3.1: ARMA(p,q) models

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Motivation

We extend ARMA models in this chapter.

Definitions

ARMA(p,q) process

$\{X_t\}$ is an ARMA(p,q) process if $\{X_t\}$ is stationary and if for every t,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and the polynomials $\phi(z) = (1 - \phi_1 z - \cdots - \phi_p z^p)$ and $\theta(z) = (1 + \theta_1 z + \cdots + \theta_q z^q)$ have no common factors.

1. X_t is an ARMA(p,q) with mean μ if we replace the above X_t with $X_t - \mu$
2. Easier to write $\phi(B)X_t = \theta(B)Z_t$
3. AR(p) if $\theta(z) = 1$
4. MA(q) if $\phi(z) = 1$

From Old to New

If we learn a little about complex numbers, we can start stating these parameter restrictions in terms of the argument to these polynomials, $z = a + bi$. This helps us because we're starting to deal with more parameters now, and because it's more common in practice. Recall $|z|^2 = z\bar{z} = a^2 + b^2$.

Before, in 2.3: $(1 - \phi_1 z)$ was our AR(1) polynomial. Setting it equal to zero yields

$$1 = \phi_1 z_1 \iff \phi_1 = 1/z_1 \iff z_1 = 1/\phi_1$$

where z_1 is the root (there's only one and it's real in this case).

The backshift operator becomes a complex number. Stationary solutions exist if and only if $\phi_1 \neq \pm 1$. Stated in terms of z now $(1 - \phi_1 z) \neq 0$ when $|z| = \pm 1$. Why the $|\cdot|$? When we have higher order AR models, the roots are often complex.

Complex Polynomials

Every degree p polynomial $\phi(z)$ can be factored into

$$\begin{aligned}\phi(z) &= 1 + \phi_1 z + \cdots \phi_p z^p \\ &= \phi_p(z - z_1)(z - z_2)\cdots(z - z_p).\end{aligned}$$

Because the coefficients $\phi_1, \ldots, \phi_p$ are real numbers, all of these roots $z_1, \ldots, z_p$, even though they are written in terms of our coefficients, are real or complex pairs.

Example:

$$\begin{aligned}\phi(z) &= 2z^3 + 0z^2 + z + 0 \\ &= 2[z^3 + 0z^2 + \frac{1}{2}z + 0] \\ &= 2(z - 0)(z - \sqrt{-1/2})(z + \sqrt{-1/2})\end{aligned}$$

Number of roots matches the degree of polynomial. One complex pair of roots, and one real root. $z_1 = 0$, $z_2 = \sqrt{-1/2}$ and $z_3 = -\sqrt{-1/2}$.

Unit Roots

Stationarity from last chapter: $(1 - \phi z) \neq 0$ when $|z| = \pm 1$

For higher order AR models, the analagous situation is that $\phi(z)$ has no "unit roots." That is, a stationary ARMA solution exists if and only if $\phi(z) = (1 - \phi_1 z - \cdots \phi_p z^p) \neq 0$ for $|z| = \pm 1$.

Whenever you plug in a z that lies on the unit circle in the complex plane, the polynomial will not be equal to 0. Our definition of ARMA requires this because we want to divide both sides by $\phi(z)$. No unit roots implies that

$$\frac{1}{\phi(z)} = \sum_{j=-\infty}^{\infty} \chi_j z^j$$

and $\sum_{j=-\infty}^{\infty} |\chi_j| < \infty$ for $1 - \delta < |z| < 1 + \delta$ with some $\delta > 0$

Causality $\rightarrow$ Stationarity

$$\frac{1}{\phi(z)} = \sum_{j=-\infty}^{\infty} \chi_j z^j$$

and $\sum_{j=-\infty}^{\infty} |\chi_j| < \infty$ for $1 - \delta < |z| < 1 + \delta$ with some $\delta > 0$

This is a fancy way of saying we can divide both sides of the equation by $\phi(B)$ to isolate X_t on the left hand side.

Notice this doesn't guarantee that it's causal. For that we need to further restrict the roots to lie outside the unit circle.

Lemmas

stationarity

A stationary solution of $\{X_t\}$ exists and is unique if and only if $|z| = 1$ implies

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p \neq 0.$$

causality

An ARMA(p,q) process $\{X_t\}$ is **causal** if and only if $|z| \leq 1$ implies

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p \neq 0.$$

causality 2

An ARMA(p,q) process $\{X_t\}$ is **causal** if and only if there exist $\{\psi_j\}$ such that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ for all t

Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

Is it stationary? Is it causal?

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$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

Is it stationary? Is it causal?

We don't care about the MA part for this.

$$1 - .5z_1 = 0 \iff z_1 = 1/.5 \iff z_1 = 2$$

$|2| \neq 0$ so a stationary solution exists. $|2| > 0$ so it is also causal.

```
polyroot(c(1, -.5))
```

Invertibility

Invertibility involves the same math with complex polynomials, but only concerns $\theta(z)$.

invertibility

An ARMA(p,q) process is $\{X_t\}$ is **invertible** if and only if there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}.$$

invertibility 2

An ARMA(p,q) process is $\{X_t\}$ is **invertible** if and only if $|z| \leq 1$ implies

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0.$$

Back to Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

Is it invertible?

Back to Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

Is it invertible?

We don't care about the AR part for this.

$$1 + .4z_1 = 0 \iff z_1 = -1/.4 \iff z_1 = -2.5$$

$|-2.5| > 0$ so it is invertible!

```
polyroot(c(1,.4))
```

Back to Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

find $\psi(z)$ and $\pi(z)$.

Back to Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

find $\psi(z)$ and $\pi(z)$.

$\chi(z)$ is the reciprocal of $\phi(z)$ iff

$$(1 + \chi_1 z + \chi_2 z^2 + \cdots)(1 - .5z) = 1$$
$$1 + z(-.5 + \chi_1) + z^2(-\chi_1 .5 + \chi_2) + \cdots = 1 + 0z + 0z^2 + \cdots$$

So $\chi_1 = 2^{-1}$, $\chi_2 = 2^{-2}, \ldots$

Back to Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

Now that we have the reciprocal, we can multiply both sides by it

$$\begin{aligned}\psi(z) = \chi(z)\theta(z) &= (1 + z/2 + z^2/4 + \cdots)(1 + .4z) \\ &= 1 + z(.4 + .5) + z^2(.4 \cdot .5 + .5^2) + \cdots\end{aligned}$$

So

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where $\psi_0 = 1$, $\psi_1 = .4 + .5$, $\psi_2 = .5(.4 + .5)$, $\psi_3 = .5^2(.4 + .5)$, ...

Back to Example 3.1.1

Checking our answers:

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where $\psi_0 = 1$, $\psi_1 = .4 + .5$, $\psi_2 = .5(.4 + .5)$, $\psi_3 = .5^2(.4 + .5)$, $\ldots$

This function ignores the 1s. Also be careful about the signs of the coefficients!

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ARMAtoMA(ar=c(.5), ma=c(.4), lag.max=3)
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Back to Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

$\xi(z)$ is the reciprocal of $\theta(z)$ iff

$$\begin{aligned} \xi(z)\theta(z) &= 1 \\ (1 + \xi_1 z + \xi_2 z^2 + \cdots)(1 + .4z) &= 1 + 0z + 0z^2 + \cdots \\ 1 + z(.4 + \xi_1) + z^2(\xi_1 .4 + \xi_2) + z^3(.4\xi_2 + \xi_3) + \cdots &= 1 + 0z + 0z^2 + \cdots \end{aligned}$$

So $\xi_0 = 1$, $\xi_1 = -.4$, $\xi_2 = .4^2$, $\xi_3 = -.4^3, \ldots$

Back to Example 3.1.1

Consider the ARMA(1,1) model

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}$$

Now that we have the reciprocal, we can multiply both sides by it

$$\begin{aligned}\pi(z) = \xi(z)\phi(z) &= (1 + -.4z + .4^2 z + \cdots)(1 - .5z) \\ &= 1 + z(-.4 - .5) + z^2(.5 \cdot .4 + .4^2) + \cdots\end{aligned}$$

So

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

where $\pi_0 = 1$, $\pi_1 = -(.4 + .5)$, $\pi_2 = -(.4 + .5)(-.4)$, $\pi_3 = -(.4 + .5)(-.4)^2$, $\ldots$

Consider the AR(2) model

$$X_t = .7X_{t-1} - .1X_{t-2} = Z_t$$

Is it causal? Find it's MA(∞) representation.

3.1.2

Consider the AR(2) model

$$X_t = .7X_{t-1} - .1X_{t-2} = Z_t$$

Is it causal? Find it's MA(∞) representation.

$$\begin{aligned}\phi(z) &= 1 - .7z + .1z^2 \\ &= .1(z^2 - 7z + 10) \\ &= .1(z-5)(z-2)\end{aligned}$$

Because both $|z_1| = |5|$ and $|z_2| = |2|$ are outside of the unit circle, this model is causal.

3.1.2

Consider the AR(2) model

$$X_t = .7X_{t-1} - .1X_{t-2} = Z_t$$

Is it causal? Find it's MA(∞) representation.

3.1.2

Consider the AR(2) model

$$X_t = .7X_{t-1} - .1X_{t-2} = Z_t$$

Is it causal? Find it's MA(∞) representation.

Find the reciprocal of $\phi(z)$ first

$$\chi(z)\phi(z) = 1$$
$$(1 + \chi_1 z + \chi_2 z^2 + \chi_3 z^3 + \cdots)(1 - .7z + .1z^2) = 1$$
$$1 + z(-.7 + \chi_1) + z^2(.1 - .7\chi_1 + \chi_2) + z^3(.1\chi_1 + -.7\chi_2 + \chi_3) + \cdots = 1$$

So $\chi_0 = 1$, $\chi_1 = .7$, $\chi_2 = .7^2 - .1$, $\chi_3 = .7\chi_2 - .1\chi_1$.

3.1.2

Consider the AR(2) model

$$X_t = .7X_{t-1} - .1X_{t-2} = Z_t$$

Is it causal? Find it's MA(∞) representation.

Then multiply both sides by this reciprocal

$$\begin{aligned} \psi(z) &= \chi(z)\theta(z) \\ &= \chi(z) \end{aligned}$$

3.1.3

Consider the ARMA(2,1) model

$$X_t - .75X_{t-1} + .5625X_{t-2} = Z_t + 1.25Z_{t-1}$$

Is it causal? Is it invertible? Hint: $.75^2 = .5625$

Consider the ARMA(2,1) model

$$X_t - .75X_{t-1} + .5625X_{t-2} = Z_t + 1.25Z_{t-1}$$

Is it causal? Is it invertible? Hint: $.75^2 = .5625$

Causality check on $\phi(z) = 1 - .75z + .5625z^2$ using quadratic formula:

$$\frac{.75 \pm \sqrt{.75^2 - 4(.5625)}}{(2)(.5625)} = \frac{.75 \pm \sqrt{.75^2\{1-4\}}}{(2)(.5625)}$$
$$= \frac{.75 \pm .75\sqrt{-3}}{(2)(.5625)} = \frac{2}{3}(1 \pm \sqrt{3}i)$$

$|z_1| = |2/3 + i2\sqrt{3}/3| = \sqrt{4/9 + 12/9} = 4/3 > 0$. $|z_2|$ is the same. So this is a causal model.

3.1.3

Consider the ARMA(2,1) model

$$X_t - .75X_{t-1} + .5625X_{t-2} = Z_t + 1.25Z_{t-1}$$

Is it causal? Is it invertible? Hint: $.75^2 = .5625$

Consider the ARMA(2,1) model

$$X_t - .75X_{t-1} + .5625X_{t-2} = Z_t + 1.25Z_{t-1}$$

Is it causal? Is it invertible? Hint: $.75^2 = .5625$

Invertibility check on $\theta(z) = 1 + 1.25z$. Clearly $|z_1| = |-4/5| <= 1$. So this is not invertible.