Unit 2: Basic Time Series Models

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Readings for Unit 2

Textbook chapters 1.2, 1.3.

This Unit

1. White noise.
2. Random walk model.
3. Autoregressive model.
4. Moving average model.
5. Mean function.
6. Measures of Dependence.

Motivation

We need to explore some of the properties of time series models to know what we are looking for.

- If we estimate an autocorrelation and it has certain properties, what models have autocorrelations consistent with such properties?

What we are doing is linking models of quantitative phenomena to the observations.

Motivation

The models that we will look at today give a rule for the current observation based on past observations or past random events. This is the **time domain** approach.

White Noise

Typically, we are thinking of a sequence of random variables that may be dependent on one another, $x_1, \ldots, x_n$. There may be times when we want to think of this as an infinite list $\ldots, x_{-1}, x_0, x_1, \ldots,$.

One model with which we are already familiar consists of a sequence of uncorrelated random variables. When the mean is zero and the sequence is indexed by time, this is usually called **white noise**.

White Noise

A sequence of random variables $x_1, x_2, \ldots, x_n$ is called **white noise** if

$$\begin{aligned} \mathrm{E}(x_t) &= 0, \\ \mathrm{Var}(x_t) &= \sigma^2, \text{(finite constant variance)} \\ \mathrm{Cov}(x_s, x_t) &= 0 \text{ for all } s \neq t. \end{aligned}$$

Note: Uncorrelated RVs does not imply they are independent. Independent RVs implies they are uncorrelated.

Gaussian White Noise

A specific example is **Gaussian white noise**; denoted by $w_1, w_2, ..., w_n$. For Gaussian white noise, all w_t are independent normal random variables, i.e. $w_t \sim \mathrm{N}(0, \sigma_w^2)$.

We'll now look at a few basic time series models: random walk, autoregressive, and moving average.

Toy Example

A simple way to model moving forward would be with an equation like

$$x_t = x_{t-1} + 1$$

for $t = 1, ..., n$.

Question: What does this model represent?

Random Walk Model

A model for analyzing trend is the **random walk model**. Your current position is determined by where you were at the last step plus the random step that you just took. So, the equation would be

$$x_t = x_{t-1} + w_t, \tag{1}$$

for $t = 1, ..., n$ and w_t are Gaussian white noises.

Random Walk Model

A nonrandom drift, δ, could also be included so that

$$x_t = \delta + x_{t-1} + w_t. \tag{2}$$

Random Walk Model

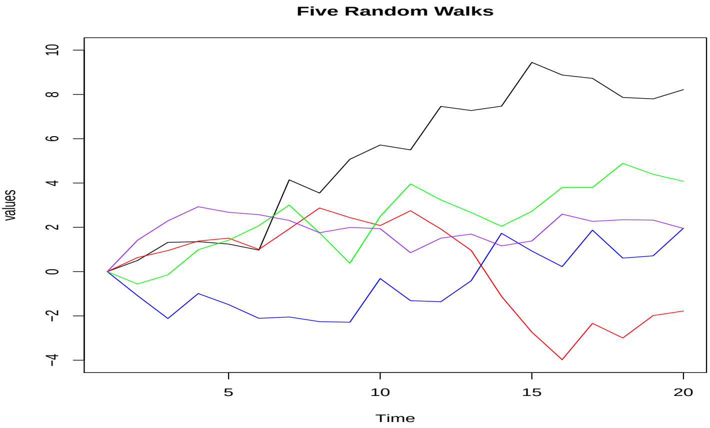
Another possible way to write (1) and (2) is

$$x_t = \sum_{i=1}^{t} w_i, \tag{3}$$

or with drift

$$x_t = \delta t + \sum_{i=1}^{t} w_i. \tag{4}$$

Random Walk Model



Any comments?

Random Walk Model

It is interesting to note that while random walks consist of a dependent sequence of random variables, it may be easily transformed into an independent sequence by looking at the sequence of "differences"

$$\nabla x_t = x_t - x_{t-1}, \tag{5}$$

where $t = 1, ..., n - 1$.

Autoregressive Models

A class of models closely related to the random walk are the **autoregressive models (AR)**. An autoregressive model is defined so that the current location is a **linear combination** of previous locations plus a random term (Gaussian white noise).

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-p} + w_t. \tag{6}$$

This is an AR(p) model.

Question: Under what condition(s) is the random walk a special case of an AR model?

Moving Average Models

There is another family of models known as **moving average (MA) models**. One way to think about these models is to take a sliding window and take a weighted average of a white noise process for everything in the window. So, start with a white noise process, $w_1, ..., w_n, ...$. Then a moving average is of the following form

$$x_t = w_t + \theta_1 w_{t-1} + ... + \theta_q w_{t-q}. \tag{7}$$

This is an MA(q) model.

Mean Function

The **mean function** is defined as

$$\mu_t = E(x_t). \tag{8}$$

Some properties of expectation:

Mean Function

Question: Derive the mean function of a random walk with drift.

Mean Function

Question: Derive the mean function of an MA(q) model.

Variance and Covariance

Recall the definition of **variance** for a random variable X:

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}[(X - \mathrm{E}X)^2] \\ &= \mathrm{E}(X^2) - (\mathrm{E}X)^2 . \end{aligned} \tag{9}$$

The **covariance** for two random variables, X and Y:

$$\mathrm{Cov}(X, Y) = \mathrm{E}[(X - \mathrm{E}X)(Y - \mathrm{E}Y)]. \tag{10}$$

The **correlation** is:

$$\mathrm{Corr}(X, Y) = \rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}. \tag{11}$$

Variance and Covariance

Some properties of variance and covariance:

Uncorrelated RVs

Two random variables X and Y are uncorrelated when $\rho(X, Y) = 0$. This implies that

Uncorrelated RVs are not Independent

Two random variables X and Y are independent when their joint density is the product of their marginal densities.

Uncorrelated RVs are not Independent

If X and Y are independent, then they are also uncorrelated.

Uncorrelated RVs are not Independent

However, if X and Y are uncorrelated, they can still be dependent.

Autocovariance Function

A common feature of time series is that the observations are dependent. The **autocovariance function** is defined as

$$\gamma(s,t) = E[(x_s - \mu_s)(x_t - \mu_t)]. \tag{12}$$

So, this is simply the covariance between x_s and x_t evaluated at all combinations. Covariance measures the strength of the **linear dependence** between random variables. A covariance that is small for s, t close together generally implies random variables that are closer to white noise. Smoother series tend to have a large autocovariance even for s and t which are far apart.

Autocovariance Function

Question: Find the autocovariance function for the random walk model.

Autocovariance Function

Question: Find the autocovariance function for the MA(2) model.

Autocorrelation Function

We also consider the **autocorrelation function** (ACF) in addition to autocovariance. The definition is natural and is given by

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}} \tag{13}$$

We will be using the ACF (13) often. The reasons why will become apparent after we discuss **stationary time series**.