Real-Time Bayesian Forecasting with State-Space Models

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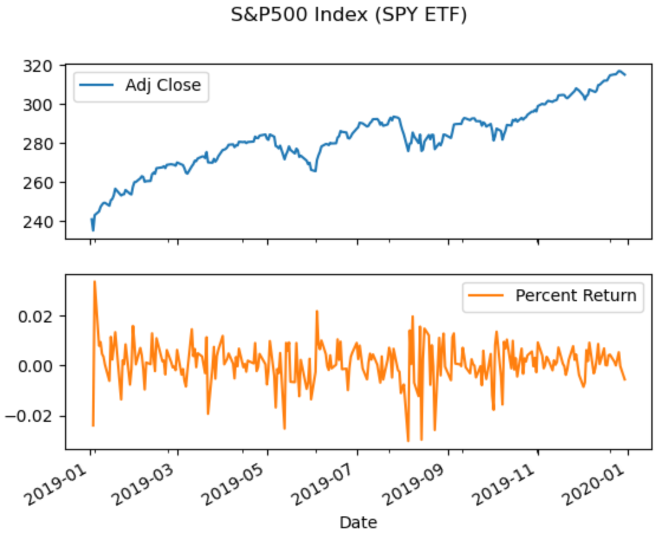
Overview

Motivation For State-Space Models

Motivation



S&P500 Index (SPY ETF)
320
300
280
260
240
Adj Close
0.02
0.00
−0.02
Percent Return
2019-01
2019-03
2019-05
2019-07
2019-09
2019-11
2020-01
Date

Our State-Space Model

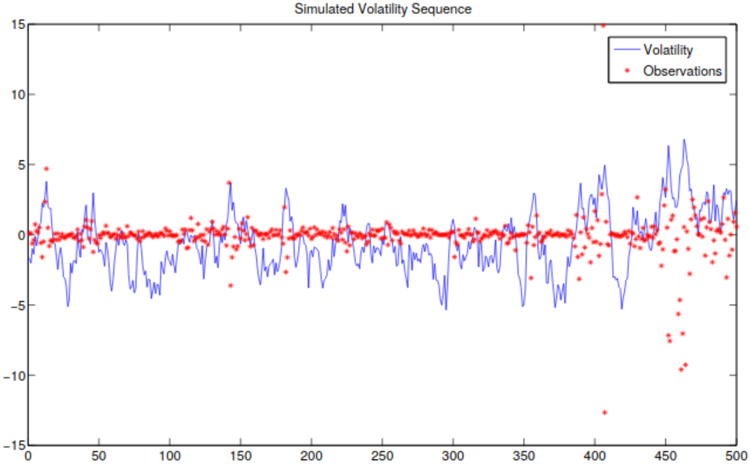
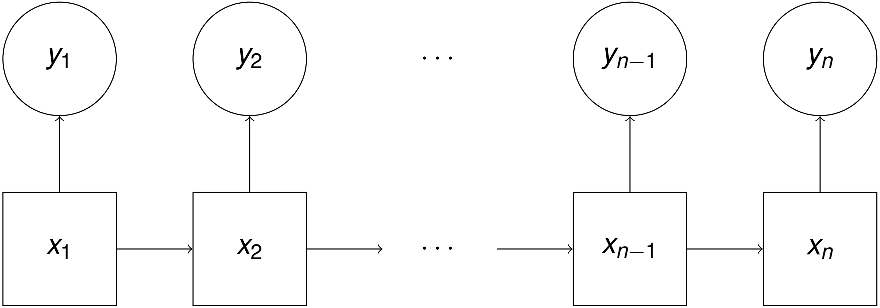


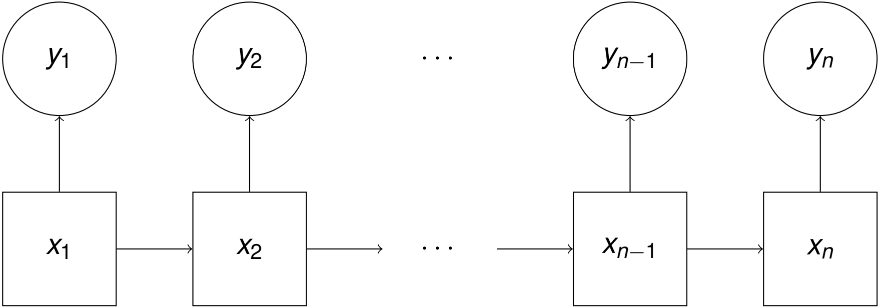
Figure: image from Doucet and Johansen, 2011

State-Space Models



vertical arrows: $g(y_t \mid x_t, \theta)$
horizontal arrows: $f(x_t \mid x_{t-1}, \theta)$

Our State-Space Model



$g(y_t \mid x_t, \theta) = \text{Normal}(0, \beta^2 e^{x_t})$
$f(x_t \mid x_{t-1}, \theta) = \text{Normal}(\phi x_{t-1}, \sigma^2)$
x_t is log-volatility, represents how "active" the market is...
(Taylor, 1982)

Forecasting in a Bayesian Way

Our Goal

Forecasting in a Bayesian way–the posterior predictive distribution:

$$p(y_{t+1} \mid y_{1:t}) = \int p(y_{t+1} \mid \theta, y_{1:t}) p(\theta \mid y_{1:t}) d\theta$$

- $y_{1:t}$ are the percent returns
- θ is the vector of model parameters for the assumed true model
- $t \to \infty$
- can't condition on unknown parameters or unknown states
- could predict further into the future, too

Our Goal

With state-space models, there are two general approaches to approximating

$$p(y_{t+1} \mid y_{1:t}) = \int p(y_{t+1} \mid \theta, y_{1:t})p(\theta \mid y_{1:t})d\theta$$

- option 1: the one step approach
- option 2: the two step approach

Option 1: A Very Difficult Task!

We want

$$
\begin{aligned}
p(y_{t+1} \mid y_{1:t}) &= \iint p(y_{t+1} \mid x_t, \theta, y_{1:t}) p(x_t, \theta \mid y_{1:t}) dx_t d\theta \\
&= \iint p(y_{t+1} \mid x_t, \theta) p(x_t, \theta \mid y_{1:t}) dx_t d\theta
\end{aligned}
$$

1. Sampling both states and parameters **from one on-line algorithm** $p(x_t, \theta \mid y_{1:t})$
2. $p(y_{t+1} \mid x_t, \theta)$ is easy to deal with

Option 2: Two Difficult Tasks in One

Option 2:

$$p(y_{t+1} \mid y_{1:t}) = \int p(y_{t+1} \mid \theta, y_{1:t}) p(\theta \mid y_{1:t}) d\theta$$

1. $p(y_{t+1} \mid \theta, y_{1:t})$ involves filtering
2. $p(\theta \mid y_{1:t})$ is standard posterior inference, but now for every t

We focus on this first, and briefly summarize option 1 later...

Option 2: The Two Step Approach

Option 2: Two Difficult Tasks in One

Option 2 was based on:

$$p(y_{t+1} \mid y_{1:t}) = \int p(y_{t+1} \mid \theta, y_{1:t}) p(\theta \mid y_{1:t}) d\theta$$

1. filtering:

$$\begin{aligned} & p(y_{t+1} \mid \theta, y_{1:t}) \\ & = \int g(y_{t+1} \mid x_{t+1}, \theta) f(x_{t+1} \mid x_t, \theta) \underbrace{p(x_t \mid y_{1:t}, \theta)}_{\text{filtering distribution}} dx_{t:t+1} \end{aligned}$$

2. $p(\theta \mid y_{1:t})$ (posterior inference for every t)

We describe filtering first...

A Quick Word on Filtering

$$p(x_t \mid y_{1:t}, \theta) = \frac{g(y_t \mid x_t, \theta) p(x_t \mid y_{1:t-1}, \theta)}{p(y_t \mid y_{1:t-1}, \theta)}$$

where $p(x_t \mid y_{1:t-1}, \theta) = \int f(x_t \mid x_{t-1}, \theta) p(x_{t-1} \mid y_{1:t-1}, \theta) dx_{t-1}$ comes from the previous filtering distribution.

On-line algorithms include

1. (Kalman, 1960) filtering for linear Gaussian models
2. Hidden Markov Model filter for finite state space models
3. approximate filtering otherwise (e.g. extended Kalman filter, unscented Kalman filter, particle filters, etc.)

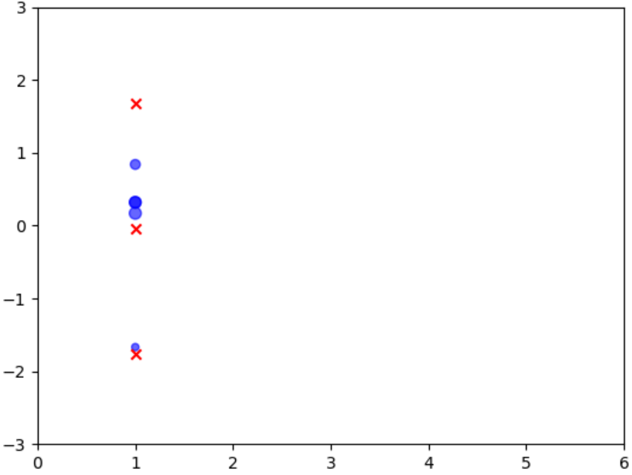
Particle Filtering (The General Strategy)

Choose a large N. Then

- sample $x_1^1, \ldots, x_1^N \overset{\text{approx. iid}}{\sim} p(x_1 \mid y_1, \theta)$
- compute weights then resample those into $x_2^1, \ldots, x_2^N \overset{\text{approx. iid}}{\sim} p(x_2 \mid y_{1:2}, \theta)$
- compute weights then resample those into $x_3^1, \ldots, x_3^N \overset{\text{approx. iid}}{\sim} p(x_3 \mid y_{1:3}, \theta)$
- etc. etc.

Particle Filtering

Red xs: true mean ± 2 std. deviations



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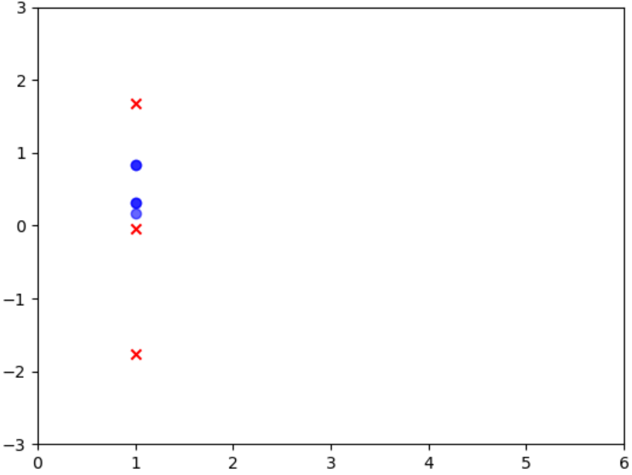

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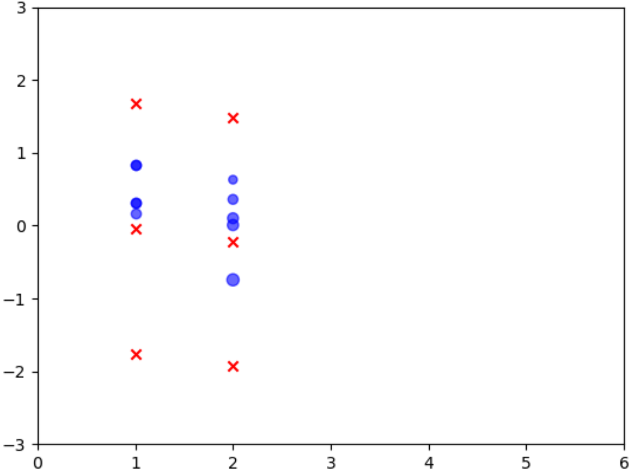

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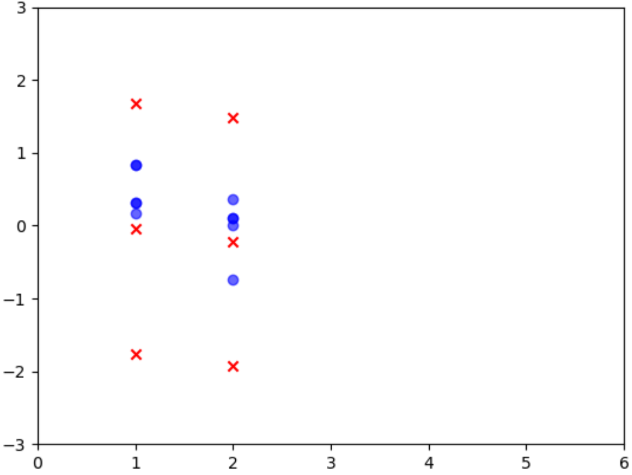

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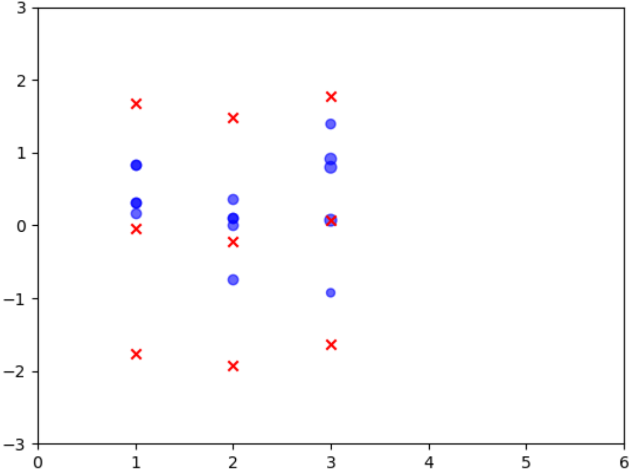

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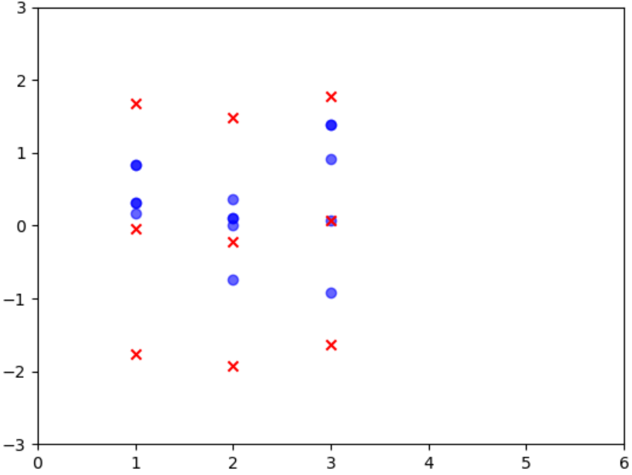

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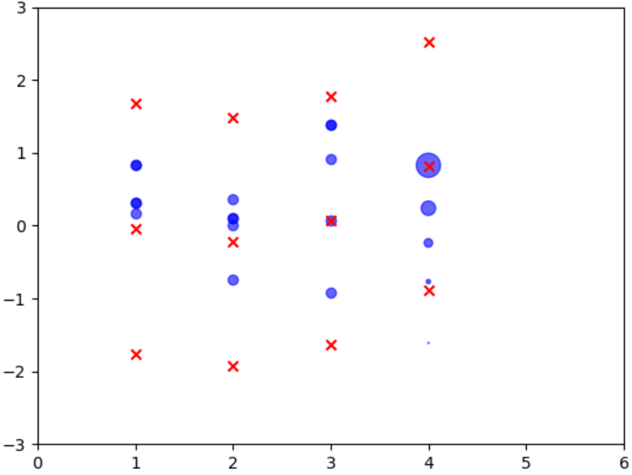

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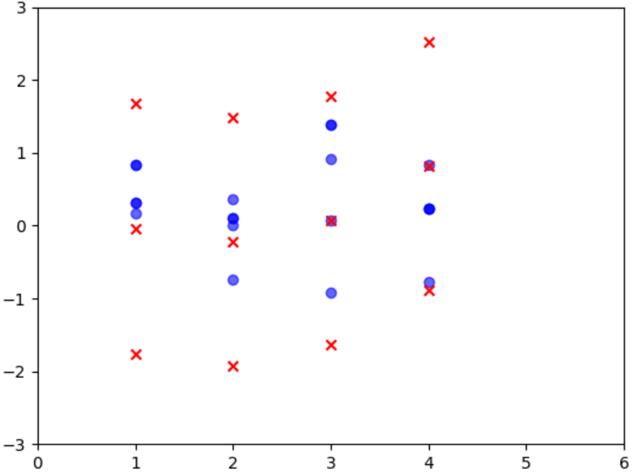

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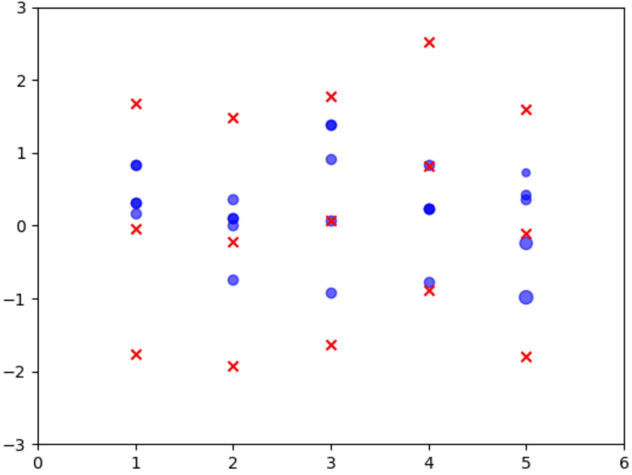

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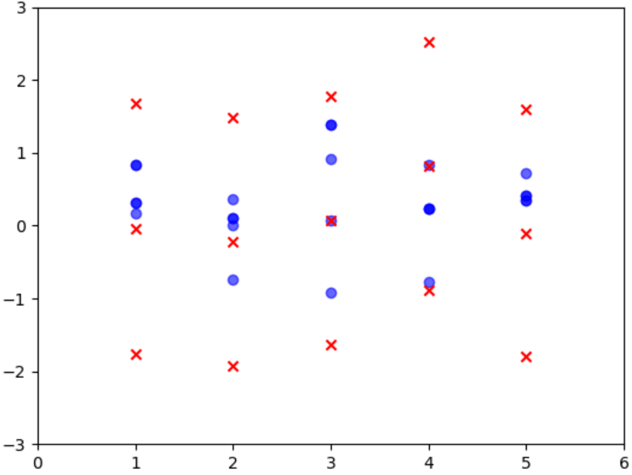


The other half of Option 2: Posteriors for θ

Option 2 was motivated by

$$p(y_{t+1} \mid y_{1:t}) = \int \overbrace{p(y_{t+1} \mid \theta, y_{1:t})}^{\text{particle filter}} \underbrace{p(\theta \mid y_{1:t})}_{?} \, d\theta$$

How do we use particle filtering, but incorporate uncertainty about θ?

The other half of Option 2: Posteriors for θ

There are many ways to sample from **one** posterior $p(\theta \mid y_{1:t})$

...but they can all be either time-consuming, difficult to tune or program, or impossible to use on data sets with large t.

Is there a way to avoid having to do this at every time step on an increasingly large data set?

Even if there was, how do we use parameter samples together with a particle filter?

This is why so much work has been done on Option 1...

Option 1: On-Line Sampling of Both States and Parameters

Option 1: On-Line Sampling From $p(x_t, \theta \mid y_{1:t})$

This is just a quick review.

It won't describe any of these in detail, but we list out a few names...

Option 1: On-Line Sampling From $p(x_t, \theta \mid y_{1:t})$

1. Run one particle filter on a model with extended state $\tilde{x}_t = (x_t, \theta)$ (Kitagawa, 1998)
2. Do the above, but exploit sufficient statistics of $p(\theta \mid x_{1:t}, y_{1:t})$ (i.e. the Storvik Filter (Storvik, 2002), (Fearnhead, 2002) and Particle Learning (Carvalho et al., 2010))
3. Run one particle filter on a model with extended state $\tilde{x}_t = (x_t, \theta_t)$, assume θ_t is an (artificial) random walk (the Liu-West Filter (Liu and West, 2001) and the original proposal in (Kitagawa, 1998))
4. Other recursive (but not on-line) algorithms: The Resample-Move algorithm (Gilks and Berzuini, 2001) and SMC^2 (Chopin, Jacob, and Papaspiliopoulos, 2013)

Option 1: On-Line Sampling From $p(x_t, \theta \mid y_{1:t})$

Option 1 approaches are all generally "fast", but they suffer from some combination of the following:

1. high variance (due to **particle degeneracy**),
2. have a nonquantifiable bias, or they are
3. not on-line.

So, what do we do...

Option 2 (With My Approach)

Back To Option 2: The Particle Swarm Filter

My approach: don't always update the posterior, and aggregate particle filters' forecasts.

$$\begin{aligned} p(y_{t+1} \mid y_{1:t}) &= \int p(y_{t+1} \mid \theta, y_{1:t}) p(\theta \mid y_{1:t}) d\theta \\ &\approx \int p(y_{t+1} \mid \theta, y_{1:t}) p(\theta \mid y_{1:s}) d\theta \end{aligned}$$

where $0 \leq s < t$. $p(\theta \mid y_{1:s})$ is either a prior, or an "old" posterior that you can't afford to update every moment.

- Sample $\{\theta^j\}_j$ **once** (or occasionally); run many particle filters with these plugged in; keep averaging predictions.

The Particle Swarm Filter: Theoretical Guarantees

Summary of some of the theoretical results in (Brown, 2021):

1. Consistency *to the proxy distribution* under strong assumptions (compact parameter space, stochastic equicontinuity of $\hat{p}(x_t \mid y_{1:t}, \theta)$);
2. asymptotic normality under the same assumptions, and again, to the proxy distribution;
3. no uniform-in-time bounds for either the bias or for the asymptotic variances (caveat emptor)

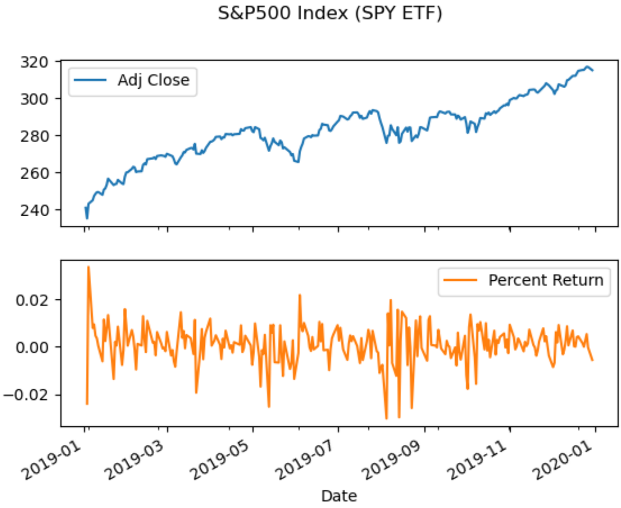
Bias

$$\overbrace{p(y_{t+1} \mid y_{1:t})}^{\text{ideal fcast density}} - \overbrace{\int p(y_{t+1} \mid \theta, y_{1:t}) p(\theta \mid y_{1:s}) d\theta}^{\text{our forecast}}$$
$$= \int p(y_{t+1} \mid \theta, y_{1:t}) \left\{ \frac{p(\theta \mid y_{1:t})}{p(\theta \mid y_{1:s})} - 1 \right\} p(\theta \mid y_{1:s}) d\theta \qquad (1)$$

- Using a prior is not encouraged (always make sure to examine forecasts produced!).
- For fixed s, $\frac{p(\theta|y_{1:t})}{p(\theta|y_{1:s})}$ (probably) grows in t.
- Occasionaly updating posterior is better than never updating.
- Easy to extend to averaging over models, too? (future work)

Numerical Experiments

The S&P 500 Index



What about options contracts on this?

Options on the S&P 500 Index

1. a **call (put) option** is a derivative contract that allows one to buy (sell) a fixed quantity at some **strike price.**
2. you can either buy or sell this contract
3. It has a finite lifetime–it expires on the **expiration date**.
4. you can buy and sell these contracts, or you can **exercise** the privelege of buying (selling) the underlying shares**.**
5. **European style** options will only allow you to exercise on the expiration date (SPX)**.**
6. **American style** options allow an exercise at any time before expiration (SPY)**.**

Options: A Few Takeaways From Mathematical Finance

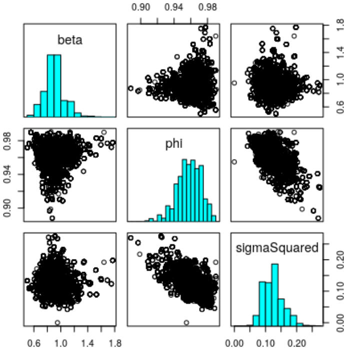
The option's price should be tightly linked with the price of the SPY, but we are not doing **pricing** here. We're not trying to come up with a hedge and identify arbitrages (if we were we would need SDEs and stochastic calculus)–we want to take bets without trading too actively.
Option prices tend to

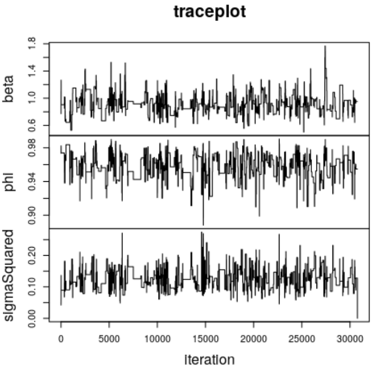
1. decrease in value **over time**
2. change in value if the underlying moves towards or away from the strike (nonlinearly with respect to the underlying)
3. decrease in value if volatility decreases

Running the Particle Swarm

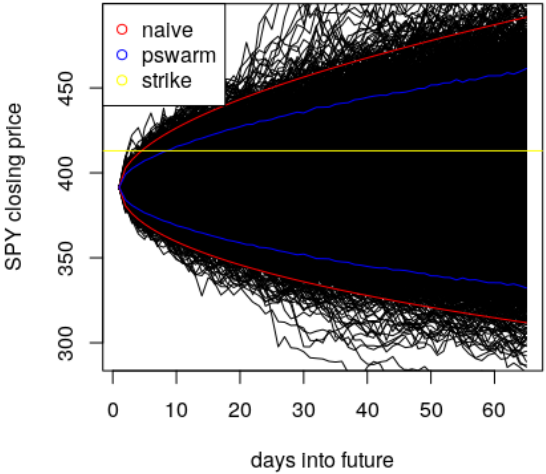
1. Use old posterior samples from a pseudo-marginal Metropolis-Hastings algorithm
2. Sample 100 parameter vectors “from” those
3. Each particle filter maintains 100 state samples
4. Each particle filter runs through the past 807 trading days (2018-01-01 until now)
5. Each particle filter is used to simulate forward in time 64 days (until expiration day)
6. Code: `github.com/tbrown122387/pf` (Brown, 2020), `github.com/tbrown122387/ssme`

Running the Particle Swarm





Running the Particle Swarm



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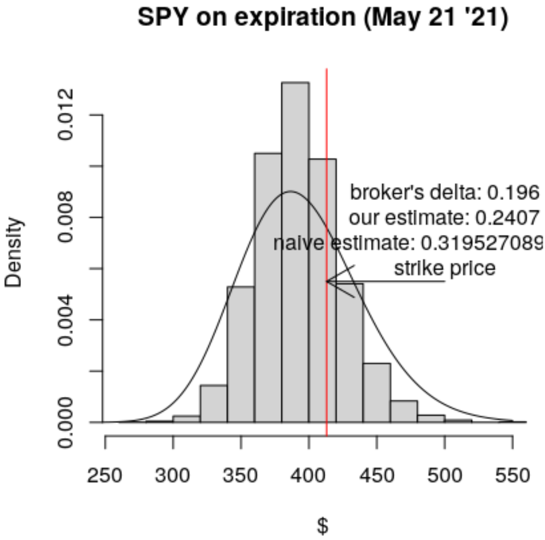

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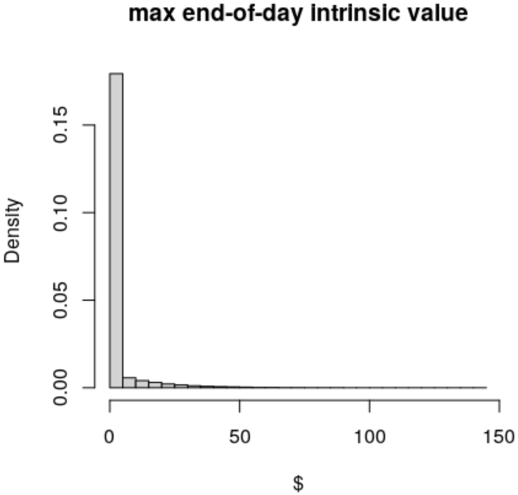
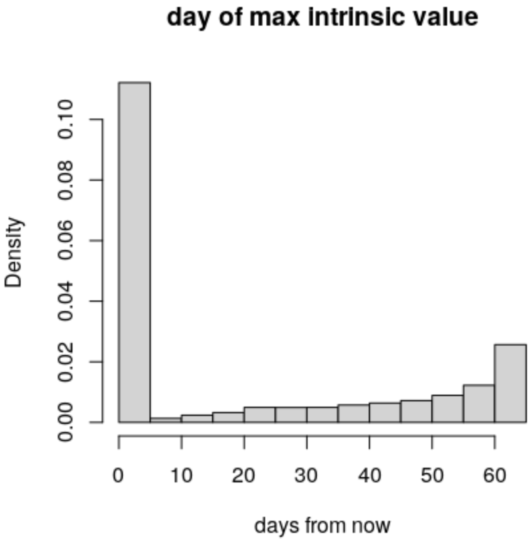


for American-style,

$2.61-$2.68 at March 18 close.

Running the Particle Swarm



Thanks!

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