STAT 7200

Introduction to Advanced Probability
Lecture 3

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"A First Look at Rigorous Probability Theory" (Jeffrey Rosenthal) A.5, Section 1

Equivalence Relations

- **Equivalence Relations** We can define relations on a set S such that for any $x, y \in S$, either $x \sim y$ (related) or $x \not\sim y$ (not related). A relation $\sim$ is an **equivalence relation** if (a) it is reflexive, $x \sim x$ for all $x \in S$; (b) it is symmetric, $x \sim y$ whenever $y \sim x$; and (c) it is transitive, $x \sim z$ if $x \sim y$ and $y \sim z$.
- We use equivalence relations to partition up a state space.
- **Example** Let N be the set of all natural number. For $m, n \in \mathrm{N}$, $m \sim n$ if and only if $|m - n|$ is an integer multiple of 2. Show that: 1) This is an equivalence relation. 2) This equivalence relation leads to the partition of N as the union of the set of even numbers and the set of odd numbers.

What is a "Reasonable" Probability Measure?

- **Uniform Measure over [0,1]** Let $X \sim Unif([0,1])$, a uniform measure over [0,1]. Then one question is, for any subset $A \subseteq [0,1]$, can we define the probability $P(A) = P(X \in A)$?
- First, as [0,1] contains all outcomes, we expect $P([0,1]) = 1$.
- Second, as the word "uniform" suggests, each x has an "equal chance" to be a realization of X. We would then conclude the probability that X equals any particular number is 0. This "uniform" principle also implies that, for any set A of the form $[a,b], (a,b), [a,b), or (a,b]$, we have $P(A) = b - a$.
- Third, if we ask what is the probability that the value of X falls into one or more disjoint intervals, naturally the answer should be the sum of probabilities of each intervals. This leads to the requirement known as "finite additivity":
  $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

What is a "Reasonable" Probability Measure?: Continued

- Fourth, we would like to extend "finite additivity" to "countable additivity" to account for the problems like:
  $1 = P([0,1]) = P(0 \cup [1, \frac{1}{2}] \cup (\frac{1}{2}, \frac{1}{4}] \cup (\frac{1}{4}, \frac{1}{8}] \cup \cdots)$
- Fifth, however, we could not expect "uncountable additivity." If this was true: we would encounter blatant paradoxes:
  $1 = P([0,1]) = P(\cup_{x \in [0,1]} \{x\}) = \sum_{x \in [0,1]} P(\{x\}) = 0.$
- Sixth, the "uniform" would also suggest that if we "shift" any subset of [0,1] by a fixed amount, the probability of the shifted subset should equal the probability of the original subset. That is, for $A \subseteq [0,1]$ and $0 \leq r < 1$, if we define its $r - \mathit{shift}$ as
  $A \oplus r = \{a + r : a \in A, a + r \leq 1\} \cup \{a + r - 1 : a \in A, a + r > 1\}.$
  Then $P(A \oplus r) = P(A)$
- However, it turns out that there is a contradiction if we assume **all** sets get a probability.

A Proof By Contradiction

Proposition 1 (1.2.6)

Let $A \subseteq [0,1]$. It is impossible to define $P(A)$ that satisfies:
1) for interval A in the forms of $[a,b], (a,b), [a,b), (a,b]$, $P(A) = b - a$;
2) countable additivity: $P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ if all A_n are disjoint.
3) $P(A \oplus r) = P(A)$.

- **Proof** Assume to the contrary that it is possible to define such a probability measure. Here we define an equivalence relation on [0,1]: $x \sim y$ if and only if $x - y$ is a rational number. This equivalence relation would induce a partition on [0,1].
- By the Axiom of Choice, define $H \subseteq [0,1]$ by picking exactly one element from each disjoint subset in the partition. We assume $0 \notin H$ as we can always replace it with 0.5 ($0 \sim 0.5$).

A Limitation of Probability Measure: continued

- **Proof: continued:** Let $r_1, r_2 \in [0,1) \cap \mathbb{Q}$. For $r_1 \neq r_2$, $H \oplus r_1$ and $H \oplus r_2$ are disjoint. If not, we can find $x \in [0,1]$ such that $x \in H \oplus r_1$ and $x \in H \oplus r_2$. Then there are $h_1 \neq h_2 \in H$ such that $x = h_1 \oplus r_1 = h_2 \oplus r_2$. As a result, $x \sim h_1$ and $x \sim h_2$, and we must accept that $h_1 \sim h_2$. However, this contradicts the definition of H.
- It is also easy to check that $[0,1] = \cup_{r \in \mathbb{Q} \cap [0,1)} (H \oplus r)]$ since each element in $[0,1]$ must be equivalent to an element in H.
- Finally, by **countable** additivity: $1 = P((0,1]) = P(\cup_{r \in \mathbb{Q} \cap [0,1]} (H \oplus r)) = \sum_{r \in \mathbb{Q} \cap [0,1)} P(H \oplus r)$. That is: $1 = \sum_{r \in \mathbb{Q} \cap [0,1)} P(H)$.
- However, this is impossible! The sum must either be 0 or ∞, but never 1.

Probability Triple

- **Probability Triple** All probability models are built on a triple $(\Omega, \mathcal{F}, P)$, the sample space, σ-algebra and probability measure.
- Ω is the sample space, the set of all possible outcomes.
- $\mathcal{F}$ is a collection of measurable subsets of Ω, a σ-algebra. For a collection $\mathcal{F}$ to be qualified as a σ-algebra , the following conditions must be met:
  a) $\emptyset \in \mathcal{F}$; b) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.
  c) If $A_1, A_2, \ldots \in \mathcal{F}$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.
  We can prove that a σ-algebra must also closed under countable intersection, finite union and finite intersection.
- $P : \mathcal{F} \to [0, 1]$ is the probability measure. It must satisfy the following axioms: a) $P(A) \geq 0$ for any $A \in \mathcal{F}$. b) $P(\Omega) = 1$; c) Countable additivity.
  More properties regarding probability measure can be found in section 2.1.

Probability Triples with Finite or Countable Sample Spaces

- Suppose the sample space is countable or finite: $\Omega = \{\omega_1, \omega_2, \ldots\}$. Then we may define probability measure in the following way:
- $\mathcal{F}$ is chosen as the power set, the collection of all possible subsets of Ω.
- Suppose $\{a_1, a_2, \ldots\}$ is any sequence with the same cardinality as Ω. Suppose further that $0 \leq a_i \leq 1, \sum_i a_i = 1$. Then we can define the probability measure as:

$$P(\omega_i) = a_i \text{ for all } \omega_i \in \Omega$$

$$P(A) = \sum_{i:\omega_i \in A} a_i$$

- It is easy to check that the above definition leads to a proper probability triple: $\mathcal{F}$ is a σ-algebra and the probability measure P defined satisfies all the axioms of probability.

Probability Triple with Uncountable Sample Space

- We have already discussed that it is impossible to define probability measure on **all** the possible subsets. Still, we expect that the σ-algebra should contain certain subsets, such as intervals.
- **Borel σ-algebra** can be defined as the smallest σ-algebra that contains all the open intervals (a, b).
- According to the fact that σ-algebra is closed under complements, unions and intersections (both finite and countable), we can show that the Borel σ-algebra must also contain other intervals as well: $(\emptyset, \{a\}, [a, b], (a, b], [a, b), [a, \infty), (a, \infty), (-\infty, a), (-\infty, a])$
- For instance, [a,b] must be a Borel Set since $[a, b] = (a, b) \cup \cap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n}) \cup \cap_{n=1}^{\infty}(b - \frac{1}{n}, b + \frac{1}{n})$

Getting ready for next class

- $\mathcal{J} :=$
  $\{$ empty, singleton, closed, open, right/ left closed (all) intervals in $[0, 1]$
- $\mathcal{B}_0 := \{$all finite unions of elements from $\mathcal{J}\}$
- $\mathcal{B}_1 := \{$all finite or countable unions of elements from $\mathcal{J}\}$

Try showing

- Exercise 2.2.3.: Prove $\mathcal{J}$ is a semialgebra.
- Exercise 2.2.5: Prove $\mathcal{B}_0$ is an algebra, but not a σ-algebra.
- Exercise 2.4.7: Prove $\mathcal{B}_1$ is still not a σ-algebra.