1.3: Some Simple Time Series Models

Taylor

University of Virginia

Definition

defn

A **time series model** for the oberved data $x_1, x_2, \ldots, x_T$, is a specification of the joint distributions of a sequence of random variables $X_1, \ldots, X_T$, of which $x_1, \ldots, x_T$ are postulated to be a realization.

It describes **all** cumulative distribution functions (CDFs) you can think of:

$$F_{X_{i_1},\ldots,X_{i_n}}(x_{i_1}, \ldots, x_{i_n}) = P(X_{i_1} \le x_{i_1}, \ldots, X_{i_n} \le x_{i_n})$$

for any n, and $i_1 < i_2 < \cdots < i_n$.

It completely defines all behavior.

Common Practice

Usually we just look at **second order properties** of a time series.

This means we look at the **first-order** means $E[X_t]$, and the **second-order** expected products (building blocks of covariances) $E[X_t X_{t+h}]$, for any $t \in T_0$, and $h = 1, 2, \ldots$.

Sometimes this is sufficient. Sometimes, for complicated time series, it is not. What follows are a few simple models where it is sufficient to just know the second-order properties. In other words, where knowing the means and variances fully specifies a model.

Example 1: IID Noise

$X_1, \ldots, X_T$ are **iid noise** if they are

1. **independent**:

$$F_{X_1,\ldots,X_T}(x_1,\ldots,x_n) = F_{X_1}(x_1) \times \cdots, F_{X_T}(x_T),$$

   for all $\{x_i\}$ and

2. **identical**:

$$F_{X_i}(x_i) = F(x_i)$$

   for all i, and

3. **noise**

$$E[X_t] = 0$$

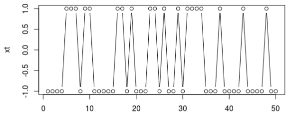
   for all t

Example 1: IID Noise

IID noise has no trend, no seasonality, and no pattern in general. Sometimes they are assumed be all be **Normally distributed**. Or sometimes they are assumed to be iid **Bernoulli random variables** (a Binomial random variable when $n = 1$).

The outcomes of a Bernoulli rv are usually coded as 1 and 0. However, to get a mean of 0, we assume the outcomes are coded as 1 and -1, and we assume $p = .5$.

```
plot(sample(x=c(1,-1), size=50,
            replace=T, prob=c(.5,.5)), type = "b",
            ylab = "xt")
```



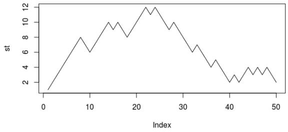
Example 2: Random Walk

A **(simple symmetric) random walk** is obtained by summing iid noise. For each $t = 1, 2, \ldots$

$$S_t = X_1 + X_2 + \cdots X_t = \sum_{i=1}^{t} X_i.$$

Usually it's convention to set $S_0 = 0$ with probability 1.

```
plot(cumsum(sample(x=c(1,-1), size=50, replace=T,
     prob=c(.5,.5))), type = "l", ylab = "st")
```



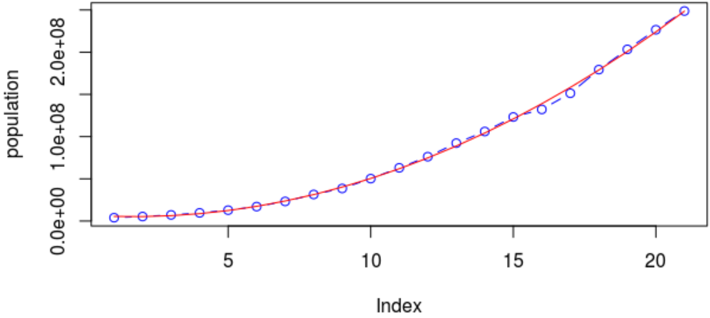
Example 3: Adding a Deterministic Trend

$$X_t = m_t + Y_t$$

1. X_t: object of interest
2. m_t: trend that we try to estimate
3. Y_t: iid noise

Example 3: Adding a Deterministic Trend

Example 1.3.4: A polynomial trend. Assume $m_t = a_0 + a_1 t + a_2 t^2$, for some unknown a_0, a_1 and a_2. See 1.3.R file.

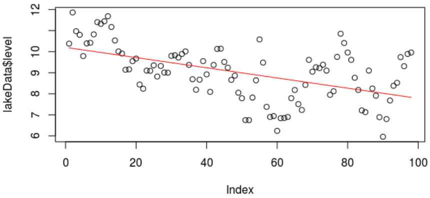


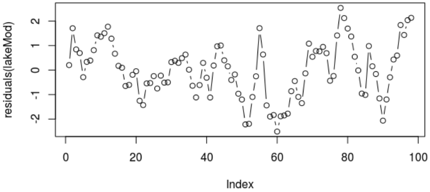
Example 3: Adding a Deterministic Trend

```
> summary(fitMod)

Call:
lm(formula = population ~ t + I(t^2), data = popData)

Residuals:
     Min       1Q   Median       3Q      Max
-6947521  -358167   436285  1481410  3391761

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  6957920    1998526   3.482  0.00266 **
t           -2159870     418437  -5.162 6.55e-05 ***
I(t^2)        650634      18472  35.223  < 2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Example 3: Adding a Deterministic Trend

Example 1.3.5: assume $m_t = a_0 + a_1 t$, for some unknown a_0 and a_1. See 1.3.R file.





Example 4: Adding a Deterministic Trend

Harmonic Regression. Adding a seasonal trend.

$$X_t = s_t + Y_t$$

1. X_t: object of interest
2. s_t: periodic function that we try to estimate
3. Y_t: iid noise

Example 4: Adding a Deterministic Trend

Some background first:

1. **period:** units: time / 1 cycle
2. **frequency:** f, units: cycles / 1 time
3. frequency and period are reciprocals
4. Units of $2\pi f$ are radians / 1 time
5. Units of $2\pi ft$ are radians
6. Sometimes people write $\lambda = 2\pi f$ (angular frequency)
7. Angle-Sum Trig Identity: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
8. Think unit circle: $\cos(0) = \cos(2\pi) = 1$, etc.

Example 4: Adding a Deterministic Trend

$$
\begin{aligned}
s_t &= a_0 + \sum_{j=1}^{k} A_j \cos(2\pi f_j t + \phi_j) && \text{amps and phase} \\
&= a_0 + \sum_{j=1}^{k} A_j \cos(2\pi f_j t)\cos(\phi_j) - A_j \sin(2\pi f_j t)\sin(\phi_j) \\
&= a_0 + \sum_{j=1}^{k} [a_j \cos(2\pi f_j t) + b_j \sin(2\pi f_j t)] && \text{use this one}
\end{aligned}
$$

Known frequencies: $f_1, \ldots, f_k$
To be estimated: $a_0, a_1, \ldots, a_k, b_1, \ldots, b_k$

Example 4: Adding a Deterministic Trend