STAT 7200

Introduction to Advanced Probability
Lecture 17

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1. Convergence Theorems
   - Exchange Different Operators
   - Moment Generating Functions

Uniformly Integrable: what does it mean

- **Uniformly Integrable** A collection of random variables $\{X_n\}$ is uniformly integrable if

$$\lim_{\alpha \to \infty} \sup_n \mathrm{E}(|X_n| 1_{|X_n| \geq \alpha}) = 0.$$

- **Uniformly Integrable** If $\{X_n\}$ is uniformly integrable, then the expectation of $\{X_n\}$ is uniformly bounded: $\sup_n \mathrm{E}(|X_n|) < \infty$. And if $\mathrm{P}(\lim_n X_n = X) = 1$, then $\mathrm{E}(|X|) < \infty$.

The Uniform Integrability Convergence Theorem

Theorem 1 (Uniform Integrability Convergence Theorem)

Let $X, X_1, X_2, \ldots$ be random variables with $\mathrm{P}(\lim_n X_n = X) = 1$, and if $\{X_n\}$ is uniformly integrable, then $\lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$.

- Proof: Let $Y_n = |X_n - X|$, then if $\lim_n \mathrm{E}(Y_n) = 0$, $\lim_n \mathrm{E}(X_n) = \mathrm{E}(X)$ by Jensen's inequality. To utilize the condition of uniform integrability, we represent $Y_n = Y_n 1_{Y_n < \alpha} + Y_n 1_{Y_n \geq \alpha}$.

- For the first part, fix α, then $|Y_n 1_{Y_n < \alpha}| \leq \alpha$, and $Y_n 1_{Y_n < \alpha} \to 0$ pointwise/wp1. Thus, by the bounded convergence theorem:

$$\lim_n \mathrm{E}(Y_n 1_{Y_n < \alpha}) = 0.$$

The Uniform Integrability Convergence Theorem: continued

- **Proof: continued** For the second part, note that if $Y_n \geq \alpha$, as $|X| + |X_n| \geq Y_n$, we must have $|X_n| \geq \alpha/2$ or $|X| \geq \alpha/2$. Thus: when $|X| \geq |X_n|$, $Y_n 1_{Y_n \geq \alpha} \leq 2|X| 1_{Y_n \geq \alpha} \leq 2|X| 1_{|X| \geq \alpha/2}$; when $|X| < |X_n|$, $Y_n 1_{Y_n \geq \alpha} \leq 2|X_n| 1_{|X_n| \geq \alpha/2}$.
  - Then $Y_n 1_{Y_n \geq \alpha} \leq 2|X_n| 1_{|X_n| \geq \alpha/2} + 2|X| 1_{|X| \geq \alpha/2}$. Consequently,

$$\sup_n \mathrm{E}(Y_n 1_{Y_n \geq \alpha}) \leq 2 \sup_n \mathrm{E}(|X_n| 1_{|X_n| \geq \alpha/2}) + 2\mathrm{E}(|X| 1_{|X| \geq \alpha/2})$$

  - By uniform integrability, the first term would go to 0 as $\alpha \to \infty$, and the second term also goes to 0 as $\mathrm{E}(|X|) < \infty$. Thus:

- $$\lim_{\alpha \to \infty} \sup_n \mathrm{E}(Y_n 1_{Y_n \geq \alpha}) = 0.$$

The Uniform Integrability Convergence Theorem: continued

- **Proof: continued** Now we have $Y_n = Y_n 1_{Y_n < \alpha} + Y_n 1_{Y_n \geq \alpha}$, and

$$\lim_n \mathrm{E}(Y_n 1_{Y_n < \alpha}) = 0, \lim_{\alpha \to \infty} \sup_n \mathrm{E}(Y_n 1_{Y_n \geq \alpha}) = 0.$$

- Then for any $\varepsilon > 0$, we can first find $\alpha_0 > 0$, so that $\sup_n \mathrm{E}(Y_n 1_{Y_n \geq \alpha_0}) < \varepsilon/2$. Then for the fixed α_0, we can find $n_0(\alpha_0)$ so that $\mathrm{E}(Y_n 1_{Y_n < \alpha_0}) < \varepsilon/2$ for all $n \geq n_0(\alpha_0)$.
- Then for any $n \geq n_0(\alpha_0)$, we have:

$$\mathrm{E}(Y_n) = \mathrm{E}(Y_n 1_{Y_n < \alpha_0}) + \mathrm{E}(Y_n 1_{Y_n \geq \alpha_0}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Then $0 \leq |E[X_n - X]| \leq \mathrm{E}(Y_n) \to 0$ as $n \to \infty$.

Exchange Differentiation and Expectation

Theorem 2 (Exchange Differentiation and Expectation)

Let $\{F_t\}_{a<t<b}$ be a collection of random variables with finite expectations on a probability triple $(\Omega, \mathcal{F}, \mathrm{P})$. Suppose further that for each ω and $t \in (a, b)$, the derivative $F_t'(\omega) = \frac{\partial}{\partial t} F_t(\omega)$ exists. Furthermore, if there is a random variable Y on the same probability triple so that $\mathrm{E}(Y) < \infty$ and $|F_t'| \leq Y$ for all $t \in (a, b)$. Then:

1. *F_t' is a random variable with finite expectation;*
2. *2) $\phi(t)$ is differentiable with finite derivative $\phi'(t) = \mathrm{E}(F_t')$ for all $t \in (a, b)$, where $\phi(t) = \mathrm{E}(F_t)$.*

Exchange Differentiation and Expectation: Proof

- The conditions in the theorem can be summarized as: derivatives are dominated by a random variable with finite expectation.
- **Proof:**

  1) To show that the derivative is a random variable, note that:

$$F'_t = \lim_{h \to 0} \frac{F_{t+h} - F_t}{h}$$

So F'_t is a random variable as it is the limit of random variables.

- Furthermore, we have $\mathrm{E}(|F'_t|) \leq \mathrm{E}(Y) < \infty$.
- By the mean value theorem, there is always a t^* between $t + h$ and t, so that $\frac{F_{t+h} - F_t}{h} = F'_{t^*}$. Then $|\frac{F_{t+h} - F_t}{h}| \leq Y$. By DCT:

$$\phi'(t) = \lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h} = \lim_{h \to 0} \mathrm{E}\big(\frac{F_{t+h} - F_t}{h}\big)$$
$$= \mathrm{E}\big(\lim_{h \to 0} \frac{F_{t+h} - F_t}{h}\big) = \mathrm{E}(F'_t).$$

Moment Generating Function

- **Moment generating function** of random variable X:

$$M_X(s) = \mathrm{E}(e^{sX}), s \in \mathrm{R}.$$

- For instance, the moment generating function of a $N(0,1)$ distributed random variable is $M_X(s) = e^{s^2/2}$.
- If $X \perp Y$, then $M_{X+Y}(s) = M_X(s)M_Y(s)$.
- We always have $M_X(0) = 1$. But for certain $s \neq 0$, $M_X(s)$ might be infinity.

MGF expansions

Theorem 3

Let X be random variable such that $M_X(s) < \infty$ for $0 < |s| < s_0$. Then $\mathrm{E}(|X^n|) < \infty$ for all n. And for $|s| < s_0$, we have:
$M_X(s) = \sum_{k=0}^{\infty} \mathrm{E}(X^k)s^k/k!$. We also have $\mathrm{E}(X^r) = M_X^{(r)}(0)$.

- **Proof:** Write the Taylor expansion $e^{sX} = \sum_{k=0}^{\infty} X^k s^k/k!$.
  For $|s| < s_0$, define $Z_n = \sum_{k=0}^{n} X^k s^k/k!$, then $\lim_n Z_n = e^{sX}$, and $|Z_n| \leq \sum_{k=0}^{n} |Xs|^k/k! \leq \sum_{k=0}^{\infty} |Xs|^k/k! = e^{|sX|} \leq e^{sX} + e^{-sX}$.
  - Furthermore, as $|s| < s_0$, $\mathrm{E}(e^{sX} + e^{-sX}) = M_X(s) + M_X(-s) < \infty$. Thus, $\mathrm{E}(Z_n) \leq \mathrm{E}(e^{sX} + e^{-sX}) < \infty$. By the dominated convergence theorem, $M_X(s) = \mathrm{E}(\lim_n Z_n) = \lim_n \mathrm{E}(Z_n) = \sum_{k=0}^{\infty} \mathrm{E}(X^k)s^k/k!$.
  - The above proof also suggests that $\mathrm{E}(|X^n|) < \infty$.

Expansion of Moment Generating Function: continued

Now about using it to generate the moments.
We rely on the fact that power series are infinitely-differentiable:

$$
\begin{aligned}
M_X^{(r)}(s) &= \left[\sum_{k=0}^{\infty} \mathrm{E}(X^k) s^k / k!\right]^{(r)} \\
&= \sum_{k=0}^{r-1} 0 + \mathrm{E}[X^r] + \sum_{k=r+1}^{\infty} \mathrm{E}(X^k) \left[k \cdots (k-r+1)\right] \frac{s^{k-r}}{k!}
\end{aligned}
$$

Then plug in $s = 0$ to get $\mathrm{E}[X^r]$.