9.3: Tests Concerning a Population Proportion

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Motivation

Recall in 8.2 we talked about large-sample confidence intervals for some θ. They were justified with a CLT argument. Basically, both of these z-like quantities were approximately standard normal rvs.

$$\frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}} \overset{\text{approx.}}{\sim} \mathcal{N}(0, 1)$$

$$\frac{\hat{\theta} - \theta}{\widehat{\sigma_{\hat{\theta}}}} \overset{\text{approx.}}{\sim} \mathcal{N}(0, 1)$$

Motivation

That means if we further assume that H_0 is true:

$$\frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} \overset{\text{approx.}}{\sim} \mathcal{N}(0, 1)$$

$$\frac{\hat{\theta} - \theta_0}{\widehat{\sigma_{\hat{\theta}}}} \overset{\text{approx.}}{\sim} \mathcal{N}(0, 1)$$

We use the first one if $\sigma_{\hat{\theta}}$ is known, and we use the second if it isn't.

a large-sample test

When we're making a hypothesis about a population proportion, though, the mean and the variance are related. For a bernoulli rv X,

$$E[X] = p, \qquad V[X] = p(1-p)$$

so for an average of independent bernoullis $\bar{X}$ we have

$$E[\bar{X}] = p, \qquad V[\bar{X}] = \frac{p(1-p)}{n}$$

For this reason $H_0 : p = p_0$ is a hypothesis about the mean and the variance.

a large-sample test

Here's our test:

1. $H_0 : p = p_0$
2. our test statistic is $Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$ (we're plugging it in in more than one place)
3. if $H_a : p > p_0$ we reject when $Z > z_\alpha$
4. if $H_a : p < p_0$ we reject when $Z < -z_\alpha$
5. if $H_a : p \neq p_0$ we reject when $Z > z_{\alpha/2}$ OR when $Z < -z_{\alpha/2}$

Example 9.11 on page 451

"Recent information suggests that obesity is an increasing problem in America among all age groups. The Associated Press (Oct. 9, 2002) reported that 1276 individuals in a sample of 4115 adults were found to be obese (a body mass index exceeding 30; this index is a measure of weight relative to height). A 1998 survey based on people's own assessment revealed that 20% of adult Americans considered themselves obese. Does the recent data suggest that the true proportion of adults who are obese is more than 1.5 times the percentage from the self-assessment survey?" We use $\alpha = .1$

Example 9.11 on page 451

1. $H_0 : p = .3 (= .2 \times 1.5)$
2. $Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{.31 - .3}{\sqrt{\frac{(.3)(.7)}{4115}}} = 1.4$
3. $z_{.1} = 1.28$

Therefore we reject H_0 in favor of H_a at a level of $\alpha = .1$

Power and Sample Size Determination

Our test statistic is

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}.$$

Under H_0:

$$E[Z] = \frac{E[\hat{p}] - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = 0.$$

and

$$V[Z] = \frac{V[\hat{p} - p_0]}{\frac{p_0(1-p_0)}{n}} = 1.$$

Power and Sample Size Determination

But when we calculate power or type 2, we don't assume H_0 is true. Let's say $p = p'$. Our test statistic is still

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}.$$

but

$$E[Z] = \frac{E[\hat{p}] - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{p' - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

and

$$V[Z] = \frac{V[\hat{p} - p_0]}{\frac{p_0(1-p_0)}{n}} = \frac{p'(1-p')/n}{p_0(1-p_0)/n}.$$

Here's an example type 2 error calculation for a right-tailed test

$$
\begin{aligned}
\beta(p') &= P(Z \leq z_\alpha | p = p') \\
&= P\left(\hat{p} \leq z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}} + p_0 | p = p'\right) \\
&= P\left(\frac{\hat{p} - p'}{\sqrt{p'(1-p')/n}} \leq \frac{z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{p'(1-p')/n}} + \frac{p_0 - p'}{\sqrt{p'(1-p')/n}}\right) \\
&= \Phi\left(\frac{z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{p'(1-p')/n}} + \frac{p_0 - p'}{\sqrt{p'(1-p')/n}}\right)
\end{aligned}
$$

Similar results are listed on page 452.

Small-Sample Tests

When n is not large, we cannot use the normal approximation to the binomial random variable X. On the one hand, this might be more accurate because we're using a more exact distribution for X.

On the other hand, however, finding rejection regions is more problematic.

Small-Sample Tests

Let's say $H_0 : p = p_0$ versus $H_a : p > p_0$. This means we reject when $X \geq c$ (for some c). What is the rejection region associated with $\alpha = .05$?

$$
\begin{aligned}
P(\text{ type 1 error }) &= P(H_0 \text{ is rejected when it's true}) \\
&= P\left[X \geq c \text{ when } X \sim \text{Binomial}(n, p_0)\ \right] \\
&= 1 - P\left[X \leq c - 1 \text{ when } X \sim \text{Binomial}(n, p_0)\ \right] \\
&\overset{\text{(set)}}{=} \alpha
\end{aligned}
$$

Sometimes there will be no such c where $P(\text{ type 1 error }) = \alpha$. As c goes farther out to the right, $P(\text{ type 1 error })$ goes down. Sometimes we will have to find the smallest such c where $P(\text{ type 1 error }) \leq \alpha$.