STAT 7200

Introduction to Advanced Probability
Lecture 5

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1. Probability Triple
   - Extension Theorem
   - Review From Last Lecture
   - $\mathbf{P}^*$ is Countably Additive over $\mathcal{M}$
   - $\mathcal{M}$ is Closed under Finite Intersections/Unions
   - $\mathcal{M}$ is Closed under Countable Unions of Disjoint Sets
   - $\mathcal{M}$ is a σ-algebra
   - $\mathcal{J} \subseteq \mathcal{M}$

"A First Look at Rigorous Probability Theory" (Jeffrey Rosenthal) Section 2.3 (continued)

Extension Theorem

Theorem 1

The Extension Theorem *Let $\mathcal{J}$ be a semialgebra of subsets of Ω,* **P** *a function from $\mathcal{J}$ to [0,1] with the following properties:*

a) $\mathbf{P}(\emptyset) = 0, \mathbf{P}(\Omega) = 1$.

b) $\mathbf{P}(\bigcup_{i=1}^{k} A_i) \geq \sum_{i=1}^{k} \mathbf{P}(A_i)$ *whenever* $A_1, \ldots, A_k \in \mathcal{J}$, $\bigcup_{i=1}^{k} A_i \in \mathcal{J}$, *and* $A_1, \ldots, A_k$ *are pairwise disjoint (finite superadditivity).*

c) $\mathbf{P}(A) \leq \sum_n \mathbf{P}(A_n)$ *whenever* $A, A_1, \ldots, A_n, \ldots \in \mathcal{J}$, *and* $A \subseteq \bigcup_n A_n$ *(countable monotonicity).*

Then there is a σ-algebra $\mathcal{M} \supseteq \mathcal{J}$ and a probability measure $\mathbf{P}^*$ on $\mathcal{M}$ so that $\mathbf{P}^*(A) = \mathbf{P}(A)$ for all $A \in \mathcal{J}$.

Constructing Probability Triples I

Goal: constructing complicated probability triples on sample space Ω

1) Select $\mathcal{J}$, a collection of subsets of Ω that forms a *semialgebra*. A semialgebra includes empty set and sample space, closed under finite intersections, and the complement of a set in $\mathcal{J}$ can be represented as the unions of disjoint sets from $\mathcal{J}$.
2) Define a function $\mathbf{P}$ from $\mathcal{J}$ to $[0,1]$ that is finitely superadditive, countably monotonic and $P(\emptyset) = 0, P(\Omega) = 1$.
3) Construct outer measure $\mathbf{P}^*$ over all subsets of Ω based on P:

$$\mathbf{P}^*(A) = \inf_{A_1, A_2 \ldots \in \mathcal{J}, A \subseteq \bigcup_i A_i} \sum_i \mathbf{P}(A_i)$$

We have shown that the outer measure is an extension of $\mathbf{P}$, is monotonic and countably subadditive over all subsets.

Constructing Probability Triples II

4) We construct a new collection of subsets, denoted as $\mathcal{M}$:

$$\mathcal{M} = \{A : A \subseteq \Omega, \mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E) = \mathbf{P}^*(E) \text{ for all } E \subseteq \Omega\}$$

   We have shown that $\mathcal{M}$ includes the empty set and sample space, and is closed under complement. And based on this definition, it is easy to see that $\mathbf{P}^*(A) = 1 - \mathbf{P}^*(A^c)$ for all $A \in \mathcal{M}$.

5) To verify that $\mathcal{M}$ is a σ-algebra and $\mathbf{P}^*$ is a proper probability measure on $\mathcal{M}$, we still need to show that $\mathcal{M}$ is closed under countable unions and $\mathbf{P}^*$ is countably additive on $\mathcal{M}$.

6) Note that, if we need to verify that whether a given set $A \in \mathcal{M}$. By the countable subadditivity of outer measure, we always have $\mathbf{P}^*(E) \leq \mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E)$ for all $E \subseteq \Omega$. Thus we only need to verify $\mathbf{P}^*(E) \geq \mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E)$ for all $E \subseteq \Omega$.

$\mathbf{P}^*$ is Countably Additive over $\mathcal{M}$

Lemma 2 (2.3.9)

For disjoint $A_1, A_2, \ldots \in \mathcal{M}$, $\mathbf{P}^*(\bigcup_n A_n) = \sum_n \mathbf{P}^*(A_n)$.

Proof: We will show the finite additivity first.

1) For disjoint $A_1, A_2 \in \mathcal{M}$, since $A_1 \in \mathcal{M}$, we have (in the definition of $\mathcal{M}$, let $E = A_1 \cup A_2$, and $A = A_1$):
$\mathbf{P}^*(A_1 \cup A_2) = \mathbf{P}^*(A_1 \cap (A_1 \cup A_2)) + \mathbf{P}^*(A_1^c \cap (A_1 \cup A_2)) = \mathbf{P}^*(A_1) + \mathbf{P}^*(A_2)$
The finite additivity would then follow as the result of induction.
2) For a countably disjoint sequence, by the finite additivity and monotonicity of $\mathbf{P}^*$, we have
$\sum_{i=1}^n \mathbf{P}^*(A_i) = \mathbf{P}^*(\bigcup_{i=1}^n A_i) \leq \mathbf{P}^*(\bigcup_n A_n)$
Furthermore, $\sum_n \mathbf{P}^*(A_n) = \lim_{n\to\infty} \sum_{i=1}^n \mathbf{P}^*(A_i) \leq \mathbf{P}^*(\bigcup_n A_n)$
3) By the countable subadditivity of $\mathbf{P}^*$, we have $\sum_n \mathbf{P}^*(A_n) \geq \mathbf{P}^*(\bigcup_n A_n)$, thus $\sum_n \mathbf{P}^*(A_n) = \mathbf{P}^*(\bigcup_n A_n)$ for disjoint sequence of $\mathcal{M}$.

$\mathcal{M}$ is Closed under Finite Intersections/Unions

Lemma 3 (2.3.10)

If $A_1, A_2, \ldots A_n \in \mathcal{M}$, then $\bigcap_{i=1}^n A_i \in \mathcal{M}$ and $\bigcup_{i=1}^n A_i \in \mathcal{M}$.

Proof: Since $\mathcal{M}$ is closed under complement, then by de Morgan's law, $\mathcal{M}$ is closed under finite unions if $\mathcal{M}$ is closed under finite intersections. So we need to show if $A, B \in \mathcal{M}$, then $A \cap B \in \mathcal{M}$.

1) For any $E \in \Omega$,

$$
\begin{aligned}
&\mathbf{P}^*(A \cap B \cap E) + \mathbf{P}^*((A \cap B)^c \cap E) \\
=&\mathbf{P}^*(A \cap B \cap E) + \mathbf{P}^*((A^c \cap B \cap E) \cup (A \cap B^c \cap E) \cup (A^c \cap B^c \cap E)) \\
\leq&\mathbf{P}^*(A \cap B \cap E) + \mathbf{P}^*(A^c \cap B \cap E) \\
+&\mathbf{P}^*(A \cap B^c \cap E) + \mathbf{P}^*(A^c \cap B^c \cap E) \\
=&\mathbf{P}^*(B \cap E) + \mathbf{P}^*(B^c \cap E) = \mathbf{P}^*(E)
\end{aligned}
$$

2) By subadditivity, $\mathbf{P}^*(E) \leq \mathbf{P}^*(A \cap B \cap E) + \mathbf{P}^*((A \cap B)^c \cap E)$. Thus $\mathbf{P}^*(E) = \mathbf{P}^*(A \cap B \cap E) + \mathbf{P}^*((A \cap B)^c \cap E)$ and we have $A \cap B \in \mathcal{M}$.

$\mathcal{M}$ is Closed under Countable Unions of Disjoint Sets: I

- To show that $\mathcal{M}$ is closed under countable unions of disjoint sets, we need the following result

Lemma 4 (2.3.11)

Let $A_1, A_2, \ldots \in \mathcal{M}$ be disjoint. Define $B_n = \bigcup_{i=1}^n A_i$, then for any $E \subseteq \Omega$, we have $\mathbf{P}^*(E \cap B_n) = \sum_{i=1}^n \mathbf{P}^*(E \cap A_i)$.

Proof: Since $B_n \in \mathcal{M}$ for all $n \in \mathbf{N}$, and note that $B_{n-1} \cap B_n = B_{n-1}$ and $B_{n-1}^c \cap B_n = A_n$, we have:

$$\begin{aligned} \mathbf{P}^*(E \cap B_n) &= \mathbf{P}^*(B_{n-1} \cap E \cap B_n) + \mathbf{P}^*(B_{n-1}^c \cap E \cap B_n) \\ &= \mathbf{P}^*(E \cap B_{n-1}) + \mathbf{P}^*(E \cap A_n) \end{aligned}$$

It is obvious that $\mathbf{P}^*(E \cap B_1) = \mathbf{P}^*(E \cap A_1)$, then the above equation would allow us to use induction to obtain $\mathbf{P}^*(E \cap B_n) = \sum_{i=1}^n \mathbf{P}^*(E \cap A_i)$.

$\mathcal{M}$ is Closed under Countable Unions of Disjoint Sets: II

Lemma 5 (2.3.13)

For disjoint $A_1, A_2, \ldots \in \mathcal{M}$, $\bigcup_n A_n \in \mathcal{M}$.

Proof: Let $B_n = \bigcup_{i=1}^n A_i$, then for any $E \subseteq \Omega$

$$\mathbf{P}^*(E) = \mathbf{P}^*(E \cap B_n) + \mathbf{P}^*(E \cap B_n^c) = \sum_{i=1}^n \mathbf{P}^*(E \cap A_i) + \mathbf{P}^*(E \cap B_n^c)$$

$$\geq \sum_{i=1}^n \mathbf{P}^*(E \cap A_i) + \mathbf{P}^*(E \cap (\bigcup_{j=1}^\infty A_j)^c)$$

Letting $n \to \infty$, we have:
$\mathbf{P}^*(E) \geq \sum_n \mathbf{P}^*(E \cap A_n) + \mathbf{P}^*(E \cap (\bigcup_n A_n)^c) \geq$
$\mathbf{P}^*(E \cap (\bigcup_n A_n)) + \mathbf{P}^*(E \cap (\bigcup_n A_n)^c)$. Thus $(\bigcup_n A_n) \in \mathcal{M}$

$\mathcal{M}$ is a σ-algebra

Lemma 6 (2.3.14)

$\mathcal{M}$ is a σ-algebra

Proof: We only need to prove that $\mathcal{M}$ is closed under countable unions. Let $A_1, A_2, \ldots \in \mathcal{M}$. For each n, define $B_n = A_n \bigcap (\bigcup_{i=1}^{n-1} A_i)^c$.

Since we already show that $\mathcal{M}$ is closed under complement, finite intersections/unions, $B_n \in \mathcal{M}$.

As $B_n \bigcap B_m = \emptyset$ for all $n \neq m$, by the previously established result, $\bigcup_n B_n \in \mathcal{M}$. However, $\bigcup_n A_n = \bigcup_n B_n$, so $\bigcup_n A_n \in \mathcal{M}$. Then $\mathcal{M}$ is closed under countable unions.

$\mathcal{J} \subseteq \mathcal{M}$

Lemma 7

$\mathcal{J} \subseteq \mathcal{M}$

Proof: We need to show that for any $A \in \mathcal{J}$, $\mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E) \leq \mathbf{P}^*(E)$ for all $E \in \Omega$.

- By the definition of outer measure and A.4.2, for any $\varepsilon > 0$, we can find $B_1, B_2 \ldots, \in \mathcal{J}$ so that $E \subseteq \bigcup_n B_n$ and $\sum_n \mathbf{P}(B_n) < \mathbf{P}^*(E) + \varepsilon$. Furthermore, by the definition of semialgebra, $A^c = \bigcup_{k=1}^K J_k$ where $J_1, J_2, \ldots J_k \in \mathcal{J}$ are pairwise disjoint. Thus:
- $\mathbf{P}^*(E \cap A) + \mathbf{P}^*(E \cap A^c) \leq \mathbf{P}^*((\bigcup_n B_n) \cap A) + \mathbf{P}^*((\bigcup_n B_n) \cap A^c)$
  $= \mathbf{P}^*(\bigcup_n (B_n \cap A)) + \mathbf{P}^*((\bigcup_n B_n) \cap (\bigcup_{k=1}^K J_k))$
  $\leq \sum_n \mathbf{P}^*(B_n \cap A) + \sum_{n,k} \mathbf{P}^*(B_n \cap J_k) = \sum_n \mathbf{P}(B_n \cap A) + \sum_{n,k} \mathbf{P}(B_n \cap J_k)$
  $= \sum_n (\mathbf{P}(B_n \cap A) + \sum_k \mathbf{P}(B_n \cap J_k)) \leq \sum_n \mathbf{P}(B_n) < \mathbf{P}^*(E) + \varepsilon$
- As ε is an arbitrary constant, we can conclude that $\mathbf{P}^*(A \cap E) + \mathbf{P}^*(A^c \cap E) \leq \mathbf{P}^*(E)$. Thus $A \in \mathcal{M}$, $\mathcal{J} \subseteq \mathcal{M}$.