4.2: Expected Values and Moment Generating Functions

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Motivation

We did this stuff (means, moments, mgfs) with discrete rvs. Now we'll do them for cts rvs. Pretty much all the results hold in the same way, but we generally replace summation with integration.

Definition

The **expected value** of a cts rv X with pdf $f(x)$ is

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

and the expectation of any function $h(X)$ (LOTUS) is

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)dx$$

Example

Recall a gaussian pdf is $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

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Let $s = x - \mu$ and $t = \mu - x$

$$\begin{aligned}
E[(X-\mu)^3] &= \int_{-\infty}^{\mu} f(x)(x-\mu)^3 dx + \int_{\mu}^{\infty} f(x)(x-\mu)^3 dx \\
&= \int_{-\infty}^{0} f(\mu+s)s^3 ds + \int_{0}^{-\infty} f(\mu-t)(-t)^3 - dt \\
&= \int_{-\infty}^{0} f(\mu+s)s^3 ds + \int_{0}^{-\infty} f(\mu-t)(t)^3 dt \\
&= \int_{-\infty}^{0} f(\mu+s)s^3 ds + \int_{0}^{-\infty} f(\mu+t)(t)^3 dt \\
&= \int_{-\infty}^{0} f(\mu+s)s^3 ds - \int_{-\infty}^{0} f(\mu+t)(t)^3 dt = 0
\end{aligned}$$

Example continued

How can we generalize this result? We didn't even use the form of the normal distribution... we just used the fact that $f(\cdot)$ is symmetric...

Definition

The **variance** of a cts rv with mean μ is

$$V[X] = E[(X - \mu)^2]$$

and this works again too:

$$V[X] = E[X^2] - (E[X])^2$$

Propositions

And these are true still as well:

$$E[aX + b] = aE[X] + b$$

$$V[aX + b] = a^2 V[X]$$

Definition

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The **moment generating function** (mgf) for a cts rv X is

$$M_X(t) = E[e^{tX}] = \int e^{tx} f(x) dx.$$

It exists if it's defined for every t in some open interval $(-\epsilon, \epsilon)$

Motivation

Often times we'll transform a random variable $Y = h(X)$, but it will be non-linear. Ideally we want $f_Y(y)$. If that's too difficult, we want $E(Y)$ and $V(Y)$.

Generally, there is no simple tool that always works in every scenario. But we can approximate it with the **delta method**

Definition

Suppose we take a transformation $h(\cdot)$ that we're going to apply to X. Suppose further that this function h is differentiable and $h'(E[X]) \neq 0$. Then

$$E[h(X)] \approx h(E[X])$$

and

$$V[h(X)] \approx h'(E[X])^2 V[X]$$

Justification

It's basically just an application of the Taylor series centered at $E[X] = \mu$

$$h(X) \approx h(\mu) + h'(\mu)(X - \mu) + \{\text{stuff we don't care about}\}$$

Then you take the mean and variance treating $h(\mu)$ and $h'(\mu)$ as constants...

Example

Similar to Example 4.17 on page 176. Let $X \sim f(x) = 2e^{-2x}$ where $x > 0$. First, find the MGF. Then use it to find the mean and variance. Finally, find (approximate) the mean and variance of $Y = \exp(X)$.

Example

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Example (continued)

finding the mgf...

Let $r = x(2-t)$, then $\frac{dr}{dx} = 2 - t$ (...also we're assuming $2 - t > 0$)

$$\begin{aligned}
M_X(t) &= \int_0^\infty 2e^{-2x}e^{tx}dx \\
&= 2\int_0^\infty e^{-x(2-t)}dx \\
&= 2\int_0^\infty e^{-r}\frac{1}{2-t}dr = \frac{2}{2-t}[-e^{-r}]_0^\infty = \frac{2}{2-t}
\end{aligned}$$

Example (continued)

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\begin{aligned}
M_X'(t) &= \frac{d}{dt}2(2-t)^{-1} \\
&= 2(-1)(2-t)^{-2}(-1) \\
&= \frac{2}{(2-t)^2}
\end{aligned}
$$

Example (continued)

$$
\begin{aligned}
M_X''(t) &= \frac{d}{dt} 2(2-t)^{-2} \\
&= 2(-2)(2-t)^{-3}(-1) \\
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So $E[X] = \frac{1}{2}$ and $V[X] = E[X^2] - (E[X])^2 = \frac{1}{4}$.

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and finally

$$E[\exp(X)] \approx \exp[E(X)]$$

and

$$V[X] \approx (\exp(E[X]))^2 \frac{1}{4} = \frac{(e^{1/2})^2}{4} = \frac{e}{4}$$

Entropy Again

Entropy is defined for continuous random variables just like it was defined for discrete ones:

$$E[-\log f(X)] = \int -\log f(x) f(x) dx.$$