Unit 18: Spectral Density

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Readings for Unit 18

Textbook chapter 4.2 (until page 175).

Last Unit

1. Introduction to Spectral Analysis.

This Unit

1. The Spectral Density: a Fourier Transform of the Autocovariance.
2. Properties of Spectral Density.

Motivation

The spectral density is a Fourier transform of the autocovariance function $\gamma(h)$. Autocovariance is in terms of **lags** whereas spectral density is in terms of **cycles**.

1. Spectral Density
2. Properties of Spectral Density
3. Worked Examples

Trigonometric Properties

Recall that

- a cosine function is **even**, i.e. $\cos(-\theta) = \cos(\theta)$.
- a sine function is **odd**, i.e. $\sin(-\theta) = -\sin(\theta)$.

Euler's Formula

Recall Euler's formula:

$$e^{i\alpha} = \cos(\alpha) + i \sin(\alpha). \tag{1}$$

Consequently,

$$\cos(\alpha) = \frac{e^{-i\alpha} + e^{i\alpha}}{2} \tag{2}$$

and

$$\sin(\alpha) = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}. \tag{3}$$

Motivating Example

Here is an example of a spectral representation of an autocovariance function.

Let $x_t = U_1 \cos(2\pi \frac{1}{4} t) + U_2 \sin(2\pi \frac{1}{4} t)$. Then

$$
\begin{aligned}
\gamma(h) &= \sigma^2 \cos(2\pi \frac{1}{4} h) \\
&= \sigma^2 \frac{e^{-i2\pi \frac{1}{4} h} + e^{i2\pi \frac{1}{4} h}}{2} \\
&= \int_{-1/2}^{1/2} e^{i2\pi\omega h} dF(\omega)
\end{aligned}
$$

where

$$
F(\omega) = \begin{cases} 0 & \omega < -\frac{1}{4} \\ \sigma^2/2 & -\frac{1}{4} \le \omega < \frac{1}{4} \\ \sigma^2 & \omega \ge \frac{1}{4} \end{cases}
$$

Proposition

This $F(\omega)$ *always exists* for *all* stationary processes.

Let x_t be stationary with an autocovariance function $\gamma(h)$. Then there exists a unique monotonically increasing function $F(\omega)$, called the **spectral distribution function**, that satisfies

1. $F(-\infty) = F(-\frac{1}{2}) = 0$ for $\omega \leq 1/2$
2. $F(\infty) = F(1/2) = \gamma(0)$ for $\omega \geq 1/2$
3. $\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h} dF(\omega)$.

Example

For ARMA models, we will also have a spectral representation of the autocovariance function, but the integral will be a smoother blend without any jumps:

$$\gamma(h) = \int_{-1/2}^{1/2} e^{i2\pi\omega h} dF(\omega) = \int_{-1/2}^{1/2} e^{i2\pi\omega h} f(\omega) d\omega$$

More specifically...

Spectral *Density*

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then it has the representation

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h} f(\omega) d\omega, \quad h = 0, \pm 1, \pm 2, \ldots. \tag{4}$$

(4) is called the **inverse transform of the spectral density**. The **spectral density** is denoted by

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\omega h}, \quad -1/2 \leq \omega \leq 1/2. \tag{5}$$

Autocovariance is in terms of **lags** whereas spectral density is in terms of **cycles**.

Spectral Density

- The spectral density, (5), provides information about the relative strengths of the various frequencies for **explaining the variation** in the time series.
- The spectral density is also called the power spectrum.
- Remember that $\gamma(h)$ completely determines the distribution for a stationary Gaussian process. So, the spectral density also completely determines the distribution for a stationary Gaussian process.

Spectral Density

Notice that when $h = 0$, from (4), we have

$$\gamma(0) = \mathrm{Var}(x_t) = \int_{-1/2}^{1/2} f(\omega) d\omega. \tag{6}$$

An interpretation of (6) is that the "total" integrated spectral density equals the variance of the time series. Thus the spectral density within a particular interval of frequencies can be viewed as the amount of the variance **explained** by those frequencies.

Derivation of Inverse Transformation of Spectral Density

Properties of Spectral Density

1. $f(\omega) \geq 0$ because $\gamma(h)$ is non-negative definite.
2. $f(\omega)$ is even, i.e. $f(\omega) = f(-\omega)$.
3. $f(\omega) = f(\omega + 1)$.

Derivation of Properties

Derivation of Properties

Spectral Density of White Noise

Spectral Density of White Noise

This means all frequencies receive equal weight. This is analogous to the spectrum of white light, where all colors enter equally in white light. (Hence the term white noise!)

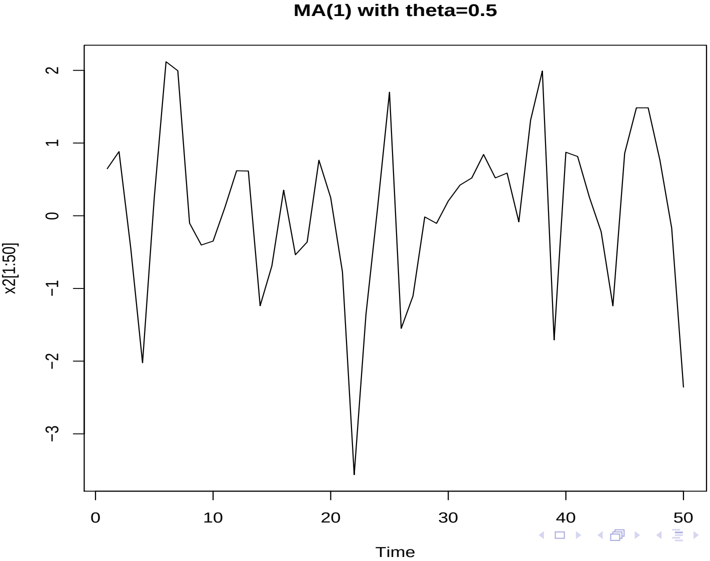
Spectral Density of MA(1)

Spectral Density of MA(1)

Question: Suppose $\theta = 0.5$ and $\sigma_w^2 = 1$, sketch the power spectrum.

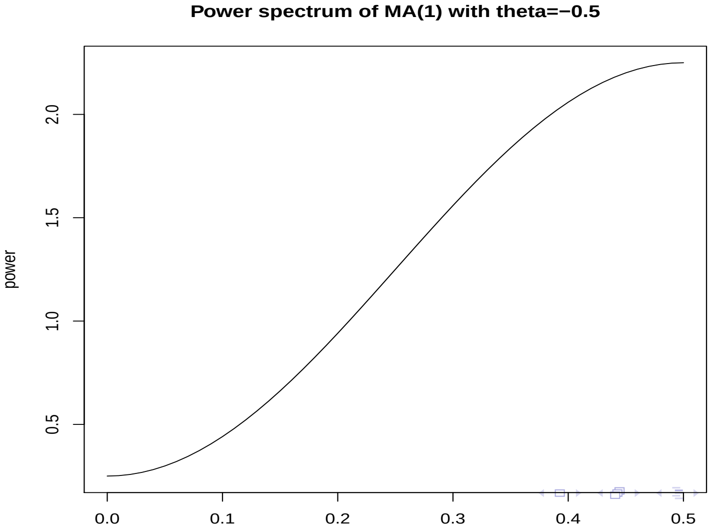
What is the implication of this power spectrum?

Time Series Plot of MA(1)

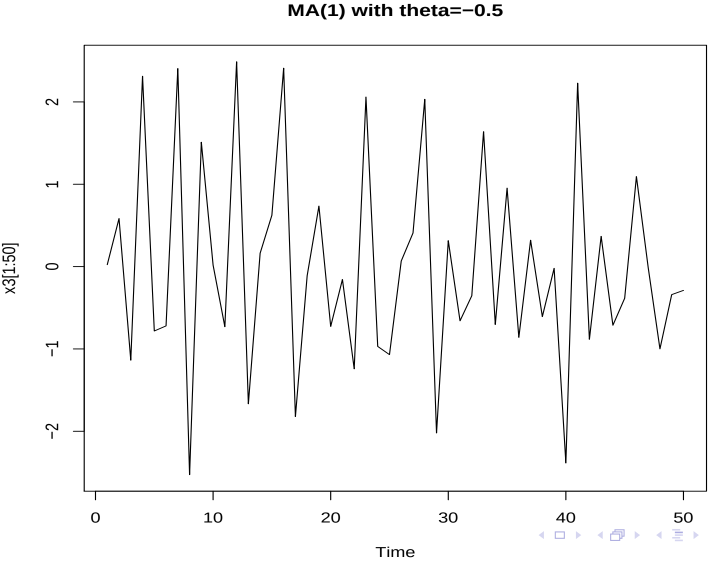


Spectral Density of MA(1)

Power spectrum of MA(1) when $\theta = -0.5$ and $\sigma_w^2 = 1$.



Time Series Plot of MA(1)



ACF of MA(1)

Question: How are the ACF plots of an MA(1) process different when $\theta = 0.5$ versus $\theta = -0.5$? Does this difference provide an explanation to the difference in their time series plots?

Spectral Density of AR(1)