Unit 22: Linear Filters

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Readings for Unit 22

Textbook chapter 4.7 (until page 214).

Last Unit

1. Smoothing periodogram to reduce variance.

Motivation

Some of the previous topics have suggested that one could "transform" a time series to modify the distribution of its spectral density or variance. In this unit we will define a linear filter and show how it can be used to extract signals from a time series.

Linear Filter

The linear filter modifies the spectral characteristics of a time series in a predictable way. Let $x_t, t = 0, \pm 1, \pm 2, \ldots,$ be a stationary **input series**, and $a_j, j = 0, \pm 1, \pm 2, \ldots,$ be a set of specified coefficients. We use the linear filter $\{a_j, j = 0, \pm 1, \pm 2, \ldots\}$ to operate on $\{x_t, t = 0, \pm 1, \pm 2, \ldots\}$ to produce an **output series**

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}. \tag{1}$$

Linear Filter

(1) is sometimes called a convolution. y_t is a linear combination of x_t's, suggesting the name "linear filter". The coefficients a_r are collectively called the **impulse response function**.

Linear Filter

Example: Recall that a causal ARMA model $\phi(B)y_t = \theta(B)w_t$ has the causal representation $y_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$. This is a special case of (1) with $a_j = 0$ for $j < 0$.

- y_t in (1) depends on all x's (both past and future) whereas causal ARMA model depends only on past values.
- In (1), we do NOT assume that x_t is a white noise series. Instead x_t can be any stationary series.

Linear Filter

Let $\gamma_x(h) = E[(x_{t+h} - Ex_{t+h})(x_t - Ex_t)]$ denote the autocovariance function of x_t, and the spectral density is denoted by

$$f_x(\omega) = \sum_{h=-\infty}^{\infty} \gamma_x(h) e^{-2\pi\omega ih}.$$

The inverse Fourier transform formula is

$$\gamma_x(h) = \int_{-0.5}^{0.5} f_x(\omega) e^{2\pi\omega ih} d\omega.$$

Linear Filter

$$
\begin{aligned}
\gamma_y(h) &= \mathrm{cov}(y_{t+h}, y_t) \\
&= \mathrm{cov}(\sum_{j\in\mathbb{Z}} a_j x_{t+h-j}, \sum_{k\in\mathbb{Z}} a_k x_{t-k}) \\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \gamma_x(h+k-j) a_j a_k \\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_{-1/2}^{1/2} f_x(\omega) e^{2\pi i\omega[h+k-j]} a_j a_k d\omega \\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_{-1/2}^{1/2} f_x(\omega) e^{2\pi i\omega h} e^{2\pi i\omega k} e^{2\pi i\omega[-j]} a_j a_k d\omega \\
&= \int_{-1/2}^{1/2} f_x(\omega) e^{2\pi i\omega h} \left(\sum_{j\in\mathbb{Z}} e^{2\pi i\omega[-j]} a_j \right) \left(\sum_{k\in\mathbb{Z}} e^{2\pi i\omega k} a_k \right) d\omega
\end{aligned}
$$

Linear Filter

So

$$\begin{aligned}\gamma_y(h) &= \int_{-1/2}^{1/2} f_x(\omega) e^{2\pi i \omega h} \left(\sum_{j\in\mathbb{Z}} e^{2\pi i \omega[-j]} a_j \right) \left(\sum_{k\in\mathbb{Z}} e^{2\pi i \omega k} a_k \right) d\omega \\ &= \int_{-1/2}^{1/2} f_x(\omega) |A_{yx}(\omega)|^2 e^{2\pi i \omega h} d\omega\end{aligned}$$

Linear Filter

Note that

$$A_{yx}(\omega) = \sum_{j=-\infty}^{\infty} a_j e^{-2\pi\omega ij} \tag{4}$$

is the Fourier transform of a_j and called the **frequency response** function. We require $\sum_{j=-\infty}^{\infty} |a_j| < \infty$ to ensure that $A_{yx}(\omega)$ is well defined.

Linear Filter

Now we compute the spectral density $f_y(\omega)$ of y_t. By the inverse Fourier transform,

$$\gamma_y(h) = \int_{-0.5}^{0.5} f_y(\omega) e^{2\pi\omega i h} d\omega. \tag{5}$$

Comparing (5) and (3), we find that

$$f_y(\omega) = \qquad\qquad\qquad\qquad . \tag{6}$$

Linear Filter

We can use (6) to compute the exact effect on the spectrum of any given filtering operation. The spectrum of the input series is changed by filtering and the effect of the change is characterized as a frequency-by-frequency multiplication by the squared magnitude of the frequency response function, $A_{yx}(\omega)$. $|A_{yx}(\omega)|^2$ is called the **power transfer** function.

Linear Filter

Suppose two filtering operations are applied to a stationary series x_t in succession, e.g.:

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j},$$

and then

$$z_t = \sum_{k=-\infty}^{\infty} b_k y_{t-k}.$$

The spectrum of the output is

$$f_z(\omega) = |A(\omega)|^2 |B(\omega)|^2 f_x(\omega).$$

Worked Example: MA(1)

Question: Consider an MA(1) process $y_t = w_t + \theta w_{t-1}$. Given that $f_w(\omega) = \sigma_w^2$, derive the spectral density of this MA(1) process using (6).

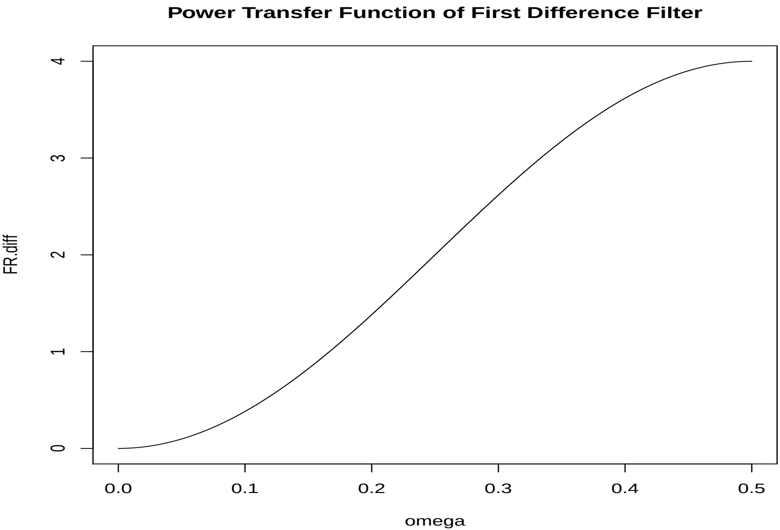
Worked Example: AR(1)

Question: Consider an AR(1) process $y_t = 0.5y_{t-1} + w_t$. Given that $f_w(\omega) = \sigma_w^2$, derive the spectral density of this AR(1) process using (6).

Worked Example: First Difference Filter

Question: Consider the first difference filter $y_t = \nabla x_t = x_t - x_{t-1}$. Derive the power transfer function for this filter and comment on the practical implications.

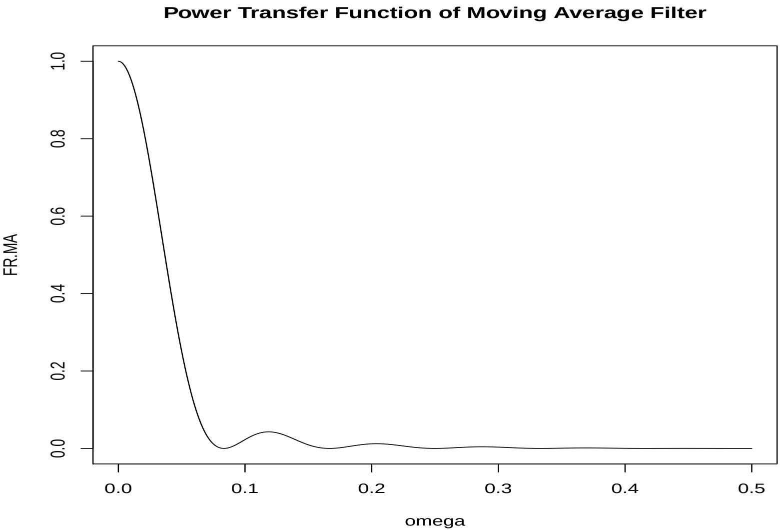
Worked Example: First Difference Filter



Worked Example: Moving Average Filter

Question: Consider the following moving average filter $y_t = \frac{1}{24}(x_{t-6} + x_{t+6}) + \frac{1}{12}\sum_{j=-5}^{5} x_{t-j}$. Derive the power transfer function for this filter and comment on the practical implications.

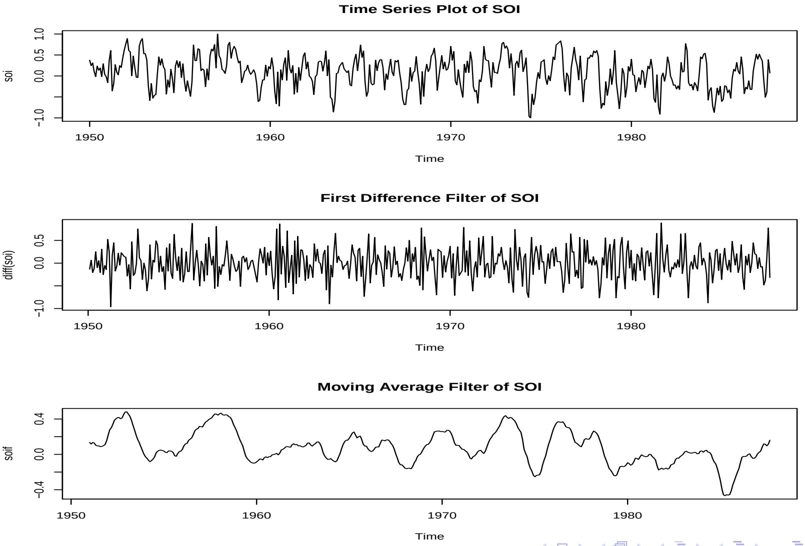
Worked Example: Moving Average Filter



Worked Example: SOI Dataset

We'll apply the first difference and 12-month moving average filters to the SOI dataset.

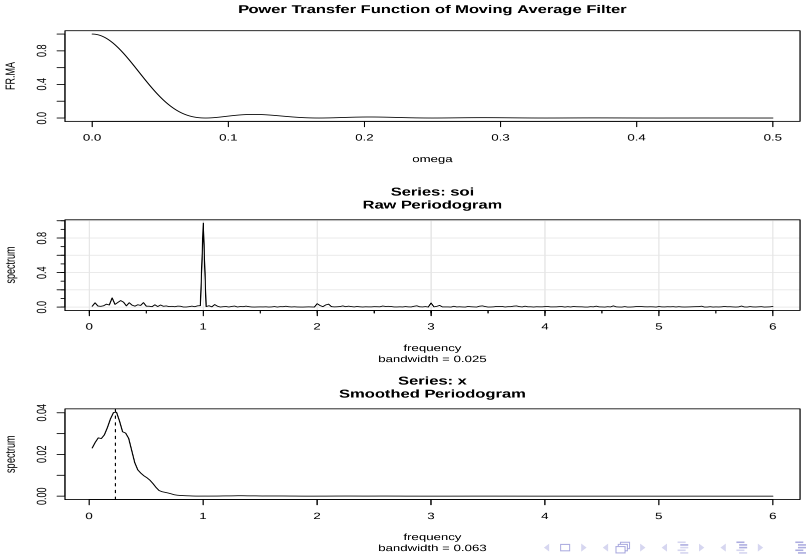
Worked Example: SOI Dataset



Worked Example: SOI Dataset

- The first difference filter retained the higher frequencies.
- The moving average filter retained the lower frequencies. Enhances the component associated with El Niño and dampens the seasonal/yearly component.

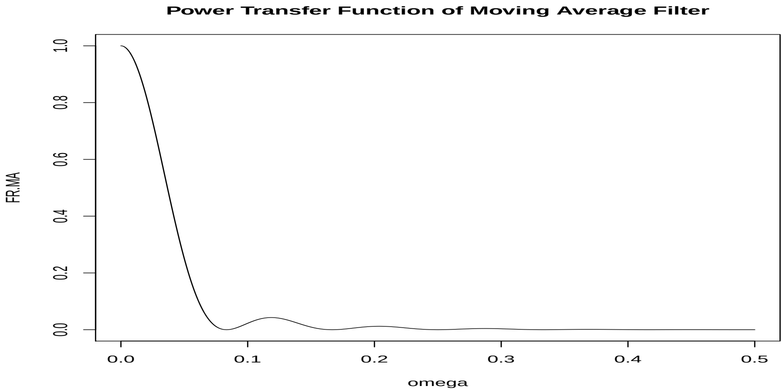
Worked Example: SOI Dataset



Worked Example: SOI Dataset

From the periodogram of the moving average filtering of the data, high frequency behavior has been removed. El Niño frequency around $1/52$.

Worked Example: SOI Dataset



For the 12-month moving average filter, frequencies higher than around 0.08 will be "cut off". Periods shorter than $1/0.08 = 12.5$ months will be dampened, and the El Niño frequency is retained.