2.6: The Wold Decomposition

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Motivation

The Wold Decomposition tells us that any stationary time series can be broken up into a deterministic part and a linear process (like an ARMA process).
In a sense, this justifies what we were doing in the first chapter: ARMA models are good for the non-deterministic part.

Definition

A **deterministic** process is one that has no prediction error. In other words, $X_t = \tilde{P}_{t-1} X_t$.

Example 1: $X_t = A\cos(\omega t) + B\sin(\omega t)$, where $\omega \in (0, \pi)$, and A and B are uncorrelated but have mean 0 and variance σ^2.

Example 2: $X_t = A + bt$

Definition

1. $X_t = A\cos(\omega t) + B\sin(\omega t)$
2. $X_t = A + bt$

It's easier to see if you write the difference equations. There's no noise!

Example 1:

$$\begin{aligned} X_t &= [2\cos(\omega)]X_{t-1} - X_{t-2} && \text{(trig identities)} \\ &= \tilde{P}_{t-1}X_t. \end{aligned}$$

Example 2:

$$\begin{aligned} X_t &= X_t - X_{t-1} + X_{t-1} \\ &= b + X_{t-1} && = \tilde{P}_{t-1}X_t. \end{aligned}$$

Notation

Two potentially confusing things in this chapter:

Here $\tilde{P}$ denotes the prediction operator for a time series with an infinite past. We skipped this section. Don't worry about the slight difference between this and P_{t-1}.

Also, deterministic does not mean non-random! This can be seen from the last two examples.

Theorem

The Wold Decomposition

If $\{X_t\}$ is a nondeterministic stationary time series, then it can be written as

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} + V_t$$

where

- $\psi_0 = 1$ and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$
- $\{Z_t\} \sim \text{WN}(0, \sigma^2)$
- $\text{Cov}(Z_s, V_t) = 0$ for all s and t
- $Z_t = \tilde{P}_t Z_t$ for all t (invertibility)
- $V_t = \tilde{P}_s V_t$ for all s and t
- $\{V_t\}$ is deterministic.

Example 1

Is the following stationary? What are the deterministic and non-deterministic pieces?

$$\begin{aligned} Y_t &= A\cos(\omega t) + B\sin(\omega t) + X_t \\ \phi(B)X_t &= \theta(B)Z_t \end{aligned}$$

Example 2

Is the following stationary? What are the deterministic and non-deterministic pieces?

$$
\begin{aligned}
Y_t &= \sum_i \left[A_i \cos(\omega_i t) + B_i \sin(\omega_i t)\right] + X_t \\
\phi(B)X_t &= \theta(B)Z_t
\end{aligned}
$$

Example 3

Is the following stationary? What are the deterministic and non-deterministic pieces?

$$Y_t = \beta_0 + \beta_1 x_t + \epsilon_t$$

$$\phi(B)\epsilon_t = \theta(B)Z_t$$