STAT 7200

Introduction to Advanced Probability
Lecture 4

Taylor R. Brown

"A First Look at Rigorous Probability Theory" (Jeffrey Rosenthal) Sections 2.1, 2.2, 2.3

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

1) Define a "probability measure" P on $\mathcal{J}$, which is a collection of subsets of Ω that forms a *semialgebra*. For instance, $\mathcal{J}$ can be chosen as the collection of all intervals.

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

1) Define a "probability measure" P on $\mathcal{J}$, which is a collection of subsets of Ω that forms a *semialgebra*. For instance, $\mathcal{J}$ can be chosen as the collection of all intervals.
2) Construct a new function P^* as an extension of P over *all* subsets of Ω. P^* is usually not a proper probability measure for all the subsets of Ω.

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

1) Define a "probability measure" P on $\mathcal{J}$, which is a collection of subsets of Ω that forms a *semialgebra*. For instance, $\mathcal{J}$ can be chosen as the collection of all intervals.
2) Construct a new function P^* as an extension of P over *all* subsets of Ω. P^* is usually not a proper probability measure for all the subsets of Ω.
3) Based on P^*, we construct a new collection of subsets, denoted as $\mathcal{M}$. $\mathcal{M}$ stands for "measurable."

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

1) Define a "probability measure" P on $\mathcal{J}$, which is a collection of subsets of Ω that forms a *semialgebra*. For instance, $\mathcal{J}$ can be chosen as the collection of all intervals.
2) Construct a new function P^* as an extension of P over *all* subsets of Ω. P^* is usually not a proper probability measure for all the subsets of Ω.
3) Based on P^*, we construct a new collection of subsets, denoted as $\mathcal{M}$. $\mathcal{M}$ stands for "measurable."
4) $\mathcal{M}$ is a σ-algebra and P^* is a proper probability measure on $\mathcal{M}$.

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

1) Define a "probability measure" P on $\mathcal{J}$, which is a collection of subsets of Ω that forms a *semialgebra*. For instance, $\mathcal{J}$ can be chosen as the collection of all intervals.
2) Construct a new function P^* as an extension of P over *all* subsets of Ω. P^* is usually not a proper probability measure for all the subsets of Ω.
3) Based on P^*, we construct a new collection of subsets, denoted as $\mathcal{M}$. $\mathcal{M}$ stands for "measurable."
4) $\mathcal{M}$ is a σ-algebra and P^* is a proper probability measure on $\mathcal{M}$.
5) $\mathcal{J} \subseteq \mathcal{M}$ and the extension from P to P^* is "unique."

Constructing Probability Triples

Here we outline the major steps in constructing (complicated) probability triples on a sample space such as $\Omega = [0, 1]$ or R.

1) Define a "probability measure" P on $\mathcal{J}$, which is a collection of subsets of Ω that forms a *semialgebra*. For instance, $\mathcal{J}$ can be chosen as the collection of all intervals.
2) Construct a new function P^* as an extension of P over *all* subsets of Ω. P^* is usually not a proper probability measure for all the subsets of Ω.
3) Based on P^*, we construct a new collection of subsets, denoted as $\mathcal{M}$. $\mathcal{M}$ stands for "measurable."
4) $\mathcal{M}$ is a σ-algebra and P^* is a proper probability measure on $\mathcal{M}$.
5) $\mathcal{J} \subseteq \mathcal{M}$ and the extension from P to P^* is "unique."

The above steps lead to a probability triple $\{\Omega, \mathcal{M}, \mathrm{P}^*\}$.

Semialgebra

- **Semialgebra**
  Let $\mathcal{J}$ be a collection of subsets of Ω. $\mathcal{J}$ is a semialgebra if
  a) $\emptyset, \Omega \in \mathcal{J}$.
  b) If $A_1, A_2, \ldots, A_k \in \mathcal{J}$, then $\bigcap_{i=1}^{k} A_i \in \mathcal{J}$. (Closed under finite intersections)
  c) If $A \in \mathcal{J}$, then there is a pairwise disjoint sequence of sets $B_1, B_2, \ldots, B_m \in \mathcal{J}$ such that $A^c = \bigcup_{i=1}^{m} B_i$.

Semialgebra

- **Semialgebra**
  Let $\mathcal{J}$ be a collection of subsets of Ω. $\mathcal{J}$ is a semialgebra if
  a) $\emptyset, \Omega \in \mathcal{J}$.
  b) If $A_1, A_2, \ldots, A_k \in \mathcal{J}$, then $\bigcap_{i=1}^{k} A_i \in \mathcal{J}$. (Closed under finite intersections)
  c) If $A \in \mathcal{J}$, then there is a pairwise disjoint sequence of sets $B_1, B_2, \ldots, B_m \in \mathcal{J}$ such that $A^c = \bigcup_{i=1}^{m} B_i$.

Proposition 1 (Exercise 2.2.3)

$\mathcal{J} = \{$ *All "intervals" contained in [0,1] (or* R$)\}$ *is a semialgebra.*

Semialgebra

- **Semialgebra**
  Let $\mathcal{J}$ be a collection of subsets of Ω. $\mathcal{J}$ is a semialgebra if
  a) $\emptyset, \Omega \in \mathcal{J}$.
  b) If $A_1, A_2, \ldots, A_k \in \mathcal{J}$, then $\bigcap_{i=1}^{k} A_i \in \mathcal{J}$. (Closed under finite intersections)
  c) If $A \in \mathcal{J}$, then there is a pairwise disjoint sequence of sets $B_1, B_2, \ldots, B_m \in \mathcal{J}$ such that $A^c = \bigcup_{i=1}^{m} B_i$.

Proposition 1 (Exercise 2.2.3)

$\mathcal{J} = \{$ *All "intervals" contained in [0,1] (or* R$)\}$ *is a semialgebra.*

- **Remark** The notion of an "interval" includes singletons and open/closed/half-open/empty intervals.

Algebra

- **Algebra**
  Let $\mathcal{B}_0$ be a collection of subsets of Ω. $\mathcal{B}_0$ is an **algebra** if
  a) $\emptyset \in \mathcal{B}_0$.
  b) If $A \in \mathcal{B}_0$, then $A^c \in \mathcal{B}_0$. (Closed under complement)
  c) If $A_1, \ldots, A_m \in \mathcal{B}_0$, then $\bigcup_{i=1}^m A_i \in \mathcal{B}_0$.

Algebra

- **Algebra**
  Let $\mathcal{B}_0$ be a collection of subsets of Ω. $\mathcal{B}_0$ is an **algebra** if
  a) $\emptyset \in \mathcal{B}_0$.
  b) If $A \in \mathcal{B}_0$, then $A^c \in \mathcal{B}_0$. (Closed under complement)
  c) If $A_1, \ldots, A_m \in \mathcal{B}_0$, then $\bigcup_{i=1}^m A_i \in \mathcal{B}_0$.
- **Remark** it's nearly the same as the definition for a σ-algebra, except the union in c) is *finite*.

Algebra

- **Algebra**
  Let $\mathcal{B}_0$ be a collection of subsets of Ω. $\mathcal{B}_0$ is an **algebra** if
  a) $\emptyset \in \mathcal{B}_0$.
  b) If $A \in \mathcal{B}_0$, then $A^c \in \mathcal{B}_0$. (Closed under complement)
  c) If $A_1, \ldots, A_m \in \mathcal{B}_0$, then $\bigcup_{i=1}^m A_i \in \mathcal{B}_0$.
- **Remark** it's nearly the same as the definition for a σ-algebra, except the union in c) is *finite*.
- **Remark** b), c) and De Morgan's law together imply further that algebras are closed under finite intersections, as well.

Algebra

- **Algebra**
  Let $\mathcal{B}_0$ be a collection of subsets of Ω. $\mathcal{B}_0$ is an **algebra** if
  a) $\emptyset \in \mathcal{B}_0$.
  b) If $A \in \mathcal{B}_0$, then $A^c \in \mathcal{B}_0$. (Closed under complement)
  c) If $A_1, \ldots, A_m \in \mathcal{B}_0$, then $\bigcup_{i=1}^m A_i \in \mathcal{B}_0$.
- **Remark** it's nearly the same as the definition for a σ-algebra, except the union in c) is *finite*.
- **Remark** b), c) and De Morgan's law together imply further that algebras are closed under finite intersections, as well.

Proposition 2 (Exercise 2.2.5)

$\mathcal{B}_0 = \{$ *All finite unions of "intervals" in [0,1] (or* R$)\}$ *is an algebra.*

Extension Theorem

Theorem 3

The Extension Theorem *Let $\mathcal{J}$ be a semialgebra of subsets of Ω and $\mathrm{P} : \mathcal{J} \to [0,1]$ such that:*

a) $\mathrm{P}(\emptyset) = 0, \mathrm{P}(\Omega) = 1$.

b) $\mathrm{P}(\bigcup_{i=1}^k A_i) \geq \sum_{i=1}^k \mathrm{P}(A_i)$ *whenever* $A_1, \dots, A_k \in \mathcal{J}$, $\bigcup_{i=1}^k A_i \in \mathcal{J}$, *and* $A_1, \dots, A_k$ *are pairwise disjoint (finite superadditivity).*

c) $\mathrm{P}(A) \leq \sum_n \mathrm{P}(A_n)$ *whenever* $A, A_1, A_2, \dots \in \mathcal{J}$, *and* $A \subseteq \bigcup_n A_n$ *(countable monotonicity).*

Then there is a σ-algebra $\mathcal{M} \supseteq \mathcal{J}$ and a countably-additive probability measure P^* on $\mathcal{M}$ such that $\mathrm{P}^*(A) = \mathrm{P}(A)$ for all $A \in \mathcal{J}$.

Outer Measure P^*

- **Remarks:** P^* is a function defined over *all* subsets of Ω (but its definition makes use of only sets from $\mathcal{J}$) but it isn't necessarily a probability measure if you're looking at all of these subsets.

Outer Measure P^*

- **Remarks:** P^* is a function defined over *all* subsets of Ω (but its definition makes use of only sets from $\mathcal{J}$) but it isn't necessarily a probability measure if you're looking at all of these subsets.
- For any $A \subseteq \Omega$

$$P^*(A) := \inf_{A_1, A_2, \ldots \in \mathcal{J}, A \subseteq \bigcup_i A_i} \sum_i P(A_i)$$

Outer Measure P^*

- **Remarks:** P^* is a function defined over *all* subsets of Ω (but its definition makes use of only sets from $\mathcal{J}$) but it isn't necessarily a probability measure if you're looking at all of these subsets.
- For any $A \subseteq \Omega$

$$P^*(A) := \inf_{A_1, A_2, \ldots \in \mathcal{J}, A \subseteq \bigcup_i A_i} \sum_i P(A_i)$$

Lemma 4

Outer measure satisfies the following properties:
a) $P^*(\emptyset) = 0$.
b) $P^*(A) \leq P^*(B)$ *if* $A \subseteq B$. *(Monotonicity)*
c) $P^*(A) = P(A)$ *if* $A \in \mathcal{J}$. *(*P^* *is an extension of* P*)*

Outer Measure P^* is Countably Subadditive

Lemma 5 (2.3.6.)

Outer measure P^* *is countably subadditive:*
$\mathsf{P}^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mathsf{P}^*(B_n)$ *for any* $B_1, B_2, \ldots \subseteq \Omega$

Outer Measure P^* is Countably Subadditive

Lemma 5 (2.3.6.)

Outer measure P^* is countably subadditive:
$P^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} P^*(B_n)$ *for any* $B_1, B_2, \ldots \subseteq \Omega$

Proof: The key is $P^*(A)$ is defined as the infimum of the countable unions of sets in $\mathcal{J}$ that "cover" A.

Outer Measure P^* is Countably Subadditive

Lemma 5 (2.3.6.)

Outer measure P^* *is countably subadditive:*
$P^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} P^*(B_n)$ *for any* $B_1, B_2, \ldots \subseteq \Omega$

Proof: The key is $P^*(A)$ is defined as the infimum of the countable unions of sets in $\mathcal{J}$ that "cover" A.

1) Given $\varepsilon > 0$, for each B_n, there must be a sequence $\{C_{nk}\}_{k=1}^{\infty}$, s.t. $C_{nk} \in \mathcal{J}$, $B_n \subseteq \bigcup_k C_{nk}$ and $\sum_k P(C_{nk}) < P^*(B_n) + \varepsilon/2^n$ (small typo in book)

Outer Measure P^* is Countably Subadditive

Lemma 5 (2.3.6.)

Outer measure P^* *is countably subadditive:*
$P^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} P^*(B_n)$ *for any* $B_1, B_2, \ldots \subseteq \Omega$

Proof: The key is $P^*(A)$ is defined as the infimum of the countable unions of sets in $\mathcal{J}$ that "cover" A.

1) Given $\varepsilon > 0$, for each B_n, there must be a sequence $\{C_{nk}\}_{k=1}^{\infty}$, s.t. $C_{nk} \in \mathcal{J}$, $B_n \subseteq \bigcup_k C_{nk}$ and $\sum_k P(C_{nk}) < P^*(B_n) + \varepsilon/2^n$ (small typo in book)
2) Since $\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n,k} C_{n,k}$, $P^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n,k} P(C_{n,k}) < \sum_n P^*(B_n) + \varepsilon$.
3) As ε is an arbitrary positive constant, we must have $P^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} P^*(B_n)$.

$\mathcal{M}$: The Measurable Sets

- Outer measure cannot always be a probability measure over *all* subsets of Ω (recall Proposition 1.2.6). Define a refined collection of subsets using P^*:

$$\mathcal{M} = \{A \subseteq \Omega : \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E) = \mathrm{P}^*(E) \text{ for all } E \subseteq \Omega\}$$

$\mathcal{M}$: The Measurable Sets

- Outer measure cannot always be a probability measure over *all* subsets of Ω (recall Proposition 1.2.6). Define a refined collection of subsets using P^*:

$$\mathcal{M} = \{A \subseteq \Omega : \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E) = \mathrm{P}^*(E) \text{ for all } E \subseteq \Omega\}$$

Lemma 6

We have the following results regarding $\mathcal{M}$:
a) $\emptyset \in \mathcal{M}, \Omega \in \mathcal{M}$
b) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$ (Closed under complement)

$\mathcal{M}$: The Measurable Sets

- Outer measure cannot always be a probability measure over *all* subsets of Ω (recall Proposition 1.2.6). Define a refined collection of subsets using P^*:

$$\mathcal{M} = \{A \subseteq \Omega : \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E) = \mathrm{P}^*(E) \text{ for all } E \subseteq \Omega\}$$

Lemma 6

We have the following results regarding $\mathcal{M}$:
a) $\emptyset \in \mathcal{M}, \Omega \in \mathcal{M}$
b) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$ (Closed under complement)

- **Remark:** We often need to verify that a given set $A \in \mathcal{M}$. By the countable subadditivity of our outer measure, we always have $\mathrm{P}^*(E) \leq \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E)$ for all $E \subseteq \Omega$. If it's easier, we only need need to verify $\mathrm{P}^*(E) \geq \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E)$ for all $E \subseteq \Omega$. This would can be achieved by using the finite superadditivity of P.

$\mathcal{M}$ and P^*

- We still need to show that $\mathcal{M}$ is a σ-algebra and P^* is a proper probability measure over $\mathcal{M}$. A review of our results so far:

$\mathcal{M}$ and P*

- We still need to show that $\mathcal{M}$ is a σ-algebra and P* is a proper probability measure over $\mathcal{M}$. A review of our results so far:
- $\mathcal{M} = \{A : A \subseteq \Omega, \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E) = \mathrm{P}^*(E)$ for all $E \subseteq \Omega\}$. We already have $\emptyset \in \mathcal{M}, \Omega \in \mathcal{M}$ and $\mathcal{M}$ is closed under complement. So we still need to prove that $\mathcal{M}$ is closed under countable unions.

$\mathcal{M}$ and P^*

- We still need to show that $\mathcal{M}$ is a σ-algebra and P^* is a proper probability measure over $\mathcal{M}$. A review of our results so far:
- $\mathcal{M} = \{A : A \subseteq \Omega, \mathrm{P}^*(A \cap E) + \mathrm{P}^*(A^c \cap E) = \mathrm{P}^*(E) \text{ for all } E \subseteq \Omega\}$. We already have $\emptyset \in \mathcal{M}, \Omega \in \mathcal{M}$ and $\mathcal{M}$ is closed under complement. So we still need to prove that $\mathcal{M}$ is closed under countable unions.
- Regarding P^*, for any $A \in \mathcal{M}$, first, $\mathrm{P}^*(A) \geq 0$; second, in the definition of $\mathcal{M}$, by choosing $E = \Omega$, we have $\mathrm{P}^*(A) = 1 - \mathrm{P}^*(A^c)$. So we still need to show that, on $\mathcal{M}$, P^* is countably additive.

$\mathcal{M}$ and P^*

- We still need to show that $\mathcal{M}$ is a σ-algebra and P^* is a proper probability measure over $\mathcal{M}$. A review of our results so far:
- $\mathcal{M} = \{A : A \subseteq \Omega, \mathsf{P}^*(A \cap E) + \mathsf{P}^*(A^c \cap E) = \mathsf{P}^*(E) \text{ for all } E \subseteq \Omega\}$. We already have $\emptyset \in \mathcal{M}, \Omega \in \mathcal{M}$ and $\mathcal{M}$ is closed under complement. So we still need to prove that $\mathcal{M}$ is closed under countable unions.
- Regarding P^*, for any $A \in \mathcal{M}$, first, $\mathsf{P}^*(A) \geq 0$; second, in the definition of $\mathcal{M}$, by choosing $E = \Omega$, we have $\mathsf{P}^*(A) = 1 - \mathsf{P}^*(A^c)$. So we still need to show that, on $\mathcal{M}$, P^* is countably additive.
- Countable additivity of P^* is shown in the next lecture, as well as the fact that $\mathcal{M}$ is a σ-field.