

# SISR Proof Sketch

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## Statement to Prove

Assume:

- Assumption 9.4.1:
  1.  $\nu(g_0) > 0$ ;
  2.  $T_k^u(x, \mathbf{X}) = \int_{\mathbf{X}} Q(x, dx') g_k(x') > 0$  (for all  $x \in \mathbf{X}$  and  $k \geq 1$ )
  3. for all  $k \geq 0$ ,  $\sup_x g_k(x) < \infty$ .
- Assumption 9.4.2
  1.  $\phi_{\nu,0} \ll \rho_0$
- Assumption 9.4.3
  1. For any time  $k \geq 1$ , and for all  $x \in \mathbf{X}$ ,  $T_k^u(x, \cdot) \ll R_k(x, \cdot)$ . Also, for all  $x \in \mathbf{X}$ , there exists a positive version  $\frac{dT_k^u(x, \cdot)}{dR_k(x, \cdot)}(x')$  such that

$$\sup_{(x, x')} \frac{dT_k^u(x, \cdot)}{dR_k(x, \cdot)}(x') < \infty$$

Then

1. If  $\{(\xi_0^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is consistent for  $(\phi_{\nu,0}, L^1(\mathbf{X}, \phi_{\nu,0}))$ , then for any  $k > 0$   $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is consistent for  $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$
2. If in addition to the above  $\{(\xi_0^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is asymptotically normal for  $(\phi_{\nu,0}, L^2(\mathbf{X}, \phi_{\nu,0}), \sigma_0, \{M_N^{1/2}\})$ , then for any  $k > 0$   $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is asymptotically normal for  $(\phi_{\nu,k}, L^2(\mathbf{X}, \phi_{\nu,k}), \sigma_k, \{M_N^{1/2}\})$  where for  $f \in L^2(\mathbf{X}, \phi_{\nu,k})$

$$\sigma_k^2(f) = \text{Var}_{\phi_{\nu,k}}(f) + \frac{\sigma_{k-1}^2 [T_{k-1}^u(f - \phi_{\nu,k}(f))] + \alpha^{-1} \eta_{k-1}^2 [(f - \phi_{\nu,k}(f))^2]}{(\phi_{\nu,k-1} T_{k-1}^u(\mathbf{X}))^2}$$

where

$$\begin{aligned} \eta_{k-1}^2(f) = & \iint \phi_{\nu,k-1}(dx) R_{k-1}(x, dx') \left\{ \frac{dT_{k-1}^u(x, \cdot)}{dR_{k-1}(x, \cdot)}(x') f(x') \right\}^2 \\ & - \iint \phi_{\nu,k-1}(dx) \{T_{k-1}^u(x, f)\}^2 \end{aligned}$$

## Proving Part 1: Consistency

Assume  $\{(\xi_{k-1}^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is consistent for  $(\phi_{\nu,k-1}, L^1(\mathbf{X}, \phi_{\nu,k-1}))$ . We want to show that this implies  $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is consistent for  $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$ .

## Step 1: Mutation

Pick  $f \in L^1(\mathbf{X}, \phi_{\nu,k})$ . We have

$$0 \leq \phi_{\nu,k}(|f|) = \frac{\phi_{\nu,k-1}T_{k-1}^u(|f|)}{\phi_{\nu,k-1}T_{k-1}^u(\mathbf{X})} < \infty.$$

Assumption 9.4.1 implies the denominator is finite, so

$$\phi_{\nu,k-1}T_{k-1}^u(|f|) < \infty$$

which means  $T_{k-1}^u(x, |f|) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$ . Also,  $1 \in L^1(\mathbf{X}, \phi_{\nu,k})$  so  $T_{k-1}^u(x, \mathbf{X}) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$  as well.

First,

$$\sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}^{N,j}) = \frac{\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})}{\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}}$$

We can rewrite the numerator as

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{ \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) - E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N] \} + \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N]$$

The second term

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N] = M_N^{-1} \sum_{j=1}^{M_N} T_{k-1}^u(\xi_{k-1}^{N,i}, f) \xrightarrow{\mathbb{P}} \phi_{\nu,k-1}T_{k-1}^u(f)$$

because  $T_{k-1}^u(x, |f|) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$ .

The first term

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{ \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) - E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N] \} \xrightarrow{\mathbb{P}} 0$$

by 9.5.7. Therefore

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \xrightarrow{\mathbb{P}} \phi_{\nu,k-1}T_{k-1}^u(f).$$

Applying exactly the same reasoning to the function 1 instead of  $f$

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} \xrightarrow{\mathbb{P}} \phi_{\nu,k-1}T_{k-1}^u(\mathbf{X}),$$

so therefore

$$\sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}^{N,j}) \xrightarrow{\mathbb{P}} \phi_{\nu,k}(f).$$

## Step 2: Selection

After the previous step, we have  $\{\tilde{\xi}_k^{N,j}, \tilde{\omega}_k^{N,j}\}_{1 \leq j \leq \tilde{M}_N}$  is consistent for  $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$ . Now we want to show the resampled values  $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is consistent for  $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$ .

Pick  $f \in L^1(\mathbf{X}, \phi_{\nu,k})$  and write the estimator as

$$M_N^{-1} \sum_{i=1}^{M_N} \left\{ f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k] \right\} + M_N^{-1} \sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k].$$

The second term

$$M_N^{-1} \sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k] = E[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}^k]$$

due to the multinomial sampling being done conditionally iid. That term can be written as

$$E[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}^k] = \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}^{N,j})$$

which is consistent using the previous section's reasoning. The first part

$$M_N^{-1} \sum_{i=1}^{M_N} \left\{ f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k] \right\}$$

converges to 0 using 9.5.7.

## Proving Part 2: Asymptotic Normality

Assume the above result, the same starting assumptions, and that  $\{(\xi_{k-1}^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is asymptotically normal for  $(\phi_{\nu,k-1}, L^2(\mathbf{X}, \phi_{\nu,k-1}), \sigma_{k-1}, \{M_N^{1/2}\})$ .

We want to show that

$\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is asymptotically normal for  $(\phi_{\nu,k}, L^2(\mathbf{X}, \phi_{\nu,k}), \sigma_k, \{M_N^{1/2}\})$ .

### Step 1: Mutation

We want to show that

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}_k^{N,j}) = \frac{\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j})}{\frac{\sum_{j'} \tilde{\omega}^{N,j'}}{\tilde{M}_N}}$$

is asymptotically normal first. Rewrite the numerator as

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \left\{ \frac{\tilde{\omega}^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) - E \left[ \frac{\tilde{\omega}^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] \right\} + \tilde{M}_N^{1/2} \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} E \left[ \tilde{\omega}^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right]$$

The second piece

$$\tilde{M}_N^{1/2} \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} E \left[ \tilde{\omega}^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] = \tilde{M}_N^{1/2} \frac{1}{M_N} \sum_{j=1}^{M_N} T_{k-1}^u(\xi_{k-1}^{N,i}, f)$$

this quantity isn't asymptotically normal, but

$$M_N^{1/2} \frac{1}{M_N} \sum_{j=1}^{M_N} T_{k-1}^u(\xi_{k-1}^{N,i}, f)$$

is as soon as we show  $T_{k-1}^u(\xi_{k-1}^{N,i}, f) \in L^2(\mathbf{X}, \phi_{\nu,k-1})$ . This is nonobvious, and the book has a few interesting lines showing why it's true. The asymptotic variance of this part is

$$\sigma_{k-1}^2 [T_{k-1}^u(f - \phi_{\nu,k}(f))] .$$

The first piece

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \left\{ \frac{\tilde{\omega}^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) - E \left[ \frac{\tilde{\omega}^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] \right\} \xrightarrow{D} \text{Normal} \left( 0, \eta_{k-1}^2 [(f - \phi_{\nu,k}(f))^2] \right)$$

using Proposition 9.5.12.

Next, we look at the joint asymptotic distribution of these two variables as a  $2 \times 1$  random vector, and use a dominated convergence theorem argument to show that they're jointly independently normal. The delta method shows that the numerator of

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}_k^{N,j}) = \frac{\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j})}{\frac{\sum_{j'} \tilde{\omega}^{N,j'}}{\tilde{M}_N}}$$

is normal.

The denominator is consistent. Rewrite it as

$$\tilde{M}_N^{-1} \sum_{j'} \left\{ \tilde{\omega}^{N,j'} - E[\tilde{\omega}^{N,j'} \mid \mathcal{F}_{k-1}^N] \right\} + \tilde{M}_N^{-1} \sum_{j'} E[\tilde{\omega}^{N,j'} \mid \mathcal{F}_{k-1}^N]$$

The second piece converges to  $\phi_{\nu,k-1} T_{k-1}^u(\mathbf{X})$  because we showed that  $T_{k-1}^u(x, \mathbf{X}) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$ . The first piece converges due to Proposition 9.5.7.

Therefore, by Slutsky's theorem,

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}_k^{N,j}) \xrightarrow{D} \text{Normal} \left( \phi_{\nu,k}(f), \frac{\sigma_{k-1}^2 [T_{k-1}^u(f - \phi_{\nu,k}(f))] + \alpha^{-1} \eta_{k-1}^2 [(f - \phi_{\nu,k}(f))^2]}{(\phi_{\nu,k-1} T_{k-1}^u(\mathbf{X}))^2} \right)$$

## Step 2: Selection

Pick  $f \in L^2(\mathbf{X}, \phi_{\nu,k})$  and write the target as

$$M_N^{1/2} \left\{ M_N^{-1} \sum_{i=1}^{M_N} f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N] \right\} + M_N^{1/2} \left\{ M_N^{-1} \sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N] \right\}$$

Focusing on the second piece first:

$$M_N^{1/2} M_N^{-1} \sum_{i=1}^{M_N} E \left[ f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_{k-1}^N \right] = M_N^{1/2} E \left[ f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}_{k-1}^N \right]$$

due to the resampling being done conditionally independently and identically, and then

$$M_N^{1/2} E \left[ f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}_{k-1}^N \right] = M_N^{1/2} \tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}_k^{N,j})$$

We showed this converges to a normal in the previous section. The first part of the sum

$$M_N^{1/2} \left\{ M_N^{-1} \sum_{i=1}^{M_N} f(\xi_k^{N,i}) - E \left[ f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N \right] \right\}$$

converges by Proposition 9.5.12. Jointly, the two summands are asymptotically indepdently normal, and using the delta method, we can prove the final result.