6.1: Statistics And Their Distributions

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- Random variables are typically denoted with letters at the end of the alphabet (e.g. X,Y, etc.)
- Before we observe a sample, the data is random $(X_1, \dots X_N)$
- The observations or data we observe are lower-case $(x_1, \ldots x_n)$
- for more info see Chapter 3

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- The same idea applies to functions of the rvs
- Example: $f(X_1, \ldots, X_n) = \frac{1}{n} \sum_i X_i$
- if we have the data already it's $f(x_1, ..., x_n) = \frac{1}{n} \sum_i x_i$
- This is what a **statistic** is: a quantity calculated from sample data.
- We're usually referring to the random variable (capital letters) because we like to talk about it's distribution/moments etc.

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Some very ubiquitous examples:

•
$$f(X_1, ..., X_n) = \frac{1}{n} \sum_i X_i = \bar{X}$$

•
$$f(X_1,...,X_n) = \frac{1}{n-1} \sum_i (X_i - \bar{X})^2 = S^2$$

•
$$f(X_1,...,X_n) = max(X_1,...,X_n) = X_{(n)}$$

•
$$f(X_1,...,X_n) = min(X_1,...,X_n) = X_{(1)}$$

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- Every sample statistic has a probability distribution
- sometimes it's easy to calculate (e.g. the mean of a bunch of normal random variables)
- sometimes it's difficult or impossible
- The probability distribution of a statistic is often called a sampling distribution
- A note on notation: I'll usually denote general probability distributions with a p or an f

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- Many analyses assume that the data are being generated from a common probability distribution
- The most tractable situation is when each data point/rv comes from the same distribution, and no data point affects any other data point
- the book calls this a random sample (page 287)
- I will just say (i.i.d) which stands for *independent and identically* distributed
- I will denote it like this $\stackrel{iid}{\sim}$
- example: $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathsf{Normal}(\mu, \sigma^2)$

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Our first example (Example 6.3 on page 290):

Let $X_1, X_2 \stackrel{iid}{\sim} \text{Exponential}(\lambda)$. What is the distribution of $T_0 = \sum X_i$? First we'll find it's cumulative distribution function (cdf) $F_{T_0}(t)$; then we'll take the derivative to get the density $f_{T_0}(t)$. Hopefully it'll be a familiar distribution again...

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$$F_{T_0}(t) = P(X_1 + X_2 \le t)$$

$$= \iint_{\{(x_1, x_2) : x_1 + x_2 \le t\}} p(x_1, x_2) dx_1 dx_2$$

$$= \iint_{\{(x_1, x_2) : x_1 + x_2 \le t\}} p(x_1) p(x_2) dx_1 dx_2$$

$$= \int_0^t \int_0^{t - x_1} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} dx_2 dx_1$$

$$= \int_0^t \lambda e^{-\lambda x_1} \left[\int_0^{t - x_1} \lambda e^{-\lambda x_2} dx_2 \right] dx_1$$

$$= \int_0^t \lambda e^{-\lambda x_1} \left[-e^{-\lambda x_2} \right]_0^{t - x_1} dx_1$$

$$= \int_0^t \left(\lambda e^{-\lambda x_1} - \lambda e^{-\lambda t} \right) dx_1$$

$$= \int_0^t \left(\lambda e^{-\lambda x_1} - \lambda e^{-\lambda t} \right) dx_1$$

$$= \int_0^t \lambda e^{-\lambda x_1} dx_1 - t\lambda e^{-\lambda t}$$

$$= \left[-e^{-\lambda x_1} \right]_0^t - t\lambda e^{-\lambda t}$$

$$= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$$

Now differentiate with respect to *t* to get the density:

$$f_{T_0}(t) = \lambda^2 t \exp(-\lambda t), t > 0$$



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Notes

A couple things:

- this is a Gamma $(2,1/\lambda)$ distribution.
- Recall that there are a few tricks that make Gamma distributions nice
- The distribution of sums of independent random variables is made easier by moment generating functions (we'll talk about these later)
- make sure you get a feel for this (a lot of the same tricks will reappear later)

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Extending example 6.3:

Let $X_1, X_2, \ldots, X_N \stackrel{iid}{\sim}$ Exponential(λ). What is the distribution of $T_0 = \sum_{i=1}^n X_i$? First we'll find it's cumulative distribution function (cdf) $F_{T_0}(t)$; then we'll take the derivative to get the density $f_{T_0}(t)$.

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$$F_{T_0}(t) = P\left(\sum_{i=1}^n X_i \le t\right)$$

$$= \iint_{\{(x_1, x_2): x_1 + x_2 \le t\}} f(x_1, \dots, x_n) dx_{1:n}$$

$$= \int_0^t \int_0^{t - \sum_{i=1}^1 x_i} \dots \int_0^{t - \sum_{i=1}^{n-1} x_i} f(s_1, \dots, s_n) ds_n ds_{n-1} \dots ds_1$$

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Alternatively, you could use the transformation theorem:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1 + X_2 \\ \vdots \\ \sum_{i=1}^n X_i \end{bmatrix}$$

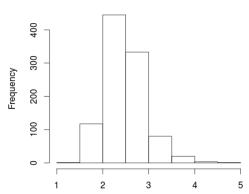
It isn't that bad to get $g(y_1, y_2, ..., y_n)$, but it's difficult to integrate out $y_1, y_2, ..., y_{n-1}$

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We can also run simulations to check our derivations.

Example: If $X_1, \ldots, X_{100} \stackrel{iid}{\sim} \text{Normal}(0, 1)$, what does the maximum $X_{(100)}$ follow?

Histogram of max(X_1, ... X_100)



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