2.4: Conditional Probability

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Motivation

Sometimes the probabilities of events change with the set of available information that we have.

Example: Let $A = \{\text{Google's share price increases tomorrow}\}$. Then $P(A) \approx .5$

But if we let $C = \{\text{the government bans access to google.com}\}$, what's $P(A \text{ given } C)$?

Conditional Probability

The notation: we'll write $P(A \text{ given } C)$ like $P(A|C)$. And here's the formula:

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

Note

1. It helps to think of this in the context of venn diagrams
2. $C \neq \varnothing$, otherwise we're dividing by 0 using this definition
3. sometimes we'll use $P(A|C)P(C) = P(A \cap C)$

Example 2.26 (pg. 76)

$$
\begin{aligned}
P(A|B \cup C) &= \frac{P(A \cap (B \cup C))}{P(B \cup C)} \\
&= \frac{P[(A \cap B) \cup (A \cap C)]}{P(B \cup C)} \\
&= \frac{P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))}{P(B \cup C)} \\
&= \frac{P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C)}{P(B \cup C)} \\
&= \frac{P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)}{P(B) + P(C) - P(B \cap C)} \\
&= .255
\end{aligned}
$$

Example

Remember we said we would use $P(A|C)P(C) = P(A \cap C)$ a lot? This is useful in situations where there is some sort of sequential thing going on, or there are several stages of some random process.

Example: you're only interested in if the stock market goes up or down. The events that it goes up on Monday, Tuesday, Wednesday, Thursday and Friday are A_1, A_2, A_3, A_4 and A_5, respectively. The changes that it goes up or goes down only depend on what happened the day before (unless it's a Monday in which case let's assume it goes up or down with probability .5). Say there's a 75% chance that it does the same thing as it did the day before. This means that there is a 25% chance it does the opposite. What's the probability the market goes up every day of the week?

Example

$$
\begin{aligned}
P(\text{goes up every day}) &= P(A_5 \cap A_4 \cap \cdots \cap A_1) \\
&= P(A_5|A_4 \cap A_3 \cap A_2 \cap A_1)P(A_4|A_3 \cap A_2 \cap A_1)\times \\
&P(A_3|A_2 \cap A_1)P(A_2|A_1)P(A_1) \\
&= P(A_5|A_4)P(A_4|A_3)P(A_3|A_2)P(A_2|A_1)P(A_1) \\
&= (.75)(.75)(.75)(.75)(.5)
\end{aligned}
$$

Motivation

Conditional probability is also a precursor for a thing called **Bayes' Theorem**. We need a few more ideas before we get there though...

If $A_1, \ldots, A_k$ are all pairwise disjoint and if $\bigcup_{i=1}^k A_i = \mathcal{S}$, then

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

Proof

Notice that

$$\begin{aligned}
P(B) &= P(B \cap \mathcal{S}) \\
&= P(B \cap (\cup_{i=1}^{k} A_i)) \\
&= P\left(\cup_{i=1}^{k} (A_i \cap B)\right) \\
&= \sum_{i=1}^{k} P\left(A_i \cap B\right) \\
&= \sum_{i=1}^{k} P\left(B|A_i\right) P(A_i)
\end{aligned}$$

Bayes' Rule

Here's Bayes' theorem:

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}$$

Note

A key take-away here is that there's a switch thing going on...the left hand side has probabilities conditioning on B, whereas the right hand side has probabilities conditioning on As.

Example

Example 2.30 pops up in STAT 2020/2120 a lot...

Let D be the event that a person has a rare disease. Let T be the event that you test positive for this disease. The bottom of page 80 gives us $P(D) = .001$, $P(T|D) = .99$ and $P(T|D') = .02$. What's the probability that a person has the disease if he/she is told that he/she has it?

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$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D')P(D')} = .047$$