Unit 3: Stationarity

Taylor R. Brown PhD

Department of Statistics, University of Virginia

Spring 2020

Readings for Unit 3

Textbook chapter 1.4, 1.5.

Last Unit

1. White noise.
2. Random walk model.
3. Autoregressive model.
4. Moving average model.
5. Mean function.
6. Measures of Dependence.

This Unit

1. Stationarity
2. Autocovariance and Autocorrelation of Stationary Time Series
3. Estimating the ACF

Motivation

In time series analysis, we would generally prefer to analyze a stationary sequence. This allows us to **better estimate** autocorrelation and other quantities of interest. One feature of stationary sequences is that they are identically distributed–but often not independent. (Though, certainly an iid sequence is stationary.) There are two types of stationarity: **strictly stationary** and **weakly stationary**.

1. Stationarity
2. Autocovariance and Autocorrelation of Stationary Time Series
3. Estimating the ACF

Strictly Stationary

A time series is **strictly stationary** if for a sequence of times $t_1, t_2, \ldots, t_k$

$$\{x_{t_1}, \ldots, x_{t_k}\}$$

has the same joint distribution as

$$\{x_{t_1+h}, \ldots, x_{t_k+h}\}$$

for every integer h. In other words,

$$P\{x_{t_1} \leq c_1, \ldots, x_{t_k} \leq c_k\} = P\{x_{t_1+h} \leq c_1, \ldots, x_{t_k+h} \leq c_k\}.$$

Location does not matter–ONLY the window size.

Weakly Stationary

A time series $\{x_t\}$ is **weakly stationary** if μ_t is **constant and does not depend on time** t, and $\gamma(s, t)$ **depends only on the distance** $|s - t|$.

From now on when we say stationary, we'll mean weakly stationary. All strongly stationary time series are also weakly stationary, but the reverse may not be the case. Most of the time we are going to be working with Gaussian time series, and in this case the two concepts coincide.

Autocovariance of Stationary Time Series

With a stationary time series, we have the following property: $\gamma(t+h, t) = \gamma(h, 0)$. So, for stationary processes we write

$$\gamma(h) = \mathrm{E}(x_{t+h} - \mu)(x_t - \mu). \tag{1}$$

We simply use the rule $\gamma(s, t) = \gamma(s - t)$. Another property of the autocovariance function when the time series is stationary is $\gamma(h) = \gamma(-h)$.

Autocorrelation of Stationary Time Series

For the autocorrelation function, we have

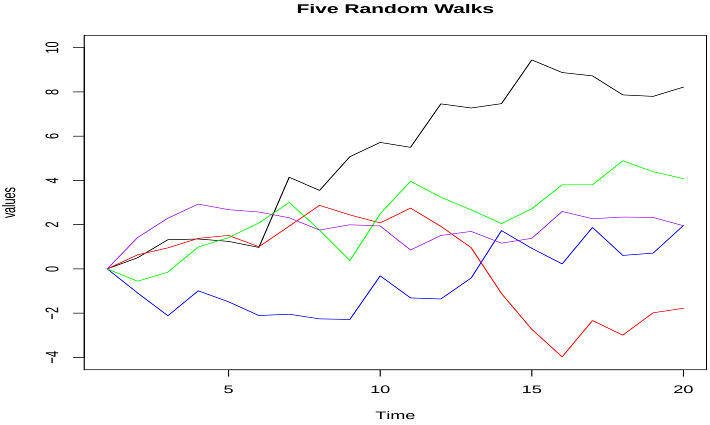
$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}. \tag{2}$$

White Noise

Question: Show that white noise is stationary.

Random Walk Process

Question: Is a random walk process $\{x_t\}$ stationary? Recall from last unit I simulated three realizations of a random walk.



MA(2) Process

Question: Show that the MA(2) process is stationary.

ACF of MA(2) Process

AR(1) Process

Question: Show that if an AR(1) process is stationary, then $|\phi_1| < 1$.

ACF of AR(1) Process

Recall Stationarity

Suppose $\{x_t\}$ is a stationary time series. Then

- Its mean is **constant**.
- Its autocovariance function is $\gamma(h) = \mathrm{E}(x_{t+h} - \mu)(x_t - \mu)$. It depends only on $h = |s - t|$. This also means that the variance, $\gamma(0)$ is constant.
- Its autocorrelation function is $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$.

Estimating the ACF

Without stationarity, we have little hope of estimating the full $\gamma(s, t)$. With stationarity, we will have **many observations** that are h apart from one another (at least when $h << n$). We now discuss how to estimate $\rho(h)$ to produce ACF plots.

Estimating the ACF

With stationarity, the (true) mean is constant. We can therefore estimate the mean using the **sample mean**

$$\bar{x} = \frac{\sum_{t=1}^{n} x_t}{n}.$$

This estimator is unbiased:

$$E(\bar{x}) = E\left(\frac{\sum_{t=1}^{n} x_t}{n}\right) = \frac{1}{n}\sum_{t=1}^{n} E(x_t) = \frac{1}{n}\sum_{t=1}^{n} \mu = \mu.$$

Estimating the ACF

Consider the **sample autocovariance function**

$$\hat{\gamma}(h) = \frac{\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(x_t - \bar{x})}{n}. \tag{3}$$

for $h = 0, 1, ..., n-1$.

For fixed h, all the random variables $y_t = (x_{t+h} - \bar{x})(x_t - \bar{x})$ have the same distribution (**due to stationarity**).

One thing to notice in the sample autocovariance function (3) is that we divide by n not $n-h$ or $n-1$. This ensures that all variance estimates (of any linear combination) are guaranteed to be positive.

Estimating the ACF

To obtain the **sample autocorrelation** we simply scale by the sample variance

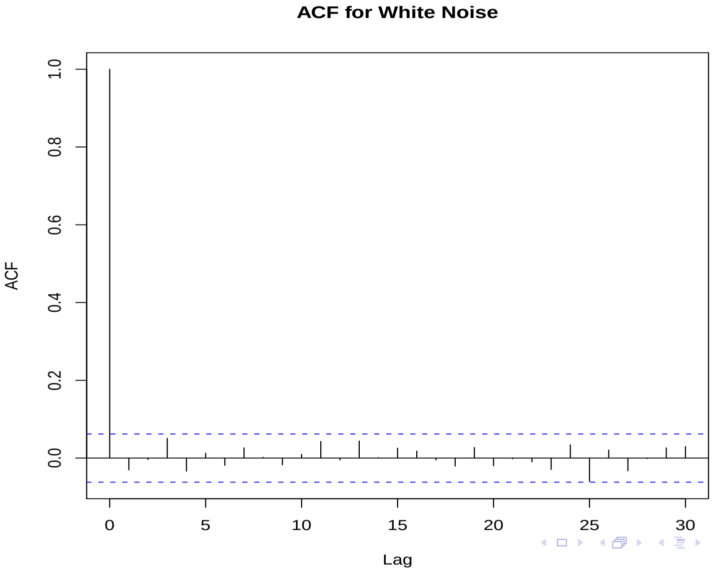
$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}. \tag{4}$$

Sample ACF

We can recognize the sample ACF of time series.

Time series	**ACF**
White noise	0
Trend	Slow decay
Periodic	Periodic
MA(q)	0 for $h > q$
AR(p)	Decays to 0 exponentially

Sample ACF for Gaussian White Noise

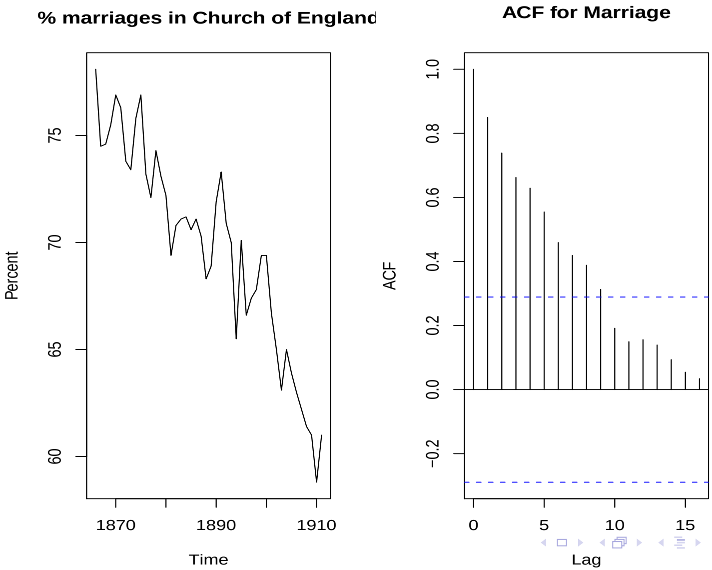


Sample ACF for Gaussian White Noise

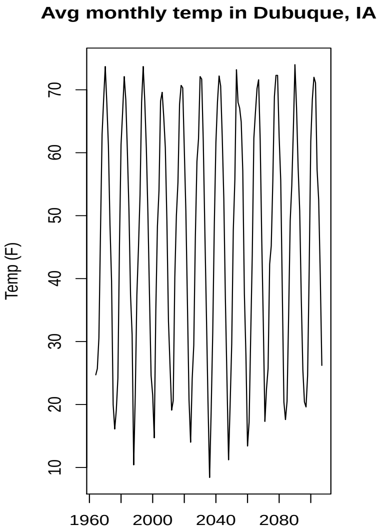
A technical results states that when the true model is white noise, $\hat{\rho}(h)$ is approximately normally distributed with zero mean and standard deviation of $1/\sqrt{n}$.

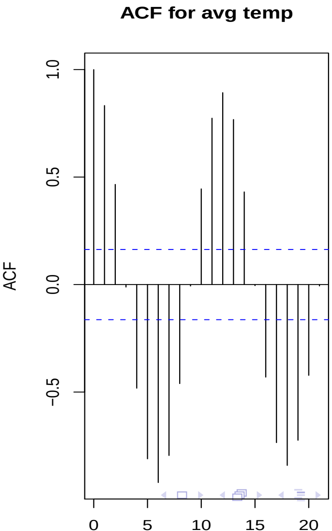
This is very useful for conducting tests concerning hypothesis about the true autocorrelation function!

Sample ACF: Marriages in Church of England

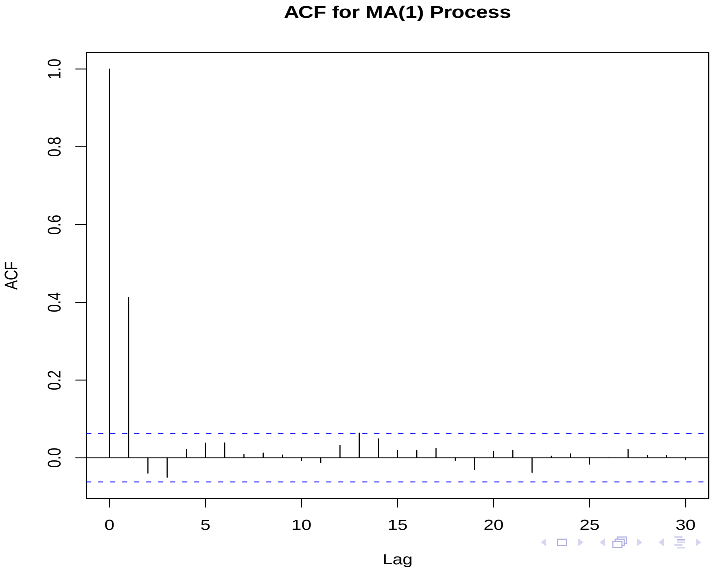


Sample ACF: Average Monthly Temperature in Dubuque, IA





Sample ACF: MA(1) Process



Sample ACF: AR(1) Process