7.2: GARCH Models

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Definitions

Recall the ARCH(p) model (Engle 1982)

ARCH(p)

$$Z_t = \sqrt{h_t} e_t, \qquad \{e_t\} \sim IID(0,1)$$

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2.$$

$\alpha_0 > 0,\ a_i \geq 0,\ p \in \mathbb{N}$

A recursive formula

In the case of ARCH(1), $h_t = \alpha_0 + \alpha_1 Z_{t-1}^2$ and

$$
\begin{aligned}
Z_t^2 &= h_t e_t^2 \\
&= [\alpha_0 + \alpha_1 Z_{t-1}^2] e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 h_{t-1} e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 [\alpha_0 + \alpha_1 Z_{t-2}^2] e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 Z_{t-2}^2 e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 [h_{t-2} e_{t-2}^2] e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 [\alpha_0 + \alpha_1 Z_{t-3}^2] e_{t-2}^2 e_{t-1}^2 e_t^2 \\
&= \left\{\alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 \alpha_0 e_{t-2}^2 e_{t-1}^2 e_t^2\right\} + \left\{\alpha_1^3 Z_{t-3}^2 e_{t-2}^2 e_{t-1}^2 e_t^2\right\} \\
&= \alpha_0 \sum_{j=0}^{n} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2 + a_1^{n+1} Z_{t-n-1}^2 e_t^2 e_{t-1}^2 \cdots e_{t-n}^2
\end{aligned}
$$

A recursive formula

$$Z_t^2 = \left(\alpha_0 \sum_{j=0}^{n} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2 \right) + \left(a_1^{n+1} Z_{t-n-1}^2 e_t^2 e_{t-1}^2 \cdots e_{t-n}^2 \right)$$

If $\alpha_1 < 1$:

- second term goes to 0 as $n \to \infty$
- first term has a limit, let's call it $\alpha_0 \sum_{j=0}^{\infty} \alpha_1^j (e_t^2 \times \cdots \times e_{t-j}^2)$

so

$$Z_t^2 = \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2$$

if we're looking at an infinitely long sequence.

Weak-Stationarity

Weakly-stationary!

$$E[Z_t] = E[\sqrt{h_t}e_t] = E[\sqrt{h_t}]E[e_t] = 0$$

Marginal variance

$$\begin{aligned}
\text{Var}[Z_t] &= E[Z_t^2] \\
&= E[\alpha_0 \sum_{j=0}^{\infty} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2] && \text{(previous slide)} \\
&= \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j = \alpha_0/(1-\alpha_1) \\
& && \text{(linearity, independence, geom. series)}
\end{aligned}$$

Autocovariance

$$\begin{aligned}
\gamma_Z(h) &= E[Z_{t+h}Z_t] = E[E(Z_{t+h}Z_t | e_s, s < t+h)] && \text{(LTE)} \\
&= E[Z_t E(Z_{t+h} | e_s, s < t+h)] = 0
\end{aligned}$$

A recursive formula

But remember that volatility is the **conditional** variance. After taking the square root on both sides of a previous formula

$$Z_t = e_t \sqrt{\alpha_0 \left(1 + \sum_{j=1}^{\infty} \alpha_1^j e_{t-1}^2 \cdots e_{t-j}^2 \right)}.$$

More recent errors get higher weights in the conditional standard deviation term.

Also note the small typo in the book.

White but not IID

The ARCH model is white noise, but not IID noise.

$$\begin{aligned}
E[Z_t^2 | Z_{1:t-1}] &= E[(\alpha_0 + \alpha_1 Z_{t-1}^2) e_t^2 | Z_{1:t-1}] \\
&= (\alpha_0 + \alpha_1 Z_{t-1}^2) E[e_t^2 | Z_{1:t-1}] \\
&= (\alpha_0 + \alpha_1 Z_{t-1}^2) \\
&\neq E[Z_t^2] = \alpha_0 / (1 - \alpha_1)
\end{aligned}$$

fun facts:

- $\{Z_t\}$ cannot be jointly Gaussian (write down the likelihood and it will factor)
- However it can be conditionally Gaussian (next slide)
- $Z_t \stackrel{d}{=} -Z_t$ (previous slide)

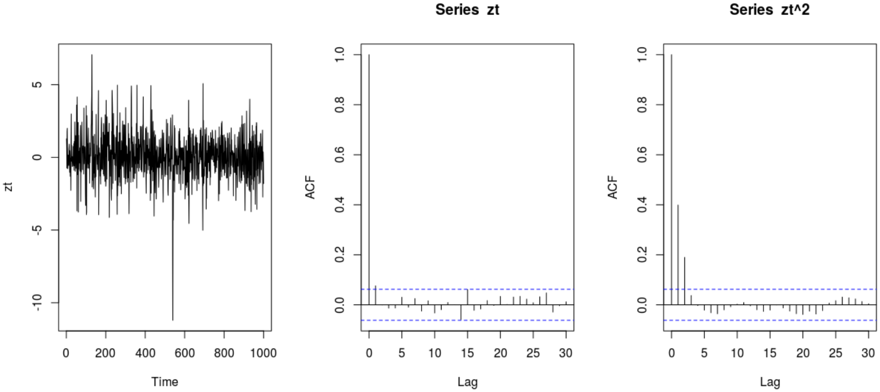
The Conditional Likelihood

Let's write down the conditional likelihood:

$$
\begin{aligned}
L &= f(z_n, z_{n-1}, \ldots, z_2 | z_1) \\
&= \prod_{t=2}^{n} f(z_t | z_{1:t-1}) \\
&= \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi h_t}} \exp\left[-\frac{z_t^2}{2h_t}\right] && \text{(cndtl nrmlty)} \\
&= \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 z_{t-1}^2)}} \exp\left[-\frac{z_t^2}{2(\alpha_0 + \alpha_1 z_{t-1}^2)}\right] && \text{(defn. } h_t\text{)}
\end{aligned}
$$

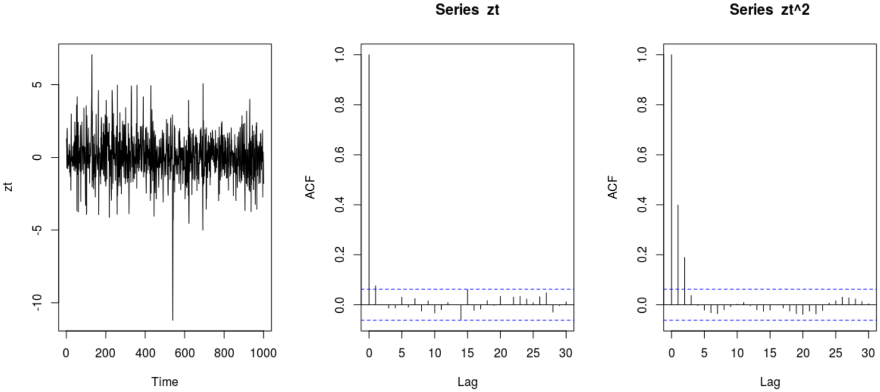
Test

Fake data or real data?



Test

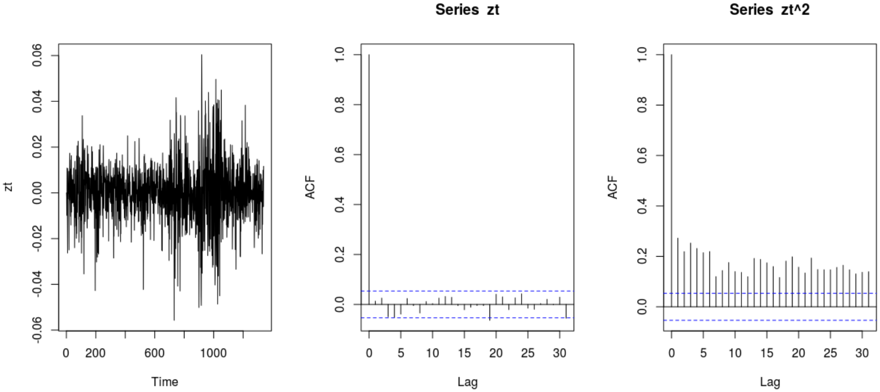
Fake data or real data?




```
spec = garchSpec(model = list(omega = 1, alpha = c(0.5),
                 beta = 0))
zt <- garchSim(spec, n = 1000)
```

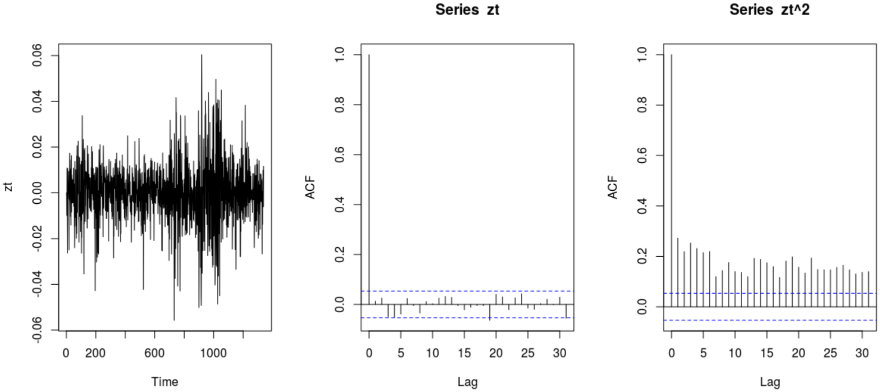
Test

Fake data or real data?



Test

Fake data or real data?



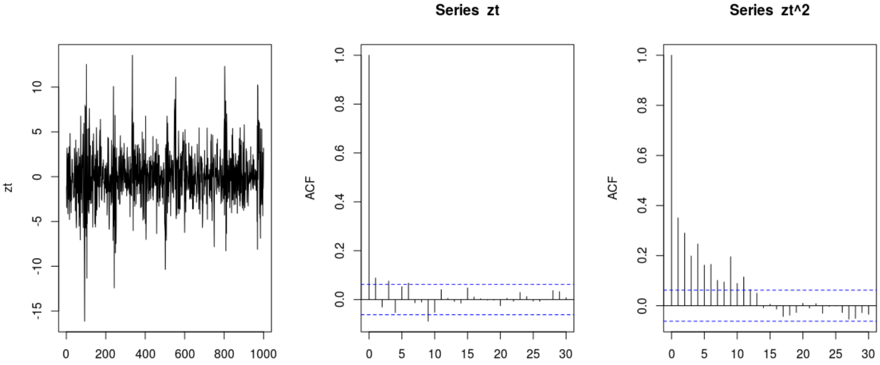

```
getSymbols(Symbols="CVX", src="google")
cvx <- CVX["2012-01-01/2017-04-28"] #remove some of the early
adCVX <- Cl(cvx)
rets <- periodReturn(adCVX, period="daily", type = "log")
```

GARCH Models

Recall the GARCH(p,q) model has the volatility as

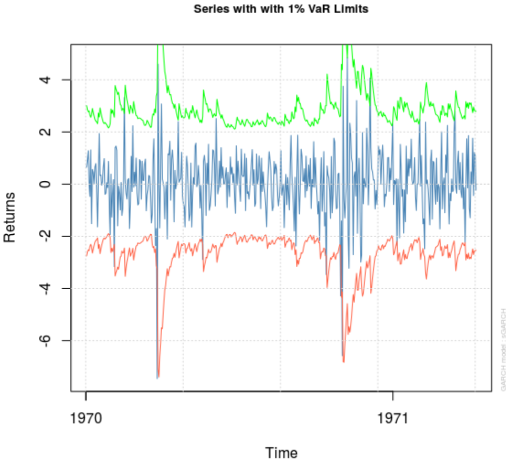
$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j},$$

where $\alpha_0 > 0$, $a_i \geq 0$, $\beta_j \geq 0$, $p \in \mathbb{N}$. This allows for autocorrelation in h_t (clustering).



Example 7.2.2

See 7.2.R for details. These are the observed returns along with 2 std.dev prediction intervals.



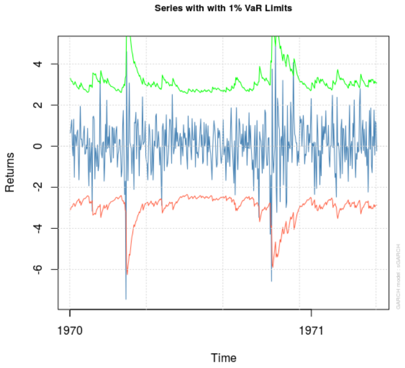
The Conditional Likelihood

Sometimes a better fit can be obtained by using t distributions for the conditional likelihood. We assume that $\sqrt{\frac{\nu}{\nu-2}}e_t \sim t_\nu$ with $\nu > 2$. The conditional likelihood:

$$\begin{aligned} L &= f(z_n, z_{n-1}, \ldots, z_2 | z_1) \\ &= \prod_{t=2}^{n} f(z_t | z_{1:t-1}) \\ &= \prod_{t=2}^{n} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\Gamma(\frac{1}{2})\sqrt{\nu-2}} \left(1 + \frac{e_t^2}{\nu-2}\right)^{-\frac{\nu+1}{2}} \end{aligned}$$

Example 7.2.2

The t distribution fit has a better AIC!




```
> (-2*likelihood(fit3))/length(xt)+2*(length(fit3@fit$coef))/l
[1] 3.094884
> (-2*likelihood(fit))/length(xt)+2*(length(fit@fit$coef))/len
[1] 3.160894
```

Example 7.2.2

Another example ARIMA(1,1) + GARCH(1,1):

$$X_t - \mu = \phi(X_{t-1} - \mu) + Z_t + \theta Z_{t-1}$$

$$Z_t = \sqrt{h_t} e_t$$

$$h_t = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}$$