2.2: Linear Processes

Taylor

University of Virginia

Motivation

Linear processes is a class that's larger than ARMA models. Knowing these well is essential to doing time series analysis. Later we'll talk about Wold's Decomposition, which says that every second-order stationary process is either a linear process or can be transformed to a linear process by subtracting a deterministic component.

Definition

Linear Time Series

The time series $\{X_t\}$ is a **Linear Process** if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

for all $t \in \mathbb{Z}$, where $Z_t \sim \text{WN}(0, \sigma^2)$, and $\{\psi_j\}$ is a sequence of constants such that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ (i.e. they're absolutely summable)

Sometimes we write this more concisely as

$$X_t = \psi(B)Z_t = \left(\sum_{j=-\infty}^{\infty} \psi_j B^j \right) Z_t.$$

Example 1

Some Linear Processes aren't always useful in finance. We don't always want to study models where X_t depends on future noise Z_{t+j}.

Example 1

Some Linear Processes aren't always useful in finance. We don't always want to study models where X_t depends on future noise Z_{t+j}.

Show how to write down the subset of these models where all future ψ_j coefficients are set to 0. Show that it is reasonable to call this class of models an MA(∞) process.

Example 1

Some Linear Processes aren't always useful in finance. We don't always want to study models where X_t depends on future noise Z_{t+j}.

Show how to write down the subset of these models where all future ψ_j coefficients are set to 0. Show that it is reasonable to call this class of models an MA(∞) process.

$$X_t = (\cdots + 0B^{-1} + \psi_0 B^0 + \psi_j B^1 + \cdots)Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

Example 2

Show that absolute summability ($\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$) guarantees that $\sum_{j=-n}^{n} \psi_j Z_{t-j}$ converges in mean square as $n \to \infty$. We only have to show Cauchy convergence!

$$
\begin{aligned}
E\left[\left|\sum_{j=-n}^{n} \psi_j Z_{t-j} - \sum_{j=-m}^{m} \psi_j Z_{t-j}\right|\right] &= E\left[\left|\sum_{m<|j|\le n} \psi_j Z_{t-j}\right|\right] \\
&\le E\left[\sum_{m<|j|\le n} |\psi_j Z_{t-j}|\right] \qquad \text{(tri-ineq.)} \\
&= E[|Z_t|] \sum_{m<|j|\le n} |\psi_j| \\
&\to 0
\end{aligned}
$$

as $n > m \to \infty$

Example 3

The space of all stationary time series is closed under taking these type of linear filters.

Theorem

Let X_t be stationary with mean 0 and have ACVF $\gamma_X(\cdot)$. Also let $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. Then

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$$

is stationary with mean 0 and has ACVF

$$\gamma_Y(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h + j - k)$$

Proof

New mean in terms of the old mean:

$$E[Y_t] = E\left[\sum_{j=-\infty}^{\infty} \psi_j X_{t-j}\right] = \sum_{j=-\infty}^{\infty} \psi_j E[X_{t-j}] = 0.$$

Proof

The new covariance function exists (check) and is equal to:

$$\gamma_Y(h) = \text{Cov}\left[\sum_{j=-\infty}^{\infty} \psi_j X_{t-j}, \sum_{k=-\infty}^{\infty} \psi_k X_{t+h-k}\right]$$
$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \text{Cov}\left[X_{t-j} X_{t+h-k}\right] \qquad (*)$$
$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h+j-k)$$

Special case: X_t is white noise. Non-zero covariance terms when $-j = h - k$! Show that we would get $\gamma_Y(h) = \gamma_X(0) \sum_{j \in \mathbb{Z}} \psi_j \psi_{j+h}$.

Successive Filtering

From $\{Y_t\}$, to $\{X_t\}$, to $\{W_t\}$. First

$$X_t = \alpha(B)Y_t = \sum_{j=-\infty}^{\infty} \alpha_j Y_{t-j},$$

then we apply another filter

$$W_t = \beta(B)X_t = \sum_{k=-\infty}^{\infty} \beta_k X_{t-k}$$

It's easiest to just treat these as polynomials:

$$W_t = \sum_{i=-\infty}^{\infty} \psi_i Y_{t-i} = \beta(B)\alpha(B)Y_t = \alpha(B)\beta(B)Y_t.$$

Show $\psi_i = \sum_j \alpha_j \beta_{i-j}$

Motivation

Why do we worry about infinite filters? We never have an infinitely large dataset, so why do we have to prove things about what happens when we do?

Sometimes it's beneficial to write certain models in this expanded way. Take for example the AR(1) model we learned earlier

$$X_t - \phi X_{t-1} = Z_t$$

where $|\phi| < 1$ and $\{Z_t\}$ is white noise. We can re-write it as

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}.$$

Motivation

$\{Z_t\}$ is white noise. We can re-write it as

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

Notice the coefficients are absolutely summable (because $|\phi| < 1$), and by the previous theorem

$$\gamma_X(h) = \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \sigma^2 = \sigma^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma^2 \phi^h}{1 - \phi^2}$$

Causality

What if $|\phi| > 1$ for an AR(1)? Then X_t is not **causal** or **future-independent**. Let's start by rewriting our AR(1) as

$$-\phi X_t = Z_{t+1} - X_{t+1}$$

becomes

$$\begin{aligned} X_t &= -\phi^{-1} Z_{t+1} + \phi^{-1} X_{t+1} \\ &= -\phi^{-1} Z_{t+1} + \phi^{-1}[-\phi^{-1} Z_{t+2} + \phi^{-1} X_{t+2}] \\ &\vdots \\ &= -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}. \end{aligned}$$

This sum converges (because $|\phi^{-1}| < 1$), however the terms are in the future.

Unit Roots

What if $|\phi| = 1$? Then X_t is not even stationary! This is a homework question.

We will generalize these results to more arbitary processes that have more complicated sets of coefficients.