3.7: The Poisson Probability Distribution

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Motivation

A lot of discrete rvs arise from simple experiments consisting of trials with a finite number of possible outcomes (e.g. binomial, multinomial, hypergeometric, negative binomial, etc). This one doesn't, but it arises from taking a limit of a binomial rv.

Definition

A rv X is said to have a **Poisson distribution** with parameter $\lambda > 0$ if its pmf is

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, \ldots$$

...it also has this mgf

$$M_X(t) = \exp\left[\lambda(e^t - 1)\right]$$

A Fact

To show that this pmf sums to 1, we need the Taylor/Maclaurin approximation to e^λ about 0

$$
\begin{aligned}
e^\lambda &= f(0) + \frac{f^{(1)}(0)(\lambda-0)^1}{1!} + \frac{f^{(2)}(0)(\lambda-0)^2}{2!} + \frac{f^{(3)}(0)(\lambda-0)^3}{3!} + \cdots \\
&= 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} + \cdots \\
&= \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}
\end{aligned}
$$

So $\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$

Fact

Also...

If $X \sim \text{Poisson}(\lambda)$ then $EX = V(X) = \lambda$

(prove this with mgfs or directly using the definition)

Proposition

Let's pretend we have an infinite number of binomial rvs $\{X_1, X_2, \ldots\}$. Let X_n be a binomial random variable with parameters n and p_n.

If we take the limit of binomial rvs with $n \to \infty$, $p_n \to 0$, and $np_n \to \lambda$, then we get the Poisson.

To prove this, we're going to take the limit of MGFs, since it's easy (the book takes the limit of pmfs)

Also, we're going to need this fact: if $a_n \to a$, then $(1 + \frac{a_n}{n})^n \to e^a$ as $n \to \infty$

Proof

Since we have a sequence of $\{X_n\}_{n=1}^{\infty}$, then we have a sequence of mgfs $\{M_{X_n}(t)\}_{n=1}^{\infty}$

$$\begin{aligned}
M_{X_n}(t) &= [1 - p_n + p_n e^t]^n \\
&= [1 + p_n(e^t - 1)]^n \\
&= [1 + \frac{np_n}{n}(e^t - 1)]^n \\
&= [1 + \frac{np_n(e^t - 1)}{n}]^n \\
&\to \exp\left[\lambda(e^t - 1)\right]
\end{aligned}$$