2.3: Counting Techniques

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Motivation

If each outcome is equally likely, and there are N distinct (think disjoint) outcomes, then the probability of any outcome O is $P(O) = \frac{1}{N}$. Then computing probabilities of events is basically just counting up how many outcomes are in that event (i.e. $P(A) = \frac{\text{num. outcomes in A}}{N}$) (Note: same thing as page 66) (also: why can we add probabilities like this?)

When you have discrete random variables, counting rules help you find out how big the sample space is and/or how many outcomes are in your event in question.

Product Rule for Ordered Pairs

If you have an ordered k-tuple of k elements $(O_1, O_2, \ldots O_k)$, where the ith element can be arranged n_i ways where $i = 1, \ldots, k$, then the total number of possible tuples is

$$n_1 \times n_2 \times \cdots \times n_k$$

Product Rule for Ordered Pairs

Example: how many 4-digit passcodes are there for your phone?

how many 4-digit passcodes are there that start with 5?

If you know that your friends passcode starts with a 5, what is the chance that you can guess it correctly in one try? You will have to assume that all the passcodes are equally likely.

Product Rule for Ordered Pairs

Note(s):

1. The author recommends drawing tree diagrams to visualize situations like these.
2. The previous scenario's drawing mechanism is sometimes described as being *with replacement* since the number of ways the ith element can occur doesn't affect subsequent or previous draws
3. now we'll talk about draws that are made *without replacement*
4. we'll do a few examples to make this idea clearer

Permutations

What if instead you were taking k things from n, and when you took an item, it couldn't be chosen again (e.g. people picking a seat at a table).

Any ordered sequence of k objects taken from a set of n distinct objects is called a **permutation**

Permutations

Example 2.2.1 from page 69: 10 teaching assistants are available. The professor needs a TA to grade exactly one problem each on a 4 problem test. How many ways can he pick TAs to grade his problems?

$$\text{number of ways} = 10 * (10-1) * (10-2) * (10-3)$$

Permutations

In general we call the number of permutatons of k things from n $P_{k,n}$. It's formula is:

$$P_{k,n} = n(n-1)\cdots(n-k+2)(n-k+1) = \frac{n!}{(n-k)!}$$

If you haven't seen factorials before: $m! = (m)(m-1)\cdots(2)(1)$

Combinations

We just highlighted the distinction between *with replacement* and *without replacement*.

Now another distinction: **ordered** and **unordered**

Example: in the previous problem the professor cared which TA graded which problem; in other words, the order mattered. What if he only cared about which TAs he chose, and he didn't care about what assignment they had?

Combinations

Given a set of n objects, any unordered subset of size k that can be formed is called a **combination**

The number of combinations of size k from n things is often denoted by the $\binom{n}{k}$, read as "n choose k" and its formula is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

a connection...

When there is *no replacement*, the connection between the number of ordered and unordered things is

$$P_{k,n} = k!\binom{n}{k}$$

Combinations

Example: how many 5-card poker hands are there in a 52 card deck?