Unit 18: Spectral Density

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Spring 2020

Readings for Unit 18

Textbook chapter 4.2 (until page 175).

Last Unit

1. Introduction to Spectral Analysis.

This Unit

1. Spectral Density: Fourier Transformation of Autocovariance.
2. Properties of Spectral Density.

Motivation

The spectral density is a Fourier transform of the autocovariance function $\gamma(h)$. Autocovariance is in terms of ____ whereas spectral density is in terms of _____.

1. Spectral Density
2. Properties of Spectral Density
3. Worked Examples

Transformation

Consider a transformation from Celsius to Fahrenheit:
$F = 1.8C + 32$.

To transform back, we use $C = \frac{F-32}{1.8}$. We don't really lose any information during the transformation.

Fourier Transform

We move between a function on the integers $\ldots, -2, -1, 0, 1, 2, \ldots$ and a function in the frequency space. So, for a function $a_t, t = \ldots, -2, -1, 0, 1, 2, \ldots$ we can move to the frequency space by taking the Fourier transform

$$A(\omega) = \sum_{t=-\infty}^{\infty} a_t e^{-2\pi i\omega t}, -0.5 \leq \omega \leq 0.5.$$

We can then go backwards using $a_t = \int_{-0.5}^{0.5} A(\omega) e^{2\pi i\omega t} d\omega$ for each t.

Spectral Density

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then it has the representation

$$\gamma(h) = \int_{-1/2}^{1/2} f(\omega) e^{2\pi i \omega h} d\omega, \quad h = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

(1) is called the ______________________________________. The ______________ is denoted by

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h}, \quad -1/2 \leq \omega \leq 1/2. \tag{2}$$

Autocovariance is in terms of ____ whereas spectral density is in terms of _____.

Spectral Density

- The spectral density, (2), provides information about the relative strengths of the various frequencies for __________ ______________ in the time series.
- The spectral density is also called the power spectrum.
- Remember that $\gamma(h)$ completely determines the distribution for a stationary Gaussian process. So, the spectral density also completely determines the distribution for a stationary Gaussian process.

Spectral Density

Notice that when $h = 0$, from (1), we have

$$\gamma(0) = \mathrm{Var}(x_t) = \int_{-1/2}^{1/2} f(\omega) d\omega. \tag{3}$$

An interpretation of (3) is that the "total" integrated spectral density equals the variance of the time series. Thus the spectral density within a particular interval of frequencies can be viewed as the amount of the variance ________ by those frequencies.

Trigonometric Properties

Recall that

- a cosine function is ____, i.e. $\cos(-\theta) = \cos(\theta)$.
- a sine function is ____, i.e. $\sin(-\theta) = -\sin(\theta)$.

Euler's Formula

Recall Euler's formula:

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha). \tag{4}$$

Consequently,

$$\cos(\alpha) = \frac{e^{-i\alpha} + e^{i\alpha}}{2} \tag{5}$$

and

$$\sin(\alpha) = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}. \tag{6}$$

Derivation of Inverse Transformation of Spectral Density

1. Spectral Density

2. Properties of Spectral Density

3. Worked Examples

Properties of Spectral Density

1. $f(\omega) \geq 0$ because $\gamma(h)$ is non-negative definite.
2. $f(\omega)$ is even, i.e. $f(\omega) = f(-\omega)$.
3. $f(\omega) = f(\omega + 1)$.

Derivation of Properties

Derivation of Properties

1 Spectral Density

2 Properties of Spectral Density

3 Worked Examples

Spectral Density of White Noise

Spectral Density of White Noise

This means all frequencies receive equal weight. This is analogous to the spectrum of white light, where all colors enter equally in white light. (Hence the term white noise!)

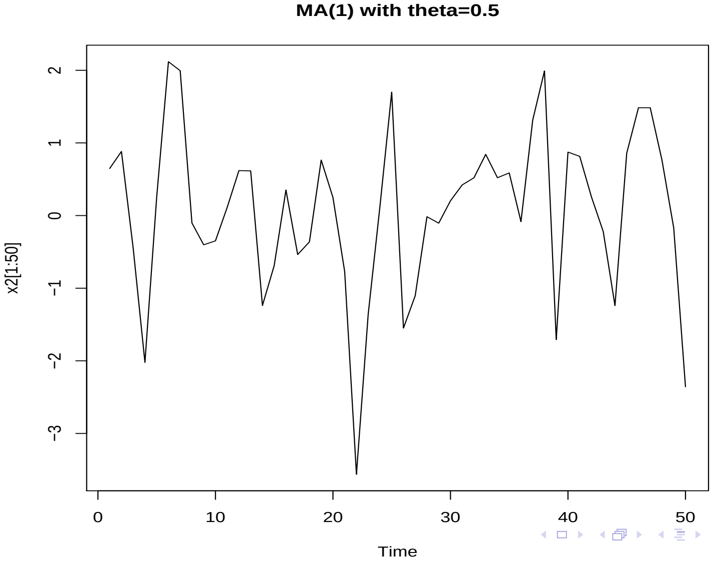
Spectral Density of MA(1)

Spectral Density of MA(1)

Question: Suppose $\theta = 0.5$ and $\sigma_w^2 = 1$, sketch the power spectrum.

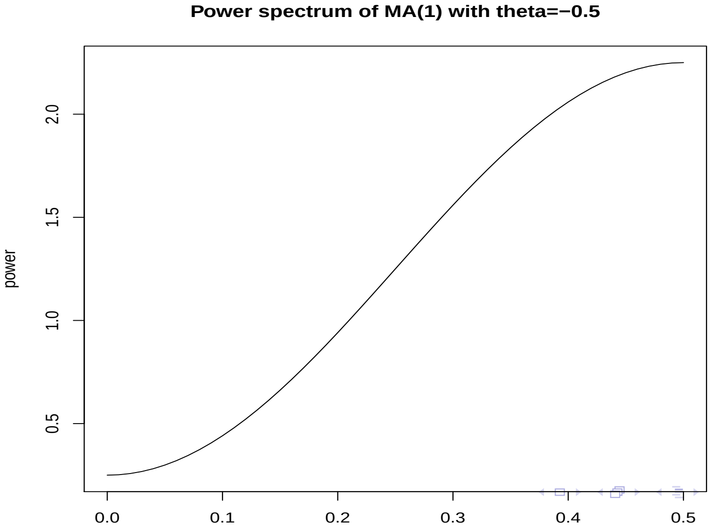
What is the implication of this power spectrum?

Time Series Plot of MA(1)

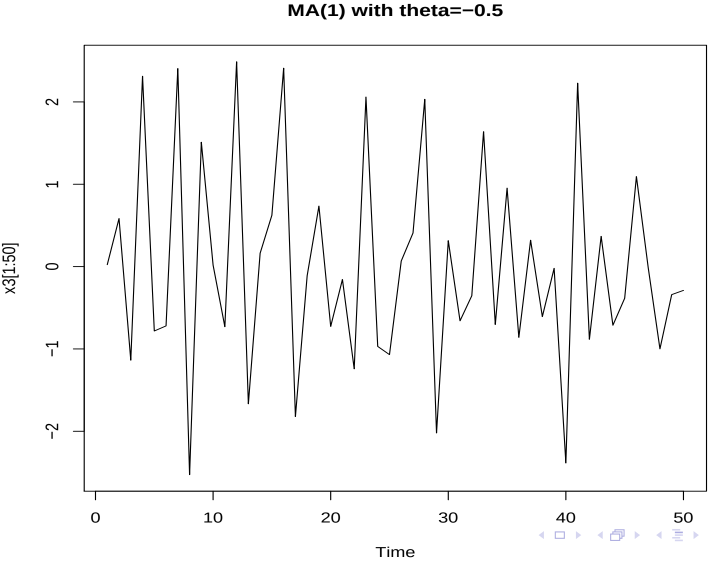


Spectral Density of MA(1)

Power spectrum of MA(1) when $\theta = -0.5$ and $\sigma_w^2 = 1$.



Time Series Plot of MA(1)



ACF of MA(1)

Question: How are the ACF plots of an MA(1) process different when $\theta = 0.5$ versus $\theta = -0.5$? Does this difference provide an explanation to the difference in their time series plots?

Spectral Density of AR(1)