EGR 8301
Control Systems Engineering

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Homework 4

Problem 1. Show that (A, B) is controllable (respectively, stabilizable) if and only if (A, BB^{T}) is controllable (respectively, stabilizable).

Hint: Use the fact that for $X \in \mathbb{R}^{n\times m}$, $\operatorname{rank} X = \operatorname{rank} XX^{\mathrm{T}} = \operatorname{rank} X^{\mathrm{T}}X$.

Problem 2. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that (A, B) is controllable if and only if $(A + \alpha I_n, B)$ is controllable for all $\alpha \in \mathbb{R}$.

Problem 3. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that (A, B) is controllable (respectively, stabilizable) if and only if $(A + BK, B)$ is controllable (respectively, stabilizable) for all $K \in \mathbb{R}^{m\times n}$.

Problem 4. Show that if $A \in \mathbb{R}^{n\times m}$ with $m \geq n$, then $AA^{\mathrm{T}} > 0$ if and only if A is a full rank matrix, that is, $\operatorname{rank} A = n$.

Problem 5. Determine the uncontrollable modes of each pair (A, B) given below

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem 6. Reduce the pair

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & 0 & -3 & 1 \\ -1 & 1 & 4 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

into controller form $A_{\mathrm{c}} = PAP^{-1}$, $B_{\mathrm{c}} = PB$. What is the transformation matrix in this case? What are the controllability indices?

Problem 7. For the system $\dot{x}(t) = Ax(t) + Bu(t)$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

determine $K \in \mathbb{R}^{2\times 4}$ so that the eigenvalues of $A + BK$ are at $-1 \pm j$ and $-2 \pm j$. Find at least two forms of K.

Problem 8. Consider the simplified equations of motion of a rigid body spacecraft in the state space form

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

Find the optimal control $u(t) = Kx(t)$ such that

$$J(x_0, K) = \int_0^\infty [x^{\mathrm{T}}(t)R_1 x(t) + u^{\mathrm{T}}(t)R_2 u(t)]\mathrm{d}t$$

is minimized, where $R_1 = \begin{bmatrix} r_{\mathrm{d}} & 0 \\ 0 & r_{\mathrm{v}} \end{bmatrix}$ and $R_2 = 1$. Compute the closed-loop dynamics and obtain the natural frequency ω_n and the damping ratio ξ of the closed-loop system in terms of r_{d} and r_{v}.

Problem 9. Consider the position control of a rotating inertia I with viscous friction B shown in the Figure below. Assume that the torque $T(t)$ is applied by a DC motor which converts an electrical input voltage $e_{\mathrm{a}}(t)$ into an output torque via the relationship $T(t) = K_{\mathrm{T}} e_{\mathrm{a}}(t)$. Using the data $B/I = 10$ and $K_{\mathrm{T}}/I = 1$, obtain

i) The state space description of the system.

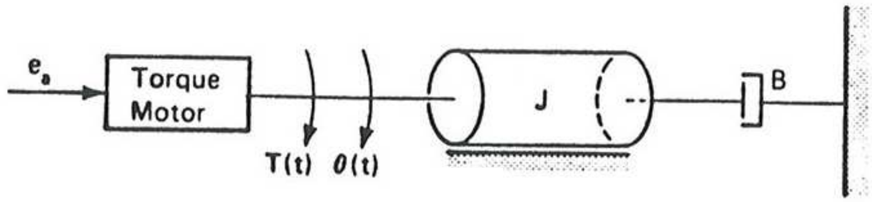
ii) The optimal control $u(t) = Kx(t)$ such that

$$J(x_0, K) = \int_0^\infty [x^{\mathrm{T}}(t) \begin{bmatrix} r_{\mathrm{v}} & 0 \\ 0 & 0 \end{bmatrix} x(t) + u^2(t)]\mathrm{d}t$$

is minimized.

iii) The closed-loop system poles.

iv) The natural frequency ω_n and the damping ratio of the closed-loop system in terms of r_{v}.



Problem 10. Consider the third-order dynamical system

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

with the cost functional

$$J(\cdot) = \int_0^\infty [x^{\mathrm{T}}(t) R_1 x(t) + u^{\mathrm{T}}(t) R_2 u(t)] \mathrm{d}t$$

where $R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $R_2 = 1$.

i) Is this system controllable? Is this system stabilizable?

ii) Using an LQR design, obtain the optimal gain $K \in \mathbb{R}^{1\times 3}$ and the closed-loop eigenvalues.

Hint: Use ARE command in MATLAB.