EGR 8301
Control Systems Engineering

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Homework 3

Problem 1. The following pair of equations corresponds to a linear dynamical system with $f_a(t)$, $t \geq 0$, being an input and $y(t)$, $t \geq 0$, being an output.

$$\begin{aligned} \ddot{y}(t) + 4\dot{y}(t) + 2y(t) &= x(t), \quad y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0, \\ \dot{x}(t) + x(t) + y(t) &= f_a(t), \quad x(0) = x_0. \end{aligned}$$

Define the state variables, derive the state-space form of the model, and determine the system matrices.

Problem 2. The model of a dynamical system is given by

$$\begin{aligned} \dot{x}_1(t) &= -3x_1(t) + 2x_2(t) + u_1(t) + 2\dot{u}_2(t), \quad x_1(0) = x_{10}, \\ \dot{x}_2(t) &= 2x_1(t) + x_2(t) + \dot{u}_1(t), \quad x_2(0) = x_{20} \\ y(t) &= x_1(t) - x_2(t) + u_2(t), \end{aligned}$$

where x_1 and x_2 are the state variables, $u_1(t)$, $u_2(t)$, $t \geq 0$, are the inputs, and $y(t)$, $t \geq 0$, is the output. Redefine the state variables in order to avoid the derivatives on the right hand side of the equations and obtain a new state-space model. Determine the system matrices.

Problem 3. A linear dynamical system with the input $u(t)$, $t \geq 0$, the output $y(t)$, $t \geq 0$, and the state variables x_1 and x_2 is characterized by the equations

$$\begin{aligned} \dot{x}_1(t) + 2\dot{x}_2(t) &= 3x_1(t) + 4x_2(t) - 5u(t), \quad x_1(0) = x_{10}, \\ \dot{x}_1(t) - \dot{x}_2(t) &= 2x_1(t) + x_2(t) + u(t), \quad x_2(0) = x_{20}, \\ y(t) &= \dot{x}_1(t) + 2x_2(t), \end{aligned}$$

Obtain the state-space form of the system and determine the system matrices.

Problem 4. Consider the dynamical system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -600 & -100 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0.$$

Obtain the transfer function for the system.

Problem 5. Obtain e^{At}, $t \geq 0$, where

$$A = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix}.$$

Problem 6. Consider the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Obtain the response $x(t)$, $t \geq 0$, when $u(t)$, $t \geq 0$, is a unit-step function. Assume $x(0) = 0$.

Problem 7. Consider the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0.$$

Assume that matrix A is nonsingular. Show that the response $x(t)$, $t \geq 0$, to a unit-ramp input $u(t) = t$ is given by

$$x(t) = e^{At}x(0) + [A^{-2}(e^{At} - I) - A^{-1}t]B, \quad t \geq 0.$$

Problem 8. For Problem 6, plot the response $x(t)$, $t \geq 0$; that is, $x_1(t)$ and $x_2(t)$ versus time, using MATLAB®.

Problem 9. Consider the single degree-of-freedom spring-mass oscillator shown in Figure 1. Write the dynamic equations of motion in state space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Assume the measured output is the position of the oscillator mass and $k = 4$ N/m and $m = 1$ kg.

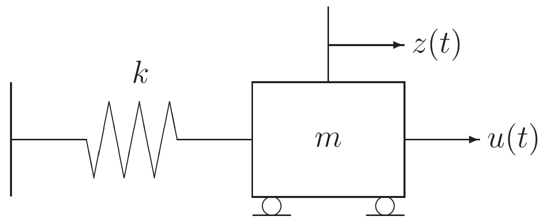

Figure 1: Spring-Mass-System

i) For $u(t) \equiv 0$ plot $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$ versus time using MATLAB®. Assume $z(0) = 1$ and $\dot{z}(0) = 1$.

ii) For the control input $u(t) = Kx(t) = [k_1\ k_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, choose values for k_1 and k_2 (numerically) so that $x(t)$ (and hence $x_1(t)$ and $x_2(t)$) goes to zero as $t \to \infty$. Plot your results using MATLAB®. This shows that you can pick your "control" force $u(t)$, $t \geq 0$, to drive the state to zero.

Problem 10. Consider the dynamical system given by

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0,$$

where $A \in \mathbb{R}^{2\times 2}$ is given by

$$A = \begin{bmatrix} -4 & -1 \\ 4.5 & -1 \end{bmatrix}.$$

Show that this system is asymptotically stable by verifying that the eigenvalues of $A \in \mathbb{R}^{2\times 2}$ have negative real parts and by showing that $e^{At} \to 0$ as $t \to \infty$. Choose several initial conditions (e.g., $x_0 = [2, 3]^{\mathrm{T}}$, $x_0 = [-5, 1]^{\mathrm{T}}$, or $x_0 = [10, -15]^{\mathrm{T}}$) to plot $x_1(t)$ and $x_2(t)$ versus time using MATLAB® and observe that $x_1(t) \to 0$ and $x_2(t) \to 0$ as $t \to \infty$.

Problem 11. Let $A \in \mathbb{R}^{2\times 2}$. Show that A is asymptotically stable if and only if $\mathrm{tr}A < 0$ and $\det A > 0$. Show this in two ways:

i) Look at the eigenvalues of A.

ii) Using Routh-Hurwitz criterion.

Note: "If and only if" statement is a necessary and sufficient condition and **must** be verified in both directions. That is, "if Condition 1 holds, then Condition 2 is satisfied" and "if Condition 2 holds, then Condition 1 is satisfied."

Problem 12. Show that if $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \mathrm{spec}\,(A)$, then $e^{At} \to 0$ as $t \to \infty$. Demonstrate this fact with a MATLAB® example.

Problem 13. If $A \in \mathbb{R}^{n\times n}$, show that

$$\det e^{At} = e^{\mathrm{tr} At}, \quad t \geq 0.$$

Do some numerical examples.

Problem 14. Consider the motion of the Van der Pol oscillator described by

$$\ddot{x}(t) + \mu(1 - x^2(t))\dot{x}(t) + x(t) = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0,$$

where $\mu \in \mathbb{R}$. Obtain the equilibrium points and determine their stability for all values of the parameter μ.

Problem 15. Consider a version of Lotka-Volterra model for a single-predator single-prey system given by

$$\begin{aligned} \dot{x}_1(t) &= \alpha x_1(t) - \gamma x_1(t)x_2(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \\ \dot{x}_2(t) &= \delta x_1(t)x_2(t) - \beta x_2(t) - \theta x_2^2(t), \quad x_2(0) = x_{20}, \end{aligned}$$

where x_1, x_2 are the populations of prey and predator, respectively, $\alpha > 0$ is the birth rate of the prey, $\beta > 0$ is the death rate of the predator, $\gamma > 0$ and $\delta > 0$ represent interactions between the two species, and $\theta > 0$ is the death rate of the predator due to direct competition within the predator population. Obtain all equilibrium points for this system and determine their stability.

Note. State variables x_1 and x_2 represent the number of species, and hence, cannot take negative values.

Extra Problems

Problem E1. Automobile suspension system: Consider the spring mass system in Figure 2 which describes part of the suspension system of an automobile. The data for this system are given as

- $m_1 = \frac{1}{4} \times \text{mass of automobile} = 375$ kg,
- $m_2 = \text{mass of one wheel} = 30$ kg,
- $k_1 = \text{spring constant} = 1500$ N/m,
- $k_2 = \text{linear spring constant of tire} = 6500$ N/m,
- $c = \text{damping constant of dashpot} = 0, 375, 750,$ and 1125 N·s/m,
- $x_1 = \text{displacement of automobile body from equilibrium position}$, m,

- $x_2 =$ displacement of wheel from equilibrium position, m,
- $\nu =$ velocity of car $= 9, 18, 27,$ or 36 m/sec.

A linear model $\dot{x}(t) = Ax(t) + Bu(t)$ for this system is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & -\frac{c}{m_1} & \frac{k_1}{m_1} & \frac{c}{m_1} \\ \frac{k_1}{m_2} & \frac{c}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix} u(t),$$

where $u(t) = \frac{1}{6}\sin(2\pi\nu t/20)$ describes the profile of the roadway.

a) Determine the eigenvalues of A for all above cases.

b) Plot the states for $t \geq 0$ when the input $u(t) = \frac{1}{6}\sin(2\pi\nu t/20)$ and $x(0) = [0, 0, 0, 0]^{\mathrm{T}}$ for all above cases. Comment your results.

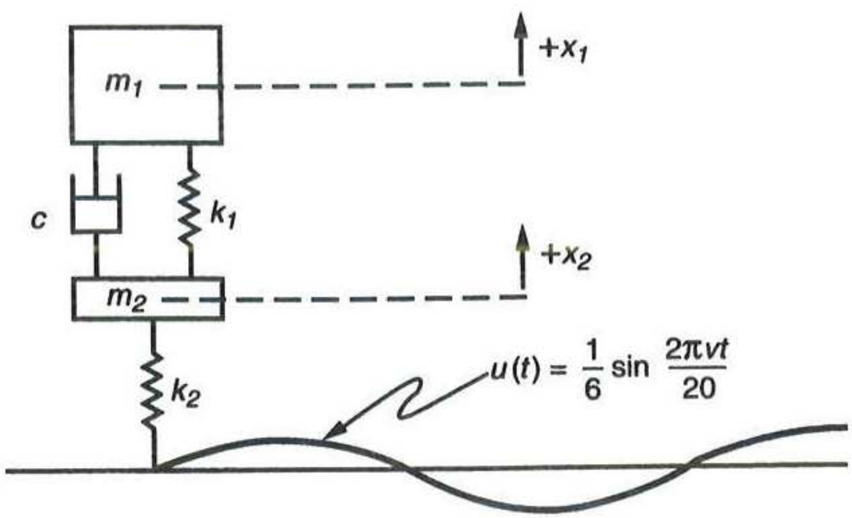

Figure 2: Model of an automobile suspension system.

Problem E2. Armature voltage-controlled DC servomotor: Figure 3 represents a simplified model of an armature voltage-controlled DC servomotor consisting of a stationary field and a rotating armature and load. With the values of the system parameters being $R_a = 2$, $L_a = 0.5$, $J = 1$, $B = 1$, $K_T = 2$, and $K_\theta = 1$, the state-space form of the system dynamics is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} e_a(t),$$

where $x_1 = \theta$ is the shaft position, $x_2 = \dot{\theta}$ is the angular velocity, $x_3 = i_a$ is the armature current, and the input $u = e_a$ is the armature voltage.

a) Determine the eigenvalues and eigenvectors of A and express $x(t)$ in terms of the modes and the initial conditions of the system when $e_a = 0$.

b) Plot the states for $t \geq 0$ when the input e_a is the unit step, and $x(0) = [0, 0, 0]^{\mathrm{T}}$. Comment your results.

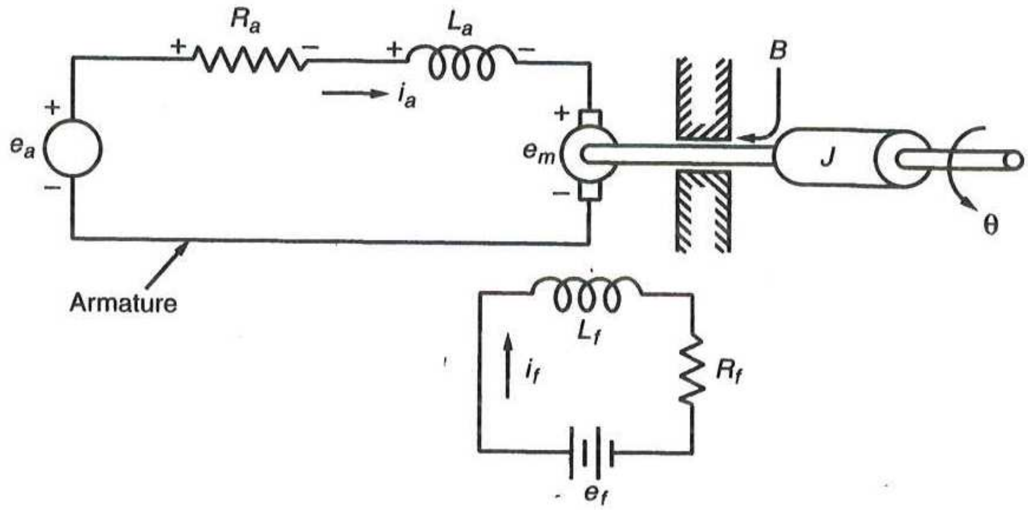

Figure 3: Electromechanical system.

Problem E3. Magnetic ball suspension system: Figure 4 depicts a schematic diagram of a ball suspension control system. The steel ball is suspended in air by the electromagnetic force generated by the electromagnet. The objective of the control is to keep the steel ball suspended at a desired equilibrium position by controlling the current $i(t)$ in the magnet coil by means of the applied voltage $v(t)$, $t \geq 0$. It can be shown that under certain simplifying assumptions, a linearized model of this system is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{2Ki_{eq}}{Ms_{eq}^2} & \frac{2Ki_{eq}^2}{Ms_{eq}^3} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} v(t).$$

$$y(t) = [0, 1, 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

A typical set of parameters is $s_{eq} = 0.01$ m, $i_{eq} = 0.125$ A, $M = 0.01058$ kg, $K = 6.5906 \times 10^{-4}$ N m^2/A^2, $R = 31.1\ \Omega$, and $L = 0.1097$ H.

a) Determine the eigenvalues and eigenvectors of A.

b) Compute transfer function.

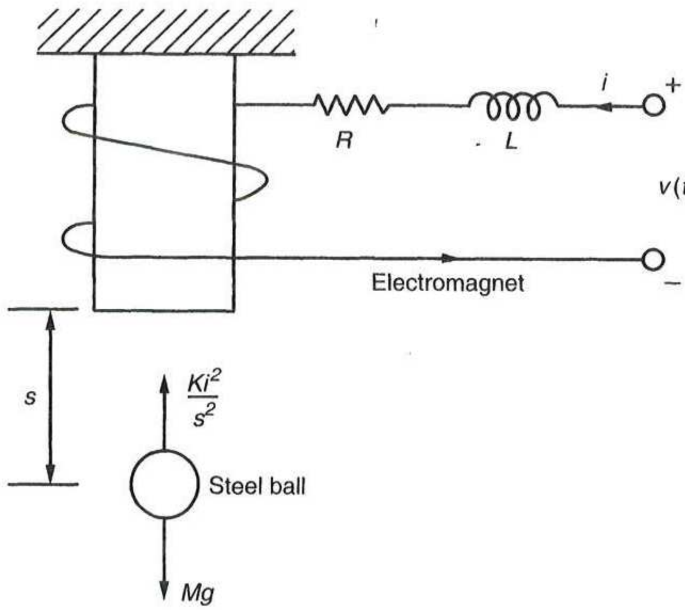

Figure 4: Magnetic ball suspension system.

c) Plot the states for $t \geq 0$ if the ball is slightly higher than the equilibrium position, namely, if $x(0) = [0, 0.0025, 0]^{\mathrm{T}}$ with $v(t) \equiv 0$. Comment your results.

Problem E4. Building subjected to an earthquake: A three-story building is modeled by a lumped mass system as shown in Figure 5. For ground acceleration $\ddot{v}$, the differential equations of motion in terms of mass displacements $[q_1, q_2, q_3]$ relative to the ground are given in state-space form as follows

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \\ \dot{x}_6(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{2c}{m_1} & \frac{k_2}{m_1} & \frac{c}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{c}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{2c}{m_2} & \frac{k_3}{m_2} & \frac{c}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{c}{m_3} & -\frac{k_3}{m_3} & -\frac{c}{m_3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} u(t),$$

where $x_1 \triangleq q_1$, $x_2 \triangleq \dot{q}_1$, $x_3 \triangleq q_2$, $x_4 \triangleq \dot{q}_2$, $x_5 \triangleq q_3$, $x_6 \triangleq \dot{q}_3$, and $u = \ddot{v}$. Let $k = 3.5025 \times 10^8$ N/m, $m = 1.0508 \times 10^6$ kg, and $c = 4.2030 \times 10^5$ N sec/m. Investigate the dynamic response of the structure due to the ground acceleration u for $\tau = 0.4$, 0.6, and 0.8 sec (see Figure5). In particular,

a) Plot the distortions

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 - x_1 \\ x_5 - x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{c} x_1, x_2, x_3, x_4, x_5, x_6 \end{array} \right]^{\mathrm{T}}.$$

If serious damage occurs when a distortion exceeds 0.08 m, will the given ground acceleration due to the earthquake cause serious damage to the building?

b) Repeat a) for different values of the damping parameter c. In particular, let $c_{new} = \alpha c_{old}$, where $\alpha = 2, 3, 10$, and repeat a) for each value of α. Also, determine the eigenvalues of A for each α and comment of your results.

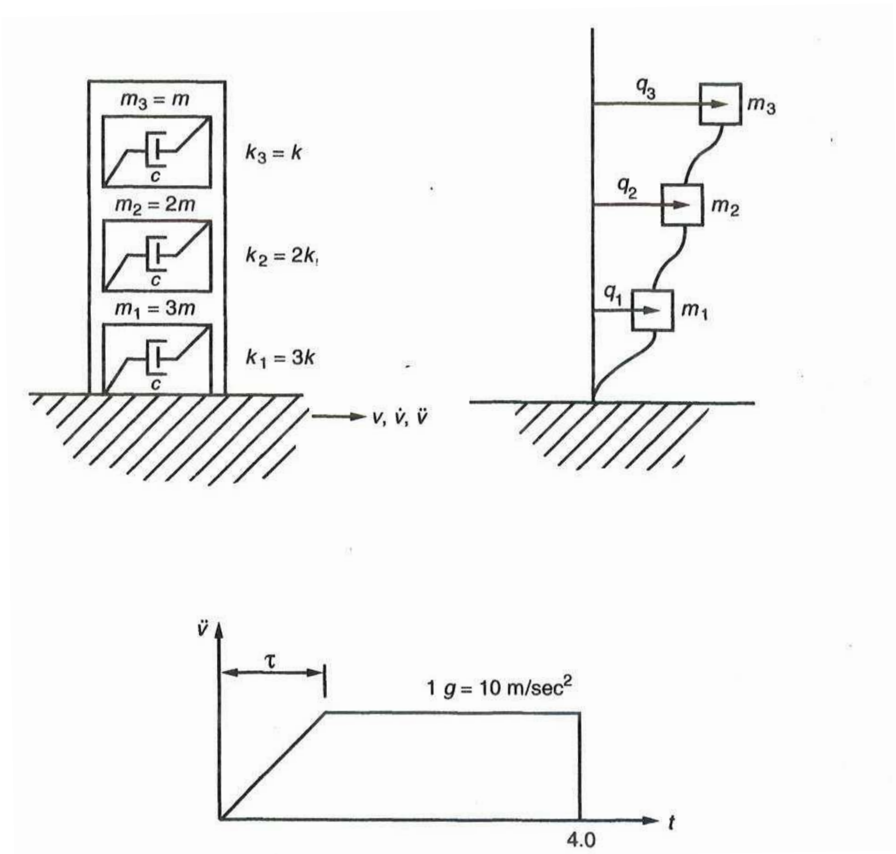

Figure 5: A model for the dynamics of a three-story structure.

Problem E5. Aircraft dynamics: For purposes of control system design, aircraft dynamics are frequently linearized about some operating condition, called a *flight regime*, where it is assumed that the aircraft velocity (Mach number) and attitude are constant. The control surfaces and engine thrust are set, or *trimmed*, to those conditions and the control system is designed to maintain these conditions, i.e., to force perturbations (deviations) from these conditions to zero.

It is customary to separate the longitudinal motion from the lateral motion since in many cases the longitudinal and lateral dynamics are only lightly coupled. As a consequence of this, the control system can be designed by considering each channel independently.

Aircraft longitudinal motion. Consider the numerical data for an actual aircraft, the AFTI-16 (a modified version of the F-16 fighter) in the landing approach configuration (speed

V=139 mph). The components of the state-space equation that describe the longitudinal motion are given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -0.0507 & -3.861 & 0 & -32.2 \\ -0.00117 & -0.5164 & 1 & 0 \\ -0.000129 & 1.4168 & -0.4932 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -0.0717 \\ -1.645 \\ 0 \end{bmatrix} u(t),$$

where the control input $u = \delta_E$ is the elevator angle and the state variables in the vector $x \triangleq [\Delta u, \alpha, q, \theta]^{\mathrm{T}}$ are the change in speed, angle of attack, pitch rate, and pitch, respectively.

The longitudinal modes of the aircraft are called *short period* and *phugoid*. The phugoid eigenvalues, which are a pair of complex conjugate eigenvalues close to the imaginary axis, cause phugoid motion, which is a slow oscillation in altitude.

a) For the state-space model that describes the aircraft longitudinal motion, determine the eigenvalues and eigenvectors of A. Express $x(t)$, when $u = 0$, in terms of the initial conditions and the modes of the system.

b) Let the elevator deflection δ_E be -1 for $t \in [0, T]$ and zero afterwards, where T may be taken to be the sampling period in your simulation. This corresponds to the maneuver made when the pilot pulls back on the stick to raise the nose of the airplane. (The minus sign conventionally represents pulling the stick back). The elevator must be restored to its original position when the desired new climbing angle is reached or the airplane will keep rotating. Plot the state for $x(0) = [0, 0, 0, 0]^{\mathrm{T}}$ and comment on your results.

c) Plot the states for $x(0) = [0, 0, 0, 0]^{\mathrm{T}}$ using a negative unit step as the elevator input. This happens when the elevator is reset to a new position in the hope of pitching the plane up *and* climbing. Comment on your results.

Aircraft lateral motion. Consider the lateral motion of a fighter aircraft traveling at a certain speed and altitude with state-space model given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -0.746 & 0.006 & -0.999 & 0.0369 \\ -12.9 & -0.746 & 0.387 & 0 \\ 4.31 & 0.024 & -0.174 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0.0012 & 0.0092 \\ 6.05 & 0.952 \\ -0.416 & -1.76 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

where the control inputs $[u_1, u_2]^{\mathrm{T}} \triangleq [\delta_A, \delta_R]^{\mathrm{T}}$ denote the aileron and rudder deflections, respectively, and the state variables in the vector $x \triangleq [\beta, p, r, \phi]^{\mathrm{T}}$ are the side-slip angle, roll rate, yaw rate, and roll angle, respectively.

a) The eigenvalues for the aircraft lateral motion consist typically of two complex conjugate eigenvalues with relatively low damping, and two real eigenvalues. The modes caused by complex eigenvalues are called *dutch-roll*. One real eigenvalue, relatively far from the origin, defines a mode called *roll subsidence*, and a real eigenvalue near the origin defines a spiral mode. The spiral mode is sometimes unstable (spiral divergence). Find the modes for the aircraft lateral motion.

b) Plot the states when $x(0) = [0, 0, 0, 0]^{\mathrm{T}}$, u_1 is the unit step and $u_2 = 0$. Repeat for $u_1 = 0$ and u_2 the unit step. Comment on your results.

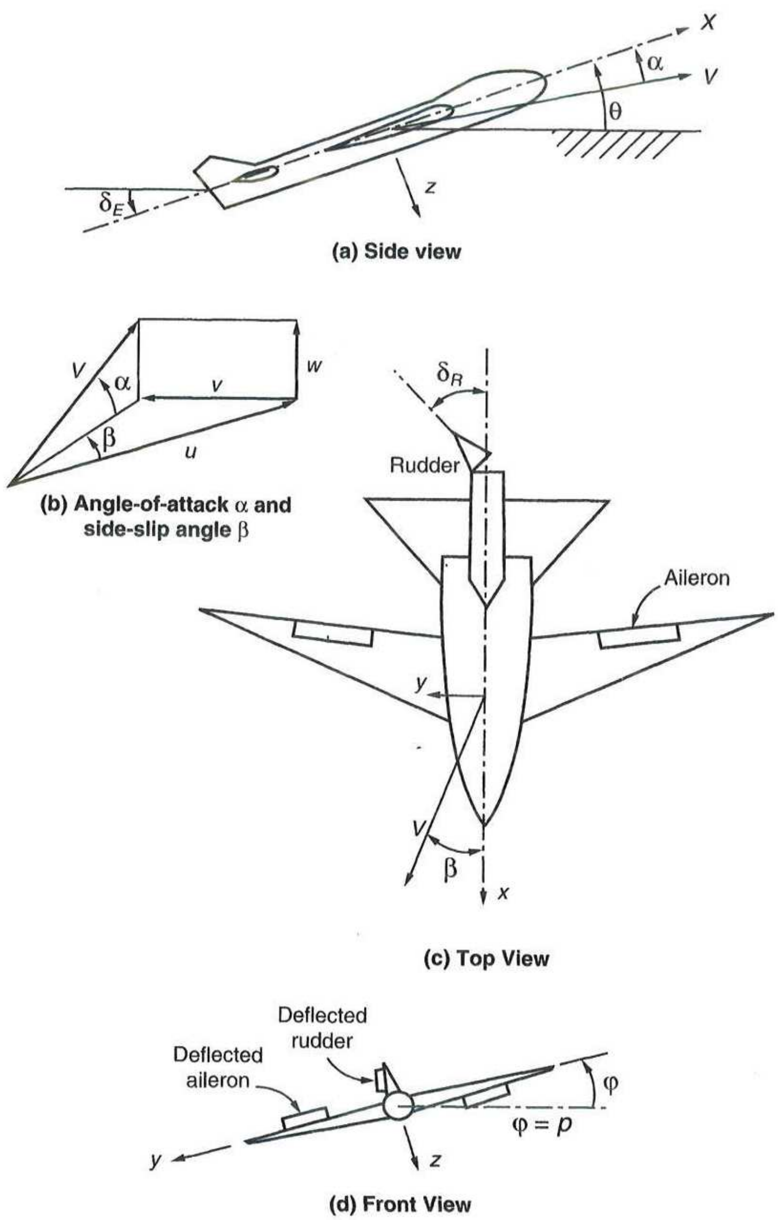

Figure 6: Aircraft dynamics.