Unit 11: ARMA Autocorrelation and Partial Autocorrelation Functions

Taylor R. Brown

Department of Statistics, University of Virginia

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Readings for Unit 11

Textbook chapter 3.3.

Last Unit

1. ARMA(p,q)
2. Condition for causality
3. Condition for invertibility
4. Condition for redundant parameters (shared roots)

This Unit

1. ACF for MA(q)
2. ACF for Causal ARMA(p,q)
3. Partial Autocorrelation Function (PACF)

Motivation

In this unit we will study the autocorrelation and partial autocorrelation functions for ARMA processes.

MA(q) Process

Let's start with an MA(q) process

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q} = \sum_{j=0}^{q} \theta_j w_{t-j},$$

where we have written $\theta_0 = 1$. Then

$$E(x_t) = \sum_{j=0}^{q} \theta_j E(w_{t-j}) = 0.$$

Autocovariance for MA(q)

The autocovariance function is

$$\begin{aligned}\gamma(h) = cov(x_t, x_{t+h}) &= E\Big[\sum_{j=0}^{q} \theta_j w_{t-j} \sum_{j'=0}^{q} \theta_{j'} w_{t+h-j'}\Big] \\ &= \sum_{j=0}^{q}\sum_{j'=0}^{q} \theta_j \theta_{j'} E(w_{t-j} w_{t+h-j'}).\end{aligned}$$

Autocovariance for MA(q)

Recall that $E(w_s w_t) = \sigma_w^2$ if $s = t$ and $E(w_s w_t) = 0$ otherwise. So we have

$$\gamma(h) = \begin{cases} \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}, & 0 \leq h \leq q, \\ 0, & h \geq q+1. \end{cases} \tag{1}$$

ACF for MA(q)

Recall that $\gamma(h) = \gamma(-h)$, so we will only need the values for $h \geq 0$. Dividing $\gamma(h)$ by $\gamma(0)$ in (1), we obtain the autocorrelation function (ACF) of an MA(q) model

$$\rho(h) = \begin{cases} \frac{\sum_{j=0}^{q-h} \theta_j \theta_{j+h}}{1+\theta_1^2+\cdots+\theta_q^2}, & 0 \leq h \leq q, \\ 0, & h \geq q+1. \end{cases} \tag{2}$$

ACF for Causal ARMA(p,q)

We have seen in (2), for MA(q) models, the ACF will be zero for lags greater than q. Moreover, because $\theta_q \neq 0$, $\rho(q) = \theta_0\theta_q/(1+\theta_1^2+\cdots+\theta_q^2) \neq 0$. Thus, the ACF provides information about the order of the dependence for a MA model. How about ARMA or AR models?

Causal ARMA(p,q)

Now we discuss causal ARMA(p, q) model

$$\phi(B)x_t = \theta(B)w_t,$$

where the roots of $\phi(z)$ are outside the unit circle. Ths means for $|z| \leq 1$, $|\phi(z)| > 0$, which means $|\phi^{-1}(z)| < \infty$.

We have the MA(∞) representation

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}, \quad \text{where} \quad \psi(z) = \frac{\theta(z)}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j. \tag{3}$$

Autocovariance for Causal ARMA(p,q)

It follows that $E(x_t) = 0$ and by (1), the autocovariance function of x_t is given by

$$\gamma(h) = cov(x_t, x_{t+h}) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}, \quad h \geq 0.$$

Autocovariance for Causal ARMA(p,q)

We won't provide an explicit formula for the ACF of an ARMA(p,q), but we will derive the system of equations you'll need to solve to get it.
Let $h \geq 0$, and $\theta_0 = 1$:

$$
\begin{aligned}
\gamma_X(h) &= \text{Cov}(X_{t+h}, X_t) \\
&= \text{Cov}(\sum_{i=1}^{p} \phi_i X_{t+h-i} + \sum_{j=0}^{q} \theta_j W_{t+h-j}, X_t) \\
&= \sum_{i=1}^{p} \phi_i \text{Cov}(X_{t+h-i}, X_t) + \sum_{j=0}^{q} \theta_j \text{Cov}\left(W_{t+h-j}, X_t\right) \\
&= \sum_{i=1}^{p} \phi_i \gamma_X(h-i) + \sum_{j=0}^{q} \theta_j \text{Cov}\left(W_{t+h-j}, \sum_{k=0}^{\infty} \psi_k W_{t-k}\right)
\end{aligned}
$$

ACVF and ACF for Causal ARMA(p,q)

Let $h \geq 0$, and $\theta_0 = 1$:

$$\begin{aligned}
\gamma_X(h) &= \sum_{i=1}^{p} \phi_i \gamma_X(h-i) + \sum_{j=0}^{q} \theta_j \text{Cov}\left(W_{t+h-j}, \sum_{k=0}^{\infty} \psi_k W_{t-k} \right) \\
&= \sum_{i=1}^{p} \phi_i \gamma_X(h-i) + \sum_{j=0}^{q} \sum_{k=0}^{\infty} \theta_j \psi_k \sigma_W^2 1(t+h-j=t-k) \\
&= \sum_{i=1}^{p} \phi_i \gamma_X(h-i) + \sigma_W^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}
\end{aligned}$$

$0 \leq j < \infty$, $k \geq 0$, and $k = j - h$. Dividing through by $\gamma(0)$ will give the equations for solving $\rho(\cdot)$.

ACVF and ACF for Causal ARMA(1,1)

For a causal ARMA(p,q), $h \geq 0$, :

$$\gamma_X(h) = \sum_{i=1}^{p} \phi_i \gamma_X(h-i) + \sigma_W^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}$$

Let's consider an ARMA(1,1). When $h \geq 2$, we have

$$\gamma(h) = \phi_1 \gamma_X(h-1)$$

This means $\gamma(h) = \phi_1^h c$ for some unknown $c \in \mathbb{R}$. To find c we consider the "initial conditions" or when $h = 0, 1$.

ACVF and ACF for Causal ARMA(1,1)

For a causal ARMA(p,q), $h \geq 0$, :

$$\gamma_X(h) = \sum_{i=1}^{p} \phi_i \gamma_X(h-i) + \sigma_W^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}$$

Still considering an ARMA(1,1), when $h = 0, 1$, we have

$$\gamma(0) = \phi_1 \gamma(-1) + \sigma_W^2 \sum_{j=0}^{1} \theta_j \psi_j \tag{4}$$

$$\gamma(1) = \phi_1 \gamma(0) + \sigma_W^2 \theta_1 \tag{5}$$

This yields (dropping the subscripts from θ_1 and ϕ_1)

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{(1+2\theta\phi+\theta^2)} \phi^{h-1}$$

Notice that there is only one part dependent on h!

ACVF for Causal AR(1)

For a causal ACVF,

$$\gamma_X(h) = \sum_{i=1}^{p} \phi_i \gamma_X(h-i) + \sigma_W^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}$$

Consider a causal AR(P) model $x_t = \phi_1 x_{t-1} + w_t$. Then

$$\gamma_X(h) = \sum_{i=1}^{p} \phi_i \gamma_X(h-i)$$

$\gamma_X(h)$ will never cut-off to 0 for any h. This makes it difficult to identify what p is.

A new idea for causal AR(p) models

We'll need this trick in the following slide...

Note we can write $X_t = \sum_{i=0}^{\infty} \psi_i W_{t-i}$

Causality means x_{t-2}, for example, only depends on $w_{t-2}, w_{t-3}, \ldots$ and hence is uncorrelated with w_{t-1} and w_t.

For $s > t$, check $\text{Cov}(w_s, x_t) = \sum_{i=0}^{\infty} \psi_j \overbrace{\text{Cov}(w_s, w_{t-i})}^{0}$.

A new idea for causal AR(p) models

For a causal AR(1) model, the γ_X and ρ_X functions don't zero out past p, so instead consider

$$\begin{aligned}
\mathrm{Cov}(x_{t+2} - \hat{x}_{t+2}, x_t - \hat{x}_t) &= \mathrm{Cov}(x_{t+2} - \phi x_{t+1}, x_t - \phi x_{t+1}) \\
&= \mathrm{Cov}(w_{t+2}, x_t - \phi x_{t+1}) \\
&= \mathrm{Cov}(w_{t+2}, x_t) - \phi \mathrm{Cov}(w_{t+2}, x_{t+1}) \\
&= 0
\end{aligned}$$

So, when considering the relationship between x_{t+2} and x_t, we first remove the linear dependence on x_{t+1}!

Notation

One way to remove linear connections is through **linear regression**.

Let $\hat{x}_{t+h}$ denote the regression of x_{t+h} on $\{x_{t+h-1}, x_{t+h-2}, \ldots, x_{t+1}\}$, which we write as

$$\hat{x}_{t+h} = \beta_1 x_{t+h-1} + \beta_2 x_{t+h-2} + \cdots + \beta_{h-1} x_{t+1}. \tag{6}$$

Here we do not include the intercept assuming the mean of x_t is zero. Otherwise, replace x_t with $x_t - \mu_x$.

Notation

In addition, let $\hat{x}_t$ denote the regression of x_t on $\{x_{t+1}, x_{t+2}, \ldots, x_{t+h-1}\}$, then

$$\hat{x}_t = \beta_1 x_{t+1} + \beta_2 x_{t+2} + \cdots + \beta_{h-1} x_{t+h-1}. \tag{7}$$

1. ACF for MA(q) Processes
2. ACF for Causal ARMA(p,q) Processes
3. **Partial Autocorrelation Function**
4. Worked Examples

Partial Autocorrelation Function

The **partial autocorrelation function (PACF)** of a stationary process x_t, denoted by ϕ_{hh}, for $h = 1, 2, \ldots$, is

$$\phi_{11} = corr(x_{t+1}, x_t) = \rho(1) \tag{8}$$

and

$$\phi_{hh} = corr(x_{t+h} - \hat{x}_{t+h}, x_t - \hat{x}_t), \quad h \geq 2. \tag{9}$$

Note that, the PACF, ϕ_{hh} is the correlation between x_{t+h} and x_t with the linear dependence of $\{x_{t+1}, \cdots, x_{t+h-1}\}$, on each, removed.

Partial Autocorrelation Function: AR(1) example

Let's go back to the example from a few slides ago, and try to calculate the PACF of a causal AR(1) model: $x_t = \phi x_{t-1} + w_t$, with $|\phi| < 1$.

By definition, $\phi_{11} = corr(x_1, x_0) = \rho(1) = \phi$.

We wrote

$$\begin{aligned} \phi_{22} &:= \mathrm{Cov}(x_{t+2} - \hat{x}_{t+2}, x_t - \hat{x}_t) \\ &= 0 \end{aligned}$$

...but why are $\hat{x}_{t+2} = \hat{x}_t = \phi x_{t+1}$?

Partial Autocorrelation Function: AR(1) example

To calculate ϕ_{22}, consider the regression of x_{t+2} on x_{t+1}, say $\hat{x}_{t+2} = \beta x_{t+1}$.

We choose β to minimize

$$E\left[(x_{t+2} - \beta x_{t+1})^2\right] = \gamma_X(0) - 2\beta\gamma_X(1) + \beta^2\gamma_X(0).$$

Taking the derivative and setting that equal to 0 yields $\beta = \phi$.

Partial Autocorrelation Function

Next, consider the regression of x_t on x_{t+1}, say $\hat{x}_t = \beta x_{t+1}$. We choose β to minimize

Partial Autocorrelation Function

In general, for a causal AR(p) model $x_h = \sum_{j=1}^{p} \phi_j x_{h-j} + w_h$.
When $h > p$, the regression of x_h on $x_{h-1}, \ldots, x_1$ is

$$\hat{x}_h = \sum_{j=1}^{p} \phi_j x_{h-j}.$$

Partial Autocorrelation Function

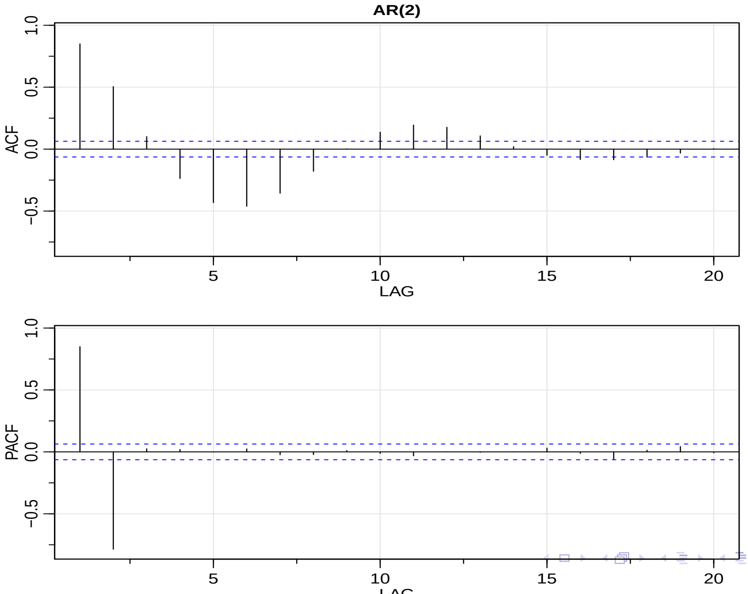
Thus, when $h > p$, by causality,

$$\phi_{hh} = corr(x_h - \hat{x}_h, x_0 - \hat{x}_0) = corr(w_h, x_0 - \hat{x}_0) = 0.$$

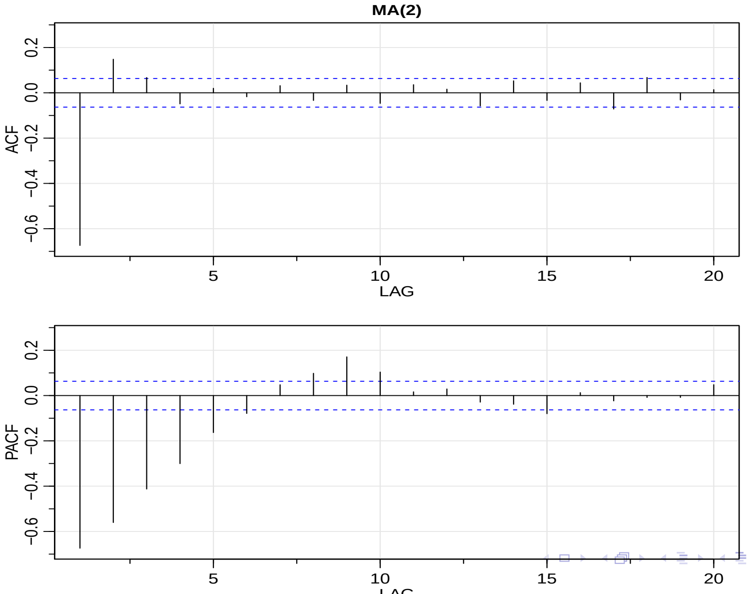
Summary

- The ACF of MA(q) model cuts off after lag q. The PACF of an AR(p) model cuts off after lag p.
- Identification of an MA(q) model is best done with ACF; identification of an AR(p) model is best done with PACF.
- The PACF between x_t and x_{t-h} is the correlation between $x_t - \hat{x}_t$ and $x_{t-h} - \hat{x}_{t-h}$. Think of it as taking the correlation between the residuals from two regression models. The dependence on all intermediate variables is removed.

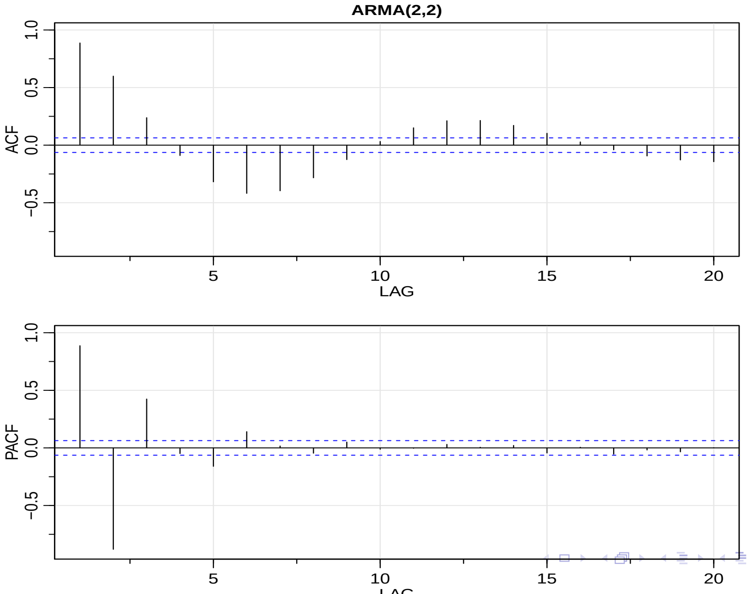
ACF and PACF of Causal AR(2)



ACF and PACF of Invertible MA(2)



ACF and PACF of Causal and Invertible ARMA(2,2)



ACF and PACF of Causal AR and Invertible MA

(From page 99, Table 3.1 of text)

	AR(p)	**MA(q)**	**ARMA(p,q)**
ACF	Decay	0 after lag q	Decay
PACF	0 after lag p	Decay	Decay

Fish Population Example

This time series from "recruit.dat" contains data on fish population in the central Pacific Ocean. The numbers represent the number of new fish in the years 1950-1987. **Question**: Based on the ACF and PACF plots, what process do you think is most likely to describe this time series?

Fish Population Example