ARCH and GARCH models

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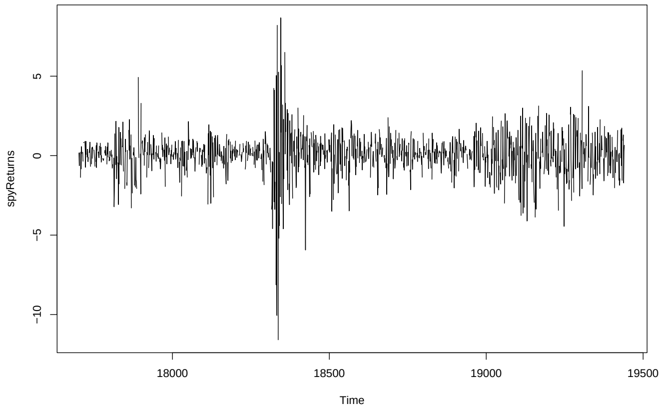
Definitions

- P_t: the closing price on day/week/month t of stock/stock index
- $X_t = \log P_t$: log-price...observed paths look very much like those of a random walk
- $Z_t = \nabla X_t$: the log return
- $100 Z_t$: percentage returns
- $h_t = \mathrm{Var}(Z_t | Z_{1:t-1})$: the conditional variance aka volatility

Motivation



Motivation

We would like models that take into account *stylized features* that appear in real-world data such as

- **tail-heaviness**
- **asymmetry**
- **volatility clustering**
- **serial dependence without correlation**

We introduce AutoRegressive Conditional Heteroscedasticity (ARCH) models, Generalized AutoRegressive Conditional Heteroscedasticity (GARCH), and stochastic volatility (SVOL) models in this slide deck.

Definition: ARCH(p)

ARCH(p) (Engle 1982)

$$Z_t = \sqrt{h_t} e_t, \qquad \{e_t\} \sim IID(0, 1)$$
$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2.$$

$\alpha_0 > 0$, $a_i \geq 0$, $p \in \mathbb{N}$

- volatility increases if we have observed big movements
- setting a_i to 0 gives us white noise model
- $\{e_t\}$ is sometimes but not always assumed to be Normal

Definition: GARCH(p,q) model

GARCH(p,q)

$$Z_t = \sqrt{h_t} e_t, \qquad \{e_t\} \sim IID(0,1)$$
$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}.$$

$\alpha_0 > 0$, $a_i \geq 0$, $\beta_j \geq 0$, $p \in \mathbb{N}$

- now volatility is a function of its own past values, in addition to the past observations
- $\{e_t\}$ may or may not be normal

Definition: Stochastic Volatility model

Stochastic Volatility Model

$$Z_t = \sqrt{h_t} e_t, \qquad \{e_t\} \sim IID(0,1)$$
$$\ln h_t = \gamma_0 + \gamma_1 \ln h_{t-1} + \eta_t, \qquad \{\eta_t\} \sim IID(0, \sigma^2).$$

where $\{\eta_t\}$ and $\{e_t\}$ are independent.

- log-volatility is an AR(1) process.
- γ_1 is usually around .95.
- more difficult to estimate (intractable likelihood)

ARCH(1)

In the case of ARCH(1), $h_t = \alpha_0 + \alpha_1 Z_{t-1}^2$ and

$$\begin{aligned}
Z_t^2 &= h_t e_t^2 \\
&= [\alpha_0 + \alpha_1 Z_{t-1}^2] e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 h_{t-1} e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 [\alpha_0 + \alpha_1 Z_{t-2}^2] e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 Z_{t-2}^2 e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 [h_{t-2} e_{t-2}^2] e_{t-1}^2 e_t^2 \\
&= \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 [\alpha_0 + \alpha_1 Z_{t-3}^2] e_{t-2}^2 e_{t-1}^2 e_t^2 \\
&= \left\{ \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_{t-1}^2 e_t^2 + \alpha_1^2 \alpha_0 e_{t-2}^2 e_{t-1}^2 e_t^2 \right\} + \left\{ \alpha_1^3 Z_{t-3}^2 e_{t-2}^2 e_{t-1}^2 e_t^2 \right\} \\
&= \alpha_0 \sum_{j=0}^{n} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2 + a_1^{n+1} Z_{t-n-1}^2 e_t^2 e_{t-1}^2 \cdots e_{t-n}^2
\end{aligned}$$

ARCH(1)

$$Z_t^2 = \left(\alpha_0 \sum_{j=0}^{n} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2 \right) + \left(a_1^{n+1} Z_{t-n-1}^2 e_t^2 e_{t-1}^2 \cdots e_{t-n}^2 \right)$$

If $\alpha_1 < 1$:

- second term goes to 0 as $n \to \infty$
- first term has a limit, let's call it $\alpha_0 \sum_{j=0}^{\infty} \alpha_1^j (e_t^2 \times \cdots \times e_{t-j}^2)$

so

$$Z_t^2 = \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2$$

if we're looking at an infinitely long sequence.

ARCH(1)

Weakly-stationary!

$$E[Z_t] = E[\sqrt{h_t} e_t] = E[\sqrt{h_t}]E[e_t] = 0$$

Marginal variance

$$\begin{aligned}
\mathrm{Var}[Z_t] &= E[Z_t^2] \\
&= E[\alpha_0 \sum_{j=0}^{\infty} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2] \qquad \text{(previous slide)} \\
&= \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j = \alpha_0/(1-\alpha_1) \\
&\qquad \text{(linearity, independence, geom. series)}
\end{aligned}$$

Autocovariance

$$\begin{aligned}
\gamma_Z(h) &= E[Z_{t+h}Z_t] = E[E(Z_{t+h}Z_t | e_s, s < t+h)] \qquad \text{(LTE)} \\
&= E[Z_t E(Z_{t+h} | e_s, s < t+h)] = 0
\end{aligned}$$

But remember that volatility in this case is the **conditional**

ARCH(1)

The ARCH(1) model is white noise, but not IID noise.

$$
\begin{aligned}
E[Z_t^2|Z_{1:t-1}] &= E[(\alpha_0 + \alpha_1 Z_{t-1}^2)e_t^2|Z_{1:t-1}] \\
&= (\alpha_0 + \alpha_1 Z_{t-1}^2)E[e_t^2|Z_{1:t-1}] \\
&= (\alpha_0 + \alpha_1 Z_{t-1}^2)
\end{aligned}
$$

This depends on Z_{t-1}. It is **not** $E[Z_t^2] = \alpha_0/(1-\alpha_1)$.

ARCH(1)

$$E[Z_t^2|Z_{1:t-1}] = (\alpha_0 + \alpha_1 Z_{t-1}^2)$$

So

$$\begin{aligned}
\gamma(1)_{Z^2} &= \mathbb{E}[Z_t^2 Z_{t-1}^2] \\
&= E\left[Z_{t-1}^2 E\left(Z_t^2 \mid Z_{1:t-1}\right)\right] \\
&= E\left[Z_{t-1}^2(\alpha_0 + \alpha_1 Z_{t-1}^2)\right] \\
&= \alpha_0 + \alpha_1 \gamma(0)_{Z^2}
\end{aligned}$$

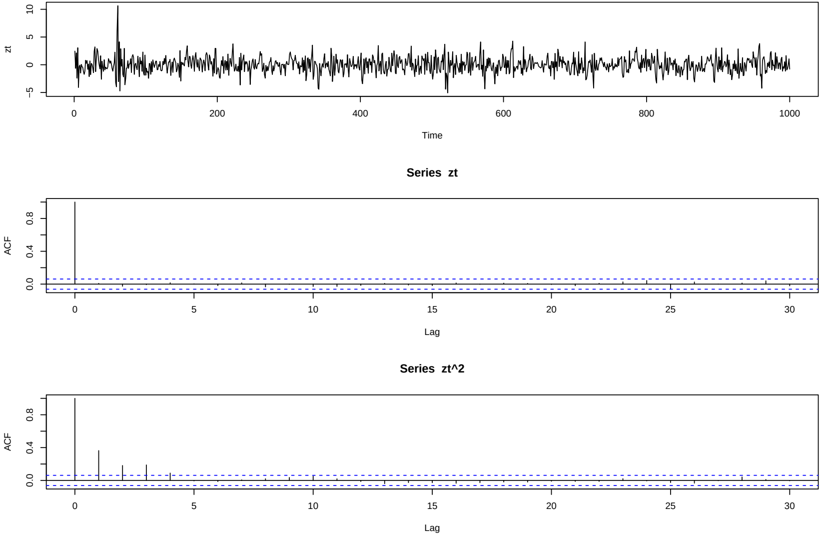
This is why we look at autocorrelation of the squared return process.

TODO: homework generalize to lag greater than 1

TODO this can be made into a hw question ˆ

Example 1

Fake data or real data?



Example 1

Fake!

```
library(fGarch)
spec <- garchSpec(model = list(omega = 1, alpha = c(0.5),
                  beta = 0))
zt <- garchSim(spec, n = 1000)
```

Example 2

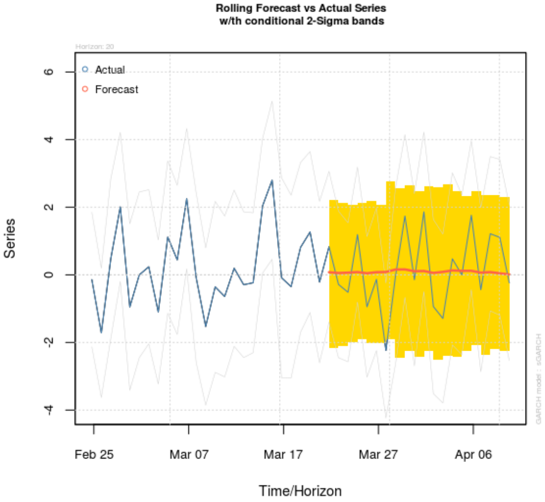
Another example ARMA(1,1) + GARCH(1,1):

$$X_t - \mu = \phi(X_{t-1} - \mu) + Z_t + \theta Z_{t-1}$$

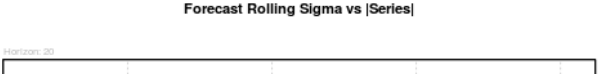
$$Z_t = \sqrt{h_t} e_t$$

$$h_t = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}$$

Example 2



Rolling Forecast vs Actual Series
w/th conditional 2-Sigma bands
Horizon: 20
Actual
Forecast
6
4
2
0
-2
-4
Series
Feb 25
Mar 07
Mar 17
Mar 27
Apr 06
Time/Horizon
GARCH model : sGARCH



Forecast Rolling Sigma vs |Series|
Horizon: 20

Sources:

Chapters 7.1,7.2 of Introduction to Time Series and Forecasting
Brockwell/Davis