

# Unit 9: ARMA Models: MA(1)

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Spring 2020

# Readings for Unit 9

Textbook chapter 3.1 (page 81 to 83).

# Last Unit

- ① AR(p) process
- ② AR(1) process
- ③ AR(1) in terms of backshift operator
- ④ AR(1) and causality

# This Unit

- ① Identifiability
- ② MA(1) in terms of backshift operator
- ③ MA(1) and invertibility
- ④ ARMA model

# Motivation

In the previous unit, we noted a condition for an  $AR(1)$  model to be causal. In this unit, we will explore another issue, this time regarding MA models. This issue is called identifiability.

## 1 MA(1) and Invertibility

## 2 ARMA Model

# MA(q) Process

Recall the MA(q) model

$$x_t = w_t + \theta_1 w_{t-1} + \dots + \theta_q w_{t-q}.$$

We can rewrite this using backshift operator as

$$x_t = \theta(B)w_t \tag{1}$$

where  $\theta(B) = 1 + \theta_1 B + \dots + \theta_q B^q$  which can be called the **moving average operator**. We already know that an MA(q) process is stationary.

# MA(1) Process

Let's look at the  $MA(1)$  model, i.e.

$$x_t = w_t + \theta w_{t-1}$$

We can then calculate the autocovariance function.

$$\gamma(0) = (1 + \theta^2)\sigma_w^2$$

and

$$\gamma(1) = \theta\sigma_w^2$$

and zero for larger lags. Assume that we simulate from a model using  $\theta = a$  and  $\sigma_w^2 = b$ , then look at the autocovariance

$$\gamma(0) = b + a^2b, \quad \gamma(1) = ab$$



# Identifiability

**Question:** Suppose we have another MA(1) model with  $\theta = 1/a$  and  $\sigma_w^2 = a^2 b$ , what is the autocovariance function for this model?

# Identifiability

Which one is “correct”? Both are correct: this is an **identifiability** problem as we have MA models that are not unique. We'll use the one that gives us an infinite order AR process.

# Identifiability

Rewrite the MA(1) model as

$$w_t = -\theta w_{t-1} + x_t.$$

This is just like the AR(1) that we saw last time but now the roles for  $x_t, w_t$  are reversed. So, we could write this as

$$\begin{aligned} w_t &= \sum_{j=0}^{\infty} (-\theta)^j x_{t-j} \\ &= \sum_{j=0}^{\infty} (-\theta)^j B^j x_t, \end{aligned} \tag{2}$$

provided that  $|\theta| < 1$ .

# MA(1) in Terms of Backshift Operator

(2) can be written as

$$w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} = \pi(B)x_t, \quad (3)$$

where  $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$  and  $\pi_j = (-\theta)^j$ .

# Identifiability

**Question:** What's an intuitive explanation as to why we want  $|\theta| < 1$  in (2)?

# Invertible MA(1)

We could write an MA(1) as

$$\begin{aligned}x_t &= w_t + \theta w_{t-1} \\&= w_t + \theta \sum_{j=0}^{\infty} (-\theta)^j x_{t-1-j} \\&= w_t - \sum_{j=1}^{\infty} (-\theta)^j x_{t-j}.\end{aligned}$$

So, we have written the MA(1) as an infinite order AR process. We require  $|\theta| < 1$  for invertibility for an MA(1). An MA model which can be written as an infinite order AR process is called a **invertible** MA. We will later extend and formalize a condition on the parameters  $\theta_1, \theta_2, \dots, \theta_q$  for an MA(q) process to be invertible.

## 1 MA(1) and Invertibility

## 2 ARMA Model

# ARMA Model

Recall that an AR(p) model is given by

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$$

or in terms of backshift operator

$$\phi(B)x_t = w_t, \quad \text{where} \quad \phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$



# ARMA Model

An MA(q) model is given by

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

or in terms of backshift operator

$$x_t = \theta(B)w_t, \quad \text{where} \quad \theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

# ARMA Model

$x_t$  is called an ARMA (p,q) model if  $x_t$  is stationary and can be written as

$$\phi(B)x_t = \theta(B)w_t.$$

Thus far, we have seen a number of issues with the general definition of ARMA(p,q) models:

- Stationary AR models that depend on the **future**.
- MA models that are not **unique** (identifiability issue).

We will next look at one more issue with ARMA(p,q) models before formalizing conditions for causality and invertibility.