3.2: The ACF and PACF of an ARMA(p,q) Process

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Motivation

This chapter is all about calculating the autocovariance function (ACVF) and the partial autocorrelation function (PACF) of *causal* ARMA processes.

First Method for Calculation of ACVF

Write the model in it's MA(∞) representation, and use the theorem we already have.

$$\phi(B)X_t = \theta(B)Z_t$$

becomes

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where $\psi(z) = \sum_{i=0}^{\infty} \psi_j z^j = \theta(z)/\phi(z)$ for any $|z| \leq 1$ (because causality).

Then

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|},$$

by theorem in section 2.2 (with $\text{Var}(Z_t) = \sigma^2$) .

Example 3.2.1

Consider a general ARMA(1,1). Let $Z_t \sim \text{WN}(0, \sigma^2)$

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

$$\begin{aligned}
\gamma(0) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \\
&= \sigma^2 \left[1 + (\theta + \phi)^2 \sum_{j=0}^{\infty} \phi^{2j} \right] \\
&= \sigma^2 \left[1 + \frac{(\theta + \phi)^2}{1 - \phi^2} \right]
\end{aligned}$$

Hint

In section 2.3 we found that for $j \geq 1$, $\psi_j = (\theta + \phi)\phi^{j-1}$

Example 3.2.1

Now find the lag-1 autocovariance for this ARMA(1,1), with $Z_t \sim \text{WN}(0, \sigma^2)$

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

$$\begin{aligned}
\gamma(1) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+1} \\
&= \sigma^2 \left[\psi_0 \psi_1 + \sum_{j=1}^{\infty} \psi_j \psi_{j+1} \right] \\
&= \sigma^2 \left[\theta + \phi + (\theta + \phi)^2 \phi \sum_{j=0}^{\infty} \phi^{2j} \right] \\
&= \sigma^2 \left[\theta + \phi + \frac{(\theta + \phi)^2 \phi}{1 - \phi^2} \right]
\end{aligned}$$

$\psi_j = (\theta + \phi)\phi^{j-1}, j \geq 1$

Example 3.2.1

Now find the lag-1 autocovariance for this ARMA(1,1), where $Z_t \sim \text{WN}(0, \sigma^2)$

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

If we plot γ, it "tails off." For $h \geq 2$

$$\begin{aligned}
\gamma(h) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \\
&= \sigma^2 \left[\psi_h + \sum_{j=1}^{\infty} \psi_j \psi_{j+h} \right] \\
&= \sigma^2 \left[(\theta + \phi)\phi^{h-1} + \sum_{j=1}^{\infty} (\theta + \phi)^2 \phi^{2j+h-2} \right] \\
&= \phi^{h-1} \gamma(1)
\end{aligned}$$

$\psi_j = (\theta + \phi)\phi^{j-1}, j \geq 1$

Example 3.2.2

Let $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Consider the MA(q) model:

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

On the other hand, ACVFs for MA processes cut off abruptly

$$\begin{aligned}
\gamma(h) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} \\
&= \sigma^2 \sum_{j=0}^{\infty} \theta_j \theta_{j+|h|} \\
&= \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}
\end{aligned}$$

Second Method for Calculation of ACVF

Multiply both sides of an ARMA(p,q) model by $X_{t-k} = \sum_{j=0}^{\infty} \psi_j Z_{t-j-k}$:

$$X_t X_{t-k} - \phi_1 X_{t-1} X_{t-k} - \cdots - \phi_p X_{t-p} X_{t-k} = \left[\sum_{l=0}^{q} \theta_l Z_{t-l}\right] \left[\sum_{j=0}^{\infty} \psi_j Z_{t-j-k}\right]$$

Then take expectations. Note that $\psi_j = 0$ for $j < 0$ and $\theta_l = 0$ for $l \notin \{0, 1, \ldots, q\}$.

Method 2

Solve the following $p+1$ equations for $\gamma(0), \gamma(1), \ldots, \gamma(p)$:

$$\gamma(k) - \phi_1 \gamma(k-1) - \cdots - \phi_p \gamma(k-p) = \sigma^2 \sum_{j=0}^{q-k} \theta_{k+j} \psi_j$$

for $0 \leq k \leq p$. The right hand side may be 0 for some k. Higher order autocovariances are also obtainable with the same formula.

Example 3.2.5 (example 3.2.3 revisited)

Consider the ARMA(1,1) from examples 3.2.1 and 3.2.3. Method 3 starts us off with:

1. $\gamma(0) - \phi\gamma(1) = \sigma^2[1 + \theta(\theta + \phi)]$
2. $\gamma(1) - \phi\gamma(0) = \sigma^2[\theta]$
3. $\gamma(k) - \phi\gamma(k-1) = 0$ for $k \geq 2$

Plugging (2) into (1) gives us

$$\gamma(0) = \sigma^2 \left[1 + \frac{(\theta + \phi)^2}{1 - \phi^2}\right]$$

(which is the same as before) and plugging this into (2) gives us

$$\gamma(1) = \sigma^2 \left[\theta + \phi + \frac{(\theta + \phi)^2\phi}{1 - \phi^2}\right].$$

For any other lag we can use (3).

Example 3.2.5 (example 3.2.3 revisited)

We can check our work with R. For more details see the code file.

```
ARMAacf(ar = c(testPhi), ma = c(testTheta))
```

Autocorrelation (ACF)

Finding the ACF from the ACVF (both sample and population acf) is easy:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

and

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

Partial Autocorrelation

Motivation: knowing that the ACF of an MA model cuts off at q is fantastic. This is because identifying the order of an MA model with sample ACFs is easy. Just find the highest nonzero lag.

Too bad ACFs don't exhibit this. But they do when you look at the *partial* autocorrelation function (PACF). The PACF for a pure AR model will cut off right at p, telling us exactly which model we need.

Partial Autocorrelation

PACF is denoted by $\alpha(h)$ where h is the lag. It gives us partial correlations. This means it gives us the correlation "controlling" for other variables.

The ACF never controlled for anything in between.

Partial Autocorrelation

PACF can be calculated with the prediction equations we used in chapter 2.5. Say we want to find $\alpha(h)$. We find the coefficients that best predict X_{t+h} in terms of $X_{t+h-1}, \ldots, X_t$. The last coefficient takes into account the intermediate variables, and "controls" for them, and so is the same as the PACF for this particular lag.

$$\hat{X}_{t+h} = \phi_{h1} X_{t+h-1} + \cdots + \phi_{hh} X_t$$

The Return of the Prediction Equations

For $j = 1, \ldots, h$ we showed $\gamma(j) = \phi_{h1}\gamma(j-1) + \cdots \phi_{hh}\gamma(j-h)$. We can rewrite this as

$$\begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(h) \end{bmatrix} = \Gamma_h \begin{bmatrix} \phi_{h1} \\ \phi_{h2} \\ \vdots \\ \phi_{hh} \end{bmatrix}$$

where $\Gamma_h = [\gamma(i-j)]_{i,j=1}^h$.

So ϕ_{hh} is the last component of $\Gamma_h^{-1}\boldsymbol{\gamma}_h$ where $\boldsymbol{\gamma}_h$ is the LHS. If we divide both sides of that equation by $\gamma(0)$, we can write everything in terms of autocorrelations. We do that procedure over and over again for each h!

Partial Autocorrelation Example: AR(p) models

I mention again that the notation for best prediction coefficients is suggestive. These two sets of ϕ are not the same. However, often times some elements are equal.

Model:

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t, \qquad \{Z_t\} \sim WN(0, \sigma^2)$$

Prediction equations for $\alpha(h)$.

$$\hat{X}_t = \phi_{h1} X_{t-1} + \cdots + \phi_{hh} X_{t-h}$$

$\alpha(h) = \phi_h$ if $h \leq p$. Otherwise $\alpha(h) = 0$. This follows from homework problem 2.15 in your last homework. It's also a new homework question!

Partial Autocorrelation

Checking our answers is possible for finding PACFs, too. We can use the same function as before to compute theorertical PACFs:

```
ARMAacf(ar = c(testPhi), ma = c(testTheta), pacf = TRUE)
```

Partial Autocorrelation

What about sample PACF? We just use the last equation and put hats on everything (we use the sample autocorrelation)

1. $\alpha(0) = 1$
2. $\hat{\alpha}(h) = \hat{\phi}_{hh}$

where $\hat{\phi}_{hh}$ is the last component of

$$\hat{\phi} = \hat{\Gamma}_h^{-1}\hat{\gamma}_h$$

```
# see code file for more
pacf(rets)
```

Partial Autocorrelation

There might be a more useful model selection strategy than just checking where you think the ACF and PACF cut off. For example, if the true model is not a pure MA or AR, then this "finding cutoffs" strategy is useless.

Now you have the computational tools to plot sample *and* theoretical ACFs and PACFS. Try more sophisticated strategies to match them up and pick a better model!