STAT 7200

Introduction to Advanced Probability
Lecture 21

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(Levy's) Continuity Theorem

Theorem 1 (The Continuity Theorem (11.1.14))

Let $\mu, \mu_1, \mu_2, \dots$ be probability measures with characteristic functions $\phi, \phi_1, \phi_2, \dots$. Then μ_n converges weakly to μ if and only if $\phi_n(t) \to \phi(t)$ for all $t \in \mathbf{R}$. That is, the weak convergence is equivalent to the pointwise convergence of characteristic functions.

- **Proof:** *(1) Weak convergence implies pointwise convergence of characteristic functions:*
- Since $cos(x)$ and $sin(x)$ are both bounded and continuous functions, then for any $t \in \mathbf{R}$:

$$\begin{aligned}\phi_n(t) &= \int cos(tx)\mu_n(dx) + i \int sin(tx)\mu_n dx \\ &\to \int cos(tx)\mu(dx) + i \int sin(tx)\mu(dx) = \phi(t),\end{aligned}$$

by the definition of weak convergence.

Continuity Theorem: Pointwise Convergence of Characteristic Function Implies Weak Convergence

- **Proof:** (2) On the other hand, if we have $\phi_n(t) \to \phi(t)$ for all $t \in \mathbf{R}$, we do not even know if the limit of $\{\mu_n\}$ exists or not.
- We will need several theorems, lemmas and corollaries to show that this is indeed true. Many of these will need their own results to prove them.

Fourier Inversion Theorem

Theorem 2 (Fourier Inversion Theorem (11.1.1))

Let μ be a Borel probability measure on $\mathbf{R}$ *with characteristic function* $\phi(t) = \int e^{itx}\mu(dx)$*. Then for* $a < b$ *and* $\mu(\{a\}) = \mu(\{b\}) = 0$*:*

$$\mu([a,b]) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)dt.$$

- **Remark 1:** We'll prove this with two Lemmas.
- **Remark 2:** The number of intervals $[a,b]$ with $\mu(\{a\}) \neq 0$ or $\mu(\{b\}) \neq 0$ is at most countable because the set $\{x : \mu(\{x\}) > 0\}$ is at most countable. That's from the previous lecture.

First Lemma to prove Fourier Inversion Theorem

Theorem 3 (Lemma 11.1.2)

For $T \geq 0$ and $a < b$

$$\int_{\mathbf{R}} \int_{-T}^{T} \left| \frac{e^{-ita} - e^{-itb}}{it} e^{itx} \right| dt \mu(dx) \leq 2T(b-a) < \infty.$$

First Lemma to prove Fourier Inversion Theorem

$$
\begin{aligned}
\left|\frac{e^{-ita} - e^{-itb}}{it} e^{itx}\right| &= \left|\frac{e^{-ita} - e^{-itb}}{it}\right| \left|e^{itx}\right| \\
&= \left|\int_a^b e^{itr} dr\right| \\
&\leq \int_a^b |e^{itr}| dr \\
&= b - a
\end{aligned}
$$

So

$$
\int_{\mathbf{R}} \int_{-T}^{T} \left|\frac{e^{-ita} - e^{-itb}}{it} e^{itx}\right| dt\mu(dx) \leq \int_{\mathbf{R}} \int_{-T}^{T} (b-a) dt\mu(dx) = 2T(b-a)
$$

Second Lemma to prove Fourier Inversion Theorem

Theorem 4 (Lemma 11.1.3)

For $T \geq 0$ and $\theta \in \mathbf{R}$

$$\lim_{T\to\infty} \int_{-T}^{T} \frac{\sin(\theta t)}{t} dt = \pi sign(\theta)$$

where $sign(\theta)$ is either 1, -1 or 0 depending on whether θ is positive, negative or 0, respectively.

We're omitting the proof because it's elementary integration, but it's fun and involves a lot of cool stuff (e.g. the sinc function, integration by parts, u-substitution, different trigonometric properties, etc.) so you should try it.

Fourier Inversion Theorem: Proof

WTS:

$$\mu([a,b]) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\phi(t)dt$$

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\phi(t)dt \\
&= \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\left(\int_{\mathbf{R}} e^{itx}\mu(dx)\right)dt \\
&= \frac{1}{2\pi}\int_{\mathbf{R}}\int_{-T}^{T} \frac{e^{it(x-a)}-e^{it(x-b)}}{it}dt\mu(dx) \qquad \text{(Fubini and first Lemma)} \\
&= \frac{1}{2\pi}\int_{\mathbf{R}}\int_{-T}^{T} \frac{i\sin(t(x-a))-i\sin(t(x-b))}{it}dt\mu(dx)
\end{aligned}$$

Fourier Inversion Theorem: Proof (continued)

Taking $T \to \infty$:

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt \\
&= \lim_{T\to\infty} \frac{1}{2\pi} \int_{\mathbf{R}} \int_{-T}^{T} \frac{\sin(t(x-a)) - \sin(t(x-b))}{t} dt \mu(dx) \\
&= \frac{1}{2\pi} \int_{\mathbf{R}} \lim_{T\to\infty} \int_{-T}^{T} \frac{\sin(t(x-a)) - \sin(t(x-b))}{t} dt \mu(dx) && \text{(DCT)} \\
&= \frac{1}{2\pi} \int_{\mathbf{R}} \pi \left[\mathrm{sign}(x-a) - \mathrm{sign}(x-b)\right] \mu(dx) && \text{(Lemma 2)} \\
&= \mu((a,b)) + \frac{1}{2}\mu(\{a\}) + \frac{1}{2}\mu(\{b\}) \\
&= \mu([a,b])
\end{aligned}
$$

where the last equality follows because $\mu(\{a\}) = \mu(\{b\}) = 0$.

Fourier Uniqueness Theorem: Characteristic Function Determines Distribution

Theorem 5 (Fourier Uniqueness Theorem)

Let X, Y be random variables. Then $\phi_X(t) = \phi_Y(t)$ if and only if $\mathcal{L}(X) = \mathcal{L}(Y)$.

- **Proof:** The "if" part is trivial. For the "only if" part, to show $\mathcal{L}(X) = \mathcal{L}(Y)$, we only need to show $\mathbf{P}(X \in I) = \mathbf{P}(Y \in I)$ for all intervals in $\mathbf{R}$ (uniqueness of extension Prop. 2.5.8).
  - By Fourier Inversion theorem, $\mathbf{P}(X \in [a, b]) = \mathbf{P}(Y \in [a, b])$ whenever $a < b$ and $\mathbf{P}(X = a) = \mathbf{P}(X = b) = \mathbf{P}(Y = a) = \mathbf{P}(Y = b) = 0$.
  - For any interval I, we can always find a sequence of closed intervals $\{[a_i, b_i]\}$ that satisfy the above conditions, and $[a_i, b_i] \to I$. Thus, we can apply the continuity of probability to show that $\mathbf{P}(X \in I) = \mathbf{P}(Y \in I)$.

Helly Selection Principle

Theorem 6 (Helly Selection Principle)

Let $\{F_n\}$ be a sequence of cdfs, each corresponding with a measure μ_n. Then there exists a subsequence F_{n_k}, and a non-decreasing, right-continuous $0 \leq F \leq 1$, such that $F_{n_k}(x) \to F(x)$ for each $x \in \mathbf{R}$ that is a continuity point of F.

The proof will remind you of the uncountability proofs.

Also, F is not necessarily a cdf.

Helly Selection Principle: proof

List out rationals $\mathbf{Q} = \{q_1, q_2, \ldots\}$. Note that $0 \leq F_n(q_1) \leq 1$ for all n. By Bolzano-Weirstrass, there exists a subsequence $l_k^{(1)}$ such that $\lim_k F_{l_k^{(1)}}(q_1)$ exists. Then there exists a subsequence of that subsequence, call it $l_k^{(2)}$, such that $\lim_k F_{l_k^{(2)}}(q_2)$ exists. Because it is a subsequence of the first one, we also have that $\lim_k F_{l_k^{(2)}}(q_1)$ exists. We can do this for each $m \in \mathbf{N}$. For each $\{l_{n_k}^{(m)}\}$, we have $\lim_k F_{l_k^{(m)}}(q_j)$ exists for all $0 < j \leq m$.

Now define $n_k = l_k^{(k)}$. These are the diagonals. However, note that all these subsequences are nested, so for any k,

- $\{n_k, n_{k+1}, \ldots\} \subseteq \{l_k^{(k)}, l_{k+1}^{(k)}, \ldots\}$
- $\{n_{k+1}, n_{k+2}, \ldots\} \subseteq \{l_{k+1}^{(k+1)}, l_{k+2}^{(k+1)}, \ldots\}$, etc.

These ensure that $\lim_k F_{n_k}(q) := G(q)$ exists for each $q \in \mathbf{Q}$. G is also clearly non-decreasing as well, just like each $F_{n,k}$.

Helly Selection Principle:proof

For each rational q, $F_{n_k}(q) \to G(q)$. G is defined on the rationals, only. Now we define

$$F(x) = \inf\{G(q) : q > x, q \in \mathbf{Q}\}$$

which is defined on the reals. It has a few properties that we'll need:

- F is non-decreasing
- $0 \leq F \leq 1$
- F is right-continuous,and
- $F(q) > G(q)$ for all $q \in \mathbf{Q}$

Next, we'll show that, for any continuity point of F, call it $x \in \mathbf{R}$, we have $F_{n_k}(x) \to F(x)$ as $k \to \infty$. Pick any $\varepsilon > 0$, then pick $r, u, s \in \mathbf{Q}$ such that $r < u < x < s$ and $F(s) - F(r) < \varepsilon$.

$$\begin{aligned} F(x) - \varepsilon &\leq F(r) \\ &= \inf\{G(q) : q > r\} \\ &= \inf\{\lim_k F_{n_k}(q) : q > r\} \end{aligned}$$

Helly Selection Principle:proof

$$
\begin{aligned}
F(x) - \varepsilon &\leq \inf_{q>r} \liminf_k F_{n_k}(q) \\
&\leq \liminf_k F_{n_k}(u) && (u > r) \\
&\leq \liminf_k F_{n_k}(x) && (x > u) \\
&\leq \limsup_k F_{n_k}(x) \\
&\leq \limsup_k F_{n_k}(s) && (s > x) \\
&= G(s) \\
&\leq F(s) \\
&\leq F(x) + \varepsilon
\end{aligned}
$$

QED

Tightness of Measure: Three More Results

Theorem 7 (11.1.10)

If $\{\mu_n\}$ is a tight sequence of prob. measures, then there exists a subsequence $\{\mu_{n_k}\}$ and a probability measure μ such that μ_{n_k} converges weakly to μ.

Theorem 8 (Corollary 11.1.11)

Let $\{\mu_n\}$ be a tight sequence of prob. measures on **R**. *Also suppose that, whenever $\mu_{n_k} \Rightarrow \nu$, then ν is always equal to μ. Then $\mu_n \Rightarrow \mu$.*

Lemma 9 (11.1.13)

Let $\{\mu_n\}$ be a sequence of probability measures on **R**, *and $\{\phi_n(t)\}$ be the characteristic functions. If there is a function g that is continuous at 0, and $\phi_n(t) \to g(t)$ for all $|t| < t_0$ ($t_0 > 0$), then $\{\mu_n\}$ is tight.*

Remember our goal is to prove the other direction of Levy's continuity theorem. Let's prove the third one first.

Tightness and Characteristic Functions

Lemma 10 (11.1.13)

Let $\{\mu_n\}$ be a sequence of probability measures on $\mathbf{R}$, *and $\{\phi_n(t)\}$ be the characteristic functions. If there is a function g that is continuous at 0, and $\phi_n(t) \to g(t)$ for all $|t| < t_0$ $(t_0 > 0)$, then $\{\mu_n\}$ is tight.*

- **Proof:** Let $y > 0$

$$\begin{aligned}
\frac{1}{y}\int_{-y}^{y}[1-\phi_n(t)]dt &= \int_{-\infty}^{\infty}\left[\frac{1}{y}\int_{-y}^{y}(1-e^{itx})dt\right]\mu_n(dx) \\
&= 2\int_{-\infty}^{\infty}(1-\frac{\sin yx}{yx})\mu_n(dx) \\
&\geq 2\int(1-\frac{1}{|yx|})\mu_n(dx) \\
&\geq \int_{|x|>2/y} 1\mu_n(dx) = \mu_n\left(\left\{x : |x| \geq \frac{2}{y}\right\}\right)
\end{aligned}$$

Tightness and Characteristic Functions: continued

- **Proof continued:** The previous discussion shows that $\mu_n[\{x : |x| \geq \frac{2}{y}\}] \leq \frac{1}{y}\int_{-y}^{y}[1-\phi_n(t)]dt$.
- Now since $g(t)$ is continuous at 0, and $g(0) = \lim_n \phi_n(0) = 1$, then for any $\varepsilon > 0$, we can always find $y_0 \in (0, t_0)$ such that:

$$|1 - g(t)| < \varepsilon/4$$

whenever $|t| < y_0$. Then

$$\begin{aligned}\left|\frac{1}{y_0}\int_{-y_0}^{y_0}[1-g(t)]dt\right| &\leq \frac{1}{y_0}\int_{-y_0}^{y_0}|1-g(t)|\,dt \\ &\leq \frac{1}{y_0}\int_{-y_0}^{y_0}\varepsilon/4dt = \varepsilon/2\end{aligned}$$

Tightness and Characteristic Functions: continued

- **Proof continued:** The previous discussion shows that
  1. $\mu_n[\{x : |x| \geq \frac{2}{y}\}] \leq \frac{1}{y}\int_{-y}^{y}[1-\phi_n(t)]dt$ and
  2. $\left|\frac{1}{y_0}\int_{-y_0}^{y_0}[1-g(t)]dt\right| \leq \varepsilon/2$

- On the other hand, as $\phi_n(t) \to g(t)$ for $|t| < t_0$, and $|\phi_n(t)| = 1$, we can apply the dominated convergence theorem:

$$\left|\int_{-y_0}^{y_0}[1-\phi_n(t)]dt - \int_{-y_0}^{y_0}[1-g(t)]dt\right| \leq \varepsilon/2$$

  for $n > N$.

- Then for all $n > N$,

$$\mu_n[\{x : |x| \geq \frac{2}{y_0}\}] \leq \frac{1}{y_0}\int_{-y_0}^{y_0}[1-\phi_n(t)]dt \leq \varepsilon$$

- It then follows that $\{\mu_n\}$ must be tight.