STAT 7200

Introduction to Advanced Probability
Lecture 6

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"A First Look at Rigorous Probability Theory" (Jeffrey Rosenthal) Section 2.4, 2.5

Extension Theorem

Theorem 1 (2.3.1.)

The Extension Theorem *Let $\mathcal{J}$ be a semialgebra of subsets of Ω, $\mathbf{P}$ a function from $\mathcal{J}$ to [0,1] with the following properties:*

a) $\mathbf{P}(\emptyset) = 0, \mathbf{P}(\Omega) = 1$.

b) $\mathbf{P}(\bigcup_{i=1}^{k} A_i) \geq \sum_{i=1}^{k} \mathbf{P}(A_i)$ *whenever* $A_1, \ldots, A_k \in \mathcal{J}$, $\bigcup_{i=1}^{k} A_i \in \mathcal{J}$, *and* $A_1, \ldots, A_k$ *are pairwise disjoint (finite superadditivity).*

c) $\mathbf{P}(A) \leq \sum_n \mathbf{P}(A_n)$ *whenever* $A, A_1, \ldots, A_n, \ldots \in \mathcal{J}$, *and* $A \subseteq \bigcup_n A_n$ *(countable monotonicity).*

Then there is a σ-algebra $\mathcal{M} \supseteq \mathcal{J}$ and a proper probability measure $\mathbf{P}^*$ on $\mathcal{M}$ so that $\mathbf{P}^*(A) = \mathbf{P}(A)$ for all $A \in \mathcal{J}$.

Application of Extension Theorem: Uniform Measure on [0,1]

- To construct the uniform measure on $[0,1]$, first we choose $\mathcal{J}$ as the collection of all intervals I in $[0,1]$, then we define $\mathbf{P}$ as $\mathbf{P}(I) =$ the length of I. Now it is easy to verify $\mathcal{J}$ is a semialgebra. If we can also verify $\mathbf{P}$ satisfies the conditions listed in the extension theorem, we can apply the extension theorem to show that we may extend $\mathbf{P}$ to the uniform measure $\mathbf{P}^*$ over $\mathcal{M} \supseteq \mathcal{J}$.
- We have $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}([0,1]) = 1$.
- For finite superadditivity, we need to show: for disjoint intervals $I_1, \ldots, I_k$ whose union $\bigcup_{i=1}^k I_i := I_0 = (a_0, b_0)$ is also an interval, we have $\mathbf{P}(\bigcup_{i=1}^k I_i) \geq \sum_{i=1}^k \mathbf{P}(I_i)$. Note that we can always reorder these intervals so that their end points (denoted by a_k, b_k) satisfy:
  $a_0 = a_1 \leq b_1 = a_2 \leq b_2 = \cdots = a_k \leq b_k = b_0$.
  Then $\sum_{i=1}^k P(I_i) = \sum_{i=1}^k (b_i - a_i) = b_k - a_1 = b_0 - a_0 = P(I_0)$.

Application of Extension Theorem: Uniform Measure on [0,1] (Exercise 2.4.3 a)

- For countable monotonicity: we should prove $\mathbf{P}(I) \leq \sum_n \mathbf{P}(I_n)$ whenever $I, I_1, I_2, \ldots$ are intervals and $I \subseteq \bigcup_{i=1}^{\infty} I_i$. First, we prove for the finite case.
- Let us suppose $\mathbf{P}(I) \leq \sum_{i=1}^{n} \mathbf{P}(I_i)$ holds for $n = k - 1$. Then when $n = k$, without loss of generality, we may assume that the length of I is positive and $a_k \leq b \leq b_k$. If $a_k \leq a$, the result follows immediately.

- Otherwise, $a < a_k \leq b \leq b_k$,
  then $I \cap I_k^c$ is still an interval and $I \cap I_k^c \subseteq \bigcup_{i=1}^{k-1} I_i$,
  $\mathbf{P}(I \cap I_k^c) \leq \sum_{i=1}^{k-1} \mathbf{P}(I_i)$.
- Thus, $\mathbf{P}(I \cap I_k^c) = a_k - a \geq a_k - a - (b_k - b) = \mathbf{P}(I) - \mathbf{P}(I_k)$,
  we have $\mathbf{P}(I) \leq \sum_{i=1}^{k} \mathbf{P}(I_i)$.

Application of Extension Theorem: Uniform Measure on [0,1] (Exercise 2.4.3 b)

- Let $I, I_1, I_2, \ldots \in \mathcal{J}$ and assume $I := [a, b]$ is closed, $\{I_i\}_i$ are all open and $I \subseteq \bigcup_{i=1}^{\infty} I_i$. We want to show that $\mathbf{P}(I) \leq \sum_{i \geq 1} \mathbf{P}(I_i)$.
- By the Heine-Borel Theorem, there exists $\{I_{i_j}\}_{j=1}^{n}$ such that $I \subseteq \bigcup_{j=1}^{n} I_{i_j}$
- By finite monotonicity (the previous slide), $\mathbf{P}(I) \leq \sum_{j=1}^{n} \mathbf{P}(I_{i_j}) \leq \sum_{i \geq 1} \mathbf{P}(I_i)$.

Application of Extension Theorem: Uniform Measure on [0,1] (Exercise 2.4.3 c)

- Let $I, I_1, I_2, \ldots \in \mathcal{J}$ and assume $I \subseteq \bigcup_{i=1}^{\infty} I_i$. We want to show that $\mathbf{P}(I) \leq \sum_{i\geq 1} \mathbf{P}(I_i)$. Here I may not be closed, and the rest may not be open.
- Assume WLOG that the length of interval I is positive. Pick any $\varepsilon < (b-a)/2$. Then $[a+\varepsilon, b-\varepsilon]$ is closed and has positive length. Also, enlarge each I_k to the open interval $(a_k - \varepsilon/2^k, b_k + \varepsilon/2^k)$. Clearly $[a_0 + \varepsilon, b_0 - \varepsilon] \subseteq \bigcup_{i=1}^{\infty}(a_i - \varepsilon/2^i, b_i + \varepsilon/2^i)$.

- By the previous slide, $b - a - 2\varepsilon \leq \sum_{i\geq 1}(b_i - a_i + \varepsilon/2^{i-1}) = \sum_{i\geq 1} \mathbf{P}(I_i) + 2\varepsilon$
- As the choice of ε is arbitrary, we have $\mathbf{P}(I) \leq \sum_{i=1}^{\infty} \mathbf{P}(I_i)$.

Lebesgue Measure

- The previous discussion allows us to apply the extension theorem to construct a probability triple $(\Omega, \mathcal{M}, \mathbf{P}^*)$ over $\Omega = [0, 1]$. The $\sigma-$algebra $\mathcal{M}$ contains all intervals and the probability measure of any interval with endpoints a, b equals $b - a$.
- This probability triple is known as the uniform distribution or the **Lebesgue Measure**, often denoted by λ. The set in $\mathcal{M}$ is called the **Lebesgue Measurable Sets**.
- On the other hand, the Borel σ-algebra $\mathcal{B}$ is defined as smallest σ-algebra that contains all the intervals $\mathcal{J}$ (denoted as $\mathcal{B} = \sigma(\mathcal{J})$). Then $\mathcal{B}$ must be a subset of $\mathcal{M}$.
- The main difference between $\mathcal{B}$ and $\mathcal{M}$ is that $\mathcal{M}$ is complete but $\mathcal{B}$ is not. The completeness is defined as if $A \in \mathcal{M}$ and $\mathbf{P}^*(A) = 0$, then any subset B of A must belong to $\mathcal{M}$ as well.

Variation of Extension Theorem

- Here we provide an alternative formulation of the Extension Theorem

Corollary 2 (2.5.4)

In the original extension theorem, the finite superadditivity condition and the countable monotonicity condition of $\mathbf{P}$ *can be replaced by the following countable additivity condition:*
$\mathbf{P}(\bigcup_n A_n) = \sum_n \mathbf{P}(A_n)$ *for disjoint* $A_1, A_2, \ldots \in \mathcal{J}$ *with* $\bigcup_n A_n \in \mathcal{J}$.

Proof: We only need to show that the countable additivity implies both finite superadditivity and countable monotonicity.

Finite superadditivity follows immediately, so we just need to worry about the countable monotonicity.

Variation of Extension Theorem: continued

Proof (continued): To prove countable monotonicity based on finite additivity, we first prove a useful property of the semialgebra:
Property of Semialgebra: If $K_1, K_2, \ldots, K_m \in \mathcal{J}$ are disjoint, then we can find $J_1, J_2, \ldots, J_K \in \mathcal{J}$ so that $K_1, K_2, \ldots, K_m, J_1, J_2, \ldots, J_K$ forms a partition of Ω.

- $K_1, K_2, \ldots, K_m, (\bigcup_{i=1}^m K_i)^c$ forms a partition of Ω. Furthermore, $(\bigcup_{i=1}^m K_i)^c = \bigcap_{i=1}^m K_i^c$, and according to the definition of semialgebra, each K_i^c is the union of finite number of disjoint subsets of $\mathcal{J}$. Here we denote $K_i^c = \bigcup_{j=1}^{n_i} I_{ij}$ where $I_{ij} \in \mathcal{J}$, and $\{I_{ij} : 1 \leq j \leq n_i\}$ are disjoint.
- $\bigcap_{i=1}^m K_i^c = \bigcap_{i=1}^m \bigcup_{j=1}^{n_i} I_{ij} = \bigcup_{f \in F, 1 \leq f(i) \leq n_i} \bigcap_{i=1}^m I_{i,f(i)}$. By the second property of semialgebras, $\bigcap_{i=1}^m J_{i,f(i)} \in \mathcal{J}$. Also notice that the union is disjoint.
- Let $K = |F|$, list out functions $f_1, f_2, \ldots, f_K$, and define $J_j := \bigcap_{i=1}^m I_{i,f_j(i)}$.
- Then $\bigcap_{i=1}^m K_i^c = \bigcup_{j=1}^K J_j$, $J_j \in \mathcal{J}$ and $K_1, K_2, \ldots, K_m, J_1, J_2, \ldots, J_K$ forms a partition of Ω.

Variation of Extension Theorem: continued

Proof (continued): Then we have the following statement:
If $K_1, K_2, \ldots, K_m \in \mathcal{J}$ are disjoint and $\bigcup_{i=1}^m K_i \subseteq B \in \mathcal{J}$, then $\mathbf{P}(B) \geq \sum_{i=1}^m \mathbf{P}(K_i)$.

- By the result in the previous slide,there exist $\{J_j\}_{j=1}^K \in \mathcal{J}$ such that $K_1, K_2, \ldots, K_m, J_1, \ldots, J_K$ forms a partition of Ω.
- $\mathbf{P}(B) = \sum_{i=1}^m \mathbf{P}(B \cap K_i) + \sum_{j=1}^K \mathbf{P}(B \cap J_j) \geq \sum_{i=1}^m \mathbf{P}(B \cap K_i) = \sum_{i=1}^m \mathbf{P}(K_i)$. The last equality follows from our countable additivity hypothesis.

Variation of Extension Theorem: continued

Proof (continued): Now let $A, A_1, A_2, \ldots \in \mathcal{J}$ such that $A \subseteq \bigcup_n A_n$. To prove countable monotonicity, we need to show $\mathbf{P}(A) \leq \sum_n \mathbf{P}(A_n)$.

- Define $B_n := A \bigcap A_n$, then $B_n \in \mathcal{J}$ and $A = \bigcup_n B_n$,
- Define $C_n := B_n \bigcap (\bigcup_{i=1}^{n-1} B_i)^c$, then $C_1, C_2, \ldots$ are disjoint and $\bigcup_n B_n = \bigcup_n C_n = A$.
- As $(\bigcup_{i=1}^{n-1} B_i)^c$ can be represented as the unions of finite disjoint subsets from $\mathcal{J}$ and $\mathcal{J}$ is closed under intersection. We should also be able to represent $C_n = \bigcup_{j=1}^{k_n} J_{nj}$ where $J_{nj} \in \mathcal{J}$ are disjoint.
- Consequently, $\mathbf{P}(A) = \mathbf{P}(\bigcup_n B_n) = \mathbf{P}(\bigcup_n C_n) = \mathbf{P}(\bigcup_{n,j} J_{nj})$. By the countable additivity, $\mathbf{P}(A) = \sum_{n,j} \mathbf{P}(J_{nj}) = \sum_n (\sum_{j=1}^{k_n} \mathbf{P}(J_{nj}))$.
- By the result we derived from previous slide, $\bigcup_{j=1}^{k_n} J_{nj} = C_n \subseteq B_n$ implies $\sum_{j=1}^{k_n} \mathbf{P}(J_{nj}) \leq \mathbf{P}(B_n)$, and $B_n \subseteq A_n$ implies $\mathbf{P}(B_n) \leq \mathbf{P}(A_n)$.
- $\mathbf{P}(A) = \sum_n (\sum_{j=1}^{k_n} \mathbf{P}(J_{nj})) \leq \sum_n \mathbf{P}(B_n) \leq \sum_n \mathbf{P}(A_n)$