proof_9.4.5

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Statement to Prove

Assume:

- Assumption 9.4.1:

1. $\nu(g_0) > 0$;
2. $T_k^u(x, \mathsf{X}) = \int_{\mathsf{X}} Q(x, dx')g_k(x') > 0$ (for all $x \in \mathsf{X}$ and $k \geq 1$)
3. for all $k \geq 0$, $\sup_x g_k(x) < \infty$.

- Assumption 9.4.2

1. $\phi_{\nu,0} \ll \rho_0$

- Assumption 9.4.3

1. For any time $k \geq 1$, and for all $x \in \mathsf{X}$, $T_k^u(x, \cdot) \ll R_k(x, \cdot)$. Also, for all $x \in \mathsf{X}$, there exists a positive version $\frac{dT_k^u(x,\cdot)}{R_k(x,\cdot)}(x')$ such that
$$\sup_{(x,x')} \frac{dT_k^u(x,\cdot)}{dR_k(x,\cdot)}(x') < \infty$$

Then

1. If $\{(\xi_0^{N,i}, 1)\}_{1\leq i \leq M_N}$ is consistent for $(\phi_{\nu,0}, L^1(\mathsf{X}, \phi_{\nu,0}))$, then for any $k > 0$ $\{(\xi_k^{N,i}, 1)\}_{1\leq i \leq M_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathsf{X}, \phi_{\nu,k}))$
2. If in addition to the above $\{(\xi_0^{N,i}, 1)\}_{1\leq i \leq M_N}$ is asymptotically normal for $(\phi_{\nu,0}, L^2(\mathsf{X}, \phi_{\nu,0}), \sigma_0, \{M_N^{1/2}\})$, then for any $k > 0$ $\{(\xi_k^{N,i}, 1)\}_{1\leq i \leq M_N}$ is asymptotically normal for $(\phi_{\nu,k}, L^2(\mathsf{X}, \phi_{\nu,k}), \sigma_k, \{M_N^{1/2}\})$ where for $f \in L^2(\mathsf{X}, \phi_{\nu,k})$

$$\sigma_k^2(f) = \mathrm{Var}_{\phi_{\nu,k}}(f) + \frac{\sigma_{k-1}^2\left[T_{k-1}^u\left(f - \phi_{\nu,k}(f)\right)\right] + \alpha^{-1}\eta_{k-1}^2\left[\left(f - \phi_{\nu,k}(f)\right)^2\right]}{\left(\phi_{\nu,k-1}T_{k-1}^u(\mathsf{X})\right)^2}$$

where

$$\begin{aligned}\eta_{k-1}^2(f) = &\iint \phi_{\nu,k-1}(dx)R_{k-1}(x,dx')\left\{\frac{dT_{k-1}^u(x,\cdot)}{dR_{k-1}(x,\cdot)}(x')f(x')\right\}^2 \\ &- \iint \phi_{\nu,k-1}(dx)\left\{T_{k-1}^u(x,f)\right\}^2\end{aligned}$$

Proving Part 1: Consistency

Assume $\{(\xi_{k-1}^{N,i}, 1)\}_{1\leq i \leq M_N}$ is consistent for $(\phi_{\nu,k-1}, L^1(\mathsf{X}, \phi_{\nu,k-1}))$. We want to show that this implies $\{(\xi_k^{N,i}, 1)\}_{1\leq i \leq M_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathsf{X}, \phi_{\nu,k}))$.

Step 1: Mutation

Pick $f \in L^1(\mathsf{X}, \phi_{\nu,k}))$. We have

$$0 \leq \phi_{\nu,k}(|f|) = \frac{\phi_{\nu,k-1} T_{k-1}^u\,(|f|)}{\phi_{\nu,k-1} T_{k-1}^u(\mathsf{X})} < \infty.$$

Assumption 9.4.1 implies the denominator is finite, so

$$\phi_{\nu,k-1} T_{k-1}^u\,(|f|) < \infty$$

which means $T_{k-1}^u\,(x, |f|) \in L^1(\mathsf{X}, \phi_{\nu,k-1})$. Also, $1 \in L^1(\mathsf{X}, \phi_{\nu,k}))$ so $T_{k-1}^u\,(x, \mathsf{X}) \in L^1(\mathsf{X}, \phi_{\nu,k-1})$ as well.
First,

$$\sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}^{N,j}) = \frac{\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})}{\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}}$$

We can rewrite the numerator as

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \left\{ \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) - E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N\right]\right\} + \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N\right]$$

The second term

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N\right] = M_N^{-1} \sum_{j=1}^{M_N} T_{k-1}^u(\xi_{k-1}^{N,i}, f) \xrightarrow{\text{P}} \phi_{\nu,k-1} T_{k-1}^u(f)$$

because $T_{k-1}^u\,(x, |f|) \in L^1(\mathsf{X}, \phi_{\nu,k-1})$.
The first term

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \left\{ \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) - E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}_{k-1}^N\right]\right\} \xrightarrow{\text{P}} 0$$

by 9.5.7. Therefore

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \xrightarrow{\text{P}} \phi_{\nu,k-1} T_{k-1}^u(f).$$

Applying exactly the same reasoning to the function 1 instead of f

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} \xrightarrow{\text{P}} \phi_{\nu,k-1} T_{k-1}^u(\mathsf{X}),$$

so therefore

$$\sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}^{N,j}) \xrightarrow{\text{P}} \phi_{\nu,k}(f).$$

Step 2: Selection

After the previous step, we have $\{\tilde{\xi}_k^{N,j}, \tilde{\omega}_k^{N,j}\}_{1\le j\le \tilde{M}_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathsf{X}, \phi_{\nu,k}))$. Now we want to show the resampled values $\{(\xi_k^{N,i}, 1)\}_{1\le i\le M_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathsf{X}, \phi_{\nu,k}))$.

Pick $f \in L^1(\mathsf{X}, \phi_{\nu,k})$ and write the estimator as

$$M_N^{-1}\sum_{i=1}^{M_N}\left\{f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k]\right\} + M_N^{-1}\sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k].$$

The second term

$$M_N^{-1}\sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k] = E[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}^k]$$

due to the multinomial sampling being done conditionally iid. That term can be written as

$$E[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}^k] = \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}^{N,j}}{\sum_{j'} \tilde{\omega}^{N,j'}} f(\tilde{\xi}^{N,j})$$

which is consistent using the previous section's reasoning. The first part

$$M_N^{-1}\sum_{i=1}^{M_N}\left\{f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}^k]\right\}$$

converges to 0 using 9.5.7.

Proving Part 1: Consistency

Assume the above result, the same starting assumptions, and that $\{(\xi_{k-1}^{N,i}, 1)\}_{1\le i\le M_N}$ is asymptotically normal for $(\phi_{\nu,k-1}, L^2(\mathsf{X}, \phi_{\nu,k-1}), \sigma_{k-1}, \{M_N^{1/2}\})$.

We want to show that

$\{(\xi_k^{N,i}, 1)\}_{1\le i\le M_N}$ is asymptotically normal for $(\phi_{\nu,k}, L^2(\mathsf{X}, \phi_{\nu,k}), \sigma_k, \{M_N^{1/2}\})$ where for $f \in L^2(\mathsf{X}, \phi_{\nu,k})$