Unit 19: Spectral Density for Causal ARMA Processes

Jeffrey Woo

Department of Statistics, University of Virginia

Spring 2020

Readings for Unit 19

Textbook chapter 4.2 (from page 176).

Last Unit

1. Spectral Density: Fourier Transformation of Autocovariance.
2. Properties of Spectral Density.

Motivation

We generalize the spectral density for causal ARMA processes.

1. Autocovariance Generating Function

2. Rational Spectrum

Causal ARMA Proccess

A zero-mean causal ARMA process can be written as

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t,$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. Therefore, the autocovariance function is

$$\begin{aligned} \gamma(h) &= \mathrm{E}(x_t x_{t+h}) \\ &= \mathrm{E}(\sum_{j=0}^{\infty} \psi_j w_{t-j} \sum_{i=0}^{\infty} \psi_i w_{t+h-i}) \\ &= \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}. \end{aligned}$$

Autocovariance Generating Function

Define the autocovariance generating function as

$$
\begin{aligned}
g(B) &= \sum_{h=-\infty}^{\infty} \gamma(h) B^h \qquad (1) \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Autocovariance Generating Function

Therefore the spectral density for a causal ARMA process can be expressed as

$$\begin{aligned} f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} \\ &= \\ &= \\ &= \end{aligned} \tag{2}$$

Autocovariance Generating Function: MA(q)

Autocovariance Generating Function: AR(p)

1. Autocovariance Generating Function
2. Rational Spectrum

Autocovariance Generating Function: ARMA(p,q)

From (2), we have

$$f(\omega) = \sigma_w^2 \left| \frac{\theta(e^{-2\pi i \omega})}{\phi(e^{-2\pi i \omega})} \right|^2 .$$

This is also called the ______________ of an ARMA(p,q).

Zeroes and Poles

Recall that every degree p polynomial $a(z)$ can be factorized as

$$a(z) = a_p(z - z_1)(z - z_2) \cdots (z - z_p)$$

where $z_1, \cdots, z_p \in \mathbb{C}$ are the roots.

Zeroes and Poles

For the MA and AR polynomials,

$$\theta(z) = \theta_q(z - z_1)(z - z_2)\cdots(z - z_q)$$

and

$$\phi(z) = \phi_p(z - p_1)(z - p_2)\cdots(z - p_p).$$

$z_1, \cdots, z_q$ are called _______. $p_1, \cdots, p_p$ are called _______.

Rational Spectrum

Therefore, the rational spectrum as expressed in (2) can be re-written as

$$\begin{aligned} f(\omega) &= \sigma_w^2 \left| \frac{\theta_q \prod_{j=1}^{q} \left(e^{-2\pi i\omega} - z_j\right)}{\phi_p \prod_{j=1}^{p} \left(e^{-2\pi i\omega} - p_j\right)} \right|^2 \\ &= \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^{q} \left|e^{-2\pi i\omega} - z_j\right|^2}{\phi_p^2 \prod_{j=1}^{p} \left|e^{-2\pi i\omega} - p_j\right|^2}. \end{aligned} \tag{3}$$

Rational Spectrum

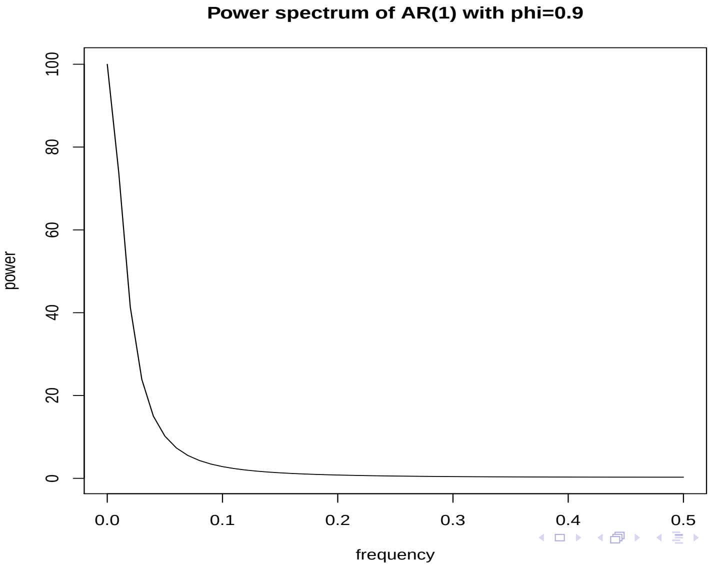
Notice that from the rational spectrum expressed in (3), the spectral density $f(\omega)$ increases as

- $e^{-2\pi i\omega}$ moves ___________ the poles p_j,
- $e^{-2\pi i\omega}$ moves ___________ the zeroes z_j.

Examples: AR(1)

Consider $\phi > 0$.

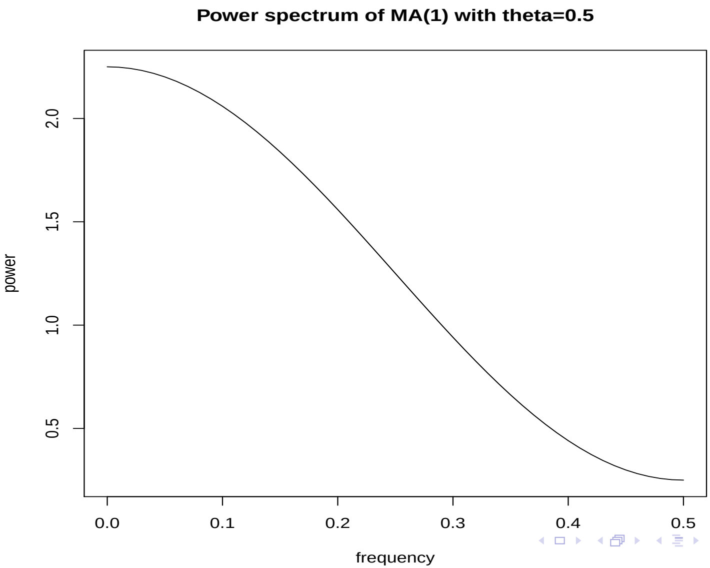
Examples: AR(1)



Examples: MA(1)

Consider $\theta > 0$.

Examples: MA(1)



Examples: AR(2)

Suppose we have the following AR(2) model:
$x_t = x_{t-1} - 0.9x_{t-2} + w_t$, where $\sigma_w^2 = 1$. The roots (poles) of the AR polynomial $\phi(z) = 0.9z^2 - z + 1$ are $p_1, p_2 = 0.555 \pm i0.8958$. Note that:

Examples: AR(2)

Using the representation (3), the spectral density is

$$f(\omega) = \frac{1}{\phi_2^2 \left| e^{-2\pi i\omega - p_1} \right|^2 \left| e^{-2\pi i\omega - p_2} \right|^2}.$$

The peaks of the spectral density for this process occurs when $e^{-2\pi i\omega}$ is near $1.054e^{-2\pi i0.16165}$.

Examples: AR(2)