3.4: Moments and Moment Generating Functions

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Motivation

The last selection dealt with expectations of transformations of our random variable like this

$$E[h(X)] = c.$$

In particular we had

$$E[(X - \mu)^2].$$

We could get number summaries for our random variable/probability distribution.

In this section we'll talk about the expectation of a certain function that looks like this:

$$E[h(X, t)] = f(t).$$

We'll find out why this is very useful.

Definition

A **central** or **moment about the mean** is the expectation of some power of the difference between X and its expected value μ. That is, for some $k \in \{1, 2, 3, \dots\}$

$$E[(X-\mu)^k]$$

A **noncentral** or **raw** or **moment about 0** is the expectation of a power of a random variable. That is, for some $k \in \{1, 2, 3, \dots\}$

$$E[X^k]$$

Example

Recall that for the discrete case, we can use the formula for $E[h(X)]$ to get a moment (say central) because

$$E[(X-\mu)^3] = E[h(X)]$$

where $h(x) = (x-\mu)^3$.

$$E[(X-\mu)^3] = \sum_x (x-\mu)^3 p(x)$$

so if there are 20 possible values for x, then we have to sum together 20 products...

Motivation

This is a partial motivation for the **moment generating function**. The **moment generating function** (mgf) is defined as

$$M_X(t) = E(\exp[tX])$$

Properties

Some properties:

1. $E[X^k] = M_X^{(k)}(0)$
2. $M_X(0) = 1$
3. MGFs uniquely determine an rv's distirbution (we will use this a lot more later)
4. If $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$

Proof

Proving (2) and (4) is left as exercise. Proof of (3) is ommitted. Proving 1:

$$\begin{aligned}\frac{d}{dt}M_X(t) &= \frac{d}{dt}Ee^{tX} \\ &= E\left[\frac{d}{dt}e^{tX}\right] \\ &= E\left[Xe^{tX}\right]\end{aligned}$$

so then $\frac{d}{dt}M_X(t)|_{t=0} = E[X]$. To find higher moments, use induction...

Example

Here's an example to see why differentiating is more convenient. Find the second raw moment of a binomial distribution with parameters n and p.
Hint: $M_X(t) = (1 - p + pe^t)^n$

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Here's an example to see why differentiating is more convenient. Find the second raw moment of a binomial distribution with parameters n and p. Hint: $M_X(t) = (1 - p + pe^t)^n$

$\frac{d^2}{dt^2} M_X(t) = n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)$ and
$\frac{d^2}{dt^2} M_X(t)|_{t=0} = n(n-1)p^2 + np = np(1-p) + n^2p^2$

contrast this doing $\sum_{x=0}^{n} p(x)x^2$