5.1: Preliminary Estimation

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Motivation

We need an initial set of parameters to start off our Maximum Likelihood fitting procedure. In this section, we learn about four Preliminary Estimation procedures.

The books recommendations are

- For pure AR models, use Yule-Walker or Burg
- For MA models, use the Innovations Algorithm or Hannan-Rissanen
- For ARMA models, use the Innovations Algorithm or Hannan-Rissanen

Yule-Walker Equations for AR models

Multiply both sides of a causal AR process by X_{t-k} where $k = 0, 1, \ldots, p$ and obtain

$$X_t X_{t-k} = \phi_1 X_{t-1} X_{t-k} + \phi_2 X_{t-2} X_{t-k} + \cdots + \phi_p X_{t-p} X_{t-k} + Z_t X_{t-k}$$

Taking expectations yields

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + \cdots \phi_p \gamma(k-p) + E[Z_t X_{t-k}].$$

Writing these all at once using matrices and vectors we have

Yule-Walker Eqns

$$\boldsymbol{\gamma}_p = \Gamma_p \boldsymbol{\phi}$$

if $k = 1, \ldots, p$ and

$$\gamma(0) = \boldsymbol{\phi}' \boldsymbol{\gamma}_p + \sigma^2$$

if $k = 0$. Recall $\boldsymbol{\gamma}_p = (\gamma(1), \ldots, \gamma(p))'$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)'$.

Sample Yule-Walker Equations

What does this have to do with estimation? Well we replace everything with sample estimates, and then solve for the unknown parameters.

$$\boldsymbol{\gamma}_p = \Gamma_p \boldsymbol{\phi}$$
$$\gamma(0) = \boldsymbol{\phi}' \boldsymbol{\gamma}_p + \sigma^2$$

becomes

Sample Yule-Walker Eqns

$$\hat{\boldsymbol{\phi}}_p = \hat{R}_p^{-1} \hat{\boldsymbol{\rho}}_p$$
$$\hat{\sigma}^2 = \hat{\gamma}(0)[1 - \hat{\boldsymbol{\rho}}_p' \hat{R}_p^{-1} \hat{\boldsymbol{\rho}}_p]$$

Sample Yule-Walker Equations

Benefits:

- The estimated parameters will always yield a causal model AR model
- $\hat{\phi} \overset{approx.}{\sim} \mathrm{N}(\phi, n^{-1}\sigma^2\Gamma_p^{-1})$ (for CIs, HTs)
- Solution pretty much always exists because we rarely have to worry about invertibility of $\hat{\Gamma}_p$ ($\hat{\gamma}(0) > 0$ sufficient condition)
- consistent parameter estimates for pure AR models

Cons come from when $q > 0$

- Equations to solve are nonlinear in $\hat{\phi}$. (We derived in earlier section to obtain covariance).
- possible non-uniqueness and non-existence

The Innovations Algorithm for MA models

Yule walker relates parameters to autocovariances/autocorrelations. The innovations algorithm relates parameters to innovations/predictions. Just like in the Yule-Walker derivation, we replace population quantities with sample quantities, and solve for the parameters.

Recall that we never derived the innovations algorithm in 2.5. We just played around with some matrices that relate. Here you literally just stick the sample autocovariances in where there were population autocovariances. The book doesn't go into any more detail than that.

The **fitted innovations MA(m) model is**

$$X_t = Z_t + \hat{\theta}_{m1} Z_{t-1} + \cdots + \hat{\theta}_{mm} Z_{t-m} \qquad \{Z_t\} \sim \text{WN}(0, \hat{v}_m)$$

where $\hat{\boldsymbol{\theta}}_m$ and v_m are obtained from the innovations algorithm with the ACVF replaced by the sample ACVF.

The Innovations Algorithm for MA models

The estimates produced from the Innovations Algorithm do not have the same properties as the YW AR estimates

1. the MA process needs to be invertible
2. there are restrictions put on the lookback window m for the innovations algorithm
   - smaller than the amount of data you used
   - but still sort of big, usually larger than q
   - $m(n) = o(n^{1/3})$

"The choice of m for any fixed sample size n can be made by increasing m until the vector $(\theta_{m1}, \ldots, \theta_{mq})'$ stabilizes. It is found in practice that there is a large range of values of m..."

The Hannan-Rissanen Algorithm for ARMA(p,q) models

Notice that an AR(p) model is just a regression model on lagged data. For an ARMA(p,q) model, it can also be seen as a regression, but some of the regressors are unobserved:

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

The main idea of the Hannan-Rissanen algorithm is estimate the error terms, and then regress X_t on lagged values and the estimated error terms.

The Hannan-Rissanen Algorithm for ARMA(p,q) models

Hannan-Rissanen Algo for ARMA(p,q) mods

Step 1: Use a Yule-Walker algorithm to estimate an AR(m) model ($m > \max(p, q)$). This gives you $\hat{\phi}_{m1}, \ldots, \hat{\phi}_{mm}$. Calculate the residuals (for $t = m + 1, \ldots, n$)

$$\hat{Z}_t = X_t - \hat{\phi}_{m1} X_{t-1} - \cdots - \hat{\phi}_{mm} X_{t-m}$$

Step 2: Run the regression

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \hat{Z}_t + \theta_1 \hat{Z}_{t-1} + \cdots + \theta_q \hat{Z}_{t-q}.$$

for $t = m + 1, \ldots, n$

H&R also suggest a third step.