Distributivity, Extended

(Tommy Burch, on and off from 08.2024 to 05.2025) (wxMaxima does not export the cleanest of latex files, so the resulting pdf contains a number of quirks and errors)

1 At or Down

This all started back in the late '90s at the UofA in a physics lab prep session (with R. McCroskey and R. Haar, among others). The discussion hinged on finding an operation (i termed it 'at' with the symbol @) over which addition was distributive. The exp / log solution below i came up with followed later the same day... Note that, for multiplcation and addition, the distributive property can be "revealed" through exponentials and their inverse, the natural logarithms:

(% i1) add(a,b) := log(exp(a)*exp(b));

$$\mathrm{add}\left(a , b\right) :=\log\left(\exp(a)\,\exp(b)\right) \tag{\% o1}$$

(% i3) add(a,b);
add(c*a,c*b);

$$b+a \tag{\% o2}$$

$$bc+ac \tag{\% o3}$$

(% i4) ratsimp(%);

$$\left(b+a\right) \,c \tag{\% o4}$$

Placing the multiplication between the exponentials in the logarithm, we arrive at the relation (addition) over which it is disributive. We can try the same with + between the exp's...

1.1 At_+

Define the @ (or "down") relation such that addition is distributive over it: a + (b @ c) = (a+b) @ (a+c)...Or: a + down(b,c) = down(a+b,a+c). Trying the following ansatz:

(% i5) at_eq : At(a,b) = log(exp(a) + exp(b));

$$\mathrm{At}\left(a\,,b\right)=\log\left(\%e^{b}+\%e^{a}\right) \tag{at_ eq}$$

(% i6) down(a,b) := radcan(log(exp(a) + exp(b)));

$$\mathrm{down}\left(a\,,b\right):=\mathrm{radcan}\left(\log\left(\exp(a)+\exp(b)\right)\right) \tag{\% o6}$$

Check:

(% i7) down(a+b,a+c);

$$\log\left(\%e^{c}+\%e^{b}\right)+a \tag{\% o7}$$

yes! (It is also an obviously commutative operation.)This should work with any base for the power / log's: e.g.,

(% i8) log10(x) := log(x) / log(10);

$$\mathrm{log10}(x):=\frac{\log\left(x\right)}{\log\left(10\right)} \tag{\% o8}$$

(% i9) radcan(log10(10ˆ (a+b) + 10ˆ (a+c)));

$$\frac{\log\left(2^{c}5^{c}+2^{b}5^{b}\right)+\left(\log\left(5\right)+\log\left(2\right)\right)a}{\log\left(5\right)+\log\left(2\right)} \tag{\% o9}$$

(% i10) logcontract(%);

$$\frac{\log\left(2^{c}5^{c}+2^{b}5^{b}\right)+\log\left(10\right)a}{\log\left(10\right)} \tag{\% o10}$$

yep! The ambiguity probably speak volumes (see, e.g., M. Burgin, arXiv:1010.3287 : there, the focus appears not to be in maintaining distributivity, unlike here, where it's the main focus). It may be worth exploring whether functions other than powers/logarithms can retain distributivity... at first glance, seemingly not.Some values with base e:

(% i11) down(0,0);

$$\log\left(2\right) \tag{\% o11}$$

(% i12) down(1,0);

$$\log\left(\%e+1\right) \tag{\% o12}$$

(%i13) down(x,0);

$$\log\left(\%e^{x}+1\right) \tag{\%o13}$$

(%i15) down(x,x);
down(x-log(2),x-log(2));

$$x+\log\left(2\right) \tag{\%o14}$$

$$x \tag{\%o15}$$

Associative:

(%i17) down(a,down(b,c));
down(down(a,b),c);

$$\log\left(\%e^{c}+\%e^{b}+\%e^{a}\right) \tag{\%o16}$$

$$\log\left(\%e^{c}+\%e^{b}+\%e^{a}\right) \tag{\%o17}$$

(%i20) diff(down(a,b),a);
diff(down(a,b),b);
diff(diff(down(a,b),b),a);

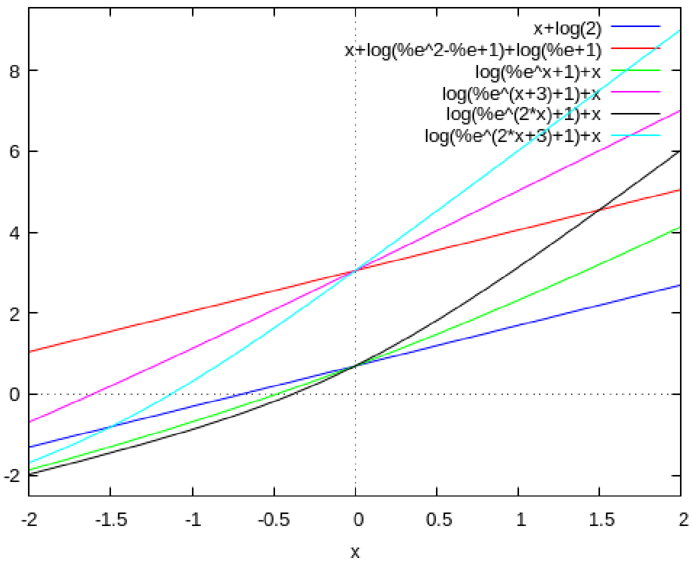
$$\frac{\%e^{a}}{\%e^{b}+\%e^{a}} \tag{\%o18}$$

$$\frac{\%e^{b}}{\%e^{b}+\%e^{a}} \tag{\%o19}$$

$$-\frac{\%e^{b+a}}{\left(\%e^{b}+\%e^{a}\right)^{2}} \tag{\%o20}$$

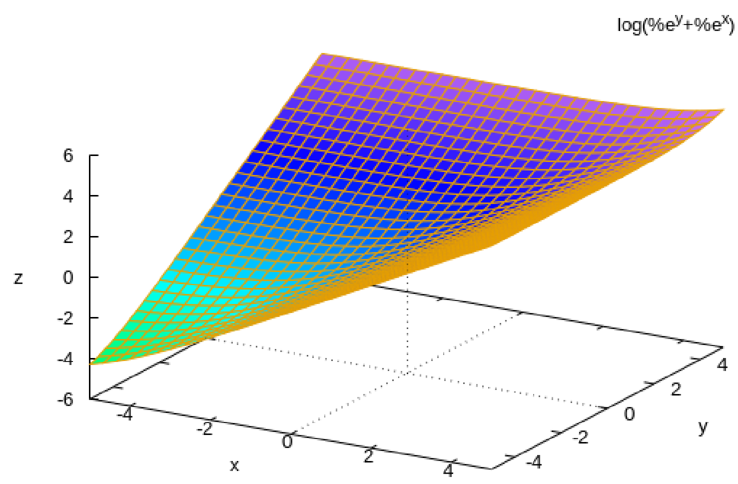
Let's have a look:

(% i21) wxplot2d ([down(x,x),down(x,x+3),down(x,2*x),down(x,2*x+3),down(x,3*x),down(x,3*x+3)], [x, -2, 2])$



(% t21)

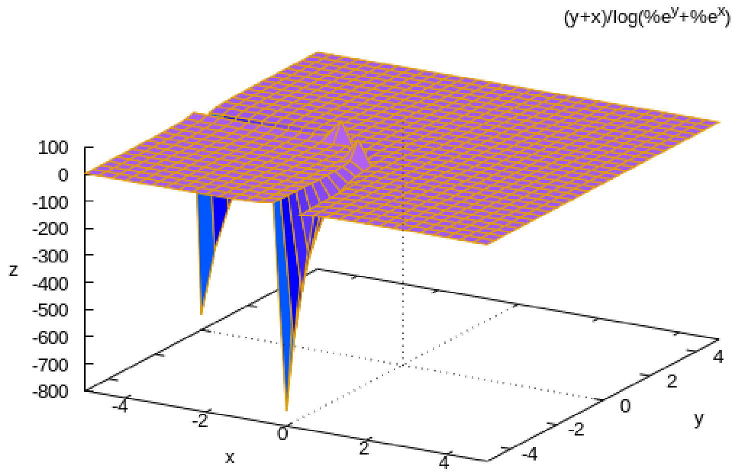
(% i22) wxplot3d (down(x,y), [x, -5, 5], [y, -5, 5])$



(% t22)

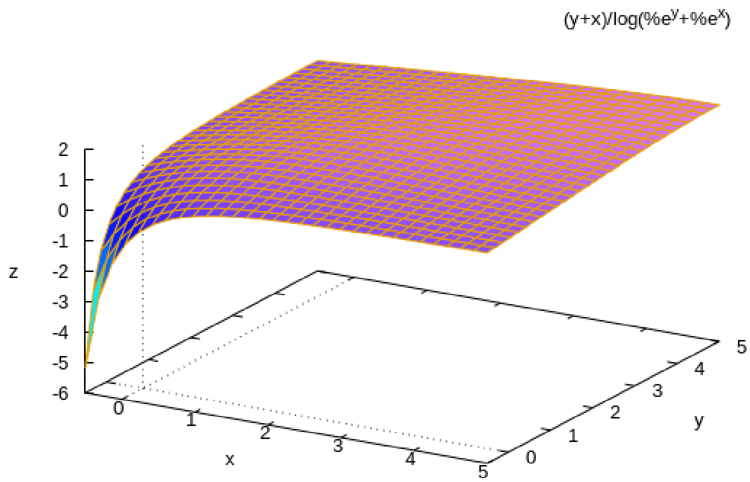
Comparing x+y with down(x,y) :

(% i23) wxplot3d ((x+y)/down(x,y), [x, -5, 5], [y, -5, 5])$



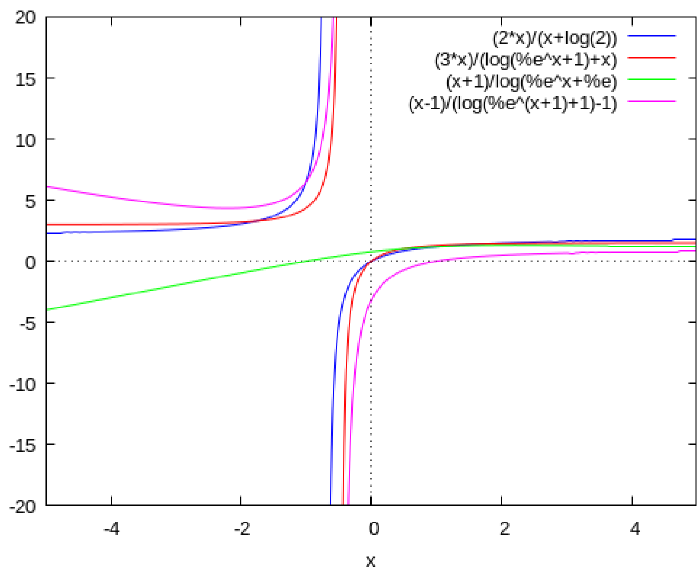
(% t23)

(% i24) wxplot3d ((x+y)/down(x,y), [x, -0.5, 5], [y, -0.5, 5])$



(% t24)

(% i25) wxplot2d ([2*x/down(x,x), 3*x/down(x,2*x),(x+1)/down(x,1),(x-1)/down(x,-1)], [x, -5, 5], [y, -20, 20])$



(% t25)

1.2 At_-,*,/

We can also consider a "down-like" operation with a minus instead a plus:

(% i26) down_min(a,b) := radcan(log(exp(a) - exp(b)));

$$\mathrm{down\_min}\left(a\,,b\right):=\mathrm{radcan}\left(\log\left(\exp(a)-\exp(b)\right)\right) \tag{\% o26}$$

Associativity:

(% i30) down(down_min(a,b),c);
down_min(down(a,c),b);
down(down_min(a,b),down_min(c,d));
down(down_min(a,d),down_min(c,b));

$$\log\left(\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o27}$$

$$\log\left(\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o28}$$

$$\log\left(-\%e^{d}+\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o29}$$

$$\log\left(-\%e^{d}+\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o30}$$

Same distributive property:

(% i31) down_min(a+b,a+c);

$$\log\left(\%e^{b}-\%e^{c}\right)+a \tag{\% o31}$$

As we've already seen, with times, we achieve addition. With divide, one achieves subtraction:

(% i33) down_div(a,b) := radcan(log(exp(a) / exp(b)));
down_div(a,b);

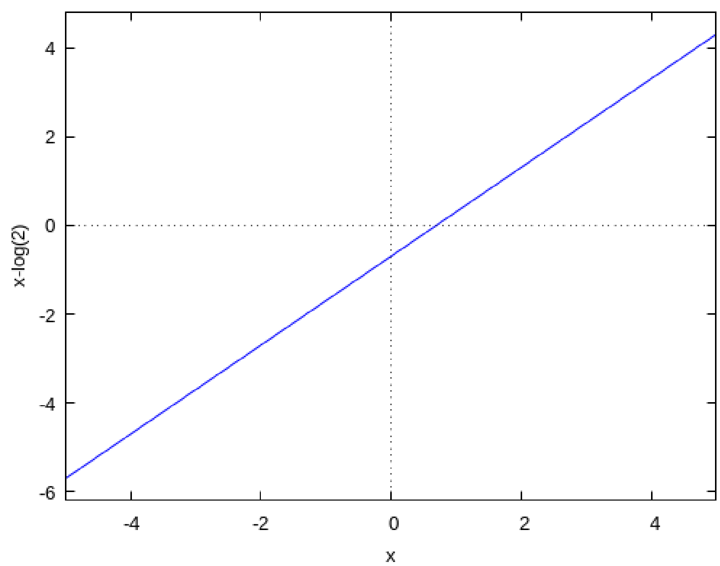
$$\mathrm{down\_div}\left(a\,,b\right):=\mathrm{radcan}\left(\log\left(\frac{\exp(a)}{\exp(b)}\right)\right) \tag{\% o32}$$

$$a-b \tag{\% o33}$$

Using @_/ or down_div for the "ratio" of + and @...

(% i35) down_div(x+x,down(x,x));
wxplot2d (2*x - down(x,x), [x, -5, 5])$

$$x - \log(2) \qquad (\% \text{ o34})$$



(% t35)

1.3 Down differentiation

Given down, down_min, and down_div, one is free to define a "down derivative". However, we must take care with the notion of infinitesimals: normally h–>0 in the definition of the derivative, but here we need h–>-∞ to approach the @_+ identity (x @ -∞ = x)...

(% i36) Df_Dx_eq : Df_Dx = limit(down_min(f(down(x,h)) , f(x)) - h , h, -∞);

$$\text{Df\_Dx} = \lim_{\to h \backslash\text{-}\backslash > -\infty} \log\left(\%e^{f\left(\log\left(\%e^{x} + \%e^{h}\right)\right)} - \%e^{f(x)}\right) - h \quad (\text{Df\_ Dx\_ eq})$$

(% i40) Df(x,h) := down_min(f(down(x,h)) , f(x));
Df(x,h);
Df_Dx(x) := ratsimp(limit(Df(x,h) - h , h, minf));
Df_Dx(x);

$$\text{Df}(x, h) := \text{down\_min}\left(f\left(\text{down}(x, h)\right), f(x)\right) \qquad (\% \text{ o37})$$

$$\log\left(\%e^{\mathrm{f}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{f}(x)}\right) \qquad (\%\ o38)$$

$$\mathrm{Df\_Dx}(x) := \mathrm{ratsimp}\left(\mathrm{limit}\left(\mathrm{Df}\left(x\,,h\right)-h\,,h\,,\text{-}\infty\right)\right) \qquad (\%\ o39)$$

$$\lim_{\to h\ \backslash\text{-}\backslash>\text{-}\infty} \log\left(\%e^{\mathrm{f}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{f}(x)}\right)-h \qquad (\%\ o40)$$

(% i43) f(x) := A*x^ b + c;
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x) := A x^{b}+c \qquad (\%\ o41)$$

$$\log\left(\%e^{A \log\left(\%e^{x}+\%e^{h}\right)^{b}}-\%e^{A x^{b}}\right)+c \qquad (\%\ o42)$$

$$(b-1)\log(x)+A x^{b}-x+c+\log(b)+\log(A) \qquad (\%\ o43)$$

(% i46) f(x) := log(A*x^ b + c);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x) := \log\left(A x^{b}+c\right) \qquad (\%\ o44)$$

$$\log\left(\log\left(\%e^{x}+\%e^{h}\right)^{b}-x^{b}\right)+\log(A) \qquad (\%\ o45)$$

$$(b-1)\log(x)-x+\log(b)+\log(A) \qquad (\%\ o46)$$

(% i49) f(x) := exp(A*x^ b + c);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x) := \exp\left(A x^{b}+c\right) \qquad (\%\ o47)$$

$$\log\left(\%e^{\%e^{A\log\left(\%e^x+\%e^h\right)^b+c}}-\%e^{\%e^{Ax^b+c}}\right) \tag{\% o48}$$

$$(b-1)\log(x)+\%e^{Ax^b+c}+Ax^b-x+c+\log(b)+\log(A) \tag{\% o49}$$

(% i52) f(x) := down(g(x) , p(x));
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=\mathrm{down}\left(\mathrm{g}(x),\mathrm{p}(x)\right) \tag{\% o50}$$

$$\log\left(\%e^{\mathrm{p}\left(\log\left(\%e^x+\%e^h\right)\right)}+\%e^{\mathrm{g}\left(\log\left(\%e^x+\%e^h\right)\right)}-\%e^{\mathrm{p}(x)}-\%e^{\mathrm{g}(x)}\right) \tag{\% o51}$$

$$\lim_{\to h \backslash\text{-}\backslash>\text{-}\infty}\log\left(\%e^{\mathrm{p}\left(\log\left(\%e^x+\%e^h\right)\right)}+\%e^{\mathrm{g}\left(\log\left(\%e^x+\%e^h\right)\right)}-\%e^{\mathrm{p}(x)}-\%e^{\mathrm{g}(x)}\right)-h \tag{\% o52}$$

So the down-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: D_Dx (e$\hat{\text{x}}$) = e$\hat{\text{x}}$.

(% i55) f(x) := bˆ (A*x+c);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=b^{Ax+c} \tag{\% o53}$$

$$\log\left(\%e^{b^{A\log\left(\%e^x+\%e^h\right)+c}}-\%e^{b^{Ax+c}}\right) \tag{\% o54}$$

$$(A\log(b)-1)\,x+\log(b)c+b^{Ax+c}+\log(\log(b))+\log(A) \tag{\% o55}$$

(% i58) f(x) := log(x);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=\log(x) \tag{\% o56}$$

$$\log\left(\log\left(\%e^{x}+\%e^{h}\right)-x\right) \tag{\% o57}$$

$$-x \tag{\% o58}$$

(% i59) kill(f);

$$\text{done} \tag{\% o59}$$

(% i60) Eig_Df_eq : Df_Dx(x) = f(x);

$$\lim_{\to h \backslash\text{-}\backslash > \text{-} \infty} \log\left(\%e^{\mathrm{f}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{f}(x)}\right)-h=\mathrm{f}(x) \tag{Eig\_ Df\_ eq}$$

As we can see from the previous solutions of Df_Dx, this DE is uniquely solved by f(x)=ex̂.

(% i61) kill(f);

$$\text{done} \tag{\% o61}$$

Alternative expressions for Df_Dx(x) (with x=log(z)):

(% i65) x = log(z);
log('diff(exp(f(log(z))) ,z,1));
log(diff(exp(f(log(z))) ,z,1));
f(x) - x + log(diff(f(x),x,1));

$$x=\log(z) \tag{\% o62}$$

$$\log\left(\frac{d}{dz}\%e^{\mathrm{f}(\log(z))}\right) \tag{\% o63}$$

$$\log\left(\%e^{\mathrm{f}(\log(z))}\left(\frac{d}{dz}\,\mathrm{f}(\log(z))\right)\right) \tag{\% o64}$$

$$\log\left(\frac{d}{dx}\,\mathrm{f}(x)\right)+\mathrm{f}(x)-x \tag{\% o65}$$

2 Down2

Heading further "down", define "three dots" (therefore), or down2, such that: a @ down2(b,c) = down2(a@b,a@c) ...Or: down(a,down2(b,c)) = down2(down(a,b),down(a,c)) ...

(% i66) td_eq : Td(a,b) = log(down(exp(a),exp(b)));

$$\mathrm{Td}\,(a\,,b) = \log\left(\log\left(\%e^{\%e^{b}} + \%e^{\%e^{a}}\right)\right) \tag{td\_ eq}$$

(% i67) down2(a,b) := radcan(log(down(exp(a),exp(b))));

$$\mathrm{down2}\,(a\,,b) := \mathrm{radcan}\,(\log\,(\mathrm{down}\,(\exp(a)\,,\exp(b)))) \tag{\% o67}$$

(% i68) down2(down(a,b),down(a,c));

$$\log\left(\log\left(\%e^{\%e^{c}} + \%e^{\%e^{b}}\right) + \%e^{a}\right) \tag{\% o68}$$

yes!down2-identity would have to be at ∞+iπ.

(% i69) down2(x,x);

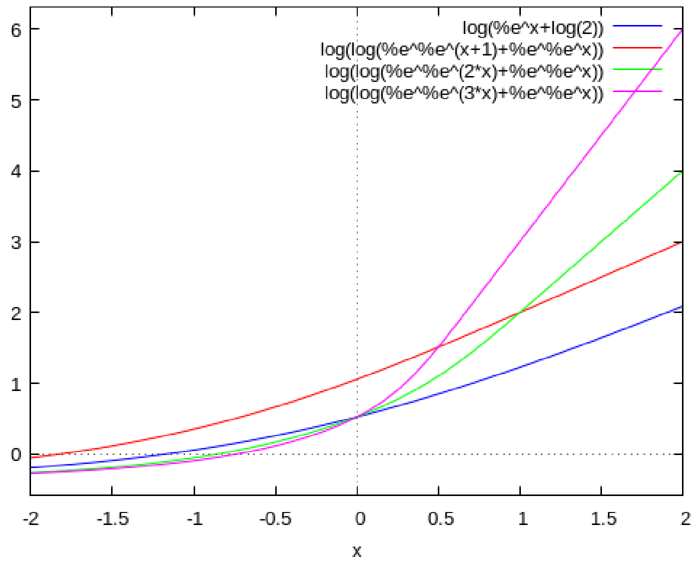
$$\log\,(\%e^{x} + \log\,(2)) \tag{\% o69}$$

(% i70) diff(down2(a,b),a);

$$\frac{\%e^{\%e^{a}+a}}{\left(\%e^{\%e^{b}} + \%e^{\%e^{a}}\right)\log\left(\%e^{\%e^{b}} + \%e^{\%e^{a}}\right)} \tag{\% o70}$$

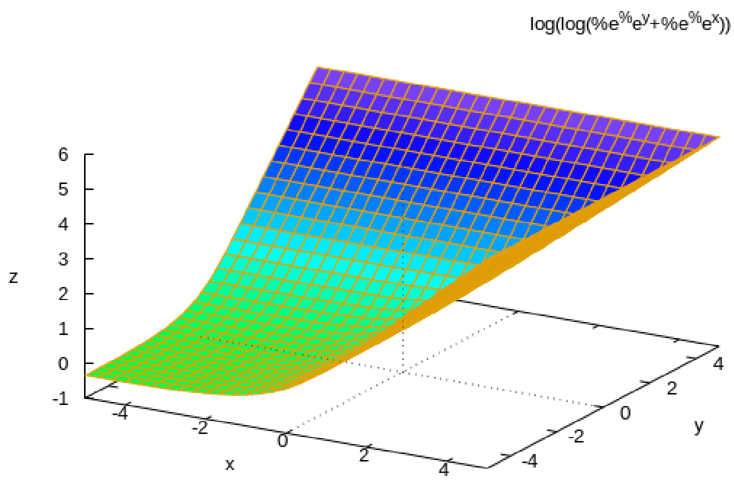
Let's take a look...

(%i71) wxplot2d ([down2(x,x),down2(x,x+1),down2(x,2*x),down2(x,3*x)], [x, -2, 2])$



(%t71)

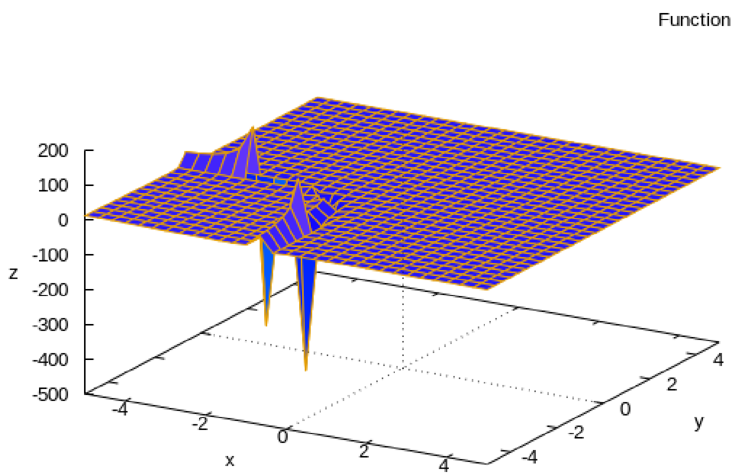
(%i72) wxplot3d (down2(x,y), [x, -5, 5], [y, -5, 5])$



(%t72)

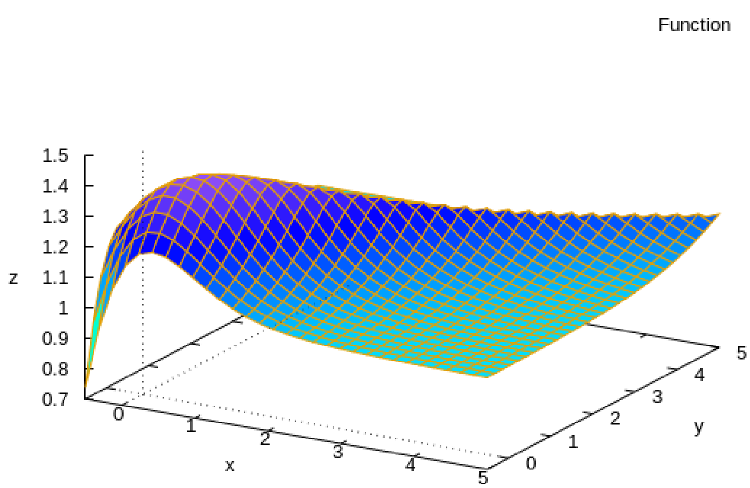
Comparing down(x,y) with down2(x,y) :

(% i73) wxplot3d (down(x,y)/down2(x,y), [x, -5, 5], [y, -5, 5])$



(% t73)

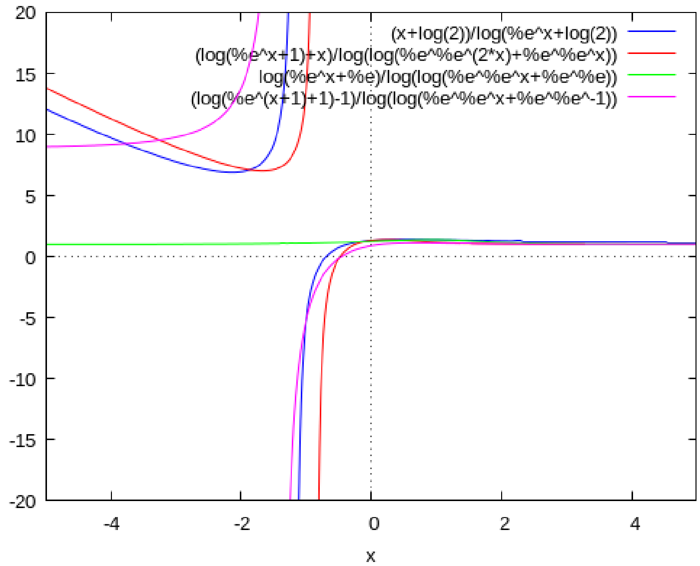
(% i74) wxplot3d (down(x,y)/down2(x,y), [x, -0.5, 5], [y, -0.5, 5])$



(% t74)

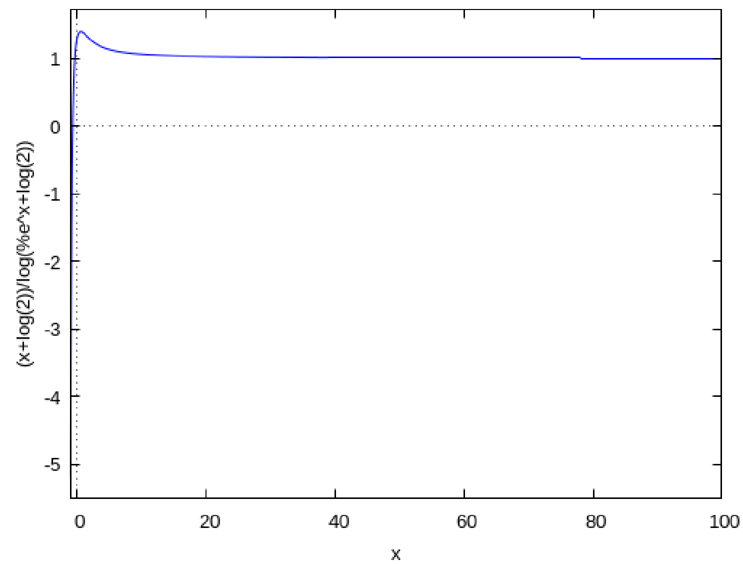
(% i75) wxplot2d ([down(x,x)/down2(x,x), down(x,2*x)/down2(x,2*x),down(x,1)/down2(x,1),down(x,-1)/down2(x,-1)], [x, -5, 5], [y, -20, 20])$

$plot2d : expression evaluates to non-numeric value somewhere in plotting range.$



(% t75)

(% i76) wxplot2d (down(x,x)/down2(x,x), [x, -1, 100])$



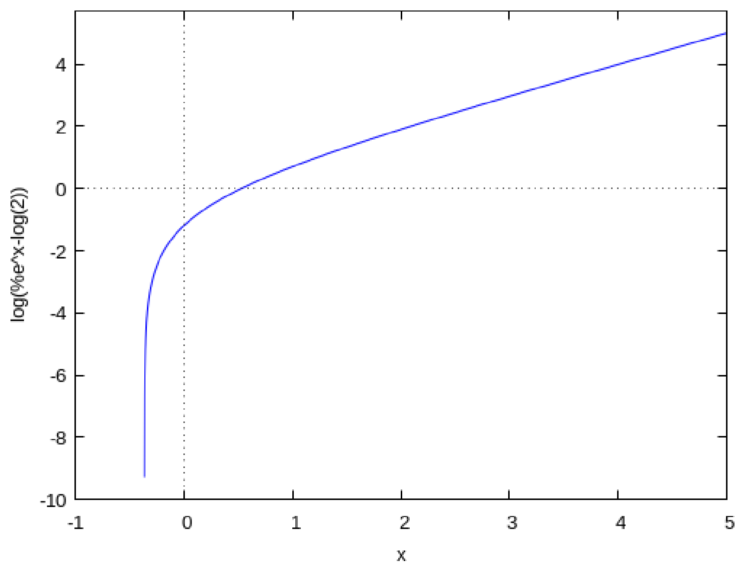
(% t76)

Using down2_div (or down_min) for the "ratio"...

(% i78) down_min(down(x,x),down2(x,x));
wxplot2d (down_min(down(x,x),down2(x,x)), [x, -1, 5])$

$\log\left(\%e^{x} - \log(2)\right) plot2d : expressionevaluatestonon{-}numericvaluesomewhereinplottingrange.$

(% o77)



(% t78)

3 Up

Let's take a look in the "up" direction for an operation which is distrubutive over multiplication. Defining "star" (or "up"): a * (b . c) = (a * b) . (a * c) ...Or: up(a,b.c) = up(a,b) . up(a,c) ...

(% i79) str_eq : Str(a,b) = exp(log(a)*log(b));

$$\mathrm{Str}\left(a , b\right) = \%e^{\log\left(a\right)\,\log\left(b\right)} \tag{str\_ eq}$$

(% i80) up(a,b) := radcan(exp(log(a)*log(b)));

$$\mathrm{up}\left(a , b\right) := \mathrm{radcan}\left(\exp\left(\log\left(a\right)\,\log\left(b\right)\right)\right) \tag{\% o80}$$

(% i81) up(a,b);

$$a^{\log\left(b\right)} \tag{\% o81}$$

(% i82) up(a,b*c);

$$a^{\log\left(c\right)+\log\left(b\right)} \tag{\% o82}$$

(%i83) up(a,b) * up(a,c);

$$a^{\log(c)+\log(b)} \tag{\% o83}$$

which is the same as the former. So, yes!Check with "down"-like operation:

(%i84) log(up(exp(a),exp(b)));

$$ab \tag{\% o84}$$

yes!up(a,b) could also be expressed with b as the base:

(%i88) is(up(a,b) = b^ log(a));
is(up(a,b) = a^ log(b));
is(up(b,a) = a^ log(b));
is(up(a,b) = up(b,a));

false (%o85)

true (%o86)

true (%o87)

true (%o88)

... although it seems Maxima has trouble (it enforces some type of alphabetization?) with that realization in the present form...?So one advantage of up(a,b), over "naive" exponentiation (a$\hat{b}$ or b$\hat{a}$), can be seen immediately: namely commutativity... up(a,b)=up(b,a). Identity for up (or up_times):

(%i89) up(%e,a);

$$a \tag{\% o89}$$

... the base %e. Some special values:

(%i91) up(x,x);
x^ log(x);

$$\%e^{\log(x)^2} \tag{\% o90}$$

$$x^{\log(x)} \tag{\% o91}$$

(% i94) diff(up(a,b),a);
diff(up(a,b),b);
diff(diff(up(a,b),a),b);

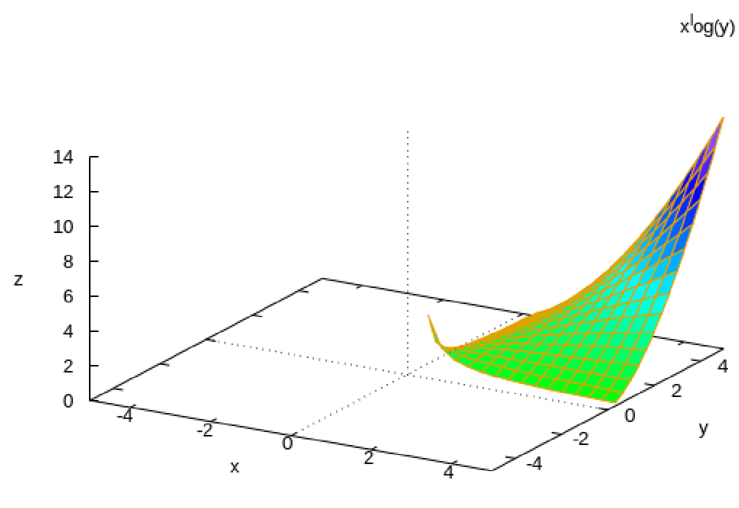
$$a^{\log(b)-1}\log(b) \tag{\% o92}$$

$$\frac{a^{\log(b)}\log(a)}{b} \tag{\% o93}$$

$$\frac{a^{\log(b)-1}\log(a)\log(b)}{b}+\frac{a^{\log(b)-1}}{b} \tag{\% o94}$$

Let's take a look:

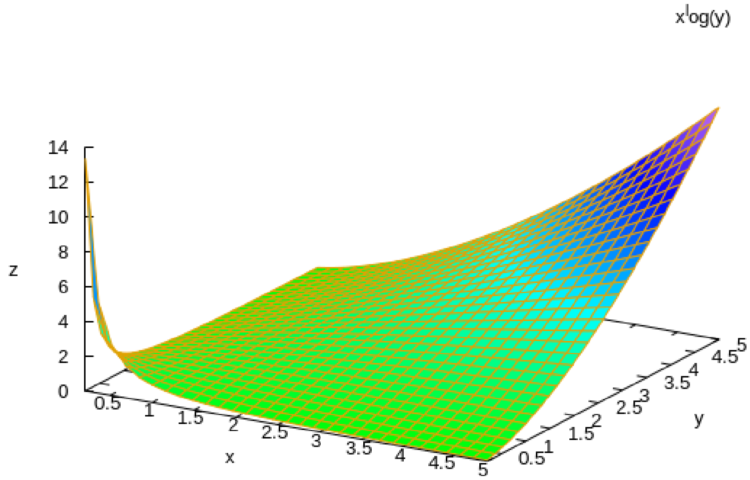
(% i95) wxplot3d (up(x,y), [x, -5, 5], [y, -5, 5])$



(% t95)

The singularity at the origin comes as no surprise. And for finite, real values, x and y should remain > 0.

(% i96) wxplot3d (up(x,y), [x, 0.2, 5], [y, 0.2, 5])$

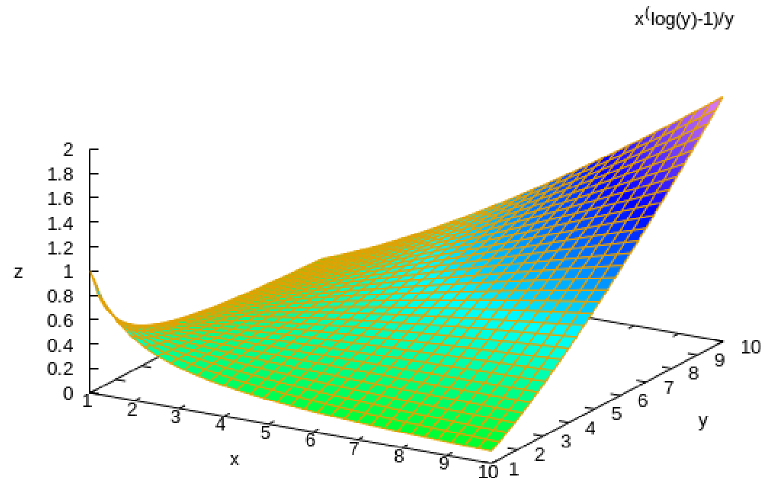


(% t96)

Compare with x.y :

(% i98) up(x,y)/(x*y);
wxplot3d (up(x,y)/(x*y), [x, 1, 10], [y, 1, 10])$

$$\frac{x^{\log(y)-1}}{y} \qquad (\% \text{ o97})$$



(% t98)

(% i101)

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up(x,x)/(x*x);
x^ (log(x)-2);
wxplot2d (log(up(x,x)/(x*x)), [x, 1, 100])$
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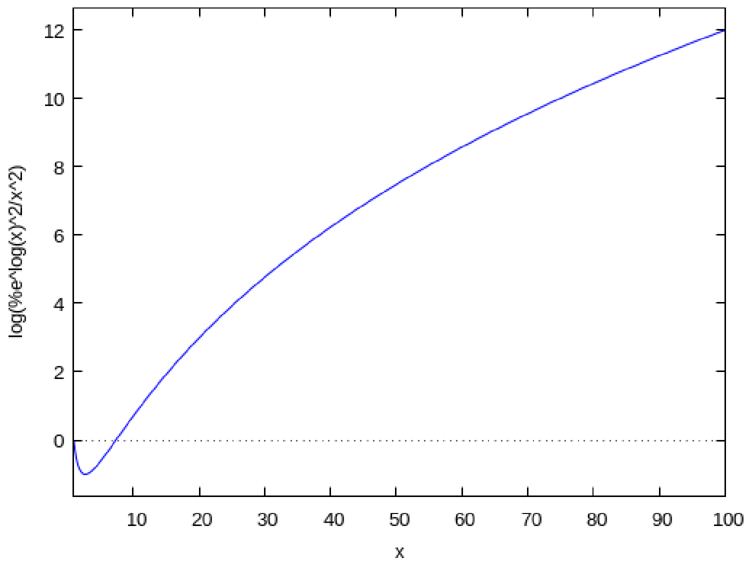
$$\frac{\%e^{\log(x)^2}}{x^2} \quad (\% \ o99)$$

$$x^{\log(x)-2} \quad (\% \ o100)$$



(% t101)

Derivative of this ratio:

(% i102) factor(diff(x^ (log(x)-2),x,1));

$$2x^{\log(x)-3}\left(\log(x)-1\right) \quad (\% \text{ o102})$$

Minimum obviously at x=e:

(% i104) up(e,e);
find_root(diff(x^ (log(x)-2),x,1), x, 1, 10);

$$\%e \quad (\% \text{ o103})$$

$$2.718281828459045 \quad (\% \text{ o104})$$

Also, up(x,x) does eventually win over x^2... at x=e^2, where up(x,x) is ...

(% i106) up(exp(2),exp(2));
float(%), numer;

$$\%e^4 \quad (\% \text{ o105})$$

$$54.59815003314423 \tag{\% o106}$$

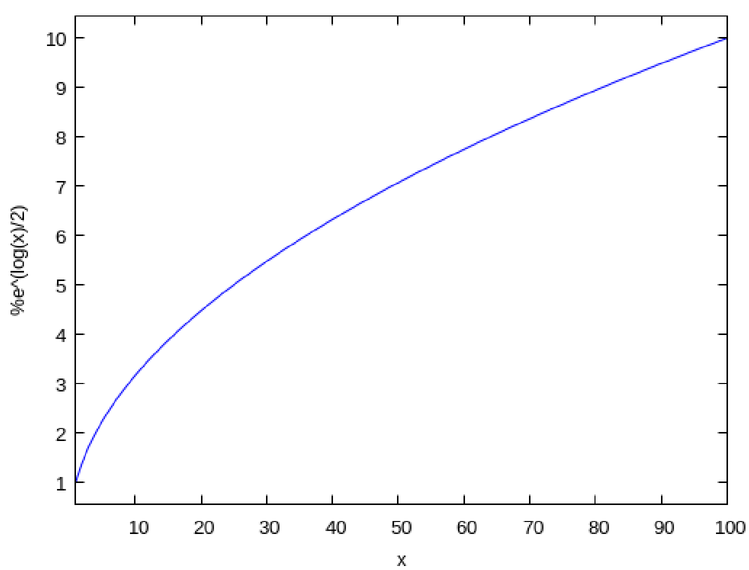
Using up_div for the "ratio"...

(% i108) exp(log(up(x,x))/log(x*x));
radcan(%);

$$\%e^{\frac{\log(x)}{2}} \tag{\% o107}$$

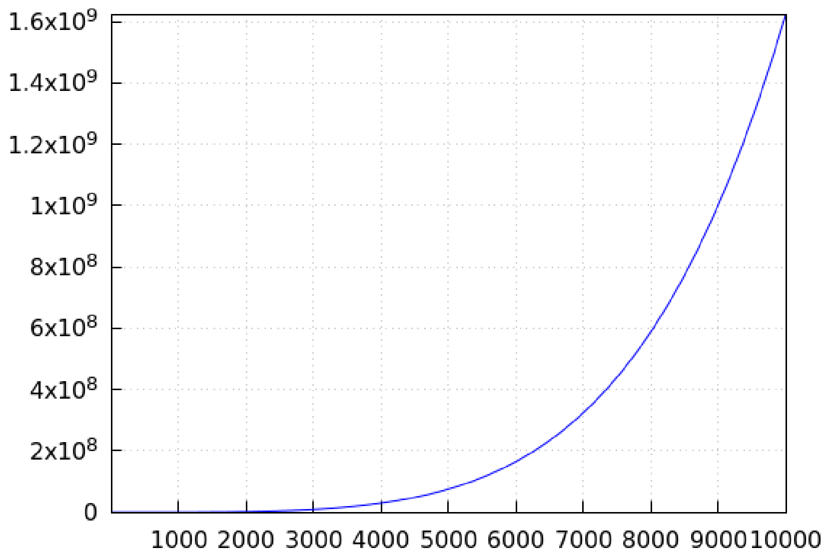
$$\sqrt{x} \tag{\% o108}$$

(% i109) wxplot2d (exp(log(up(x,x))/log(x*x)), [x, 1, 100])$



(% t109)

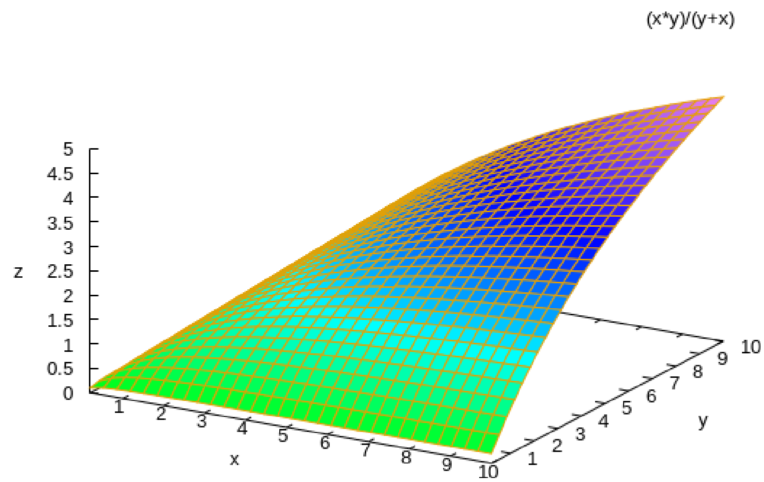
(% i110) wxdraw2d(grid = true, nticks = 1000, parametric(t^ 2 , up(t,t) , t,1,100))$



(% t110)

And x.y with x+y :

(% i111) wxplot3d (x*y/(x+y), [x, 0.2, 10], [y, 0.2, 10])$



(% t111)

Possible relations to polylogarithms?:

(% i115) integrate(log(t)/(t-1), t, 1, x);
integrate(log(t)/(t+1), t, 1, x);
integrate(log(t)/t, t, 1, x);
integrate(log(y)/t, t, 1, x);

"Is "

$$x-1$$

" positive, negative or zero?"

positive;

$$\log(1-x)\log(x)+\mathrm{li}_2(x)-\frac{\pi^2}{6} \quad (\%\ o112)$$

"Is "

$$x-1$$

" positive, negative or zero?"

positive;

$$\log(x)\log(x+1) + \mathrm{li}_2(-x) + \frac{\pi^2}{12} \tag{\% o113}$$

"Is "

$x - 1$

" positive, negative or zero?"

positive;

$$\frac{\log(x)^2}{2} \tag{\% o114}$$

"Is "

$x - 1$

" positive, negative or zero?"

positive;

$$\log(x)\log(y) \tag{\% o115}$$

3.1 Up differentiation

To avoid problems with alphabetical preferences...

(% i117)
```
up(x,h);
up(a,h);
```

$$h^{\log(x)} \tag{\% o116}$$

$$a^{\log(h)} \tag{\% o117}$$

we'll use 'a' as the independent variable here.Working with up_/, up_+ as times, up_- as divide, and the new "up_+"-identity, h->1...

(% i121)

```
Uf(a,h) := f(a*h) / f(a) ;
Uf(a,h);
Uf_Ua(a) := radcan( limit( exp( log(Uf(a,h)) / log(h) ) , h, 1, plus) );
Uf_Ua(a);
```

$$\mathrm{Uf}\,(a\,,h) := \frac{\mathrm{f}\,(ah)}{\mathrm{f}(a)} \tag{\% o118}$$

$$\frac{\mathrm{f}\,(ah)}{\mathrm{f}(a)} \tag{\% o119}$$

$$\mathrm{Uf\_Ua}(a) := \mathrm{radcan}\left(\mathrm{limit}\left(\exp\left(\frac{\log\left(\mathrm{Uf}\,(a\,,h)\right)}{\log\,(h)}\right),h\,,1\,,\mathrm{plus}\right)\right) \tag{\% o120}$$

$$\lim_{\rightarrow h \text{ \textbackslash-\textbackslash> } 1+} \frac{\mathrm{f}\,(ah)^{\frac{1}{\log\,(h)}}}{\mathrm{f}(a)^{\frac{1}{\log\,(h)}}} \tag{\% o121}$$

(% i123)

```
Uf_Ua(a) := ratsimp( limit( Uf(a,h)^ (1 / log(h))  , h, 1, plus));
Uf_Ua(a);
```

$$\mathrm{Uf\_Ua}(a) := \mathrm{ratsimp}\left(\mathrm{limit}\left(\mathrm{Uf}\,(a\,,h)^{\frac{1}{\log\,(h)}}\,,h\,,1\,,\mathrm{plus}\right)\right) \tag{\% o122}$$

$$\lim_{\rightarrow h \text{ \textbackslash-\textbackslash> } 1+} \left(\frac{\mathrm{f}\,(ah)}{\mathrm{f}(a)}\right)^{\frac{1}{\log\,(h)}} \tag{\% o123}$$

(% i126)

```
f(a) := n*a + c;
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := na + c \tag{\% o124}$$

$$\frac{ahn + c}{an + c} \tag{\% o125}$$

$$\%e^{\frac{an}{an+c}} \qquad (\%\ o126)$$

(% i130)

```
assume(y>1);
f(a) := a^ y + c;
Uf(a,h);
Uf_Ua(a);
```

$$[y > 1] \qquad (\%\ o127)$$

$$\mathrm{f}(a) := a^y + c \qquad (\%\ o128)$$

$$\frac{(ah)^y + c}{c + a^y} \qquad (\%\ o129)$$

$$\%e^{\frac{a^y y}{c+a^y}} \qquad (\%\ o130)$$

(% i133)

```
f(a) := exp(b*a^ y + c);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := \exp\left(ba^y + c\right) \qquad (\%\ o131)$$

$$\%e^{b(ah)^y - a^y b} \qquad (\%\ o132)$$

"Is "

y

" an "

integer

"?"

$no;$

$$\%e^{a^y b y} \quad (\%o133)$$

(%i136)
```
f(a) := b^ (a^ y + c);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := b^{a^y + c} \quad (\%o134)$$

$$b^{(a h)^y - a^y} \quad (\%o135)$$

"Is "

y

" an "

integer

"?"

no;

"Is "

$y - 1$

" positive, negative or zero?"

positive;

$$\%e^{a^y \log(b) y} \quad (\%o136)$$

(%i139)
```
f(a) := exp(b^ (a^ y + c));
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := \exp\left(b^{a^y + c}\right) \quad (\%o137)$$

$$\%e^{b^{(ah)^y+c}-b^{c+a^y}} \quad (\%\ o138)$$

"Is "

c

" an "

integer

"?"

no;

$$\%e^{a^y b^{c+a^y}\log(b) y} \quad (\%\ o139)$$

(% i142)

```
f(a) := g(a) * p(a);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a):=\mathrm{g}(a)\,\mathrm{p}(a) \quad (\%\ o140)$$

$$\frac{\mathrm{g}(ah)\,\mathrm{p}(ah)}{\mathrm{g}(a)\,\mathrm{p}(a)} \quad (\%\ o141)$$

$$\lim_{\to h \backslash\text{-}\backslash> 1+} \left(\frac{\mathrm{g}(ah)\,\mathrm{p}(ah)}{\mathrm{g}(a)\,\mathrm{p}(a)}\right)^{\frac{1}{\log(h)}} \quad (\%\ o142)$$

So the up-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: U_Ua (eâ) = eâ.

(% i145)

```
f(a) := b^ exp(a^ y + c);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a):=b^{\exp(a^y+c)} \quad (\%\ o143)$$

$$b^{\%e^{(ah)^y+c}-\%e^{c+a^y}} \quad (\%\ o144)$$

"Is "

c

" an "

integer

"?"

$no;$

$$\%e^{a^y \log(b) y \%e^{c+a^y}} \qquad (\%\ o145)$$

(% i148)
```
f(a) := log(a);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := \log(a) \qquad (\%\ o146)$$

$$\frac{\log(a h)}{\log(a)} \qquad (\%\ o147)$$

$$\%e^{\frac{1}{\log(a)}} \qquad (\%\ o148)$$

(% i149)
```
kill(f);
```

done (% o149)

Alternative expressions for Uf_Ua(a) (with a=exp(z)):

(% i154)
```
a = exp(z);
exp('diff( log(f(exp(z))) ,z,1));
exp(diff( log(f(exp(z))) ,z,1));
exp( a * diff(f(a),a,1) / f(a) );
exp('diff( log(f(a)) ,log(a),1));
```

$$a = \%e^{z} \qquad (\%\ o150)$$

$$\%e^{\frac{d}{dz}\log\left(\mathrm{f}(\%e^{z})\right)} \quad (\%\ o151)$$

$$\%e^{\frac{\frac{d}{dz}\mathrm{f}\left(\%e^{z}\right)}{\mathrm{f}(\%e^{z})}} \quad (\%\ o152)$$

$$\%e^{\frac{a\left(\frac{d}{da}\mathrm{f}(a)\right)}{\mathrm{f}(a)}} \quad (\%\ o153)$$

$$\%e^{\frac{d}{d\log(a)}\log\left(\mathrm{f}(a)\right)} \quad (\%\ o154)$$

This leads to an alternative expression for the "normal" derivative, d/dx, in terms of the down derivative:

(% i158)
```
exp(x + 'Df_Dx(x) - f(x));
z = log(x); g(x) = log(f(x));
exp('Dg_Dz(x));
```

$$\%e^{\mathrm{Df\_Dx}(x)-\mathrm{f}(x)+x} \quad (\%\ o155)$$

$$z = \log(x) \quad (\%\ o156)$$

$$\mathrm{g}(x) = \log\left(\mathrm{f}(x)\right) \quad (\%\ o157)$$

$$\%e^{\mathrm{Dg\_Dz}(x)} \quad (\%\ o158)$$

4 Up2

Now to be distributive over "up": up2(a , up(b,c)) = up(up2(a,b) , up2(a,c)) ...

(% i159)
```
up2(a,b) := radcan( exp( up(log(a),log(b)) ) );
```

$$\mathrm{up2}(a\,,b) := \mathrm{radcan}\left(\exp\left(\mathrm{up}\left(\log(a)\,,\log(b)\right)\right)\right) \quad (\%\ o159)$$

(% i161) up(up2(a,b),up2(a,c));
up2(a,up(b,c));

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \quad (\% \text{ o160})$$

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \quad (\% \text{ o161})$$

Bingo! And obviously also commutative, unlike say, tetration (ŷ x or x̂ y), which is neither. Identity for up2 (or up2_times):

(% i162) up2(%eˆ %e,a);

$$a \quad (\% \text{ o162})$$

... or base to the power of the base. So one can imagine that for up3 it would be e^{e^e}... etc.

(% i163) diff(up2(a,b),a);

$$\frac{\log(a)^{\log(\log(b))-1}\,\%e^{\log(a)^{\log(\log(b))}}\,\log(\log(b))}{a} \quad (\% \text{ o163})$$

(% i166) up2(x,x);
exp(log(x)ˆ log(x));
xˆ (log(x)ˆ (log(x)-1));

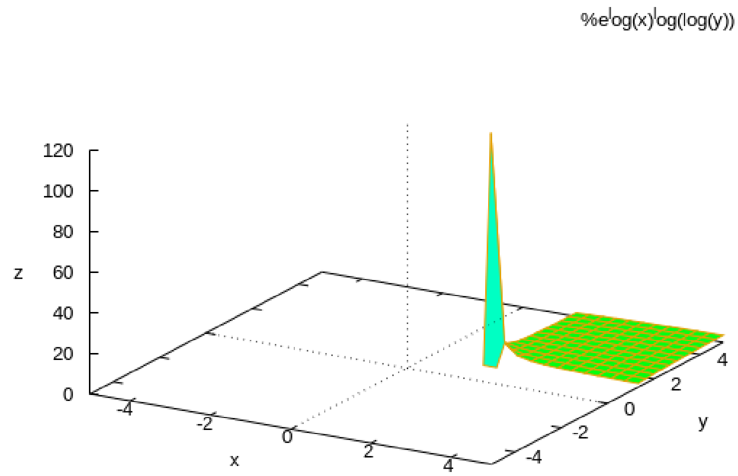
$$\%e^{\%e^{\log(\log(x))^2}} \quad (\% \text{ o164})$$

$$\%e^{\log(x)^{\log(x)}} \quad (\% \text{ o165})$$

$$x^{\log(x)^{\log(x)-1}} \quad (\% \text{ o166})$$

which is up(x,x)̂[log(x)̂(log(x)-2)]. Taking a look...

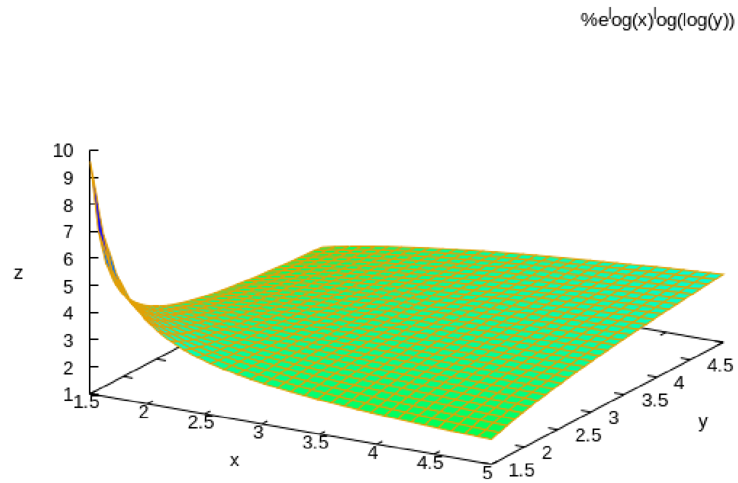
(%i167) wxplot3d (up2(x,y), [x, -5, 5], [y, -5, 5])$



(%t167)

Problems now at x or y = 1.

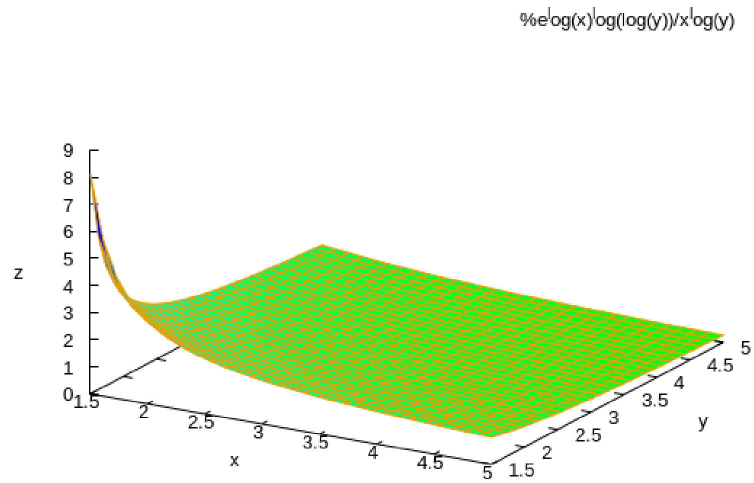
(%i168) wxplot3d (up2(x,y), [x, 1.5, 5], [y, 1.5, 5])$



(%t168)

Comparing up2(x,y) with up(x,y) :

(%i169) wxplot3d (up2(x,y)/up(x,y), [x, 1.5, 5], [y, 1.5, 5])$



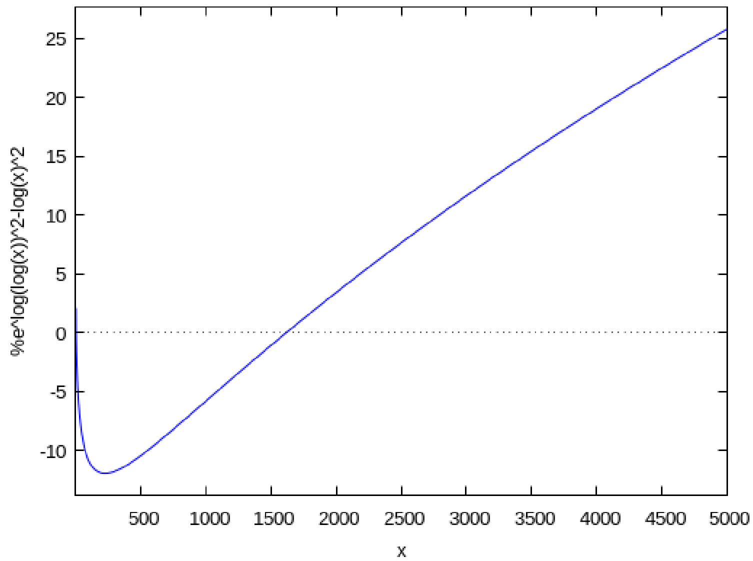
$$(\% \; t169)$$

(%i173) up2(x,x)/up(x,x);
log(up2(x,x)/up(x,x));
log(x)ˆ log(log(x))-log(x)ˆ 2;
wxplot2d (log(up2(x,x)/up(x,x)), [x, 1.5, 5000])$

$$\%e^{\%e^{\log\left(\log\left(x\right)\right)^2}-\log\left(x\right)^2} \qquad (\% \; o170)$$

$$\%e^{\log\left(\log\left(x\right)\right)^2}-\log\left(x\right)^2 \qquad (\% \; o171)$$

$$\log\left(x\right)^{\log\left(\log\left(x\right)\right)}-\log\left(x\right)^2 \qquad (\% \; o172)$$



(% t173)

(% i174) factor(diff(exp(log(x)ˆ log(log(x))-log(x)ˆ 2),x,1));

$$\frac{2\%e^{\log(x)^{\log(\log(x))}-\log(x)^2}\left(\log(x)^{\log(\log(x))}\log(\log(x))-\log(x)^2\right)}{x\log(x)} \quad (\%\ o174)$$

(% i175) find_root(''(%), x, 100, 300);

225.248329275192 (% o175)

Note also that up2(x,x) does eventually win over up(x,x)... at x=e(ˆe²), where up2(x,x) is...

(% i177) up2(exp(exp(2)),exp(exp(2)));
float(%), numer;

$$\%e^{\%e^4} \quad (\%\ o176)$$

$$5.148435562634557\,10^{23} \quad (\%\ o177)$$

Holy Avogadro, batman! Using up2_div for the "ratio"...

(%i179)
```
up_div(a,b) := radcan( exp(log(a) / log(b)) );
up2_div(a,b) := radcan( exp( up_div(log(a),log(b)) ) );
```

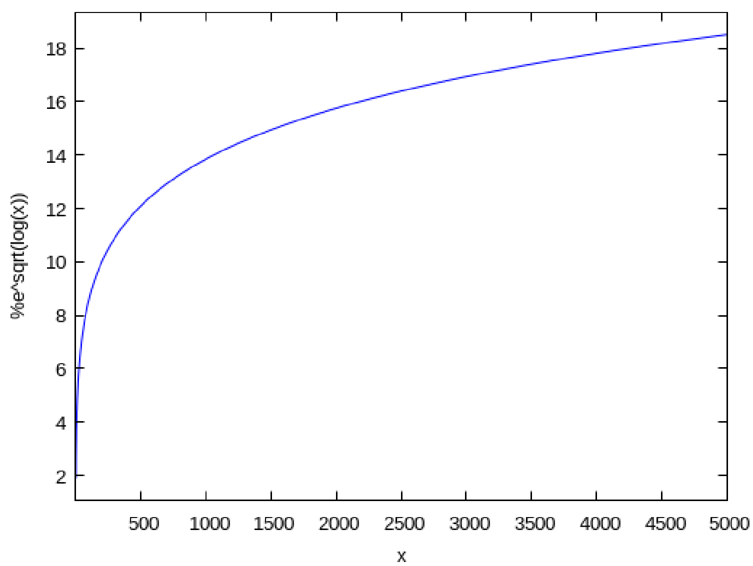
$$\mathrm{up\_div}\left(a\,,b\right):=\mathrm{radcan}\left(\exp\left(\frac{\log\left(a\right)}{\log\left(b\right)}\right)\right) \quad (\%\,o178)$$

$$\mathrm{up2\_div}\left(a\,,b\right):=\mathrm{radcan}\left(\exp\left(\mathrm{up\_div}\left(\log\left(a\right),\log\left(b\right)\right)\right)\right) \quad (\%\,o179)$$

(%i180)
```
up2_div(up2(x,x),up(x,x));
```

$$\%e^{\sqrt{\log\left(x\right)}} \quad (\%\,o180)$$

(%i181)
```
wxplot2d (up2_div(up2(x,x),up(x,x)), [x, 1.5, 5000])$
```

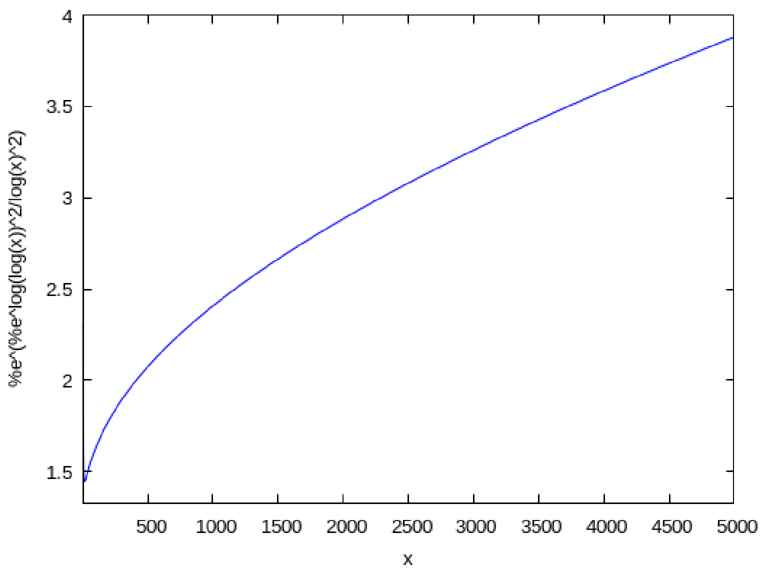


(% t181)

(%i184)
```
up_div(up2(x,x),up(x,x));
exp(log(x)^ (log(log(x))-2));
wxplot2d (up_div(up2(x,x),up(x,x)), [x, 3, 5000])$
```

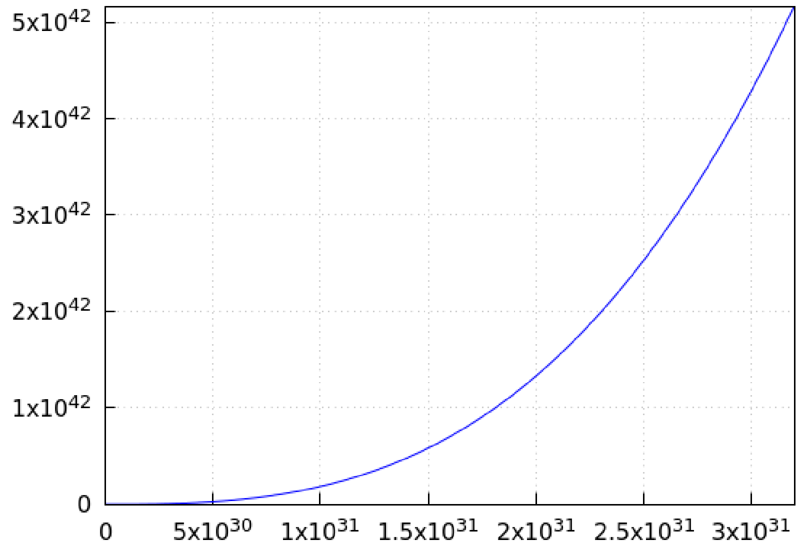
$$\%e^{\frac{\%e^{\log\left(\log\left(x\right)\right)^2}}{\log\left(x\right)^2}} \quad (\%\,o182)$$

$\%e^{\log(x)^{\log(\log(x))-2}}$ (%o183)



(%t184)

(%i185) wxdraw2d(grid = true, nticks = 1000, parametric(up(t,t) , up2(t,t) , t,1.5,5000))$



(%t185)