Univariate Time Series Models

Taylor R. Brown

Last class

The last two classes we learned about different types of financial instruments.

Now we will start discussing how to model univariate series.

The models we discuss are useful in their own rite, and they are also building blocks of more complicated models.

Definition

$\{X_t\}$ is **strictly stationary** if for any h, and any selection of k time points $(t_1, \ldots, t_k)$:

$$F_{X_{t_1+h},\ldots,X_{t_k+h}}(a_1, \ldots, a_k) = F_{X_{t_1},\ldots,X_{t_k}}(a_1, \ldots, a_k),$$

for any $a_1, \ldots, a_k$.

Intuitively this means that the distribution of a bunch of time points does NOT depend on where they are in time, only on how they are spaced apart from one another.

More often we will be concerned with *weak* stationarity.

Definition

Let $\{X_t\}$ be a time series with $E[X_t] < \infty$ for each t.

The **mean function** is defined as

$$\mu_X(t) = E[X_t]$$

The **covariance function** is defined on pairs of integral time points r, s as

$$\begin{aligned}
\gamma_X(r, s) &= \mathrm{Cov}(X_r, X_s) \\
&= E[(X_r - E[X_r])(X_s - E[X_s])] \\
&= E[X_r X_s] - E[X_r]E[X_s].
\end{aligned}$$

Definition

$\{X_t\}$ is **weakly stationary** if

- $\mu_X(t)$ is constant or free of t,
- $\gamma_X(t, t+h)$ is independent or free of t for each h

Intuitively: mean doesn't change, and covariances only depend on the lags.

From now on, when we say "stationary," we mean this type.

Definition

The **autocovariance function** for $\{X_t\}$ is defined as

$$\gamma_X(h) = \mathrm{Cov}(X_{t+h}, X_t).$$

"Auto" means "self"

The **autocorrelation function** for $\{X_t\}$ is defined as

$$\rho_X(h) = \mathrm{Cor}(X_{t+h}, X_t) = \frac{\mathrm{Cov}(X_{t+h}, X_t)}{\sqrt{\mathrm{Var}(X_t)\mathrm{Var}(X_{t+h})}} = \frac{\gamma_X(h)}{\gamma_X(0)}.$$

Because we are only defining this function for stationary series

Properties

The covariance operator is **bilinear**:

- $\mathrm{Cov}(aX, Y) = a\mathrm{Cov}(X, Y)$
- $\mathrm{Cov}(X, aY) = a\mathrm{Cov}(X, Y)$
- $\mathrm{Cov}(X + Z, Y) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(Z, Y)$
- $\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$

"bi" means "two" or "both" (linear in both arguments)

Properties

Independence implies (is stronger than) 0 correlation/covariance

$$\begin{aligned}\gamma_X(h) &= E[X_t X_{t+h}] - E[X_t]E[X_{t+h}] \\ &= E[X_t]E[X_{t+h}] - E[X_t]E[X_{t+h}] \\ &= 0\end{aligned}$$

We use these properties a lot when we look at autocovariance functions for different models.

Example 1: IID Noise

Let $\{X_t\}$ be IID noise with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Then:

$$\gamma_X(h) = E[X_{t+h}X_t] = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

- This is stationary.
- We are not saying what the distribution is!
- From now on we write $X_t \overset{iid}{\sim} \text{IID}(0, \sigma^2)$

Example 2: White Noise

Let $\{X_t\}$ be uncorrelated but not necessarily independent with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Then:

$$\gamma_X(h) = E[X_{t+h}X_t] = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

- This is stationary.
- We are not saying what the distribution is!
- From now on we write $X_t \overset{iid}{\sim} \text{WN}(0, \sigma^2)$
- All IID Noise is White Noise, but not all White Noise is IID Noise.

Example 3: Random Walk

Let $\{X_t\}$ be uncorrelated but not necessarily independent with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Define the random walk as $S_t = \sum_{i=1}^{t} X_i$. Then:

$$\begin{aligned}
\gamma_X(t+h, t) &= E[S_{t+h} S_t] \\
&= E\left[\left\{\sum_{i=1}^{t+h} X_i\right\}\left\{\sum_{i=1}^{t} X_i\right\}\right] \\
&= E\left[\left\{\sum_{i=1}^{t} X_i\right\}\left\{\sum_{i=1}^{t} X_i\right\}\right] \\
&= \sum_{i=1}^{t} E[X_i^2] \\
&= t\sigma^2.
\end{aligned}$$

Example 3: Random Walk

Let $\{X_t\}$ be uncorrelated but not necessarily independent with $E[X_t] = 0$ and $\text{Var}(X_t) = E[X_t^2] = \sigma^2 < \infty$. Define the random walk as $S_t = \sum_{i=1}^{t} X_i$. Then:

$$\gamma_X(h) = t\sigma^2.$$

- This is not stationary!
- We are not saying what the distribution is!
- From now on we write $X_t \overset{iid}{\sim} \text{WN}(0, \sigma^2)$
- All IID Noise is White Noise, but not all White Noise is IID Noise.

Example 4: First-Order Moving Average MA(1)

Let $\{Z_t\} \sim \text{WN}(0, \sigma^2)$. Define $\{X_t\}$ as

$$X_t = Z_t + \theta Z_{t-1}$$

with $t \in \mathbb{Z}$, and $\theta \in \mathbb{R}$.

$E[X_t] = 0$ for all t by linearity, and

$$\begin{aligned}
\gamma_X(h) &= E[X_{t+h} X_t] \\
&= E[(Z_{t+h} + \theta Z_{t+h-1})(Z_t + \theta Z_{t-1})] \\
&= E[Z_{t+h} Z_t] + \theta E[Z_{t+h} Z_{t-1}] + \theta E[Z_{t+h-1} Z_t] + \theta^2 E[Z_{t+h-1} Z_{t-1}] \\
&= (1 + \theta^2)\gamma_Z(h) + \theta \gamma_Z(h+1) + \theta \gamma_Z(h-1) \\
&= \begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \sigma^2 \theta & h = \pm 1 \\ 0 & |h| > 1 \end{cases}
\end{aligned}$$

Example 4: First-Order Moving Average MA(1)

$E[X_t] = 0$ for all t by linearity, and

$$\begin{aligned}
\rho_X(h) &= \frac{\gamma_X(h)}{\gamma_X(0)} \\
&= \begin{cases} 1 & \\ \theta/(1+\theta^2) & h = \pm 1 \\ 0 & |h| > 1 \end{cases}
\end{aligned}$$

- This is stationary

Example 5: First-Order Autoregression AR(1)

Let $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Also, assume $-1 < \phi < 1$, and $E[Z_t X_s] = 0$ for $s < t$. Define $\{X_t\}$ as

$$X_t = \phi X_{t-1} + Z_t \qquad (*)$$

with $t \in \mathbb{Z}$.

1. $E[X_t] = \phi E[X_{t-1}]$ for all t, by linearity.
2. And if $h > 0$

$$\begin{aligned} \gamma_X(h) &= E[X_{t+h} X_t] \\ &= E[(\phi X_{t+h-1} + Z_{t+h})(X_t + Z_t)] \\ &= \phi \gamma_X(h-1) \end{aligned}$$

3. $\gamma_X(0) = E[(\phi X_{t-1} + Z_t)^2] = \phi^2 \gamma_X(0) + \sigma^2 \iff \gamma_X(0) = \frac{\sigma^2}{1-\phi^2}$

Example 5: First-Order Autoregression AR(1)

Let $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Also, assume $-1 < \phi < 1$, and $E[Z_t X_s] = 0$ for $s < t$. Define $\{X_t\}$ as

$$X_t = \phi X_{t-1} + Z_t \qquad (*)$$

with $t \in \mathbb{Z}$.

- If one has mean 0, they all do, so we assume they all have mean 0.
- Clearly γ_X is symmetric, i.e. $\gamma_X(h) = \gamma_X(-h)$

Under these assumptions, $\{X_t\}$ is stationary with $\mu_X(t) = 0$, $\gamma_X(h) = \phi^{|h|} \frac{\sigma^2}{(1-\phi^2)}$ and $\rho_X(h) = \phi^{|h|}$.

The Sample ACVF and ACF

We never have the true/population autocovariance or autocorrelation function. So far we are just theorizing about made up models.

Enter the **sample autocovariance and autocorrelation functions**. They estimate γ_X and ρ_X.

The Sample ACVF and ACF

The **sample mean** of the data $x_1, \ldots, x_n$ is $\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$.

The **sample autocovariance function** for the data $x_1, \ldots, x_n$ is

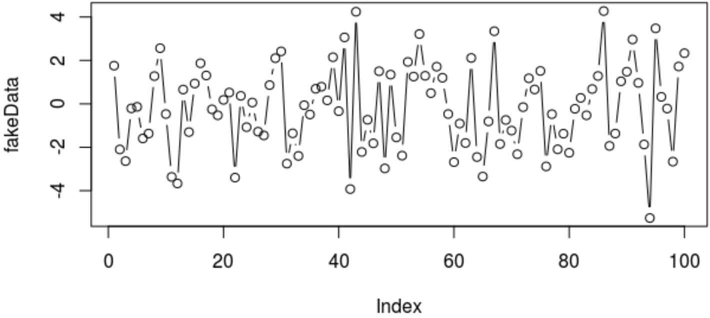
$$\hat{\gamma}(h) = \frac{1}{n}\sum_{i=1}^{n-|h|}(x_{t+|h|} - \bar{x})(x_t - \bar{x}), \qquad -n < h < n.$$

The **sample autocorrelation function** for the data $x_1, \ldots, x_n$ is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \qquad -n < h < n.$$

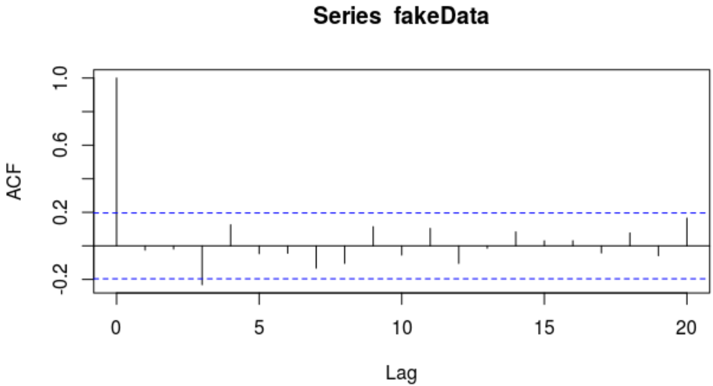
Test Yourself

What model is appropriate for the following data?



Test Yourself

What about if we look at the sample autocorrelation?



The bounds are 95% confidence intervals, which means we should expect to see about $(100 - 95)\%$ of the data to be accidentally outside this range.

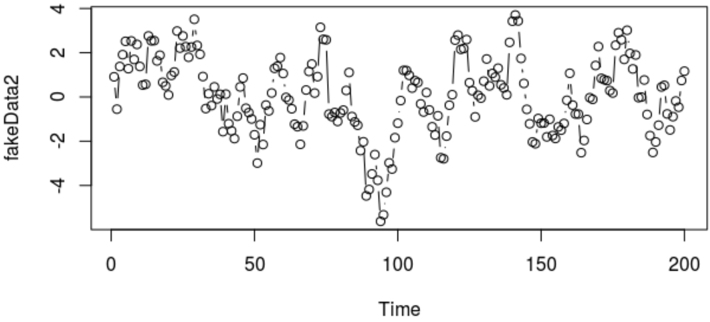
Test Yourself

Answer: it was IID Gaussian Noise.

```
fakeData <- rnorm(n=100,mean = 0, sd = 2)
plot(fakeData, type = "b")
acf(fakeData, type = "correlation")
```

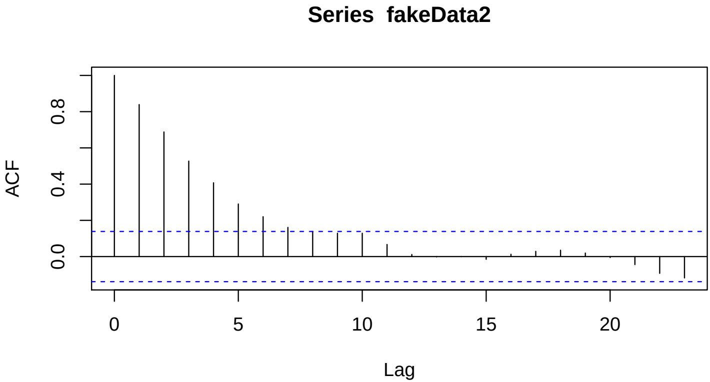
Test Yourself

Round 2:



Test Yourself

Looking at the sample autocorrelation?



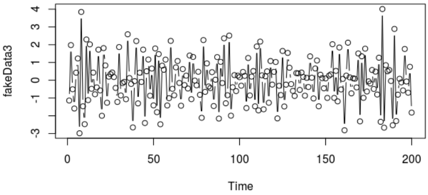
Test Yourself

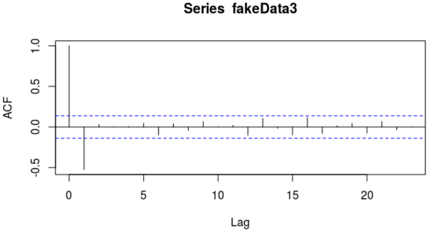
Answer: it was AR(1) with Gaussian Noise.

```
arima.sim(list(ar=c(.9)), n = 200)
plot(fakeData2, type = "b")
acf(fakeData2, type = "correlation")
```

Test Yourself

Another one:





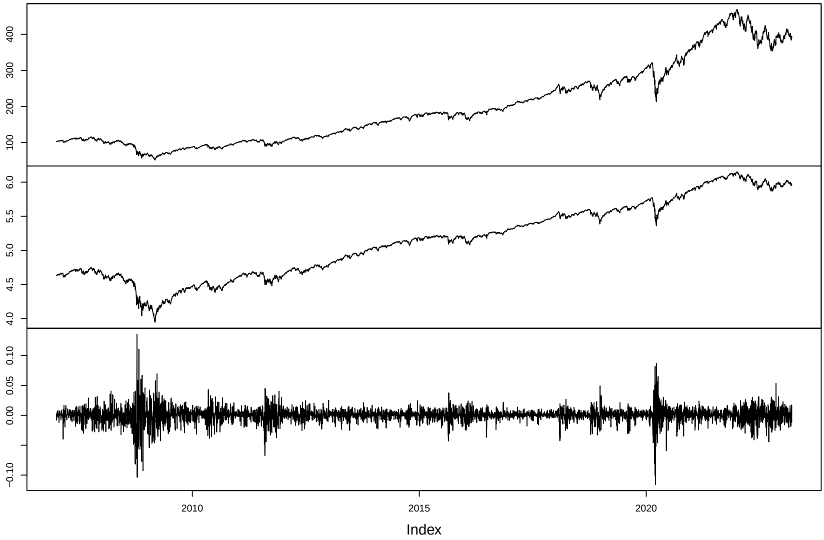
Test Yourself

Answer: MA(1) with Gaussian Noise

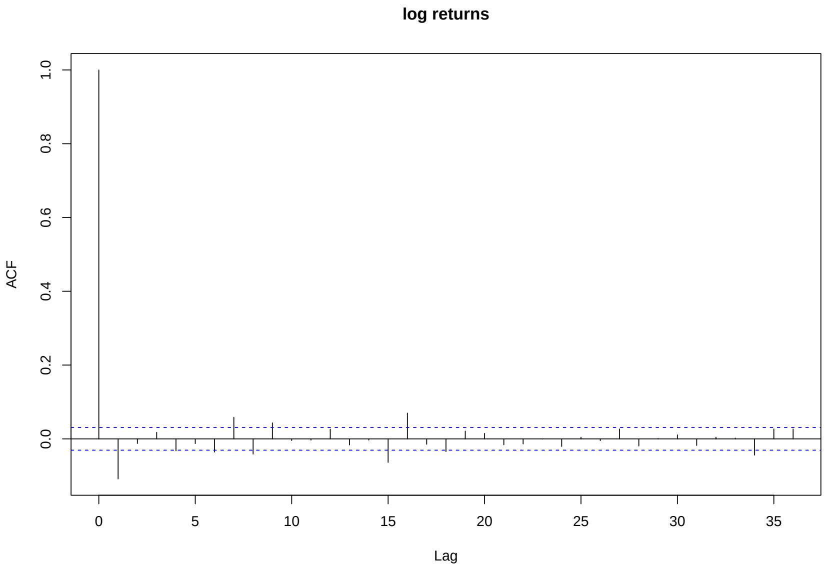
```
fakeData3 <- arima.sim(list(ma=c(-.9)), n = 200)
plot(fakeData3, type = "b")
acf(fakeData3)
```

Test Yourself

Daily close prices of S&P500 ETF (SPY)

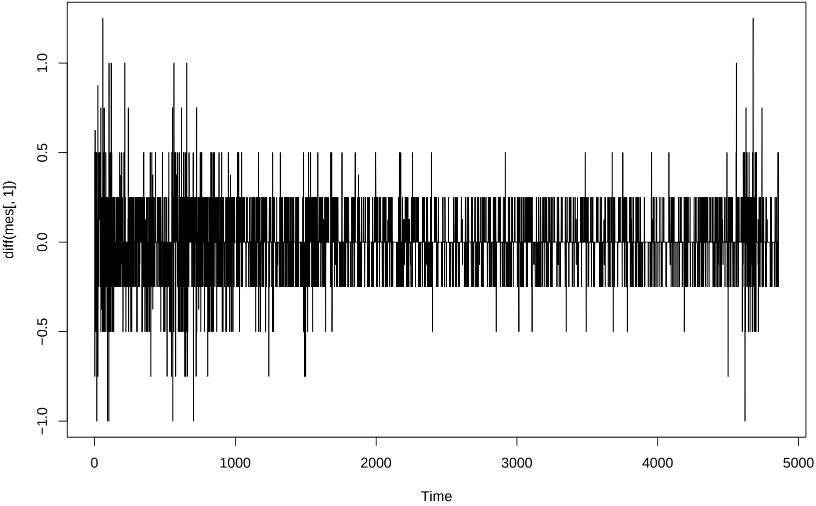


Test Yourself



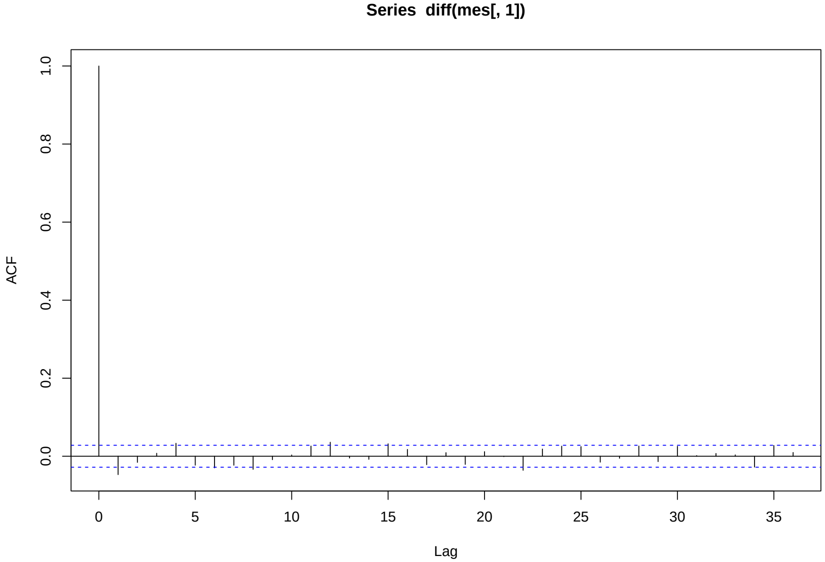
Test Yourself

One day of high frequency micro emini s&p 500 futures



Test Yourself

High frequency micro emini s&p 500 futures



Sources:

Chapter 1.4 of Introduction to Time Series and Forecasting
Brockwell/Davis