2.4: Properties of the Sample Mean and the Sample Autocorrelation Function

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Motivation

X_t is characterized by μ and $\gamma(\cdot)$. This chapter talks about model-free estimators of these quantities. Both the formulas for the estimators, and their sampling distributions.

Estimating the Mean

We estimate the mean with $\bar{X} = n^{-1} \sum_i X_i$. It is unbiased

$$E[\bar{X}] = n^{-1}(E[X_1] + \cdots + E[X_n]) = \mu$$

by linearity of $E[\cdot]$ and stationarity, and it's mean squared error (MSE) is

$$\begin{aligned}
\mathrm{MSE}(\bar{X}) &= \mathrm{Var}(\bar{X}) && \text{defn} \\
&= \mathrm{Cov}\left(\sum_{i=1}^{n} n^{-1} X_i, \sum_{j=1}^{n} n^{-1} X_j\right) && \text{defn} \\
&= n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) && \text{bilinearity of cov} \\
&= n^{-2} \sum_{h=-(n-1)}^{n-1} (n - |h|)\gamma_X(h) && \text{count diagonally : } h = i - j \\
&= n^{-1} \sum_{h} \left(1 - \frac{|h|}{n}\right) \gamma_X(h)
\end{aligned}$$

Note

1.)

$$\mathrm{Var}(\bar{X}) = n^{-1} \sum_{h=-(n-1)}^{(n-1)} \left(1 - \frac{|h|}{n}\right) \gamma_X(h) \to 0$$

as $n \to \infty$ if $\gamma(h) \to 0$, and

2.)

$$n\mathrm{Var}(\bar{X}) \to \sum_{h=-\infty}^{\infty} \gamma(h)$$

as $n \to \infty$ if $\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$

Proof

Inference for μ

We are usually interested if $\mu > 0$ or not. This affects our decision on whether or not to buy an asset. To construct confidence intervals and perform hypothesis tests, we need the sampling distribution of $\bar{X}$.

If our time series is weakly stationary then

$$\bar{X} \overset{\text{approx.}}{\sim} \text{Normal}\left(\mu, n^{-1} \sum_{h=-n}^{n} \gamma_X(h)\right).$$

for large n. Or, if we assume all our noise terms are Normally distributed, then

$$\bar{X} \sim \text{Normal}\left(\mu, n^{-1} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_X(h)\right)$$

exactly. However, we usually don't know the true autocovariance function.

Inference for μ

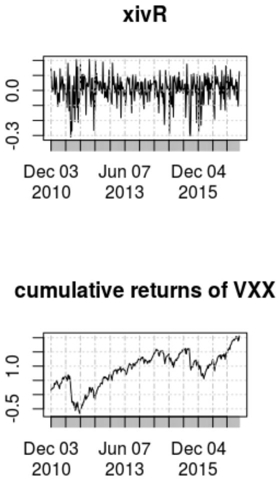
We can estimate $V^2 = \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_X(h)$ with

$$\hat{V}^2 = \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \hat{\gamma}_X(h).$$

A $(1-\alpha)\%$ confidence interval is $\bar{x} \pm z_{\alpha/2}\sqrt{v^2/n}$, and a hypothesis test against the null of $H_0 : \mu = 0$ can use the test statistic $\sqrt{n}\bar{x}/\hat{v}$ (rejection region depends on the alternative hypothesis).

Example

$\bar{X} = 0.006153473!$



Example

So we're interested in μ. The previous theorems are useful because they tell us our estimates get very close to the true mean, but also that future average returns will be arbitrarily close to the true mean. Buying and holding is profitable if $\mu > 0$.

95% Confidence interval:

```
> xbar - zAlphaOverTwo * sqrt(asympVar) #lower
[1] -0.0273102
> xbar + zAlphaOverTwo * sqrt(asympVar) #upper
[1] 0.03961714
```

Try different approximations for the standard deviation, try raising α, or try lower confidence intervals.

Estimating γ and ρ

If we have a time series, knowing about the mean is great. However, we can increase the accuracy of our predictions if we also learn about the time structure via $\gamma(\cdot)$ or $\rho(\cdot)$.

Recall that

$$\hat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-|h|} (X_{t+|h|} - \bar{X}_n)(X_t - \bar{X}_n)$$

and

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$

NND of Autocovariance

Proving a matrix is non-negative definite often involves finding a square root matrix. When you can do that, writing the quadratic form becomes a square norm that can't be negative.

Often times the autocovariance matrix is positive definite (> 0), too (page 52).

Lastly, notice for h large or n small, higher order autocovariance estimates are unreliable. "Jenkins (1976), p. 33 who suggest that n should be at least about 50 and $h \leq n/4$."

Also, we always divide by n. If we divided sums of squares by $(n - h)$, this proof wouldn't work.

NND of Autocovariance

Notice that

$$\hat{\Gamma}_k = n^{-1} T T'$$

where

$$\hat{\Gamma}_k = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \cdots & \hat{\gamma}(k-1) \\ \hat{\gamma}(1) & \hat{\gamma}_{(}0) & \cdots & \hat{\gamma}(k-2) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\gamma}(k-1) & \hat{\gamma}(k-1) & \cdots & \hat{\gamma}(0) \end{bmatrix}$$

and T is equal to

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & X_1 - \bar{X} & X_2 - \bar{X} & \cdots & X_k - \bar{X} \\ 0 & \cdots & 0 & X_1 - \bar{X} & X_2 - \bar{X} & \cdots & X_k - \bar{X} & 0 \\ \vdots & & & & & & & \vdots \\ 0 & X_1 - \bar{X} & X_2 - \bar{X} & \cdots & X_k - \bar{X} & 0 & \cdots & 0 \end{bmatrix}$$

So for any weight vector w, $w'\hat{\Gamma}_k w = n^{-1}(T'w)'(T'w) = n^{-1}c'c \geq 0$.

Bartlett's Formula

The sampling distribution of $\hat{\rho}$ is very difficult to find exactly. Approximately, when n is large, though, we have $\hat{\boldsymbol{\rho}}_k = (\hat{\rho}(1), \ldots, \hat{\rho}_k)'$

$$\hat{\boldsymbol{\rho}}_k \overset{\text{approx.}}{\sim} \mathrm{N}(\boldsymbol{\rho}, n^{-1}\boldsymbol{W})$$

The coefficients of $\boldsymbol{W}$ are given by **Barlett's formula** (not reproduced here). Two versions of this formula are given on page 53.

We used this formula once before: chapter 1.6 tested residuals for independence (example 2.4.2 mentions this). Under the null that $\boldsymbol{\rho} = \mathbf{0}$, the asymptotic covariance matrix simplifies to $n^{-1}I_n$.