8.2: Second-Order Properties of Multivariate Time Series Examples

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Motivation

We analyze vector-valued time series from a second-order point of view.

Example 8.2.1

Let's focus on stationary time series. Recall the autocovariance function:
$\Gamma(h) = E[(\mathbf{X}_{t+h} - \boldsymbol{\mu})(\mathbf{X}_t - \boldsymbol{\mu})']$

$$\begin{bmatrix} X_{t,1} \\ X_{t,2} \end{bmatrix} = \begin{bmatrix} Z_t \\ Z_t \end{bmatrix} + \begin{bmatrix} 0 \\ .75Z_{t-10} \end{bmatrix}$$

1. $\Gamma(-10) = ?$
2. $\Gamma(0) = ?$
3. $\Gamma(10) = ?$

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$$\Gamma(-10) = \begin{bmatrix} 0 & .75 \\ 0 & .75 \end{bmatrix}$$

$$\Gamma(0) = \begin{bmatrix} 1 & 1 \\ 1 & 1.5625 \end{bmatrix}$$

$$\Gamma(10) = \begin{bmatrix} 0 & 0 \\ .75 & .75 \end{bmatrix}$$

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$$\Gamma(-10) = \begin{bmatrix} 0 & .75 \\ 0 & .75 \end{bmatrix} \qquad R(-10) = \begin{bmatrix} 0 & .6 \\ 0 & .48 \end{bmatrix}$$

$$\Gamma(0) = \begin{bmatrix} 1 & 1 \\ 1 & 1.5625 \end{bmatrix} \qquad R(0) = \begin{bmatrix} 1 & .8 \\ .8 & 1 \end{bmatrix}$$

$$\Gamma(10) = \begin{bmatrix} 0 & 0 \\ .75 & .75 \end{bmatrix} \qquad R(-10) = \begin{bmatrix} 0 & 0 \\ .6 & .48 \end{bmatrix}$$

The Autocovariance Function

Basic Properties of Γ

1. $\Gamma(h) = \Gamma'(-h)$
2. for all i, j, $|\gamma_{ij}(h)| \leq \sqrt{\gamma_{ii}(0)\gamma_{jji}(0)}$
3. $\gamma_{ii}(h)$ is element i's autocovariance function
4. $\sum_{j=1}^{n} \sum_{k=1}^{n} \mathbf{a}_j' \Gamma(j-k) \mathbf{a}_k \geq 0$

Try proving these.

Vector-valued noise

White Noise

The m-variate series $\{\mathbf{Z}_t\}$ is **white noise with mean** 0 **and covariance matrix** Σ, written

$$\{\mathbf{Z}_t\} \sim WN(\mathbf{0}, \Sigma),$$

if $\{\mathbf{Z}_t\}$ is stationary with mean vector $\mathbf{0}$ and covariance matrix function

$$\Gamma(h) = \begin{cases} \Sigma, & h = 0 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

IID Noise

The m-variate series $\{\mathbf{Z}_t\}$ is **iid noise with mean** 0 **and covariance matrix** Σ, written

$$\{\mathbf{Z}_t\} \sim iid(\mathbf{0}, \Sigma),$$

if $\{\mathbf{Z}_t\}$ are independent and identically distributed with mean vector $\mathbf{0}$ and covariance matrix Σ.

More definitions

linear process

The m-variate series $\{\mathbf{X}_t\}$ is a **linear process** if it has the representation

$$\mathbf{X}_t = \sum_{j=-\infty}^{\infty} C_j \mathbf{Z}_{t-j}, \qquad \{\mathbf{Z}_t\} \sim WN(\mathbf{0}, \Sigma)$$

where $\{C_j\}$ is a sequence of matrices whose components are absolutely summable.

This is stationary and has covariance matrix function

$$\Gamma(h) = \sum_{j=-\infty}^{\infty} C_{j+h} \Sigma C_j'.$$

(This is a HW question)

MA(∞)

MA(∞) process

An MA(∞) process is a linear process with $C_j = 0$ for $j < 0$.

$$\mathbf{X}_t = \sum_{j=0}^{\infty} C_j \mathbf{Z}_{t-j}, \qquad \{\mathbf{Z}_t\} \sim WN(\mathbf{0}, \Sigma)$$