6.3: Unit Roots in Time Series Models

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Motivation

"...a root near 1 of the autoregressive polynomial suggests that the data should be differenced before fitting an ARMA model, whereas a root near 1 of the moving-average polynomial indicates that the data were overdifferenced."

Testing non-stationarity

Assume an AR(1) model: $X_t - \mu = \phi_1(X_{t-1} - \mu) + Z_t$.

We want to test $H_0 : \phi_1 = 1$. Under the null hypothesis, our time series is a random walk (not stationary).

The standard large sample asymptotic distributions for our estimators of ϕ_1 only apply when $|\phi_1| < 1$.

Q: What do we do?

Test: $H_0 : \phi_1 = 1$

A: We take the difference!
Our AR(1) model: $X_t - \mu = \phi_1(X_{t-1} - \mu) + Z_t$
becomes:

$$\begin{aligned}\nabla X_t &= \mu + \phi_1(X_{t-1} - \mu) + Z_t - X_{t-1} \\ &= \mu(1 - \phi_1) + (\phi_1 - 1)X_{t-1} + Z_t \\ &= \phi_0^* + \phi_1^* X_{t-1} + Z_t\end{aligned}$$

Now we test the null hypothesis: $H_0 : \phi_1^* = 0$.

Test: $H_0 : \phi_1 = 1$

Even though we test the null hypothesis: $H_0 : \phi_1^* = 0$, we estimate the full regression model

$$\nabla X_t = \phi_0^* + \phi_1^* X_{t-1} + Z_t$$

We get the familiar t-test statistic (only it doesn't follow a t-distribution under the null this time)

$$\hat{\tau}_\mu = \frac{\hat{\phi}_1^*}{\sqrt{\frac{S^2}{\sum_{t=2}^n (X_{t-1} - \bar{X})^2}}}$$

where $S^2 = \sum_{t=2}^n (\nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* X_{t-1})^2 / (n - 1 - 2)$

Test: $H_0 : \phi_1 = 1$

Dickey and Fuller derived the asymptotic distribution of this quantity.

$$\hat{\tau}_\mu = \frac{\hat{\phi}_1^*}{\sqrt{\frac{S^2}{\sum_{t=2}^n (X_{t-1} - \bar{X})^2}}}$$

where $S^2 = \sum_{t=2}^n (\nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* X_{t-1})^2 / (n-3)$

You reject the null when this thing is small. The 0.01, 0.05, and 0.10 quantiles of the limit distribution of $\hat{\tau}_\mu$ are -3.43, -2.86, and -2.57 (see Table 8.5.2 of Fuller 1976 or R code printout)

See 6.3.R

Example

When you assume the true model is a higher order AR process, the test is called the *augmented* Dickey-Fuller test.

A homework question asks you to show that the AR(p) model

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \cdots + \phi_p(X_{t-p} - \mu) + Z_t$$

can be written as

$$\nabla X_t = \phi_0^* + \phi_1^* X_{t-1} + \phi_2^* \nabla X_{t-1} + \cdots + \phi_p^* \nabla X_{t-p+1} + Z_t$$

Example 1

Recall that the null is non-stationary with a unit root, and the test is a "left-tailed" test. We generate data with a unit root, so the data shouldn't reject the hypothesis most of the time.

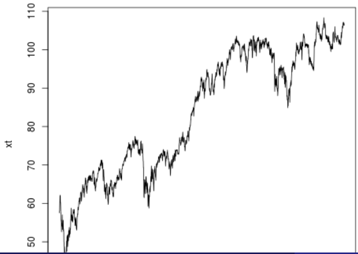
```
library(urca)

# generate data from an arima(1,1,0) model
fakeData <- arima.sim(model = list(ar=c(.9), order = c(1,1,0))
plot.ts(fakeData)
summary(ur.df(fakeData, type = "drift", lags=1)) # regular dic
summary(ur.df(fakeData, type = "drift", lags=3)) # augmented c
```

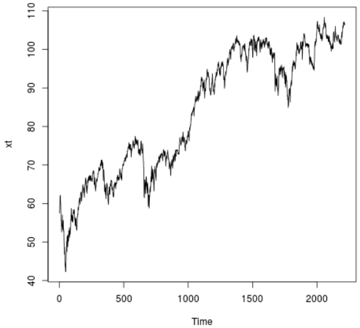
Example 2

Let's pretend you've bought one nasdaq share and sold short one S&P. This looks like a nice portfolio because it appears to trend up, and you don't have to re-balance your weights (it's always 1 to 1, so no transaction costs).

```
getSymbols(c("SPY","QQQQ"), src = "google")
xt <- SPY$SPY.Close - QQQQ$QQQQ.Close
xt <- xt[500:2714] # :)
plot.ts(xt)
```



Example 2: Is there Mean-Reversion?



This X_t doesn't look stationary, but is it a "stochastic trend"

$$\phi(B)(1-B)(X_t - \mu t) = \theta(B)Z_t$$

or a "deterministic trend"

$$\phi(B)(X_t - [\beta_0 + \beta_1 t]) = \theta(B)Z_t?$$

Note that we can write the last one as a regression plus ARMA error if (if

Example

```
myTest <- ur.df(xt, "trend") # note "trend"
summary(myTest)

Value of test-statistic is: -3.0016 4.9831 4.616

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.96 -3.41 -3.12
phi2  6.09  4.68  4.03
phi3  8.27  6.25  5.34
```

Unit Roots in the MA polynomial

Why are unit roots bad in the MA polynomial?

Reason 1: if $\phi(B)X_t = \theta(B)Z_t$ is the true model, a causal and invertible ARMA(p,q), then ∇X_t is a non-invertible ARMA(p,q+1).

$$\begin{aligned}
\phi(B)\nabla X_t &= \phi(B)X_t - \phi(B)X_{t-1} \\
&= \theta(B)Z_t - \theta(B)Z_{t-1} \\
&= \theta(B)(1-B)Z_t
\end{aligned}$$

...test of overdifferencing...

Unit Roots in the MA polynomial

Reason 2: Which model is correct? Consider again the difference between a stochastic trend, and a deterministic trend.

$$\nabla^k X_t = a + V_t$$

or

$$X_t = c_0 + c_1 t + \cdots + c_k t^k + W_t$$

The former has unit roots, while the latter has none.

Unit Roots in the MA polynomial

Simplify by setting $k = 1$ which is what happens most of the time in financial time series anyways

$$X_t = c + X_{t-1} + Z_t \iff \nabla X_t = c + Z_t$$

and

$$X_t = a + ct + Z_t \iff \nabla X_t = c + Z_t - Z_{t-1}$$

If you run one simulation for each, they look very similar! However, the second one has a unit root.

Watch out for overdifferencing!