STAT 7200

Introduction to Advanced Probability
Lecture 2

Taylor R. Brown

"A First Look at Rigorous Probability Theory" (Jeffrey Rosenthal)
Sections A.3 and A.4

Limits: Limits of Sequences of Real Numbers

- **Limit of A Sequence of Real Numbers** A sequence of real numbers $x_1, x_2, \ldots$ converges to another real number x if, given any $\varepsilon > 0$, there is a $N \in \mathrm{N}$, such that $n > N$ implies $|x_n - x| < \varepsilon$.
  We denote this as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
- **Example** Show that $\lim_{n\to\infty} \frac{1}{n^k} = 0$.
- **Proof** First, choose an arbitrary $\varepsilon > 0$.
  Set $N := \left\lceil \frac{1}{\varepsilon^{1/k}} \right\rceil$. Then $n > N$ guarantees $|\frac{1}{n^k} - 0| < \varepsilon$.

Sequences that Converge to Infinity and Sequences without Limits

- **Converges to Infinity** A sequence of real numbers $x_1, x_2, \ldots$ converges to infinity if for any $M \in \mathrm{R}$, there is a $N \in \mathrm{N}$, such that $n > N$ implies $X_n > M$. We write this as $\lim_{n \to \infty} x_n = \infty$. We define the convergence to negative infinity in a similar fashion.
- **Example** $n^2 \to \infty$.
- There are sequences that do not have a finite or infinite limit (e.g. $0, 1, 0, 1, 0, 1, \cdots$, which oscillates between 0 and 1). These do not converge to anything, finite or infinite.

Properties of Limits

Theorem 1

If $\lim_{n\to\infty} x_n = x$, *and* $\lim_{n\to\infty} y_n = y$, *then*
1) For any a, $\lim_{n\to\infty} ax_n = ax$; *2)* $\lim_{n\to\infty}(x_n + y_n) = x + y$;
3) $\lim_{n\to\infty}(x_n y_n) = xy$; *4) If* $x > 0$, *then* $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$.

- **Proof** We only consider the situation in which both limits are finite.

  2): By definition, given any $\varepsilon > 0$, there are $N_1, N_2 \in \mathrm{N}$, such that $|x_n - x| < \varepsilon/2$ for $n > N_1$ and $|y_n - y| < \varepsilon/2$ for $n > N_2$.

  Now we let $N^* = \max(N_1, N_2)$, then for any $n > N^*$, $|x_n - x| < \varepsilon/2$ *and* $|y_n - y| < \varepsilon/2$.

  Furthermore, for $n > N^*$, we have,
  $|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
  Thus, $\lim_{n\to\infty}(x_n + y_n) = x + y$.

Properties of Limits (continued)

- 3) $\lim_{n\to\infty}(x_n y_n) = xy$
- **Proof:** The intuition is to show $|x_n y_n - xy|$ can be arbitrarily small for large enough n. This can be shown by the following inequality: $|x_n y_n - xy| = |x_n y_n - xy_n + xy_n - xy| \leq |y_n||x_n - x| + |x||y_n - y|$, in which $|y_n|$ approaches y, and $|x_n - x|$, $|y_n - y|$ approaches 0 for large n. A rigorous proof for the case $x \neq 0$ is shown below:

  a) For any $\varepsilon > 0$, there is $N_1 \in \mathrm{N}$ such that for any $n > N_1$, $|y_n - y| < \varepsilon/(2|x|)$.

  b) Choose any constant $\delta > 0$. Then there is $N_2 \in \mathrm{N}$ such that for any $n > N_2$, $|y_n - y| < \delta$, which further implies $|y_n| < |y| + \delta$.

  c) For the same $\varepsilon > 0$, there is $N_3 \in \mathrm{N}$ such that for any $n > N_3$, $|x_n - x| < \varepsilon/(2(|y| + \delta))$.

  d) Now we let $N^* = \max(N_1, N_2, N_3)$, then for any $n > N^*$,

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - xy_n + xy_n - xy| \leq |y_n||x_n - x| + |x||y_n - y| \\ &\leq (|y| + \delta)\varepsilon/(2(|y| + \delta)) + |x|\varepsilon/(2|x|) = \varepsilon \end{aligned}$$

  Thus, $\lim_{n\to\infty}(x_n y_n) = xy$.

Squeeze Theorem

Theorem 2

Suppose that we have three sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *that satisfy* $a_n \leq b_n \leq c_n$ *for all* n *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. *Then* $\lim_{n\to\infty} b_n = L$

- **Proof** For any $\varepsilon > 0$, there are $N_1, N_2 \in \mathrm{N}$, such that $|a_n - L| < \varepsilon$ for $n > N_1$ and $|c_n - L| < \varepsilon$ for $n > N_2$.

  Now we let $N^* = \max(N_1, N_2)$, then for any $n > N^*$, $|a_n - L| < \varepsilon$ *and* $|c_n - L| < \varepsilon$. These two inequalities further imply $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$.

  Thus, for $n > N^*$, $|b_n - L| < \varepsilon$. We have $\lim_{n\to\infty} b_n = L$
- **Example** $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ since $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

Limits Preserve Order

Theorem 3

Suppose that we have two sequences $\{a_n\}$, $\{b_n\}$ that satisfy $a_n \leq b_n$ for all n. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then $L \leq M$.

- **Proof** Assume to the contrary that $L > M$. Pick $\varepsilon > 0$ such that $M + \varepsilon < L - \varepsilon$ (e.g. $\varepsilon = (L - M)/4$)

  For this same $\varepsilon > 0$, pick $N \in \mathrm{N}$ such that $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$ for $n > N$. However, these two inequalities imply $a_n > L - \varepsilon > M + \varepsilon > b_n$ when $n > N$, which contradicts the hypothesis that $a_n \leq b_n$ for all n. Thus, $L \leq M$.

Sums of Infinite Sequences

- For a sequence $x_1, x_2, \ldots$, we define its sum as

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{i=1}^{n} x_i$$

- This boils down to a different sequence: the partial sums $s_n := \sum_{i=1}^{n} x_i$.
- For nonnegative sequences, the limit is either finite or infinite.
- **Examples**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty; \; \sum_{n=1}^{\infty} \frac{1}{n!} = e.$$

Sums of Infinite Sequences

Theorem 4

1) If $\sum_{n=1}^{\infty} x_n$ converges, then for every $\varepsilon > 0$, there is an $N \in \mathrm{N}$ such that $|\sum_{k=n+1}^{\infty} x_k| < \varepsilon$ for all $n > N$
2) Let $\{x_n\}_{n \in \mathrm{N}}$ and $\{y_n\}_{n \in \mathrm{N}}$ be two sequences of real numbers with $|x_n| < y_n$ for all n. If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges and $|\sum_{n=1}^{\infty} x_n| < \sum_{n=1}^{\infty} y_n$

Bounds and Limits

- A set $A \subseteq R$ is **bounded above (or below)** if there is a real number M such that $a \leq M$ (or $a \geq M$) for all $a \in A$. A set that is bounded above and below is called **bounded**.

Proposition 5

If $\lim_{n\to\infty} x_n = x$, *then* $\{x_n : n \in \mathrm{N}\}$ *is bounded.*

- **Proof** Choose $\varepsilon = 1$. Because $\lim_{n\to\infty} x_n = x$, we can find a large N where $|x_n - x| < 1$ for any $n > N$.

  Let $M = \max\{x_1, x_2, \cdots, x_N, x+1\}$, $L = \min\{x_1, x_2, \cdots, x_N, x-1\}$. Clearly $L \leq x_n \leq M$ for all $n \in \mathrm{N}$.

Supremum and Infimum

- **Supremum** For any nonempty subset A of R that is bounded above, the **supremum** or **least upper bound** is the number L such that 1) $a \leq L$ for all $a \in A$. 2) For any other upper bound L' of A, $L' \geq L$. The supremum of A is denoted by $\sup A$.
- **Infimum** Similarly, we can also define the **infimum** or **greatest lower bound** for any nonempty subset A of R that is bounded below as $\inf A$
- **Example**
  1) $\inf\{0, 1, 2, 3, \ldots\} = 0$;
  2) $\sup\{1/2, 2/3, 3/4, \ldots, n/(n+1), \ldots\} = 1$.
- **Exercise** Show that, if A and B are two nonempty subset of R, $A \subseteq B$, and if the corresponding suprema and infima exist, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

Properties of Supremum and Infimum

- Every nonempty subset of R that is bounded above has a supremum. Similarly, every nonempty subset R that is bounded below has an infimum.
- If a nonempty set A is not bounded below, we will denote $\inf A = -\infty$. Similarly, if A is not bounded above, $\sup A = \infty$.

Proposition 6

If A is a non-empty set that is bounded below. Then for any $\varepsilon > 0$, there is $a \in A$ with $\inf A \leq a < \inf A + \varepsilon$

- **Proof** If such a does not exist, then for all $a \in A$, we have $a \geq \inf A + \varepsilon$. That is, $\inf A + \varepsilon$ is a lower bound of A. However, by definition $\inf A$ is the greatest lower bound of A and we reach a contradiction.

Towards the Bolzano-Weierstrass Theorem

Lemma 7

A monotone increasing sequence that is bounded above converges (to a finite value). A monotone decreasing sequence that is bounded below converges (to a finite value).

- **Proof** Suppose that sequence $x_1, x_2, \ldots$ is a monotone increasing sequence that is bounded above. Then $x_n \to \sup\{x_n : n \in \mathrm{N}\}$. Why?

  For any $\varepsilon > 0$, since $L - \varepsilon$ can not be an upper bound of $\{x_n\}$, there must be a natural number N such that $x_N > L - \varepsilon$.

  However, since $\{x_n\}$ is a increasing sequence, for all $n > N$, $L \geq x_n \geq x_N > L - \varepsilon$.

  The inequality above suggests that $|x_n - L| < \varepsilon$ for all $n > N$. Thus, $\lim_{n\to\infty} x_n = L$.

Towards the Bolzano-Weierstrass Theorem

Lemma 8

Every real sequence x_n has a monotone subsequence x_{n_k}.

- **Proof** Define $S = \{n : x_m > x_n, \forall m > n\}$. This is either countably infinite or finite. If it's the first, write it as $\{n_1, n_2, \ldots\}$. Clearly x_{n_k} is monotone in this case.
- Suppose S is finite now. That means it's bounded, so there exists n_1 greater than all elements of S. This means $n_1 \notin S$. In other words, $n_1 \in S^c$.
- Looking at the definition of S, we see there exists $n_2 > n_1$ such that $x_{n_2} \leq x_{n_1}$. As $n_2 \notin S$, we can find $n_3, n_4, \ldots$. This mean x_{n_k} is monotonically decreasing.

The Bolzano-Weierstrass Theorem

Theorem 9

Every bounded real sequence x_n has a convergent subsequence x_{n_k}.

- **Proof** Just use the previous two lemmas.

Limit Superior and Limit Inferior

- **Limit Superior and Limit Inferior** For $x_1, x_2, \ldots$, the **limit inferior** is defined as $\liminf_{n\to\infty} x_n = \lim_{n\to\infty}(\inf_{m\geq n} x_m)$ the **limit superior** is defined as $\limsup_{n\to\infty} x_n = \lim_{n\to\infty}(\sup_{m\geq n} x_m)$
- **Exercise** Find the limit superior and limit inferior for $0, 1, 0, 1, \cdots$?
- Both limit superior and limit inferior exist (maybe infinity). For this, note that both $\{\inf_{m\geq n} x_m\}_{n=1}^{\infty}$ and $\{\sup_{m\geq n} x_m\}_{n=1}^{\infty}$ are monotone sequences.

Proposition 10

$\inf_n x_n \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup_n x_n$

Limit Superior, Limit Inferior and Limit

Theorem 11

$\lim_{n\to\infty} x_n$ *exists if and only if* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$

- **Proof** Let $\{v_n : v_n = \inf_{m\geq n} x_m\}$ and $\{u_n : u_n = \sup_{m\geq n} x_m\}$, then $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} v_n$ and $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} u_n$. Note that for all n, we have $v_n \leq x_n \leq u_n$.

  1) "if" part: By the Squeeze Theorem,
  if $\lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n = x$, we must have $\lim_{n\to\infty} x_n = x$.

  2) "only if" part: If $\lim_{n\to\infty} x_n = x$, then for any ε, there is a $N \in \mathrm{N}$, such that for $n > N$, $x - \varepsilon < x_n < x + \varepsilon$.

  Consequently, we deduce that, for $n > N$, $x - \varepsilon \leq v_n \leq u_n \leq x + \varepsilon$. Thus, $x - \varepsilon \leq \lim_{n\to\infty} v_n \leq \lim_{n\to\infty} u_n \leq x + \varepsilon$. Furthermore, since ε is arbitrary. we must have $x \leq \lim_{n\to\infty} v_n \leq \lim_{n\to\infty} u_n \leq x$. Thus, $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Example

- **Problem:** Let $\{x_n\}_{n\in \mathrm{N}}$ and $\{y_n\}_{n\in \mathrm{N}}$ be two sequences of real numbers with $y_n \geq 0$ for all n such that $\limsup_{n\to\infty} \frac{|x_n|}{y_n} < \infty$ and $\sum_{n=1}^{\infty} y_n < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges .
- **Proof:** The key here is to show that $|x_n|$ is bounded by y_n times a positive constant.

  Since $\limsup_{n\to\infty} \frac{|x_n|}{y_n} = \lim_{n\to\infty}(\sup_{m\geq n} \frac{|x_m|}{y_m})$ converges, $\sup_n \frac{|x_n|}{y_n}$ must be finite and positive.

  Assuming that $\sup_n \frac{|x_n|}{y_n} = M > 0$, then for any n, $\frac{|x_n|}{y_n} \leq M$, and $|x_n| \leq My_n$.

  However, as $\sum_{n=1}^{\infty} y_n < \infty$, $\sum_{n=1}^{\infty} My_n$ is also finite. That is, $|x_n|$ is bounded by a sequence whose sum converges, then $\sum_{n=1}^{\infty} x_n$ also converges and $|\sum_{n=1}^{\infty} x_n| \leq M \sum_{n=1}^{\infty} y_n$.

Exchange Summation and Limit

Theorem 12

Let $\{x_{nk}\}_{n,k\in\mathrm{N}}$ be a collection of real numbers, such that $\lim_{n\to\infty} x_{nk} = a_k$ for each fixed k. If $\sum_{k=1}^{\infty} \sup_n |x_{nk}| < \infty$, then
$\lim_{n\to\infty} \sum_{k=1}^{\infty} x_{nk} = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lim_{n\to\infty} x_{nk}$

- **Proof** For any fixed k, $|a_k| = |\lim_{n\to\infty} x_{nk}| \le \sup_n |x_{nk}|$, so $\sum_{k=1}^{n} |a_k| < \infty$.
  We now need to prove that
  $|\sum_{k=1}^{\infty} x_{nk} - \sum_{k=1}^{\infty} a_k| = |\sum_{k=1}^{\infty}(x_{nk} - a_k)|$ is smaller than any $\varepsilon > 0$ for large n. To achieve this, we should break this sum into two parts:
  $|\sum_{k=1}^{\infty}(x_{nk} - a_k)| \le |\sum_{k=1}^{K}(x_{nk} - a_k)| + |\sum_{k=K+1}^{\infty}(x_{nk} - a_k)|$.
  1) For the second sum, note that
  $|\sum_{k=K+1}^{\infty}(x_{nk} - a_k)| \le \sum_{k=K+1}^{\infty} |x_{nk} - a_k| \le 2\sum_{k=K+1}^{\infty} \sup_n |x_{nk}|$.
  However, since $\sum_{k=1}^{\infty} \sup_n |x_{nk}| < \infty$, we should be able to choose K big enough such that $\sum_{k=K+1}^{\infty} \sup_n |x_{nk}| < \varepsilon/4$.

Exchange Sum and Limit: continued

- **Proof: continued** Our goal is to show that $|\sum_{k=1}^{\infty}(x_{nk}-a_k)| \leq |\sum_{k=1}^{K}(x_{nk}-a_k)| + |\sum_{k=K+1}^{\infty}(x_{nk}-a_k)| < \varepsilon$ for big n, and we have already proved that we can choose K big enought such that $|\sum_{k=K+1}^{\infty}(x_{nk}-a_k)| < \varepsilon/2$.

  2) For the first sum, since $|\sum_{k=1}^{K}(x_{nk}-a_k)| \leq \sum_{k=1}^{K}|(x_{nk}-a_k)|$, and $\lim_{n\to\infty} x_{nk} = a_k$. Then for each $1 \leq k \leq K$, we can find $N_K \in \mathrm{N}$ such that for $n > N_k$, $|x_{nk}-a_k| < \varepsilon/(2K)$.

  If we choose $N^* = \max(N_1, N_2, \cdots, N_K)$, then for all $n > N^*$, $\sum_{k=1}^{K}|(x_{nk}-a_k)| < \sum_{k=1}^{K}\varepsilon/(2K) = \varepsilon/2$.

  3) Now combine the results in both 1) and 2), we conclude that $|\sum_{k=1}^{\infty}(x_{nk}-a_k)| < \varepsilon$ for $n > N^*$. Thus, $\lim_{n\to\infty}\sum_{k=1}^{\infty} x_{nk} = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\lim_{n\to\infty} x_{nk}$. That is, the exact order of taking limit with respect to n and summing over k does not matter.