Separation Axioms: Characterizations and Properties

Contents

1. Introduction

Although the definitions presented here for ”regular” and ”T3” are not uncommon, there is significant variation in the literature: some authors switch the definitions of ”regular” and ”T3” as they are used here, or use both terms interchangeably. This article uses the term ”regular” freely, but will usually say ”regular Hausdorff”, which is unambiguous, instead of the less precise ”T3”.

CHATGPT

These notes provide a historically grounded and technically precise account of the development of the classical separation axioms in general topology, including the weaker symmetry/preregularity axioms usually denoted by R_0 and R_1. The text is written for use as lecture notes: it mixes careful formal definitions, short proofs or proof sketches where illuminating, and historically annotated commentary identifying the central contributors and the chronology of ideas. The emphasis is on giving instructors and advanced students both rigorous material and context for how modern terminology came into being.

Overview and goals

These notes aim to do the following:

- Present formal definitions of the most commonly used separation axioms (from T_0 up to T_6, and variants such as $T_{2\frac{1}{2}}$ and complete regularity), together with R_0 and R_1.
- Trace the historical origin of each axiom or property: who introduced it, in what year or work, and how the modern name evolved.
- Provide a compact list of standard theorems linking the axioms and a few remarks about their mathematical significance.
- Offer a short annotated bibliography (primary and standard texts) for further reading.

Short timeline of key milestones

- **Early 20th century: abstraction of topological ideas.** Maurice Fréchet emphasized abstract spaces (1906), initiating broader axiomatizations of ’spaces’ where notions of open sets and convergence could be studied without metrics.
- **1914: Hausdorff’s Grundzüge der Mengenlehre.** Felix Hausdorff synthesized set theory and topology into a systematic treatment; his presentation popularized the separation property now called Hausdorff or T_2 and laid groundwork for the modern axiomatic approach.
- **1920s: Urysohn, Alexandroff, and others.** Urysohn’s work on normality, the lemma that now bears his name, and related metrization ideas further clarified stronger separation properties and demonstrated their consequences (e.g. extension theorems, metrization results).
- **Mid 20th century: formal naming and weaker variants.** Kolmogorov’s name became attached to the weakest axiom (T_0), while notions equivalent or closely related to the modern R_0 and R_1 were introduced and studied by Shanin (circa 1943) and later by Davis (1961), among others. Subsequent decades saw fuller systematic catalogs of the ‘T-axioms’ in textbooks by Kelley, Willard, Munkres, and others.

Definitions (formal) and short commentary

Throughout, let $(X, \mathcal{T})$ be a topological space. We assume familiarity with the basic language (open/closed sets, closure $\overline{A}$, neighbourhoods, continuous maps).

Distinguishability and separation (common vocabulary)

Definition 1.1 (Topological distinguishability)**.** Two points $x, y \in X$ are said to be topologically distinguishable if there exists an open set containing one but not the other. Equivalently, $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Definition 1.2 (Separated points)**.** Two points $x, y \in X$ are separated if there exist neighbourhoods U of x and V of y with $U \cap V = \emptyset$.

Classical T-axioms

We list a standard ordering and the equivalent historical names that one encounters in textbooks. Modern practice often places a colloquial prefix T on many axioms, with the convention that some authors require T_0 as a baseline for higher T_n and others do not; below we state each axiom as an independent property.

Definition 1.3 (T_0 (Kolmogorov))**.** *X is a T_0-space if for every pair of distinct points $x, y \in X$ at least one of them has an open neighbourhood not containing the other.*

Definition 1.4 (T_1 (Fréchet / accessible))**.** *X is a T_1-space if for every pair of distinct points x, y each has an open neighbourhood not containing the other. Equivalently, every singleton $\{x\}$ is closed.*

Definition 1.5 (R_0 (symmetric / Shanin))**.** A space X is R_0 if any two topologically distinguishable points are separated (i.e. each lies in an open set not containing the other). Equivalently, points lying in the same closure of singletons cannot be separated by open sets in opposite ways; another characterization: every open set is a union of closures of singletons.

Definition 1.6 (R_1 (preregular / Davis))**.** A space X is R_1 if any two topologically distinguishable points can be separated by neighbourhoods (so R_1 is the preregular analogue of Hausdorff: Hausdorff = $T_0 + R_1$). In other words, R_1 asserts the usual disjoint-neighbourhood separation only for pairs that are already topologically distinguishable.

Remarks on conventions

Different authors choose slightly different baseline conventions: some require a T_0 hypothesis before naming higher axioms (T_3 as 'regular + T_1'), some include the T_1 requirement in the definition of regular or normal. In lecture notes it is best to state your convention upfront and keep it consistent; the rest of this document uses the convention of treating the separation properties as independent properties and indicating explicitly when T_0 or T_1 is appended.

Historical notes (detailed)

This section gives more detailed historical remarks, arranged by person and theme. Dates refer to original publications where possible.

Maurice Fréchet (1906)

Maurice Fréchet introduced the idea of 'abstract spaces' in his 1906 thesis and early work: instead of restricting attention to metric or Euclidean examples, he proposed studying spaces in which convergence and neighborhood were axiomatized. The terminology 'Fréchet space' later came to denote certain topological vector spaces, while the name 'Fréchet topology' is sometimes historically used for the accessibility property that later became T_1; hence T_1 is often called the Fréchet axiom in older texts.

Felix Hausdorff (1914)

In his influential 1914 book Grundzüge der Mengenlehre Hausdorff gave one of the first systematic expositions that treated topological ideas axiomatically. Hausdorff introduced and investigated properties that include what is now called Hausdorff or T_2 separation. His text propelled the shift from metric-specific arguments to an axiomatic point of view in which separation axioms were natural structural hypotheses.

Kolmogorov and the T_0 name

Andrey Kolmogorov's influence on topology is multifold; his name became attached to the weakest separation requirement, T_0, which captures a minimal distinguishability condition. Kolmogorov's work in the 1930s and expositions by later authors fixed the eponym.

Urysohn, Tietze, and the function-separation perspective

Pavel Urysohn (whose work was published in the mid-1920s) developed basic existence results for continuous functions separating closed sets in normal spaces (Urysohn's lemma). Heinrich Tietze earlier proved extension results for metric spaces (Tietze extension theorem). Together these results convinced topologists that separation by continuous functions — not merely by disjoint open sets — is a central and fruitful concept; it led to the modern hierarchy involving complete regularity, Tychonoff spaces, and related function-space techniques.

Shanin and Davis: R_0 and R_1

The symmetric or R_0 axiom was explicitly identified by N. A. Shanin in the 1940s in investigations of separation in the soviet literature; later, A. S. Davis in 1961 reintroduced and popularized a preregularity notion later called R_1. These axioms filled a conceptual gap: they let researchers state 'Hausdorff-like' or 'regular-like' properties while avoiding the requirement that all points be topologically distinguishable, which is necessary in many pathological or quotient-space constructions.

Textbook consolidation and the modern picture

From the mid-20th century onward, standard topology textbooks (Kelley, 1955; Willard, 1970; Munkres, 1975 and later editions) collated the various separation axioms, standardized definitions, and emphasized relationships and counterexamples. These texts are the primary source for the modern, widely used hierarchy of separation axioms and for pedagogical ordering in lectures.

Key theorems and relations (compact list)

We list commonly used implications; proofs can be found in standard texts.

- T_2 implies T_1 implies T_0.
- T_2 is equivalent to $T_0 + R_1$.
- Regularity + T_1 gives the usual T_3 notion (often called regular Hausdorff).
- Normality + T_1 gives T_4; metric spaces are T_4.
- Urysohn's lemma: normal T_1 spaces allow separation of disjoint closed sets by continuous functions.
- Tietze extension theorem: in normal T_1 spaces continuous functions defined on closed sets extend to the whole space.

Pedagogical recommendations

When teaching these concepts to students:

- Start with examples and counterexamples rather than only axiomatic implications; classical counterexamples (cofinite, cocountable, lower limit, Moore, and particular point topologies) illuminate the independence of axioms.
- Be explicit about the conventions: whether T_1 is assumed when one says 'regular' or 'normal'.
- Use the R_0/R_1 discussion when motivating quotient constructions: quotient spaces often fail T_0 or T_1 but sometimes meet R_0 or R_1 and one must know how theorems adapt.

Annotated bibliography (select)

- Hausdorff, F., Grundzüge der Mengenlehre, 1914. (Foundational; early axiomatic exposition.)
- Fréchet, M., 1906 thesis material. (Introduction of abstract spaces.)

- Kolmogorov, A. N., 1930s work on topology (T0 attribution).
- Urysohn, P., papers (early 1920s; Urysohn's lemma, normality discussions).
- Tietze, H., early 20th century, extension theorem for metric spaces.
- Shanin, N. A., 1943, on separation in topological spaces (introduction of symmetric or R_0 idea).
- Davis, A. S., 1961, works relating to preregular spaces (R_1).
- Munkres, J. R., Topology: A First Course, later editions. (Pedagogical consolidation of the separation axioms.)
- Kelley, J. L., General Topology, 1955. (Authoritative modern treatment.)

DeepSeek

This note provides a detailed historical context and an exhaustive mathematical exposition of the hierarchy of separation axioms (the Trennungsaxiome) in point-set topology. We trace their development from the early 20th century, starting with the foundational work of Felix Hausdorff, through the refinements of Heinrich Tietze, Pavel Urysohn, and others. Special emphasis is placed on the often-overlooked symmetry axioms R_0 and R_1, which provide a crucial link between the lower separation axioms and the stronger notions of regularity and complete regularity. This exposition includes a comprehensive analysis of the T_5 and T_6 axioms, often omitted from introductory texts, and integrates numerous characterizations using filters, nets, and closure operators, as found in advanced literature.

Introduction: The Motivation for Separation

The fundamental concept in topology is to formalize the notion of "closeness" without relying on a distance metric. Early topological spaces, as defined by Fréchet, were often too general to prove meaningful theorems analogous to those in analysis. A central problem was finding conditions under which points and closed sets can be "separated" by disjoint open sets or continuous functions. This led to the creation of a hierarchy of conditions now known as the separation axioms.

Historical Genesis: 1914–1925

The story begins with Felix Hausdorff's groundbreaking 1914 book, Grundzüge der Mengenlehre. In this work, he gave the first essentially modern definition of a topological space based on neighborhoods. He then imposed an extra condition:

Definition 1.7 (Hausdorff's Original Axiom)**.** For any two distinct points x and y, there exist neighborhoods U_x and U_y such that $U_x \cap U_y = \emptyset$.

This is our modern T_2 axiom. Hausdorff's work was profoundly influential, and for a time, the term "Hausdorff space" was synonymous with "topological space." However, it soon became clear that many useful spaces did not satisfy this strong condition (e.g., the Zariski topology, which is not T_2), while many weaker conditions were still strong enough to prove important results.

In the early 1920s, several mathematicians worked on refining these notions:

- **Heinrich Tietze (1921)** and **Kazimierz Kuratowski** explored properties that would allow for the extension of continuous functions (a problem leading to $T_{3.5}$ and T_4).
- **Pavel Urysohn (1923)**, in his famous work on metrization theorems, explicitly used and formalized what we now call **regularity** and **normality**. His famous lemma is a cornerstone of the theory.
- **Heinrich Tietze (1923)** published his extension theorem for normal spaces (T_4).

The systematic classification into the now-standard T_i hierarchy is largely attributed to the influential books of **Andrey Kolmogorov** (1930s) and especially **John L. Kelley** in his 1955 book General Topology. The "T" is often thought to stand for "Trennungsaxiom" (German for separation axiom), though its exact origin is debated. The axioms T_5 (completely normal Hausdorff) and T_6 (perfectly normal Hausdorff) emerged as further refinements to address the behavior of subspaces and the existence of certain continuous functions.

The Standard Hierarchy of T-Axioms and Related Concepts

The axioms are defined incrementally. It is crucial to remember that higher T_i numbers imply stronger separation conditions, but the converse is not true. The following table, while extensive, is not exhaustive but captures many key definitions and equivalent characterizations, particularly those involving filters, nets, and the R_0/R_1 properties.

The R_0 and R_1 Axioms: A Deeper Look

The R_0 and R_1 axioms, formalized later than the standard T_i axioms, provide a more granular decomposition of separation properties. They focus on the symmetry of the topology and the separation of points from each other.

Theorem 1.8 (The Fundamental Link via R_0/R_1)**.** The R_1 and R_0 axioms connect directly to the standard hierarchy:

- A space is T_1 **if and only if** it is both T_0 and R_0.
- A space is T_2 (Hausdorff) **if and only if** it is both T_0 and R_1.
- More generally, a space is T_2 **if and only if** it is R_1 and T_1 (the T_0 condition is implied by T_1).

This theorem shows that R_0 is the property that ensures "points are closed" when combined with distinguishability (T_0), while R_1 is the property that ensures "distinguishable points can be separated" when combined with distinguishability (T_0).

The Higher Axioms: T_5 and T_6

The T_5 and T_6 axioms address the hereditary nature of normality and the strong interplay between closed sets and continuous functions.

Definition 1.9 (Completely Normal (T_5))**.** A space is **completely normal** if every subspace is normal. A T_5 space is a completely normal T_1 space (and hence Hausdorff).

Remark 1.10. This is equivalent to the statement that for any two separated sets A and B (i.e., $A \cap \overline{B} = \overline{A} \cap B = \emptyset$), there exist disjoint open sets containing them. This is a stronger condition than normality, which only requires this for sets that are already closed *and* disjoint. The classic example is the Tychonoff plank with a corner point removed, which is normal but not completely normal (as its subspace, the original plank, is not normal).

Definition 1.11 (Perfectly Normal (T_6))**.** A space is **perfectly normal** if it is normal and every closed set is a G_δ set (a countable intersection of open sets). A T_6 space is a perfectly normal T_1 space.

Remark 1.12. Perfect normality is a very strong separation property. It implies that not only can disjoint closed sets be separated by a function (Urysohn's Lemma), but that each closed set is precisely the zero set of some continuous real-valued function. Every metric space is perfectly normal. Perfect normality implies complete normality ($T_6 \Rightarrow T_5$).

2. Preliminaries

Throughout this study, let the notations $\mathbb{Z}^+$, $\mathbb{N}$, and $\mathbb{R}$ represent the sets of all positive integers, nonnegative integers, and real numbers, respectively. Moreover, let I be an index set and $I_n := \{1, 2, 3, \ldots, n\}$, for all $n \in \mathbb{Z}^+$.

Definition 2.1. Let X be a set and $\tau \subseteq P(X)$. Then, τ is called a topology on X if the following conditions hold:

i. $\emptyset, X \in \tau$

ii. If $U_i \in \tau$, for all $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$

iii. If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$

Moreover, the ordered pair (X, τ) is called a topological space.

Throughout this study, let (X, τ) and (Y, υ) be two topological space.

Definition 2.2. (X, τ) be a topological space and $A \subseteq X$. If $A \in \tau$, then A is called a τ-open (briefly an open) set in X. Moreover, if $X \setminus A \in \tau$, then A is called a τ-closed (briefly a closed) set in X.

Throughout this study, let τ^c represent the set of all closed sets in X.

Definition 2.3. Let (X, τ) be a topological space and let $x \in X$ and $A \subseteq X$.

- The family of open neighborhoods of x is defined as follows:

$$\tau(x) := \{U \in \tau : x \in U\}$$

- The family of neighborhoods of x is defined as follows:

$$N(x) := \{N \subseteq X : \exists G \in \tau \ni x \in G \wedge G \subseteq N\}$$

- The family of closed neighborhoods of x is defined as follows:

$$\tau^c(x) := \{F \in \tau^c : \exists G \in \tau \ni x \in G \wedge G \subseteq F\}$$

Similarly,

- The family of open neighborhoods of A is defined as follows:

$$\tau(A) := \{U \in \tau : A \subseteq U\}$$

- The family of neighborhoods of A is defined as follows:

$$N(A) := \{N \subseteq X : \exists G \in \tau \ni A \subseteq G \subseteq N\}$$

- The family of closed neighborhoods of A is defined as follows:

$$\tau^c(A) := \{F \in \tau^c : \exists G \in \tau \ni A \subseteq G \subseteq F\}$$

Definition 2.4. Let (X, τ) be a topological space and $A \subseteq X$. Then, the closure of A, denoted by $\mathrm{cl}(A)$, is defined as follows:

$$\mathrm{cl}(A) := \bigcap_{\substack{F \in \tau^c \\ A \subseteq F}} F$$

Equivalently, $\mathrm{cl}(A)$ is the smallest closed set in X containing A.

Definition 2.5. Let (X, τ) be a topological space and let $A \subseteq X$. The interior of A, denoted by $\mathrm{int}(A)$, is defined as

$$\mathrm{int}(A) =:= \bigcup_{\substack{U \in \tau \\ U \subseteq A}} U$$

Equivalently, $\mathrm{int}(A)$ is the largest open set in X contained in A.

Definition 2.6. Let (X, τ) be a topological space and $A \subseteq X$. Then, the family

$$\tau_A = \{U \cap A : U \in \tau\}$$

is a topology on A and called the subspace topology. Moreover, the pair (A, τ_A) is called a subspace of (X, τ).

Definition 2.7. A property P is said to be hereditary if for every topological space (X, τ) having P and every subspace (A, τ_A) of (X, τ), the subspace (A, τ_A) also has P.

Definition 2.8. Let (X, τ) and (Y, υ) be two topological spaces. A function $f : X \to Y$ is called continuous if $f^{-1}(V) \in \tau$, for all $V \in \upsilon$.

Definition 2.9. Let (X, τ) and (Y, υ) be two topological spaces. A function $f : X \to Y$ is called closed if $f(F) \in \upsilon^c$, for all $F \subseteq \tau^c$.

Definition 2.10. Let (X, τ) and (Y, υ) be topological spaces. A function $f : X \to Y$ is called open if $f(U) \in \upsilon$, for all $U \subseteq \tau$.

Definition 2.11. Let (X, τ) and (Y, υ) be topological spaces. A function $f : X \to Y$ is called a homeomorphism if it is bijective, continuous, and its inverse $f^{-1} : Y \to X$ is continuous. If $f : X \to Y$ is a homeomorphism, then the topological spaces (X, τ) and (Y, υ) are said to be homeomorphic.

Definition 2.12. Let $\{(X_i, \tau_i) : i \in I\}$ be a family of topological spaces and let $X = \prod\limits_{i \in I} X_i$ be their product equipped with the product topology τ, where τ is the smallest topology on X such that for all $j \in I$, the canonical projection

$$\begin{aligned} \pi_j : \quad X \quad &\to X_j \\ (x_i)_{i \in I} &\mapsto x_j \end{aligned}$$

is continuous.

Definition 2.13. A property P is called a topological property/homeomorphically invariant if whenever a topological space (X, τ) has P and (X, τ) is homeomorphic to a topological space (Y, υ), then (Y, υ) also has P.

Definition 2.14. Let (X, τ) be a topological space and $A, B \subseteq X$. Then, the sets A and B are said to be strongly separated if $\mathrm{cl}(A) \cap B = \emptyset$ and $A \cap \mathrm{cl}(B) = \emptyset$.

Definition 2.15. Let (X, τ) be a topological space and let $\equiv$ be an equivalence relation on X. Denote the set of equivalence classes by $X/{\equiv}$, i.e., let $X/{\equiv} = \{[x] : x \in X\}$ where $[x] = \{y \in X : x \equiv y\}$, and consider the canonical projection $\pi : X \to X/{\equiv}$ defined by $\pi(x) = [x]$. The quotient topology $\tau_\equiv$ on $X/{\equiv}$ is defined by

$$\tau_\equiv = \{U \subseteq X/{\equiv} : \pi^{-1}(U) \in \tau\}$$

Equivalently, $\tau_\equiv$ is the finest topology on $X/{\equiv}$ that makes the projection π continuous. Moreover, the ordered pair $(X/{\equiv}, \tau_\equiv)$ is called the quotient topological space. In particular, the ordered pair $(X/{\sim}, \tau_\sim)$ is called the Kolmogorov quotient topological space. Here, $x \sim y \Leftrightarrow x$ and y are topologically indistinguishable.

Definition 2.16. Let (X, τ) be a topological space and $x \in X$. Then, the kernel of x, denoted by $\ker(\{x\})$, is defined as follows:

$$\ker(\{x\}) := \{y \in X : x \in \mathrm{cl}(\{y\})\} = \bigcap_{\substack{U \in \tau \\ x \in U}} U$$

Definition 2.17. Let (X, τ) be a topological space and $\mathcal{B} \subseteq \tau$. Then, $\mathcal{B}$ is called a basis for τ if every open set $U \in \tau$ can be written as a union of elements of $\mathcal{B}$, i.e.

$$U = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq U}} B$$

Definition 2.18. Let (X, τ) be a topological space and $\mathcal{S} \subseteq P(X)$. Then, $\mathcal{S}$ is called a subbasis for τ if the collection of all finite intersections of elements of $\mathcal{S}$ forms a basis for τ.

Definition 2.19. Let (X, τ) be a topological space and $\mathcal{S}$ be a a subbasis for τ. Then, $\mathcal{S}$ is called an open subbase if $S \in \tau$, for all $S \in \mathcal{S}$, i.e., $\mathcal{S} \subseteq \tau$ and called a closed subbase if $S \in \tau^c$, for all $S \in \mathcal{S}$, i.e., $\mathcal{S} \subseteq \tau^c$.

Definition 2.20. Let (X, τ) be a topological space and $x \in X$. A collection $\mathcal{B}_x$ of open sets in X is called an open local base at x if

$$\mathcal{B}_x \subseteq \tau(x) \quad \text{and} \quad \text{for all } U \in \tau(x), \text{ there exists a} V \in \mathcal{B}_x \text{ such that } V \subseteq U$$

Definition 2.21. Let (X, τ) be a topological space and $x \in X$. A collection $\mathcal{N}_x$ of subsets of X is called a local base at x if

$$\mathcal{N}_x \subseteq N(x) \quad \text{and} \quad \text{for all } N \in N(x), \text{ there exists an } M \in \mathcal{N}_x \text{ such that } M \subseteq N$$

where $N(x)$ denotes the family of neighborhoods of x.

Definition 2.22. A topological space (X, τ) is called first-countable if for all $x \in X$, there exists a countable local base $\mathcal{N}_x$ at x.

Definition 2.23. A topological space (X, τ) is called second-countable if there exists a countable basis $\mathcal{B}$ for τ.

Definition 2.24. Let (X, τ) be a topological space. Then, a collection $\mathcal{U} \subseteq \tau$ is called an open cover of X if

$$X \subseteq \bigcup_{U \in \mathcal{U}} U$$

Definition 2.25. Let (X, τ) be a topological space and $\mathcal{U}$ be a cover of X. Then, a subcollection $\mathcal{V} \subseteq \mathcal{U}$ is called a subcover if

$$X \subseteq \bigcup_{V \in \mathcal{V}} V$$

Definition 2.26. A topological space (X, τ) is called a Lindelöf space if every open cover of X has a countable subcover.

Definition 2.27. A collection of sets $\{A_i : i \in I\}$ is called mutually disjoint if $A_i \cap A_j = \emptyset$, for all $i, j \in I$ with $i \neq j$.

Definition 2.28. Let A and B be two sets. Then, it is said to be that A meets B, or that A and B intersect if $A \cap B \neq \emptyset$.

Definition 2.29. The Sierpiński space is the topological space (X, τ) where $X = \{0, 1\}$ and $\tau = \{\emptyset, \{1\}, X\}$. Equivalently, it is the unique topological space with two points in which exactly one singleton is closed.

Definition 2.30. Let X be a nonempty set. Then, a function $d : X \times X \to [0, \infty)$ is called a metric on X if for all $x, y, z \in X$, the following conditions hold:

i. $d(x, y) = 0$ if and only if $x = y$

ii. $d(x, y) = d(y, x)$

iii. $d(x, z) \leq d(x, y) + d(y, z)$

Moreover, the ordered pair (X, d) is called a metric space.

Definition 2.31. Let (X, d) be a metric space. Then, the topology induced by d on X is defined as

$$\tau_d = \{U \subseteq X : \text{for all } x \in U, \text{ there exists } r > 0 \text{ such that } B_d(x, r) \subseteq U\}$$

where

$$B_d(x, r) = \{y \in X : d(x, y) < r\}$$

denotes the open ball of radius $r > 0$ centered at $x \in X$.

Definition 2.32. Let $F \subseteq P(X)$ and $F \neq \emptyset$. If the following conditions hold, then the family F is called a filter in X:

i. $\emptyset \notin F$

ii. $A \cap B \in F$, for all $A, B \in F$

iii. If $A \in F$ and $A \subseteq B$, then $B \in F$

Throughout this section, let $\mathcal{F}(X)$ denote the set of all filters in a nonempty set X.

Definition 2.33. Let D be a nonempty set and $\leq$ be a relation on d. Then, the relation $\leq$ is called a direction on D if the following conditions hold:

i. Reflexivity: For all $\alpha \in D$, $\alpha \preceq \alpha$

ii. Transitivity: For all $\alpha, \beta, \gamma \in D$, if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$

iii. Directedness: For all $\alpha, \beta \in D$, there exists a $\gamma \in D$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$

Moreover, the ordered pair $(D, \preceq)$ is called a directed set.

Definition 2.34. Let $(D, \preceq)$ be a directed set and $x : D \to X$ be a function. Then, the function x is called a net in X indexed by the directed set D, and for brevity, it is denoted by $(x_\alpha)_{\alpha \in D}$.

Throughout this section, let $\mathcal{N}(D, X)$ denote the set of all nets in X indexed by a directed set D.

Definition 2.35. Let (X, τ) be a topological space and $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$. Then, the net $(x_\alpha)_{\alpha \in D}$ is said to converge to a point $x \in X$, denoted $x_\alpha \to x$ or $x \in \lim x_\alpha$, if for all $N \in N(x)$, there exists an $\alpha_0 \in D$ such that $x_\alpha \in N$, for all $\alpha_0 \preceq \alpha$.

Definition 2.36. Let (X, τ) be a topological space and $F \in \mathcal{F}(X)$ be a filter on X. Then, the filter F is said to converge to a point $x \in X$, denoted $F \to x$ or $x \in \lim F$, if $N(x) \subseteq F$.

Definition 2.37. Let (X, τ) be a topological space, $F \in \mathcal{F}(X)$, and $x \in X$. Then, the point x is called a closure point of the filter F if $N \cap A \neq \emptyset$, for all $N \in N(x)$ and for all $A \in F$. The set of all closure points of $\mathcal{F}$ is denoted by $\mathrm{cl}(F)$.

Definition 2.38. Let (X, τ) be a topological space, $F \in \mathcal{F}(X)$, and $x \in X$. Then, the point x is called a cluster point of the filter F if for all $N \in N(x)$, there exists an $A \in F$ such that $N \cap A \neq \emptyset$. The set of all cluster points of F is denoted by $\mathrm{clust}(F)$.

Definition 2.39. Let (X, τ) be a topological space, $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$, and $x \in X$. Then, the point x is called a cluster point of the net $(x_\alpha)_{\alpha \in D}$ if for all $N \in N(x)$ and for all $\alpha \in D$, there exists $\alpha \preceq \beta$ such that $x_\beta \in N$. The set of all cluster points of (x_α) is denoted by $\mathrm{clust}(x_\alpha)$.

3. Definitions, Characterizations, and Properties

3.1. Topologically Indistinguishable

x and y are topologically indistinguishable	$\overset{\text{def}}{\Leftrightarrow}$	For all $U \in \tau$, $x \in U \Leftrightarrow y \in U$
	$\Leftrightarrow$	For all $K \in \tau^c$, $x \in K \Leftrightarrow y \in K$
	$\Leftrightarrow$	$N(x) = N(y)$
	$\Leftrightarrow$	$\text{cl}(\{x\}) = \text{cl}(\{y\})$ (or equivalently, $x \in \text{cl}(\{y\})$ and $y \in \text{cl}(\{x\})$)
	$\Leftrightarrow$	$\ker(\{x\}) = \ker(\{y\})$
	$\Leftrightarrow$	For all $F \in \mathcal{F}(X)$, $F \to x \Leftrightarrow F \to y$
	$\Leftrightarrow$	For all $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$, $x_\alpha \to x \Leftrightarrow x_\alpha \to y$
	$\Leftrightarrow$	For all $F \in \mathcal{F}(X)$, $x \in \text{clust}(F) \Leftrightarrow y \in \text{clust}(F)$
	$\Leftrightarrow$	For all $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$, $x \in \text{clust}(x_\alpha) \Leftrightarrow y \in \text{clust}(x_\alpha)$

i. The relation indistinguishable is an equivalence relation on a topological space.

ii. In a topological space, if $\{x\}$ and $\{y\}$ are strongly separated sets, then they are topologically distinguishable.

iii. In a topological space, if x and y are topologically distinguishable, then they are distinct, i.e., $x \neq y$.

iv. If $f : X \to Y$ is a continuous function and $x, y \in X$ such that x and y are topologically indistinguishable, then $f(x)$ and $f(y)$ are topologically indistinguishable.

v. Two elements in a product space are topologically indistinguishable if and only if each of their components are topologically indistinguishable.

3.2. T_0 Spaces

(X,τ) is a T_0 (Kolmogorov) space or τ is a T_0 (Kolmogorov) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, x and y are topologically distinguishable
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau(x)$ such that $y \notin U$ or there exists a $V \in \tau(y)$ such that $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $K \in \tau^c(x)$ such that $y \notin K$ or there exists an $F \in \tau^c(y)$ such that $x \notin F$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an open local base $\mathcal{B}_x$ such that there exists an $U \in \mathcal{B}_x$ satisfying the condition $y \notin U$ or there exists an open local base $\mathcal{B}_y$ such that there exists an $V \in \mathcal{B}_y$ satisfying the condition $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $N \in N(x)$ such that $y \notin N$ or there exists a $M \in N(y)$ such that $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a local base $\mathcal{N}_x$ such that there exists an $N \in \mathcal{N}_x$ satisfying the condition $y \notin N$ or there exists a local base $\mathcal{N}_y$ such that there exists an $M \in \mathcal{N}_y$ satisfying the condition $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$ (or equivalently, $x \notin \mathrm{cl}(\{y\})$ or $y \notin \mathrm{cl}(\{x\})$)
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $x \notin \mathrm{acc}(\{y\})$ or $y \notin \mathrm{acc}(\{x\})$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $\ker(\{x\}) \neq \ker(\{y\})$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $F \in \mathcal{F}(X)$ such that $F \to x$ and $F \nrightarrow y$ or $F \nrightarrow x$ and $F \to y$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$ such that $x_\alpha \to x$ and $x_\alpha \nrightarrow y$ or $x_\alpha \nrightarrow x$ and $x_\alpha \to y$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $F \in \mathcal{F}(X)$ such that $x \in \mathrm{clust}(F)$ and $y \notin \mathrm{clust}(F)$ or $x \notin \mathrm{clust}(F)$ and $y \in \mathrm{clust}(F)$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $(x_\alpha)_{\alpha \in D} \in \mathcal{N}(D, X)$ such that $x \in \mathrm{clust}(x_\alpha)$ and $y \notin \mathrm{clust}(x_\alpha)$ or $x \notin \mathrm{clust}(x_\alpha)$ and $y \in \mathrm{clust}(x_\alpha)$

i. Being a T_0 space is a topological property.

ii. Being a T_0 space is a hereditary property.

iii. If (X,τ) is a T_0 space and $\tau \subseteq \upsilon$, then (X,υ) is a T_0 space.

iv. $(\mathbb{N},\tau_l)$ and $(\mathbb{N},\tau_u)$ are T_0 spaces. Here,

$$\tau_l = \{\emptyset, \mathbb{N}, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,4,5\}, \dots\}$$

and

$$\tau_u = \{\emptyset, \mathbb{N}, \{2,3,4,5,6,\dots\}, \{3,4,5,6,\dots\}, \{4,5,6,\dots\}, \{5,6,\dots\}, \dots\}$$

v. For all $i \in I$, (X_i,τ_i) is a T_0 space if and only if $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is a T_0 space.

vi. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i,τ_i) is a T_0 space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a T_0 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

vii. If $f : (X,\tau) \to (Y,\upsilon)$ is closed and surjective and (X,τ) is a T_0 space, then (Y,υ) is a T_0 space.

viii. If $f : (X, \tau) \to (Y, \upsilon)$ is open and surjective and (X, τ) is a T_0 space, then (Y, υ) is a T_0 space.

ix. If $f : (X, \tau) \to (X, \upsilon)$ is closed identity function such that $\tau \subseteq \upsilon$ and (X, τ) is a T_0 space, then (X, υ) is a T_0 space.

x. $(X/{\sim}, \tau_{\sim})$ is a T_0 space.

xi. The topological space induced by a metric space is a T_0 space.

3.3. R_0 Spaces

(X,τ) is an R_0 (a symmetric) space or τ is an R_0 (a symmetric) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all $x, y \in X$, $x \in \mathrm{cl}(\{y\}) \Leftrightarrow y \in \mathrm{cl}(\{x\})$
	$\Leftrightarrow$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exist $N \in N(x)$ and $N \in N(y)$ such that $y \notin N$ and $x \notin M$
	$\Leftrightarrow$	For all $x \in X$ and for all $U \in \tau(x)$, $\mathrm{cl}(\{x\}) \subseteq U$
	$\Leftrightarrow$	For all $x \in X$ and for all $N \in N(x)$, $\mathrm{cl}(\{x\}) \subseteq N$
	$\Leftrightarrow$	For all $x \in X$ and $K \in \tau^c$ such that $x \notin K$, $K \cap \mathrm{cl}(\{x\}) = \emptyset$
	$\Leftrightarrow$	For all $x \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : x$ and y are topologically indistinguishable$\}$
	$\Leftrightarrow$	The family $\{\mathrm{cl}(\{x\}) : x \in X\}$ is a partition of X
	$\Leftrightarrow$	The family $\{\ker(\{x\}) : x \in X\}$ is a partition of X
	$\Leftrightarrow$	For all $x \in X$, $\mathrm{cl}(\{x\}) \subseteq \ker(\{x\})$
	$\Leftrightarrow$	The specialization preorder ($x \leq y \Leftrightarrow x \in \mathrm{cl}(\{y\}) \Leftrightarrow \mathrm{cl}\{x\}) \subseteq \mathrm{cl}(\{y\})$) on X is symmetric (and therefore an equivalence relation)
	$\Leftrightarrow$	For all $U \in \tau$, there exists a family $\{K_i \in \tau^c : i \in I\}$ such that $U = \bigcup_{i \in I} K_i$
	$\Leftrightarrow$	For all $K \in \tau^c$, there exists a family $\{U_i \in \tau(K) : i \in I\}$ such that $K = \bigcap_{i \in I} U_i$
	$\Leftrightarrow$	For all $x \in X$, the fixed ultrafilter at x converges only to the points that are topologically indistinguishable from x
	$\Leftrightarrow$	For all $A \subseteq X$ and for all $U \in \tau$ satisfying the condition $A \cap U \neq \emptyset$, there exists an $K \in \tau^c$ such that $A \cap K \neq \emptyset$ and $K \subseteq U$.

i. Being an R_0 space is a hereditary property.

ii. For all $i \in I$, (X_i, τ_i) is an R_0 space if and only if $\left(\prod_{i \in I} X_i, \prod_{i \in I} \tau_i\right)$ is an R_0 space.

iii. (X,τ) is an R_0 space if and only if $(X/\sim, \tau_\sim)$ is an R_0 space.

iv. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is an R_0 space if and only if $\left(\bigcup_{i \in I} X_i, \tau\right)$ is an R_0 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

v. The indiscrete topological space (X,τ) such that $X \geq \emptyset$ is an R_0 space.

vi. If (X,τ) and (X,υ) are two topological spaces, (X,τ) is an R_0 space, and $\tau < \upsilon$, then $\tau \subseteq \upsilon$. Here, $\tau < \upsilon$ if and only if every τ-cover $\mathcal{U}$ of X has a υ-refinement $\mathcal{V}$, i.e., there exists a υ-cover $\mathcal{V}$ of X such that for all $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $V \subseteq U$.

vii. If (X,τ) and (X,υ) are two topological spaces and (X,τ) is an R_0 space, then $\tau < \upsilon$ if and only if $\tau \subseteq \upsilon$.

viii. If (X,τ) is an R_0 space, $A \subseteq X$, A is closed in X, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$, then (X,τ_A) is an R_0 space.

ix. If (X,τ) is an R_0 space, $f : (X,\tau) \to (Y,\upsilon)$ is closed, continuous, and surjective, and $f^{-1}(y)$ is a finite set, for all $y \in Y$, then (Y,υ) is an R_0 space.

x. If (X, τ) is an R_0 space and for all $F \in \tau^c$, there exists a countable open base of F for $\tau(F)$, then every closed set in X can be written as the intersection of countably many open sets.

xi. If (X, τ) is an R_0 space and for all $F \in \tau^c$, there exists a countable open base of F for $\tau(F)$, then (X, τ) is a first countable space.

xii. If (X, τ) is an R_0 space, then $\mathrm{cl}(\{x\})$ is compact, for all $x \in X$.

3.4. T_1 Spaces

(X,τ) is a T_1 (accessible or Fréchet) space or τ is a T_1 (accessible or Fréchet) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $y \notin U$ and $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist open local bases $\mathcal{B}_x$ and $\mathcal{B}_y$ such that there exist $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ satisfying the conditions $y \notin U$ and $x \notin V$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist $N \in N(x)$ and $M \in N(y)$ such that $y \notin N$ and $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist open local bases $\mathcal{N}_x$ and $\mathcal{N}_y$ such that there exist $N \in \mathcal{N}_x$ and $M \in \mathcal{N}_y$ satisfying the conditions $y \notin N$ and $x \notin M$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, $\{x\}$ and $\{y\}$ are strongly separated sets.
	$\Leftrightarrow$	(X,τ) is a T_0 space and an R_0 space.
	$\Leftrightarrow$	For all $x \in X$, $\{x\} \in \tau^c$
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap\limits_{U \in \tau(x)} U$
	$\Leftrightarrow$	For all $A \subseteq X$, $A = \bigcap\limits_{U \in \tau(A)} U$
	$\Leftrightarrow$	For all $x \in X$ and for all $\mathcal{B}_x$, $\{x\} = \bigcap\limits_{U \in \mathcal{B}_x} U$
	$\Leftrightarrow$	Every set in X that can be written as the intersection of countably many open sets is closed in X
	$\Leftrightarrow$	Every finite subset of X is closed.
	$\Leftrightarrow$	Every subset of X whose complement is a finite set is open.
	$\Leftrightarrow$	For the cofinite topological space (X,υ), the identity mapping from (X,τ) to (X,υ) is continuous.
	$\Leftrightarrow$	For all $x \in X$, the fixed ultrafilter at x converges only to x.
	$\Leftrightarrow$	For every subset $A \subseteq X$ and for every point $x \in X$, $x \in \operatorname{acc}(A)$ if and only if for every $U \in \tau(x)$, U contains infinitely many points of A.
	$\Leftrightarrow$	Each continuous function from the Sierpinski space to X is a constant function.

i. Every T_1 space is an R_0 space.

ii. Every T_1 space is a T_0 space.

iii. The only T_1 space (X,τ) such that X is a finite set is the discrete topological space.

iv. (X,τ) is an R_0 space if and only if $(X/\!\sim, \tau_\sim)$ is a T_1 space.

v. Being a T_1 space is a topological property.

vi. Being a T_1 space is a hereditary property.

vii. If (X,τ) is a T_1 space and $\tau \subseteq \upsilon$, then (X,υ) is a T_1 space.

viii. For all $i \in I$, (X_i, τ_i) is a T_1 space if and only if $\left(\prod\limits_{i\in I} X_i, \prod\limits_{i\in I} \tau_i\right)$ is a T_1 space.

ix. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a T_1 space if and only if $\left(\bigcup_{i \in I} X_i, \tau\right)$ is a T_1 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

x. $(X/\equiv, \tau_\equiv)$ is a T_1 space if and only if $X/\equiv \subseteq \tau^c$.

xi. If $f : (X, \tau) \to (Y, \upsilon)$ is injective and continuous and (Y, υ) is a T_1 space, then (X, τ) is a T_1 space.

xii. If (X, τ) is a T_1 space, $A \subseteq X$, $x \in \mathrm{acc}(A)$, and $F \subseteq X$ is a finite set, then $x \in \mathrm{acc}(A \setminus F)$.

xiii. If (X, τ) is a T_1 space and $A \subseteq X$, then $\mathrm{acc}(A)$ is closed.

xiv. If (X, τ) is a T_1 space and $x \in X$, then $\mathrm{acc}(\{x\})$ is closed.

xv. If (X, τ) is a T_1 space and $A \subseteq X$ is a finite set, then $\mathrm{acc}(A) = \emptyset$.

xvi. If (X, τ) is a T_1 space and $A \subseteq X$, then $\mathrm{acc}(\mathrm{acc}(A)) \subseteq \mathrm{acc}(A)$.

xvii. If (X, τ) is a T_1 space and $A \subseteq X$, then $\mathrm{cl}(\mathrm{acc}(A)) = \mathrm{acc}(A) = \mathrm{acc}(\mathrm{cl}(A))$.

xviii. If (X, τ) is a T_1 space, $x \in X$, $\mathcal{B}_x$ is a open local base, then for all $y \in X$ such that $x \neq y$, there exists a $U \in \mathcal{B}_x$ such that $y \notin U$.

xix. If (X, τ) is a first countable and a T_1 space, $x \in X$, $A \subseteq X$, and $x \in \mathrm{acc}(A)$, then there exists a sequence consisting of distinct points in $A \setminus \{x\}$ and converging to x.

xx. If (X, τ) is a first countable and a T_1 space and $x \in X$, then there exist countably many open sets U_i such that $\{x\} = \bigcap_{i \in \mathbb{N}} U_i$.

xxi. If (X, τ) is a T_1 space such that X has at least two elements and $\mathcal{B}$ is a base for (X, τ), then $\mathcal{B} \setminus \{X\}$ is a base for (X, τ).

xxii. The cofinite topological space (X, τ) such that X is an infinite set is a T_1 space.

xxiii. For any set X, the smallest T_1 topology is the cofinite topology on X.

xxiv. For any infinite set X, the topological space (X, τ), where τ is the smallest T_1 topology is a connected space.

xxv. A T_1 space is a countably compact space if and only if every infinite set of the space has an accumulation point.

xxvi. A T_1 space is a countably compact space if and only if every infinite open cover of the space has a proper subcover.

xxvii. For every T_1 space (X, τ), $|X| \leq 2^{\min\{|\mathcal{B}| \,:\, \mathcal{B} \text{ is a base of } X\}}$.

xxviii. If $f : (X, \tau) \to (Y, \upsilon)$ is closed and surjective and (X, τ) is a T_1 space, then (Y, υ) is a T_1 space.

xxix. If $f : (X, \tau) \to (Y, \upsilon)$ is open and surjective and (X, τ) is a T_1 space, then (Y, υ) is a T_1 space.

xxx. If $f : (X, \tau) \to (X, \upsilon)$ is closed identity function such that $\tau \subseteq \upsilon$ and (X, τ) is a T_1 space, then (X, υ) is a T_1 space.

xxxi. The topological space induced by a metric space is a T_1 space.

xxxii. If every convergent sequence on a topological space has a unique limit, then the space is a T_1 space.

xxxiii. Every continuous function from the indiscrete topological space (X, τ) to a T_1 space (X, υ) is a constant function.

xxxiv. If, for all $x \in X$, there exists a $N \in N(x)$ such that N is a T_1 space with respect to the induced topology on (X, τ), then (X, τ) is a T_1 space.

xxxv. If for all distinct points $x, y \in X$, there exists a continuous function $f : X \to \mathbb{R}$ and distinct points $a, b \in \mathbb{R}$ such that $f(x) = a$ and $f(y) = b$, then the space is a T_1 space.

3.5. R_1 Spaces

(X,τ) is an R_1 (a preregular) space or τ is an R_1 (a preregular) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exists a $N \in N(x)$ and a $M \in N(y)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.
	$\Leftrightarrow$	For all $x, y \in X$, x and y are topologically distinguishable implies that there exists a $N \in N(\mathrm{cl}(\{x\}))$ and a $M \in N(\mathrm{cl}(\{y\}))$ such that $N \cap M = \emptyset$.
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist $U, V \in \tau$ such that $x \in U$, $\mathrm{cl}(\{y\}) \subseteq V$, and $U \cap V = \emptyset$.
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau$ such that $x \in U$ and $\mathrm{cl}(U) \subseteq X \setminus \mathrm{cl}(\{y\})$.
	$\Leftrightarrow$	For all $x \in X$, $\mathrm{cl}(\{x\}) = \bigcap_{K \in \tau^c(x)} K$
	$\Leftrightarrow$	For all $x, y \in X$, every neighborhood of x meets every neighborhood of y implies that $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, $x \in \mathrm{clust}(N(y))$ implies that $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, $y \in \mathrm{clust}(N(x))$ implies that $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, there exists a filter $F \in \mathcal{F}(X)$ such that $F \to x$ and $F \to y$ implies $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, there exists a net $(x_\alpha)_{\alpha\in D} \in \mathcal{N}(D, X)$ such that $x_\alpha \to x$ and $x_\alpha \to y$ implies $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, there exists a filter $F \in \mathcal{F}(X)$ such that $F \neq P(X)$, $F \to x$, and $F \to y$ implies $y \in \mathrm{cl}(x)$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \forall N \in N(x), \forall M \in N(Y), N \cap M \neq \emptyset\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : x \in \mathrm{clust}(N(y))\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : y \in \mathrm{clust}(N(x))\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \exists F \in \mathcal{F}(X) \ni F \to x \wedge F \to y\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \exists (x_\alpha)_{\alpha\in D} \in \mathcal{N}(D, X) \ni x_\alpha \to x \wedge x_\alpha \to y\}$.
	$\Leftrightarrow$	For all $x, y \in X$, $\mathrm{cl}(\{x\}) = \{y \in X : \exists F \in \mathcal{F}(X) \ni F \neq P(X) \wedge F \to x \wedge F \to y\}$.
	$\Leftrightarrow$	For all $x \in X$ and for all $F \in \mathcal{F}(X)$, if $x \in \lim F$, then $\mathrm{cl}(\{x\}) = \lim F$.
	$\Leftrightarrow$	For all $x \in X$ and for all $(x_\alpha)_{\alpha\in D} \in \mathcal{N}(D, X)$, if $x \in \lim x_\alpha$, then $\mathrm{cl}(\{x\}) = \lim x_\alpha$.
	$\Leftrightarrow$	For all $x, y \in X$ and for all $F \in \mathcal{F}(X)$, if $x, y \in \lim F$, then x and y are topologically indistinguishable.
	$\Leftrightarrow$	For all $x, y \in X$ and for all $(x_\alpha)_{\alpha\in D} \in \mathcal{N}(D, X)$, if $x, y \in \lim x_\alpha$, then x and y are topologically indistinguishable.

i. Every R_1 space is an R_0 space.

ii. Being an R_1 space is a hereditary property.

iii. Being an R_1 space is a topological property.

iv. For all $i \in I$, (X_i, τ_i) is an R_1 space if and only if $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is an R_1 space.

v. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is an R_1 space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is an R_1 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

vi. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{x\}) \cap A = \emptyset$, there exists an $N \in N(x)$ and an $M \in N(A)$ such that $N \cap M = \emptyset$.

vii. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{x\}) \cap A = \emptyset$, there exists an $U \in \tau(x)$ and an $V \in \tau(A)$ such that $U \cap V = \emptyset$.

viii. In an R_1 space (X, τ), for all compact sets $A \subseteq X$ and for all closed and compact sets $B \subseteq X$ satisfying the condition $A \cap B = \emptyset$, there exists an $U \in \tau(A)$ and an $V \in \tau(B)$ such that $U \cap V = \emptyset$.

ix. In an R_1 space (X, τ), if a compact set $A \subseteq X$ is contained in a $U \in \tau$, then $\mathrm{cl}(A) \subseteq U$.

x. In an R_1 space (X, τ), $\mathrm{cl}(A)$ is compact, for all compact sets $A \subseteq X$. More generally, if A is compact and $A \subseteq B \subseteq \mathrm{cl}(A)$, then B is compact.

xi. In an R_1 space (X, τ), $\mathrm{cl}(S)$ is compact, for all compact sets $A \subseteq X$ and for all $S \subseteq X$ with $S \subseteq A$.

xii. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\{x\} \cap \mathrm{cl}(A) = \emptyset$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xiii. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\ker(\{x\}) \neq \ker(\{y\})$, for all $y \in A$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xiv. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$, for all $y \in A$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xv. In an R_1 space (X, τ), for all $x \in X$ and for all compact sets $A \subseteq X$ satisfying the condition $\mathrm{cl}(\{y\}) \subseteq A$, for all $y \in A$, there exists a $U \in \tau(x)$ and a $V \in \tau(A)$ such that $U \cap V = \emptyset$.

xvi. In an R_1 space (X, τ), for all compact sets $A \subseteq X$, $\mathrm{cl}(A) = \bigcup_{x\in A} \mathrm{cl}(\{x\})$.

xvii. If $f, g : (X, \tau) \to (Y, \upsilon)$ are continuous and (Y, υ) is an R_1 space, then the set $\{x \in X : \mathrm{cl}(\{f(x)\}) = \mathrm{cl}(\{g(x)\})\}$ is closed in X.

xviii. (X, τ) is an R_1 space if and only if $(X/\!\sim, \tau_\sim)$ is an R_1 space.

3.6. T_2 Spaces

(X,τ) is a T_2 (separated or Hausdorff) space or τ is a T_2 (separated or Hausdorff) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $U \cap V = \emptyset$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist $N \in N(x)$ and $M \in N(y)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all distinct and nonempty subsets A and B of X, there exist distinct open sets U and V such that $A \cap U \neq \emptyset$ and $B \cap V \neq \emptyset$.
	$\Leftrightarrow$	Every convergent net on X has a unique limit.
	$\Leftrightarrow$	Every convergent filter on X has a unique limit.
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap\limits_{U \in \tau(x)} \mathrm{cl}(U)$
	$\Leftrightarrow$	For all $x \in X$ and for all $\mathcal{B}_x$, $\{x\} = \bigcap\limits_{U \in \mathcal{B}_x} \mathrm{cl}(U)$
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap\limits_{K \in \tau^c(x)} K$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $N \in N(x)$ such that $y \notin \mathrm{cl}(N)$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau(x)$ such that $y \notin \mathrm{cl}(U)$
	$\Leftrightarrow$	For all $x \in X$, $\{x\} = \bigcap\limits_{N \in N(x)} \mathrm{cl}(N)$
	$\Leftrightarrow$	For all $x \in X$ and for all $\mathcal{N}_x$, $\{x\} = \bigcap\limits_{N \in \mathcal{N}_x} \mathrm{cl}(N)$
	$\Leftrightarrow$	The set $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$.
	$\Leftrightarrow$	Every convergent filter has a unique closure point, which coincides with the point to which the filter converges.
	$\Leftrightarrow$	(X,τ) is a T_0 space and an R_1 space.
	$\Leftrightarrow$	(X,τ) is a T_1 space and an R_1 space.
	$\Leftrightarrow$	For all continuous function $f : D \to X$ where D is a dense set of a topological space (Y, υ), there exists at most one continuous function $g : Y \to X$ such that $g_D = f$.

i. Every T_2 space is an R_1 space.

ii. Every T_2 space is an R_0 space.

iii. Every T_2 space is a T_1 space.

iv. Every T_2 space is a T_0 space.

v. Being a T_2 space is a hereditary property.

vi. Being a T_2 is a topological property.

vii. For all $i \in I$, (X_i, τ_i) is a T_2 space if and only if $\left(\prod\limits_{i\in I} X_i, \prod\limits_{i\in I} \tau_i\right)$ is a T_2 space.

viii. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a T_2 space if and only if $\left(\bigcup_{i \in I} X_i, \tau\right)$ is a T_2 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

ix. If (X, τ) is a T_2 space and $\tau \subseteq \upsilon$, then (X, υ) is a T_2 space.

x. Every discrete topological space is a T_2 space.

xi. In a T_2 space, every convergent sequence on X has a unique limit.

xii. In a T_2 space, every convergent sequence has a unique cluster point, which coincides with the point to which the sequence converges.

xiii. In an R_1 space, if every convergent sequence on X has a unique limit, then it is a T_2 space.

xiv. In a first countable space, if every convergent sequence on X has a unique limit, then it is a T_2 space.

xv. (X, τ) is an R_1 space if and only if $(X/\sim, \tau_\sim)$ is a T_2 space.

xvi. If $(X/\equiv, \tau_\equiv)$ is a T_2 space, then $\equiv$ is closed in $X \times X$.

xvii. If (X, τ) is a T_2 space and $\equiv$ is closed in $X \times X$, then $(X/\equiv, \tau_\equiv)$ is a T_2 space.

xviii. If $\pi : X \to X/\equiv$ is an open function and $\equiv$ is closed in $X \times X$, then $(X/\equiv, \tau_\equiv)$ is a T_2 space.

xix. In a T_2 space, for all compact subsets A and for all $x \in X \setminus A$, there exists a $N \in N(x)$ and $M \in N(A)$ such that $N \cap M = \emptyset$.

xx. In a T_2 space, every compact subset of X is closed.

xxi. Let S be a subset of a T_2 space. Then, S is compact if and only if S is closed and S is contained in a compact set.

xxii. In a compact T_2 space, every closed set in X is compact.

xxiii. Let S be a subset of a compact T_2 space. Then S is compact if and only if S is closed.

xxiv. In an R_1 space, if every compact subset of X is closed, then it is a T_2 space.

xxv. If $f : (X, \tau) \to (Y, \upsilon)$ is injective and continuous and (Y, υ) is a T_2 space, then (X, τ) is a T_2 space.

xxvi. If $f, g : (X, \tau) \to (Y, \upsilon)$ are continuous and (Y, υ) is a T_2 space, then the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

xxvii. If $f, g : (X, \tau) \to (Y, \upsilon)$ are continuous, (Y, υ) is a T_2 space, and there exists a dense subset D of X such that $f(D) = g(D)$, then $f = g$.

xxviii. If $f : (X, \tau) \to (Y, \upsilon)$ is continuous and (Y, υ) is a T_2 space, then the graph of f, i.e., $\{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

xxix. Let $f : (X, \tau) \to (Y, \upsilon)$ be continuous and (Y, υ) be a T_2 space. Consider the following relation on X: $x_1 \equiv x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then, $(X/\equiv, \tau_\equiv)$ is a T_2 space and $\{(x_1, x_2) : x_1 \equiv x_2\}$ is closed in $X \times X$.

xxx. Let $f : (X, \tau) \to (Y, \upsilon)$ be continuous and surjective, (X, τ) be a compact space, and (Y, υ) be a T_2 space. Consider the following relation on X: $x_1 \equiv x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then, the function $f^* : X/\equiv \to Y$ defined by $f^*([x]) = f(x)$ is a homeomorphism.

xxxi. If $f : (X, \tau) \to (Y, \upsilon)$ is an open and surjective function and $\{(x_1, x_2) : x_1 \equiv x_2\}$, where $x_1 \equiv x_2 \Leftrightarrow f(x_1) = f(x_2)$, is closed in $X \times X$, then (Y, υ) is a T_2 space.

xxxii. Let $f : (X, \tau) \to (Y, \upsilon)$ be continuous. If (X, τ) is compact and (Y, υ) is a T_2 space, then f is a closed function.

xxxiii. Let $f : (X, \tau) \to (Y, \upsilon)$ be bijective and continuous. If (X, τ) is compact and (Y, υ) is a T_2 space, then f is a homeomorphism.

xxxiv. If A and B are disjoint compact subsets of a T_2 space, then there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

xxxv. If A and B are disjoint compact subsets of a T_2 space, then there exist disjoint neighborhoods N and M such that $A \subseteq N$ and $B \subseteq M$.

xxxvi. If (X, τ) is a T_2 space such that every proper closed subspace is compact, then (X, τ) is compact.

xxxvii. If (X, τ) is a T_2 space such that X is a finite set, then τ is the discrete topology and (X, τ) is homeomorphic to a subspace of $[0, 1]$.

xxxviii. If (X, τ) is a T_2 space, $n \in \mathbb{Z}^+$, and x_1, x_2, ..., and x_n are distinct elements in X, then there exist mutually disjoint open sets U_1, U_2, ..., U_n such that $x_1 \in U_1$, $x_2 \in U_2$, ..., $x_n \in U_n$.

xxxix. If (X, τ) is a T_2 space such that X is a infinite set. Then, there exists a class of mutually disjoint open sets such that the class is an infinite set.

xl. If (X, τ) is a T_2 space and $f : X \to X$ is a continuous mapping satisfying the condition $f \circ f = f$, then $f(X)$ is closed in X. In other words, if (X, τ) is a T_2 space and A is a subset of X satisfying the condition that there exists a continuous and surjective function $f : X \to A$ such that $f(x) = x$, for all $x \in A$, then A is closed in X.

xli. If (X, τ) is a T_2 space, (X, υ) is a compact space, and $\tau \subseteq \upsilon$, then $\tau = \upsilon$.

xlii. If (X, τ) is a compact space, (Y, υ) is a T_2 space, and $f : X \to Y$ is surjective and continuous, then $U \in \upsilon$, for all $U \subseteq Y$ satisfying the condition $f^{-1}(U) \in \tau$.

xliii. If (X, τ) is a compact space, (Y, υ) is a T_2 space, and $f : X \to Y$ is surjective and continuous, then $\left\{U \subseteq Y : f^{-1}(U) \in \tau\right\}$ is a topology on Y.

xliv. Let (X, τ) be a compact space, (Y, υ) be a T_2 space, and $f : X \to Y$ be surjective and continuous. Then, B is closed in Y if and only if $f^{-1}(B)$ is closed in X.

xlv. If (X, τ) is a compact space, (Y, υ) is a T_2 space, and $f : X \to Y$ is surjective and continuous, then $U \in \upsilon$, for all $U \subseteq Y$ satisfying the condition $f^{-1}(U) \in \tau$.

xlvi. If $f : (X, \tau) \to (Y, \upsilon)$ is closed and bijective and (X, τ) is a T_2 space, then (Y, υ) is a T_2 space.

xlvii. If $f : (X, \tau) \to (Y, \upsilon)$ is open and bijective and (X, τ) is a T_2 space, then (Y, υ) is a T_2 space.

xlviii. If $f : (X, \tau) \to (X, \upsilon)$ is the identity function such that $\tau \subseteq \upsilon$ and (X, τ) is a T_2 space, then (X, υ) is a T_2 space.

xlix. If $f : (X, \tau) \to (Y, \upsilon)$ is closed and surjective, $f^{-1}(\{y\})$ is compact, for all $y \in Y$, (X, τ) is a T_2 space, then (Y, υ) is a T_2 space.

l. The topological space induced by a metric space is a T_2 space.

li. If (X, τ), (X, υ), and $(X, \tau \cap \upsilon)$ are T_2 spaces, then the set $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$ with respect to $\tau \times \upsilon$.

lii. There are T_2 spaces (X, τ) and (X, υ) such that the set $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$ with respect to $\tau \times \upsilon$.

liii. If (X, τ) and (X, υ) are T_2 spaces, and for all disjoint sets $U, V \in \tau$, there exist disjoint sets $U^*, V^* \in \upsilon$ such that $U \subseteq U^*$ and $V \subseteq V^*$ and vice versa, then $(X, \tau \cap \upsilon)$ is a T_2 space.

liv. For any nonempty set X, there exists a T_2 space (Y, υ) such that $Y = \bigcup_{x \in X} Y_x$, where the sets Y_x are mutually disjoints and dense in Y (Hint: $Y = X \times \mathbb{Q}$, $Y_x = \{x\} \times \mathbb{Q}$, and $\upsilon = \left\{\bigcup_{i \in I} X \times (I_i \cap \mathbb{Q}) : I_i \text{ is an interval in } \mathbb{R}\right\}$ such that I is an index set).

lv. Every topological space is the continuous open image of a T_2 space.

lvi. If for all distinct points $x, y \in X$, there exists a continuous function $f : X \to \mathbb{R}$ and distinct points $a, b \in \mathbb{R}$ such that $f(x) = a$ and $f(y) = b$, then the space is a T_2 space.

lvii. If, for all $x \in X$, there exists an $F \in \tau^c(x)$ such that F is a T_2 space with respect to the induced topology on (X, τ), then (X, τ) is a T_2 space.

lviii. For every topological spaces, there exists a T_2 space such that whose quotient space is the topological space.

lix. If (X, τ) is a compact T_2 space, the following are equivalent:

- The quotient space $(X/\equiv, \tau_{\equiv})$ is a T_2 space.
- The canonical projection map π is a closed function.
- $\{(x, y) : \pi(x) = \pi(y)\}$ is closed in $X \times X$.

3.7. $T_{2\frac{1}{2}}$ Spaces

(X,τ) is a $T_{2\frac{1}{2}}$ (Urysohn) space or τ is a $T_{2\frac{1}{2}}$ (Urysohn) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exists a $K \in \tau^c(x)$ and an $L \in \tau^c(y)$ such that $K \cap L = \emptyset$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a $U \in \tau(x)$ and a $V \in \tau(y)$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists an $N \in N(x)$ and an $M \in N(y)$ such that $\mathrm{cl}(N) \cap \mathrm{cl}(M) = \emptyset$

i. Every $T_{2\frac{1}{2}}$ space is a T_2 space.

ii. Every $T_{2\frac{1}{2}}$ space is a T_1 space.

iii. Every $T_{2\frac{1}{2}}$ space is a T_0 space.

iv. Every $T_{2\frac{1}{2}}$ space is a R_1 space.

v. Every $T_{2\frac{1}{2}}$ space is a R_0 space.

vi. Being a $T_{2\frac{1}{2}}$ space is a hereditary property.

vii. Being a $T_{2\frac{1}{2}}$ space is a topological property.

viii. For all $i \in I$, (X_i, τ_i) is a $T_{2\frac{1}{2}}$ space if and only if $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is a $T_{2\frac{1}{2}}$ space.

ix. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a $T_{2\frac{1}{2}}$ space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a $T_{2\frac{1}{2}}$ space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

x. If $f : (X,\tau) \to (Y,\upsilon)$ is closed and surjective and (X,τ) is a $T_{2\frac{1}{2}}$ space, then (Y,υ) is a $T_{2\frac{1}{2}}$ space.

xi. If $f : (X,\tau) \to (Y,\upsilon)$ is open and surjective and (X,τ) is a $T_{2\frac{1}{2}}$ space, then (Y,υ) is a $T_{2\frac{1}{2}}$ space.

xii. If $f : (X,\tau) \to (X,\upsilon)$ is closed identity function such that $\tau \subseteq \upsilon$ and (X,τ) is a $T_{2\frac{1}{2}}$ space, then (X,υ) is a $T_{2\frac{1}{2}}$ space.

xiii. The topological space induced by a metric space is a $T_{2\frac{1}{2}}$ space.

3.8. Completely T_2 Spaces

(X,τ) is a completely (functionally) T_2 space or τ is a completely (functionally) T_2 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ with $x \in f^{-1}(\{0\})$ and $y \in f^{-1}(\{1\})$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ and there exist $\alpha, \beta \in (0,1)$ satisfying the condition $\alpha < \beta$ such that $x \in f^{-1}([0,\alpha))$ and $y \in f^{-1}((\beta, 1])$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$
	$\Leftrightarrow$	(X,τ) is a T_2 space, and for all disjoint compact subsets A and B of X, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$
	$\Leftrightarrow$	For all distinct points $x, y \in X$, there exist two continuous functions $f, g : X \to \mathbb{R}$ such that $x \in \operatorname{int}(\{z \in X : f(z) = 0\})$, $y \in \operatorname{int}(\{z \in X : g(z) = 0\})$, and $\{z \in X : f(z) = 0\} \cap \{z \in X : g(z) = 0\} = \emptyset$

i. Every completely T_2 space is a R_1 space.

ii. Every completely T_2 space is a R_0 space.

iii. Every completely T_2 space is a $T_{2\frac{1}{2}}$ space.

iv. Every completely T_2 space is a T_2 space.

v. Every completely T_2 space is a T_1 space.

vi. Every completely T_2 space is a T_0 space.

vii. If (X,τ) is a topological space and $x \equiv y :\Leftrightarrow \forall f \in C_{\mathbb{R}}(X), f(x) = f(y)$, where $C_{\mathbb{R}}(X) := \{f : X \to \mathbb{R} : f \text{ is continuous}\}$, then $(X/{\equiv}, \tau_{\equiv})$ is a completely T_2 space.

viii. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a completely T_2 space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a completely T_2 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

ix. The topological space induced by a metric space is a completely T_2 space.

3.9. Regular Spaces

(X,τ) is a regular space or τ is a regular topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exist $U, V \in \tau$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exists a $U \in \tau(x)$ such that $\mathrm{cl}(U) \cap F = \emptyset$
	$\Leftrightarrow$	For all $A, B \subseteq X$ satisfying the condition $A \setminus \mathrm{cl}(B) \neq \emptyset$, there exists a $U \in \tau$ such that $\mathrm{cl}(B) \subseteq U$ and $A \setminus \mathrm{cl}(U) \neq \emptyset$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exist $N \in N(x)$ and $N \in N(F)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c$ such that $x \notin F$, there exist $U \in \tau(x)$ and $V \in \tau(F)$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$
	$\Leftrightarrow$	For all $x \in X$ and for all $U \in \tau(x)$, there exists a $V \in \tau(x)$ such that $\mathrm{cl}(V) \subseteq U$
	$\Leftrightarrow$	For all $x \in X$ and for all $N \in N(x)$, there exists an $M \in N(x)$ such that $\mathrm{cl}(M) \subseteq N$
	$\Leftrightarrow$	For all $x \in X$ and for all $N \in N(x)$, there exists an $F \in \tau^c(x)$ such that $F \subseteq N$
	$\Leftrightarrow$	Every point in X has a neighborhood base consisting of closed sets.
	$\Leftrightarrow$	For all $F \in \tau^c$, $F = \bigcap\limits_{K \in \tau^c(F)} K$

i. Every regular space is an R_1 space.

ii. Every regular space is an R_0 space.

iii. In a topological space, if every open set is a closed set, then the space is a regular space.

iv. Being a regular space is a hereditary property.

v. Being a regular is a topological property.

vi. For all $i \in I$, (X_i, τ_i) is a regular space if and only if $\left(\prod\limits_{i\in I} X_i, \prod\limits_{i\in I} \tau_i\right)$ is a regular space.

vii. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a regular space if and only if $\left(\bigcup\limits_{i\in I} X_i, \tau\right)$ is a regular space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

viii. (X,τ) is a regular space if and only if $(X/\sim, \tau_\sim)$ is a regular space.

ix. If (X,τ) is a regular space, then for all $A, B \subseteq X$ satisfying the condition that there exists an $N \in N(A)$ such that $N \cap B = \emptyset$ or there exists an $M \in N(B)$ such that $A \cap M = \emptyset$, there exists an $N^* \in N(A)$ and an $M^* \in N(B)$ such that $N^* \cap M^* = \emptyset$.

x. In a regular space (X,τ), for all compact sets $A \subseteq X$ and for all $N \in N(A)$, there exists an $F \in \tau^c(A)$ such that $F \subseteq N$.

xi. If (X,τ) is a regular space, $A \subseteq X$, A is closed in X, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$, then (X, τ_A) is a regular space.

xii. If (X,τ) is a regular space and A is an infinite subset of X, then there exist open sets U_1, U_2, $\dots$ in X such that $\mathrm{cl}(U_n) \cap \mathrm{cl}(U_m) = \emptyset$, for all $n, m \in \mathbb{N}$ with $n \neq m$, and $U_n \cap A \neq \emptyset$, for all $n \in \mathbb{N}$.

xiii. The topological space induced by a metric space is a regular space.

xiv. If, for all $x \in X$, there exists an $F \in \tau^c(x)$ such that F is a regular space with respect to the induced topology on (X,τ), then (X,τ) is a regular space.

xv. If, for all $x \in X$, there exists a $U \in \tau(x)$ such that U is a regular space with respect to the induced topology on (X, τ), then (X, τ) is a regular space.

xvi. In a regular space (X, τ), for all $F \in \tau^c$, $F = \bigcap_{U \in \tau(F)} U$.

xvii. In a regular space (X, τ), if every net in $S \subseteq X$ has a cluster point in X, then $\mathrm{cl}(S)$ is compact.

xviii. In a regular space (X, τ), if every proper filter on X containing $S \subseteq X$ has a cluster point in X, then $\mathrm{cl}(S)$ is compact.

3.10. T_3 Spaces

(X,τ) is a T_3 space or τ is a T_3 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_0 space and a regular space

i. Every T_3 space is a R_1 space.

ii. Every T_3 space is a R_0 space.

iii. Every T_3 space is a regular space.

iv. Every T_3 space is a $T_{2\frac{1}{2}}$ space.

v. Every T_3 space is a T_2 space.

vi. Every T_3 space is a T_1 space.

vii. Every T_3 space is a T_0 space.

viii. (X,τ) is a regular space if and only if $(X/\sim,\tau_\sim)$ is a T_3 space.

ix. Being a T_3 space is a hereditary property.

x. Being a T_3 is a topological property.

xi. For all $i\in I$, (X_i,τ_i) is a T_3 space if and only if $\left(\prod\limits_{i\in I} X_i, \prod\limits_{i\in I}\tau_i\right)$ is a T_3 space.

xii. Let $i\neq j\Rightarrow X_i\cap X_j=\emptyset$, for all $i,j\in I$. Then, for all $i\in I$, (X_i,τ_i) is a T_3 space if and only if $\left(\bigcup\limits_{i\in I} X_i,\tau\right)$ is a T_3 space. Here, $G\in\tau$ if and only if $G\cap X_i\in\tau_i$, for all $i\in I$.

xiii. For every T_3 space (X,τ), $|X|\leq 2^{\min\{|A|\,:\,A\text{ is a dense set in }X\}}$.

xiv. If $f:(X,\tau)\to(Y,\upsilon)$ is closed, open, continuous, and surjective and (X,τ) is a T_3 space, then (Y,υ) is a T_3 space.

xv. If A is a compact subset of a T_3 space and B is a closed subset of the space with $A\cap B=\emptyset$, then there exist disjoint open sets U and V such that $A\subseteq U$ and $B\subseteq V$.

xvi. The topological space induced by a metric space is a T_3 space.

xvii. Every compact T_2 space is a T_3 space.

xviii. In a T_3 space, for all distinct $x,y\in X$, there exist $U\in\tau(x)$ and $V\in\tau(y)$ such that $\mathrm{cl}(U)\cap\mathrm{cl}(V)=\emptyset$.

3.11. Completely Regular Spaces

(X,τ) is a completely regular space or τ is a completely regular topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(F) = \{1\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 1$ and $f(F) = \{0\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [a,b]$ with $f(x) = a$ and $f(F) = \{b\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $F \in \tau^c \setminus \{\emptyset\}$ such that $x \notin F$, there exists a continuous function $f : X \to [a,b]$ with $f(x) = b$ and $f(F) = \{a\}$
	$\Leftrightarrow$	There exists a closed subbase $\mathcal{S}$ of X such that for each $x \in X$ and for each $F \in \mathcal{S}$ satisfying the condition $x \notin F$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(F) = \{1\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $U \in \tau(x) \setminus \{X\}$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(X \setminus U) = \{1\}$
	$\Leftrightarrow$	For each $x \in X$ and for each $N \in N(x) \setminus \{X\}$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(X \setminus N) = \{1\}$
	$\Leftrightarrow$	Every singleton in X has a neighborhood base consisting of sets N such that for each N, there exists a continuous function $f : X \to \mathbb{R}$ with $N = f^{-1}(\{0\})$.
	$\Leftrightarrow$	Every closed set in X is an intersection of sets A such that for each A, there exists a continuous function $f : X \to \mathbb{R}$ with $A = f^{-1}(\{0\})$. (The family of the sets A such that for each A, there exists a continuous function $f : X \to \mathbb{R}$ with $A = f^{-1}(\{0\})$ is a basis for τ^c)
	$\Leftrightarrow$	The family of the sets A such that for each A, there exists a continuous function $f : X \to \mathbb{R}$ with $A = X \setminus f^{-1}(\{0\})$ is a basis for τ

i. Every completely regular space is an R_1 space.

ii. Every completely regular space is an R_0 space.

iii. Every completely regular space is a regular space.

iv. Every completely regular space is a completely T_2 space.

v. Every completely regular space is a $T_{2\frac{1}{2}}$ space.

vi. Every completely regular space is a T_2 space.

vii. Every completely regular space is a T_1 space.

viii. Every completely regular space is a T_0 space.

ix. Every T_0 and completely regular space is a T_3 space.

x. In a topological space, if every open set is a closed set, then the space is a completely regular space.

xi. Being a completely regular space is a hereditary property.

xii. Being a completely regular is a topological property.

xiii. For all $i \in I$, (X_i, τ_i) is a completely regular space if and only if $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is a completely regular space.

xiv. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a completely regular space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a completely regular space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

xv. If (X, τ) is a completely regular space and a T_1 space, then for all distinct points $x, y \in X$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

xvi. In a completely regular space (X, τ), for all compact sets $A \subseteq X$ and for all $N \in N(A)$, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{1\}$ and $f(X \setminus N) = \{0\}$.

xvii. If (X, τ) is a completely regular space, $A \subseteq X$, A is closed in X, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$, then (X, τ_A) is a completely regular space.

xviii. The topological space induced by a metric space is a completely regular space.

xix. (X, τ) is a completely regular space if and only if $(X/{\sim}, \tau_{\sim})$ is a completely regular space.

3.12. $T_{3\frac{1}{2}}$ Spaces

(X,τ) is a $T_{3\frac{1}{2}}$ (T_π or Tikhonov) space or τ is a $T_{3\frac{1}{2}}$ (T_π or Tikhonov) topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_0 space and a completely regular space
	$\Leftrightarrow$	(X,τ) is homeomorphic to a subspace of a compact Hausdorff space.

i. Every $T_{3\frac{1}{2}}$ space is an R_1 space.

ii. Every $T_{3\frac{1}{2}}$ space is an R_0 space.

iii. Every $T_{3\frac{1}{2}}$ space is a completely regular space.

iv. Every $T_{3\frac{1}{2}}$ space is a T_3 space.

v. Every $T_{3\frac{1}{2}}$ space is a regular space.

vi. Every $T_{3\frac{1}{2}}$ space is a completely T_2 space.

vii. Every $T_{3\frac{1}{2}}$ space is a $T_{2\frac{1}{2}}$ space.

viii. Every $T_{3\frac{1}{2}}$ space is a T_2 space.

ix. Every $T_{3\frac{1}{2}}$ space is a T_1 space.

x. Every $T_{3\frac{1}{2}}$ space is a T_0 space.

xi. Being a $T_{3\frac{1}{2}}$ space is a hereditary property.

xii. Being a $T_{3\frac{1}{2}}$ space is a topological property.

xiii. (X,τ) is a completely regular space if and only if $(X/\sim,\tau_\sim)$ is a $T_{3\frac{1}{2}}$ space.

xiv. For all $i\in I$, (X_i,τ_i) is a $T_{3\frac{1}{2}}$ space if and only if $\left(\prod_{i\in I} X_i, \prod_{i\in I}\tau_i\right)$ is a $T_{3\frac{1}{2}}$ space.

xv. Let $i\neq j\Rightarrow X_i\cap X_j=\emptyset$, for all $i,j\in I$. Then, for all $i\in I$, (X_i,τ_i) is a $T_{3\frac{1}{2}}$ space if and only if $\left(\bigcup_{i\in I} X_i,\tau\right)$ is a $T_{3\frac{1}{2}}$ space. Here, $G\in\tau$ if and only if $G\cap X_i\in\tau_i$, for all $i\in I$.

xvi. In a $T_{3\frac{1}{2}}$ space (X,τ), for all distinct points $x,y\in X$, if there exists a continuous function $f:X\to\mathbb{R}$ such that $f(x)\neq f(y)$.

xvii. If $f:(X,\tau)\to(Y,\upsilon)$ is closed, open, continuous, and surjective and (X,τ) is a $T_{3\frac{1}{2}}$ space, then (Y,υ) is a $T_{3\frac{1}{2}}$ space.

xviii. If A is a compact subset of a $T_{3\frac{1}{2}}$ space and B is a closed subset of the space with $A\cap B=\emptyset$, then there exists a continuous function $f:X\to[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$.

xix. The topological space induced by a metric space is a $T_{3\frac{1}{2}}$ space.

3.13. Normal Spaces

(X,τ) is a normal space or τ is a normal topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all disjoint sets $F, K \in \tau^c$, there exist $U, V \in \tau$ such that $F \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c$, there exist $U, V \in \tau$ such that $F \subseteq U$, $K \subseteq V$, and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$
	$\Leftrightarrow$	For all strongly separated sets $F, K \in \tau$ that can be written as the union of countably many closed sets, there exist $U, V \in \tau$ such that $F \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c$, there exists a $U \in \tau$ such that $F \subseteq U$ and $\mathrm{cl}(U) \cap K = \emptyset$
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c$, there exists an $N \in N(F)$ and an $M \in N(K)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all $K \in \tau^c$ and for all $N \in N(K)$, there exists an $M \in N(K)$ such that $\mathrm{cl}(M) \subseteq N$
	$\Leftrightarrow$	For all $K \in \tau^c$ and for all $U \in \tau(K)$, there exists an $V \in \tau(K)$ such that $\mathrm{cl}(V) \subseteq U$
	$\Leftrightarrow$	For all open sets $U, V \in \tau$ satisfying the condition $X = U \cup V$, there exist $F, K \in \tau^c$ such that $F \subseteq U$, $K \subseteq V$, and $X = F \cup K$
	$\Leftrightarrow$	For all open sets $U, V \in \tau$ satisfying the condition $X = U \cup V$, there exist $U^*, V^* \in \tau$ such that $\mathrm{cl}(U^*) \subseteq U$, $\mathrm{cl}(V^*) \subseteq V$, and $X = \mathrm{cl}(U^*) \cup \mathrm{cl}(V^*)$
	$\Leftrightarrow$	(Urysohn's Lemma) For all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to [0,1]$ such that $f(F) = \{0\}$ and $f(K) = \{1\}$
	$\Leftrightarrow$	(A Generalization of Urysohn's Lemma) For all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to [a,b]$ such that $f(F) = \{a\}$ and $f(K) = \{b\}$.
	$\Leftrightarrow$	For all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to \mathbb{R}$ and distinct points $a, b \in \mathbb{R}$ such that $f(F) = \{a\}$ and $f(K) = \{b\}$.
	$\Leftrightarrow$	(Tietze's Extension Theorem) If Y is a nonempty closed subspace of X and $f : Y \to [0,1]$ (or $f : Y \to \mathbb{R}$) is a continuous function, then there exists a continuous function $g : X \to [0,1]$ (or $g : X \to \mathbb{R}$) such that $g_{\mid Y} = f$.
	$\Leftrightarrow$	Every closed set in X has a neighborhood base consisting of closed sets.
	$\Leftrightarrow$	For all $U \in \tau$ and for all $K \in \tau^c$ satisfying the condition $K \subseteq U$, there exists an $F \in \tau^c(K)$ such that $K \subseteq U$

i. Every discrete space and every indiscrete space is a normal space.

ii. Every normal and R_0 space is a completely regular space.

iii. Every compact R_1 space is a normal space.

iv. Every compact T_2 space is a normal space.

v. Every compact regular space is a normal space.

vi. Every normal and regular space is a completely regular space.

vii. Being a normal space is a topological property.

viii. In a topological space, if every open set is a closed set, then the space is a normal space.

ix. Every closed subspace of a normal space is a normal space.

x. Every subspace of a normal space that can be written as the union of countably many closed sets is a normal space.

xi. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a normal space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a normal space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

xii. If $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is a normal space, then for all $i \in I$, (X_i, τ_i) is a normal space.

xiii. Let $f : X \to Y$ be a surjective function. Then, (Y, υ) is a normal space if and only if $\left(X, f^{-1}(\upsilon)\right)$ is a normal space. Here, $f^{-1}(\upsilon) := \left\{f^{-1}(V) \subseteq X : V \in \upsilon\right\}$.

xiv. Every regular and Lindelöf space is a normal space.

xv. Every second countable and regular space is a normal space.

xvi. If (X, τ) and (Y, υ) are two topological space, (X, τ) is a normal space, and $f : X \to Y$ is a closed, continuous, and surjective function, then (Y, υ) is a normal space.

xvii. For all $F \in \tau^c$ and for all $U \in \tau(F)$, if there exists a family $\{U_i \in \tau : i \in \mathbb{N}\}$ such that $F \subseteq \bigcup_{i\in\mathbb{N}} U_i$ and $\mathrm{cl}(U_i) \subseteq U$, for all $i \in \mathbb{N}$, then (X, τ) is a normal space.

xviii. In a normal space (X, τ), if $F \in \tau^c$ that can be written as the intersection of countably many open sets and $K \in \tau^c$ satisfying the condition $F \cap K = \emptyset$, then there exists a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$.

xix. Let (X, τ) be a normal space, $A \in \tau^c$, and $\tau_A = \{U \cup (V \cap A) : U, V \in \tau\}$. Then, (X, τ_A) is a normal space if and only if $X \setminus A$ is a normal space with respect to the induced topology on (X, τ).

xx. The topological space induced by a metric space is a normal space.

3.14. T_4 Spaces

(X,τ) is a T_4 space or τ is a T_4 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a normal space
	$\Leftrightarrow$	(X,τ) is a T_2 space and a normal space

i. Every T_4 space is an R_1 space.

ii. Every T_4 space is an R_0 space.

iii. Every T_4 space is a normal space.

iv. Every T_4 space is a $T_{3\frac{1}{2}}$ space.

v. Every T_4 space is a completely regular space.

vi. Every T_4 space is a T_3 space.

vii. Every T_4 space is a regular space.

viii. Every T_4 space is a completely T_2 space.

ix. Every T_4 space is a $T_{2\frac{1}{2}}$ space.

x. Every T_4 space is a T_2 space.

xi. Every T_4 space is a T_1 space.

xii. Every T_4 space is a T_0 space.

xiii. Being a T_4 space is a topological property.

xiv. Every closed subspace of a T_4 space is a T_4 space.

xv. If $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is a T_4 space, then for all $i \in I$, (X_i, τ_i) is a T_4 space.

xvi. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a T_4 space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a T_4 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

xvii. Every compact T_2 space is a T_4 space.

xviii. Every second countable and T_3 space is a T_4 space.

xix. Every Lindelöf and T_3 space is a T_4 space.

xx. Every countable T_3 space is a T_4 space.

xxi. The topological space induced by a metric space is a T_4 space.

xxii. If (X,τ) is a T_4 space and $\mathcal{B}$ is a base for (X,τ), then for all $U \in \mathcal{B}$ and for all $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in \mathrm{cl}(V)$ and $\mathrm{cl}(V) \subseteq U$.

xxiii. In a T_1 space, if there exists exactly one nonisolated point in the space, then the space is a T_4 space.

xxiv. In a T_2 space, if there exists a finitely number of nonisolated points in the space, then the space is a T_4 space.

xxv. In a T_4 space, if the family $\{U_1, U_2, \ldots, U_n\}$ is an open cover of X, then there exists a closed cover $\{F_1, F_2, \ldots, F_n\}$ of X such that $F_i \subseteq U_i$, for all $i \in I_n$.

xxvi. If (X,τ) is a T_4 space, $Y \subseteq X$, and there exist countably many closed sets F_i such that $Y = \bigcup_{i\in\mathbb{N}} F_i$, then (Y, τ_Y) is a T_4 space.

xxvii. Let (X, τ) be a T_4 space and $F \in \tau^c$. Then, there exist countably many open sets U_i such that $F = \bigcap_{i \in \mathbb{N}} U_i$ if and only if there exists a continuous function $f : X \to [0, 1]$ such that $F = f^{-1}(\{0\})$.

xxviii. Let (X, τ) be a T_4 space and $U \in \tau$. Then, there exist countably many closed sets F_i such that $U = \bigcup_{i \in \mathbb{N}} F_i$ if and only if there exists a continuous function $f : X \to [0, 1]$ such that $U = f^{-1}((0, 1])$.

xxix. If $f : (X, \tau) \to (Y, \upsilon)$ is closed, continuous, and surjective and (X, τ) is a T_4 space, then (Y, υ) is a T_4 space.

xxx. In a T_2 space (X, τ), the following are equivalent:

- The space (X, τ) is compact.
- For all (Y, υ), the projection $p : X \times Y \to Y$ is a closed and continuous function.
- For all T_4 spaces (Y, υ), the projection $p : X \times Y \to Y$ is a closed and continuous function.

xxxi. In a T_1 space (X, τ), if, for all disjoint sets $F, K \in \tau^c \setminus \{\emptyset\}$, there exists a continuous function $f : X \to [0, 1]$ such that $f(F) = \{0\}$ and $f(K) = \{1\}$, then the space is a T_4 space.

xxxii. In a T_1 space (X, τ), if, for all $K \in \tau^c$ and for all $U \in \tau(K)$, there exists an $V \in \tau(K)$ such that $\mathrm{cl}(V) \subseteq U$, then the space is a T_4 space.

xxxiii. In a T_1 space (X, τ), if, for all $K \in \tau^c$ and for all $N \in N(K)$, there exists an $M \in N(K)$ such that $\mathrm{cl}(M) \subseteq N$, then the space is a T_4 space.

xxxiv. In a T_4 space, for all distinct $F, K \in \tau^c$, there exist $U \in \tau(F)$ and $V \in \tau(K)$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

3.15. Completely Normal Spaces

(X, τ) is a completely (hereditarily) normal space or τ is a completely (hereditarily) normal topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all strongly separated sets $A, B \subseteq X$, there exists an $N \in N(A)$ and an $M \in N(B)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all strongly separated sets $A, B \subseteq X$, there exists a $U \in \tau(A)$ and a $V \in \tau(B)$ such that $U \cap V = \emptyset$
	$\Leftrightarrow$	For all $A, B \subseteq X$ satisfying the condition that there exists a $U \in \tau(A)$ and a $V \in \tau(B)$ such that $U \cap B = \emptyset$ and $A \cap V = \emptyset$, there exists an $N \in N(A)$ and an $M \in N(B)$ such that $N \cap M = \emptyset$
	$\Leftrightarrow$	For all $A, B \subseteq X$ satisfying the condition that there exists an $N \in N(A)$ and an $M \in N(B)$ such that $N \cap B = \emptyset$ and $A \cap M = \emptyset$, there exists an $N^* \in N(A)$ and an $M^* \in N(B)$ such that $N^* \cap M^* = \emptyset$
	$\Leftrightarrow$	Every subspace of (X, τ) is a normal space.
	$\Leftrightarrow$	Every open subspace of (X, τ) is a normal space.
	$\Leftrightarrow$	For all $A \subseteq X$ and for all $U \in \tau(A)$ satisfying the condition $\mathrm{cl}(A) \subseteq U$, there exists an $F \in \tau^c(A)$ such that $F \subseteq U$.
	$\Leftrightarrow$	For all $F, K \in \tau^c$, there exist $F^*, K^* \in \tau^c$ such that $F^* \cup K^* = X$, $F^* \cap (F \cup K) = F$, and $K^* \cap (F \cup K) = K$

i. Every completely normal space is an R_1 space.

ii. Every completely normal space is an R_0 space.

iii. Every completely normal space is a T_4 space.

iv. Every completely normal space is a normal space.

v. Every completely normal space is a $T_{3\frac{1}{2}}$ space.

vi. Every completely normal space is a completely regular space.

vii. Every completely normal space is a T_3 space.

viii. Every completely normal space is a regular space.

ix. Every completely normal space is a completely T_2 space.

x. Every completely normal space is a $T_{2\frac{1}{2}}$ space.

xi. Every completely normal space is a T_2 space.

xii. Every completely normal space is a T_1 space.

xiii. Every completely normal space is a T_0 space.

xiv. Being a completely normal space is a hereditary property.

xv. Being a completely normal space is a topological property.

xvi. If $f : X \to Y$ is a function, υ is a topology on Y, and (Y, υ) is a completely normal space, then $\left(X, f^{-1}(\upsilon)\right)$ is a completely normal space. Here, $f^{-1}(\upsilon) := \left\{f^{-1}(V) \subseteq X : V \in \upsilon\right\}$.

xvii. If $f : X \to Y$ is a surjective function, υ is a topology on Y, and $\left(X, f^{-1}(\upsilon)\right)$ is a completely normal space, then (Y, υ) is a completely normal space. Here, $f^{-1}(\upsilon) := \left\{f^{-1}(V) \subseteq X : V \in \upsilon\right\}$.

xviii. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a completely normal space if and only if $\left(\bigcup_{i \in I} X_i, \tau\right)$ is a completely normal space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

xix. Every second countable and T_3 space is a completely normal space.

xx. In a completely normal space (X, τ), for all strongly separated subsets A and B of X, there exists a $U \in \tau$ such that $A \subseteq U$ and $\mathrm{cl}(U) \cap B = \emptyset$.

xxi. In a completely normal space (X, τ), for all $A, B \subseteq X$ satisfying the condition $A \cap \mathrm{cl}(B) = \emptyset$, there exists a $U \in \tau$ such that $A \subseteq U$, $U \cap \mathrm{cl}(B) = \emptyset$, and $\mathrm{cl}(U) \cap B \subseteq \mathrm{cl}(A)$.

xxii. In a completely normal space (X, τ), for all mutually strongly separated subsets A_1, A_2, ..., A_n of X, there exist $U_1, U_2, \ldots, U_n \in \tau$ such that $A_i \subseteq U_i$ and $\mathrm{cl}(A_i) \cap \mathrm{cl}(A_j) = \mathrm{cl}(U_i) \cap \mathrm{cl}(U_j)$, for all $i, j \in I_n$ with $i \neq j$.

xxiii. The topological space induced by a metric space is a completely normal space.

3.16. T_5 Spaces

(X,τ) is a T_5 space or τ is a T_5 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a completely normal space
	$\Leftrightarrow$	Every subspace of (X,τ) is a T_4 space.
	$\Leftrightarrow$	(X,τ) is a T_2 space and a completely normal space

i. Every T_5 space is an R_1 space.

ii. Every T_5 space is an R_0 space.

iii. Every T_5 space is a completely normal space.

iv. Every T_5 space is a T_4 space.

v. Every T_5 space is a normal space.

vi. Every T_5 space is a $T_{3\frac{1}{2}}$ space.

vii. Every T_5 space is a completely regular space.

viii. Every T_5 space is a T_3 space.

ix. Every T_5 space is a regular space.

x. Every T_5 space is a completely T_2 space.

xi. Every T_5 space is a $T_{2\frac{1}{2}}$ space.

xii. Every T_5 space is a T_2 space.

xiii. Every T_5 space is a T_1 space.

xiv. Every T_5 space is a T_0 space.

xv. The topological space induced by a metric space is a T_5 space.

xvi. Being a T_5 space is a hereditary property.

xvii. Being a T_5 space is a topological property.

xviii. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a T_5 space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a T_5 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

xix. If $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is a T_5 space, then for all $i \in I$, (X_i, τ_i) is a T_5 space.

3.17. Perfectly Normal Spaces

(X, τ) is a perfectly normal space or τ is a perfectly normal topology on X	$\overset{\text{def}}{\Leftrightarrow}$	For all disjoint sets $F, K \in \tau^c$, there exists a continuous function $f : X \to [0, 1]$ with $F = f^{-1}(\{0\})$ and $K = f^{-1}(\{1\})$
	$\Leftrightarrow$	(X, τ) is a normal space and every open set in X can be written as the union of countably many closed sets.
	$\Leftrightarrow$	(X, τ) is a normal space and every closed set in X can be written as the intersection of countably many open sets.
	$\Leftrightarrow$	For all $U \in \tau$, there exists a continuous function $f : X \to [0, 1]$ such that $X \setminus U = f^{-1}(\{0\})$
	$\Leftrightarrow$	For all $F \in \tau^c$, there exists a continuous function $f : X \to [0, 1]$ such that $F = f^{-1}(\{0\})$

i. Every perfectly normal space is an R_1 space.

ii. Every perfectly normal space is an R_0 space.

iii. Every perfectly normal space is a completely normal space.

iv. Every perfectly normal space is a T_4 space.

v. Every perfectly normal space is a normal space.

vi. Every perfectly normal space is a $T_{3\frac{1}{2}}$ space.

vii. Every perfectly normal space is a completely regular space.

viii. Every perfectly normal space is a T_3 space.

ix. Every perfectly normal space is a regular space.

x. Every perfectly normal space is a completely T_2 space.

xi. Every perfectly normal space is a $T_{2\frac{1}{2}}$ space.

xii. Every perfectly normal space is a T_2 space.

xiii. Every perfectly normal space is a T_1 space.

xiv. Every perfectly normal space is a T_0 space.

xv. The topological space induced by a metric space is a perfectly normal space.

xvi. Being a perfectly normal space is a hereditary property.

xvii. If $f : (X, \tau) \to (Y, \upsilon)$ is closed, continuous, and surjective and (X, τ) is a perfectly normal space, then (Y, υ) is a perfectly normal space.

xviii. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a perfectly normal space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a perfectly normal space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

xix. In a perfectly normal space (X, τ), for all $F \in \tau^c$, there exists a continuous function $f : X \to [0, 1]$ with $F = f^{-1}(\{0\})$.

xx. If (X, τ) is a normal and an R_0 space and for all $F \in \tau^c$, there exists a countable open base of F for $\tau(F)$, then (X, τ) is a perfectly normal space.

3.18. T_6 Spaces

(X,τ) is a T_6 space or τ is a T_6 topology on X	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a perfectly normal space
	$\Leftrightarrow$	(X,τ) is a T_4 space and every open set in X can be written as the union of countably many closed sets.
	$\Leftrightarrow$	(X,τ) is a T_4 space and every closed set in X can be written as the intersection of countably many open sets.
	$\Leftrightarrow$	(X,τ) is a T_2 space and a perfectly normal space

i. Every T_6 space is an R_1 space.

ii. Every T_6 space is an R_0 space.

iii. Every T_6 space is a perfectly normal space.

iv. Every T_6 space is a T_5 space.

v. Every T_6 space is a completely normal space.

vi. Every T_6 space is a T_4 space.

vii. Every T_6 space is a normal space.

viii. Every T_6 space is a $T_{3\frac{1}{2}}$ space.

ix. Every T_6 space is a completely regular space.

x. Every T_6 space is a T_3 space.

xi. Every T_6 space is a regular space.

xii. Every T_6 space is a completely T_2 space.

xiii. Every T_6 space is a $T_{2\frac{1}{2}}$ space.

xiv. Every T_6 space is a T_2 space.

xv. Every T_6 space is a T_1 space.

xvi. Every T_6 space is a T_0 space.

xvii. The topological space induced by a metric space is a T_6 space.

xviii. Let $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, for all $i, j \in I$. Then, for all $i \in I$, (X_i, τ_i) is a T_6 space if and only if $\left(\bigcup_{i\in I} X_i, \tau\right)$ is a T_6 space. Here, $G \in \tau$ if and only if $G \cap X_i \in \tau_i$, for all $i \in I$.

xix. If $\left(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i\right)$ is a T_6 space, then for all $i \in I$, (X_i, τ_i) is a T_6 space.

4. Conclusion

Şart gerektirmeden doğrudan geçişler

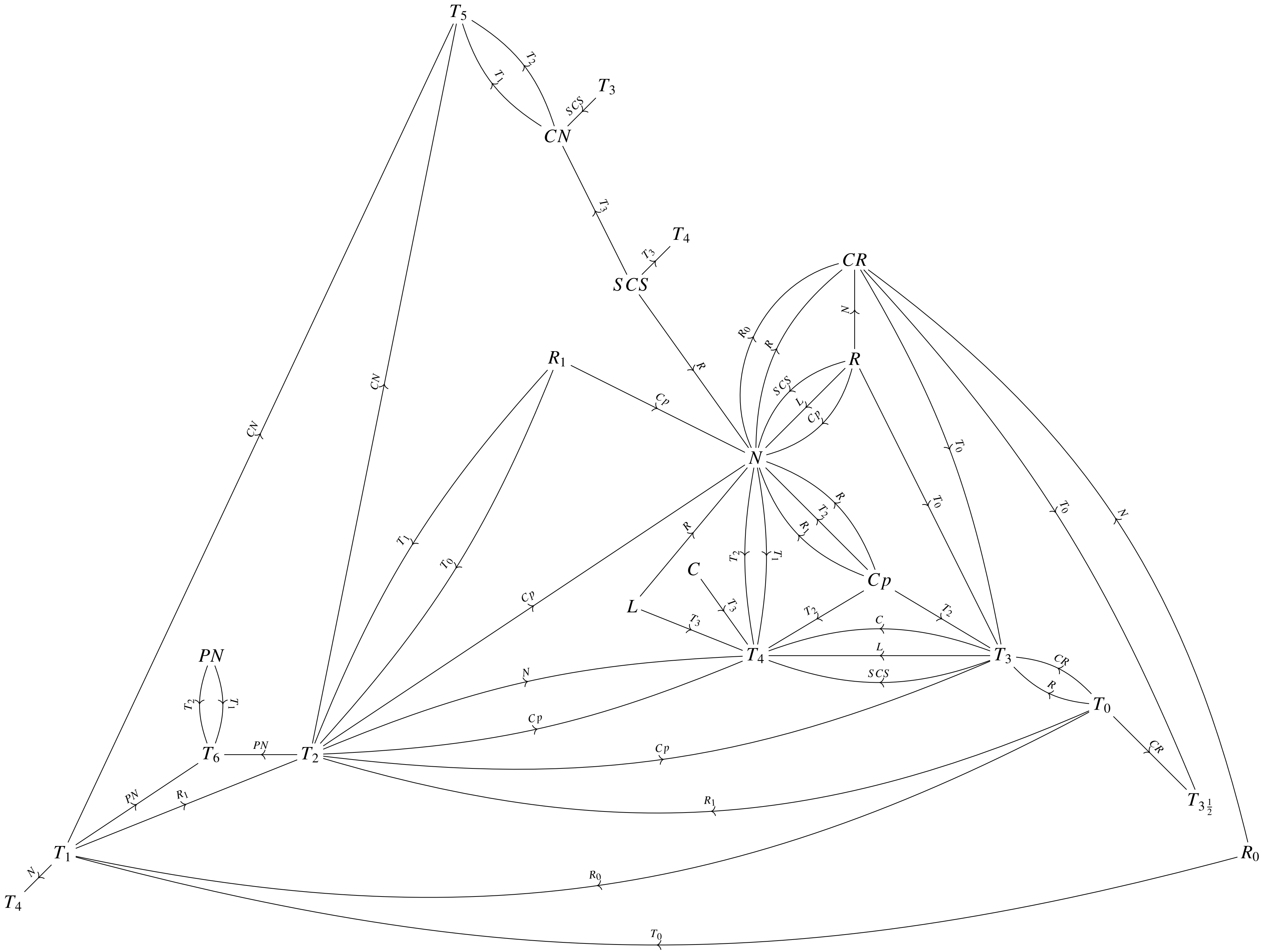
$$
\begin{tikzcd}
 & & PN \arrow[dr] & & N & & CR \arrow[r] \arrow[dr] & R \arrow[rr] & & R_1 \arrow[r] & R_0 & \\
M \arrow[r] & T_6 \arrow[ur] \arrow[dr] & & CN \arrow[r] & T_4 \arrow[u] \arrow[r] & T_{3\frac{1}{2}} \arrow[ur] \arrow[dr] & & CT_2 \arrow[dr] & & & & \\
 & & T_5 \arrow[ur] & & & & T_3 \arrow[rr] \arrow[d] & & T_{2\frac{1}{2}} \arrow[r] & T_2 \arrow[r] \arrow[uu] & T_1 \arrow[r] \arrow[uu] & T_0 \\
 & & & & & & R & & & & &
\end{tikzcd}
$$

caption: Implication map of common separation/regularity axioms. Eşdeğerlikler ve koşullu çıkarımlar (ör. čompacẗ) ok üzerinde etiketlenmiştir.

caption: Hierarchy of Separation Axioms

Provide short descriptions of standard examples (cofinite topology, indiscrete, discrete, lower limit topology, particular-point topology, Sierpiński space) and how they satisfy or fail various axioms. (This appendix can be extended with proofs and diagrams as needed for lecture handouts.)

The historical development of the separation axioms reflects the evolution of topological thought from concrete metric-like spaces to the abstract and general framework we use today. The R_0 and R_1 axioms, though less famous, provide essential conceptual clarity by decomposing the T_1 and T_2 conditions. The full hierarchy, extending to T_6, captures increasingly refined notions of separation, regularity, and normality, which are crucial for advanced topics in topology, analysis, and functional analysis. Understanding these properties and their numerous equivalent characterizations—especially through the lenses of filters, nets, and closure operations—is fundamental for any deep engagement with the subject.



T_5
T_3
CN
T_4
SCS
CR
R_1
R
N
C
Cp
L
T_4
T_3
PN
T_0
T_6
T_2
$T_{3\frac{1}{2}}$
T_1
R_0
T_4

	T_0	R_0	T_1	R_1	T_2	$T_{2\frac{1}{2}}$	CT_2	R	T_3	CR	$T_{3\frac{1}{2}}$	N	T_4	CN	T_5	PN	T_6
Closed-Hereditary Property	+																
Open-Hereditary Property	+																
Hereditary Property	+																
Injective Property	+																
Surjective Property	+																
Topological Property	+																
Productive Property	+																
Projective Property	+																
Disjoint Sum Property	+																
Inverse Image Property	+																
Divisible Property	+																

Table 19. Example Table

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