Separation Axioms: Characterizations and Properties

Burak Arslan and Tuğçe Aydın

Version of December 23, 2025

Contents

1 Introduction **3**

2 Preliminaries **5**

3 Definitions, Characterizations, and Properties **10**

3.1 Topologically Indistinguishable 10

3.2 T_0 Spaces 11

3.3 R_0 Spaces 13

3.4 T_1 Spaces 15

3.5 R_1 Spaces 17

3.6 T_2 Spaces 19

3.7 $T_{2\frac{1}{2}}$ Spaces 23

3.8 Completely T_2 Spaces 24

3.9 Regular Spaces 25

3.10 T_3 Spaces 27

3.11 Completely Regular Spaces 28

3.12 $T_{3\frac{1}{2}}$ Spaces 30

3.13 Normal Spaces 31

3.14 T_4 Spaces 33

3.15 Completely Normal Spaces 35

3.16 T_5 Spaces 37

3.17 Perfectly Normal Spaces 38

3.18 T_6 Spaces 39

4 Conclusion **40**

References **43**

1. Introduction

One of the central aims of general topology is to formalize the intuitive notion of “closeness” and “separation” without relying on a distance function. Once the concept of a topological space was abstracted in the early twentieth century and defined solely in terms of open sets (or equivalent axiomatic formulations), that is, within a minimal framework consisting only of the most basic and natural axioms required to describe openness and neighborhoods, it became apparent that such a framework is often too weak to recover many classical results from analysis and geometry. In particular, without imposing additional assumptions, distinct points may fail to be distinguishable, limits of sequences or nets need not be unique, and fundamental theorems relying on the separation of points or closed sets by neighborhoods or continuous functions may no longer hold. The systematic study of conditions that control how points, closed sets, and more general subsets can be separated led to the development of what are now called the separation axioms, traditionally referred to in the German literature as Trennungsaxiome.

Historically, the modern theory begins with the work of Fréchet in 1906 [18], who emphasized abstract spaces defined via convergence and neighborhoods, and culminates in Hausdorff’s 1914 treatise “Grundzüge der Mengenlehre” [21]. Hausdorff introduced the requirement that distinct points admit disjoint neighborhoods, a condition that later called as the Hausdorff or T_2 axiom. For a time, this property was regarded as almost synonymous with being a topological space. Subsequent developments, however, revealed that many mathematically significant examples fail to be Hausdorff, while still satisfying weaker separation properties sufficient for large parts of the theory. This realization motivated the formulation and systematic study of weaker axioms.

The resulting hierarchy, commonly denoted by T_0, T_1, T_2, . . ., reflects increasingly strong forms of distinguishability and separation. At the lowest level, T_0 captures the minimal requirement that distinct points are topologically distinguishable. The T_1 condition strengthens this by using that each point admits an open neighborhood excluding the other, and, equivalently, that singletons are closed. Hausdorff separation is then defined as the condition that distinct points have disjoint open neighborhoods. Beyond this point, subsequent axioms address the separation of points from closed sets, or of closed sets from each other, leading to notions such as regularity, completely regularity, normality, completely normality, and perfectly normality, which play a decisive role in function-theoretic results, such as extension theorems.

A subtle but conceptually significance refinement of this hierarchy is provided by the regularity axioms usually denoted R_0 [13,38] and R_1 [13,47]. These properties make explicit the symmetry and preregularity implicit in topological distinguishability and reveal how the classical T-axioms are built from simpler separation principles. Here, R_0 formalizes the idea that each of topologically indistinguishable points admits a neighborhood excluding the other, while R_1 captures a Hausdorff-like separation restricted to pairs of points that are already distinguishable. From this perspective, natural equivalences emerge: T_1 can be decomposed into T_0 together with R_0, and the Hausdorff condition can be viewed as T_0 combined with R_1, as well as T_1 and R_1. These equivalences demonstrates that R_0 is the property that ensures “points are closed” when combined with distinguishability (T_0), while R_1 is the property that ensures “distinguishable points can be separated” when combined with distinguishability (T_0). These refinements are not merely theoretical; they are indispensable when dealing with quotient spaces and other constructions, in which T_0 or T_1 may fail, but meaningful separation phenomena persist.

After 1906, as the theory matured through the contributions of Tietze, Urysohn, Alexandroff, Hopf, and Kolmogorov, attention increasingly shifted toward stronger axioms governing closed sets and continuous functions. Normality addresses the separation of disjoint closed sets by neighborhoods and underlies classical function-theoretic results such as Urysohn’s lemma and the Tietze extension theorem. Complete regularity is conceptually different: it requires that a point and a closed set not containing it can be separated by a real-valued continuous function, a property that is fundamental for the theory of Tychonoff spaces and for describing topological spaces in terms of their families of continuous real-valued functions. Even stronger conditions, traditionally labeled completely and perfectly normal, address hereditary normality and the representation of closed sets as zero sets, (that is, sets of the form $f^{-1}(\{0\})$ for some continuous function defined from the relevant topological space to $[0, 1]$) or G_δ-sets, that is, as intersections of countably many open sets. Although these higher axioms are often omitted from elementary treatments, they are essential for a complete picture of separation phenomena and for understanding the fine structure of non-metrizable spaces.

Finally, it is important to emphasize that the terminology surrounding separation axioms is not entirely uniform across the literature. In particular, the usage of terms such as "regular" and "T_3" varies: some authors incorporate a T_1 assumption into the definition of regularity, while others treat regularity as an independent property and reserve expressions like "regular Hausdorff" or T_3 for the combined condition. Throughout these notes, separation properties are treated as independent notions, and additional hypotheses such as T_0 or T_1 are stated explicitly when required. This approach avoids ambiguity and reflects modern practice in advanced topology.

These lecture notes are designed to provide a coherent conceptual framework for the separation axioms. Emphasis is placed not only on cricual equivalences of the axioms, but also on the logical relationships between them. The goal is to equip the reader with a clear understanding of how these axioms distinguish points and sets within a topological space, ranging from point–point separation to point–set and set–set separation, and how these notions are logically related.

The remainder of these lecture notes is organized as follows: Section 2 introduces the basic notions and terminology used throughout the text. Section 3 presents the concept of topological indistinguishability together with the seventeen separation axioms $\big($$T_0$, R_0, T_1, R_1, T_2, $T_{2\frac{1}{2}}$, completely T_2, regular, T_3, completely regular, $T_{3\frac{1}{2}}$, normal, T_4, completely normal, T_5, perfectly normal, and T_6 spaces$\big)$ considered in this study, formulated in a manner consistent with the adopted terminology. This section also includes equivalences, logical implications, and several fundamental properties associated with these axioms. The final section is devoted to a comparative discussion of the separation axioms. In this context, two diagrams are provided to illustrate the hierarchy of the axioms under no additional assumptions and under specific supplementary conditions, respectively. In addition, a table summarizing which properties are satisfied or not satisfied by the considered separation axioms is included.

All definitions, equivalences, and properties used in this study have been collected from the sources listed in References and presented in a consistent and unified manner. The development of this material is ongoing, and readers who intend to use the results presented here are encouraged to verify the statements directly from the original definitions. It should also be noted that the relevant literature contains many intermediate variants of separation axioms beyond those considered in this study; however, such variants lie outside the scope of the present notes. Finally, the properties discussed here are restricted to those requiring only basic knowledge of topology; more advanced topics involving compactness, paracompactness, uniform spaces, or metric spaces are not addressed. Therefore, these notes are intended as an introductory reference on separation axioms and as a foundational resource for readers who wish to pursue more advanced studies in this area.

2. Preliminaries

Throughout this study, let the notations $\mathbb{Z}^+$, $\mathbb{N}$, and $\mathbb{R}$ represent the sets of all positive integers, nonnegative integers, and real numbers, respectively. Moreover, let I be an index set and $I_n := \{1, 2, 3, \ldots, n\}$, for all $n \in \mathbb{Z}^+$.

Definition 2.1. Let X be a set and $\tau \subseteq P(X)$. Then, τ is called a topology on X if the following conditions hold:

i. $\emptyset, X \in \tau$

ii. If $U_i \in \tau$, for all $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$

iii. If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$

Moreover, the ordered pair (X, τ) is called a topological space.

Throughout this study, let (X, τ) and (Y, υ) be two topological space.

Definition 2.2. (X, τ) be a topological space and $A \subseteq X$. If $A \in \tau$, then A is called a τ-open (briefly an open) set in X. Moreover, if $X \setminus A \in \tau$, then A is called a τ-closed (briefly a closed) set in X.

Throughout this study, let τ^c represent the set of all closed sets in X.

Definition 2.3. Let (X, τ) be a topological space and let $x \in X$ and $A \subseteq X$.

- The family of open neighborhoods of x is defined as follows:

$$\tau(x) := \{U \in \tau : x \in U\}$$

- The family of neighborhoods of x is defined as follows:

$$N(x) := \{N \subseteq X : \exists G \in \tau \ni x \in G \wedge G \subseteq N\}$$

- The family of closed neighborhoods of x is defined as follows:

$$\tau^c(x) := \{F \in \tau^c : \exists G \in \tau \ni x \in G \wedge G \subseteq F\}$$

Similarly,

- The family of open neighborhoods of A is defined as follows:

$$\tau(A) := \{U \in \tau : A \subseteq U\}$$

- The family of neighborhoods of A is defined as follows:

$$N(A) := \{N \subseteq X : \exists G \in \tau \ni A \subseteq G \subseteq N\}$$

- The family of closed neighborhoods of A is defined as follows:

$$\tau^c(A) := \{F \in \tau^c : \exists G \in \tau \ni A \subseteq G \subseteq F\}$$

Definition 2.4. Let (X, τ) be a topological space and $A \subseteq X$. Then, the closure of A, denoted by $\mathrm{cl}(A)$, is defined as follows:

$$\mathrm{cl}(A) := \bigcap_{\substack{F \in \tau^c \\ A \subseteq F}} F$$

Equivalently, $\mathrm{cl}(A)$ is the smallest closed set in X containing A.

Definition 2.5. Let (X, τ) be a topological space and let $A \subseteq X$. The interior of A, denoted by $\mathrm{int}(A)$, is defined as

$$\mathrm{int}(A) =:= \bigcup_{\substack{U \in \tau \\ U \subseteq A}} U$$

Equivalently, $\mathrm{int}(A)$ is the largest open set in X contained in A.

Definition 2.6. Let (Y, υ) be a topological space, $X \neq \emptyset$, and $f : X \to Y$ be a function. Then, the topological space $\left(X, f^{-1}(\upsilon)\right)$ is called the preimage space of the topological space (Y, υ) under the function f. Here, $f^{-1}(\upsilon) := \left\{f^{-1}(V) \subseteq X : V \in \upsilon\right\}$.

Definition 2.7. Let (X, τ) be a topological space and $A \subseteq X$. Then, the family

$$\tau_A = \{U \cap A : U \in \tau\}$$

is a topology on A and called the subspace topology. Moreover, the pair (A, τ_A) is called a subspace of (X, τ).

Definition 2.8. Let (X, τ) be a topological space and (A, τ_A) be a subspace of (X, τ). If A is closed in X, then (A, τ_A) is called a closed subspace of (X, τ).

Definition 2.9. Let (X, τ) and (Y, υ) be two topological spaces. A function $f : X \to Y$ is called continuous if $f^{-1}(V) \in \tau$, for all $V \in \upsilon$.

Definition 2.10. Let (X, τ) and (Y, υ) be two topological spaces. A function $f : X \to Y$ is called closed if $f(F) \in \upsilon^c$, for all $F \subseteq \tau^c$.

Definition 2.11. Let (X, τ) and (Y, υ) be topological spaces. A function $f : X \to Y$ is called open if $f(U) \in \upsilon$, for all $U \subseteq \tau$.

Definition 2.12. Let (X, τ) and (Y, υ) be topological spaces. A function $f : X \to Y$ is called a homeomorphism if it is bijective, continuous, and its inverse $f^{-1} : Y \to X$ is continuous. If $f : X \to Y$ is a homeomorphism, then the topological spaces (X, τ) and (Y, υ) are said to be homeomorphic.

Definition 2.13. Let $\{(X_i, \tau_i) : i \in I\}$ be a family of topological spaces. Then, the topological space $\left(\prod\limits_{i \in I} X_i, \tau\right)$ is called the product space of the family $\{(X_i, \tau_i) : i \in I\}$. Here, τ is the smallest topology on X such that for all $j \in I$, the canonical projection

$$\begin{aligned} \pi_j : \quad X &\to X_j \\ (x_i)_{i \in I} &\mapsto x_j \end{aligned}$$

is continuous.

Definition 2.14. Let (X, τ) be a topological space and $A, B \subseteq X$. Then, the sets A and B are said to be strongly separated if $\mathrm{cl}(A) \cap B = \emptyset$ and $A \cap \mathrm{cl}(B) = \emptyset$.

Definition 2.15. Let (X, τ) be a topological space and let $\equiv$ be an equivalence relation on X. Denote the set of equivalence classes by $X/{\equiv}$, i.e., let $X/{\equiv} = \{[x] : x \in X\}$ where $[x] = \{y \in X : x \equiv y\}$, and consider the canonical projection $\pi : X \to X/{\equiv}$ defined by $\pi(x) = [x]$. The quotient topology $\tau_{\equiv}$ on $X/{\equiv}$ is defined by

$$\tau_{\equiv} = \{U \subseteq X/{\equiv} : \pi^{-1}(U) \in \tau\}$$

Equivalently, $\tau_{\equiv}$ is the finest topology on $X/{\equiv}$ that makes the projection π continuous. Moreover, the ordered pair $(X/{\equiv}, \tau_{\equiv})$ is called the quotient topological space. In particular, the ordered pair $(X/{\sim}, \tau_{\sim})$ is called the Kolmogorov quotient topological space. Here, $x \sim y \Leftrightarrow x$ and y are topologically indistinguishable.

Definition 2.16. Let (X, τ) be a topological space and $x \in X$. Then, the kernel of x, denoted by $\ker(\{x\})$, is defined as follows:

$$\ker(\{x\}) := \{y \in X : x \in \mathrm{cl}(\{y\})\} = \bigcap_{\substack{U \in \tau \\ x \in U}} U$$

Definition 2.17. Let (X, τ) be a topological space and $\mathcal{B} \subseteq \tau$. Then, $\mathcal{B}$ is called a basis for τ if every open set $U \in \tau$ can be written as a union of elements of $\mathcal{B}$, i.e.

$$U = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq U}} B$$

Definition 2.18. Let (X, τ) be a topological space and $\mathcal{S} \subseteq P(X)$. Then, $\mathcal{S}$ is called a subbasis for τ if the family of all finite intersections of elements of $\mathcal{S}$ forms a basis for τ.

Definition 2.19. Let (X, τ) be a topological space and $\mathcal{S}$ be a a subbasis for τ. Then, $\mathcal{S}$ is called an open subbase if $S \in \tau$, for all $S \in \mathcal{S}$, i.e., $\mathcal{S} \subseteq \tau$ and called a closed subbase if $S \in \tau^c$, for all $S \in \mathcal{S}$, i.e., $\mathcal{S} \subseteq \tau^c$.

Definition 2.20. Let (X, τ) be a topological space and $x \in X$. A family $\mathcal{B}_x$ of open sets in X is called an open local base at x if

$$\mathcal{B}_x \subseteq \tau(x) \quad \text{and} \quad \text{for all } U \in \tau(x), \text{ there exists a} V \in \mathcal{B}_x \text{ such that } V \subseteq U$$

Definition 2.21. Let (X, τ) be a topological space and $x \in X$. A family $\mathcal{N}_x$ of subsets of X is called a local base at x if

$$\mathcal{N}_x \subseteq N(x) \quad \text{and} \quad \text{for all } N \in N(x), \text{ there exists an } M \in \mathcal{N}_x \text{ such that } M \subseteq N$$

where $N(x)$ denotes the family of neighborhoods of x.

Definition 2.22. A topological space (X, τ) is called first-countable if for all $x \in X$, there exists a countable local base $\mathcal{N}_x$ at x.

Definition 2.23. A topological space (X, τ) is called second-countable if there exists a countable basis $\mathcal{B}$ for τ.

Definition 2.24. Let (X, τ) be a topological space. Then, a family $\mathcal{U} \subseteq \tau$ is called an open cover of X if

$$X \subseteq \bigcup_{U \in \mathcal{U}} U$$

Definition 2.25. Let (X, τ) be a topological space and $\mathcal{U}$ be a cover of X. Then, a subfamily $\mathcal{V} \subseteq \mathcal{U}$ is called a subcover if

$$X \subseteq \bigcup_{V \in \mathcal{V}} V$$

Definition 2.26. A topological space (X, τ) is called a Lindelöf space if every open cover of X has a countable subcover.

Definition 2.27. A family of sets $\{A_i : i \in I\}$ is called mutually disjoint if $A_i \cap A_j = \emptyset$, for all $i, j \in I$ with $i \neq j$.

Definition 2.28. Let $\{(X_i, \tau_i) : i \in I\}$ be a family of topological spaces such that $\{X_i : i \in I\}$ is mutually disjoint. Then, the topological space $\left(\coprod_{i \in I} X_i, \tau\right)$ is called the topological-sum/disjoint-sum space of the family $\{(X_i, \tau_i) : i \in I\}$. Here, $\coprod_{i \in I} X_i := \bigcup_{i \in I} X_i$ and $\tau := \left\{G \in \coprod_{i \in I} X_i : \forall i \in I, G \cap X_i \in \tau_i\right\}$.

Definition 2.29. Let A and B be two sets. Then, it is said to be that A meets B, or that A and B intersect if $A \cap B \neq \emptyset$.

Definition 2.30. The Sierpiński space is the topological space (X, τ) where $X = \{0, 1\}$ and $\tau = \{\emptyset, \{1\}, X\}$. Equivalently, it is the unique topological space with two points in which exactly one singleton is closed.

Definition 2.31. Let X be a nonempty set. Then, a function $d : X \times X \to [0, \infty)$ is called a metric on X if for all $x, y, z \in X$, the following conditions hold:

i. $d(x, y) = 0$ if and only if $x = y$

ii. $d(x, y) = d(y, x)$

iii. $d(x, z) \leq d(x, y) + d(y, z)$

Moreover, the ordered pair (X, d) is called a metric space.

Definition 2.32. Let (X, d) be a metric space. Then, the topology induced by d on X is defined as

$$\tau_d = \{U \subseteq X : \text{for all } x \in U, \text{ there exists } r > 0 \text{ such that } B_d(x, r) \subseteq U\}$$

where

$$B_d(x, r) = \{y \in X : d(x, y) < r\}$$

denotes the open ball of radius $r > 0$ centered at $x \in X$.

Definition 2.33. Let $F \subseteq P(X)$ and $F \neq \emptyset$. If the following conditions hold, then the family F is called a filter in X:

i. $\emptyset \notin F$

ii. $A \cap B \in F$, for all $A, B \in F$

iii. If $A \in F$ and $A \subseteq B$, then $B \in F$

Throughout this section, let $\mathcal{F}(X)$ denote the set of all filters in a nonempty set X.

Definition 2.34. Let D be a nonempty set and $\preceq$ be a relation on d. Then, the relation $\preceq$ is called a direction on D if the following conditions hold:

i. Reflexivity: For all $\alpha \in D$, $\alpha \preceq \alpha$

ii. Transitivity: For all $\alpha, \beta, \gamma \in D$, if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$

iii. Directedness: For all $\alpha, \beta \in D$, there exists a $\gamma \in D$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$

Moreover, the ordered pair $(D, \preceq)$ is called a directed set.

Definition 2.35. Let $(D, \preceq)$ be a directed set and $x : D \to X$ be a function. Then, the function x is called a net in X indexed by the directed set D, and for brevity, it is denoted by $(x_\alpha)_{\alpha \in D}$.

Throughout this section, let $\mathcal{N}(D, X)$ denote the set of all nets in X indexed by a directed set D.

Definition 2.36. Let (X, τ) be a topological space and $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$. Then, the net $(x_\alpha)_{\alpha \in D}$ is said to converge to a point $x \in X$, denoted $x_\alpha \to x$ or $x \in \lim x_\alpha$, if for all $N \in N(x)$, there exists an $\alpha_0 \in D$ such that $x_\alpha \in N$, for all $\alpha_0 \preceq \alpha$.

Definition 2.37. Let (X, τ) be a topological space and $F \in \mathcal{F}(X)$ be a filter on X. Then, the filter F is said to converge to a point $x \in X$, denoted $F \to x$ or $x \in \lim F$, if $N(x) \subseteq F$.

Definition 2.38. Let (X, τ) be a topological space, $F \in \mathcal{F}(X)$, and $x \in X$. Then, the point x is called a closure point of the filter F if $N \cap A \neq \emptyset$, for all $N \in N(x)$ and for all $A \in F$. The set of all closure points of $\mathcal{F}$ is denoted by $\mathrm{cl}(F)$.

Definition 2.39. Let (X, τ) be a topological space, $F \in \mathcal{F}(X)$, and $x \in X$. Then, the point x is called a cluster point of the filter F if for all $N \in N(x)$, there exists an $A \in F$ such that $N \cap A \neq \emptyset$. The set of all cluster points of F is denoted by $\mathrm{clust}(F)$.

Definition 2.40. Let (X, τ) be a topological space, $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$, and $x \in X$. Then, the point x is called a cluster point of the net $(x_\alpha)_{\alpha \in D}$ if for all $N \in N(x)$ and for all $\alpha \in D$, there exists $\alpha \preceq \beta$ such that $x_\beta \in N$. The set of all cluster points of (x_α) is denoted by $\mathrm{clust}(x_\alpha)$.

Definition 2.41. A property P is said to be

i. hereditary if, whenever a topological space (X, τ) has P, then every subspace (A, τ_A) of (X, τ) also has P.

ii. closed-hereditary if, whenever a topological space (X, τ) has P, then every closed subspace (A, τ_A) of (X, τ) also has P.

iii. topological/homeomorphically invariant if, whenever a topological space (X, τ) has P and (X, τ) is homeomorphic to a topological space (Y, υ), then (Y, υ) also has P.

iv. refinement-preserved if, whenever a topological space (X, τ) has P, then every topology υ on X satisfying $\tau \subseteq \upsilon$ also yields a space (X, υ) having P.

v. closed-surjective-map-preserved if, whenever $f : (X, \tau) \to (Y, \upsilon)$ is closed and surjective and (X, τ) has P, then (Y, υ) also has P.

vi. open-surjective-map-preserved if, whenever $f : (X, \tau) \to (Y, \upsilon)$ is open and surjective and (X, τ) has P, then (Y, υ) also has P.

vii. closed-continuous-surjective-map-preserved if, whenever $f : (X, \tau) \to (Y, \upsilon)$ is continuous, closed, and surjective, and (X, τ) has P, then (Y, υ) also has P.

viii. closed-open-continuous-surjective-map-preserved if, whenever $f : (X, \tau) \to (Y, \upsilon)$ is continuous, surjective, open, and closed, and (X, τ) has P, then (Y, υ) also has P.

ix. closed-bijective-map-preserved if, whenever $f : (X, \tau) \to (Y, \upsilon)$ is closed and bijective, and (X, τ) has P, then (Y, υ) also has P.

x. open-bijective-map-preserved if, whenever $f : (X, \tau) \to (Y, \upsilon)$ is open and bijective, and (X, τ) has P, then (Y, υ) also has P.

xi. productive if, whenever each topological space (X_i, τ_i) in a family $\{(X_i, \tau_i) : i \in I\}$ has P, then the product space $\left(\prod_{i \in I} X_i, \tau\right)$ of the family also has P.

xii. projective if, whenever the product space $\left(\prod_{i \in I} X_i, \tau\right)$ of a family of topological spaces $\{(X_i, \tau_i) : i \in I\}$ has P, then the factor space (X_i, τ_i) also has P, for every $i \in I$.

xiii. disjoint-sum-preserved if, whenever each topological space (X_i, τ_i) in a family $\{(X_i, \tau_i) : i \in I\}$ such that $\{X_i : i \in I\}$ is mutually disjoint has P, then the disjoint-sum space $\left(\coprod_{i \in I} X_i, \tau\right)$ also has P.

xiv. disjoint-summand-preserved if, whenever the disjoint-sum space $\left(\coprod_{i \in I} X_i, \tau\right)$ of a family of topological spaces $\{(X_i, \tau_i) : i \in I\}$ such that $\{X_i : i \in I\}$ is mutually disjoint has P, then the summand space (X_i, τ_i) also has P, for every $i \in I$.

xv. preimage-map-preserved if, whenever $f : X \to Y$ is a function and a topological space (Y, υ) has P, then the preimage space $\left(X, f^{-1}(\upsilon)\right)$ of (Y, υ) under f also has P.

xvi. surjective-preimage-map-preserved if, whenever $f : X \to Y$ is surjective and a topological space (Y, υ) has P, then the preimage space $\left(X, f^{-1}(\upsilon)\right)$ of (Y, υ) under f also has P.

xvii. Kolmogorov-quotient-preserved if, whenever a topological space (X, τ) has P, then its Kolmogorov quotient space $(X/\sim, \tau_\sim)$ also has P.

3. Definitions, Characterizations, and Properties

3.1. Topologically Indistinguishable

x and y are topologically indistinguishable	$\overset{\text{def}}{\Leftrightarrow}$	For all $U \in \tau$, $x \in U \Leftrightarrow y \in U$
	$\Leftrightarrow$	For all $K \in \tau^c$, $x \in K \Leftrightarrow y \in K$
	$\Leftrightarrow$	$N(x) = N(y)$
	$\Leftrightarrow$	$\text{cl}(\{x\}) = \text{cl}(\{y\})$ (or equivalently, $x \in \text{cl}(\{y\})$ and $y \in \text{cl}(\{x\})$)
	$\Leftrightarrow$	$\ker(\{x\}) = \ker(\{y\})$
	$\Leftrightarrow$	For all $F \in \mathcal{F}(X)$, $F \to x \Leftrightarrow F \to y$
	$\Leftrightarrow$	For all $(x_\alpha)_{\alpha\in D} \in \mathcal{N}(D, X)$, $x_\alpha \to x \Leftrightarrow x_\alpha \to y$
	$\Leftrightarrow$	For all $F \in \mathcal{F}(X)$, $x \in \text{clust}(F) \Leftrightarrow y \in \text{clust}(F)$
	$\Leftrightarrow$	For all $(x_\alpha)_{\alpha\in D} \in \mathcal{N}(D, X)$, $x \in \text{clust}(x_\alpha) \Leftrightarrow y \in \text{clust}(x_\alpha)$

i. The relation indistinguishable is an equivalence relation on a topological space.

ii. In a topological space, if $\{x\}$ and $\{y\}$ are strongly separated sets, then they are topologically distinguishable.

iii. In a topological space, if x and y are topologically distinguishable, then they are distinct, i.e., $x \neq y$.

iv. If $f : X \to Y$ is a continuous function and $x, y \in X$ such that x and y are topologically indistinguishable, then $f(x)$ and $f(y)$ are topologically indistinguishable.

v. Two elements in a product space are topologically indistinguishable if and only if each of their components are topologically indistinguishable.

3.2. T_0 Spaces

(X, τ) is a T_0 (Kolmogorov) space or τ is a T_0 (Kolmogorov) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, x and y are topologically distinguishable
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau(x)$ such that $y \notin U$ or there exists a $V \in \tau(y)$ such that $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $K \in \tau^c(x)$ such that $y \notin K$ or there exists an $F \in \tau^c(y)$ such that $x \notin F$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an open local base $\mathcal{B}_x$ such that there exists an $U \in \mathcal{B}_x$ satisfying the condition $y \notin U$ or there exists an open local base $\mathcal{B}_y$ such that there exists an $V \in \mathcal{B}_y$ satisfying the condition $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $N \in N(x)$ such that $y \notin N$ or there exists a $M \in N(y)$ such that $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a local base $\mathcal{N}_x$ such that there exists an $N \in \mathcal{N}_x$ satisfying the condition $y \notin N$ or there exists a local base $\mathcal{N}_y$ such that there exists an $M \in \mathcal{N}_y$ satisfying the condition $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $\text{cl}(\{x\}) \neq \text{cl}(\{y\})$ (or equivalently, $x \notin \text{cl}(\{y\})$ or $y \notin \text{cl}(\{x\})$)
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $x \notin \text{acc}(\{y\})$ or $y \notin \text{acc}(\{x\})$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $\ker(\{x\}) \neq \ker(\{y\})$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $F \in \mathcal{F}(X)$ such that $F \to x$ and $F \nrightarrow y$ or $F \nrightarrow x$ and $F \to y$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$ such that $x_\alpha \to x$ and $x_\alpha \nrightarrow y$ or $x_\alpha \nrightarrow x$ and $x_\alpha \to y$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $F \in \mathcal{F}(X)$ such that $x \in \text{clust}(F)$ and $y \notin \text{clust}(F)$ or $x \notin \text{clust}(F)$ and $y \in \text{clust}(F)$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$ such that $x \in \text{clust}(x_\alpha)$ and $y \notin \text{clust}(x_\alpha)$ or $x \notin \text{clust}(x_\alpha)$ and $y \in \text{clust}(x_\alpha)$

i. Being a T_0 space is a topological property.

ii. Being a T_0 space is a hereditary property.

iii. If (X, τ) is a T_0 space and $\tau \subseteq \upsilon$, then (X, υ) is a T_0 space.

iv. $(\mathbb{N}, \tau_l)$ and $(\mathbb{N}, \tau_u)$ are T_0 spaces. Here,

$$\tau_l = \{\emptyset, \mathbb{N}, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \ldots\}$$

and

$$\tau_u = \{\emptyset, \mathbb{N}, \{2, 3, 4, 5, 6, \ldots\}, \{3, 4, 5, 6, \ldots\}, \{4, 5, 6, \ldots\}, \{5, 6, \ldots\}, \ldots\}$$

v. For all $i \in I$, (X_i, τ_i) is a T_0 space if and only if $\left(\prod_{i \in I} X_i, \tau\right)$ is a T_0 space.

vi. For all $i \in I$, (X_i, τ_i) is a T_0 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i \in I} X_i, \tau\right)$ is a T_0 space.

vii. If $f : (X, \tau) \to (Y, \upsilon)$ is closed and surjective and (X, τ) is a T_0 space, then (Y, υ) is a T_0 space.

viii. If $f : (X, \tau) \to (Y, \upsilon)$ is open and surjective and (X, τ) is a T_0 space, then (Y, υ) is a T_0 space.

ix. If $f : (X, \tau) \to (X, \upsilon)$ is closed identity function such that $\tau \subseteq \upsilon$ and (X, τ) is a T_0 space, then (X, υ) is a T_0 space.

x. $(X/\sim, \tau_\sim)$ is a T_0 space.

xi. The topological space induced by a metric space is a T_0 space.

3.3. R_0 Spaces

(X,τ) is an R_0 (a symmetric) space or τ is an R_0 (a symmetric) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all $x, y \in X$, $x \in \mathrm{cl}(\{y\}) \Leftrightarrow y \in \mathrm{cl}(\{x\})$
	$\Leftrightarrow$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exist $N \in N(x)$ and $N \in N(y)$ such that $y \notin N$ and $x \notin M$
	$\Leftrightarrow$	For all $x \in X$ and for all $U \in \tau(x)$, $\mathrm{cl}(\{x\}) \subseteq U$
	$\Leftrightarrow$	For all $x \in X$ and for all $N \in N(x)$, $\mathrm{cl}(\{x\}) \subseteq N$
	$\Leftrightarrow$	For all $x \in X$ and $K \in \tau^c$ such that $x \notin K$, $K \cap \mathrm{cl}(\{x\}) = \emptyset$
	$\Leftrightarrow$	For all $x \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : x$ and y are topologically indistinguishable$\}$
	$\Leftrightarrow$	The family $\{\mathrm{cl}(\{x\}) : x \in X\}$ is a partition of X
	$\Leftrightarrow$	The family $\{\ker(\{x\}) : x \in X\}$ is a partition of X
	$\Leftrightarrow$	For all $x \in X$, $\mathrm{cl}(\{x\}) \subseteq \ker(\{x\})$
	$\Leftrightarrow$	The specialization preorder ($x \leq y \Leftrightarrow x \in \mathrm{cl}(\{y\}) \Leftrightarrow \mathrm{cl}\{x\}) \subseteq \mathrm{cl}(\{y\})$) on X is symmetric (and therefore an equivalence relation)
	$\Leftrightarrow$	For all $U \in \tau$, there exists a family $\{K_i \in \tau^c : i \in I\}$ such that $U = \bigcup_{i \in I} K_i$
	$\Leftrightarrow$	For all $K \in \tau^c$, there exists a family $\{U_i \in \tau(K) : i \in I\}$ such that $K = \bigcap_{i \in I} U_i$
	$\Leftrightarrow$	For all $x \in X$, the fixed ultrafilter at x converges only to the points that are topologically indistinguishable from x
	$\Leftrightarrow$	For all $A \subseteq X$ and for all $U \in \tau$ satisfying the condition $A \cap U \neq \emptyset$, there exists an $K \in \tau^c$ such that $A \cap K \neq \emptyset$ and $K \subseteq U$.

i. Being an R_0 space is a hereditary property.

ii. (X,τ) is an R_0 space if and only if $(X/\sim, \tau_\sim)$ is an R_0 space.

iii. For all $i \in I$, (X_i, τ_i) is an R_0 space if and only if $\left(\prod_{i \in I} X_i, \tau\right)$ is an R_0 space.

iv. For all $i \in I$, (X_i, τ_i) is an R_0 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i \in I} X_i, \tau\right)$ is an R_0 space.

v. The indiscrete topological space (X,τ) such that $X \geq \emptyset$ is an R_0 space.

vi. If (X,τ) and (X,υ) are two topological spaces, (X,τ) is an R_0 space, and $\tau < \upsilon$, then $\tau \subseteq \upsilon$. Here, $\tau < \upsilon$ if and only if every τ-cover $\mathcal{U}$ of X has a υ-refinement $\mathcal{V}$, i.e., there exists a υ-cover $\mathcal{V}$ of X such that for all $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $V \subseteq U$.

vii. If (X,τ) and (X,υ) are two topological spaces and (X,τ) is an R_0 space, then $\tau < \upsilon$ if and only if $\tau \subseteq \upsilon$.

viii. If (X,τ) is an R_0 space, $A \subseteq X$, A is closed in X, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$, then (X, τ_A) is an R_0 space.

ix. If (X,τ) is an R_0 space, $f : (X,\tau) \to (Y,\upsilon)$ is closed, continuous, and surjective, and $f^{-1}(y)$ is a finite set, for all $y \in Y$, then (Y,υ) is an R_0 space.

x. If (X,τ) is an R_0 space and for all $F \in \tau^c$, there exists a countable open base of F for $\tau(F)$, then every closed set in X can be written as the intersection of countably many open sets.

xi. If (X, τ) is an R_0 space and for all $F \in \tau^c$, there exists a countable open base of F for $\tau(F)$, then (X, τ) is a first countable space.

xii. If (X, τ) is an R_0 space, then $\mathrm{cl}(\{x\})$ is compact, for all $x \in X$.

3.4. T_1 Spaces

(X,τ) is a T_1 (accessible or Fréchet) space or τ is a T_1 (accessible or Fréchet) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $y \notin U$ and $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist open local bases $\mathcal{B}_x$ and $\mathcal{B}_y$ such that there exist $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ satisfying the conditions $y \notin U$ and $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist $N \in N(x)$ and $M \in N(y)$ such that $y \notin N$ and $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist open local bases $\mathcal{N}_x$ and $\mathcal{N}_y$ such that there exist $N \in \mathcal{N}_x$ and $M \in \mathcal{N}_y$ satisfying the conditions $y \notin N$ and $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $\{x\}$ and $\{y\}$ are strongly separated sets.
	$\Leftrightarrow$	(X,τ) is a T_0 space and an R_0 space.
	$\Leftrightarrow$	For all $x \in X$, $\{x\} \in \tau^c$
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap\limits_{U \in \tau(x)} U$
	$\Leftrightarrow$	For all $A \subseteq X$, $A = \bigcap\limits_{U \in \tau(A)} U$
	$\Leftrightarrow$	For all $x \in X$ and for all $\mathcal{B}_x$, $\{x\} = \bigcap\limits_{U \in \mathcal{B}_x} U$
	$\Leftrightarrow$	Every set in X that can be written as the intersection of countably many open sets is closed in X
	$\Leftrightarrow$	Every finite subset of X is closed.
	$\Leftrightarrow$	Every subset of X whose complement is a finite set is open.
	$\Leftrightarrow$	For the cofinite topological space (X,υ), the identity mapping from (X,τ) to (X,υ) is continuous.
	$\Leftrightarrow$	For all $x \in X$, the fixed ultrafilter at x converges only to x.
	$\Leftrightarrow$	For every subset $A \subseteq X$ and for every point $x \in X$, $x \in \text{acc}(A)$ if and only if for every $U \in \tau(x)$, U contains infinitely many points of A.
	$\Leftrightarrow$	Each continuous function from the Sierpinski space to X is a constant function.

i. Every T_1 space is an R_0 space.

ii. Every T_1 space is a T_0 space.

iii. The only T_1 space (X,τ) such that X is a finite set is the discrete topological space.

iv. (X,τ) is an R_0 space if and only if $(X/\!\sim, \tau_\sim)$ is a T_1 space.

v. Being a T_1 space is a topological property.

vi. Being a T_1 space is a hereditary property.

vii. If (X,τ) is a T_1 space and $\tau \subseteq \upsilon$, then (X,υ) is a T_1 space.

viii. For all $i \in I$, (X_i,τ_i) is a T_1 space if and only if $\left(\prod\limits_{i \in I} X_i, \tau\right)$ is a T_1 space.

ix. For all $i \in I$, (X_i, τ_i) is a T_1 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i \in I} X_i, \tau\right)$ is a T_1 space.

x. $(X/\equiv, \tau_\equiv)$ is a T_1 space if and only if $X/\equiv \subseteq \tau^c$.

xi. If $f : (X, \tau) \to (Y, \upsilon)$ is injective and continuous and (Y, υ) is a T_1 space, then (X, τ) is a T_1 space.

xii. If (X, τ) is a T_1 space, $A \subseteq X$, $x \in \mathrm{acc}(A)$, and $F \subseteq X$ is a finite set, then $x \in \mathrm{acc}(A \setminus F)$.

xiii. If (X, τ) is a T_1 space and $A \subseteq X$, then $\mathrm{acc}(A)$ is closed.

xiv. If (X, τ) is a T_1 space and $x \in X$, then $\mathrm{acc}(\{x\})$ is closed.

xv. If (X, τ) is a T_1 space and $A \subseteq X$ is a finite set, then $\mathrm{acc}(A) = \emptyset$.

xvi. If (X, τ) is a T_1 space and $A \subseteq X$, then $\mathrm{acc}(\mathrm{acc}(A)) \subseteq \mathrm{acc}(A)$.

xvii. If (X, τ) is a T_1 space and $A \subseteq X$, then $\mathrm{cl}(\mathrm{acc}(A)) = \mathrm{acc}(A) = \mathrm{acc}(\mathrm{cl}(A))$.

xviii. If (X, τ) is a T_1 space, $x \in X$, $\mathcal{B}_x$ is a open local base, then for all $y \in X$ such that $x \neq y$, there exists a $U \in \mathcal{B}_x$ such that $y \notin U$.

xix. If (X, τ) is a first countable and a T_1 space, $x \in X$, $A \subseteq X$, and $x \in \mathrm{acc}(A)$, then there exists a sequence consisting of distinct points in $A \setminus \{x\}$ and converging to x.

xx. If (X, τ) is a first countable and a T_1 space and $x \in X$, then there exist countably many open sets U_i such that $\{x\} = \bigcap_{i \in \mathbb{N}} U_i$.

xxi. If (X, τ) is a T_1 space such that X has at least two elements and $\mathcal{B}$ is a base for (X, τ), then $\mathcal{B} \setminus \{X\}$ is a base for (X, τ).

xxii. The cofinite topological space (X, τ) such that X is an infinite set is a T_1 space.

xxiii. For any set X, the smallest T_1 topology is the cofinite topology on X.

xxiv. For any infinite set X, the topological space (X, τ), where τ is the smallest T_1 topology is a connected space.

xxv. A T_1 space is a countably compact space if and only if every infinite set of the space has an accumulation point.

xxvi. A T_1 space is a countably compact space if and only if every infinite open cover of the space has a proper subcover.

xxvii. For every T_1 space (X, τ), $|X| \leq 2^{\min\{|\mathcal{B}| \,:\, \mathcal{B} \text{ is a base of } X\}}$.

xxviii. If $f : (X, \tau) \to (Y, \upsilon)$ is closed and surjective and (X, τ) is a T_1 space, then (Y, υ) is a T_1 space.

xxix. If $f : (X, \tau) \to (Y, \upsilon)$ is open and surjective and (X, τ) is a T_1 space, then (Y, υ) is a T_1 space.

xxx. If $f : (X, \tau) \to (X, \upsilon)$ is closed identity function such that $\tau \subseteq \upsilon$ and (X, τ) is a T_1 space, then (X, υ) is a T_1 space.

xxxi. The topological space induced by a metric space is a T_1 space.

xxxii. If every convergent sequence on a topological space has a unique limit, then the space is a T_1 space.

xxxiii. Every continuous function from the indiscrete topological space (X, τ) to a T_1 space (X, υ) is a constant function.

xxxiv. If, for all $x \in X$, there exists a $N \in N(x)$ such that N is a T_1 space with respect to the induced topology on (X, τ), then (X, τ) is a T_1 space.

xxxv. If for all distinct points $x, y \in X$, there exists a continuous function $f : X \to \mathbb{R}$ and distinct points $a, b \in \mathbb{R}$ such that $f(x) = a$ and $f(y) = b$, then the space is a T_1 space.

3.5. R_1 Spaces

(X,τ) is an R_1 (a preregular) space or τ is an R_1 (a preregular) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exists a $N \in N(x)$ and a $M \in N(y)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.
	$\Leftrightarrow$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exists a $N \in N(\mathrm{cl}(\{x\}))$ and a $M \in N(\mathrm{cl}(\{y\}))$ such that $N \cap M = \emptyset$.
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist $U, V \in \tau$ such that $x \in U$, $\mathrm{cl}(\{y\}) \subseteq V$, and $U \cap V = \emptyset$.
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau$ such that $x \in U$ and $\mathrm{cl}(U) \subseteq X \setminus \mathrm{cl}(\{y\})$.
	$\Leftrightarrow$	For all $x \in X$, $\mathrm{cl}(\{x\}) = \bigcap_{K \in \tau^c(x)} K$
	$\Leftrightarrow$	For all $x, y \in X$, every neighborhood of x meets every neighborhood of y implies that $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, $x \in \mathrm{clust}(N(y))$ implies that $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, $y \in \mathrm{clust}(N(x))$ implies that $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, there exists a filter $F \in \mathcal{F}(X)$ such that $F \to x$ and $F \to y$ implies $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, there exists a net $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$ such that $x_\alpha \to x$ and $x_\alpha \to y$ implies $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, there exists a filter $F \in \mathcal{F}(X)$ such that $F \neq P(X)$, $F \to x$, and $F \to y$ implies $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \forall N \in N(x), \forall M \in N(Y), N \cap M \neq \emptyset\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : x \in \mathrm{clust}(N(y))\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : y \in \mathrm{clust}(N(x))\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \exists F \in \mathcal{F}(X) \ni F \to x \wedge F \to y\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \exists (x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X) \ni x_\alpha \to x \wedge x_\alpha \to y\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \exists F \in \mathcal{F}(X) \ni F \neq P(X) \wedge F \to x \wedge F \to y\}$.
	$\Leftrightarrow$	For all $x \in X$ and for all $F \in \mathcal{F}(X)$, if $x \in \lim F$, then $\mathrm{cl}(\{x\}) = \lim F$.
	$\Leftrightarrow$	For all $x \in X$ and for all $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$, if $x \in \lim x_\alpha$, then $\mathrm{cl}(\{x\}) = \lim x_\alpha$.
	$\Leftrightarrow$	For all $x, y \in X$ and for all $F \in \mathcal{F}(X)$, if $x, y \in \lim F$, then x and y are topologically indistinguishable.
	$\Leftrightarrow$	For all $x, y \in X$ and for all $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$, if $x, y \in \lim x_\alpha$, then x and y are topologically indistinguishable.

i. Every R_1 space is an R_0 space.

ii. Being an R_1 space is a hereditary property.

iii. Being an R_1 space is a topological property.

iv. For all $i \in I$, (X_i, τ_i) is an R_1 space if and only if $\left(\prod_{i\in I} X_i, \tau\right)$ is an R_1 space.

v. For all $i \in I$, (X_i, τ_i) is an R_1 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is an R_1 space.

vi. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{x\}) \cap A = \emptyset$, there exists an $N \in N(x)$ and an $M \in N(A)$ such that $N \cap M = \emptyset$.

vii. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{x\}) \cap A = \emptyset$, there exists an $U \in \tau(x)$ and an $V \in \tau(A)$ such that $U \cap V = \emptyset$.

viii. In an R_1 space (X, τ), for all compact sets $A \subseteq X$ and for all closed and compact sets $B \subseteq X$ satisfying the condition $A \cap B = \emptyset$, there exists an $U \in \tau(A)$ and an $V \in \tau(B)$ such that $U \cap V = \emptyset$.

ix. In an R_1 space (X, τ), if a compact set $A \subseteq X$ is contained in a $U \in \tau$, then $\mathrm{cl}(A) \subseteq U$.

x. In an R_1 space (X, τ), $\mathrm{cl}(A)$ is compact, for all compact sets $A \subseteq X$. More generally, if A is compact and $A \subseteq B \subseteq \mathrm{cl}(A)$, then B is compact.

xi. In an R_1 space (X, τ), $\mathrm{cl}(S)$ is compact, for all compact sets $A \subseteq X$ and for all $S \subseteq X$ with $S \subseteq A$.

xii. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\{x\} \cap \mathrm{cl}(A) = \emptyset$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xiii. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\ker(\{x\}) \neq \ker(\{y\})$, for all $y \in A$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xiv. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$, for all $y \in A$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xv. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{y\}) \subseteq A$, for all $y \in A$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xvi. In an R_1 space (X, τ), for all compact sets $A \subseteq X$, $\mathrm{cl}(A) = \bigcup_{x\in A} \mathrm{cl}(\{x\})$.

xvii. If $f, g : (X, \tau) \to (Y, \upsilon)$ are continuous and (Y, υ) is an R_1 space, then the set $\{x \in X : \mathrm{cl}(\{f(x)\}) = \mathrm{cl}(\{g(x)\})\}$ is closed in X.

xviii. (X, τ) is an R_1 space if and only if $(X/{\sim}, \tau_{\sim})$ is an R_1 space.

3.6. T_2 Spaces

(X,τ) is a T_2 (separated or Hausdorff) space or τ is a T_2 (separated or Hausdorff) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $U \cap V = \emptyset$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist $N \in N(x)$ and $M \in N(y)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all distinct and nonempty subsets A and B of X, there exist distinct open sets U and V such that $A \cap U \neq \emptyset$ and $B \cap V \neq \emptyset$.
	$\Leftrightarrow$	Every convergent net on X has a unique limit.
	$\Leftrightarrow$	Every convergent filter on X has a unique limit.
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap_{U \in \tau(x)} \mathrm{cl}(U)$
	$\Leftrightarrow$	For all $x \in X$ and for all $\mathcal{B}_x$, $\{x\} = \bigcap_{U \in \mathcal{B}_x} \mathrm{cl}(U)$
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap_{K \in \tau^c(x)} K$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $N \in N(x)$ such that $y \notin \mathrm{cl}(N)$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau(x)$ such that $y \notin \mathrm{cl}(U)$
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap_{N \in N(x)} \mathrm{cl}(N)$
	$\Leftrightarrow$	For all $x \in X$ and for all $\mathcal{N}_x$, $\{x\} = \bigcap_{N \in \mathcal{N}_x} \mathrm{cl}(N)$
	$\Leftrightarrow$	The set $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.
	$\Leftrightarrow$	Every convergent filter has a unique closure point, which coincides with the point to which the filter converges.
	$\Leftrightarrow$	(X,τ) is a T_0 space and an R_1 space.
	$\Leftrightarrow$	(X,τ) is a T_1 space and an R_1 space.
	$\Leftrightarrow$	For all continuous function $f : D \to X$ where D is a dense set of a topological space (Y, υ), there exists at most one continuous function $g : Y \to X$ such that $g_D = f$.

i. Every T_2 space is an R_1 space.

ii. Every T_2 space is an R_0 space.

iii. Every T_2 space is a T_1 space.

iv. Every T_2 space is a T_0 space.

v. Being a T_2 space is a hereditary property.

vi. Being a T_2 is a topological property.

vii. For all $i \in I$, (X_i, τ_i) is a T_2 space if and only if $\left(\prod_{i \in I} X_i, \tau\right)$ is a T_2 space.

viii. For all $i \in I$, (X_i, τ_i) is a T_2 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a T_2 space.

ix. If (X, τ) is a T_2 space and $\tau \subseteq \upsilon$, then (X, υ) is a T_2 space.

x. Every discrete topological space is a T_2 space.

xi. In a T_2 space, every convergent sequence on X has a unique limit.

xii. In a T_2 space, every convergent sequence has a unique cluster point, which coincides with the point to which the sequence converges.

xiii. In an R_1 space, if every convergent sequence on X has a unique limit, then it is a T_2 space.

xiv. In a first countable space, if every convergent sequence on X has a unique limit, then it is a T_2 space.

xv. (X, τ) is an R_1 space if and only if $(X/\sim, \tau_\sim)$ is a T_2 space.

xvi. If $(X/\equiv, \tau_\equiv)$ is a T_2 space, then $\equiv$ is closed in $X \times X$.

xvii. If (X, τ) is a T_2 space and $\equiv$ is closed in $X \times X$, then $(X/\equiv, \tau_\equiv)$ is a T_2 space.

xviii. If $\pi : X \to X/\equiv$ is an open function and $\equiv$ is closed in $X \times X$, then $(X/\equiv, \tau_\equiv)$ is a T_2 space.

xix. In a T_2 space, for all compact subsets A and for all $x \in X \setminus A$, there exists a $N \in N(x)$ and $M \in N(A)$ such that $N \cap M = \emptyset$.

xx. In a T_2 space, every compact subset of X is closed.

xxi. Let S be a subset of a T_2 space. Then, S is compact if and only if S is closed and S is contained in a compact set.

xxii. In a compact T_2 space, every closed set in X is compact.

xxiii. Let S be a subset of a compact T_2 space. Then S is compact if and only if S is closed.

xxiv. In an R_1 space, if every compact subset of X is closed, then it is a T_2 space.

xxv. If $f : (X, \tau) \to (Y, \upsilon)$ is injective and continuous and (Y, υ) is a T_2 space, then (X, τ) is a T_2 space.

xxvi. If $f, g : (X, \tau) \to (Y, \upsilon)$ are continuous and (Y, υ) is a T_2 space, then the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

xxvii. If $f, g : (X, \tau) \to (Y, \upsilon)$ are continuous, (Y, υ) is a T_2 space, and there exists a dense subset D of X such that $f(D) = g(D)$, then $f = g$.

xxviii. If $f : (X, \tau) \to (Y, \upsilon)$ is continuous and (Y, υ) is a T_2 space, then the graph of f, i.e., $\{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

xxix. Let $f : (X, \tau) \to (Y, \upsilon)$ be continuous and (Y, υ) be a T_2 space. Consider the following relation on X: $x_1 \equiv x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then, $(X/\equiv, \tau_\equiv)$ is a T_2 space and $\{(x_1, x_2) : x_1 \equiv x_2\}$ is closed in $X \times X$.

xxx. Let $f : (X, \tau) \to (Y, \upsilon)$ be continuous and surjective, (X, τ) be a compact space, and (Y, υ) be a T_2 space. Consider the following relation on X: $x_1 \equiv x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then, the function $f^* : X/\equiv \to Y$ defined by $f^*([x]) = f(x)$ is a homeomorphism.

xxxi. If $f : (X, \tau) \to (Y, \upsilon)$ is an open and surjective function and $\{(x_1, x_2) : x_1 \equiv x_2\}$, where $x_1 \equiv x_2 \Leftrightarrow f(x_1) = f(x_2)$, is closed in $X \times X$, then (Y, υ) is a T_2 space.

xxxii. Let $f : (X, \tau) \to (Y, \upsilon)$ be continuous. If (X, τ) is compact and (Y, υ) is a T_2 space, then f is a closed function.

xxxiii. Let $f : (X, \tau) \to (Y, \upsilon)$ be bijective and continuous. If (X, τ) is compact and (Y, υ) is a T_2 space, then f is a homeomorphism.

xxxiv. If A and B are disjoint compact subsets of a T_2 space, then there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

xxxv. If A and B are disjoint compact subsets of a T_2 space, then there exist disjoint neighborhoods N and M such that $A \subseteq N$ and $B \subseteq M$.

xxxvi. If (X,τ) is a T_2 space such that every proper closed subspace is compact, then (X,τ) is compact.

xxxvii. If (X,τ) is a T_2 space such that X is a finite set, then τ is the discrete topology and (X,τ) is homeomorphic to a subspace of $[0,1]$.

xxxviii. If (X,τ) is a T_2 space, $n \in \mathbb{Z}^+$, and $x_1, x_2, \ldots,$ and x_n are distinct elements in X, then there exist mutually disjoint open sets $U_1, U_2, \ldots, U_n$ such that $x_1 \in U_1$, $x_2 \in U_2, \ldots, x_n \in U_n$.

xxxix. If (X,τ) is a T_2 space such that X is a infinite set. Then, there exists a class of mutually disjoint open sets such that the class is an infinite set.

xl. If (X,τ) is a T_2 space and $f : X \to X$ is a continuous mapping satisfying the condition $f \circ f = f$, then $f(X)$ is closed in X. In other words, if (X,τ) is a T_2 space and A is a subset of X satisfying the condition that there exists a continuous and surjective function $f : X \to A$ such that $f(x) = x$, for all $x \in A$, then A is closed in X.

xli. If (X,τ) is a T_2 space, (X,υ) is a compact space, and $\tau \subseteq \upsilon$, then $\tau = \upsilon$.

xlii. If (X,τ) is a compact space, (Y,υ) is a T_2 space, and $f : X \to Y$ is surjective and continuous, then $U \in \upsilon$, for all $U \subseteq Y$ satisfying the condition $f^{-1}(U) \in \tau$.

xliii. If (X,τ) is a compact space, (Y,υ) is a T_2 space, and $f : X \to Y$ is surjective and continuous, then $\left\{U \subseteq Y : f^{-1}(U) \in \tau\right\}$ is a topology on Y.

xliv. Let (X,τ) be a compact space, (Y,υ) be a T_2 space, and $f : X \to Y$ be surjective and continuous. Then, B is closed in Y if and only if $f^{-1}(B)$ is closed in X.

xlv. If (X,τ) is a compact space, (Y,υ) is a T_2 space, and $f : X \to Y$ is surjective and continuous, then $U \in \upsilon$, for all $U \subseteq Y$ satisfying the condition $f^{-1}(U) \in \tau$.

xlvi. If $f : (X,\tau) \to (Y,\upsilon)$ is closed and bijective and (X,τ) is a T_2 space, then (Y,υ) is a T_2 space.

xlvii. If $f : (X,\tau) \to (Y,\upsilon)$ is open and bijective and (X,τ) is a T_2 space, then (Y,υ) is a T_2 space.

xlviii. If $f : (X,\tau) \to (X,\upsilon)$ is the identity function such that $\tau \subseteq \upsilon$ and (X,τ) is a T_2 space, then (X,υ) is a T_2 space.

xlix. If $f : (X,\tau) \to (Y,\upsilon)$ is closed and surjective, $f^{-1}(\{y\})$ is compact, for all $y \in Y$, (X,τ) is a T_2 space, then (Y,υ) is a T_2 space.

l. The topological space induced by a metric space is a T_2 space.

li. If (X,τ), (X,υ), and $(X,\tau \cap \upsilon)$ are T_2 spaces, then the set $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$ with respect to $\tau \times \upsilon$.

lii. There are T_2 spaces (X,τ) and (X,υ) such that the set $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$ with respect to $\tau \times \upsilon$.

liii. If (X,τ) and (X,υ) are T_2 spaces, and for all disjoint sets $U, V \in \tau$, there exist disjoint sets $U^*, V^* \in \upsilon$ such that $U \subseteq U^*$ and $V \subseteq V^*$ and vice versa, then $(X,\tau \cap \upsilon)$ is a T_2 space.

liv. For any nonempty set X, there exists a T_2 space (Y,υ) such that $Y = \bigcup_{x \in X} Y_x$, where the sets Y_x are mutually disjoints and dense in Y (Hint: $Y = X \times \mathbb{Q}$, $Y_x = \{x\} \times \mathbb{Q}$, and $\upsilon = \left\{\bigcup_{i \in I} X \times (I_i \cap \mathbb{Q}) : I_i \text{ is an interval in } \mathbb{R}\right\}$ such that I is an index set).

lv. Every topological space is the continuous open image of a T_2 space.

lvi. If for all distinct points $x, y \in X$, there exists a continuous function $f : X \to \mathbb{R}$ and distinct points $a, b \in \mathbb{R}$ such that $f(x) = a$ and $f(y) = b$, then the space is a T_2 space.

lvii. If, for all $x \in X$, there exists an $F \in \tau^c(x)$ such that F is a T_2 space with respect to the induced topology on (X,τ), then (X,τ) is a T_2 space.

lviii. For every topological spaces, there exists a T_2 space such that whose quotient space is the topological space.

lix. If (X,τ) is a compact T_2 space, the following are equivalent:

- The quotient space $(X/\equiv, \tau_\equiv)$ is a T_2 space.
- The canonical projection map π is a closed function.
- $\{(x, y) : \pi(x) = \pi(y)\}$ is closed in $X \times X$.

3.7. $T_{2\frac{1}{2}}$ Spaces

(X,τ) is a $T_{2\frac{1}{2}}$ (Urysohn) space or τ is a $T_{2\frac{1}{2}}$ (Urysohn) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exists a $K \in \tau^c(x)$ and an $L \in \tau^c(y)$ such that $K \cap L = \emptyset$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau(x)$ and a $V \in \tau(y)$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $N \in N(x)$ and an $M \in N(y)$ such that $\mathrm{cl}(N) \cap \mathrm{cl}(M) = \emptyset$

i. Every $T_{2\frac{1}{2}}$ space is a T_2 space.

ii. Every $T_{2\frac{1}{2}}$ space is a T_1 space.

iii. Every $T_{2\frac{1}{2}}$ space is a T_0 space.

iv. Every $T_{2\frac{1}{2}}$ space is a R_1 space.

v. Every $T_{2\frac{1}{2}}$ space is a R_0 space.

vi. Being a $T_{2\frac{1}{2}}$ space is a hereditary property.

vii. Being a $T_{2\frac{1}{2}}$ space is a topological property.

viii. For all $i \in I$, (X_i, τ_i) is a $T_{2\frac{1}{2}}$ space if and only if $\left(\prod_{i\in I} X_i, \tau\right)$ is a $T_{2\frac{1}{2}}$ space.

ix. For all $i \in I$, (X_i, τ_i) is a $T_{2\frac{1}{2}}$ space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a $T_{2\frac{1}{2}}$ space.

x. If $f : (X,\tau) \to (Y,\upsilon)$ is closed and surjective and (X,τ) is a $T_{2\frac{1}{2}}$ space, then (Y,υ) is a $T_{2\frac{1}{2}}$ space.

xi. If $f : (X,\tau) \to (Y,\upsilon)$ is open and surjective and (X,τ) is a $T_{2\frac{1}{2}}$ space, then (Y,υ) is a $T_{2\frac{1}{2}}$ space.

xii. If $f : (X,\tau) \to (X,\upsilon)$ is closed identity function such that $\tau \subseteq \upsilon$ and (X,τ) is a $T_{2\frac{1}{2}}$ space, then (X,υ) is a $T_{2\frac{1}{2}}$ space.

xiii. The topological space induced by a metric space is a $T_{2\frac{1}{2}}$ space.

3.8. Completely T_2 Spaces

(X,τ) is a completely (functionally) T_2 space or τ is a completely (functionally) T_2 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ with $x \in f^{-1}(\{0\})$ and $y \in f^{-1}(\{1\})$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ and there exist $\alpha, \beta \in (0,1)$ satisfying the condition $\alpha < \beta$ such that $x \in f^{-1}([0,\alpha))$ and $y \in f^{-1}((\beta,1])$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$
	$\Leftrightarrow$	(X,τ) is a T_2 space, and for all disjoint compact subsets A and B of X, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist two continuous functions $f, g : X \to \mathbb{R}$ such that $x \in \text{int}(\{z \in X : f(z) = 0\})$, $y \in \text{int}(\{z \in X : g(z) = 0\})$, and $\{z \in X : f(z) = 0\} \cap \{z \in X : g(z) = 0\} = \emptyset$

i. Every completely T_2 space is a R_1 space.

ii. Every completely T_2 space is a R_0 space.

iii. Every completely T_2 space is a $T_{2\frac{1}{2}}$ space.

iv. Every completely T_2 space is a T_2 space.

v. Every completely T_2 space is a T_1 space.

vi. Every completely T_2 space is a T_0 space.

vii. If (X,τ) is a topological space and $x \equiv y :\Leftrightarrow \forall f \in C_{\mathbb{R}}(X), f(x) = f(y)$, where $C_{\mathbb{R}}(X) := \{f : X \to \mathbb{R} : f \text{ is continuous}\}$, then $(X/\equiv, \tau_{\equiv})$ is a completely T_2 space.

viii. For all $i \in I$, (X_i, τ_i) is a completely T_2 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i \in I} X_i, \tau\right)$ is a completely T_2 space.

ix. The topological space induced by a metric space is a completely T_2 space.

3.9. Regular Spaces

(X,τ) is a regular space or τ is a regular topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exist $U, V \in \tau$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exists a $U \in \tau(x)$ such that $\mathrm{cl}(U) \cap F = \emptyset$
	$\Leftrightarrow$	For all $A, B \subseteq X$ satisfying the condition $A \setminus \mathrm{cl}(B) \neq \emptyset$, there exists a $U \in \tau$ such that $\mathrm{cl}(B) \subseteq U$ and $A \setminus \mathrm{cl}(U) \neq \emptyset$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exist $N \in N(x)$ and $N \in N(F)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exist $U \in \tau(x)$ and $V \in \tau(F)$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$
	$\Leftrightarrow$	For all $x \in X$ and for all $U \in \tau(x)$, there exists a $V \in \tau(x)$ such that $\mathrm{cl}(V) \subseteq U$
	$\Leftrightarrow$	For all $x \in X$ and for all $N \in N(x)$, there exists an $M \in N(x)$ such that $\mathrm{cl}(M) \subseteq N$
	$\Leftrightarrow$	For all $x \in X$ and for all $N \in N(x)$, there exists an $F \in \tau^c(x)$ such that $F \subseteq N$
	$\Leftrightarrow$	Every point in X has a neighborhood base consisting of closed sets.
	$\Leftrightarrow$	For all $F \in \tau^c$, $F = \bigcap\limits_{K \in \tau^c(F)} K$

i. Every regular space is an R_1 space.

ii. Every regular space is an R_0 space.

iii. In a topological space, if every open set is a closed set, then the space is a regular space.

iv. Being a regular space is a hereditary property.

v. Being a regular is a topological property.

vi. For all $i \in I$, (X_i, τ_i) is a regular space if and only if $\left(\prod\limits_{i \in I} X_i, \tau\right)$ is a regular space.

vii. For all $i \in I$, (X_i, τ_i) is a regular space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod\limits_{i \in I} X_i, \tau\right)$ is a regular space.

viii. (X, τ) is a regular space if and only if $(X/{\sim}, \tau_{\sim})$ is a regular space.

ix. If (X, τ) is a regular space, then for all $A, B \subseteq X$ satisfying the condition that there exists an $N \in N(A)$ such that $N \cap B = \emptyset$ or there exists an $M \in N(B)$ such that $A \cap M = \emptyset$, there exists an $N^* \in N(A)$ and an $M^* \in N(B)$ such that $N^* \cap M^* = \emptyset$.

x. In a regular space (X, τ), for all compact sets $A \subseteq X$ and for all $N \in N(A)$, there exists an $F \in \tau^c(A)$ such that $F \subseteq N$.

xi. If (X, τ) is a regular space, $A \subseteq X$, A is closed in X, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$, then (X, τ_A) is a regular space.

xii. If (X, τ) is a regular space and A is an infinite subset of X, then there exist open sets U_1, U_2, $\ldots$ in X such that $\mathrm{cl}(U_n) \cap \mathrm{cl}(U_m) = \emptyset$, for all $n, m \in \mathbb{N}$ with $n \neq m$, and $U_n \cap A \neq \emptyset$, for all $n \in \mathbb{N}$.

xiii. The topological space induced by a metric space is a regular space.

xiv. If, for all $x \in X$, there exists an $F \in \tau^c(x)$ such that F is a regular space with respect to the induced topology on (X, τ), then (X, τ) is a regular space.

xv. If, for all $x \in X$, there exists a $U \in \tau(x)$ such that U is a regular space with respect to the induced topology on (X, τ), then (X, τ) is a regular space.

xvi. In a regular space (X, τ), for all $F \in \tau^c$, $F = \bigcap_{U \in \tau(F)} U$.

xvii. In a regular space (X, τ), if every net in $S \subseteq X$ has a cluster point in X, then $\mathrm{cl}(S)$ is compact.

xviii. In a regular space (X, τ), if every proper filter on X containing $S \subseteq X$ has a cluster point in X, then $\mathrm{cl}(S)$ is compact.

3.10. T_3 Spaces

(X,τ) is a T_3 space or τ is a T_3 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_0 space and a regular space

i. Every T_3 space is a R_1 space.

ii. Every T_3 space is a R_0 space.

iii. Every T_3 space is a regular space.

iv. Every T_3 space is a $T_{2\frac{1}{2}}$ space.

v. Every T_3 space is a T_2 space.

vi. Every T_3 space is a T_1 space.

vii. Every T_3 space is a T_0 space.

viii. (X,τ) is a regular space if and only if $(X/{\sim},\tau_{\sim})$ is a T_3 space.

ix. Being a T_3 space is a hereditary property.

x. Being a T_3 is a topological property.

xi. For all $i \in I$, (X_i,τ_i) is a T_3 space if and only if $\left(\prod_{i\in I} X_i, \tau\right)$ is a T_3 space.

xii. For all $i \in I$, (X_i,τ_i) is a T_3 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a T_3 space.

xiii. For every T_3 space (X,τ), $|X| \le 2^{\min\{|A| \,:\, A \text{ is a dense set in } X\}}$.

xiv. If $f : (X,\tau) \to (Y,\upsilon)$ is closed, open, continuous, and surjective and (X,τ) is a T_3 space, then (Y,υ) is a T_3 space.

xv. If A is a compact subset of a T_3 space and B is a closed subset of the space with $A \cap B = \emptyset$, then there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

xvi. The topological space induced by a metric space is a T_3 space.

xvii. Every compact T_2 space is a T_3 space.

xviii. In a T_3 space, for all distinct $x, y \in X$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

3.11. Completely Regular Spaces

(X,τ) is a completely regular space or τ is a completely regular topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(F) = \{1\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 1$ and $f(F) = \{0\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [a,b]$ with $f(x) = a$ and $f(F) = \{b\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [a,b]$ with $f(x) = b$ and $f(F) = \{a\}$
	$\Leftrightarrow$	There exists a closed subbase $\mathcal{S}$ of X such that for each $x \in X$ and for each $F \in \mathcal{S}$ satisfying the condition $x \notin F$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(F) = \{1\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $U \in \tau(x) \setminus \{X\}$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(X \setminus U) = \{1\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $N \in N(x) \setminus \{X\}$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(X \setminus N) = \{1\}$
	$\Leftrightarrow$	Every singleton in X has a neighborhood base consisting of sets N such that for each N, there exists a continuous function $f : X \to \mathbb{R}$ with $N = f^{-1}(\{0\})$.
	$\Leftrightarrow$	Every closed set in X is an intersection of sets A such that for each A, there exists a continuous function $f : X \to \mathbb{R}$ with $A = f^{-1}(\{0\})$. (The family of the sets A such that for each A, there exists a continuous function $f : X \to \mathbb{R}$ with $A = f^{-1}(\{0\})$ is a basis for τ^c)
	$\Leftrightarrow$	The family of the sets A such that for each A, there exists a continuous function $f : X \to \mathbb{R}$ with $A = X \setminus f^{-1}(\{0\})$ is a basis for τ

i. Every completely regular space is an R_1 space.

ii. Every completely regular space is an R_0 space.

iii. Every completely regular space is a regular space.

iv. Every completely regular space is a completely T_2 space.

v. Every completely regular space is a $T_{2\frac{1}{2}}$ space.

vi. Every completely regular space is a T_2 space.

vii. Every completely regular space is a T_1 space.

viii. Every completely regular space is a T_0 space.

ix. Every T_0 and completely regular space is a T_3 space.

x. In a topological space, if every open set is a closed set, then the space is a completely regular space.

xi. Being a completely regular space is a hereditary property.

xii. Being a completely regular is a topological property.

xiii. For all $i \in I$, (X_i, τ_i) is a completely regular space if and only if $\left(\prod_{i\in I} X_i, \tau\right)$ is a completely regular space.

xiv. For all $i \in I$, (X_i, τ_i) is a completely regular space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a completely regular space.

xv. If (X, τ) is a completely regular space and a T_1 space, then for all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

xvi. In a completely regular space (X, τ), for all compact sets $A \subseteq X$ and for all $N \in N(A)$, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{1\}$ and $f(X \setminus N) = \{0\}$.

xvii. If (X, τ) is a completely regular space, $A \subseteq X$, A is closed in X, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$, then (X, τ_A) is a completely regular space.

xviii. The topological space induced by a metric space is a completely regular space.

xix. (X, τ) is a completely regular space if and only if $(X/\sim, \tau_\sim)$ is a completely regular space.

3.12. $T_{3\frac{1}{2}}$ Spaces

(X, τ) is a $T_{3\frac{1}{2}}$ (T_π or Tikhonov) space or τ is a $T_{3\frac{1}{2}}$ (T_π or Tikhonov) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X, τ) is a T_0 space and a completely regular space
	$\Leftrightarrow$	(X, τ) is homeomorphic to a subspace of a compact Hausdorff space.

i. Every $T_{3\frac{1}{2}}$ space is an R_1 space.

ii. Every $T_{3\frac{1}{2}}$ space is an R_0 space.

iii. Every $T_{3\frac{1}{2}}$ space is a completely regular space.

iv. Every $T_{3\frac{1}{2}}$ space is a T_3 space.

v. Every $T_{3\frac{1}{2}}$ space is a regular space.

vi. Every $T_{3\frac{1}{2}}$ space is a completely T_2 space.

vii. Every $T_{3\frac{1}{2}}$ space is a $T_{2\frac{1}{2}}$ space.

viii. Every $T_{3\frac{1}{2}}$ space is a T_2 space.

ix. Every $T_{3\frac{1}{2}}$ space is a T_1 space.

x. Every $T_{3\frac{1}{2}}$ space is a T_0 space.

xi. Being a $T_{3\frac{1}{2}}$ space is a hereditary property.

xii. Being a $T_{3\frac{1}{2}}$ space is a topological property.

xiii. (X, τ) is a completely regular space if and only if $(X/\sim, \tau_\sim)$ is a $T_{3\frac{1}{2}}$ space.

xiv. For all $i \in I$, (X_i, τ_i) is a $T_{3\frac{1}{2}}$ space if and only if $\left(\prod_{i\in I} X_i, \tau\right)$ is a $T_{3\frac{1}{2}}$ space.

xv. For all $i \in I$, (X_i, τ_i) is a $T_{3\frac{1}{2}}$ space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a $T_{3\frac{1}{2}}$ space.

xvi. In a $T_{3\frac{1}{2}}$ space (X, τ), for all distinct points $x, y \in X$, if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$.

xvii. If $f : (X, \tau) \to (Y, \upsilon)$ is closed, open, continuous, and surjective and (X, τ) is a $T_{3\frac{1}{2}}$ space, then (Y, υ) is a $T_{3\frac{1}{2}}$ space.

xviii. If A is a compact subset of a $T_{3\frac{1}{2}}$ space and B is a closed subset of the space with $A \cap B = \emptyset$, then there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

xix. The topological space induced by a metric space is a $T_{3\frac{1}{2}}$ space.

3.13. Normal Spaces

(X, τ) is a normal space or τ is a normal topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all disjoint sets $F, K \in \tau^c$, there exist $U, V \in \tau$ such that $F \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c$, there exist $U, V \in \tau$ such that $F \subseteq U$, $K \subseteq V$, and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$
	$\Leftrightarrow$	For all strongly separated sets $F, K \in \tau$ that can be written as the union of countably many closed sets, there exist $U, V \in \tau$ such that $F \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c$, there exists a $U \in \tau$ such that $F \subseteq U$ and $\mathrm{cl}(U) \cap K = \emptyset$
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c$, there exists an $N \in N(F)$ and an $M \in N(K)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all $K \in \tau^c$ and for all $N \in N(K)$, there exists an $M \in N(K)$ such that $\mathrm{cl}(M) \subseteq N$
	$\Leftrightarrow$	For all $K \in \tau^c$ and for all $U \in \tau(K)$, there exists an $V \in \tau(K)$ such that $\mathrm{cl}(V) \subseteq U$
	$\Leftrightarrow$	For all open sets $U, V \in \tau$ satisfying the condition $X = U \cup V$, there exist $F, K \in \tau^c$ such that $F \subseteq U$, $K \subseteq V$, and $X = F \cup K$
	$\Leftrightarrow$	For all open sets $U, V \in \tau$ satisfying the condition $X = U \cup V$, there exist $U^*, V^* \in \tau$ such that $\mathrm{cl}(U^*) \subseteq U$, $\mathrm{cl}(V^*) \subseteq V$, and $X = \mathrm{cl}(U^*) \cup \mathrm{cl}(V^*)$
	$\Leftrightarrow$	(Urysohn's Lemma) For all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to [0, 1]$ such that $f(F) = \{0\}$ and $f(K) = \{1\}$
	$\Leftrightarrow$	(A Generalization of Urysohn's Lemma) For all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to [a, b]$ such that $f(F) = \{a\}$ and $f(K) = \{b\}$.
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to \mathbb{R}$ and distinct points $a, b \in \mathbb{R}$ such that $f(F) = \{a\}$ and $f(K) = \{b\}$.
	$\Leftrightarrow$	(Tietze's Extension Theorem) If Y is a nonempty closed subspace of X and $f : Y \to [0, 1]$ (or $f : Y \to \mathbb{R}$) is a continuous function, then there exists a continuous function $g : X \to [0, 1]$ (or $g : X \to \mathbb{R}$) such that $g_{\mid Y} = f$.
	$\Leftrightarrow$	Every closed set in X has a neighborhood base consisting of closed sets.
	$\Leftrightarrow$	For all $U \in \tau$ and for all $K \in \tau^c$ satisfying the condition $K \subseteq U$, there exists an $F \in \tau^c(K)$ such that $K \subseteq U$

i. Every discrete space and every indiscrete space is a normal space.

ii. Every normal and R_0 space is a completely regular space.

iii. Every compact R_1 space is a normal space.

iv. Every compact T_2 space is a normal space.

v. Every compact regular space is a normal space.

vi. Every normal and regular space is a completely regular space.

vii. Being a normal space is a topological property.

viii. In a topological space, if every open set is a closed set, then the space is a normal space.

ix. Every closed subspace of a normal space is a normal space.

x. Every subspace of a normal space that can be written as the union of countably many closed sets is a normal space.

xi. For all $i \in I$, (X_i, τ_i) is a normal space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a normal space.

xii. If $\left(\prod_{i\in I} X_i, \tau\right)$ is a normal space, then for all $i \in I$, (X_i, τ_i) is a normal space.

xiii. Let $f : X \to Y$ be a surjective function. Then, (Y, υ) is a normal space if and only if $\left(X, f^{-1}(\upsilon)\right)$ is a normal space.

xiv. Every regular and Lindelöf space is a normal space.

xv. Every second countable and regular space is a normal space.

xvi. If (X, τ) and (Y, υ) are two topological space, (X, τ) is a normal space, and $f : X \to Y$ is a closed, continuous, and surjective function, then (Y, υ) is a normal space.

xvii. For all $F \in \tau^c$ and for all $U \in \tau(F)$, if there exists a family $\{U_i \in \tau : i \in \mathbb{N}\}$ such that $F \subseteq \bigcup_{i\in\mathbb{N}} U_i$ and $\mathrm{cl}(U_i) \subseteq U$, for all $i \in \mathbb{N}$, then (X, τ) is a normal space.

xviii. In a normal space (X, τ), if $F \in \tau^c$ that can be written as the intersection of countably many open sets and $K \in \tau^c$ satisfying the condition $F \cap K = \emptyset$, then there exists a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.

xix. Let (X, τ) be a normal space, $A \in \tau^c$, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$. Then, (X, τ_A) is a normal space if and only if $X \setminus A$ is a normal space with respect to the induced topology on (X, τ).

xx. The topological space induced by a metric space is a normal space.

3.14. T_4 Spaces

(X,τ) is a T_4 space or τ is a T_4 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a normal space
	$\Leftrightarrow$	(X,τ) is a T_2 space and a normal space

i. Every T_4 space is an R_1 space.

ii. Every T_4 space is an R_0 space.

iii. Every T_4 space is a normal space.

iv. Every T_4 space is a $T_{3\frac{1}{2}}$ space.

v. Every T_4 space is a completely regular space.

vi. Every T_4 space is a T_3 space.

vii. Every T_4 space is a regular space.

viii. Every T_4 space is a completely T_2 space.

ix. Every T_4 space is a $T_{2\frac{1}{2}}$ space.

x. Every T_4 space is a T_2 space.

xi. Every T_4 space is a T_1 space.

xii. Every T_4 space is a T_0 space.

xiii. Being a T_4 space is a topological property.

xiv. Every closed subspace of a T_4 space is a T_4 space.

xv. If $\left(\prod_{i\in I} X_i, \tau\right)$ is a T_4 space, then for all $i \in I$, (X_i, τ_i) is a T_4 space.

xvi. For all $i \in I$, (X_i, τ_i) is a T_4 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a T_4 space.

xvii. Every compact T_2 space is a T_4 space.

xviii. Every second countable and T_3 space is a T_4 space.

xix. Every Lindelöf and T_3 space is a T_4 space.

xx. Every countable T_3 space is a T_4 space.

xxi. The topological space induced by a metric space is a T_4 space.

xxii. If (X,τ) is a T_4 space and $\mathcal{B}$ is a base for (X,τ), then for all $U \in \mathcal{B}$ and for all $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in \mathrm{cl}(V)$ and $\mathrm{cl}(V) \subseteq U$.

xxiii. In a T_1 space, if there exists exactly one nonisolated point in the space, then the space is a T_4 space.

xxiv. In a T_2 space, if there exists a finitely number of nonisolated points in the space, then the space is a T_4 space.

xxv. In a T_4 space, if the family $\{U_1, U_2, \dots, U_n\}$ is an open cover of X, then there exists a closed cover $\{F_1, F_2, \dots, F_n\}$ of X such that $F_i \subseteq U_i$, for all $i \in I_n$.

xxvi. If (X,τ) is a T_4 space, $Y \subseteq X$, and there exist countably many closed sets F_i such that $Y = \bigcup_{i\in\mathbb{N}} F_i$, then (Y, τ_Y) is a T_4 space.

xxvii. Let (X, τ) be a T_4 space and $F \in \tau^c$. Then, there exist countably many open sets U_i such that $F = \bigcap_{i \in \mathbb{N}} U_i$ if and only if there exists a continuous function $f : X \to [0, 1]$ such that $F = f^{-1}(\{0\})$.

xxviii. Let (X, τ) be a T_4 space and $U \in \tau$. Then, there exist countably many closed sets F_i such that $U = \bigcup_{i \in \mathbb{N}} F_i$ if and only if there exists a continuous function $f : X \to [0, 1]$ such that $U = f^{-1}((0, 1])$.

xxix. If $f : (X, \tau) \to (Y, \upsilon)$ is closed, continuous, and surjective and (X, τ) is a T_4 space, then (Y, υ) is a T_4 space.

xxx. In a T_2 space (X, τ), the following are equivalent:

- The space (X, τ) is compact.
- For all (Y, υ), the projection $p : X \times Y \to Y$ is a closed and continuous function.
- For all T_4 spaces (Y, υ), the projection $p : X \times Y \to Y$ is a closed and continuous function.

xxxi. In a T_1 space (X, τ), if, for all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to [0, 1]$ such that $f(F) = \{0\}$ and $f(K) = \{1\}$, then the space is a T_4 space.

xxxii. In a T_1 space (X, τ), if, for all $K \in \tau^c$ and for all $U \in \tau(K)$, there exists an $V \in \tau(K)$ such that $\mathrm{cl}(V) \subseteq U$, then the space is a T_4 space.

xxxiii. In a T_1 space (X, τ), if, for all $K \in \tau^c$ and for all $N \in N(K)$, there exists an $M \in N(K)$ such that $\mathrm{cl}(M) \subseteq N$, then the space is a T_4 space.

xxxiv. In a T_4 space, for all distinct $F, K \in \tau^c$, there exist $U \in \tau(F)$ and $V \in \tau(K)$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

3.15. Completely Normal Spaces

(X, τ) is a completely (hereditarily) normal space or τ is a completely (hereditarily) normal topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all strongly separated sets $A, B \subseteq X$, there exists an $N \in N(A)$ and an $M \in N(B)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all strongly separated sets $A, B \subseteq X$, there exists a $U \in \tau(A)$ and a $V \in \tau(B)$ such that $U \cap V = \emptyset$
	$\Leftrightarrow$	For all $A, B \subseteq X$ satisfying the condition that there exists a $U \in \tau(A)$ and a $V \in \tau(B)$ such that $U \cap B = \emptyset$ and $A \cap V = \emptyset$, there exists an $N \in N(A)$ and an $M \in N(B)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all $A, B \subseteq X$ satisfying the condition that there exists an $N \in N(A)$ and an $M \in N(B)$ such that $N \cap B = \emptyset$ and $A \cap M = \emptyset$, there exists an $N^* \in N(A)$ and an $M^* \in N(B)$ such that $N^* \cap M^* = \emptyset$
	$\Leftrightarrow$	Every subspace of (X, τ) is a normal space.
	$\Leftrightarrow$	Every open subspace of (X, τ) is a normal space.
	$\Leftrightarrow$	For all $A \subseteq X$ and for all $U \in \tau(A)$ satisfying the condition $\mathrm{cl}(A) \subseteq U$, there exists an $F \in \tau^c(A)$ such that $F \subseteq U$.
	$\Leftrightarrow$	For all $F, K \in \tau^c$, there exist $F^*, K^* \in \tau^c$ such that $F^* \cup K^* = X$, $F^* \cap (F \cup K) = F$, and $K^* \cap (F \cup K) = K$

i. Every completely normal space is an R_1 space.

ii. Every completely normal space is an R_0 space.

iii. Every completely normal space is a T_4 space.

iv. Every completely normal space is a normal space.

v. Every completely normal space is a $T_{3\frac{1}{2}}$ space.

vi. Every completely normal space is a completely regular space.

vii. Every completely normal space is a T_3 space.

viii. Every completely normal space is a regular space.

ix. Every completely normal space is a completely T_2 space.

x. Every completely normal space is a $T_{2\frac{1}{2}}$ space.

xi. Every completely normal space is a T_2 space.

xii. Every completely normal space is a T_1 space.

xiii. Every completely normal space is a T_0 space.

xiv. Being a completely normal space is a hereditary property.

xv. Being a completely normal space is a topological property.

xvi. If $f : X \to Y$ is a function, υ is a topology on Y, and (Y, υ) is a completely normal space, then $\left(X, f^{-1}(\upsilon)\right)$ is a completely normal space.

xvii. If $f : X \to Y$ is a surjective function, υ is a topology on Y, and $\left(X, f^{-1}(\upsilon)\right)$ is a completely normal space, then (Y, υ) is a completely normal space.

xviii. For all $i \in I$, (X_i, τ_i) is a completely normal space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a completely normal space.

xix. Every second countable and T_3 space is a completely normal space.

xx. In a completely normal space (X, τ), for all strongly separated subsets A and B of X, there exists a $U \in \tau$ such that $A \subseteq U$ and $\mathrm{cl}(U) \cap B = \emptyset$.

xxi. In a completely normal space (X, τ), for all $A, B \subseteq X$ satisfying the condition $A \cap \mathrm{cl}(B) = \emptyset$, there exists a $U \in \tau$ such that $A \subseteq U$, $U \cap \mathrm{cl}(B) = \emptyset$, and $\mathrm{cl}(U) \cap B \subseteq \mathrm{cl}(A)$.

xxii. In a completely normal space (X, τ), for all mutually strongly separated subsets A_1, A_2, ..., A_n of X, there exist $U_1, U_2, \ldots, U_n \in \tau$ such that $A_i \subseteq U_i$ and $\mathrm{cl}(A_i) \cap \mathrm{cl}(A_j) = \mathrm{cl}(U_i) \cap \mathrm{cl}(U_j)$, for all $i, j \in I_n$ with $i \neq j$.

xxiii. The topological space induced by a metric space is a completely normal space.

3.16. T_5 Spaces

(X,τ) is a T_5 space or τ is a T_5 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a completely normal space
	$\Leftrightarrow$	Every subspace of (X,τ) is a T_4 space.
	$\Leftrightarrow$	(X,τ) is a T_2 space and a completely normal space

i. Every T_5 space is an R_1 space.

ii. Every T_5 space is an R_0 space.

iii. Every T_5 space is a completely normal space.

iv. Every T_5 space is a T_4 space.

v. Every T_5 space is a normal space.

vi. Every T_5 space is a $T_{3\frac{1}{2}}$ space.

vii. Every T_5 space is a completely regular space.

viii. Every T_5 space is a T_3 space.

ix. Every T_5 space is a regular space.

x. Every T_5 space is a completely T_2 space.

xi. Every T_5 space is a $T_{2\frac{1}{2}}$ space.

xii. Every T_5 space is a T_2 space.

xiii. Every T_5 space is a T_1 space.

xiv. Every T_5 space is a T_0 space.

xv. The topological space induced by a metric space is a T_5 space.

xvi. Being a T_5 space is a hereditary property.

xvii. Being a T_5 space is a topological property.

xviii. If $\left(\prod_{i\in I} X_i, \tau\right)$ is a T_5 space, then for all $i \in I$, (X_i, τ_i) is a T_5 space.

xix. For all $i \in I$, (X_i, τ_i) is a T_5 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i, \tau\right)$ is a T_5 space.

3.17. Perfectly Normal Spaces

(X, τ) is a perfectly normal space or τ is a perfectly normal topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all disjoint sets $F, K \in \tau^c$, there exists a continuous function $f : X \to [0, 1]$ with $F = f^{-1}(\{0\})$ and $K = f^{-1}(\{1\})$
	$\Leftrightarrow$	(X, τ) is a normal space and every open set in X can be written as the union of countably many closed sets.
	$\Leftrightarrow$	(X, τ) is a normal space and every closed set in X can be written as the intersection of countably many open sets.
	$\Leftrightarrow$	For all $U \in \tau$, there exists a continuous function $f : X \to [0, 1]$ such that $X \setminus U = f^{-1}(\{0\})$
	$\Leftrightarrow$	For all $F \in \tau^c$, there exists a continuous function $f : X \to [0, 1]$ such that $F = f^{-1}(\{0\})$

i. Every perfectly normal space is an R_1 space.

ii. Every perfectly normal space is an R_0 space.

iii. Every perfectly normal space is a completely normal space.

iv. Every perfectly normal space is a T_4 space.

v. Every perfectly normal space is a normal space.

vi. Every perfectly normal space is a $T_{3\frac{1}{2}}$ space.

vii. Every perfectly normal space is a completely regular space.

viii. Every perfectly normal space is a T_3 space.

ix. Every perfectly normal space is a regular space.

x. Every perfectly normal space is a completely T_2 space.

xi. Every perfectly normal space is a $T_{2\frac{1}{2}}$ space.

xii. Every perfectly normal space is a T_2 space.

xiii. Every perfectly normal space is a T_1 space.

xiv. Every perfectly normal space is a T_0 space.

xv. The topological space induced by a metric space is a perfectly normal space.

xvi. Being a perfectly normal space is a hereditary property.

xvii. If $f : (X, \tau) \to (Y, \upsilon)$ is closed, continuous, and surjective and (X, τ) is a perfectly normal space, then (Y, υ) is a perfectly normal space.

xviii. For all $i \in I$, (X_i, τ_i) is a perfectly normal space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i \in I} X_i, \tau\right)$ is a perfectly normal space.

xix. In a perfectly normal space (X, τ), for all $F \in \tau^c$, there exists a continuous function $f : X \to [0, 1]$ with $F = f^{-1}(\{0\})$.

xx. If (X, τ) is a normal and an R_0 space and for all $F \in \tau^c$, there exists a countable open base of F for $\tau(F)$, then (X, τ) is a perfectly normal space.

3.18. T_6 Spaces

(X,τ) is a T_6 space or τ is a T_6 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a perfectly normal space
	$\Leftrightarrow$	(X,τ) is a T_4 space and every open set in X can be written as the union of countably many closed sets.
	$\Leftrightarrow$	(X,τ) is a T_4 space and every closed set in X can be written as the intersection of countably many open sets.
	$\Leftrightarrow$	(X,τ) is a T_2 space and a perfectly normal space

i. Every T_6 space is an R_1 space.

ii. Every T_6 space is an R_0 space.

iii. Every T_6 space is a perfectly normal space.

iv. Every T_6 space is a T_5 space.

v. Every T_6 space is a completely normal space.

vi. Every T_6 space is a T_4 space.

vii. Every T_6 space is a normal space.

viii. Every T_6 space is a $T_{3\frac{1}{2}}$ space.

ix. Every T_6 space is a completely regular space.

x. Every T_6 space is a T_3 space.

xi. Every T_6 space is a regular space.

xii. Every T_6 space is a completely T_2 space.

xiii. Every T_6 space is a $T_{2\frac{1}{2}}$ space.

xiv. Every T_6 space is a T_2 space.

xv. Every T_6 space is a T_1 space.

xvi. Every T_6 space is a T_0 space.

xvii. The topological space induced by a metric space is a T_6 space.

xviii. If $\left(\prod_{i\in I} X_i,\tau\right)$ is a T_6 space, then for all $i \in I$, (X_i,τ_i) is a T_6 space.

xix. For all $i \in I$, (X_i,τ_i) is a T_6 space such that $\{X_i : i \in I\}$ is mutually disjoint if and only if $\left(\coprod_{i\in I} X_i,\tau\right)$ is a T_6 space.

4. Conclusion

In this study, basic historical information related to the separation axioms under consideration is first provided. Afterward, the definitions and properties appearing in the sources in References related to these separation axioms are provided within a consistent framework. The definitions, examples, and properties of the separation axioms are not limited to those in this study. This draft can be extended by exploring and systematizing other topological concepts – such as paracompact, uniform, or metric spaces – and their relationships with the separation axioms. Based on the gathered information, Figure 1 illustrates the implications among the separation axioms without any additional conditions, Figure 2 presents the condition-dependent implications, and Table 1 provides information on whether the considered separation axioms satisfy the properties of being closed-hereditary, hereditary, refinement-preserved, closed-surjective-map-preserved, open-surjective-map-preserved, closed-continuous-surjective-map-preserved, closed-open-continuous-surjective-map-preserved, closed-bijective-map-preserved, open-bijective-map-preserved, topological, productive, projective, disjoint sum, disjoint summand, preimage-map-preserved, surjective-preimage-map-preserved, and Kolmogorov quotient.

In Figure 1, the implications among the separation axioms considered in this study are presented. In this figure, the expression

$$A \to B$$

means "If (X, τ) is an A space, then it is a B space." In particular, the expression

$$M \to T_6$$

means "If (X, τ) is the topological space induced by a metric space, then it is a T_6 space." Moreover, the abbreviations used in the figure are as follows: Perfectly normal (PN), completely normal (CN), normal (N), completely regular (CR), regular (R), and completely T_2 (CT_2).

$$\begin{tikzcd}
 & & PN \arrow[dr] & & N & & CR \arrow[r] \arrow[dr] & R \arrow[rr] & & R_1 \arrow[r] & R_0 & \\
M \arrow[r] & T_6 \arrow[ur] \arrow[dr] & & CN \arrow[r] & T_4 \arrow[u] \arrow[r] & T_{3\frac{1}{2}} \arrow[ur] \arrow[dr] & & CT_2 \arrow[dr] & & & & \\
 & & T_5 \arrow[ur] & & & & T_3 \arrow[rr] \arrow[d] & & T_{2\frac{1}{2}} \arrow[r] & T_2 \arrow[r] \arrow[uu] & T_1 \arrow[r] \arrow[uu] & T_0 \\
 & & & & & & R & & & & &
\end{tikzcd}$$

Figure 1. Hierarchy of the separation axioms considered in this study

In Figure 2, some condition-dependent implications related to the separation axioms considered in this study are presented. In this figure, the expression

$$A \xrightarrow{B} C$$

means "If (X, τ) is A and B spaces, then it is a C space." Here, B is an additional condition required for A to imply C. For example,

$$\begin{tikzcd}
PN \arrow[r, bend left, "T_1"] \arrow[r, bend right, "T_2"'] & T_6
\end{tikzcd}$$

means "If (X, τ) is perfectly normal and T_1 spaces, then it is a T_6 space" and "If (X, τ) is perfectly normal and T_2 spaces, then it is a T_6 space." In addition to the abbreviations in Figure 1, the abbreviations used in Figure 2 are as follows: Compact (Cp), Lindelöf (L), Second Countable Space (SCS), and Countable (C).

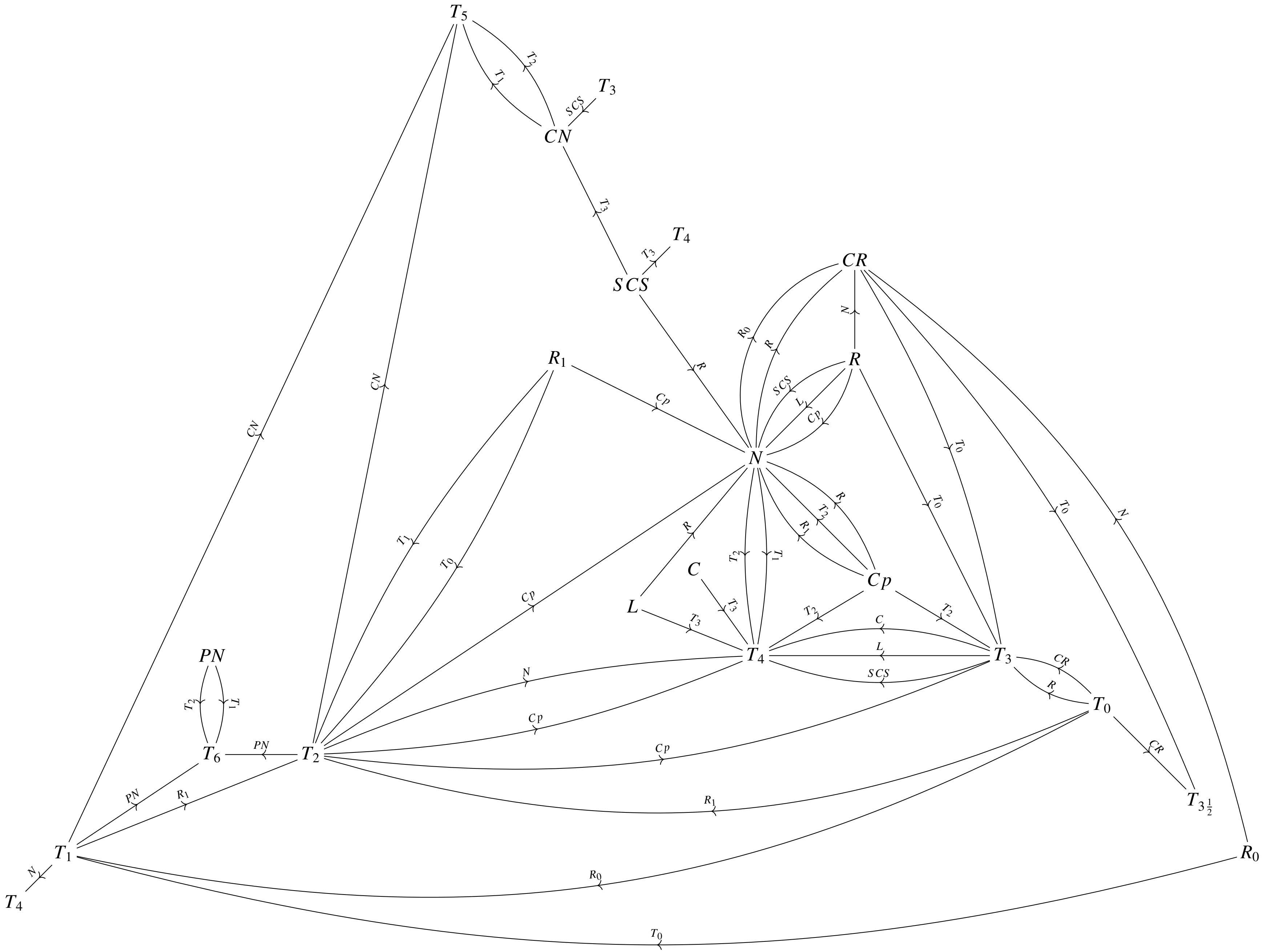

Figure 2. Implications under specific conditions concerning the separation axioms considered in this study

Table 1. Properties concerning the considered separation axioms

	T_0	R_0	T_1	R_1	T_2	$T_{2\frac{1}{2}}$	CT_2	R	T_3	CR	$T_{3\frac{1}{2}}$	N	T_4	CN	T_5	PN	T_6
Closed-Hereditary Property	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	
Hereditary Property	+	+	+	+	+	+		+	+	+	+	–	–	+	+	+	
Refinement-Preserved Property	+		+		+												
Closed-Surjective-Map-Preserved Property	+		+		–	+											
Open-Surjective-Map-Preserved Property	+		+		–	+											
Closed-Continuous-Surjective-Map-Preserved Property	+		+			+						+	+			+	
Closed-Open-Continuous-Surjective-Map-Preserved Property	+		+			+			+		+	+	+			+	
Closed-Bijective-Map-Preserved Property	+		+		+	+											
Open-Bijective-Map-Preserved Property	+		+		+	+											
Topological Property	+		+	+	+	+		+	+	+	+	+	+	+	+		
Productive Property	+	+	+	+	+	+		+	+	+	+	–	–		–		–
Projective Property	+	+	+	+	+	+		+	+	+	+	+	+		+		+
Disjoint Sum Property	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Disjoint Summand Property	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Preimage-Map-Preserved Property														+			
Surjective-Preimage-Map-Preserved Property												+		+			
Kolmogorov Quotient Property	+	+	+	+	+			+	+	+	+						

"+" indicates that the corresponding separation axiom satisfies the relevant property, whereas "–" indicates that it does not.

References

[1] Alexandroff, P., & Hopf, H. (1935). Topologie I. *Springer*.

[2] Alexandroff, P., & Urysohn, P. (1924). Zur Theorie der topologischen Räume. *Mathematische Annalen*, 92, 258–266.

[3] Alexandroff, P., & Urysohn, P. (1929). Mémoire sur les espaces topologiques compacts. *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam*.

[4] Arkhangel'skii, A. V., & Fedorchuk, V. V. (1990). General topology I: Basic concepts and constructions dimension theory. *Springer*.

[5] Arkhangel'skii, A. V., & Ponomarev, V. I. (1987). Fundamentals of general topology: Problems and exercises. *Springer*.

[6] Aslım, G. (2011). Genel topoloji, 5th Edition, *Ege University*.

[7] Baum, J. D. (1964). Elements of point-set topology, *Prentice Hall*.

[8] Bourbaki, N. (1967). Elements of mathematics: General topology, Part 1, *Addison-Wesley Publishing Company*.

[9] Bourbaki, N. (1967). Elements of mathematics: General topology, Part 2, *Addison-Wesley Publishing Company*.

[10] Čech, E. (1932). Sur la dimension des espaces parf aitement normaux, *Bull. Intern. Acad. Tcheque Sci.* 33, 38–55.

[11] Croom, F. H. (2008). Principles of topology. *Cengage Learning*.

[12] Császár, Á. (1978). General topology. *Adam Hilger*.

[13] Davis, A. S. (1961). Indexed systems of neighborhoods for general topological spaces. *The American Mathematical Monthly*, 68 (9), 886–894.

[14] Dube, T. (1974). A note on R_0-topological spaces. *Matematički Vesnik*, 11 (26), 203–208

[15] Dube, T. (1982). A note on R_1-topological spaces. *Periodica Mathematica Hungarica*, 13 (4), 267–271

[16] Engelking, R. (1989). General topology. *Heldermann Verlag*.

[17] Ercan, Z. (2017). A characterization of completely Hausdorff spaces. *Mathematical Proceedings of the Royal Irish Academy*, 117A (1), 1–4.

[18] Fréchet, M. (1906). Sur quelques points du calcul fonctionnel, *Rend. del Circ. Mat. di Palermo* 22, 1–72.

[19] Fréchet, M. (1926). Les espaces abstraits, *Gauthier Villars*.

[20] Gamelin, T. W., & Greene, R. E. (1999). Introduction to topology, 2nd Edition, *Dover Publications*.

[21] Hausdorff, F. (1914). Grundzüge der Mengenlehre. *Veit & Company*.

[22] Hausdorff, F. (2005). Set theory, *American Mathematical Society Chelsea Publishing*.

[23] Hausdorff, F. (1935). Gestufte Riiume, *Fund. Math.* 25, 486–502.

[24] Ilgaz, A. (1987). Topolojiye giriş. *Marmara University*.

[25] Karaca, İ. (2021). Teorik ve Uygulama Alanlarıyla Topoloji. *Palme Yayınevi*.

[26] Kelley, J. L. (1975). General topology. *Springer*.

[27] Koçak, M. (2021). Genel Topolojiye Giriş ve Problem Çözümleri. *Nisan Kitabevi*.

[28] Kuratowski, K. (1966). *Topology*. Vol. I, *Academic Press*.

[29] Lipschutz, S. (1965). Schaum's outline of theory and problems of general topology. *McGraw-Hill*.

[30] Mendelson, B. (1990). Introduction to topology. 3rd Edition, *Dover Publications*.

[31] Morris, S. A. (2024). Topology without tears. https://www.topologywithouttears.net/

[32] Munkres, J. R. (2014). Topology. *Pearson Education Limited*.

[33] Murdeshwar, M. G., & Naimpally, S. A. (1966). R_1-topological spaces. *Canadian Mathematical Bulletin*, 9 (4), 521–523.

[34] Murdeshwar, M. G., & Naimpally, S. A. (1966). Semi-Hausdorff spaces. *Canadian Mathematical Bulletin*, 9 (3), 353–356.

[35] Naimpally, S. A. (1967). On R_0-topological spaces. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica* 10, 53–54

[36] Riesz, F. (1907). Die Genesis des Raumbegriffs, *Math. und Naturwiss. Berichte aus Ungarn* 24, 309–353.

[37] Schechter, E. (1997). Handbook of analysis and its foundations, *Academic Press*.

[38] Shanin, N. A. (1943). On separation in topological spaces. *Doklady Akademii Nauk SSSR*, 38, 110–113.

[39] Steen, L. A., & Seebach, J. A. (1978). Counterexamples in topology. *Springer*.

[40] Sutherland, W. A. (2009). Introduction to metric & topological spaces. *Oxford University Press*.

[41] Tietze, H. (1923). Beiträge zur allgemeinen Topologie. I, *Math. Ann.* 88, 290–312.

[42] Tychonoff, A. (1930). Über die topologische Erweiterung von Räumen, *Math. Ann.* 102, 544–561.

[43] Urysohn, P. (1925). Über die Mächtigkeit der zusammenhängenden Mengen, *Math. Ann.* 94, 262–295.

[44] Vietoris, L. (1921). Stetige Mengen, *Monatsh. für Math. und Phys.* 31, 173–204.

[45] Willard, S. (2004). General topology. *Dover Publications*.

[46] Wikipedia (2002). Separation axiom. Accessed Dec 23, 2025. Retrieved from https://en.wikipedia.org/wiki/Separation_axiom

[47] Yang, C.-T. (1954). On paracompact spaces. *Proceedings of the American Mathematical Society*, 5, 185–189.

[48] Yüksel, Ş. (2010). Genel topoloji. 7th Edition, *Eğitim Akademi*.