STAT 7200

Introduction to Advanced Probability
Lecture 4

Taylor R. Brown

"A First Look at Rigorous Probability Theory" (Jeffrey Rosenthal)
Sections 2.1, 2.2, 2.3

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

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The above steps lead to a probability triple $\{\Omega, \mathcal{M}, \mathrm{P}^*\}$.

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  a) $\emptyset, \Omega \in \mathcal{J}$.
  b) If $A_1, A_2, \ldots, A_k \in \mathcal{J}$, then $\bigcap_{i=1}^{k} A_i \in \mathcal{J}$. (Closed under finite intersections)
  c) If $A \in \mathcal{J}$, then there is a pairwise disjoint sequence of sets $B_1, B_2, \ldots, B_m \in \mathcal{J}$ such that $A^c = \bigcup_{i=1}^{m} B_i$.

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- **Remark** The notion of an "interval" includes singletons and open/closed/half-open/empty intervals.

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Proposition 2 (Exercise 2.2.5)

$\mathcal{B}_0 = \{$ *All finite unions of "intervals" in [0,1] (or* R$)\}$ *is an algebra.*

Extension Theorem

Theorem 3

The Extension Theorem *Let $\mathcal{J}$ be a semialgebra of subsets of Ω and* $\mathrm{P} : \mathcal{J} \to [0,1]$ *such that:*

a) $\mathrm{P}(\emptyset) = 0, \mathrm{P}(\Omega) = 1$.

b) $\mathrm{P}(\bigcup_{i=1}^k A_i) \geq \sum_{i=1}^k \mathrm{P}(A_i)$ *whenever* $A_1, \ldots, A_k \in \mathcal{J}$, $\bigcup_{i=1}^k A_i \in \mathcal{J}$, *and* $A_1, \ldots, A_k$ *are pairwise disjoint (finite superadditivity).*

c) $\mathrm{P}(A) \leq \sum_n \mathrm{P}(A_n)$ *whenever* $A, A_1, A_2, \ldots \in \mathcal{J}$, *and* $A \subseteq \bigcup_n A_n$ *(countable monotonicity).*

Then there is a σ-algebra $\mathcal{M} \supseteq \mathcal{J}$ and a countably-additive probability measure P^* on $\mathcal{M}$ such that $\mathrm{P}^*(A) = \mathrm{P}(A)$ for all $A \in \mathcal{J}$.

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Lemma 4

Outer measure satisfies the following properties:
a) $P^*(\emptyset) = 0$.
b) $P^*(A) \leq P^*(B)$ *if* $A \subseteq B$. *(Monotonicity)*
c) $P^*(A) = P(A)$ *if* $A \in \mathcal{J}$. *(*P^* *is an extension of* P*)*

Outer Measure P^* is Countably Subadditive

Lemma 5 (2.3.6.)

Outer measure P^* *is countably subadditive:*
$\mathrm{P}^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mathrm{P}^*(B_n)$ *for any* $B_1, B_2, \ldots \in \Omega$

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1) Given $\varepsilon > 0$, for each B_n, there must be a sequence $\{C_{nk}\}_{k=1}^{\infty}$, s.t. $C_{nk} \in \mathcal{J}$, $B_n \subseteq \bigcup_k C_{nk}$ and $\sum_k P(C_{nk}) < P^*(B_n) + \varepsilon/2^n$ (small typo in book)

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2) Since $\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n,k} C_{n,k}$,
$\mathrm{P}^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n,k} \mathrm{P}(C_{n,k}) < \sum_n \mathrm{P}^*(B_n) + \varepsilon$.
3) As ε is an arbitrary positive constant, we must have
$\mathrm{P}^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mathrm{P}^*(B_n)$.

$\mathcal{M}$: The Measurable Sets

- Outer measure cannot always be a probability measure over *all* subsets of Ω (recall Proposition 1.2.6). Define a refined collection of subsets using P^*:

$$\mathcal{M} = \{A \subseteq \Omega : \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E) = \mathrm{P}^*(E) \text{ for all } E \subseteq \Omega\}$$

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- **Remark:** We often need to verify that a given set $A \in \mathcal{M}$. By the countable subadditivity of our outer measure, we always have $\mathrm{P}^*(E) \leq \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E)$ for all $E \subseteq \Omega$. If it's easier, we only need need to verify $\mathrm{P}^*(E) \geq \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E)$ for all $E \subseteq \Omega$. This would can be achieved by using the finite superadditivity of P.

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- Regarding P^*, for any $A \in \mathcal{M}$, first, $\mathrm{P}^*(A) \geq 0$; second, in the definition of $\mathcal{M}$, by choosing $E = \Omega$, we have $\mathrm{P}^*(A) = 1 - \mathrm{P}^*(A^c)$. So we still need to show that, on $\mathcal{M}$, P^* is countably additive.

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- Countable additivity of P^* is shown in the next lecture, as well as the fact that $\mathcal{M}$ is a σ-field.