5.4: Transformations of Random Variables

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Motivation

"In the previous chapter we discussed the problem of starting with a single random variable X, forming some function of X, such as X^2 or e^X, to obtain a new random variable $Y = h(X)$, and investigating the distribution of this new random variable. We now generalize this scenario by starting with more than a single random variable."

Notation

A few notes on notation:

1. $f(x_1, x_2)$ is the original pdf
2. $g(y_1, y_2)$ is the pdf of the two new rvs
3. $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$
4. $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$

...and in this section we're always talking about cts rvs...

Motivation

Remember how in the univariate case, we needed that thing $\left|\frac{d}{dy}g^{-1}(y)\right|$?

Now we have v_1 and v_2, and we can differentiate with respect to y_1 and y_2.

We can make this into a 2x2 matrix:

$$\begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix}.$$

Main Theorem

Let $T = \{(y_1, y_2) : g(y_1, y_2) > 0\}$ and suppose that all the partial derivatives we write down exist for every (y_1, y_2) in T, and they are continuous. Let

$$M = \begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix}.$$

Then $\det(M)$ is called the **Jacobian**, and the new joint pdf for Y_1, Y_2 is

$$g(y_1, y_2) = f[v_1(y_1, y_2), v_2(y_1, y_2)]|\det(M)|$$

Example

Example 5.25 on page 267: Let's start off with X_1 and X_2 two independent exponential random variables both with parameter λ. Then $f_{X_1,X_2}(x_1, x_2) = \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2}$. What's the distribution for $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$?

Example

Example 5.25 on page 267: Let's start off with X_1 and X_2 two independent exponential random variables both with parameter λ. Then $f_{X_1,X_2}(x_1,x_2) = \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2}$. What's the distribution for $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1+X_2)$?
Original transformations are

$$\left[\begin{array}{c} Y_1 \\ Y_2 \end{array}\right] = \left[\begin{array}{c} X_1 + X_2 \\ \frac{X_1}{X_1+X_2} \end{array}\right]$$

backwards transformations are

$$\left[\begin{array}{c} X_1 \\ X_2 \end{array}\right] = \left[\begin{array}{c} Y_1 Y_2 \\ Y_1(1-Y_2) \end{array}\right]$$

Example (continued)

Now the jacobian

$$\begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ (1-y_2) & -y_1 \end{bmatrix}$$

So

$$|\det(M)| = |-y_1y_2 - y_1(1-y_2)| = |y_1|$$

...plug all this in to our formula and we get

$$\begin{aligned} g(y_1, y_2) &= \lambda^2 e^{-\lambda y_1 y_2} e^{-\lambda y_1 (1-y_2)} |y_1|, \qquad 0 < y_1 < \infty, 0 < y_2 < 1 \\ &= \lambda^2 e^{-\lambda y_1} y_1 \, 1(0 < y_1) 1(0 < y_2 < 1) \end{aligned}$$

Let's start off with $f_{X_1X_2}(x_1, x_2) = x_1 + x_2, \quad 0 < x_1, x_2 < 1$. We ultimately want the distribution of X_1X_2. We need an auxiliary random variable, though.

Example 5.26 on page 268

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Original transformations are

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1X_2 \\ X_2 \end{bmatrix}$$

backwards transformations are

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Y_1/Y_2 \\ Y_2 \end{bmatrix}$$

Now the jacobian

$$\begin{bmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1/y_2 & -y_1(y_2)^{-2} \\ 0 & 1 \end{bmatrix}$$

So

$$|\det(M)| = |1/y_2|$$

...plug all this in to our formula and we get

$$\begin{aligned} g(y_1, y_2) &= \left(\frac{y_1}{y_2} + y_2\right)\frac{1}{y_2}, \quad 0 < \frac{y_1}{y_2} < 1, 0 < y_2 < 1 \\ &= \left(\frac{y_1}{y_2} + y_2\right)\frac{1}{y_2}, \quad 0 < y_1 < y_2 < 1 \end{aligned}$$

We aren't finished yet. We have

$$g(y_1, y_2) = \left(\frac{y_1}{y_2} + y_2\right)\frac{1}{y_2}, \qquad 0 < y_1 < y_2 < 1$$

but we want $g_1(y_1)$.

$$\begin{aligned}
g_1(y_1) &= \int_{y_1}^{1} g(y_1, y_2) dy_2 \\
&= \int_{y_1}^{1} \left(\frac{y_1}{y_2} + y_2\right)\frac{1}{y_2} dy_2 \\
&= \int_{y_1}^{1} y_1 y_2^{-2} + 1 dy_2 \\
&= -y_1 y_2^{-1}|_{y_2=y_1}^{y_2=1} + y_2|_{y_2=y_1}^{y_2=1} \\
&= -y_1(1 - \frac{1}{y_1}) + (1 - y_1) = 2(1 - y_1), \qquad 0 < y_1 < 1
\end{aligned}$$

Why does this work?

Recall $g(y_1, y_2)dy_1dy_2 = f[v_1(y_1, y_2), v_2(y_1, y_2)]|\det(M)|dy_1dy_2$. We need

$$\iint f[v_1(y_1, y_2), v_2(y_1, y_2)]|\det(M)|dy_1dy_2 = \iint f(x_1, x_2)dx_1dx_2$$

(integrating both sides gives you the same probabilities (or expected values)

Think of these as the sum of volumes of a bunch of prisms.

1. heights are the same: $f(x_1, x_2) = f[v_1(y_1, y_2), v_2(y_1, y_2)]$
2. bases are the same: $|\det(M)|dy_1dy_2 = dx_1dx_2$