

Unit 19: Spectral Density for Causal ARMA Processes

Taylor R. Brown

Department of Statistics, University of Virginia

Fall 2020

Readings for Unit 19

Textbook chapter 4.2 (from page 176).

Last Unit

- 1 Spectral Density: Fourier Transformation of Autocovariance.
- 2 Properties of Spectral Density.

Motivation

We generalize the spectral density for causal ARMA processes.

1 Spectral Density of Causal ARMA models

2 Rational Spectrum

Causal ARMA Process

A zero-mean causal ARMA process can be written as

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B)w_t.$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. Therefore, the autocovariance function is

$$\begin{aligned}\gamma(h) &= E(x_t x_{t+h}) \\ &= E\left(\sum_{j=0}^{\infty} \psi_j w_{t-j} \sum_{i=0}^{\infty} \psi_i w_{t+h-i}\right) \\ &= \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}.\end{aligned}$$

Spectral Density of Causal ARMA models

Therefore the spectral density for a causal ARMA process can be expressed as

$$\begin{aligned} f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} \\ &= \\ &= \\ &= \end{aligned} \tag{1}$$

Example: MA(q)

Example: AR(p)

Spectral Density of Causal ARMA models

From (1), we have

$$f(\omega) = \sigma_w^2 \left| \frac{\theta(e^{-2\pi i\omega})}{\phi(e^{-2\pi i\omega})} \right|^2.$$

This is also called the **rational spectrum** of an ARMA(p,q).

Zeroes and Poles

Recall that every degree p polynomial $a(z)$ can be factorized as

$$a(z) = a_p(z - z_1)(z - z_2) \cdots (z - z_p)$$

where $z_1, \dots, z_p \in \mathbb{C}$ are the roots.

Zeroes and Poles

For the MA and AR polynomials,

$$\theta(z) = \theta_q(z - z_1)(z - z_2) \cdots (z - z_q)$$

and

$$\phi(z) = \phi_p(z - p_1)(z - p_2) \cdots (z - p_p).$$

z_1, \dots, z_q are called **zeroes**. p_1, \dots, p_p are called **poles**.

Rational Spectrum

Therefore, the rational spectrum as expressed in (1) can be re-written as

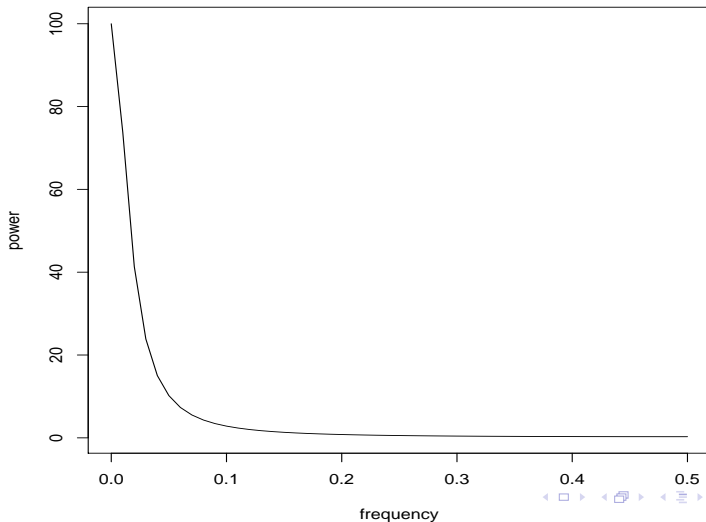
$$\begin{aligned} f(\omega) &= \sigma_w^2 \left| \frac{\theta_q \prod_{j=1}^q (e^{-2\pi i\omega} - z_j)}{\phi_p \prod_{j=1}^p (e^{-2\pi i\omega} - p_j)} \right|^2 \\ &= \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^q |e^{-2\pi i\omega} - z_j|^2}{\phi_p^2 \prod_{j=1}^p |e^{-2\pi i\omega} - p_j|^2}. \end{aligned} \quad (2)$$

Examples: AR(1)

Consider $\phi > 0$.

Examples: AR(1)

Power spectrum of AR(1) with $\phi=0.9$

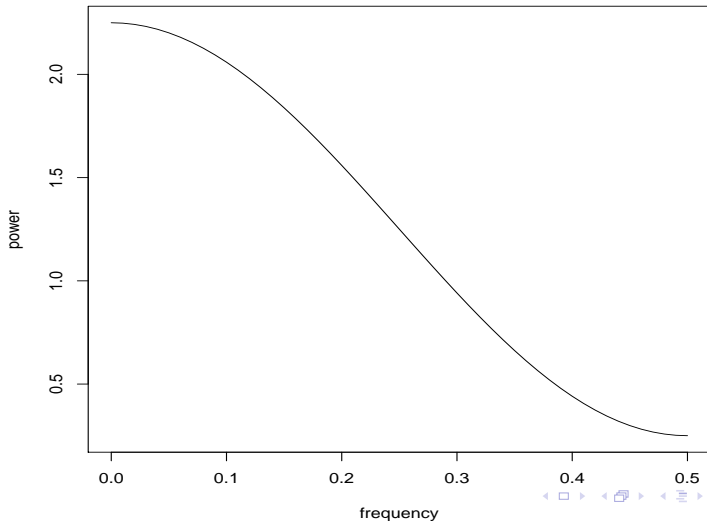


Examples: MA(1)

Consider $\theta > 0$.

Examples: MA(1)

Power spectrum of MA(1) with theta=0.5



Examples: AR(2)

Suppose we have the following AR(2) model:

$x_t = x_{t-1} - 0.9x_{t-2} + w_t$, where $\sigma_w^2 = 1$. The roots (poles) of the AR polynomial $\phi(z) = 0.9z^2 - z + 1$ are $p_1, p_2 = 0.555 \pm i0.8958$. Note that:

Examples: AR(2)

Using the representation (2), the spectral density is

$$f(\omega) = \frac{1}{\phi_2^2 |e^{-2\pi i\omega} - p_1|^2 |e^{-2\pi i\omega} - p_2|^2}.$$

The peaks of the spectral density for this process occurs when $e^{-2\pi i\omega}$ is near $1.054e^{-2\pi i0.16165}$.

Examples: AR(2)

Power spectrum of AR(2) with $\phi = c(-1, 0.9)$

