1.4: Stationary Models and the Autocorrelation Function

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Definition

defn

$\{X_t\}$ is **strictly stationary** if for any h, and any selection of k time points $(t_1, \ldots, t_k)$:

$$F_{X_{t_1+h}, \ldots, X_{t_k+h}}(a_1, \ldots, a_k) = F_{X_{t_1}, \ldots, X_{t_k}}(a_1, \ldots, a_k),$$

for any $a_1, \ldots, a_k$.

Intuitively this means that the distribution of a bunch of time points does NOT depend on where they are in time, only on how they are spaced apart from one another.

More often we will be concerned with *weak* stationarity.

Definition

Let $\{X_t\}$ be a time series with $E[X_t] < \infty$ for each t.

mean function

The **mean function** is defined as

$$\mu_X(t) = E[X_t]$$

covariance function

The **covariance function** is defined on pairs of integral time points r, s as

$$\begin{aligned}\gamma_X(r,s) &= \mathrm{Cov}(X_r, X_s) \\ &= E[(X_r - E[X_r])(X_s - E[X_s])] \\ &= E[X_r X_s] - E[X_r]E[X_s].\end{aligned}$$

Definition

weak stationarity

$\{X_t\}$ is **weakly stationary** if

1. $\mu_X(t)$ is constant or free of t
2. $\gamma_X(t, t+h)$ is independent or free of t for each h

Intuitively: mean doesn't change, and covariances only depend on the lags.

From now on, when we say "stationary," we mean this type.

Definition

The ACVF

The **autocovariance function** for $\{X_t\}$ is defined as

$$\gamma_X(h) = \text{Cov}(X_{t+h}, X_t).$$

"Auto" means "self"

The ACF

The **autocorrelation function** for $\{X_t\}$ is defined as

$$\rho_X(h) = \text{Cor}(X_{t+h}, X_t) = \frac{\text{Cov}(X_{t+h}, X_t)}{\sqrt{\text{Var}(X_t)\text{Var}(X_{t+h})}} = \frac{\gamma_X(h)}{\gamma_X(0)}.$$

Because we are only defining this function for stationary series

Properties

Property 1

The covariance operator is **bilinear**:

- $\text{Cov}(aX, Y) = a\text{Cov}(X, Y)$
- $\text{Cov}(X, aY) = a\text{Cov}(X, Y)$
- $\text{Cov}(X + Z, Y) = \text{Cov}(X, Y) + \text{Cov}(Z, Y)$
- $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$

"bi" means "two" or "both" (linear in both arguments)

Property 2

Independence implies (is stronger than) 0 correlation/covariance

$$\gamma_X(h) = E[X_t X_{t+h}] - E[X_t]E[X_{t+h}] = E[X_t]E[X_{t+h}] - E[X_t]E[X_{t+h}] = 0$$

We use these properties a lot when we look at autocovariance functions for different models.

Example 1: IID Noise

Let $\{X_t\}$ be IID noise with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Then:

$$\gamma_X(h) = E[X_{t+h}X_t] = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

- This is stationary.
- We are not saying what the distribution is!
- From now on we write $X_t \stackrel{iid}{\sim} \text{IID}(0, \sigma^2)$

Example 2: White Noise

Let $\{X_t\}$ be uncorrelated but not necessarily independent with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Then:

$$\gamma_X(h) = E[X_{t+h}X_t] = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

- This is stationary.
- We are not saying what the distribution is!
- From now on we write $X_t \overset{iid}{\sim} \text{WN}(0, \sigma^2)$
- All IID Noise is White Noise, but not all White Noise is IID Noise.

Example 3: Random Walk

Let $\{X_t\}$ be uncorrelated but not necessarily independent with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Define the random walk as $S_t = \sum_{i=1}^{t} X_i$. Then:

$$\begin{aligned}
\gamma_X(t+h, t) &= E[S_{t+h} S_t] \\
&= E\left[\left\{\sum_{i=1}^{t+h} X_i\right\}\left\{\sum_{i=1}^{t} X_i\right\}\right] \\
&= E\left[\left\{\sum_{i=1}^{t} X_i\right\}\left\{\sum_{i=1}^{t} X_i\right\}\right] \\
&= \sum_{i=1}^{t} E[X_i^2] \\
&= t\sigma^2.
\end{aligned}$$

Example 3: Random Walk

Let $\{X_t\}$ be uncorrelated but not necessarily independent with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Define the random walk as $S_t = \sum_{i=1}^{t} X_i$. Then:

$$\gamma_X(h) = t\sigma^2.$$

- This is NOT stationary.
- We are not saying what the distribution is!
- But $E[S_t] = E[\sum_{i=1}^{t} X_t] = \sum_{i=1}^{t} E[X_t] = 0$ by linearity.
- no "pattern" but variance increasing with horizon

Example 4: First-Order Moving Average MA(1)

Let $\{Z_t\} \sim \text{WN}(0, \sigma^2)$. Define $\{X_t\}$ as

$$X_t = Z_t + \theta Z_{t-1}$$

with $t \in \mathbb{Z}$, and $\theta \in \mathbb{R}$.

$E[X_t] = 0$ for all t by linearity, and

$$\begin{aligned}
\gamma_X(h) &= E[X_{t+h}X_t] \\
&= E[(Z_{t+h} + \theta Z_{t+h-1})(Z_t + \theta Z_{t-1})] \\
&= E[Z_{t+h}Z_t] + \theta E[Z_{t+h}Z_{t-1}] + \theta E[Z_{t+h-1}Z_t] + \theta^2 E[Z_{t+h-1}Z_{t-1}] \\
&= (1+\theta^2)\gamma_Z(h) + \theta\gamma_Z(h+1) + \theta\gamma_Z(h-1) \\
&= \begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \sigma^2\theta & h = \pm 1 \\ 0 & |h| > 1 \end{cases}
\end{aligned}$$

Example 4: First-Order Moving Average MA(1)

$E[X_t] = 0$ for all t by linearity, and

$$\begin{aligned}\rho_X(h) &= \frac{\gamma_X(h)}{\gamma_X(0)} \\ &= \begin{cases} 1 & \\ \theta/(1+\theta^2) & h = \pm 1 \\ 0 & |h| > 1 \end{cases}\end{aligned}$$

- This is stationary

Example 5: First-Order Autoregression AR(1)

Let $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Also, assume $-1 < \phi < 1$, and $E[Z_t X_s] = 0$ for $s < t$. Define $\{X_t\}$ as

$$X_t = \phi X_{t-1} + Z_t \tag{*}$$

with $t \in \mathbb{Z}$.

1. $E[X_t] = \phi E[X_{t-1}]$ for all t, by linearity.
2. And if $h > 0$

$$\begin{aligned}\gamma_X(h) &= E[X_{t+h}X_t] \\ &= E[(\phi X_{t+h-1} + Z_{t+h})(X_t + Z_t)] \\ &= \phi\gamma_X(h-1)\end{aligned}$$

3. $\gamma_X(0) = E[(\phi X_{t-1} + Z_t)^2] = \phi^2\gamma_X(0) + \sigma^2 \iff \gamma_X(0) = \frac{\sigma^2}{1-\phi^2}$

Example 5: First-Order Autoregression AR(1)

Let $\{Z_t\} \sim \text{WN}(0, \sigma^2)$. Also, assume $-1 < \phi < 1$, and $E[Z_t X_s] = 0$ for $s < t$. Define $\{X_t\}$ as

$$X_t = \phi X_{t-1} + Z_t \qquad (*)$$

with $t \in \mathbb{Z}$.

1. If one has mean 0, they all do, so we assume they all have mean 0.
2. Clearly γ_X is symmetric, i.e. $\gamma_X(h) = \gamma_X(-h)$

AR(1) stationarity

Under these assumptions, $\{X_t\}$ is stationary with $\mu_X(t) = 0$, $\gamma_X(h) = \phi^{|h|} \frac{\sigma^2}{(1-\phi^2)}$ and $\rho_X(h) = \phi^{|h|}$.

The Sample ACVF and ACF

We never have the true/population autocovariance or autocorrelation function. So far we are just theorizing about made up models.

Enter the **sample autocovariance and autocorrelation functions**. They estimate γ_X and ρ_X.

The Sample ACVF and ACF

The **sample mean** of the data $x_1, \ldots, x_n$ is $\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$.

The sample autocovariance function

The **sample autocovariance function** for the data $x_1, \ldots, x_n$ is

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{i=1}^{n-|h|}(x_{t+|h|} - \bar{x})(x_t - \bar{x}), \qquad -n < h < n.$$

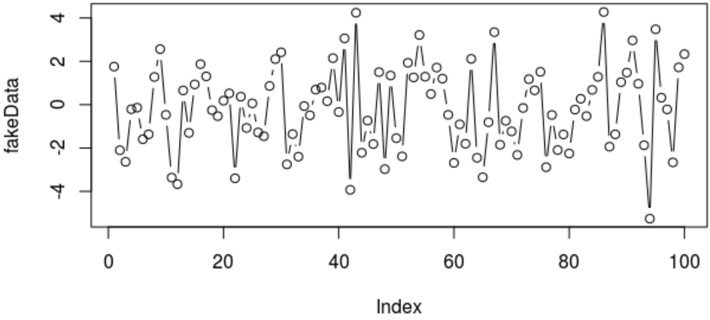
The sample autocorrelation function

The **sample autocorrelation function** for the data $x_1, \ldots, x_n$ is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \qquad -n < h < n.$$

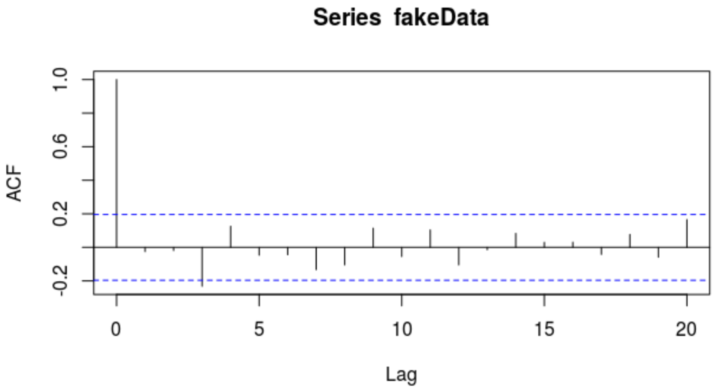
Test Yourself 1

What model is appropriate for the following data?



Test Yourself 1

What about if we look at the sample autocorrelation?



The bounds are 95% confidence intervals, which means we should expect to see about $(100 - 95)\%$ of the data to be accidentally outside this range.

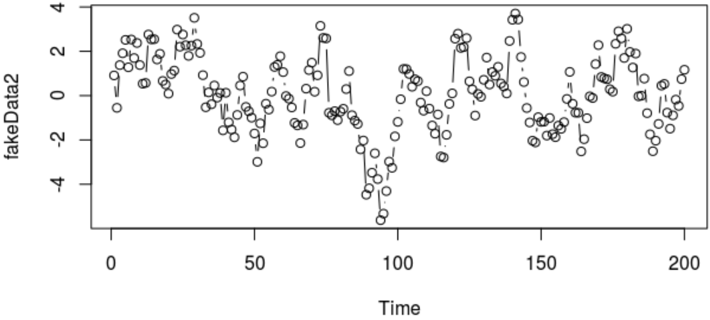
Test Yourself 1

Answer: it was IID Gaussian Noise.

```
fakeData <- rnorm(n=100,mean = 0, sd = 2)
plot(fakeData, type = "b")
acf(fakeData, type = "correlation")
```

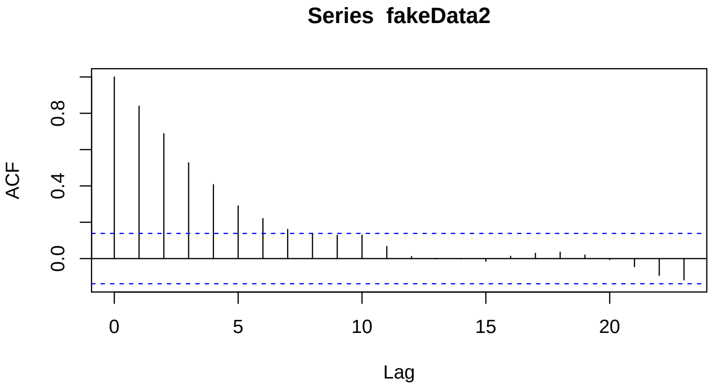
Test Yourself 2

Round 2:



Test Yourself 2

Looking at the sample autocorrelation?



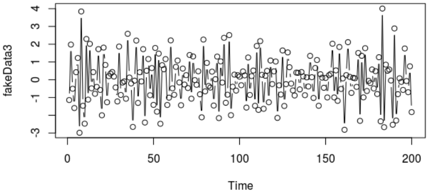
Test Yourself 2

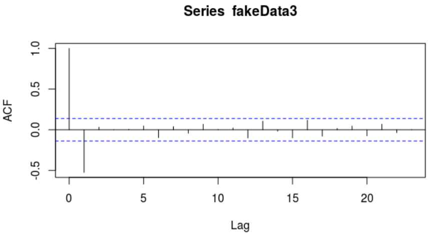
Answer: it was AR(1) with Gaussian Noise.

```
arima.sim(list(ar=c(.9)), n = 200)
plot(fakeData2, type = "b")
acf(fakeData2, type = "correlation")
```

Test Yourself 3

Last one:





Test Yourself 3

Answer: MA(1) with Gaussian Noise

```
fakeData3 <- arima.sim(list(ma=c(-.9)), n = 200)
plot(fakeData3, type = "b")
acf(fakeData3)
```