2.3: Introduction to ARMA Processes

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Motivation

We introduce Autoregressive Moving Average Processes (ARMA) by doing some examples. Through these examples we start talking about some of the "weird" things that can happen with these, which supports our hypothesis that time series modelling is more difficult than vanilla regression.

Definition

ARMA(1,1)

The time series $\{X_t\}$ is an **ARMA(1,1)** process if it is stationary and it satisfies

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $\phi + \theta \neq 0$

this can be written as $\phi(B)X_t = \theta(B)(Z_t)$

Restrictions on Parameters

Why $\phi + \theta \neq 0$? Say $|\phi| < 1$, then

$$\begin{aligned}
X_t &= \phi^{-1}(B)\theta(B)Z_t \\
&= (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)(1 + \theta B)Z_t \\
&= [1 + (\theta + \phi)B + (\phi\theta + \phi^2)B^2 + (\phi^2\theta + \phi^3)B^3 \cdots]Z_t \\
&= Z_t + \left[(\theta + \phi)\sum_{j=1}^{\infty} \phi^{j-1}B^j\right] Z_t
\end{aligned}$$

Show that if $|\phi| > 1$ then

$$X_t = -\theta\phi^{-1}Z_t - (\theta + \phi)\sum_{j=1}^{\infty} \phi^{-(j+1)} Z_{t+j}$$

Summary and New Term

Note:

1. Stationary solution exists iff $\phi \neq \pm 1$
2. Solution is **causal** iff $|\phi| < 1$
3. Solution is **noncausal** if $|\phi| > 1$

New definition: invertibility.

1. X_t is **causal** if it can be written in terms of $Z_s, s \leq t$
2. X_t is **invertible** if Z_t can be written in terms of $X_s, s \leq t$

Invertibility Demo

ARMA(1,1) is invertible iff $|\theta| < 1$.

$$
\begin{aligned}
Z_t &= \theta^{-1}(B)\phi(B)X_t \\
&= (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \cdots)(1 - \phi B) \\
&= [1 - (\phi + \theta)B + (\theta\phi + \theta^2)B^2 + \cdots]X_t \\
&\vdots \\
&= X_t - (\theta + \phi)\sum_{j=1}^{\infty}(-\theta)^{j-1}X_{t-j}
\end{aligned}
$$

Show if $|\theta| > 1$ then it isn't invertible and

$$
Z_t = -\phi\theta^{-1}X_t + (\theta + \phi)\sum_{j=1}^{\infty}(-\theta)^{-(j-1)}X_{t+j}.
$$

Summary

1. $\phi + \theta \neq 0$
2. Stationary solution exists iff $|\phi| \neq 1$.
3. Solution is **causal** iff $|\phi| < 1$
4. Solution is **invertible** if $|\theta| < 1$

The errors and data are linear combinations of each other. Causal and invertible models are preferable because the alternatives don't really "make sense," especially in the context of financial data.