4.4: The Gamma Distribution and Its Relatives

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Motivation

Normal distributions are probably the most popular but are defined for rvs that take on values on any part of the real line $(-\infty, \infty)$. Gamma rvs are defined for only positive numbers. Also, they include a lot of useful specific distributions.

Definition

This is a special function; it isn't a pdf or cdf. We use it a lot whenever we work with integrals for gamma-ish pdfs.

Definition

For $\alpha > 0$, the **gamma function** $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-z} z^{\alpha-1} dz$$

Note: z is just a dummy variable. This is a function in α

Properties

1. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$
2. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

So it generalizes the factorial function for non-integral arguments, and it comes up with "special numbers" a lot.

Note: you can always just use wolfram alpha or some other symbolic computational thing to evaluate the gamma function

Definition

Definition

A cts rv X is a **Gamma Distribution** if it has pdf

$$f(x;\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

with x, α and β all being positive.

it's MGF is

$$M_X(t) = (1-\beta t)^{-\alpha}$$

Proof

It has simple-to-remember mean and variance

$$EX = \alpha\beta$$

$$VX = \alpha\beta^2$$

Also you can multiply gamma rvs by a constant and they're still gamma (you can't add constants though)

$$X \sim \text{Gamma}(\alpha, \beta) \to cX \sim \text{Gamma}(\alpha, c\beta)$$

Proof

Sometimes "recognizing different gamma densities" is even easier than "recognizing gamma functions"

$$
\begin{aligned}
EX &= \int_0^\infty x \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{(\alpha+1)-1} e^{-x/\beta} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{1} \left[\frac{1}{\beta^{\alpha+1}\Gamma(\alpha+1)} \int_0^\infty x^{(\alpha+1)-1} e^{-x/\beta} dx \right] \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{1} \\
&= \beta \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \\
&= \beta\alpha \frac{\Gamma(\alpha)}{\Gamma(\alpha)}
\end{aligned}
$$

Proof

Let's start out with $X \sim \text{Gamma}(\alpha, \beta)$. We want to show that $Y = cX$ follows a $\text{Gamma}(\alpha, c\beta)$ distribution. This is easiest to do with mgfs..

$$M_Y(t) = E[e^{tY}] = E[e^{tcX}] = E[e^{(tc)X}] = M_X(tc) = (1 - \beta ct)^{-\alpha}$$

Proof

Another way to prove it:

$$F_Y(y) = P(Y \leq y) = P(cX \leq y) = P(X \leq y/c) = F_X(y/c)$$

So $F_Y(y) = F_X(y/c)$.
So $f_Y(y) = f_X(y/c)\frac{1}{c}$ (by the chain rule)

Some special cases

A **chi-squared distribution** with parameter ν is the same as a Gamma$(\nu/2, 2)$
So it's density is $f(x;\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2} x^{\nu/2-1}$

A **exponential distribution** with parameter λ is the same as a Gamma$(1, \frac{1}{\lambda})$
So it's density is $f(y;\lambda) = \lambda e^{-y\lambda}$

Exercise

This is similar to example 4.30 on page 199. Let $T \sim \text{Exponential}(\lambda)$ denote the waiting time until a call arrives (in days). Find it's cdf. Then use it's cdf to find the probability that we wait more than 2 days for a call (use $\lambda = .5$).

Exercise

$$P(T \leq t) = \int_0^t \lambda e^{-\lambda x} dx = [-e^{-\lambda x}]|_{x=0}^{x=t} = -e^{-\lambda t} + 1$$

So $F_T(t) = 1 - e^{-\lambda t}$. Then

$$P(T > 2) = 1 - P(T \leq 2) = 1 - [1 - e^{-\lambda 2}] = e^{-\lambda 2}$$

So if we use $\lambda = \frac{1}{2}$, then $P(T > 2) = \frac{1}{e}$