

Unit 9: ARMA Models: MA(1)

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Readings for Unit 9

Textbook chapter 3.1 (page 83 to 86).

Last Unit

- 1 AR(p) process
- 2 AR(1) process
- 3 AR(1) in terms of backshift operator
- 4 AR(1) and causality

This Unit

- 1 Identifiability
- 2 MA(1) in terms of backshift operator
- 3 MA(1) and invertibility
- 4 ARMA model

Motivation

In the previous unit, we noted a condition for an $AR(1)$ model to be causal. In this unit, we will explore another issue, this time regarding MA models. This issue is called identifiability.

1 MA(1) and Invertibility

2 ARMA Model

MA(q) Process

The MA(q) model is defined as

$$x_t = w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}.$$

We can rewrite this using backshift operator as

$$x_t = \theta(B)w_t \tag{1}$$

where $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$ which is called the **moving average operator**. We already know that an MA(q) process is stationary.

MA(1) Process

Let's look at the $MA(1)$ model, i.e.

$$x_t = w_t + \theta w_{t-1}$$

We can then calculate the autocovariance function.

$$\gamma(0) = (1 + \theta^2)\sigma_w^2$$

and

$$\gamma(1) = \theta\sigma_w^2$$

and zero for larger lags.

Identifiability

Question: Suppose we have another MA(1) model with $\tilde{\theta} = 1/\theta$ and $\tilde{\sigma}_w^2 = \theta^2 \sigma_w^2$, what is the autocovariance function for this model?

Identifiability

Which one is “correct”? Both are correct: this is an **identifiability** problem as we have MA models that are not unique. We'll use the one that gives us an infinite order AR process.

Identifiability

Rewrite the MA(1) model as

$$w_t = -\theta w_{t-1} + x_t.$$

This is just like the AR(1) that we saw last time but now the roles for x_t , w_t are reversed. So, we could write this as

$$\begin{aligned} w_t &= \sum_{j=0}^{\infty} (-\theta)^j x_{t-j} \\ &= \sum_{j=0}^{\infty} (-\theta)^j B^j x_t, \end{aligned} \tag{2}$$

provided that $|\theta| < 1$.

MA(1) in Terms of Backshift Operator

(2) can be written as

$$w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} = \pi(B)x_t, \quad (3)$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$ and $\pi_j = (-\theta)^j$.

Identifiability

Question: What's an intuitive explanation as to why we want $|\theta| < 1$ in (2)?

Invertible MA(1)

We could write an MA(1) as

$$\begin{aligned}x_t &= w_t + \theta w_{t-1} \\&= w_t + \theta \sum_{j=0}^{\infty} (-\theta)^j x_{t-1-j} \\&= w_t - \sum_{j=1}^{\infty} (-\theta)^j x_{t-j}.\end{aligned}$$

So, we have written the MA(1) as an infinite order AR process. We require $|\theta| < 1$ for invertibility for an MA(1). An MA model which can be written as an infinite order AR process is called a **invertible** MA. We will later extend and formalize a condition on the parameters $\theta_1, \theta_2, \dots, \theta_q$ for an MA(q) process to be invertible.

1 MA(1) and Invertibility

2 ARMA Model

ARMA Model

Recall that an AR(p) model is given by

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$$

or in terms of backshift operator

$$\phi(B)x_t = w_t, \quad \text{where} \quad \phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

ARMA Model

An MA(q) model is given by

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

or in terms of backshift operator

$$x_t = \theta(B)w_t, \quad \text{where} \quad \theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

ARMA Model

x_t is called an ARMA (p,q) model if x_t is stationary and can be written as

$$\phi(B)x_t = \theta(B)w_t.$$

Thus far, we have seen a number of issues with the general definition of ARMA(p,q) models:

- Stationary AR models that depend on the **future**.
- MA models that are not **unique** (identifiability issue).

We will next look at one more issue with ARMA(p,q) models before formalizing conditions for causality and invertibility.