STAT 7200

Introduction to Advanced Probability
Lecture 4

Taylor R. Brown

1. Probability Triples
   - Extension Theorem
     - Constructing Probability Triples
     - Semialgebra
     - Algebra
     - Extension Theorem
     - Outer Measure $\mathbf{P}^*$
     - Outer Measure $\mathbf{P}^*$ is Countably Subadditive
     - $\mathcal{M}$: The Measurable Sets
     - $\mathcal{M}$ and $\mathbf{P}^*$

"A First Look at Rigorous Probability Theory" (Jeffrey Rosenthal)
Sections 2.1, 2.2, 2.3

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or $\mathbf{R}$.

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The above steps lead to a probability triple $\{\Omega, \mathcal{M}, \mathbf{P}^*\}$.

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  a) $\emptyset, \Omega \in \mathcal{J}$.
  b) If $A_1, A_2, \cdots, A_k \in \mathcal{J}$, then $\bigcap_{i=1}^{k} A_i \in \mathcal{J}$. (Closed under finite intersections)
  c) If $A \in \mathcal{J}$, then there is a pairwise disjoint sequence of sets $B_1, B_2, \cdots, B_m \in \mathcal{J}$ so that $A^c = \bigcup_{i=1}^{m} B_i$.

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Proposition 1 (Exercise 2.2.3)

$\mathcal{J} = \{$ All "intervals" contained in [0,1] (or $\mathbf{R}$)$\}$ is a semialgebra.

- **Remark** The notion of an "interval" includes singletons and open/closed/half-open/empty intervals.

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  Let $\mathcal{B}_0$ be a collection of subsets of Ω. $\mathcal{B}_0$ is an **algebra** if
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Proposition 2 (Exercise 2.2.5)

$\mathcal{B}_0 = \{$ *All finite unions of "intervals" in [0,1] (or* $\mathbf{R}$) $\}$ *is an algebra.*

Extension Theorem

Theorem 3

The Extension Theorem *Let $\mathcal{J}$ be a semialgebra of subsets of Ω and $\mathbf{P} : \mathcal{J} \to [0,1]$ such that:*

a) $\mathbf{P}(\emptyset) = 0, \mathbf{P}(\Omega) = 1$.

b) $\mathbf{P}(\bigcup_{i=1}^k A_i) \geq \sum_{i=1}^k \mathbf{P}(A_i)$ *whenever* $A_1, \cdots, A_k \in \mathcal{J}$, $\bigcup_{i=1}^k A_i \in \mathcal{J}$, *and* $A_1, \cdots, A_k$ *are pairwise disjoint (finite superadditivity).*

c) $\mathbf{P}(A) \leq \sum_n \mathbf{P}(A_n)$ *whenever* $A, A_1, \cdots, A_n, \cdots \in \mathcal{J}$, *and* $A \subset \bigcup_n A_n$ *(countable monotonicity).*

Then there is a σ-algebra $\mathcal{M} \supseteq \mathcal{J}$ and a countably-additive probability measure $\mathbf{P}^*$ on $\mathcal{M}$ so that $\mathbf{P}^*(A) = \mathbf{P}(A)$ for all $A \in \mathcal{J}$.

Outer Measure $\mathbf{P}^*$

- **Remarks:** $\mathbf{P}^*$ is a function defined over *all* subsets of Ω (but its definition makes use of only sets from $\mathcal{J}$) but it isn't necessarily a probability measure if you're looking at all of these subsets.

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- For any $A \subseteq \Omega$

$$\mathbf{P}^*(A) := \inf_{A_1, A_2 \cdots, \in \mathcal{J}, A \subset \bigcup_i A_i} \sum_i \mathbf{P}(A_i)$$

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Lemma 4

Outer measure satisfies the following properties:
a) $\mathbf{P}^*(\emptyset) = 0$.
b) $\mathbf{P}^*(A) \leq \mathbf{P}^*(B)$ *if* $A \subseteq B$. *(Monotonicity)*
c) $\mathbf{P}^*(A) = \mathbf{P}(A)$ *if* $A \in \mathcal{J}$. *(*$\mathbf{P}^*$ *is an extension of* $\mathbf{P}$*)*

Outer Measure $\mathbf{P}^*$ is Countably Subadditive

Lemma 5 (2.3.6.)

Outer measure $\mathbf{P}^*$ *is countably subadditive:*
$\mathbf{P}^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mathbf{P}^*(B_n)$ *for any* $B_1, B_2, \cdots \in \Omega$

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1) Given $\varepsilon > 0$, for each B_n, there must be a sequence $\{C_{nk}\}_{k=1}^{\infty}$, s.t. $C_{nk} \in \mathcal{J}$, $B_n \subseteq \bigcup_k C_{nk}$ and $\sum_k \mathbf{P}(C_{nk}) < \mathbf{P}^*(B_n) + \varepsilon/2^n$ (small typo in book)

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2) Since $\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n,k} C_{n,k}$,
$\mathbf{P}^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n,k} \mathbf{P}(C_{n,k}) < \sum_n \mathbf{P}^*(B_n) + \varepsilon$.
3) As ε is an arbitrary positive constant, we must have
$\mathbf{P}^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mathbf{P}^*(B_n)$.

$\mathcal{M}$: The Measurable Sets

- Outer measure cannot always be a probability measure over *all* subsets of Ω (recall Proposition 1.2.6). Define a refined collection of subsets using $\mathbf{P}^*$:

$$\mathcal{M} = \{A \subseteq \Omega : \mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E) = \mathbf{P}^*(E) \text{ for all } E \subseteq \Omega\}$$

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We have the following results regarding $\mathcal{M}$:
a) $\emptyset \in \mathcal{M}, \Omega \in \mathcal{M}$
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- **Remark:** We often need to verify that a given set $A \in \mathcal{M}$. By the countable subadditivity of our outer measure, we always have $\mathbf{P}^*(E) \leq \mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E)$ for all $E \subset \Omega$. If it's easier, we only need need to verify $\mathbf{P}^*(E) \geq \mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E)$ for all $E \subseteq \Omega$. This would can be achieved by using the finite superadditivity of $\mathbf{P}$.

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- Regarding $\mathbf{P}^*$, for any $A \in \mathcal{M}$, first, $\mathbf{P}^*(A) \geq 0$; second, in the definition of $\mathcal{M}$, by choosing $E = \Omega$, we have $\mathbf{P}^*(A) = 1 - \mathbf{P}^*(A^c)$. So we still need to show that, on $\mathcal{M}$, $\mathbf{P}^*$ is countably additive.

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- Countable additivity of $\mathbf{P}^*$ is shown in the next lecture, as well as the fact that $\mathcal{M}$ is a sigma-field.