4.3: The Normal Distribution

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Motivation

"The normal distribution is the most important one in all of probability and statistics." (first paragraph on page 179)

Definition

Definition

a cts rv X has a **normal distribution** with parameters μ and σ^2 if it has the pdf

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

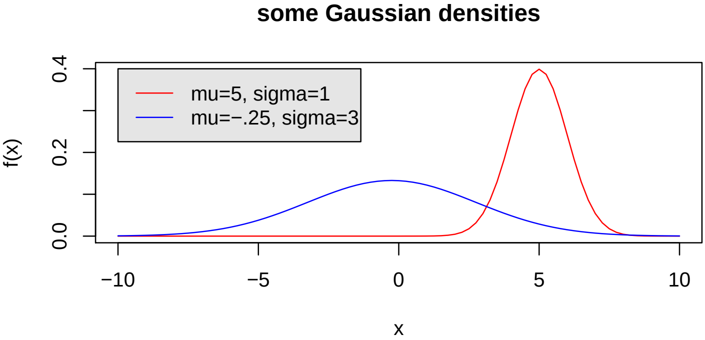
where $-\infty < \mu < \infty$, $\sigma^2 > 0$ and $-\infty < x < \infty$

shorthand: $X \sim \mathcal{N}(\mu, \sigma^2)$

Definition

Here's a picture of two examples of this function

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$



Definition

a special instance of the normal distribution is the **standard normal distribution**. You just set $\mu = 0$ and $\sigma^2 = 1$

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right]$$

where $-\infty < z < \infty$

shorthand: $Z \sim \mathcal{N}(0, 1)$

It's convention to use capital Z when we're talking about standard normal rvs

Note

When we're talking about probabilities, we can "do algebra" inside the parentheses. E.g:

$$P(a \leq \mu + Z\sigma \leq b) = P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$$

or

$$P\left(\frac{1}{X} > -c\right) = P\left(X > \frac{1}{-c}\right)$$

with $X, c > 0$ for the last one.

Notation

A note on notation:

z_α denotes the $1 - \alpha \times 100$th percentile for a standard normal distribution

Proposition

We'll generalize this a bit later, but if

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

then

$$aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

That's why we go back and forth between $X = \mu + \sigma Z$ and $Z = \frac{X-\mu}{\sigma}$ a lot. They are both normal, but sometimes one will have more convenient parameters than the other.

MGFs

The normal mgf for $\mathcal{N}(\mu, \sigma^2)$ is

$$M_X(t) = \exp\left[\mu t + \sigma^2 t^2/2\right]$$

MGFs

Proof:

$$
\begin{aligned}
M_Z(t) &= E[e^{zt}] \\
&= \int e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \int \frac{1}{\sqrt{2\pi}} e^{-(z^2-2tz)/2} dz \\
&= \int \frac{1}{\sqrt{2\pi}} e^{-(z^2-2tz+t^2-t^2)/2} dz \\
&= e^{t^2/2} \int \frac{1}{\sqrt{2\pi}} e^{-(z^2-2tz+t^2)/2} dz \\
&= e^{t^2/2} \int \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2} dz = e^{t^2/2}
\end{aligned}
$$

Then

$$M_X(t) = E[e^{t(\mu+Z\sigma)}] = e^{t\mu} E[e^{(t\sigma)Z}] = e^{t\mu} M_Z(t\sigma) = \exp\left[\mu t + \sigma^2 t^2/2\right]$$

example

Let R denote a random yearly return on a financial asset.
$R \sim \mathcal{N}(\mu, \sigma^2)$
We initially invest \$1.
The value of our investment after a year is $I_1 = e^R$
What is the expected value of our investment after a year?

example

We're looking for $E[e^{tR}]$, so we can plug in $t = 1$. This particular MGF is:

$$E[e^{tR}] = M_R(t) = \exp\left[\mu t + \sigma^2 t^2/2\right].$$

where the last equality follows from two slides ago.

So instead of finding moments of R, or using it to find out what distribution transformations of R follow, we can use MGFs for this too (in this literal way).

In chapter 5 we will go further and find the distribution of this random variable.