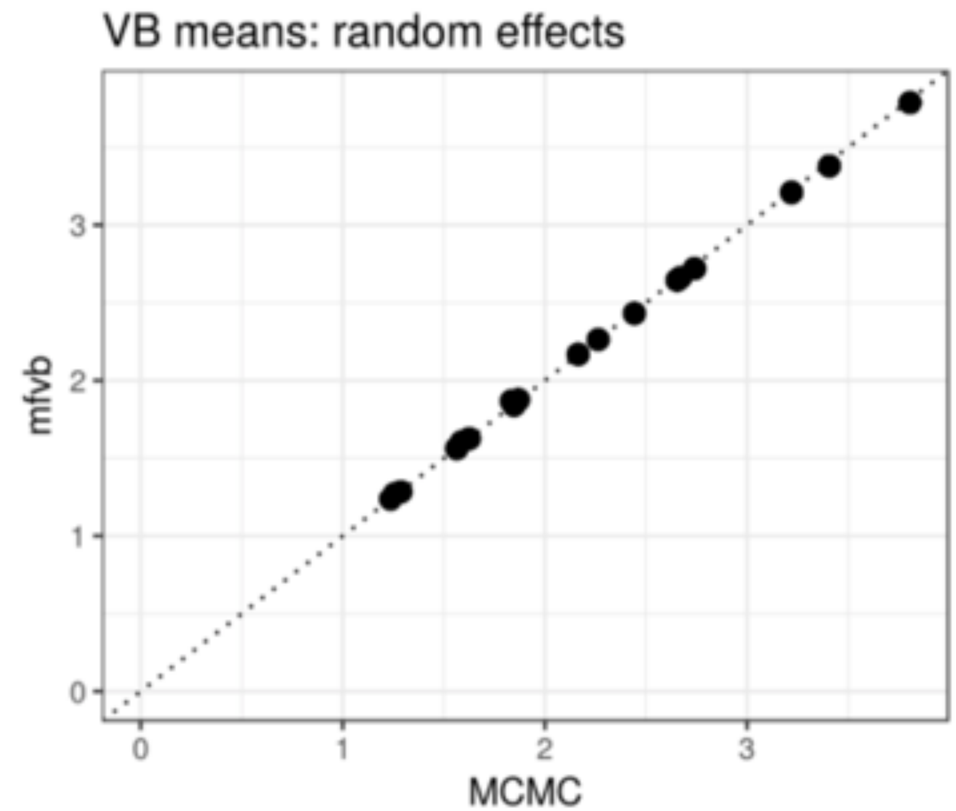

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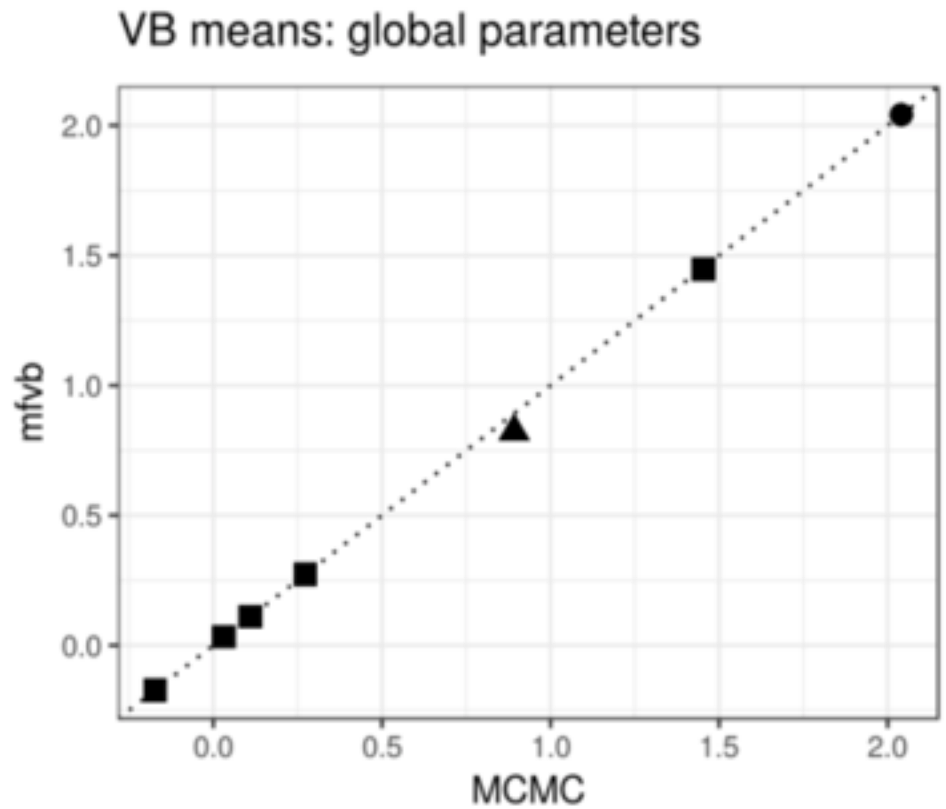

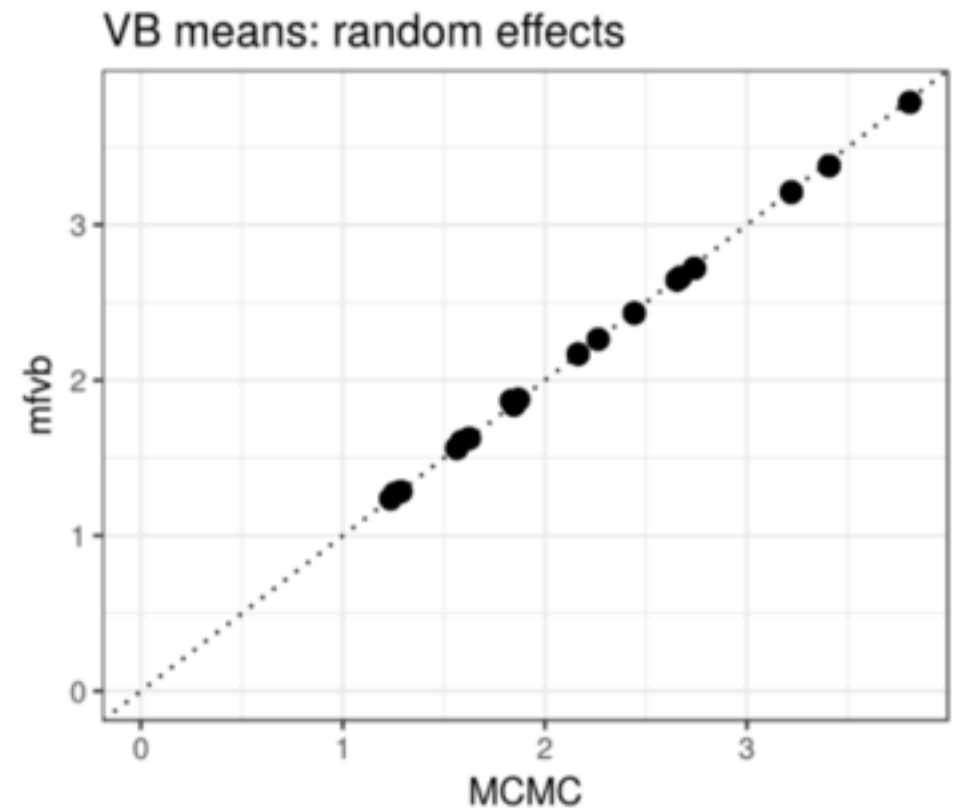

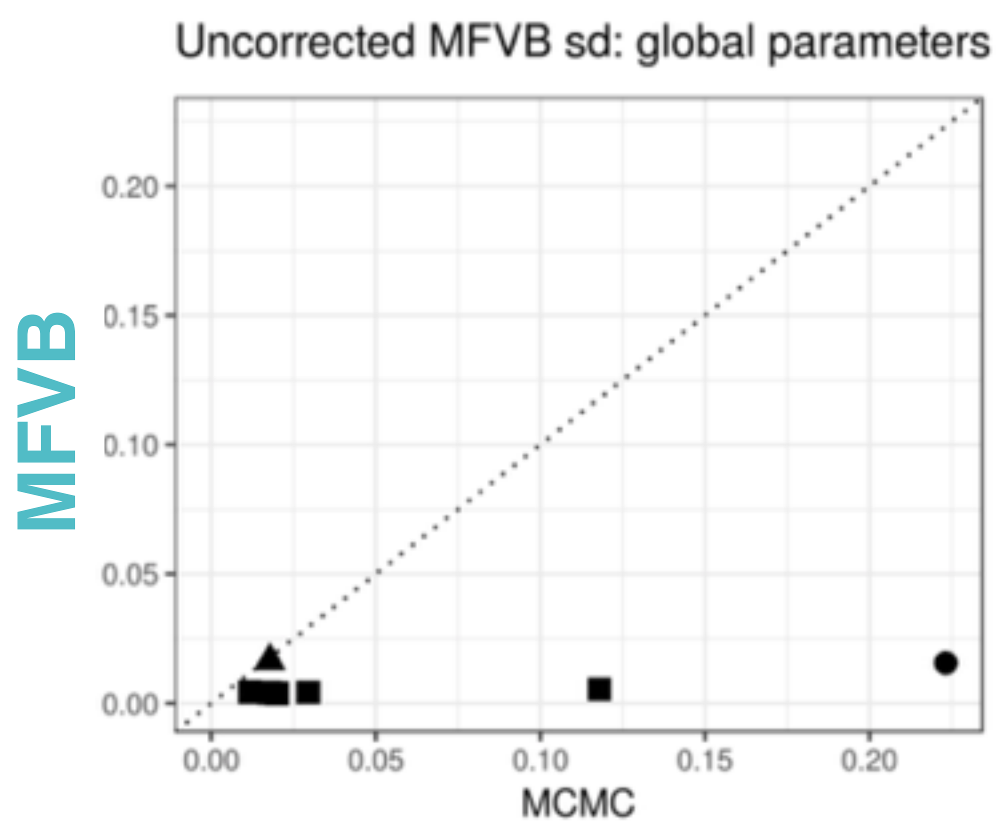

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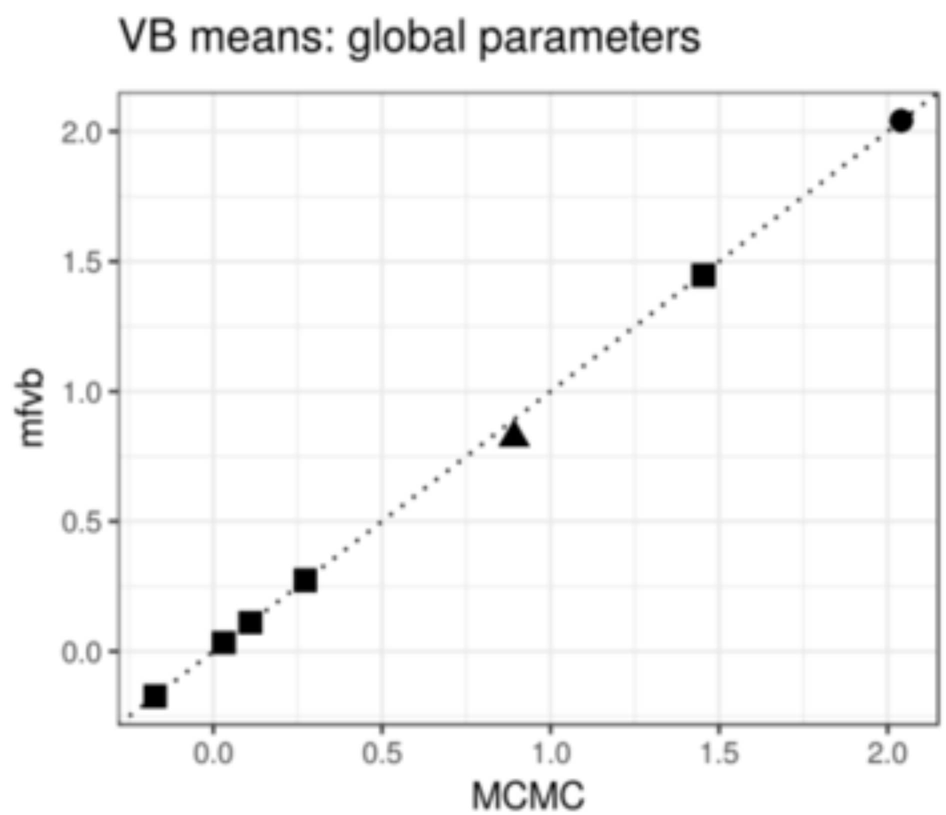
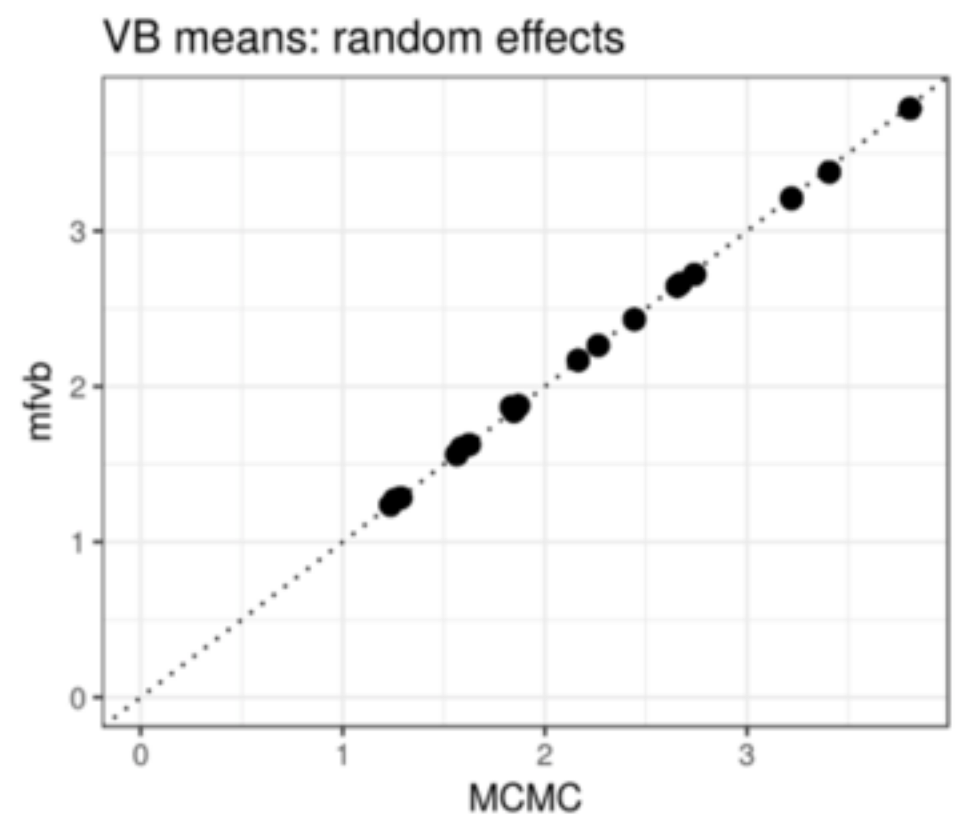
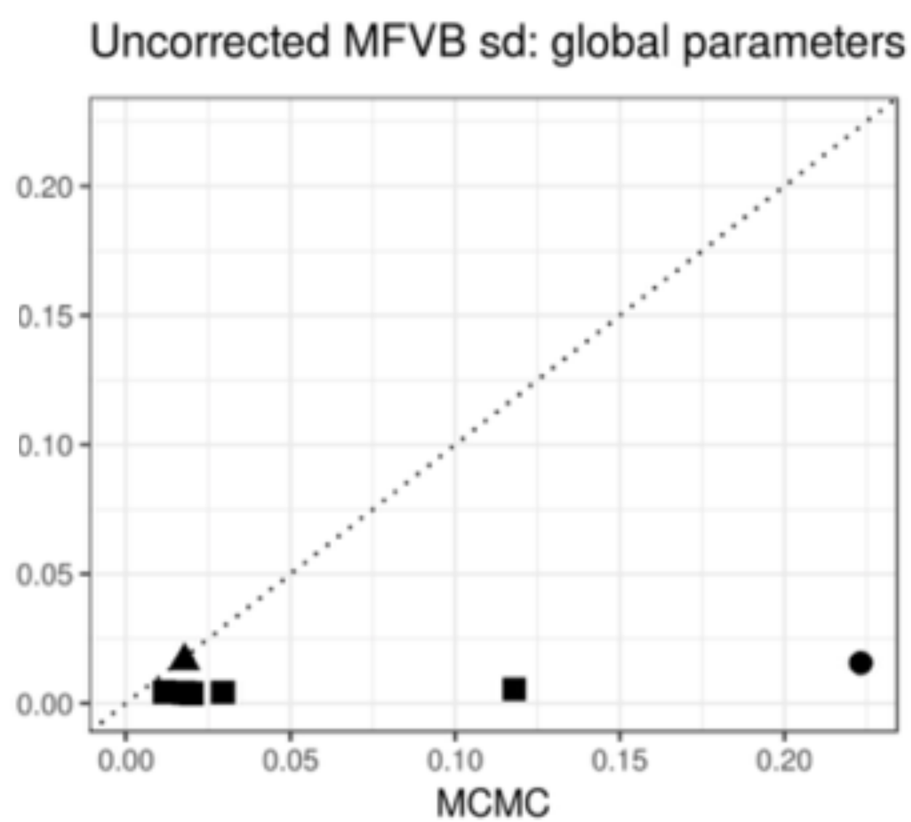
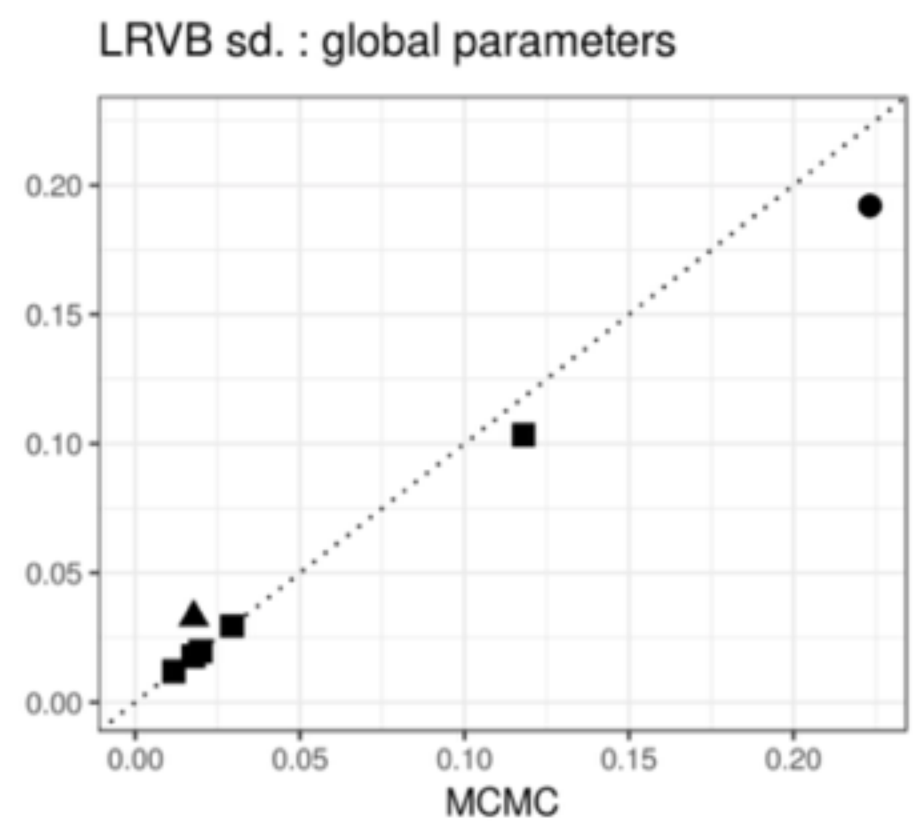




MFVB



LRVB



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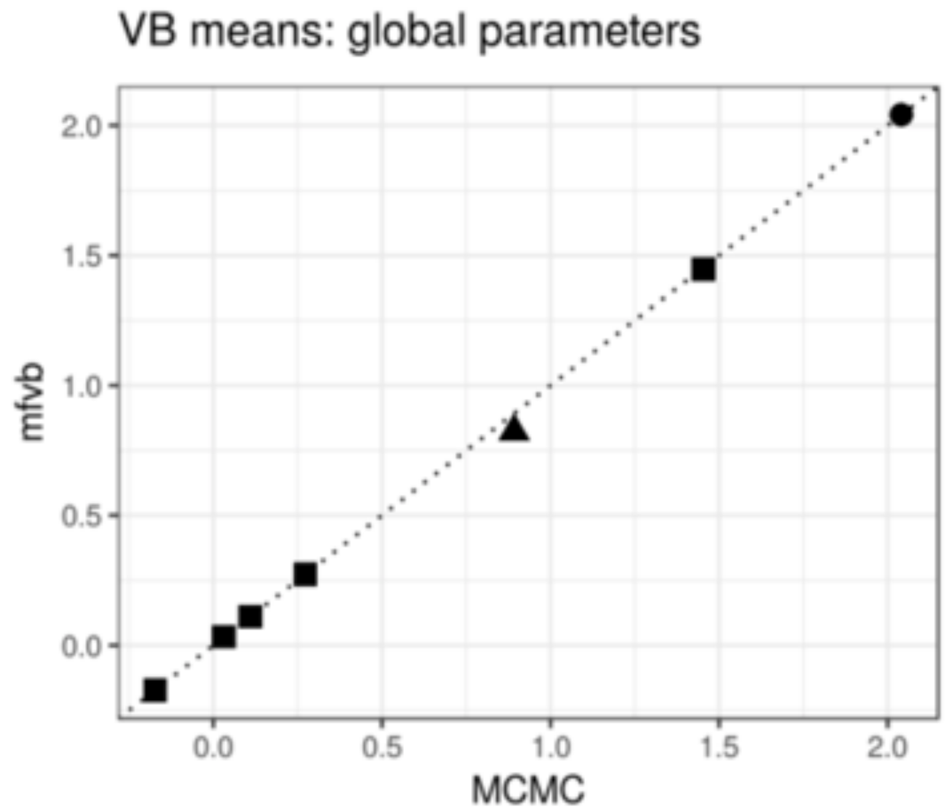

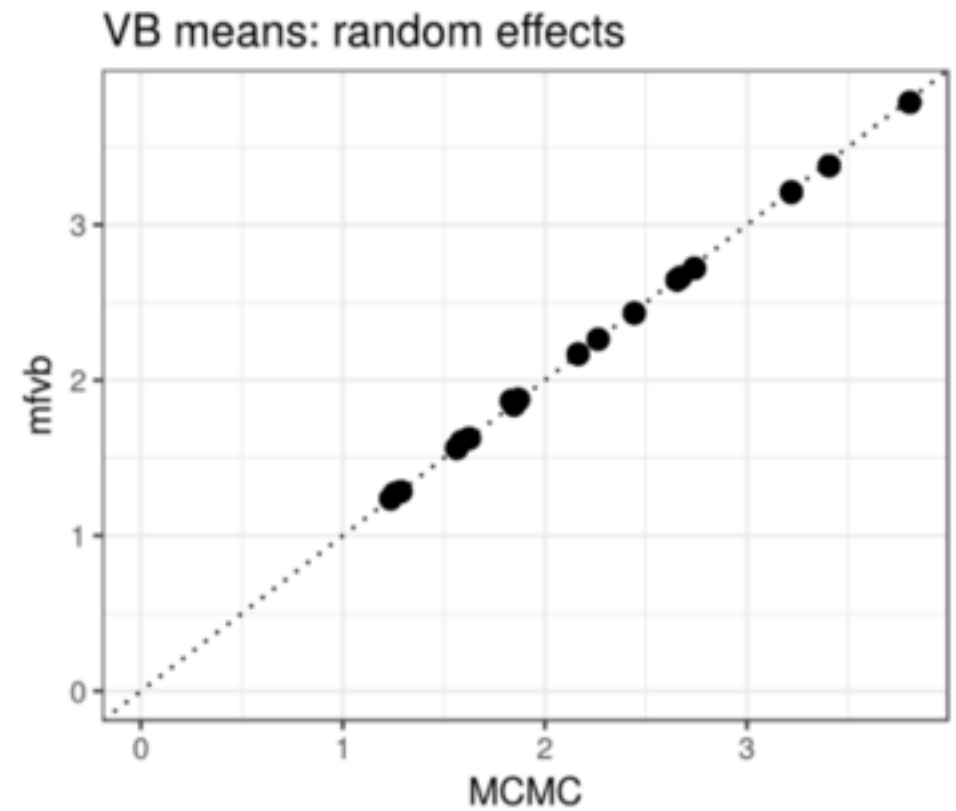

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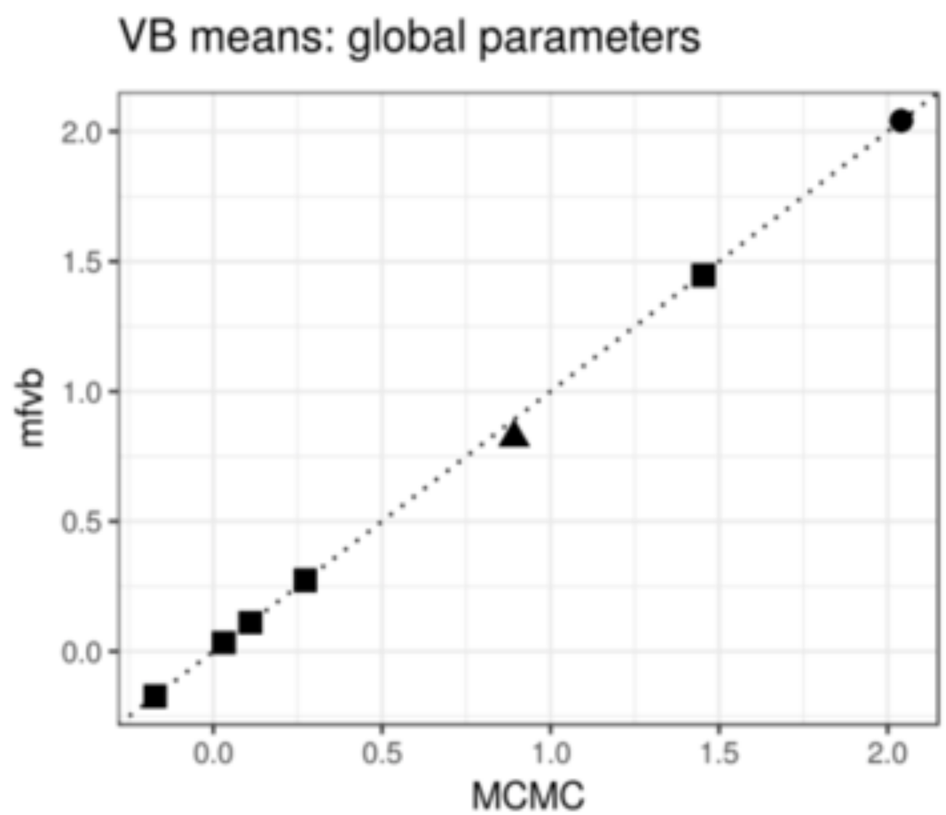

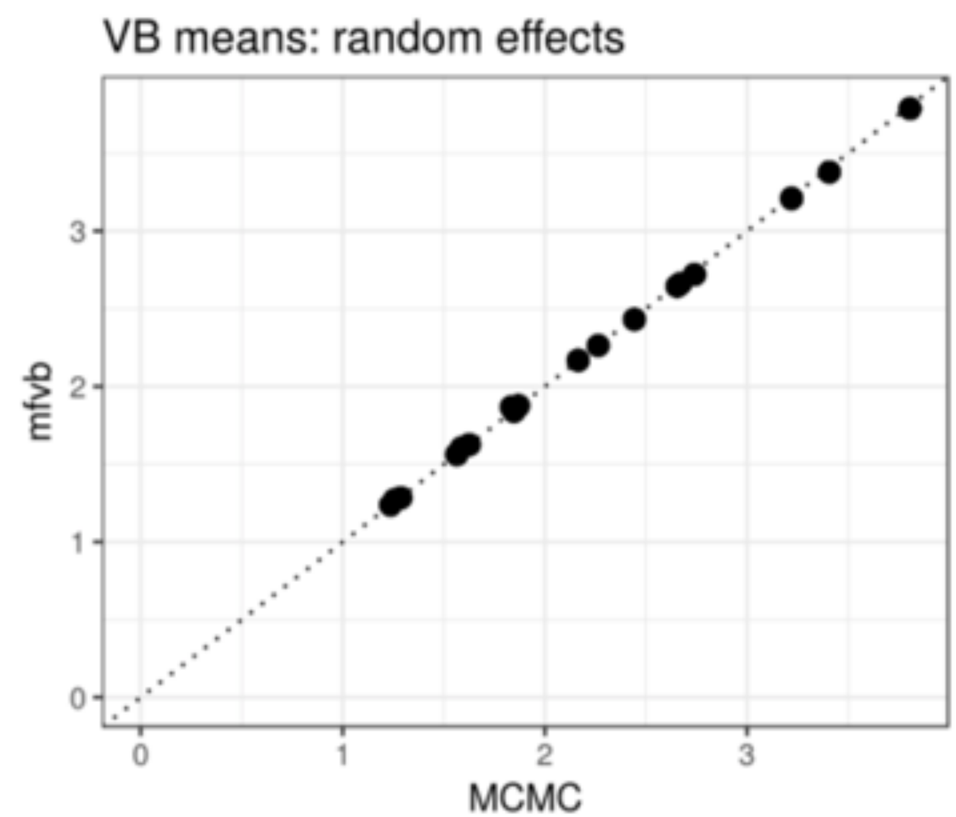

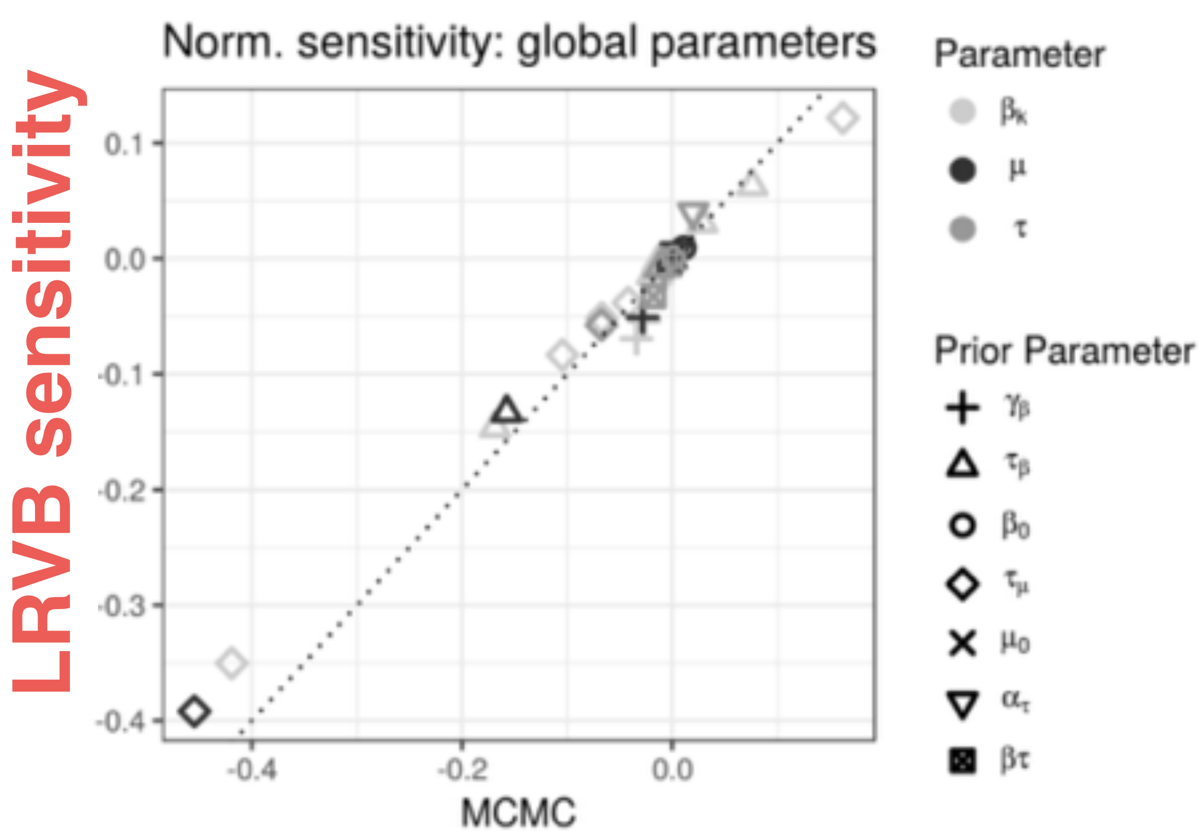

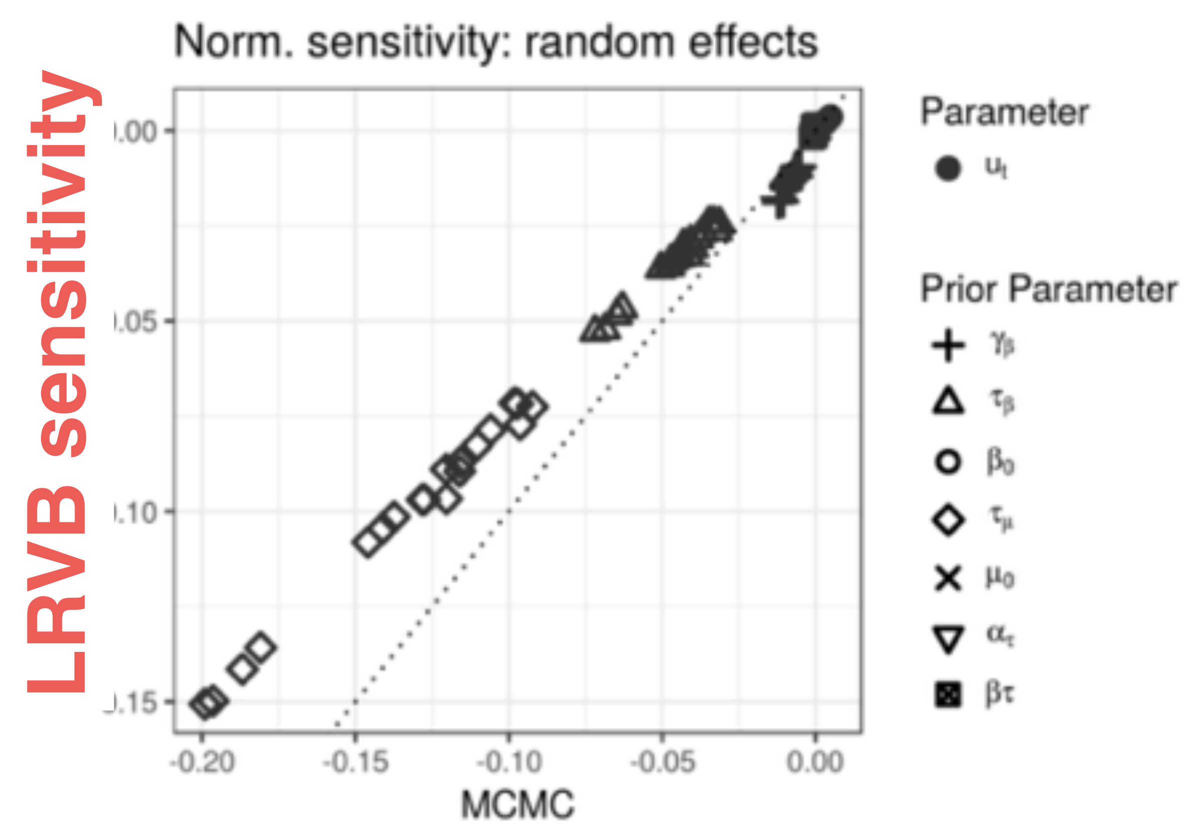

Posterior means: revisited

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[Giordano, Broderick, Jordan 2015]

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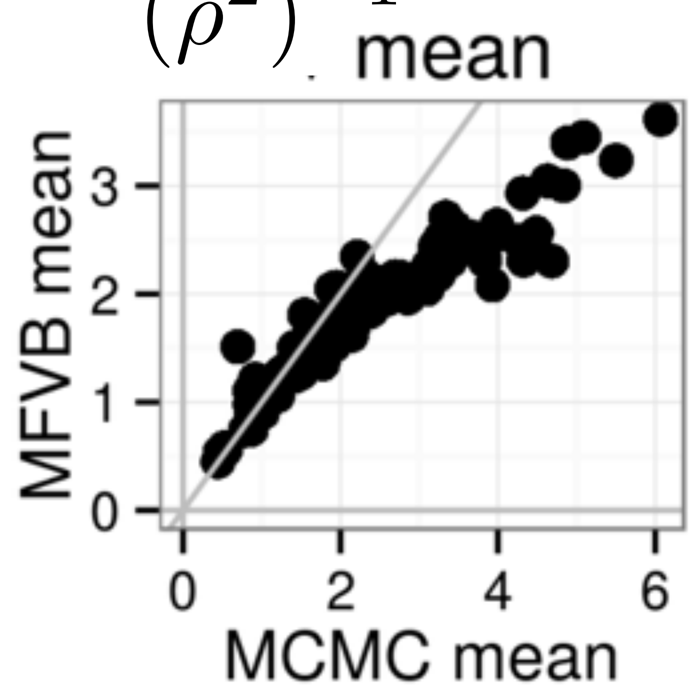

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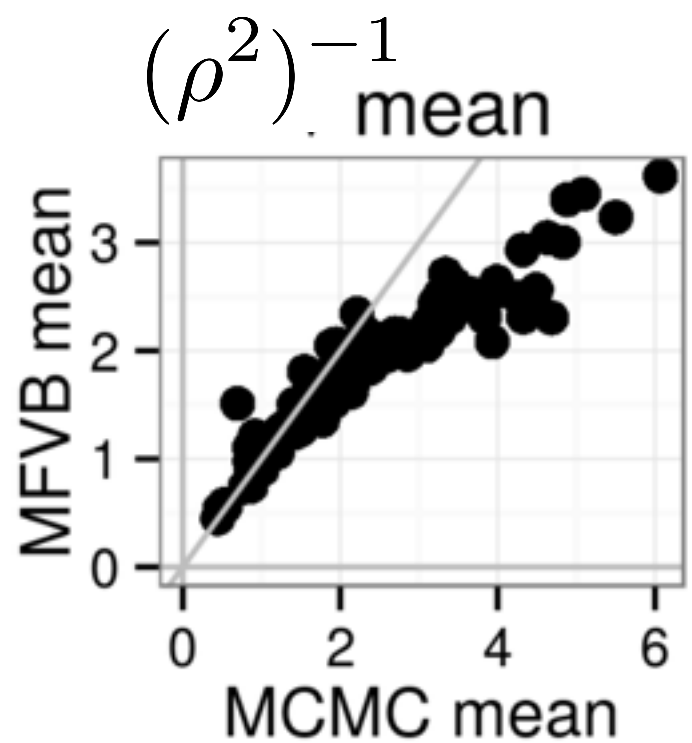

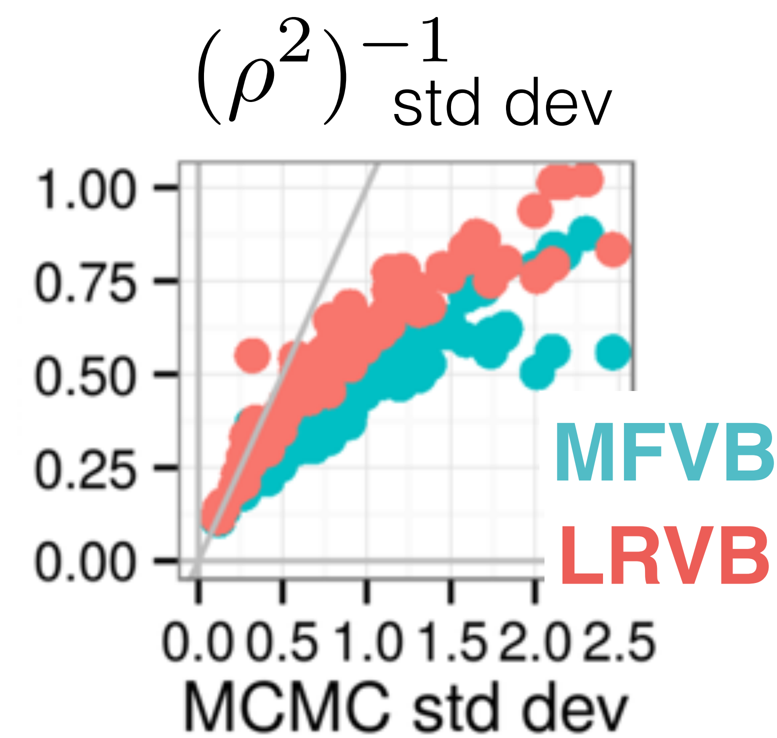

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Conclusions

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- Data summarization for scalability (Part III)

[Giordano, Broderick, Jordan 2015, 2017]

References (1/2)

R Giordano, T Broderick, and MI Jordan. Linear response methods for accurate covariance estimates from mean field variational Bayes. *NIPS* 2015.

R Giordano, T Broderick, R Meager, JH Huggins, and MI Jordan. Fast robustness quantification with variational Bayes. *ICML Workshop on #Data4Good: Machine Learning in Social Good Applications*, 2016. ArXiv:1606.07153.

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Code links in the papers

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