# 8.4: Multivariate ARMA Processes

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# Motivation

In this section we introduce ARMA models for vectors $\mathbf{X}_t$.

# Example 8.4.1

a vector-valued AR(1).

$$\mathbf{X}_t = \Phi \mathbf{X}_{t-1} + \mathbf{Z}_t, \qquad \mathbf{Z}_t \sim WN(0, \Sigma)$$

or

$$\begin{bmatrix} X_{t,1} \\ X_{t,2} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} X_{t-1,1} \\ X_{t-1,2} \end{bmatrix} + \begin{bmatrix} Z_{t,1} \\ Z_{t,2} \end{bmatrix}$$

or

$$X_{t,1} = \Phi_{11} X_{t-1,1} + \Phi_{12} X_{t-1,2} + Z_{t,1}$$
$$X_{t,2} = \Phi_{21} X_{t-1,1} + \Phi_{22} X_{t-1,2} + Z_{t,2}$$

$$\begin{aligned}
\mathbf{X}_t &= \Phi \mathbf{X}_{t-1} + \mathbf{Z}_t \\
&= \Phi^2 \mathbf{X}_{t-2} + \Phi \mathbf{Z}_{t-1} + \mathbf{Z}_t \\
&= \Phi^3 \mathbf{X}_{t-3} + \Phi^2 \mathbf{Z}_{t-2} + \Phi \mathbf{Z}_{t-1} + \mathbf{Z}_t \\
&\vdots \\
&= \sum_{j=1}^{\infty} \Phi^j Z_{t-j}
\end{aligned}$$

as long as the eigenvalues of $\Phi$ are less than 1.

# Example 8.4.1

Why does $\mathbf{\Phi}^k \mathbf{X}_{t-k} \to \mathbf{0}$ as long as the eigenvalues of $\mathbf{\Phi}$ are less than 1?

Assume for simplicity that the eigenvectors $q_1, q_2, \ldots, q_n$ are linearly independent (the proof is more complicated if they are not). We can write

$$\mathbf{\Phi} q_i = \lambda_i q_i$$

as

$$\mathbf{\Phi} Q = Q \Lambda$$

or

$$\mathbf{\Phi} = Q \Lambda Q^{-1}$$

where $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

Finally, notice that $\mathbf{\Phi}^k = Q \Lambda^k Q^{-1} \to Q \mathbf{0} Q^{-1} = \mathbf{0}$.

# Example 8.4.1

$q_i \neq 0$ is an eigenvector with eigenvalue $\lambda_i < 1$ iff

$$\mathbf{\Phi} q_i = \lambda_i q_i$$

iff

$$(I\lambda_i - \mathbf{\Phi})q_i = 0$$

iff

$$\det(I\lambda_i - \mathbf{\Phi}) = 0$$

iff

$$\det(I - \mathbf{\Phi} z) = 0 \qquad z = 1/\lambda_i > 0$$

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The last equation is the determinant of a matrix-valued polynomial.

# Definition

## Vector ARMA

$\{\mathbf{X}_t\}$ is an **ARMA(p,q) process** if $\{\mathbf{X}_t\}$ is stationary and if for every $t$,

$$\mathbf{X}_t - \mathbf{\Phi}_1 \mathbf{X}_{t-1} - \cdots - \mathbf{\Phi}_p \mathbf{X}_{t-p} = \mathbf{Z}_t + \mathbf{\Theta}_1 \mathbf{Z}_{t-1} + \cdots + \mathbf{\Theta}_q \mathbf{Z}_{t-q},$$

where $\{\boldsymbol{Z}_t\} \sim \text{WN}(0, \mathbf{\Sigma})$. $\{\mathbf{X}_t\}$ is an ARMA(p,q) process with mean $\boldsymbol{\mu}$ if $\{\mathbf{X}_t - \boldsymbol{\mu}\}$ is an ARMA(p,q) process.

We can write $\mathbf{\Phi}(B)\mathbf{X}_t = \mathbf{\Theta}\mathbf{Z}_t$ where $\mathbf{\Phi}(z) = I - \mathbf{\Phi}_1 z - \cdots \mathbf{\Phi}_p z^p$ and $\mathbf{\Theta}(z) = I + \mathbf{\Theta}_1 z + \cdots + \mathbf{\Theta}_q z^q$.

# Definition

## Causal Vector ARMA model

An ARMA(p,q) process $\{\mathbf{X}_t\}$ is **causal** if there exist matrices $\{\boldsymbol{\Psi}_j\}$ with absolutely summable components such that

$$\mathbf{X}_t = \sum_{j=0}^{\infty} \Psi_j \mathbf{Z}_{t-j} \qquad \text{for all } t.$$

Equivalently:

$$\det(\Phi(z)) \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1.$$

# Definition

## Invertible Vector ARMA model

An ARMA(p,q) process $\{\mathbf{X}_t\}$ is **invertible** if there exist matrices $\{\boldsymbol{\Pi}_j\}$ with absolutely summable components such that

$$\mathbf{Z}_t = \sum_{j=0}^{\infty} \boldsymbol{\Pi}_j \mathbf{X}_{t-j} \qquad \text{for all } t.$$

Equivalently:

$$\det(\boldsymbol{\Theta}(z)) \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| \leq 1.$$

# Definition

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These definitions are analagous to the univariate case. Roots outside are good!

# Remark

Can a vector AR(1) be equivalent to a vector MA(1)? This isn't possible for univariate ARMA models. Hint: consider the vector-AR(1) model with $\mathbf{\Phi} = \left[ \begin{array}{cc} 0 & .5 \\ 0 & 0 \end{array} \right]$.

# Remark

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There is no identifiability problem if you restrict your attention to AR models (VAR models). From now on, we only deal with these.

# Theoretical ACF of causal VAR models

For a causal VAR model

$$\mathbf{X}_t = \sum_{j=0}^{\infty} \mathbf{\Psi}_j \mathbf{Z}_{t-j},$$

the ACF is

$$\Gamma(h) = \sum_{j=0}^{\infty} \mathbf{\Psi}_{j+h} \mathbf{\Sigma} \mathbf{\Psi}_j.$$

# Applied Example

See code for a VAR modelling example.