

SISR Proof Sketch

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Statement to Prove

Assume:

- Assumption 9.4.1:

1. $\nu(g_0) > 0$;
2. $T_k^u(x, \mathbf{X}) = \int_{\mathbf{X}} Q(x, dx') g_k(x') > 0$ (for all $x \in \mathbf{X}$ and $k \geq 1$)
3. for all $k \geq 0$, $\sup_x g_k(x) < \infty$.

- Assumption 9.4.2

1. $\phi_{\nu,0} \ll \rho_0$

- Assumption 9.4.3

1. For any time $k \geq 1$, and for all $x \in \mathbf{X}$, $T_k^u(x, \cdot) \ll R_k(x, \cdot)$. Also, for all $x \in \mathbf{X}$, there exists a positive version $\frac{dT_k^u(x, \cdot)}{R_k(x, \cdot)}(x')$ such that

$$\sup_{(x, x')} \frac{dT_k^u(x, \cdot)}{dR_k(x, \cdot)}(x') < \infty$$

Then

1. If $\{(\xi_0^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is consistent for $(\phi_{\nu,0}, L^1(\mathbf{X}, \phi_{\nu,0}))$, then for any $k > 0$ $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$
2. If in addition to the above $\{(\xi_0^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is asymptotically normal for $(\phi_{\nu,0}, L^2(\mathbf{X}, \phi_{\nu,0}), \sigma_0, \{M_N^{1/2}\})$, then for any $k > 0$ $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is asymptotically normal for $(\phi_{\nu,k}, L^2(\mathbf{X}, \phi_{\nu,k}), \sigma_k, \{M_N^{1/2}\})$ where for $f \in L^2(\mathbf{X}, \phi_{\nu,k})$

$$\sigma_k^2(f) = \text{Var}_{\phi_{\nu,k}}(f) + \frac{\sigma_{k-1}^2 [T_{k-1}^u(f - \phi_{\nu,k}(f))] + \alpha^{-1} \eta_{k-1}^2 [(f - \phi_{\nu,k}(f))^2]}{(\phi_{\nu,k-1} T_{k-1}^u(\mathbf{X}))^2}$$

where

$$\begin{aligned} \eta_{k-1}^2(f) = & \iint \phi_{\nu,k-1}(dx) R_{k-1}(x, dx') \left\{ \frac{dT_{k-1}^u(x, \cdot)}{dR_{k-1}(x, \cdot)}(x') f(x') \right\}^2 \\ & - \iint \phi_{\nu,k-1}(dx) \{T_{k-1}^u(x, f)\}^2 \end{aligned}$$

Proving Part 1: Consistency

Assume $\{(\xi_{k-1}^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is consistent for $(\phi_{\nu,k-1}, L^1(\mathbf{X}, \phi_{\nu,k-1}))$. We want to show that this implies $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$.

Step 1: Mutation

Pick $f \in L^1(\mathbf{X}, \phi_{\nu,k})$. We have

$$0 \leq \phi_{\nu,k}(|f|) = \frac{\phi_{\nu,k-1}T_{k-1}^u(|f|)}{\phi_{\nu,k-1}T_{k-1}^u(\mathbf{X})} < \infty.$$

Assumption 9.4.1 implies the denominator is finite, so

$$\phi_{\nu,k-1}T_{k-1}^u(|f|) < \infty$$

which means $T_{k-1}^u(x, |f|) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$. Also, $1 \in L^1(\mathbf{X}, \phi_{\nu,k})$ so $T_{k-1}^u(x, \mathbf{X}) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$ as well.

First,

$$\sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\sum_{j'} \tilde{\omega}_k^{N,j'}} f(\tilde{\xi}_k^{N,j}) = \frac{\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j})}{\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}_k^{N,j}}$$

We can rewrite the numerator as

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \left\{ \tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) - E \left[\tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] \right\} + \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E \left[\tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right]$$

The second term

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E \left[\tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] = M_N^{-1} \sum_{j=1}^{M_N} T_{k-1}^u(\xi_{k-1}^{N,i}, f) \xrightarrow{\mathbb{P}} \phi_{\nu,k-1}T_{k-1}^u(f)$$

because $T_{k-1}^u(x, |f|) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$.

The first term

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \left\{ \tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) - E \left[\tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] \right\} \xrightarrow{\mathbb{P}} 0$$

by 9.5.7. Therefore

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) \xrightarrow{\mathbb{P}} \phi_{\nu,k-1}T_{k-1}^u(f).$$

Applying exactly the same reasoning to the function 1 instead of f

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}_k^{N,j} \xrightarrow{\mathbb{P}} \phi_{\nu,k-1}T_{k-1}^u(\mathbf{X}),$$

so therefore

$$\sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\sum_{j'} \tilde{\omega}_k^{N,j'}} f(\tilde{\xi}_k^{N,j}) \xrightarrow{\mathbb{P}} \phi_{\nu,k}(f).$$

Step 2: Selection

After the previous step, we have $\{\tilde{\xi}_k^{N,j}, \tilde{\omega}_k^{N,j}\}_{1 \leq j \leq \tilde{M}_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$. Now we want to show the resampled values $\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is consistent for $(\phi_{\nu,k}, L^1(\mathbf{X}, \phi_{\nu,k}))$.

Pick $f \in L^1(\mathbf{X}, \phi_{\nu,k})$ and write the estimator as

$$M_N^{-1} \sum_{i=1}^{M_N} \left\{ f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N] \right\} + M_N^{-1} \sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N].$$

The second term

$$M_N^{-1} \sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N] = E[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}_k^N]$$

due to the multinomial sampling being done conditionally iid. That term can be written as

$$E[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}_k^N] = \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\sum_{j'} \tilde{\omega}_k^{N,j'}} f(\tilde{\xi}_k^{N,j})$$

which is consistent using the previous section's reasoning. The first part

$$M_N^{-1} \sum_{i=1}^{M_N} \left\{ f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N] \right\}$$

converges to 0 using 9.5.7.

Proving Part 2: Asymptotic Normality

Assume the above result, the same starting assumptions, and that $\{(\xi_{k-1}^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is asymptotically normal for $(\phi_{\nu,k-1}, L^2(\mathbf{X}, \phi_{\nu,k-1}), \sigma_{k-1}, \{M_N^{1/2}\})$.

We want to show that

$\{(\xi_k^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is asymptotically normal for $(\phi_{\nu,k}, L^2(\mathbf{X}, \phi_{\nu,k}), \sigma_k, \{M_N^{1/2}\})$.

Step 1: Mutation

We want to show that

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\sum_{j'} \tilde{\omega}_k^{N,j'}} f(\tilde{\xi}_k^{N,j}) = \frac{\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j})}{\frac{\sum_{j'} \tilde{\omega}_k^{N,j'}}{\tilde{M}_N}}$$

is asymptotically normal first. Rewrite the numerator as

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \left\{ \frac{\tilde{\omega}_k^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) - E \left[\frac{\tilde{\omega}_k^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] \right\} + \tilde{M}_N^{1/2} \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} E \left[\tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right]$$

The second piece

$$\tilde{M}_N^{1/2} \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} E \left[\tilde{\omega}_k^{N,j} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] = \tilde{M}_N^{1/2} \frac{1}{M_N} \sum_{j=1}^{M_N} T_{k-1}^u(\xi_{k-1}^{N,i}, f)$$

this quantity isn't asymptotically normal, but the rescaled version

$$M_N^{1/2} \frac{1}{M_N} \sum_{j=1}^{M_N} T_{k-1}^u(\xi_{k-1}^{N,i}, f)$$

is as soon as we show $T_{k-1}^u(\xi_{k-1}^{N,i}, f) \in L^2(\mathbf{X}, \phi_{\nu,k-1})$. This is nonobvious, and the book has a few interesting lines showing why it's true. The asymptotic variance of this part is

$$\sigma_{k-1}^2 [T_{k-1}^u(f - \phi_{\nu,k}(f))].$$

The first piece

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \left\{ \frac{\tilde{\omega}_k^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) - E \left[\frac{\tilde{\omega}_k^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j}) \mid \mathcal{F}_{k-1}^N \right] \right\} \xrightarrow{D} \text{Normal} \left(0, \eta_{k-1}^2 [(f - \phi_{\nu,k}(f))^2] \right)$$

using Proposition 9.5.12.

Next, we look at the joint asymptotic distribution of these two variables as a 2×1 random vector, and use a dominated convergence theorem argument to show that they're jointly independently normal. The delta method shows that the numerator of

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\sum_{j'} \tilde{\omega}_k^{N,j'}} f(\tilde{\xi}_k^{N,j}) = \frac{\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\tilde{M}_N} f(\tilde{\xi}_k^{N,j})}{\frac{\sum_{j'} \tilde{\omega}_k^{N,j'}}{\tilde{M}_N}}$$

is normal.

The denominator is consistent. Rewrite it as

$$\tilde{M}_N^{-1} \sum_{j'} \left\{ \tilde{\omega}_k^{N,j'} - E[\tilde{\omega}_k^{N,j'} \mid \mathcal{F}_{k-1}^N] \right\} + \tilde{M}_N^{-1} \sum_{j'} E[\tilde{\omega}_k^{N,j'} \mid \mathcal{F}_{k-1}^N]$$

The second piece converges to $\phi_{\nu,k-1} T_{k-1}^u(\mathbf{X})$ because we showed that $T_{k-1}^u(x, \mathbf{X}) \in L^1(\mathbf{X}, \phi_{\nu,k-1})$. The first piece converges to 0 due to Proposition 9.5.7.

Therefore, by Slutsky's theorem,

$$\tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\sum_{j'} \tilde{\omega}_k^{N,j'}} f(\tilde{\xi}_k^{N,j}) \xrightarrow{D} \text{Normal} \left(\phi_{\nu,k}(f), \frac{\sigma_{k-1}^2 [T_{k-1}^u(f - \phi_{\nu,k}(f))] + \alpha^{-1} \eta_{k-1}^2 [(f - \phi_{\nu,k}(f))^2]}{(\phi_{\nu,k-1} T_{k-1}^u(\mathbf{X}))^2} \right)$$

Step 2: Selection

Pick $f \in L^2(\mathbf{X}, \phi_{\nu,k})$ and write the target as

$$M_N^{1/2} \left\{ M_N^{-1} \sum_{i=1}^{M_N} f(\xi_k^{N,i}) - E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N] \right\} + M_N^{1/2} \left\{ M_N^{-1} \sum_{i=1}^{M_N} E[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N] \right\}$$

Focusing on the second piece first:

$$M_N^{1/2} M_N^{-1} \sum_{i=1}^{M_N} E \left[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_{k-1}^N \right] = M_N^{1/2} E \left[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}_{k-1}^N \right]$$

due to the resampling being done conditionally independently and identically, and then

$$M_N^{1/2} E \left[f(\xi_k^{N,1}) \mid \tilde{\mathcal{F}}_{k-1}^N \right] = M_N^{1/2} \tilde{M}_N^{1/2} \sum_{j=1}^{\tilde{M}_N} \frac{\tilde{\omega}_k^{N,j}}{\sum_{j'} \tilde{\omega}_k^{N,j'}} f(\tilde{\xi}_k^{N,j})$$

We showed this converges to a normal in the previous section. The first part of the sum

$$M_N^{1/2} \left\{ M_N^{-1} \sum_{i=1}^{M_N} f(\xi_k^{N,i}) - E \left[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N \right] \right\}$$

converges by Proposition 9.5.12. It's asymptotic variance is

$$\text{plim}_{N \rightarrow \infty} \sum_{i=1}^{M_N} \frac{\text{Var} \left[f(\xi_k^{N,i}) \mid \tilde{\mathcal{F}}_k^N \right]}{M_N^{-1}} = \text{Var}_{\phi_{\nu,k}}(f)$$

Jointly, we can use a dominated convergence theorem argument to show the two summands are asymptotically indepdently normal. Finally, we use the delta method, we can prove the final result.