1.5: Estimation and Elimination of Trend and Seasonal Components

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Motivation

All the material from last chapter was only for **stationary** time series only!

Before you use all of that:

1. plot the series (does it look stationary?)
2. Does it need to be broken up into homogeneous segments?
3. Do we need to discard outliers? Incorrectly recorded?
4. Do we need to decompose the data into trend, seasonal component and stationary component?
5. Do we need to transform the data? Take the logarithm? Take Differences? Take percents changes?

Definition

Approach 1: Classical Decomposition Model

The **classical decomposition model** for a series $\{X_t\}$ is written as

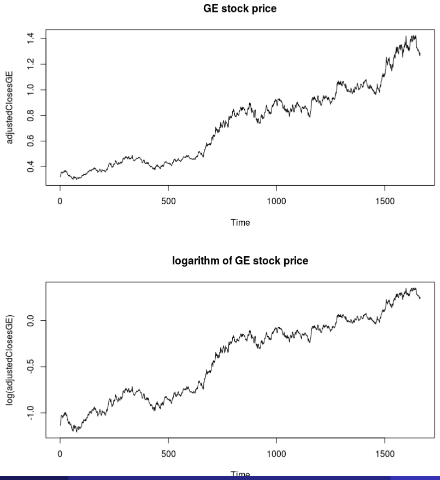
$$X_t = m_t + s_t + Y_t,$$

where

1. m_t is the slowly changing trend component
2. s_t is the seasonal component with known frequency/period
3. Y_t is weakly stationary noise (something from last chapter)

Nota Bene

Transforming the data by taking the log of each x_t is commonly done if the seasonal and noise fluctuations appear to increase with the level of the process.



Definition

With financial time series, people are usually interested in returns

- continuously compounded: $R_t = \log(X_t) - \log(X_{t-1})$
- compounded weekly, daily, etc.: $S_t = \left(\frac{X_t - X_{t-1}}{X_{t-1}}\right) = X_t / X_{t-1} - 1$

Running "Wealth":

- Taking the log of each time point is looking at the cumulative sum of the first one
- You can take the arithmetic everage of the log returns $\sum_{t=1}^{T} R_t / T$
- If you don't look at continuously compounded returns, you want the geometric average: $(\prod_{t=1}^{T} [1 + S_t])^{1/T}$
- We usually take the log

Fitting the trend?

If our model is

$$X_t = m_t + s_t + Y_t,$$

how do we get an idea of m_t and s_t?

First, let's assume $s_t = 0$.

Method 1: Smoothing with finite moving average filter

Assume the model is

$$X_t = m_t + Y_t, \qquad t = 1, \dots, n$$

with $E[Y_t] = 0$, then define

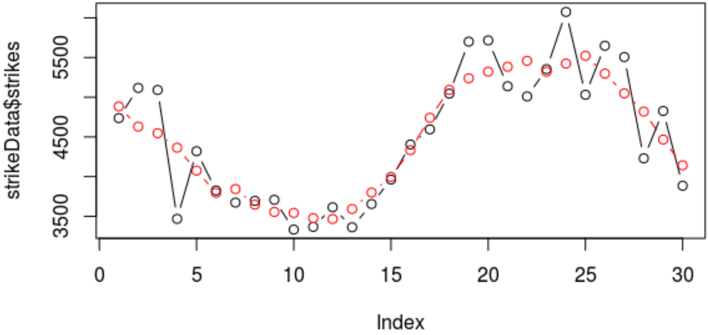
$$W_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j}.$$

Why? Because the mean shows up better and the noise is attenuated.

$$W_t = \frac{1}{2q+1} \sum_{j=-q}^{q} m_{t-j} + \frac{1}{2q+1} \sum_{j=-q}^{q} Y_{t-j} \approx \frac{1}{2q+1} \sum_{j=-q}^{q} m_{t-j}$$

for $q+1 \leq t \leq n-q$ because $\frac{1}{2q+1} \sum_{j=-q}^{q} Y_{t-j} \to 0$ by the law of large numbers.

Method 1: Smoothing with finite moving average filter



Be careful though. Applying **linear filters** might wash away the trend component too!

Method 2: Exponential Smoothing

Assume the model is

$$X_t = m_t + Y_t, \qquad t = 1, \ldots, n$$

with $E[Y_t] = 0$ again. Then pick $\alpha \in [0, 1]$. Finally define the mean estimate recursively:

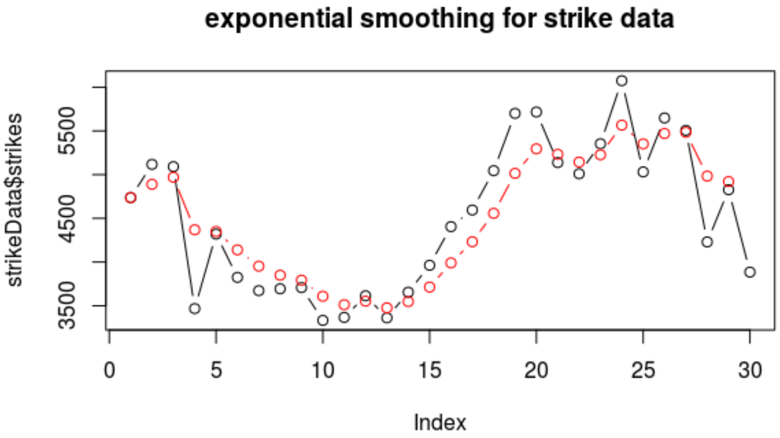
$$\hat{m}_t = \alpha X_t + (1 - \alpha)\hat{m}_{t-1}$$

with the initial condition $\hat{m}_1 = x_1$.

They call it exponential smoothing because the weights decrease exponentially:

$$\begin{aligned}\hat{m}_t &= \alpha X_t + (1-\alpha)[\alpha X_{t-1} + (1-\alpha)\hat{m}_{t-2}] \\ &= \alpha X_t + \alpha(1-\alpha)X_{t-1} + (1-\alpha)^2[\alpha X_{t-2} + (1-\alpha)\hat{m}_{t-3}] \\ &= \alpha \sum_{j=0}^{t-2}(1-\alpha)^j X_{t-j} + (1-\alpha)^{t-1}X_1\end{aligned}$$

Method 2: Exponential Smoothing



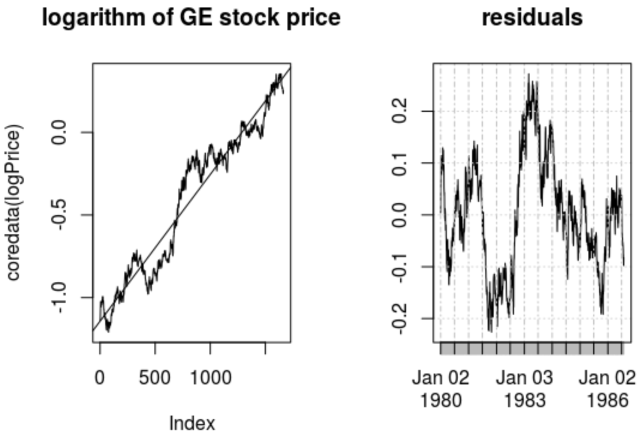
Methods 3 & 4

We can also

1. Smooth by fitting a Fourier series that lacks high frequency components
2. Smooth by fitting a polynomial to the data

More on these later...

Method 4



Equivalent to assuming

$$\log(P_t) = \beta_0 + \beta_1 t + W_t \text{ OR } \log(P_t) - \log(P_{t-1}) = \beta_1 + W_t - W_{t-1}$$

Preliminary Definitions

The Backshift Operator

The **backshift operator**, denoted by B, means

$$BX_t = X_{t-1}.$$

Note that $B^2 = BX_{t-1} = X_{t-2}$ and so on.

The Differencing Operator

The **differencing operator**, denoted by ∇, means

$$\nabla X_t = X_t - X_{t-1} = (1 - B)X_t$$

Note that $\nabla^2 X_t = \nabla(1 - B)X_t = (1 - 2B + B^2)X_t = X_t - 2X_{t-1} + X_{t-2}$, and so on.

Approach 2: Trend Elimination by Differencing

The **differencing method** attempts to detrend the data by differencing the time series, and fitting a stationary model on the differenced series $\nabla^k X_t$, for some k.

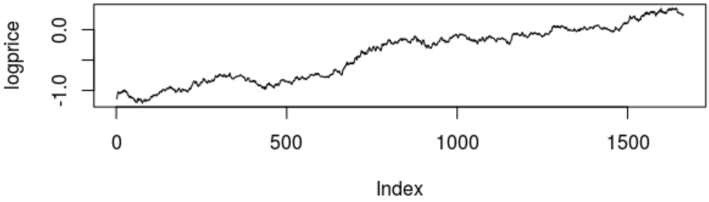
Example: Assume $X_t = a_0 + a_1 t + a_2 t^2 + Y_t$.

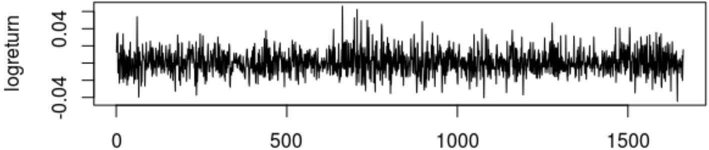
$$\nabla X_t = a_1 + 2a_2 t - a_2 + \nabla Y_t$$
$$\nabla^2 X_t = 2a_2 + \nabla^2 Y_t$$

So we would model $\nabla^2 X_t$ with a stationary time series model from last chapter.

Differencing Example

Differencing once gives us the log returns.





Fitting the trend?

If our model is

$$X_t = m_t + s_t + Y_t,$$

how do we get an idea of m_t and s_t?

Now, let's assume $s_t \neq 0$. Specifically, let $s_{t+d} = s_t$ and $\sum_{j=1}^{d} s_j = 0$, for some period d.

Method S1: Estimation of Trend and Seasonal Components

If our model is

$$X_t = m_t + s_t + Y_t,$$

Step 1: Estimate m_t with $\hat{m}_t$. d is the period.

- If $d = 2q$, then $\hat{m}_t = (.5x_{t-q} + x_{t-q+1} + \cdots + x_{t+q-1} + .5x_{t+q})/d$
- If $d = 2q + 1$, then $\hat{m}_t = (\sum_{i=-q}^{q} x_{t-i})/(2q + 1)$

for $q < t \leq n - q$

Method S1: Estimation of Trend and Seasonal Components

If our model is

$$X_t = m_t + s_t + Y_t,$$

Step 2: Estimate s_t with $\hat{s}_t$. For times $k = 1, \ldots, d$

- $w_k = \text{mean}_j(x_{k+jd} - \hat{m}_{k+jd})$
- $\hat{s}_k = w_k - d^{-1} \sum_{i=1}^{d} w_i$
- $\hat{s}_l = \hat{s}_{l-d}$ for $l > d$

for $q < k + jd \leq n - q$

Method S1: Estimation of Trend and Seasonal Components

If our model is

$$X_t = m_t + s_t + Y_t,$$

Step 3: Deseasonalize the data

- $d_t = x_t - \hat{s}_t$

for $0 < t \leq n$

Step 4: Re-estimate the trend m_t

- Use a method that we talked about in the prior section on the series d_t

Method S2: Elimination of Trend and Seasonal Components by Differencing

If our model is

$$X_t = m_t + s_t + Y_t,$$

then

$$\nabla_d X_t = m_t - m_{t-d} + Y_t - Y_{t-d}$$

where ∇_d is the lag-d difference, (not ∇^d!)

In R:

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diff(logRets, lag=12)
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Note that $\nabla \nabla_d X_t = \nabla_d \nabla X_t$, but it's recommended that you apply ∇_d first (left side).