Univariate Time Series Models

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Last class

Last class we previewed some examples of univariate time series models.

- AR(1): $X_t - \phi X_{t-1} = Z_t$
- MA(1): $X_t = Z_t + \theta Z_{t-1}$

Now we'll explore how to TODO

Linear Processes

Before we define linear processes, we define B: **the backshift operator**

$$BY_t = Y_{t-1}$$

We will take a complex polynomial, and think of it as a function of the backshift operator.

For example:

$$\psi(z) = 1 + \psi z + \psi_2 z^2 + \cdots$$

$$\psi(B) = 1 + \psi B + \psi_2 B^2 + \cdots$$

Linear Processes

The time series $\{X_t\}$ is a **linear process** if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

for all $t \in \mathbb{Z}$, where $Z_t \sim \mathrm{WN}(0, \sigma^2)$, and $\{\psi_j\}$ is a sequence of constants such that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ (i.e. they're absolutely summable)

Write this more concisely as

$$X_t = \psi(B) Z_t = \left(\sum_{j=-\infty}^{\infty} \psi_j B^j \right) Z_t.$$

Linear Processes

Some Linear Processes aren't always useful in finance.

We don't always want to study models where X_t depends on future noise Z_{t+j}.

We can assume our model has all its future ψ_j coefficients set to 0.

$$X_t = (\cdots + 0B^{-1} + \psi_0 B^0 + \psi_j B^1 + \cdots)Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

This is an MA(∞) process, which is more general than the MA(1) we discussed yesterday.

Example 1

An AR(1) can be written as a MA(∞) if $|\phi| < 1$.

When an AR process can be written as an MA process, we call the AR process **causal**.

$$X_t = \phi X_{t-1} + Z_t$$

$$\phi(B) X_t = Z_t$$

$$X_t = \phi^{-1}(B) Z_t$$

We just have to find the $\phi^{-1}(z)$ complex polynomial...

Example 1

Call $\phi^{-1}(z) = \psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \cdots$

Solve

$$1 = 1 + 0z + 0z^2 + \cdots = (1 - \phi z)(1 + \psi_1 z + \psi_2 z^2 + \cdots)$$

- $0 = \psi_1 - \phi$
- $0 = \psi_2 - \phi\psi_1$
- etc. etc.

So

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

and these coefficients are absolutely summable: $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

Example 2

An MA(1) can be written as an AR(∞) if $|\theta| < 1$.

When an MA process can be written as an AR process, we call the MA process **invertible**.

$$X_t = Z_t + \theta Z_{t-1}$$

$$X_t = \theta(B) Z_t$$

$$\theta^{-1}(B) X_t = Z_t$$

We just have to find the $\theta^{-1}(z) := \pi(z)$ complex polynomial, and verify that the coefficients are absolutely summable. TODO make this homework.

Why are Causality and Invertibility Important

Causality and Invertibility make the model **identifiable**.

For any model that doesn't satisfy these assumptions, you can always find a different model with different parameters, that has the **same** autocovariance function.

Also, a bunch of theorems will assume these conditions.

Also, they "make more sense" in the context of financial data.

ARMA models

The time series $\{X_t\}$ is an **ARMA(1,1)** process if it is stationary and it satisfies

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$ and $\phi + \theta \neq 0$.

It is causal and invertible if $|\phi| < 1$ and $|\theta| < 1$, respectively.

This can be written as $\phi(B)X_t = \theta(B)(Z_t)$

There are also ARMA(p, q) models, but we will focus on the cases where $p, q \in \{0, 1\}$ for simplicity.

Restrictions on Parameters

Why $\phi + \theta \neq 0$? Say $|\phi| < 1$, then

$$\begin{aligned}
X_t &= \phi^{-1}(B)\theta(B)Z_t \\
&= (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)(1 + \theta B)Z_t \\
&= [1 + (\theta + \phi)B + (\phi\theta + \phi^2)B^2 + (\phi^2\theta + \phi^3)B^3 \cdots]Z_t \\
&= Z_t + \left[(\theta + \phi)\sum_{j=1}^{\infty} \phi^{j-1} B^j\right] Z_t
\end{aligned}$$

If $|\phi| > 1$ then

$$X_t = -\theta\phi^{-1}Z_t - (\theta + \phi)\sum_{j=1}^{\infty} \phi^{-(j+1)} Z_{t+j}$$

The Sample Mean

The ARMA(1,1)

$$\phi(B)X_t = \theta(B)Z_t$$

has a zero mean. If you want to give it a nonzero mean, write it like this:

$$\phi(B)(X_t - \mu) = \theta(B)Z_t$$

not this

$$\phi(B)X_t + c = \theta(B)Z_t$$

The Sample Mean

If $\{X_t\}$ are some returns, estimating the mean is important because it's the average rate of return.

We can estimate the mean with

$$\bar{X} = n^{-1} \sum_i X_i.$$

It is unbiased

$$E[\bar{X}] = n^{-1}(E[X_1] + \cdots + E[X_n]) = \mu$$

by linearity of $E[\cdot]$ and stationarity.

This assumes the window of data you're using are all distributed from the same model (i.e. no "regime shifts.")

The Sample Autocovariance

If we have a time series, knowing about the mean is great. However, we can increase the accuracy of our predictions if we also learn about the time structure via $\gamma(\cdot)$ or $\rho(\cdot)$.

Recall that

$$\hat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-|h|} (X_{t+|h|} - \bar{X}_n)(X_t - \bar{X}_n)$$

and

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$

The Sample Mean

The mean's mean squared error (MSE) is

$$\begin{aligned}
\mathrm{MSE}(\bar{X}) &= \mathrm{Var}(\bar{X}) && \text{defn} \\
&= \mathrm{Cov}\left(\sum_{i=1}^{n} n^{-1} X_i, \sum_{j=1}^{n} n^{-1} X_j\right) && \text{defn} \\
&= n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) && \text{bilinearity of cov} \\
&= n^{-2} \sum_{h=-(n-1)}^{n-1} (n - |h|)\gamma_X(h) && \text{count diagonally : } h = i - j \\
&= n^{-1} \sum_{h} \left(1 - \frac{|h|}{n}\right) \gamma_X(h)
\end{aligned}$$

The Sample Mean

As we get more and more data, if the data are from the same process, then the estimate of μ gets more and more precise.

For a fixed set of data, we can estimate the uncertainty of the estimate with the asymptotic variance. This expression is a function of the **autocovariance function,** which is an unknown quantity that needs to be estimated too.

In practice, estimating the mean return from historical data is *extremely tricky* because you never know if the mean is changing or if your model is true! Using a rolling window is common in practice, but you never really know how to choose the window size.

The Sample Mean

1.)

$$\mathrm{Var}(\bar{X}) = n^{-1} \sum_{h=-(n-1)}^{(n-1)} \left(1 - \frac{|h|}{n}\right) \gamma_X(h) \to 0$$

as $n \to \infty$ if $\gamma(h) \to 0$, and

2.)

$$n\mathrm{Var}(\bar{X}) \to \sum_{h=-\infty}^{\infty} \gamma(h)$$

as $n \to \infty$ if $\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$

▸ Proof

The Sample Mean

If our time series is weakly stationary then

$$\bar{X} \overset{\text{approx.}}{\sim} \text{Normal}\left(\mu, n^{-1} \sum_{h=-n}^{n} \gamma_X(h)\right).$$

for large n. Or, if we assume all our noise terms are Normally distributed, then

$$\bar{X} \sim \text{Normal}\left(\mu, n^{-1} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_X(h)\right)$$

exactly. However, we usually don't know the true autocovariance function.

The Sample Mean

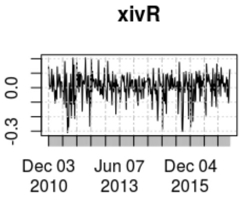
We can estimate $V^2 = \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_X(h)$ with

$$\hat{V}^2 = \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \hat{\gamma}_X(h).$$

A $(1-\alpha)\%$ confidence interval is $\bar{x} \pm z_{\alpha/2}\sqrt{v^2/n}$, and a hypothesis test against the null of $H_0 : \mu = 0$ can use the test statistic $\sqrt{n}\bar{x}/\hat{v}$ (rejection region depends on the alternative hypothesis).

Example

Consider the VXX exchange traded fund? Is it a good investment?



Example

The average return $\bar{X} = 0.006153473$. It's positive, so that's a good start.

```
> xbar - zAlphaOverTwo * sqrt(asympVar) #lower
[1] -0.0273102
> xbar + zAlphaOverTwo * sqrt(asympVar) #upper
[1] 0.03961714
```

Sources:

Chapter 2.2,2.3,2.4 of Introduction to Time Series and Forecasting
Brockwell/Davis