Introduction

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Welcome!

The purpose of the class is to expose you to a *wide* variety of mathematical concepts used in the financial industry.

Different participants in different markets will specialize in different types of techniques.

Let's start with descriptions of a few different financial instruments.

Time Value of Money

\$100 now is worth more than \$100 tomorrow. Why? Because if you had it now, you could invest it.

If there is some *risk-free* (annualized) interest rate r, then after one year \$100 becomes $\$100(1+r)$. After n years it becomes $\$100(1+r)^n$

$$\text{Future Value } n \text{ years ahead} = \text{Present Value } \times (1+r)^n$$

or equivalently

$$\frac{\text{Future Value } n \text{ years ahead}}{(1+r)^n} = \text{Present Value}$$

Time Value of Money

If the rate is compounded m times per year, the value of \$100 now in n years will be

$$100 \times \left(1 + \frac{r}{m}\right)^{mn}$$

If the rate is compounded continuously, the value of \$100 now in n years will be

$$\lim_{m \to \infty} 100 \times \left(1 + \frac{r}{m}\right)^{mn} = 100e^{rn}$$

Bond Markets

Bonds are financial instruments represents one party lending money to another.

Governments and companies sell/issue bonds to borrow money.

Investors buy bonds to lend money.

Bond Markets

Bond holders may receive *coupons* or they may not, and that depends on what kind of bond we're talking about.

A **zero-coupon bond** does not pay coupons. The buyer buys a bond at a discount, and receives the full *face value* at the *maturity date* (assuming the buyer doesn't sell it ahead of that time).

A **coupon bond** pays the face value at the maturity date, but it also pays intermittent extra payments, usually twice a year.

Bond Markets

Yield is a percent measurement of how much money a bondholder makes *if he/she holds the bond until maturity.* It can be calculated in a few ways.

Yield to maturity is the most common because it takes into account **the time value of money**.

Bond Markets

Yield to maturity is the y that satisfies

$$P = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_k + F}{(1+y)^k}$$

where

- F is face value
- k is the number of coupon payments received over the life of the bond
- P is the price of the bond now
- C_i is a coupon payment i step ahead in the future

Bond Markets

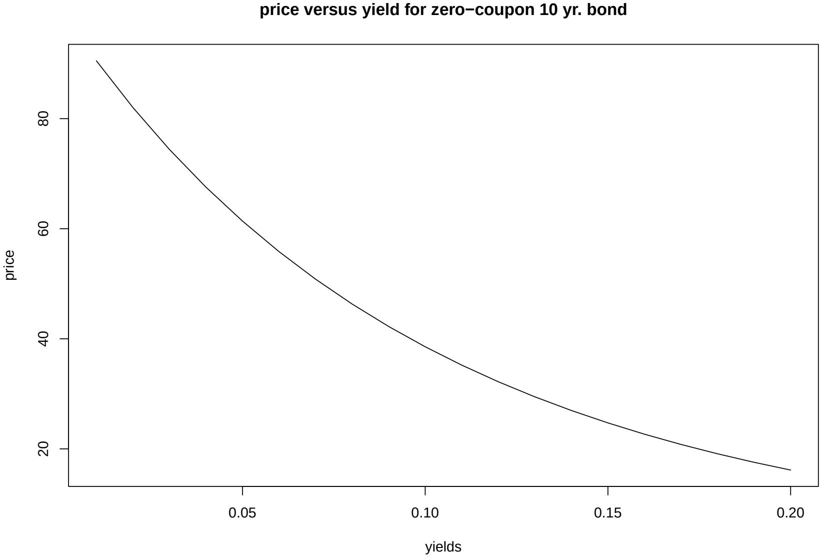
$$P = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_k + F}{(1+y)^k}$$

Example 1: 10-year zero-coupon has a face value of 100 has a yield-to-maturity of 10%.

- $P = \frac{100}{(1.1)^10} = \$38.55$

If we had the price P but didn't know the yield y, we could solve for that instead: $(1+y)^k = F/P \implies y = (F/P)^{1/k} - 1$

Bond Markets



In general, yields and prices move in opposite directions, and bonds with longer maturities are more sensitive to yield movements.

Bond Markets

Suppose you couldn't hold it until maturity, though. Say after five years you had to sell it.

Suppose further that the yield (to-maturity) was 5% at that point in the future.

At that point in time, $\tilde{P} = \frac{100}{(1.05)^5} = \$78.35$. You bought low (\$38.55) and sold high (\$78.35). Nice!

However, it could've gone the other way on you. Future interest rates will always be unknown and *random*. You either model the randomness, or you always hold until maturity.

Bond Markets

$$P = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_k + F}{(1+r)^k}$$

Example 2: 3-year zero-coupon has a face value of 100 and is selling for $90 right now. It pays coupons at a rate of 5% (annualized), twice a year. What's the yield?

- $C_i = \frac{.05}{2} 100 = 2.5$
- $k = 3 * 2 = 6$ (twenty semi-annual periods)

Solve for y:

$$90 = \frac{2.5}{1+y} + \frac{2.5}{(1+y)^2} + \cdots + \frac{2.5}{(1+y)^6} + \frac{100}{(1+y)^6}$$

Bond Markets

Solve for y:

$$90 = \frac{2.5}{1+y} + \frac{2.5}{(1+y)^2} + \cdots + \frac{2.5}{(1+y)^6} + \frac{100}{(1+y)^6}$$

or equivalently find the root of

$$f(y) = 90 - \left[\frac{2.5}{1+y} + \frac{2.5}{(1+y)^2} + \cdots + \frac{2.5}{(1+y)^6} + \frac{100}{(1+y)^6}\right]$$

Bond Markets

```
paymentsPerYear <- 2
numYears <- 3
couponRate <- .05
faceValue <- 100
discountPrice <- 97.29
numPeriods <- paymentsPerYear*numYears

f <- function(y){
  coupons <- rep(couponRate*faceValue/paymentsPerYear,
                 numPeriods)
  discountPrice -
    sum(coupons/((1+y)^(1:numPeriods))) -
    faceValue/((1+y)^(numPeriods))
}
uniroot(f, c(0,1))
```

Bond Markets

```
## $root
## [1] 0.02999521
##
## $f.root
## [1] -0.003957356
##
## $iter
## [1] 5
##
## $init.it
## [1] NA
##
## $estim.prec
## [1] 6.103516e-05
```

Bond Markets

We solved this for y:

$$90 = \frac{2.5}{1+y} + \frac{2.5}{(1+y)^2} + \cdots + \frac{2.5}{(1+y)^6} + \frac{100}{(1+y)^6}$$

However, y here is the semi-annual rate. If we wanted to get the annualized yield r,

$$90 = \frac{2.5}{(1+\frac{r}{2})} + \frac{2.5}{(1+\frac{r}{2})^2} + \cdots + \frac{2.5}{(1+\frac{r}{2})^6} + \frac{100}{(1+\frac{r}{2})^6}$$

so we just double y and get 6%.

Bond Markets

Nothing about this has to be random. You buy a bond, you hold to maturity, you know what you'll get. Unless...

1. the borrower declares bankruptcy or restructures,
2. you sell the bond to someone else before the maturity date, or
3. the value of the currency gets devalued, or...

Later we will discuss how to address some of these topics. We will model default probabilities, and we will model the random time-evolution of yields.

Bond Markets

If you might not hold to maturity, it's worth knowing how sensitive the price is to interest rates:

A bond's *duration* measures, roughly, how long you'll have to wait to get your money back. It also how much your bond price will change for a one point change in yield.

Bond Markets

Instead of

$$P = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_k + F}{(1+r)^k}$$

let's write the price of a bond as

$$P = \sum_{i=1}^{n} C_i e^{-rt_i}$$

Then

$$-\frac{dP}{dr} = \sum_{i=1}^{k} C_i e^{-rt_i} t_i$$

Duration is defined as

$$D := -\frac{dP}{dr} P^{-1} = \sum_{i=1}^{k} \left[\frac{C_i e^{-rt_i}}{P} \right] t_i$$

Bond Markets

Duration is positive. Its unit is years.

If the yield increases by one percent, the percent change in a bond price is $-D$. :(

If you're *forecasting* rate increases, you want to reduce your duration. If you're right, less pain.

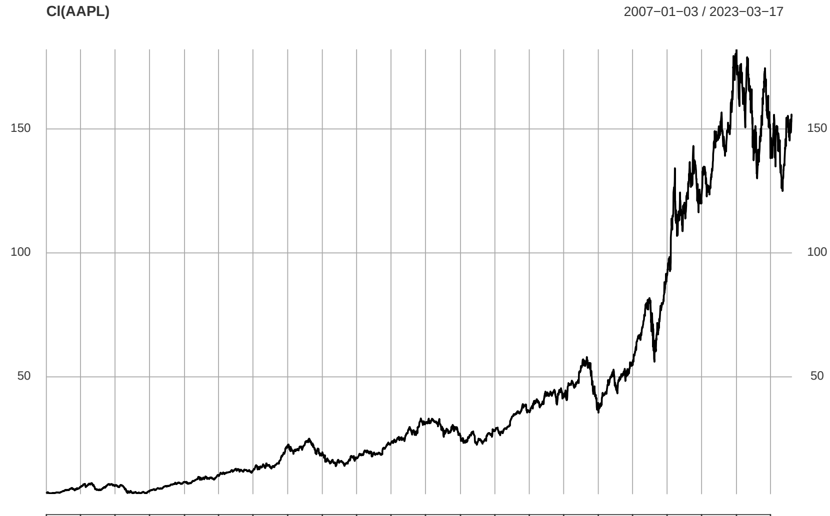
If you're *forecasting* rate decreases, you want to increase your duration. If you're right, bigger reward.

There's also a thing called *convexity*, which is related to the second derivative of the bond price...

Equities

Shares of *stock* represent ownership in a company.

Companies sell stock to raise money. Investors buy stock in the hopes that it will be worth more later.



Equities

If you buy a one share of US stock at time t with price P_t, and then sell it at P_{t+1}, you made (or lost) $P_{t+1} - P_t$.

Usually we express this as a percentage of the amount of money you initially invested.

If there are no dividends, the *one period arithmetic return* is

$$R_t := \frac{P_{t+1} - P_t}{P_t}$$

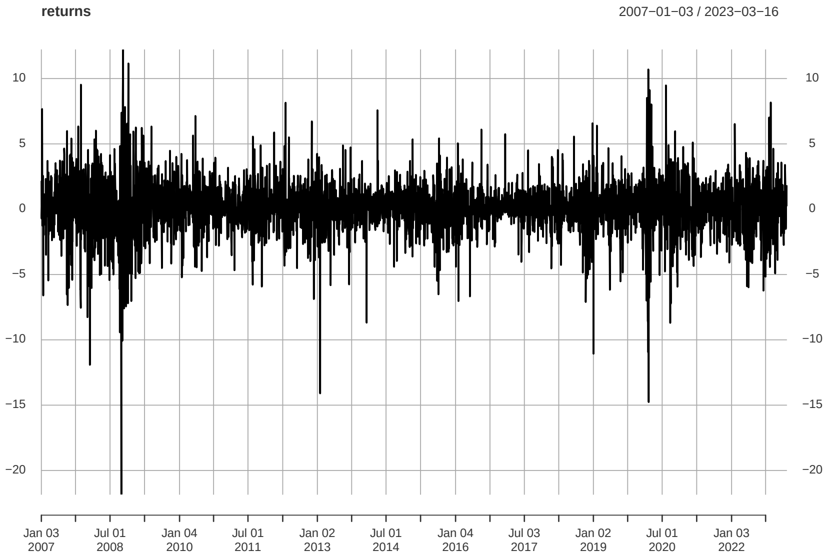
(sometimes it is scaled by 100)

If you do receive a dividend d_{t+1}, then the return is

$$R_t := \frac{P_{t+1} - P_t + d_t}{P_t}$$

A lot of times we just plot the "Adjusted" close, which is adjusted for all corporate actions, and is easier to model.

Equities


returns
2007−01−03 / 2023−03−16
10
5
0
−5
−10
−15
−20
Jan 03 2007
Jul 01 2008
Jan 04 2010
Jul 01 2011
Jan 02 2013
Jul 01 2014
Jan 04 2016
Jul 03 2017
Jan 02 2019
Jul 01 2020
Jan 03 2022

Equities

Notice that

$$1 + R_t = \frac{P_{t-1}}{P_{t-1}} + \frac{P_t - P_{t-1}}{P_{t-1}} = P_t / P_{t-1}$$

If you start with V_t dollars at time t, then after two periods you'll have

$$V_t(1 + R_{t+1})(1 + R_{t+2}) = \underbrace{\frac{V_t}{P_t}}_{\text{number of shares}} \underbrace{P_t}_{\text{share price}} \left(\frac{P_{t+1}}{P_t}\right) \left(\frac{P_{t+2}}{P_{t+1}}\right)$$

In general, after n periods, your rate of return is

$$\prod_{i=1}^{n} (1 + R_{t+i})$$

What is the distribution of this random variable?

Equities

We also sometimes use the continuously compounded rate of return:

$$R_t := \log \frac{P_{t+1}}{P_t} = \log P_{t+1} - \log P_t$$

If you invest V_t at time t, then the random value of the position at time $t+2$ is

$$V_t \exp(R_{t+1}) \exp(R_{t+2})$$

We are interested in forecasting R_{t+i}. Does it tend to be positive? How volatile are these returns? What are the *drivers* of these returns?

Currencies

Different countries have different currencies.

Buying goods abroad will first require you to change your local currency into a foreign one.

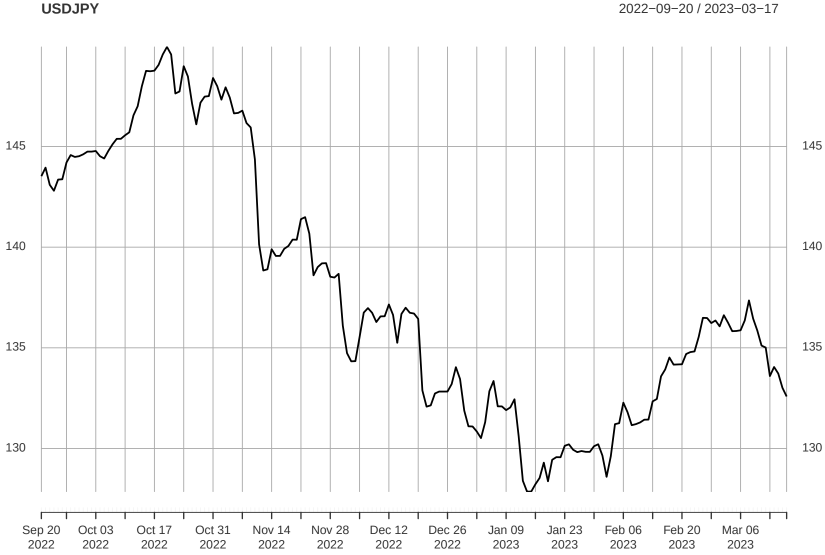
After accepting payment for selling goods abroad, you will have to convert this payment back to your local currency.

The value of one currency in terms of another changes over time.

For example, if the EUR/USD exchange rate is 1.0664, that means one EURO is equivalent (at that moment) to 1.0664 USD.

Currencies

```
## [1] "USD/JPY"
```



Commodities

The price of commodities changes over time as well: different types of oil, metals, and agricultural products can all be bought and sold.

Commodities

```
## [1] "XAU/USD"
```

Sources:

Options, Futures and Other Derivatives Chapter 4

An Introduction to Analysis of Financial Data with R (Ruey Tsay)

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