8.7: Cointegration

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Motivation

1. What happens when the roots of $\det[\Phi(z)]$ are not all outside the unit circle?
2. What happens if several series share a stochastic trend, or in other words, if there is a long-run economic equilibrium between several non-stationary variables?

Supplementary materials: chapters 6.3 of "New Introduction to Multiple Time Series Analysis" by Helmut Lütkepohl.

Example 8.7.1

If

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{t} Z_t \\ \sum_{i=1}^{t} Z_t + W_t \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_t \\ W_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ W_{t-1} \end{bmatrix} + \cdots$$

then

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_t \\ W_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ W_{t-1} \end{bmatrix}$$

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then

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$\det[\Phi(z)] = \det[I - Iz] = (1-z)(1-z)$. The roots are $z_1 = z_2 = 1$.

Example 8.7.1

So let's try differencing all the series.

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_t \\ W_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ W_{t-1} \end{bmatrix}$$

becomes

$$\nabla \begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_t \\ W_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ W_{t-1} \end{bmatrix}$$

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becomes

$$\triangledown \begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_t \\ W_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ W_{t-1} \end{bmatrix}$$

$\det[\boldsymbol{\Theta}(z)] = 1 + z$. This process is not invertible. Note that you need to re-define the noise as $(Z_t, Z_t + W_t)'$ so that the leading coefficient matrix is an identity matrix.

Example 8.7.1

$$
\begin{aligned}
\begin{bmatrix} X_t \\ Y_t \end{bmatrix} &= \begin{bmatrix} \sum_{i=1}^{t} Z_t \\ \sum_{i=1}^{t} Z_t + W_t \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_t \\ W_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ W_{t-1} \end{bmatrix} + \cdots
\end{aligned}
$$

Notice that $X_t = Y_t + W_t$, so $X_t \approx Y_t$. They are each separately nonstationary, but they are "tied together." If you didn't difference them, but instead looked at the spread between them, this would be an AR(0) model.

Definitions

We call a K-dimensional process Y_t **integrated of order** d, written as $Y_t \sim I(d)$, if $\nabla^d Y_t$ is causal and stationary and $\nabla^{d-1} Y_t$ is not. Usually we will be interested in the case of $d = 1$.

An $I(d)$ process Y_t is called **cointegrated** if there exists atleast one (there may be more) linear combination $\beta' Y_t$, $\beta \neq 0$ which is integrated of order less than d.

In the last example $d = 1$ and $\beta' = (1, -1)$. Often there will be more than 1 linear combination (e.g. $\boldsymbol{\beta}$). And these vectors are not unique.

Definitions

VECM

Let $\mathbf{Y}_t$ be a K-dimensional, vector-valued time series. A **vector error correction model** (VECM) can be written as

$$\nabla Y_t = \boldsymbol{\alpha}\boldsymbol{\beta}' Y_{t-1} + \Gamma_1 \nabla Y_{t-1} + U_t,$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are $K \times r$ matrices.

This is like a VAR(1) model, but note that the differences depend additionally on linear combinations of the levels: $\boldsymbol{\beta}' Y_{t-1}$.

Alternatively you could think of this as a nonstationary VAR(2) model (see next slide).

VAR vs VECM

Start with a non-stationary VAR(2)

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + U_t$$

difference once

$$\begin{aligned}
Y_t - Y_{t-1} &= -IY_{t-1} + A_1 Y_{t-1} + A_2 Y_{t-2} + U_t \\
&= -IY_{t-1} + A_1 Y_{t-1} + A_2 Y_{t-1} - A_2 Y_{t-1} + A_2 Y_{t-2} + U_t \\
&= -(I - A_1 - A_2) Y_{t-1} - A_2 \triangledown Y_{t-1} + W_t \\
&= \boldsymbol{\alpha} \boldsymbol{\beta}' Y_{t-1} + \Gamma_1 \triangledown Y_{t-1} + U_t
\end{aligned}$$

Note that -$(I - A_1 - A_2) = \boldsymbol{\alpha}\boldsymbol{\beta}'$ is singular because det$[\boldsymbol{\Phi}(z)]$ has a unit root.

A Second Definition

The K-dimensional VAR(p)

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + U_t$$

is **cointegrated of rank** r if

$$\mathbf{\Pi} \stackrel{\text{def}}{=} -(I - A_1 - \cdots - A_p)$$

has rank $0 < r < K$, and thus can be written as $\alpha\beta'$ with each of these matrices being of dimension $K \times r$. The matrix β is called the **cointegration matrix** or **matrix of cointegrating vectors** and α is called the **loadings matrix**.

A Little Algebra

Assume the K-dimensional VAR(p)

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + U_t$$

is cointegrated of rank r. Then

$$\begin{aligned}
\nabla Y_t &= -IY_{t-1} + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + U_t \\
&= -IY_{t-1} + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + U_t \\
&\qquad \pm A_2 Y_{t-1} \pm \cdots \pm A_p Y_{t-1} \\
&= \mathbf{\Pi} Y_{t-1} + (-A_2 - A_3 - \cdots - A_p) Y_{t-1} \\
&\qquad A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + U_t \\
&= \mathbf{\Pi} Y_{t-1} + \mathbf{\Gamma}_1 \nabla Y_{t-1} + \cdots + \mathbf{\Gamma}_{p-1} \nabla Y_{t-p+1}
\end{aligned}$$

where $A_1 = \mathbf{\Pi} + I + \mathbf{\Gamma}_1$, $A_p = -\mathbf{\Gamma}_{p-1}$ and $A_i = \mathbf{\Gamma}_i - \mathbf{\Gamma}_{i-1}$ for $i = 2, \ldots, p-1$.

Interpretation of $\beta' Y_{t-1}$

In

$$\nabla Y_t = \boldsymbol{\alpha}\boldsymbol{\beta}' Y_{t-1} + \Gamma_1 \nabla Y_{t-1} + U_t,$$

$\boldsymbol{\alpha}\boldsymbol{\beta}' Y_{t-1}$ must be causal and stationary. Therefore

$$[\boldsymbol{\alpha}'\boldsymbol{\alpha}]^{-1}\boldsymbol{\alpha}'\boldsymbol{\alpha}\boldsymbol{\beta}' Y_{t-1} = \boldsymbol{\beta}' Y_{t-1}$$

must be causal and stationary.

Think of this as a lower-dimensional series of "spreads" between financial time series. They are not differenced, but they are linearly-combined in 1 or more ways.

Nota Bene: Non-Uniqueness of Cointegrating Relationships

The matrix $\mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$ is not unique. Take any invertible matrix Q. Clearly

$$\boldsymbol{\alpha}\boldsymbol{\beta}' = \boldsymbol{\alpha} Q Q^{-1} \boldsymbol{\beta}',$$

so we can always set $\boldsymbol{\alpha}^* = \boldsymbol{\alpha} Q$ and $\boldsymbol{\beta}^* = \boldsymbol{\beta} Q^{-1}$