A Super Brief Introduction to Multivariate Garch Models

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Motivation

How do we model changing volatility for multiple assets?

Supplementary materials: chapters 16.3 of "New Introduction to Multiple Time Series Analysis" by Helmut Lütkepohl.

Motivation

Suppose that $u_t = (u_{1t}, \ldots, u_{Kt})'$ is a return vector of K stocks. Assume

$$u_t = \Sigma_{t|t-1}^{1/2} \epsilon_t$$

where $\epsilon_t \sim iid(0, I_K)$ is a matrix-valued function of $u_{t-1}, u_{t-2}, \ldots$. The volatility matrix is:

$$\begin{aligned}
\mathrm{Var}\left[u_t \mid u_{1:t-1}\right] &= \mathrm{Var}\left[\Sigma_{t|t-1}^{1/2} \epsilon_t \mid u_{1:t-1}\right] \\
&= \Sigma_{t|t-1}^{1/2} \,\mathrm{Var}\left[\epsilon_t \mid u_{1:t-1}\right] \Sigma_{t|t-1}^{1/2'} \\
&= \Sigma_{t|t-1}^{1/2} \,\mathrm{Var}\left[\epsilon_t\right] \Sigma_{t|t-1}^{1/2'} \\
&= \Sigma_{t|t-1}^{1/2} \Sigma_{t|t-1}^{1/2'} = \Sigma_{t|t-1}
\end{aligned}$$

Definition

Assume that $\Sigma_{t|t-1}^{1/2}$ is symmetric.

Multivariate ARCH(q)

$$u_t = \Sigma_{t|t-1}^{1/2} \epsilon_t$$

$$\text{vech}\left(\Sigma_{t|t-1}^{1/2}\right) = \gamma_0 + \Gamma_1 \text{vech}(u_{t-1} u'_{t-1}) + \cdots + \Gamma_q \text{vech}(u_{t-q} u'_{t-q})$$

...what's vech?

Explanation

Recall that a covariance matrix (and this square root covariance matrix) is symmetric. That means it's redundant to model all K^2 terms. There are only $K + \binom{K}{2} = \frac{2K}{2} + \frac{K!}{(K-2)!2} = \frac{2K+K^2-K}{2} = K(K+1)/2$ unique elements.

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$$\text{vech}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} a \\ d \\ g \\ e \\ h \\ i \end{bmatrix}$$

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Well, we only take the vech of symmetric matrices...

Example: a bivariate ARCH(1)

The short way to write an ARCH(1) is:

$$\text{vech}\left(\Sigma_{t|t-1}^{1/2}\right) = \gamma_0 + \Gamma_1 \,\text{vech}(u_{t-1}u_{t-1}'))$$

$$\begin{aligned}
\text{vech}\left(\Sigma_{t|t-1}^{1/2}\right) &= \text{vech}\left(\begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{bmatrix}\right) \\
&= \begin{bmatrix} \sigma_{11}^2 \\ \sigma_{12}^2 \\ \sigma_{22}^2 \end{bmatrix} \\
&= \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \begin{bmatrix} u_{t-1,1}^2 \\ u_{t-1,1}u_{t-1,2} \\ u_{t-1,2}^2 \end{bmatrix}
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\end{aligned}$$

Which coefficients would you zero-out to get two independent ARCH(1) processes?

The Number of Parameters

How many parameters does this model have?

Multivariate ARCH(q)

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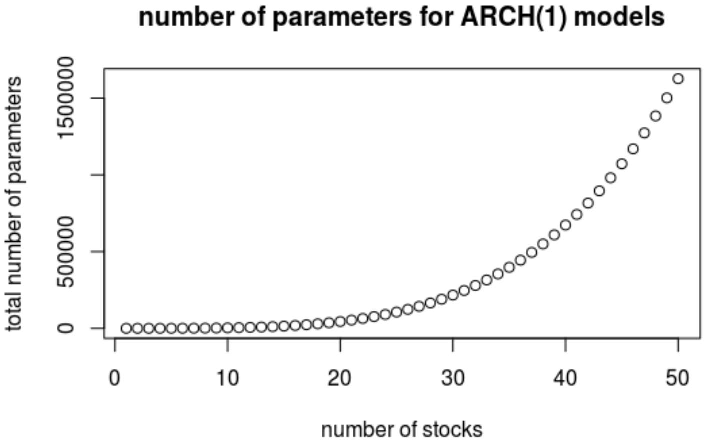
1. intercept: $K(K+1)/2$
2. Γ_i: $[K(K+1)/2]^2$
3. total: $K(K+1)/2 + q[K(K+1)/2]^2$...wow

The Number of Parameters



Restrictions

1. restrict each Γ_i to be diagonal...$K(K+1)/2 + qK(K+1)/2$
2. individual ARCH(q) models: $K + qK$
3. BEKK model: $\Sigma_{t|t-1} = C_0^* C_0^{*'} + \Gamma_1^{*'} u_{t-1} u_{t-1}' \Gamma_1^* + \cdots + \Gamma_q^{*'} u_{t-q} u_{t-q}' \Gamma_q^*$ ...$K(K+1)/2 + qK^2$ parameters

the last one ensures positive-definiteness, too

Definition

Assume that $\Sigma_{t|t-1}^{1/2}$ is symmetric.

Multivariate GARCH(q,m)

$$u_t = \Sigma_{t|t-1}^{1/2} \epsilon_t$$

$$\text{vech}\left(\Sigma_{t|t-1}^{1/2}\right) = \gamma_0 + \sum_{i=1}^{q} \Gamma_i \, \text{vech}(u_{t-i} u'_{t-i}) + \sum_{j=1}^{m} G_j \, \text{vech}(\Sigma_{t-i|t-i-1}^{1/2})$$

Definition

Idea: separate estimation of the correlations and the variances...

Dynamic Conditional Correlation model (DCC)

Let h_t be the vector of conditional variances

$$\Sigma_{t|t-1} = D_t R_t D_t$$

where $D_t = \text{diag}(\sqrt{h_{11,t}}, \ldots, \sqrt{h_{KK,t}})$ is a diagonal matrix, and R_t is a positive-definite correlation matrix.

$$h_t = \omega + \sum_{i=1}^{p} A_i u_{t-i} \odot u_{t-i} + \sum_{j=1}^{q} B_i h_{t-i}$$

$$\mathbf{Q}_t = \bar{\mathbf{Q}} + a(\epsilon_t \epsilon'_{t-1} - \bar{\mathbf{Q}}) + b(\mathbf{Q}_{t-1} - \bar{\mathbf{Q}})$$

$$R_t = \text{diag}(\mathbf{Q}_t)^{-1/2} \mathbf{Q}_t \text{diag}(\mathbf{Q}_t)^{-1/2}$$

with $a + b < 1$.

Example

Say at time t you have $\Sigma_{t+1|t}$ $\mu_{t+1|t}$. Risk and reward of your future return are

$$\text{Var}(w' r_{t+1} | r_{1:t}) = w' \Sigma_{t+1|t} w$$

and

$$E[w' r_{t+1} | r_{1:t}] = w' \mu_{t+1|t}$$

respectively.

You can use Lagrange multipliers to minimize $w' \Sigma_{t+1|t} w$ subject to $w' \mathbf{1} = 1$, and it will yield

$$w_{\text{opt}} = \frac{\Sigma_{t+1|t}^{-1} \mathbf{1}}{\mathbf{1}' \Sigma_{t+1|t}^{-1} \mathbf{1}}.$$

NB: There are a lot of other options (e.g. take into account expected return, disallowing short-selling, investing in risk-free bonds, other-types of constraints, utility functions, taking into account transaction costs, etc. etc.). This is way outside the scope of this class and probably statistics in general...this is just for illustrative purposes.

Example R code

```
getMinVarPortfolio <- function(sigma){
  n <- nrow(sigma)
  ones <- matrix(rep(1,n), ncol=1)
  denom <- t(ones) %*% solve(sigma) %*% ones
  solve(sigma) %*% ones / denom[1]
}
```