Appendix A
Sea Water Density According to UNESCO Formula

Density of sea water for a given temperature T in the range $0 < T < 40\,^\circ\mathrm{C}$, salinity S in the range $0 < S < 42\,\mathrm{PSU}$ and pressure p is determined from the equation of state (UNESCO 1981):

$$\rho = \frac{\rho(S, T, 0)}{1 - \dfrac{p}{K(S, T, p)}} \tag{A.1}$$

where $K(T, S, p)$ is module of sea water compressibility.

Sea water density is determined using the following algorithm:

- calculation of the SMOW density (*Standard Mean Ocean Water*):

$$\rho_{SMOW} = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4 + a_5 T^5 \tag{A.2}$$

where:

$$\left.\begin{array}{rcr} a_0 &=& 999.842\,594 \\ a_1 &=& 6.793\,953 \times 10^{-2} \\ a_2 &=& -9.095\,290 \times 10^{-3} \\ a_3 &=& 1.001\,685 \times 10^{-4} \\ a_4 &=& -1.120\,083 \times 10^{-6} \\ a_5 &=& 6.536\,332 \times 10^{-9} \end{array}\right\} \tag{A.3}$$

- calculation of sea water density at the normal atmospheric pressure (*normal atmosphere*) ($p = 0$):

$$\rho(S, T, 0) = \rho_{SMOW} + B_1 S + C_1 S^{1.5} + d_0\, S^2 \tag{A.4}$$

with:

$$B_1 = b_0 + b_1 T + b_2 T^2 + b_3 T^3 + b_4 T^4 \tag{A.5}$$

S.R. Massel, *Internal Gravity Waves in the Shallow Seas*,
GeoPlanet: Earth and Planetary Sciences, DOI 10.1007/978-3-319-18908-6

$$C_1 = c_0 + c_1 T + c_2 T^2 \tag{A.6}$$

where:

$$\left.\begin{aligned} b_0 &= 8.2449 \times 10^{-1} \\ b_1 &= -4.0899 \times 10^{-3} \\ b_2 &= 7.6438 \times 10^{-5} \\ b_3 &= -8.2467 \times 10^{-7} \\ b_4 &= 5.3875 \times 10^{-9} \end{aligned}\right\} \tag{A.7}$$

$$\left.\begin{aligned} c_0 &= -5.7246 \times 10^{-3} \\ c_1 &= 1.0227 \times 10^{-4} \\ c_2 &= -1.6546 \times 10^{-6} \\ d_0 &= 4.8314 \times 10^{-4} \end{aligned}\right\} \tag{A.8}$$

- determination of the compressibility module at pressure $p = 0$:

$$K(S, T, 0) = K_w + F_1 S + G_1 S^{1.5} \tag{A.9}$$

coefficients K_w, F_1 and G_1 are given by:

$$K_w = e_0 + e_1 T + e_2 T^2 + e_3 T^3 + e_4 T^4 \tag{A.10}$$

where:

$$\left.\begin{aligned} e_0 &= 19\,652.210\,000 \\ e_1 &= 148.420\,600 \\ e_2 &= -2.327\,105 \\ e_3 &= 1.360\,477 \times 10^{-2} \\ e_4 &= -5.155\,288 \times 10^{-5} \end{aligned}\right\} \tag{A.11}$$

$$F_1 = f_0 + f_1 T + f_2 T^2 + f_3 T^3 \tag{A.12}$$

where:

$$\left.\begin{aligned} f_0 &= 54.674\,600 \\ f_1 &= -0.603\,459 \\ f_2 &= 1.099\,870 \times 10^{-2} \\ f_3 &= -6.167\,000 \times 10^{-5} \end{aligned}\right\} \tag{A.13}$$

and

$$G_1 = g_0 + g_1 T + g_2 T^2 \tag{A.14}$$

where:

$$\left.\begin{aligned} g_0 &= 7.9440 \times 10^{-2} \\ g_1 &= 1.6483 \times 10^{-2} \\ g_2 &= -5.3009 \times 10^{-4} \end{aligned}\right\} \tag{A.15}$$

- determination of final compressibility module of the sea water:

$$K(S, T, p) = K(S, T, 0) + A_1 p + B_2 p^2 \tag{A.16}$$

with:

$$A_1 = A_w + (i_0 + i_1 T + i_2 T^2)\, S + j_0\, S^{1.5} \tag{A.17}$$

$$A_w = h_0 + h_1 T + h_2 T^2 + h_3 T^3 \tag{A.18}$$

where:

$$\left.\begin{aligned} h_0 &= 3.239\,90 \\ h_1 &= 1.437\,13 \times 10^{-3} \\ h_2 &= 1.160\,92 \times 10^{-4} \\ h_3 &= -5.779\,05 \times 10^{-7} \end{aligned}\right\} \tag{A.19}$$

$$\left.\begin{aligned} i_0 &= 2.283\,80 \times 10^{-3} \\ i_1 &= -1.098\,10 \times 10^{-5} \\ i_2 &= -1.607\,80 \times 10^{-6} \\ j_0 &= 1.910\,75 \times 10^{-4} \end{aligned}\right\} \tag{A.20}$$

and

$$B_2 = B_w + (m_0 + m_1 T + m_2 T^2)\, S \tag{A.21}$$

where:

$$B_w = k_0 + k_1 T + k_2 T^2 \tag{A.22}$$

with:

$$\left.\begin{aligned} k_0 &= 8.509\,35 \times 10^{-5} \\ k_1 &= -6.122\,93 \times 10^{-6} \\ k_2 &= 5.278\,70 \times 10^{-8} \end{aligned}\right\} \tag{A.23}$$

$$\left.\begin{aligned} m_0 &= -9.9348 \times 10^{-7} \\ m_1 &= 2.0816 \times 10^{-8} \\ m_2 &= 9.1697 \times 10^{-10} \end{aligned}\right\} \tag{A.24}$$

- final determination of the sea water density $\rho(S, T, p)$ from Eq. (A.1):

Example 1: salinity $S = 8$ PSU, temperature $T = 10\,°\mathrm{C}$, pressure $p = 0$ bar.
As the pressure $p = 0$, the density $\rho(S, T, 0)$ is given by Eq. (A.4). Thus we have:

– $\rho_{SMOW} = 999.702\,09$ from Eq. (A.2),
– $\rho_w(8, 10, 0) = 1005.946\,59$ from Eq. (A.4).

Example 2: salinity $S = 8$ PSU, temperature $T = 10\,°\mathrm{C}$, pressure $p = 10$ bar.
Similarly to example above we obtain:

– $\rho_{SMOW} = 999.702\,09$ from Eq. (A.2),
– $\rho(S, T, 0) = 1005.946\,59$ from Eq. (A.4),
– $K_w = 20\,916.794\,90$ from Eq. (A.10),
– $A_1 = 3.285\,74$ from Eq. (A.17),
– $B_2 = 0.000\,20$ from Eq. (A.21),
– $K = 21\,351.408\,20$ from Eq. (A.16),
– $\rho(8, 10, 10) = 1006.417\,97$ from Eq. (A.1).

Reference

UNESCO (1981) Tenth report of the joint panel on oceanographic tables and standards. UNESCO Technical Papers in Marine Science, Paris, 25 p

Appendix B
Determination of Isopycnal Displacements by Moving CTD Sensors

The method of up and down movement of the towed CTD sensor has been applied for transects in the Southern Baltic waters for many years. When a ship r/v *Oceania* moves with velocity C_s, the CTD sensor falls from the sea surface to the sea bottom and rises from bottom to surface, and again moves down to the sea bottom in the repeated cycles (see Fig. B.1). At each station, when CTD sensor was on the sea surface (or on the sea bottom), the latitude φ and longitude λ, as well as the time t were known. Between two following stations, the ship velocity was only approximately constant. To find the local ship velocity, let us transform the geographic coordinates (φ, λ) into local Cartesian coordinates (x, y), where x is coordinate along east-west direction and y is coordinate along south-north direction.

As the Earth is not a perfect sphere, the radius of the main circle inclined at $\varphi = 55.25°$ becomes:

$$R(\varphi) = \sqrt{\frac{(a^2 \cos\varphi)^2 + (b^2 \sin\varphi)^2}{(a \cos\varphi)^2 + (b \sin\varphi)^2}} \tag{B.1}$$

where $a = 6378.1370$ km is an equator radius and $b = 6356.7523$ km is a polar radius. For $\varphi = 55.25°$ from Eq. (B.1) we obtain $R(55.25°) = 6363.7398$ km. Using the $R(55.25°)$ value we obtain the following relationships between geographical and local Cartesian coordinates:

$$\Delta\lambda = 1° = \frac{\pi \cdot R(55.25°) \cdot \cos(55.25°)}{180} = 63.309\,\text{km}$$

or

$$\Delta\lambda = 1' = \frac{\pi \cdot R(55.25°) \cdot \cos(55.25°)}{60 \cdot 180} = 1.055\,\text{km}$$

S.R. Massel, *Internal Gravity Waves in the Shallow Seas*,
GeoPlanet: Earth and Planetary Sciences, DOI 10.1007/978-3-319-18908-6

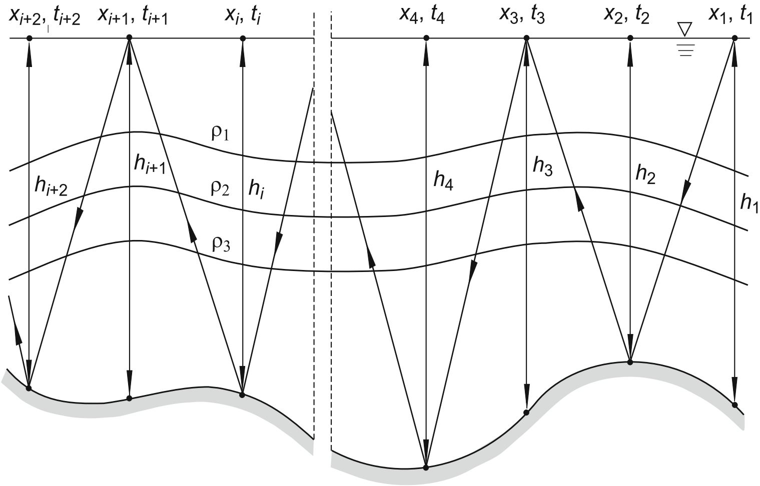

Fig. B.1 CTD sensor zig-zag movement through the water column

and

$$\Delta\varphi = 1^\circ = \frac{\pi b}{180} = 110.946\,\mathrm{km}$$

or

$$\Delta\varphi = 1' = \frac{\pi b}{60 \cdot 180} = 1.849\,\mathrm{km}$$

The ship moved approximately along latitude $\varphi = 55.25^\circ$ and the deviation from this line was around $\pm 3'$ during the whole transect. Therefore for simplicity we neglect this deviation and assume that the transect coincides with the latitude $\varphi = 55.25^\circ$.

The mean local ship velocity between stations i and $i + 1$ is:

$$C_{i,i+1} = \frac{\Delta\lambda}{\Delta t} = \frac{1055 \cdot (\lambda_{i+1} - \lambda_i)}{60 \cdot (t_{i+1} - t_i)} \tag{B.2}$$

where longitude λ_i is expressed in geographical minutes, time t_i is in minutes, and ship velocity is in $\mathrm{m\,s^{-1}}$.

To assess the position of the given isopycnal displacements in the water column, values of temperature T, salinity S or density ρ at a given point $P_{i,i+1}$, located between stations i and $i + 1$, are required. Let us first determine distance $x_{P_{i,i+1}}$ and time $t_{P_{i,i+1}}$, corresponding to the situation when CTD sensor reaches a selected point $P_{i,i+1}$ with required density ρ, moving down from the sea surface. From geometrical relationships (see Fig. B.2) we have:

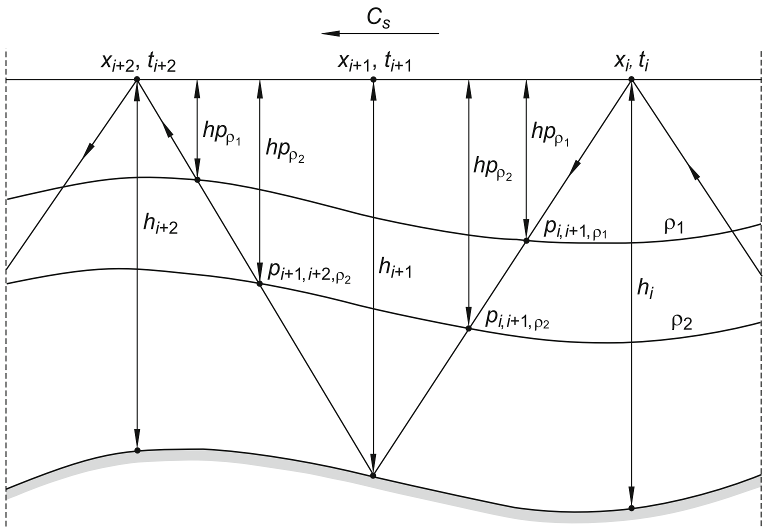

Fig. B.2 CTD sensor crossing the required isopycnal

$$x_{P_{i,i+1}} = \frac{h_{P_{i,i+1}}}{h_{i+1}} (x_{i+1} - x_i) + x_i \tag{B.3}$$

The corresponding time $t_{P_{i,i+1}}$ is given by:

$$t_{P_{i,i+1}} = \frac{h_{P_{i,i+1}}}{h_{i+1}} (t_{i+1} - t_i) + t_i \tag{B.4}$$

Similarly for CTD sensor moving up we obtain:

$$x_{P_{i,i+1}} = x_{i+1} - \frac{h_{P_{i,i+1}}}{h_i} (x_{i+1} - x_i) \tag{B.5}$$

and for time $t_{P_{i,i+1}}$ we have:

$$t_{P_{i,i+1}} = t_{i+1} - \frac{h_{P_{i,i+1}}}{h_i} (t_{i+1} - t_i) \tag{B.6}$$

Using formulae given in the Eqs. (B.3)–(B.6) we can determine functions $T(x, z, t)$ and $S(x, z, t)$, as well as isopycnals $\rho(x, z, t)$ along the experimental transect.

In general, the depth of submergence of a given isopycnal of density ρ is a function of x and t, i.e. $h_\rho = h_\rho(x, t)$. However, when CTD sensor is moving with approximately constant speed C_s, both variables x and t are related as $x = C_s t$. Let us consider the case of a ship moving westward from some point P_0 towards the Słupsk Sill (see Fig. 7.1). The information on the isopycnal displacements can

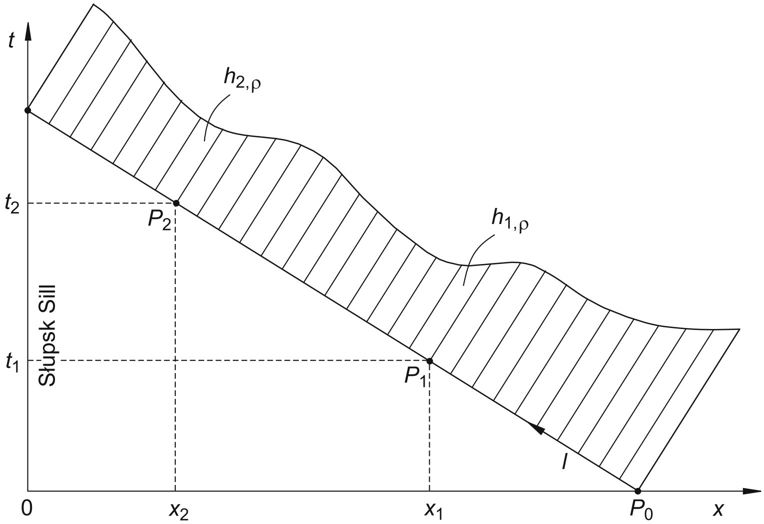

Fig. B.3 Changing of isopycnal displacements along the transect in the two dimensional space (x, t)

be obtained along the cross-section where $x = C_s\, t$. This is schematically illustrated in Fig. B.3. At distance x_1 and corresponding time t_1 the ship is at point P_1 where the given isopycnal ρ is located at water depth $h_{1,\,\rho}$. Similarly, at point P_2 we have isopycnal submergence $h_{2,\,\rho}$. To keep uniform dimensions on both axes, we define the distance l along the cross-section in the two-dimensional space $h = f(x, t)$ as follows:

$$l = \sqrt{[x^2 + (C_s t)^2]} \tag{B.7}$$

It should be noted that dependence of the isopycnal submergence h_ρ on time t is validated by the assumption that internal waves profile in the form:

$$\zeta \approx \exp[i(kx \pm \omega t)] = \exp[ik(x \pm C_0\, t)] \tag{B.8}$$

where C_0 is the wave phase speed and sign $(\pm)$ reflects the possible different wave directions. Using fact that $x = C_s\, t$, we obtain:

$$\zeta \sim \exp[ik(C_s \pm C_0)t] \tag{B.9}$$

For long waves reproduced by solitary waves or by cnoidal waves, the isopycnal displacements are functions of $\cosh(x_1)$ or $cn(x_1)$, where the arguments $x_1 \sim \alpha(x - Ut)$. When the sensor is moving with speed C_s, the arguments become $x_1 \sim \alpha(\pm C_s - U)t$, where parameter α depends on wave amplitude and water depth.

Index

A
Arkona Basin, 132, 142, 143, 150
Aspect ratio, 48
Australian North West Shelf, 4, 6, 87, 88, 111, 120

B
Baltic Proper, 142, 150
Baltic seiches, 134, 135
Beddies, 133
Bornholm Basin, 132
Bornholm Deep, 143
Boussinesq approximation, 13

C
Coefficient
  dispersion, 36, 69, 134
  nonlinear, 36, 69, 134
Complete elliptic integrals, 39, 100
Critical angle, 93, 111

D
Danish Straits, 131
Density structure
  exponential density profile, 46, 51, 65, 122
  other specific densities, 59
  three-layer sea, 53, 58
  two pycnoclines, 64
  two-layer sea, 26
Diapycnal mixing, 83, 84, 133
Dispersion relation, 17, 19, 27, 30, 34, 46, 60, 97, 129

E
Energy flux, 19, 47, 99, 113
Equations
  available potential energy, 80, 85
  continuity, 11, 12, 56
  determinant, 54, 63
  diffusion, 83
  extended Korteweg-de Vries (eKdV), 36, 74
  generalised Korteweg-de Vries, 102
  Hamilton, 96
  hydrostatics, 13
  kinetic energy, 77, 79
  Korteweg-de Vries (KdV), 37, 69
  momentum, 11, 81
  potential energy, 77–79, 81, 83
  rotated extended Korteweg-de Vries (reKdV), 104
  Taylor-Goldstein, 21
  UNESCO formula, 7

F
Fluid element states
  neutral, 10
  stable, 9
  unstable, 10
Frequency
  angular frequency of Earth rotation, 13
  extrinsic, 96
  inertial, 13, 16
  intrinsic, 96
  Väisälä-Brunt frequency, 8, 10, 16, 46, 51, 54, 55, 66, 102
Froude number, 82, 83, 110
Functions
  autocorrelation, 121

S.R. Massel, *Internal Gravity Waves in the Shallow Seas*,
GeoPlanet: Earth and Planetary Sciences, DOI 10.1007/978-3-319-18908-6

complex analytical signal, 127
cosine-elliptic, 40
Dirac’s delta, 121
eigenfunction, 63
instantaneous amplitude, 127
instantaneous frequency, 127
instantaneous phase, 127
Jacobian elliptic, 39, 70, 72
Kronecker delta, 121
stream function, 14, 20, 90, 92
velocity potential function, 14, 26, 27

G
Gdańsk Basin, 143
Gdańsk Deep, 143
Gotland Basin, 132, 134
Gotland Deep, 55, 57, 64, 89, 132, 149
Gulf of Bothnia, 132
Gulf of Finland, 132
Gulf of Gdańsk, 132

H
High gradient zones
halocline, 8
pycnocline, 8
thermocline, 8

I
Indonesian Throughflow, 5
Inner product, 115
Iribarren number, 94
Isolines
isohalines, 8
isopycnals, 8
isotherms, 8
Isopycnal displacements, 50, 72, 76, 122, 141, 157
Isopycnal submergence, 143, 160
Isotherm displacements, 136, 137

L
Landsort Deep, 131
Local-scale energy density, 138

M
Middle Atlantic Bight, 5
Moving CTD sensors, 142

O
Orthogonality condition, 62

R
Ray method, 95
Research vessel *Oceania*, 143, 147, 157
Richardson number, 21
Rossby radius, 116, 149, 150

S
Seas
Andaman, 4, 6, 87, 110
Baltic, 131, 132, 134, 135
Black, 6
Japan, 6
Kaspiyan, 6
North, 132
South China, 5, 105
Sulu, 4, 87
White, 6
Seawater properties
density, 7, 58, 153
potential density, 7, 51
pressure, 7
salinity, 7
temperature, 7
Spectra, 119
frequency, 121, 122, 136
Garrett-Munk, 119, 128
wavenumber, 122
Station Hel, 135
Strait of Gibraltar, 1, 4
Słupsk Furrow, 132, 134, 135, 143
Słupsk Sill, 132, 134, 143

T
Thermistor chain, 135
Transforms
Fourier, 119, 123
Fourier-Stjeltjes, 119, 120
Hilbert, 119, 126, 139
Wavelet, 123, 124, 138
dilation parameter, 124
global energy spectrum, 126
local-scale energy density, 126
Morlet wavelet, 124
mother wavelet, 124
time-scale energy density, 126
total energy, 126
translation parameter, 124

U
UNESCO formula, 139, 153
U.S.S. Thresher tragedy, 32

V
Variational approach, 61, 70
Velocities
  group velocity, 18, 53
  phase velocity, 27, 30, 38, 40, 48, 53, 55, 63, 149
  wave velocity components, 27, 31, 38, 46, 49, 55, 71, 74, 93, 100, 139

W
Water state
  baroclinic, 8
  barotropic, 8
Wave action, 98
Wave amplitude, 100, 101
Wave breaking, 95, 105
Wave generation
  by local disturbances, 109
  wind action, 112
Wave induced mixing, 81
Wave vector, 18, 19, 90, 91
Waves
  cnoidal, 36, 39, 40, 70
  diurnal tides, 6
  inertial, 7
  interfacial, 27, 33, 35, 36
  internal tides, 4
  reflected, 90
  semi-diurnal tides, 6
  short-period, 6
  solitary, 3, 5, 36, 38, 71
Wind stress, 112, 114, 115