Unit 19: Spectral Density for Causal ARMA Processes

Jeffrey Woo

Department of Statistics, University of Virginia

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Readings for Unit 19

Textbook chapter 4.2 (from page 176).

Last Unit

1. Spectral Density: Fourier Transformation of Autocovariance.
2. Properties of Spectral Density.

Motivation

We generalize the spectral density for causal ARMA processes.

1 Autocovariance Generating Function

2 Rational Spectrum

Causal ARMA Proccess

A zero-mean causal ARMA process can be written as

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t.$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. Therefore, the autocovariance function is

$$\begin{aligned}
\gamma(h) &= \mathrm{E}(x_t x_{t+h}) \\
&= \mathrm{E}(\sum_{j=0}^{\infty} \psi_j w_{t-j} \sum_{i=0}^{\infty} \psi_i w_{t+h-i}) \\
&= \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}.
\end{aligned}$$

Autocovariance Generating Function

Define the autocovariance generating function as

$$
\begin{aligned}
g(B) &= \sum_{h=-\infty}^{\infty} \gamma(h) B^h \qquad (1)\\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Autocovariance Generating Function

Therefore the spectral density for a causal ARMA process can be expressed as

$$
\begin{aligned}
f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} \\
&= \\
&= \\
&=
\end{aligned}
\tag{2}
$$

Autocovariance Generating Function: MA(q)

Autocovariance Generating Function: AR(p)

1 Autocovariance Generating Function

2 Rational Spectrum

Autocovariance Generating Function: ARMA(p,q)

From (2), we have

$$f(\omega) = \sigma_w^2 \left| \frac{\theta(e^{-2\pi i\omega})}{\phi(e^{-2\pi i\omega})} \right|^2 .$$

This is also called the **rational spectrum** of an ARMA(p,q).

Zeroes and Poles

Recall that every degree p polynomial $a(z)$ can be factorized as

$$a(z) = a_p(z - z_1)(z - z_2)\cdots(z - z_p)$$

where $z_1, \cdots, z_p \in \mathbb{C}$ are the roots.

Zeroes and Poles

For the MA and AR polynomials,

$$\theta(z) = \theta_q(z - z_1)(z - z_2) \cdots (z - z_q)$$

and

$$\phi(z) = \phi_p(z - p_1)(z - p_2) \cdots (z - p_p).$$

$z_1, \cdots, z_q$ are called **zeroes**. $p_1, \cdots, p_p$ are called **poles**.

Rational Spectrum

Therefore, the rational spectrum as expressed in (2) can be re-written as

$$
\begin{aligned}
f(\omega) &= \sigma_w^2 \left| \frac{\theta_q \prod_{j=1}^{q} \left(e^{-2\pi i\omega} - z_j\right)}{\phi_p \prod_{j=1}^{p} \left(e^{-2\pi i\omega} - p_j\right)} \right|^2 \\
&= \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^{q} \left|e^{-2\pi i\omega} - z_j\right|^2}{\phi_p^2 \prod_{j=1}^{p} \left|e^{-2\pi i\omega} - p_j\right|^2}. \qquad (3)
\end{aligned}
$$

Rational Spectrum

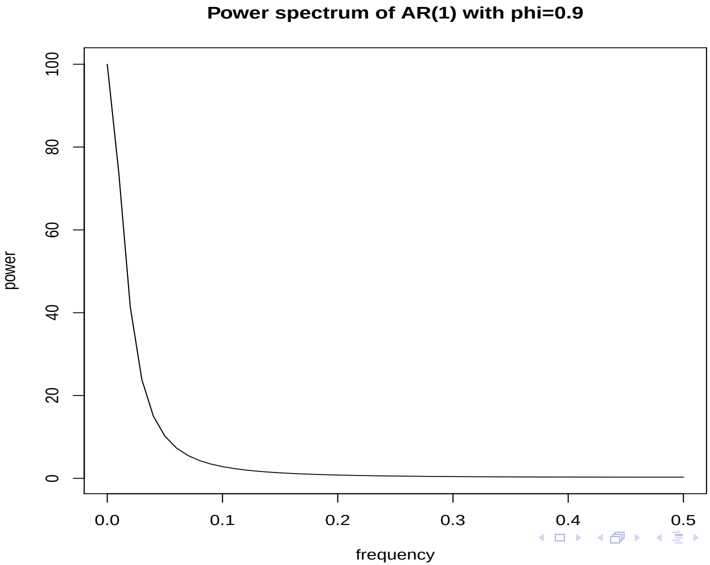
Notice that from the rational spectrum expressed in (3), the spectral density $f(\omega)$ increases as

- $e^{-2\pi i\omega}$ moves **closer to** the poles p_j,
- $e^{-2\pi i\omega}$ moves **further from** the zeroes z_j.

Examples: AR(1)

Consider $\phi > 0$.

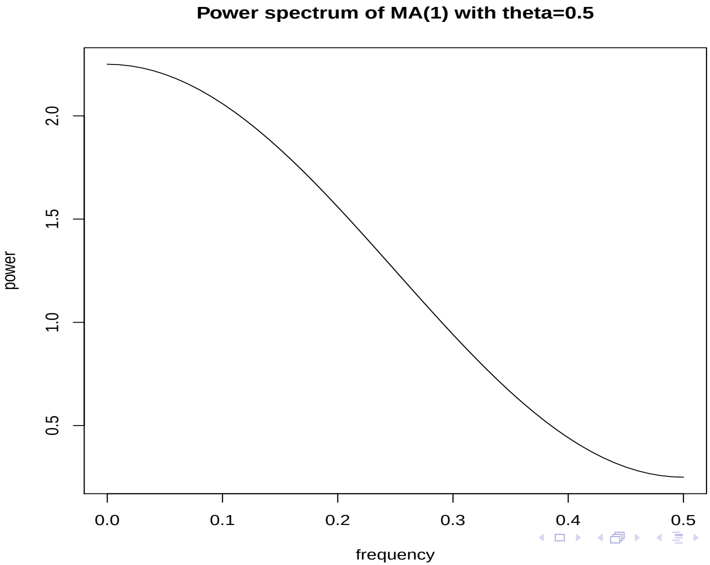
Examples: AR(1)



Examples: MA(1)

Consider $\theta > 0$.

Examples: MA(1)



Examples: AR(2)

Suppose we have the following AR(2) model:
$x_t = x_{t-1} - 0.9x_{t-2} + w_t$, where $\sigma_w^2 = 1$. The roots (poles) of the AR polynomial $\phi(z) = 0.9z^2 - z + 1$ are $p_1, p_2 = 0.555 \pm i0.8958$. Note that:

Examples: AR(2)

Using the representation (3), the spectral density is

$$f(\omega) = \frac{1}{\phi_2^2 \left|e^{-2\pi i\omega} - p_1\right|^2 \left|e^{-2\pi i\omega} - p_2\right|^2}.$$

The peaks of the spectral density for this process occurs when $e^{-2\pi i\omega}$ is near $1.054e^{-2\pi i 0.16165}$.

Examples: AR(2)