Unit 19: Spectral Density for Causal ARMA Processes

Taylor R. Brown

Department of Statistics, University of Virginia

Fall 2020

Readings for Unit 19

Textbook chapter 4.2 (from page 176).

Last Unit

1. Spectral Density: Fourier Transformation of Autocovariance.
2. Properties of Spectral Density.

Motivation

We generalize the spectral density for causal ARMA processes.

1. Autocovariance Generating Function

2. Rational Spectrum

Causal ARMA Proccess

A zero-mean causal ARMA process can be written as

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t.$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. Therefore, the autocovariance function is

$$\begin{aligned}
\gamma(h) &= \mathrm{E}(x_t x_{t+h}) \\
&= \mathrm{E}\left(\sum_{j=0}^{\infty} \psi_j w_{t-j} \sum_{i=0}^{\infty} \psi_i w_{t+h-i}\right) \\
&= \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}.
\end{aligned}$$

Spectral Density of Causal ARMA models

Therefore the spectral density for a causal ARMA process can be expressed as

$$
\begin{aligned}
f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} \\
&= \\
&= \\
&= \qquad (1)
\end{aligned}
$$

Example: MA(q)

Example: AR(p)

Autocovariance Generating Function: ARMA(p,q)

From (1), we have

$$f(\omega) = \sigma_w^2 \left| \frac{\theta(e^{-2\pi i \omega})}{\phi(e^{-2\pi i \omega})} \right|^2 .$$

This is also called the **rational spectrum** of an ARMA(p,q).

Zeroes and Poles

Recall that every degree p polynomial $a(z)$ can be factorized as

$$a(z) = a_p(z - z_1)(z - z_2) \cdots (z - z_p)$$

where $z_1, \cdots, z_p \in \mathbb{C}$ are the roots.

Zeroes and Poles

For the MA and AR polynomials,

$$\theta(z) = \theta_q(z - z_1)(z - z_2)\cdots(z - z_q)$$

and

$$\phi(z) = \phi_p(z - p_1)(z - p_2)\cdots(z - p_p).$$

$z_1, \cdots, z_q$ are called **zeroes**. $p_1, \cdots, p_p$ are called **poles**.

Rational Spectrum

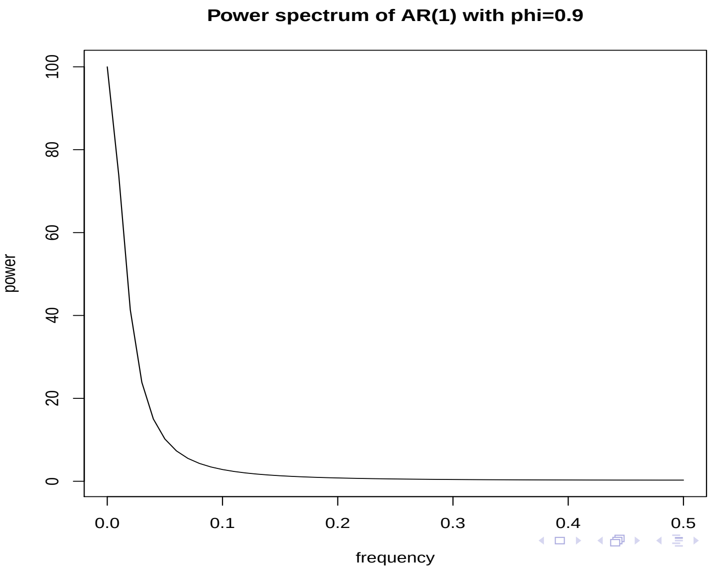
Therefore, the rational spectrum as expressed in (1) can be re-written as

$$
\begin{aligned}
f(\omega) &= \sigma_w^2 \left| \frac{\theta_q \prod_{j=1}^{q} \left(e^{-2\pi i\omega} - z_j\right)}{\phi_p \prod_{j=1}^{p} \left(e^{-2\pi i\omega} - p_j\right)} \right|^2 \\
&= \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^{q} \left|e^{-2\pi i\omega} - z_j\right|^2}{\phi_p^2 \prod_{j=1}^{p} \left|e^{-2\pi i\omega} - p_j\right|^2}. \qquad (2)
\end{aligned}
$$

Examples: AR(1)

Consider $\phi > 0$.

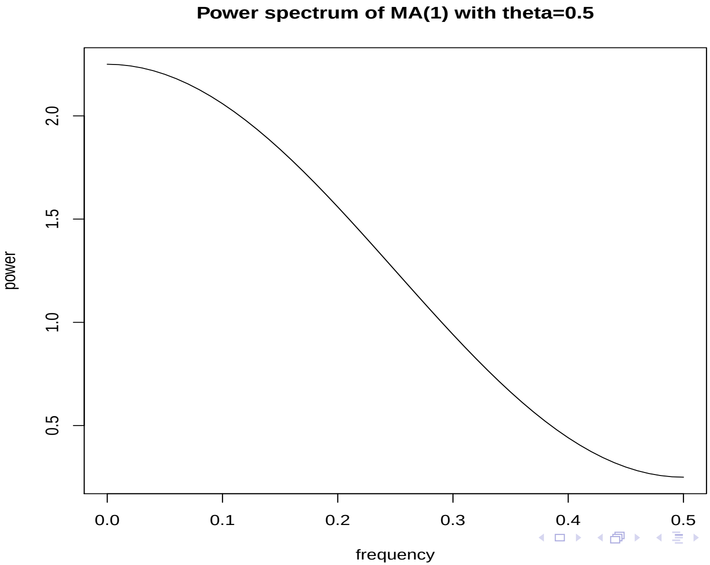
Examples: AR(1)



Examples: MA(1)

Consider $\theta > 0$.

Examples: MA(1)



Examples: AR(2)

Suppose we have the following AR(2) model:
$x_t = x_{t-1} - 0.9x_{t-2} + w_t$, where $\sigma_w^2 = 1$. The roots (poles) of the AR polynomial $\phi(z) = 0.9z^2 - z + 1$ are $p_1, p_2 = 0.555 \pm i0.8958$. Note that:

Examples: AR(2)

Using the representation (2), the spectral density is

$$f(\omega) = \frac{1}{\phi_2^2 \left| e^{-2\pi i\omega} - p_1 \right|^2 \left| e^{-2\pi i\omega} - p_2 \right|^2}.$$

The peaks of the spectral density for this process occurs when $e^{-2\pi i\omega}$ is near $1.054 e^{-2\pi i 0.16165}$.

Examples: AR(2)