STAT 7200
Introduction to Advanced Probability
Lecture 24

Taylor R. Brown

1. Weak Convergence
   - Characteristic Function
   - Continuity Theorem
   - Central Limit Theorem

Triangular Array

- **Triangular Arrays:** A collection of random variables
  $\{Z_{nk} : n \geq 1, 1 \leq k \leq r_n\}$ so that the r.v.s. in each row are
  independent.

$$
\begin{array}{llll}
Z_{11}, \cdots, & Z_{1r_1} & & \\
Z_{21}, \cdots, & \cdots, & Z_{2r_2} & \\
\cdots & & & \\
Z_{n1}, \cdots, & \cdots, & \cdots & Z_{nr_n} \\
\cdots & & &
\end{array}
$$

- We will assume that $E(Z_{nk}) = 0$ and $Var(Z_{nk}) = \sigma_{nk}^2$. Set
  $S_n = Z_{n1} + \cdots + Z_{nr_n}$, and $s_n^2 = Var(S_n) = \sigma_{n1}^2 + \cdots + \sigma_{nr_n}^2$. Then
  the question is, under what condition, the distribution of S_n/s_n would
  converge to $N(0,1)$?

Lyapunov Condition Implies Lindeberg Condition

- For a triangular array $\{Z_{nk}\}$, if for some $\delta > 0$

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{E(Z_{nk}^{2+\delta})}{s_n^{2+\delta}} = 0,$$

then for any $\varepsilon > 0$.

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{r_n} E(Z_{nk}^2 1_{|Z_{nk}| \geq \varepsilon s_n}) = 0.$$

- **Proof:**

$$\frac{1}{s_n^2} \sum_{k=1}^{r_n} E(Z_{nk}^2 1_{|Z_{nk}| \geq \varepsilon s_n}) \leq \frac{1}{s_n^2} \sum_{k=1}^{r_n} E[Z_{nk}^2 (\frac{Z_{nk}}{\varepsilon s_n})^\delta 1_{|Z_{nk}| \geq \varepsilon s_n}]$$

$$= \frac{1}{\varepsilon^\delta} \sum_{k=1}^{r_n} E(\frac{Z_{nk}^{2+\delta}}{s_n^{2+\delta}} 1_{|Z_{nk}| \geq \varepsilon s_n})$$

$$\leq \frac{1}{\varepsilon^\delta} \sum_{k=1}^{r_n} E(\frac{Z_{nk}^{2+\delta}}{s_n^{2+\delta}}) \to 0.$$

Infinitely Divisible Distributions

- The limit of the triangular array is not restricted to the Normal distribution.
- **Example:** If $Z_{nk} \sim Bern(\lambda/n)$, then $\mathcal{L}(S_n)$ would converge weakly to $Poisson(\lambda)$.
- Generally speaking, in the triangular array, if:
    1. $\sup_n s_n^2 < \infty$;
    2. the uniformly asymptotically negligible (u.a.n.) (or uniform smallness) condition holds:
    $$u_n = \frac{\max_{1 \leq k \leq n_r} \sigma_{nk}^2}{s_n^2} \to 0$$
    3. The weak limit of $\mathcal{L}(S_n)$ exists,

    then the limit must be an **infinitely divisible** distribution.

Infinitely Divisible Distributions

- **Infinitely Divisible Distribution** A distribution μ is called infinitely divisible if for any $n \in \mathbb{N}$, we can find i.i.d. random variables $X_1, X_2, \cdots, X_n$ such that $X_1 + X_2 + \cdots + X_n \sim \mu$.
- Examples of infinitely divisible distributions are: Normal, Poisson, Cauchy...
- On the other hand, an infinitely divisible distribution must be the limit of some triangular array.

On the Proof of CLT

- Our proof of the classical version of CLT relies on the following fact: For any random variables X with finite second moments:

$$\phi(t) = 1 + it\mathsf{E}(X) - \frac{t^2}{2}\mathsf{E}(X^2) + o(|t^2|).$$

- Here we will discuss why the remainder term is of the order $o(|t^2|)$.

On the Proof of CLT: continued

- In Taylor's inequality states:

$$|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}| \leq \min\{\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\}$$

For the proof, see the document I uploaded to Collab. Also, see the Billingsley textbook I referenced.

- If the n-th moment of random variable X is finite, replace x with Xt, then take expectations:

$$|\phi(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!} E(X^k)| \leq E\big[\min\{\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\}\big]$$

On the Proof of CLT: continued

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$$|\phi(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!} \mathsf{E}(X^k)| \leq \mathsf{E}\big[\min\{\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\}\big]$$

- And

$$\lim_{|t|\to 0} \mathsf{E}\big[\min\{\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\}\big]/|t|^n = \lim_{|t|\to 0} \mathsf{E}\big[\min\{\frac{|t||X|^{n+1}}{(n+1)!}, \frac{2|X|^n}{n!}\}\big]$$
$$= \mathsf{E}\big[\lim_{|t|\to 0} \min\{\frac{|t||X|^{n+1}}{(n+1)!}, \frac{2|X|^n}{n!}\}\big] = 0$$

The exchange of limit and expectation is justified based on the dominated convergence theorem. Thus:
$$\phi(t) = \sum_{k=0}^{n} \frac{(it)^k}{k!} \mathsf{E}(X^k) + o(|t^n|)$$

Proof to Lindeberg Central Limit Theorem

Theorem 1 (Lindeberg Central Limit Theorem)

For triangular array $\{Z_{nk}\}$, if the following Lindeberg condition holds:

$$\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{k=1}^{r_n} \mathsf{E}(Z_{nk}^2 1_{|Z_{nk}| \geq \varepsilon s_n}) = 0,$$

for any $\varepsilon > 0$, then $\mathcal{L}(S_n/s_n)$ converges weakly to $N(0,1)$.

- **Proof:** WLOG, we may assume $s_n^2 = 1$ (otherwise we can just define $Z_{nk}^* = Z_{nk}/s_n$. Then our goal is to show that, if $\lim_{n \to \infty} \sum_{k=1}^{r_n} \mathsf{E}(Z_{nk}^2 1_{|Z_{nk}| \geq \varepsilon}) = 0$, $\mathcal{L}(S_n)$ converges weakly to $N(0,1)$.
- Based on the continuity theorem, we only need to show:

$$\phi_{S_n}(t) = \prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) \to e^{-t^2/2}.$$

Proof to Lindeberg Central Limit Theorem: continued

- **continued:** By the expansion of characteristic function, we have:

$$\prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) = \prod_{k=1}^{r_n} [1 - \frac{t^2}{2}\sigma_{nk}^2 + o(|t|^2)].$$

- As $1 = s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2$,

$$e^{-\frac{t^2}{2}} = \prod_{k=1}^{r_n} e^{-\frac{t^2}{2}\sigma_{nk}^2}.$$

- We may investigate

$$|\prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) - e^{-\frac{t^2}{2}}| \le |\prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) - \prod_{k=1}^{r_n} (1 - \frac{t^2}{2}\sigma_{nk}^2)|$$

$$+ |\prod_{k=1}^{r_n} (1 - \frac{t^2}{2}\sigma_{nk}^2) - \prod_{k=1}^{r_n} e^{-\frac{t^2}{2}\sigma_{nk}^2}|$$

- If both terms on the right side of inequality converges to 0 as $n \to \infty$, we can prove $\phi_{S_n}(t) = \prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) \to e^{-t^2/2}$.

Proof to Lindeberg Central Limit Theorem: continued

- **continued:** To establish the limit behaviors of the aforementioned terms, we need the following lemma:

Lemma 2

Let $z_1, \cdots, z_m$, $w_1, \cdots, w_m$ be complex numbers, and $|z_k| \leq 1, |w_k| \leq 1$ for $1 \leq k \leq n$. Then:

$$|z_1 \cdots z_m - w_1 \cdots w_m| \leq \sum_{k=1}^{m} |z_k - w_k|$$

- When $m = 1$, the equality clearly holds. For $m + 1$,

$$|z_1 \cdots z_m z_{m+1} - w_1 \cdots w_m w_{m+1}|$$
$$= |z_1 \cdots z_m z_{m+1} - w_1 \cdots w_m z_{m+1} + w_1 \cdots w_m z_{m+1} - w_1 \cdots w_m w_{m+1}|$$
$$\leq |z_1 \cdots z_m - w_1 \cdots w_m||z_{m+1}| + |w_1 \cdots w_m||z_{m+1} - w_{m+1}|$$
$$\leq \sum_{k=1}^{m+1} |z_k - w_k|$$

Proof to Lindeberg Central Limit Theorem: continued

- **continued:** Now apply the lemma, we have:

$$|\prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) - \prod_{k=1}^{r_n}(1 - \frac{t^2}{2}\sigma_{nk}^2)| \leq \sum_{k=1}^{r_n} |\phi_{Z_{nk}}(t) - 1 + \frac{t^2}{2}\sigma_{nk}^2|$$

- Based on our discussion of the expansion of characteristic function, $|\phi(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!}E(X^k)| \leq E[\min\{\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\}]$, thus:

$$|\phi_{Z_{nk}}(t) - 1 + \frac{t^2}{2}\sigma_{nk}^2| \leq E[\min\{|tZ_{nk}|^3, |tZ_{nk}|^2\}]$$

- For fixed t, $|tZ_{nk}|^3 \leq |tZ_{nk}|^2$ if Z_{nk} is small enough. Then for small $\varepsilon > 0$,

$$E[\min\{|tZ_{nk}|^3, |tZ_{nk}|^2\}] \leq E[|tZ_{nk}|^3 1_{|Z_{nk}|<\varepsilon}] + E[|tZ_{nk}|^2 1_{|Z_{nk}|\geq\varepsilon}]$$
$$\leq |t|^3 \varepsilon \sigma_{nk}^2 + |t|^2 E[|Z_{nk}|^2 1_{|Z_{nk}|\geq\varepsilon}]$$

Proof to Lindeberg Central Limit Theorem: continued

- **continued:** Therefore,

$$
\begin{aligned}
|\prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) - \prod_{k=1}^{r_n}(1 - \frac{t^2}{2}\sigma_{nk}^2)| &\leq \sum_{k=1}^{r_n} |\phi_{Z_{nk}}(t) - 1 + \frac{t^2}{2}\sigma_{nk}^2| \\
&= \varepsilon|t|^3 \sum_{k=1}^{r_n} \sigma_{nk}^2 + |t|^2 \sum_{k=1}^{r_n} \mathsf{E}[|Z_{nk}|^2 1_{|Z_{nk}| \geq \varepsilon}] \\
&= \varepsilon|t|^3 + |t|^2 \sum_{k=1}^{r_n} \mathsf{E}[|Z_{nk}|^2 1_{|Z_{nk}| \geq \varepsilon}]
\end{aligned}
$$

- As the choice of ε is arbitrary, , and for fixed $\varepsilon > 0$, the second term converges to 0 as n goes to infinity. We can then conclude that, for any fixed t,

$$
|\prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) - \prod_{k=1}^{r_n}(1 - \frac{t^2}{2}\sigma_{nk}^2)| \to 0,
$$

as $n \to \infty$.

Proof to Lindeberg Central Limit Theorem: continued

- **continued:** Now we may turn our attention to:

$$|\prod_{k=1}^{r_n}(1 - \frac{t^2}{2}\sigma_{nk}^2) - \prod_{k=1}^{r_n} e^{-\frac{t^2}{2}\sigma_{nk}^2}| \le \sum_{k=1}^{r_n}|e^{-\frac{t^2}{2}\sigma_{nk}^2} - 1 + \frac{t^2}{2}\sigma_{nk}^2|$$

- But for any complex number z,

$$|e^z - 1 - z| = |\sum_{k=2}^{\infty}\frac{z^k}{k!}| \le |z^2|\sum_{k=2}^{\infty}\frac{|z^{k-2}|}{(k-2)!} = |z^2||e^{|z|}|.$$

- Thus,

$$|\prod_{k=1}^{r_n}(1 - \frac{t^2}{2}\sigma_{nk}^2) - \prod_{k=1}^{r_n} e^{-\frac{t^2}{2}\sigma_{nk}^2}| \le \sum_{k=1}^{r_n}\frac{|t|^4}{4}\sigma_{nk}^4 e^{\frac{|t|^2}{2}\sigma_{nk}^2}$$

$$\le \frac{|t|^4}{4}u_n e^{\frac{|t|^2}{2}u_n}\sum_{k=1}^{r_n}\sigma_{nk}^2 = \frac{|t|^4}{4}u_n e^{\frac{|t|^2}{2}u_n}.$$

where $u_n = \max_{1 \le k \le r_n}\sigma_{nk}^2$.

Proof to Lindeberg Central Limit Theorem: continued

- **continued:** Apply the Lindeberg condition, we can show that,
  $u_n = \max_{1 \leq k \leq r_n} \sigma_{nk}^2 \to 0$ as $n \to \infty$ (uniform smallness condition).

- For any $\varepsilon > 0$, we have:

$$\sigma_{nk}^2 = E(Z_{nk}^2) = E(Z_{nk}^2 1_{|Z_{nk}| < \varepsilon}) + E(Z_{nk}^2 1_{|Z_{nk}| \geq \varepsilon}) \leq \varepsilon^2 + E(Z_{nk}^2 1_{|Z_{nk}| \geq \varepsilon})$$

- Thus,

$$u_n \leq \varepsilon^2 + \sum_{k=1}^{n_r} E(Z_{nk}^2 1_{|Z_{nk}| \geq \varepsilon})$$

- By Lindeberg condition, we have $u_n \to 0$ as $n \to \infty$., then,

$$|\prod_{k=1}^{r_n} (1 - \frac{t^2}{2}\sigma_{nk}^2) - \prod_{k=1}^{r_n} e^{-\frac{t^2}{2}\sigma_{nk}^2}| \leq \frac{|t|^4}{4} u_n e^{\frac{|t|^2}{2} u_n} \to 0.$$

- Combine the results above, we have: $|\prod_{k=1}^{r_n} \phi_{Z_{nk}}(t) - e^{-\frac{t^2}{2}}| \to 0$, as $n \to \infty$. Thus, $\mathcal{L}(S_n)$ converges weakly to $N(0, 1)$.

Lindeberg Condition and Central Limit Theorem

- As we have discussed, Lindeberg condition is a sufficient condition for the central limit theorem. Conversely, we might ask, if the central limit theorem holds for the triangular array, would the Lindeberg condition hold? That is, is the Lindeberg condition a necessary condition for the CLT?

- Generally speaking, Lindeberg condition is not necessary. However, if the u.a.n. condition holds ($u_n = \max_{1 \leq k \leq r_n} \frac{\sigma_{nk}^2}{s_n^2} \to 0$, in the proof of Lindeberg CLT, we also show that Lindeberg condition implies u.a.n. condition), then Lindeberg condition is necessary.

Theorem 3 (Lindeberg-Feller Theorem)

For triangular array $\{Z_{nk}\}$, if the u.a.n. condition holds, then the Lindeberg condtion is necessary and sufficient for the validity of the central limit theorem: $\mathcal{L}(S_n/s_n)$ converges weakly to $N(0,1)$.