Separation Axioms: Characterizations and Implications

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(X, τ) is a T_0 space	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct $x, y \in X$, there exists a $U \in \tau$ such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$
	$\Leftrightarrow$	For all distinct $x, y \in X$, $x \notin \{y\}'$ or $y \in \{x\}'$
(X, τ) is a R_0 space	$\overset{\text{def}}{\Leftrightarrow}$	For all $x, y \in X$, $x \in \overline{\{y\}} \Rightarrow y \in \overline{\{x\}}$
	$\Leftrightarrow$	For all $x, y \in X$, $\overline{\{x\}} = \overline{\{y\}}$
	$\Leftrightarrow$	For all $x \in X$, $\overline{\{x\}} = \{y \in X : \mathcal{N}_x = \mathcal{N}_y\}$
	$\Leftrightarrow$	The Kolmogorov quotient (ayırt edilemezlik denklik bağıntısına göre elde edilen parçalanış) of X is a T_1 space
	$\Leftrightarrow$	The specialization preorder ($x \leq y \Leftrightarrow x \in \overline{\{y\}} \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$) on X is symmetric (and therefore an equivalence relation)
	$\Leftrightarrow$	The family $\left\{\overline{\{x\}} : x \in X\right\}$ is a partition of X
	$\Leftrightarrow$	For all $x \in X$ and $F \in \tau^k$ such that $x \notin F$, $F \cap \overline{\{x\}} = \emptyset$
	$\Leftrightarrow$	For all $N \in \mathcal{N}_x$, $\overline{\{x\}} \subseteq N$
	$\Leftrightarrow$	For all $U \in \tau$, $\exists \{F_i : i \in I\} \subseteq \tau^k \ni U = \bigcup_{i \in I} F_i$
	$\Leftrightarrow$	For all $x \in X$, the fixed ultrafilter at x converges only to the points y such that $\mathcal{N}_x = \mathcal{N}_y$
(X, τ) is a T_1 space	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct $x, y \in X$, there exists $U, V \in \tau$ such that $x \in U$, $y \notin U$, $x \notin V$, and $y \in V$
	$\Leftrightarrow$	(X, τ) is a T_0 space and a R_0 space
	$\Leftrightarrow$	For all $x \in X$, $\{x\} \in \tau^k$
	$\Leftrightarrow$	For all $A \subseteq X$, $A = \bigcap_{A \subseteq U \in \tau} U$
	$\Leftrightarrow$	Every finite set of X is closed
	$\Leftrightarrow$	Every set of X whose complement is a finite set is open
	$\Leftrightarrow$	For all $x \in X$, the fixed ultrafilter at x converges only to x
	$\Leftrightarrow$	For every subset $S \subseteq X$ and every point $x \in X$, x is a limit point of S if and only if every open neighborhood of x contains infinitely many points of S
	$\Leftrightarrow$	Each map from the Sierpinski space, which has two points and only one of which is closed, to X is trivial (constant sanırım).
	$\Leftrightarrow$	The map from the Sierpinski space to the single point has the lifting property with respect to the map from X to the single point. Yani, $f : S \to \{x\}$ and $g : X \to \{x\}$ iki fonksiyon olmak üzere bir $l : S \to X$ sürekli fonksiyonu vardır.
(X, τ) is a R_1 space	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct $x, y \in X$, if $x \in \overline{\{y\}}$ or $y \in \overline{\{x\}}$, then there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	xxx
(X, τ) is a T_2 space	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct $x, y \in X$, there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$

	$\Leftrightarrow$	xxx
(X,τ) is a $T_{2\frac{1}{2}}$ space	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct $x, y \in X$, there exist $F, K \in \tau^k$ such that $x \in F$, $y \in K$, and $F \cap K = \emptyset$
	$\Leftrightarrow$	xxx
(X,τ) is a completely T_2 space	$\overset{\text{def}}{\Leftrightarrow}$	For all distinct $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$
	$\Leftrightarrow$	xxx
(X,τ) is a regular space	$\overset{\text{def}}{\Leftrightarrow}$	For each $x \in X$ and for each $F \in \tau^k$ such that $x \notin F$, there exist $U, V \in \tau$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	xxx
(X,τ) is a T_3 space	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_0 space and a regular space
	$\Leftrightarrow$	xxx
(X,τ) is a completely regular space	$\overset{\text{def}}{\Leftrightarrow}$	For each $x \in X$ and for each $F \in \tau^k$ such that $x \notin F$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(F) = 1$
	$\Leftrightarrow$	xxx
(X,τ) is a $T_{3\frac{1}{2}}$ space	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_0 space and a completely regular space
	$\Leftrightarrow$	xxx
(X,τ) is a normal space	$\overset{\text{def}}{\Leftrightarrow}$	For all disjoint sets $F, K \in \tau^k$, there exist $U, V \in \tau$ such that $F \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	xxx
(X,τ) is a T_4 space	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a normal space
	$\Leftrightarrow$	xxx
(X,τ) is a completely normal space	$\overset{\text{def}}{\Leftrightarrow}$	For all strongly separated sets $A, B \subseteq X$, there exist $U, V \in \tau$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$
	$\Leftrightarrow$	xxx
(X,τ) is a T_5 space	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_1 space and a completely normal space
	$\Leftrightarrow$	xxx
(X,τ) is a perfectly normal space	$\overset{\text{def}}{\Leftrightarrow}$	For all disjoint set $F, K \in \tau^k$, there exists a continuous function $f : X \to [0,1]$ with $F = f^{-1}(\{0\})$ and $K = f^{-1}(\{1\})$
	$\Leftrightarrow$	xxx
(X,τ) is a T_6 space	$\overset{\text{def}}{\Leftrightarrow}$	(X,τ) is a T_0 space and a perfectly normal space
	$\Leftrightarrow$	xxx