STAT 7200

Introduction to Advanced Probability
Lecture 16

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1. Convergence Theorems

Recap: The Monotone Convergence Theorem

Theorem 1 (Monotone Convergence Theorem)

Let $X_1, X_2, \ldots$ be a sequence of increasing non-negative random variables that converges to random variable X almost surely, then $\mathrm{E}(X_n) \to \mathrm{E}(X)$.

- In this lecture we will explore other sufficient conditions, including the dominated condition and the uniformly integrable condition. We'll start, however, with an import tool we'll need called Fatou's Lemma.

Fatou's Lemma

- The overall goal is to establish the equality: $\lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$.
- This equality can only be valid if $\lim_n \mathrm{E}(X_n)$ exists. However, there is no general guarantee that $\mathrm{E}(X_1), \mathrm{E}(X_2), \cdots$ converges.
- The MCT provides a condition where $\lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$. This chapter also mentioned the "Bounded Convergence Theorem," but we skipped chapter 7.
- For general cases, $\limsup_n \mathrm{E}(X_n)$ and $\liminf_n \mathrm{E}(X_n)$ always exist. As $\liminf_n \mathrm{E}(X_n) \leq \limsup_n \mathrm{E}(X_n)$, if we can show that $\limsup_n \mathrm{E}(X_n) \leq \mathrm{E}(X) \leq \liminf_n \mathrm{E}(X_n)$, we may conclude that $\lim_n \mathrm{E}(X_n) = \limsup_n \mathrm{E}(X_n) = \lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$
- We need the following famous lemma as the tool for studying $\limsup_n \mathrm{E}(X_n)$ and $\liminf_n \mathrm{E}(X_n)$.

Theorem 2 (Fatou's Lemma)

If $X_n \geq C > -\infty$ for all n ($\{X_n\}$ is bounded below) then

$$\mathrm{E}(\liminf_n X_n) \leq \liminf_n \mathrm{E}(X_n)$$

Fatou's Lemma: Proof

- **Proof:** The goal is to show $\mathrm{E}(\liminf_n X_n) \leq \liminf_n \mathrm{E}(X_n)$ when $X_n \geq C > -\infty$ for all n.
  Let $Y_n = \inf_{k \geq n} X_k$, then $C \leq Y_n \leq X_n$, then $\liminf_n \mathrm{E}(X_n) \geq \liminf_n \mathrm{E}(Y_n)$.
- Furthermore, as $\{Y_n\}$ forms an increasing sequence, its limit always exists. We define $Y = \lim_n Y_n = \liminf_n X_n$, then $\{Y_n\} \nearrow Y$. By the monotone convergence theorem:
  $\liminf_n \mathrm{E}(Y_n) = \lim_n \mathrm{E}(Y_n - C + C) = \lim_n \mathrm{E}(Y_n - C) + C = \mathrm{E}(Y - C) + C = \mathrm{E}(Y) = \mathrm{E}(\liminf_n X_n)$.
- **Remark** We often apply Fatou's lemma when $X_n \to X$, in this case, $\mathrm{E}(X) \leq \liminf_n \mathrm{E}(X_n)$.

The Dominated Convergence Theorem

Theorem 3 (Dominated Convergence Theorem)

Let $X, X_1, X_2, \ldots$ *be random variables with* $\mathrm{P}(\lim_n X_n = X) = 1$, *and if there is another random variable* Y *so that* $\mathrm{E}(Y) < \infty$ *and* $|X_n| \leq Y$ *for all* n, *then* $\lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$.

- **Proof** By the previous discussion, to prove $\lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$, we only need to show that $\limsup_n \mathrm{E}(X_n) \leq \mathrm{E}(X) \leq \liminf_n \mathrm{E}(X_n)$. This goal can be achieved using Fatou's lemma, and the "bounded below" condition will be established through the fact that $|X_n| \leq Y$.
- As $Y + X_n \geq 0$, by Fatou's lemma, $\mathrm{E}(Y)+\mathrm{E}(X) = \mathrm{E}(Y+X) \leq \liminf \mathrm{E}(Y+X_n) = \mathrm{E}(Y)+\liminf_n \mathrm{E}(X_n)$. Thus $\mathrm{E}(X) \leq \liminf_n \mathrm{E}(X_n)$.
- Similarly, as $Y - X_n \geq 0$, applying Fatou's lemma, we have $\mathrm{E}(-X) \leq \liminf_n \mathrm{E}(-X_n)$. That is $\mathrm{E}(X) \geq \limsup_n \mathrm{E}(X_n)$

Uniformly Integrable Condition

- For random variable X and any $\alpha > 0$, we may define the "truncated tail" of X as $|X|1_{|X|\geq\alpha}$. Then it is clear that $|X|1_{|X|\geq\alpha} \to 0$ as $\alpha \to \infty$ with probability 1.
- Furthermore, since $|X|1_{|X|\geq\alpha} \leq |X|$, if $\mathrm{E}(|X|) < \infty$, we can apply the dominate convergence theorem: $\lim_{\alpha\to\infty} \mathrm{E}(|X|1_{|X|\geq\alpha}) = \mathrm{E}(0) = 0$.
- For a sequence of random variables $\{X_n\}$ with $\mathrm{E}(|X_n|) < \infty$, although the expectation of the "truncated tail" of each X_n would converges to 0 as the threshold $\alpha \to \infty$. The convergence speed for each n might be different. To establish a convergence theorem for $\mathrm{E}(X_n)$, it is often necessary to make sure that such convergence speed is fast enough for each n. For this purpose, we propose the following conditions:
- **Uniformly Integrable** A collection of random variables $\{X_n\}$ is uniformly integrable if

$$\lim_{\alpha\to\infty} \sup_n \mathrm{E}(|X_n|1_{|X_n|\geq\alpha}) = 0.$$

Consequence of Uniformly Integrable Condition

Proposition 4

If $\{X_n\}$ *is uniformly integrable, then* $\sup_n \mathrm{E}(|X_n|) < \infty$. *And if* $\mathrm{P}(\lim_n X_n = X) = 1$, *then* $\mathrm{E}(|X|) < \infty$.

- **Proof:** As $|X_n| = |X_n| 1_{X_n < \alpha} + |X_n| 1_{X_n \geq \alpha}$, thus:

$$\begin{aligned} \sup_n \mathrm{E}(|X_n|) &= \sup_n \mathrm{E}(|X_n| 1_{X_n < \alpha} + |X_n| 1_{X_n \geq \alpha}) \\ &\leq \alpha + \sup_n \mathrm{E}(|X_n| 1_{X_n \geq \alpha}) \end{aligned}$$

  - Since $\lim_{\alpha \to \infty} \sup_n \mathrm{E}(|X_n| 1_{|X_n| \geq \alpha}) = 0$, we may choose a particular α_0 so that $\sup_n \mathrm{E}(|X_n| 1_{|X_n| \geq \alpha_0}) < 1$. Then $\sup_n \mathrm{E}(|X_n|) \leq \alpha_0 + 1 < \infty$.
  - By Fatou's lemma, $\mathrm{E}(|X|) \leq \liminf_n \mathrm{E}(|X_n|) \leq \sup_n \mathrm{E}(|X_n|) < \infty$.

The Uniform Integrability Convergence Theorem

Theorem 5 (Uniform Integrability Convergence Theorem)

Let $X, X_1, X_2, \ldots$ *be random variables with* $\mathrm{P}(\lim_n X_n = X) = 1$*, and if* $\{X_n\}$ *is uniformly integrable, then* $\lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$*.*