

DISTRIBUTIVITY, EXTENDED

(Tommy Burch, on and off from 08.2024 to 05.2025)

1 At or Down

This all started back in the late '90s at the UofA in a physics lab prep session (with R. McCroskey and R. Haar, among others). The discussion hinged on finding an operation (i termed it 'at' with the symbol @) over which addition was distributive. The exp / log solution below i came up with followed later the same day... Note that, for multiplication and addition, the distributive property can be "revealed" through exponentials and their inverse, the natural logarithms:

$$\begin{aligned} \text{(\% i1) } \text{add(a,b) := log(exp(a)*exp(b));} \\ \text{add(a,b) := log(exp(a)exp(b))} \end{aligned} \quad (\% \text{ o1})$$

$$\begin{aligned} \text{(\% i3) } \text{add(a,b);} \\ \text{add(c*a,c*b);} \\ b + a \end{aligned} \quad (\% \text{ o2})$$

$$bc + ac \quad (\% \text{ o3})$$

$$\begin{aligned} \text{(\% i4) } \text{ratsimp(\%);} \\ (b + a) c \end{aligned} \quad (\% \text{ o4})$$

Placing the multiplication between the exponentials in the logarithm, we arrive at the relation (addition) over which it is distributive. We can try the same with + between the exp's...

1.1 At_+

Define the @ (or "down") relation such that addition is distributive over it: $a + (b @ c) = (a+b) @ (a+c)$... Or: $a + \text{down}(b,c) = \text{down}(a+b,a+c)$. Trying the following ansatz:

$$\begin{aligned} \text{(\% i5) } \text{at_eq : At(a,b) = log(exp(a) + exp(b));} \\ \text{At(a,b) = log(\%e^b + \%e^a)} \end{aligned} \quad (\text{at_eq})$$

(% i6) down(a,b) := radcan(log(exp(a) + exp(b)));

$$\text{down}(a, b) := \text{radcan}(\log(\exp(a) + \exp(b))) \quad (\% \text{ o6})$$

Check:

(% i7) down(a+b,a+c);

$$\log\left({\%e}^c + {\%e}^b\right) + a \quad (\% \text{ o7})$$

yes! (It is also an obviously commutative operation.) This should work with any base for the power / log's: e.g.,

(% i8) log10(x) := log(x) / log(10);

$$\log_{10}(x) := \frac{\log(x)}{\log(10)} \quad (\% \text{ o8})$$

(% i9) radcan(log10(10^ (a+b) + 10^ (a+c)));

$$\frac{\log\left(2^c 5^c + 2^b 5^b\right) + (\log(5) + \log(2)) a}{\log(5) + \log(2)} \quad (\% \text{ o9})$$

(% i10) logcontract(%);

$$\frac{\log\left(2^c 5^c + 2^b 5^b\right) + \log(10) a}{\log(10)} \quad (\% \text{ o10})$$

yep! The ambiguity probably speak volumes (see, e.g., M. Burgin, arXiv:1010.3287 : there, the focus appears not to be in maintaining distributivity, unlike here, where it's the main focus). It may be worth exploring whether functions other than powers/logarithms can retain distributivity... at first glance, seemingly not. Some values with base e:

(% i11) down(0,0);

$$\log(2) \quad (\% \text{ o11})$$

(% i12) down(1,0);

$$\log({\%e} + 1) \quad (\% \text{ o12})$$

(% i13) down(x,0);

$$\log({\%e}^x + 1) \quad (\% \text{ o13})$$

```
(% i15) down(x,x);
        down(x-log(2),x-log(2));
```

$$x + \log(2) \quad (\% \text{ o14})$$

$$x \quad (\% \text{ o15})$$

Associative:

```
(% i17) down(a,down(b,c));
        down(down(a,b),c);
```

$$\log\left({\%e}^c + {\%e}^b + {\%e}^a\right) \quad (\% \text{ o16})$$

$$\log\left({\%e}^c + {\%e}^b + {\%e}^a\right) \quad (\% \text{ o17})$$

```
(% i20) diff(down(a,b),a);
        diff(down(a,b),b);
        diff(diff(down(a,b),b),a);
```

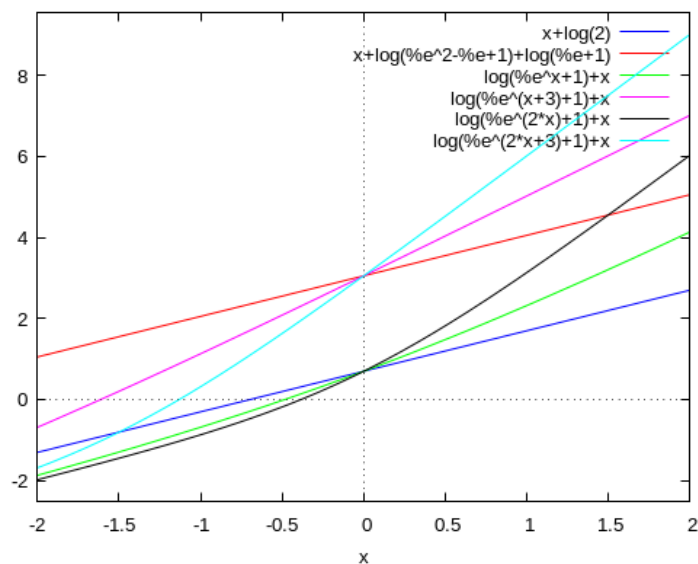
$$\frac{{\%e}^a}{{\%e}^b + {\%e}^a} \quad (\% \text{ o18})$$

$$\frac{{\%e}^b}{{\%e}^b + {\%e}^a} \quad (\% \text{ o19})$$

$$-\frac{{\%e}^{b+a}}{\left({\%e}^b + {\%e}^a\right)^2} \quad (\% \text{ o20})$$

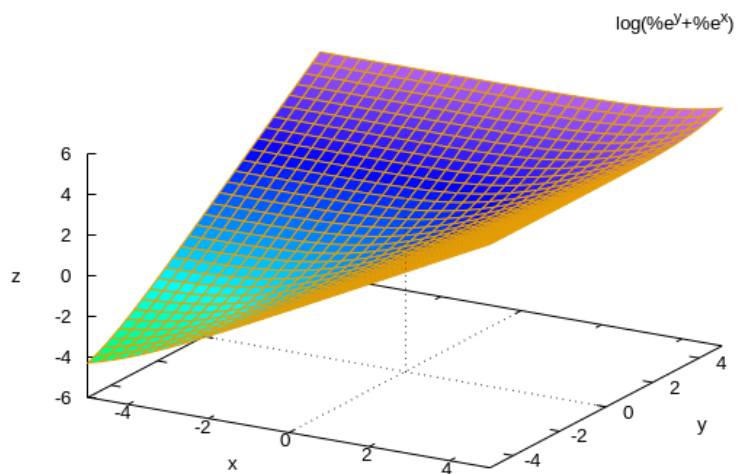
Let's have a look:

```
(% i21) wxplot2d ([down(x,x),down(x,x+3),down(x,2*x),down(x,2*x+3),down(x,3*x),down(x,3*x+3)],
[x, -2, 2])$
```



(% t21)

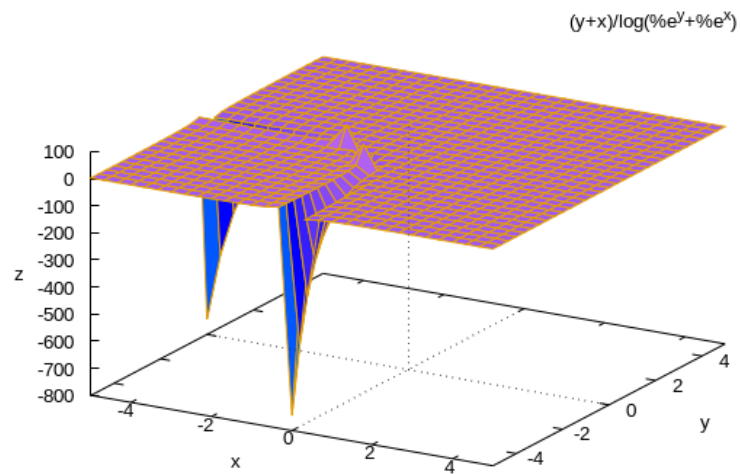
```
(% i22) wxplot3d (down(x,y), [x, -5, 5], [y, -5, 5])$
```



(% t22)

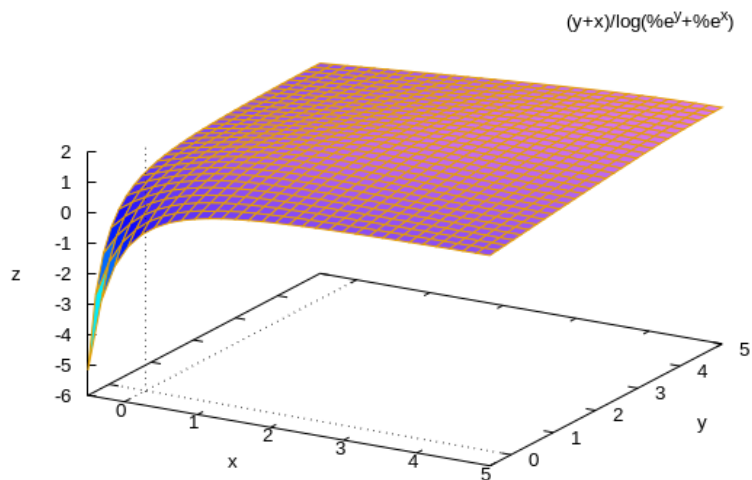
Comparing $x+y$ with $\text{down}(x,y)$:

(% i23) `wxplot3d ((x+y)/down(x,y), [x, -5, 5], [y, -5, 5])$`



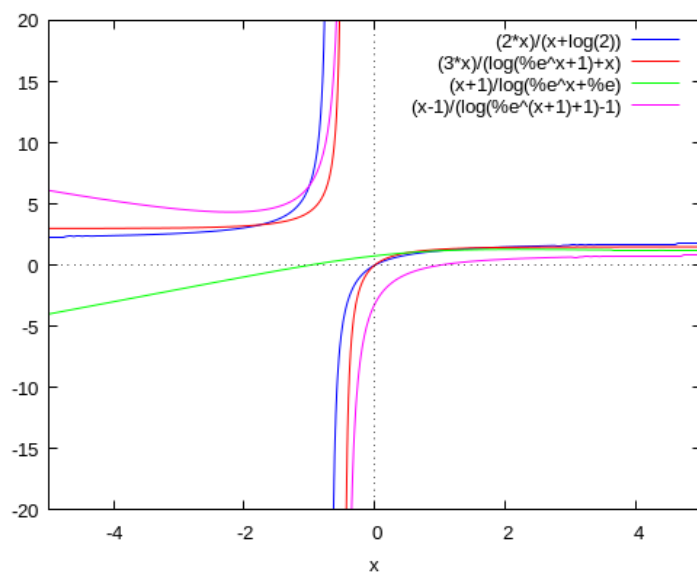
(% t23)

(% i24) wxplot3d ((x+y)/down(x,y), [x, -0.5, 5], [y, -0.5, 5])\$



(% t24)

(% i25) wxplot2d ([2*x/down(x,x), 3*x/down(x,2*x),(x+1)/down(x,1),(x-1)/down(x,-1)], [x, -5, 5], [y, -20, 20])\$



(% t25)

1.2 At₋,*,/

We can also consider a "down-like" operation with a minus instead a plus:

(% i26) `down_min(a,b) := radcan(log(exp(a) - exp(b)));`

$$\text{down_min}(a, b) := \text{radcan}(\log(\exp(a) - \exp(b))) \quad (\% \text{ o26})$$

Associativity:

(% i30) `down(down_min(a,b),c);`
`down_min(down(a,c),b);`
`down(down_min(a,b),down_min(c,d));`
`down(down_min(a,d),down_min(c,b));`

$$\log\left(\%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o27})$$

$$\log\left(\%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o28})$$

$$\log\left(-\%e^d + \%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o29})$$

$$\log\left(-\%e^d + \%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o30})$$

Same distributive property:

(% i31) `down_min(a+b,a+c);`

$$\log\left(\%e^b - \%e^c\right) + a \quad (\% \text{ o31})$$

As we've already seen, with times, we achieve addition. With divide, one achieves subtraction:

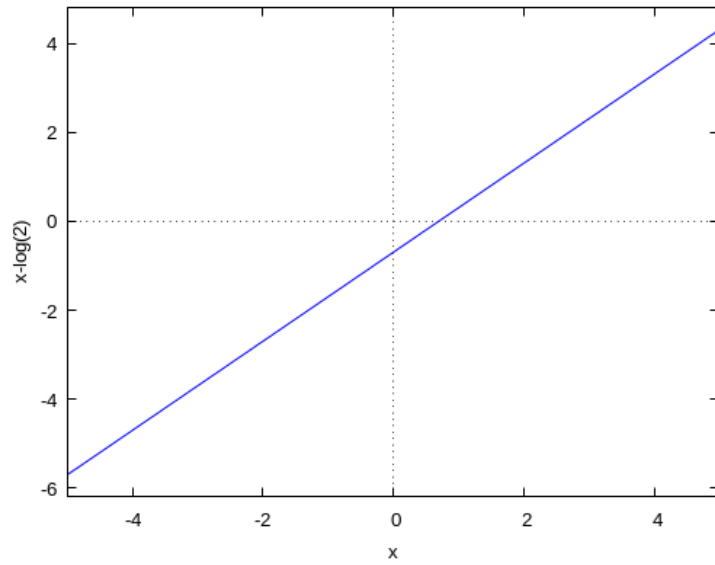
(% i33) `down_div(a,b) := radcan(log(exp(a) / exp(b)));`
`down_div(a,b);`

$$\text{down_div}(a, b) := \text{radcan}\left(\log\left(\frac{\exp(a)}{\exp(b)}\right)\right) \quad (\% \text{ o32})$$

$$a - b \quad (\% \text{ o33})$$

Using @_/ or down_div for the "ratio" of + and @...

```
(% i35) down_div(x+x,down(x,x));
wxplot2d (2*x - down(x,x), [x, -5, 5])$
x - log(2)                                     (% o34)
```



(% t35)

1.3 Down differentiation

Given down, down_min, and down_div, one is free to define a "down derivative". However, we must take care with the notion of infinitesimals: normally $h \rightarrow 0$ in the definition of the derivative, but here we need $h \rightarrow -\infty$ to approach the @_+ identity ($x @ -\infty = x$)...

```
(% i36) Df_Dx_eq : Df_Dx = limit( down_min( f(down(x,h)) , f(x) ) - h , h, -infinity);
```

$$Df_Dx = \lim_{h \searrow -\infty} \log \left(\frac{e^{f(\log(e^x + e^h))}}{e^{f(x)}} \right) - h \quad (Df_Dx_eq)$$

```
(% i40) Df(x,h) := down_min( f(down(x,h)) , f(x) );
Df(x,h);
Df_Dx(x) := ratsimp( limit( Df(x,h) - h , h, minf) );
Df_Dx(x);
```

```
Df(x,h) := down_min( f(down(x,h)) , f(x))                                     (% o37)
```


$$\log \left(\%e^{\text{f}(\log(\%e^x + \%e^h))} - \%e^{\text{f}(x)} \right) \quad (\% \text{ o38})$$

$$\text{Df_Dx}(x) := \text{ratsimp}(\text{limit}(\text{Df}(x, h) - h, h, -\infty)) \quad (\% \text{ o39})$$

$$\lim_{\rightarrow h \setminus \setminus > -\infty} \log \left(\%e^{\text{f}(\log(\%e^x + \%e^h))} - \%e^{\text{f}(x)} \right) - h \quad (\% \text{ o40})$$

$$\begin{aligned} (\% \text{ i43}) \quad & \text{f(x)} := \text{A}^*\text{x}^b + \text{c}; \\ & \text{Df(x,h)}; \\ & \text{Df_Dx(x)}; \end{aligned}$$

$$\text{f}(x) := Ax^b + c \quad (\% \text{ o41})$$

$$\log \left(\%e^{A \log(\%e^x + \%e^h)^b} - \%e^{Ax^b} \right) + c \quad (\% \text{ o42})$$

$$(b-1) \log(x) + Ax^b - x + c + \log(b) + \log(A) \quad (\% \text{ o43})$$

$$\begin{aligned} (\% \text{ i46}) \quad & \text{f(x)} := \log(\text{A}^*\text{x}^b + \text{c}); \\ & \text{Df(x,h)}; \\ & \text{Df_Dx(x)}; \end{aligned}$$

$$\text{f}(x) := \log(Ax^b + c) \quad (\% \text{ o44})$$

$$\log \left(\log \left(\%e^x + \%e^h \right)^b - x^b \right) + \log(A) \quad (\% \text{ o45})$$

$$(b-1) \log(x) - x + \log(b) + \log(A) \quad (\% \text{ o46})$$

$$\begin{aligned} (\% \text{ i49}) \quad & \text{f(x)} := \exp(\text{A}^*\text{x}^b + \text{c}); \\ & \text{Df(x,h)}; \\ & \text{Df_Dx(x)}; \end{aligned}$$

$$\text{f}(x) := \exp(Ax^b + c) \quad (\% \text{ o47})$$

$$\log \left(e^{e^{A \log(e^x + e^h)^{b+c}}} - e^{e^{Ax^b+c}} \right) \quad (\% \text{ o48})$$

$$(b-1) \log(x) + e^{Ax^b+c} + Ax^b - x + c + \log(b) + \log(A) \quad (\% \text{ o49})$$

(% i52) f(x) := down(g(x) , p(x));
 Df(x,h);
 Df_ Dx(x) ;
 f(x) := down(g(x) , p(x)) \quad (\% \text{ o50})

$$\log \left(e^{p(\log(e^x + e^h))} + e^{g(\log(e^x + e^h))} - e^{p(x)} - e^{g(x)} \right) \quad (\% \text{ o51})$$

$$\lim_{h \searrow -\infty} \log \left(e^{p(\log(e^x + e^h))} + e^{g(\log(e^x + e^h))} - e^{p(x)} - e^{g(x)} \right) - h \quad (\% \text{ o52})$$

So the down-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: $D_Dx(e^x) = e^x$.

(% i55) f(x) := b^ (A*x+c);
 Df(x,h);
 Df_ Dx(x) ;
 f(x) := b^{Ax+c} \quad (\% \text{ o53})

$$\log \left(e^{b^{A \log(e^x + e^h)^{b+c}}} - e^{b^{Ax+c}} \right) \quad (\% \text{ o54})$$

$$(A \log(b) - 1)x + \log(b)c + b^{Ax+c} + \log(\log(b)) + \log(A) \quad (\% \text{ o55})$$

(% i58) f(x) := log(x);
 Df(x,h);
 Df_ Dx(x) ;
 f(x) := log(x) \quad (\% \text{ o56})

$$\log \left(\log \left(e^x + e^h \right) - x \right) \quad (\% \text{ o57})$$

$$-x \quad (\% \text{ o58})$$

(% i59) kill(f);

done (\% o59)

(% i60) Eig_Df_eq : Df_Dx(x) = f(x);

$$\lim_{h \rightarrow -\infty} \log \left(e^{f(\log(e^x + e^h))} - e^{f(x)} \right) - h = f(x) \quad (\text{Eig_Df_eq})$$

As we can see from the previous solutions of Df_Dx, this DE is uniquely solved by f(x)=x.

(% i61) kill(f);

done (\% o61)

Alternative expressions for Df_Dx(x) (with x=log(z)):

(% i65) x = log(z);
log(diff(exp(f(log(z))) ,z,1));
log(diff(exp(f(log(z))) ,z,1));
f(x) - x + log(diff(f(x),x,1));

$$x = \log(z) \quad (\% \text{ o62})$$

$$\log \left(\frac{d}{dz} e^{f(\log(z))} \right) \quad (\% \text{ o63})$$

$$\log \left(e^{f(\log(z))} \left(\frac{d}{dz} f(\log(z)) \right) \right) \quad (\% \text{ o64})$$

$$\log \left(\frac{d}{dx} f(x) \right) + f(x) - x \quad (\% \text{ o65})$$

2 Down2

Heading further "down", define "three dots" (therefore), or down2, such that: a
 @ down2(b,c) = down2(a@b,a@c) ...Or: down(a,down2(b,c)) = down2(down(a,b),down(a,c))
 ...

(% i66) td_eq : Td(a,b) = log(down(exp(a),exp(b)));

$$\text{Td}(a, b) = \log \left(\log \left({}_0e^{{}_0e^b} + {}_0e^{{}_0e^a} \right) \right) \quad (\text{td_eq})$$

(% i67) down2(a,b) := radcan(log(down(exp(a),exp(b))));

$$\text{down2}(a, b) := \text{radcan} \left(\log \left(\text{down} \left(\exp(a), \exp(b) \right) \right) \right) \quad (\% \text{ o67})$$

(% i68) down2(down(a,b),down(a,c));

$$\log \left(\log \left({}_0e^{{}_0e^c} + {}_0e^{{}_0e^b} \right) + {}_0e^a \right) \quad (\% \text{ o68})$$

yes!down2-identity would have to be at $\infty + i\pi$.

(% i69) down2(x,x);

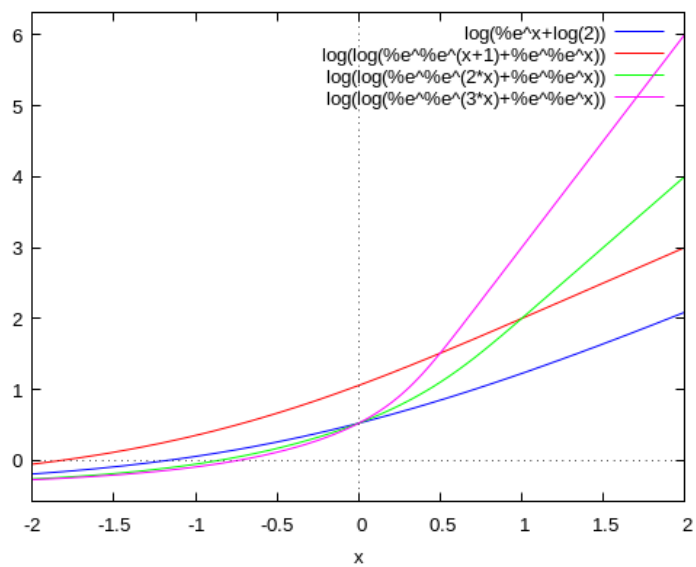
$$\log({}_0e^x + \log(2)) \quad (\% \text{ o69})$$

(% i70) diff(down2(a,b),a);

$$\frac{{}_0e^{{}_0e^a + a}}{\left({}_0e^{{}_0e^b} + {}_0e^{{}_0e^a} \right) \log \left({}_0e^{{}_0e^b} + {}_0e^{{}_0e^a} \right)} \quad (\% \text{ o70})$$

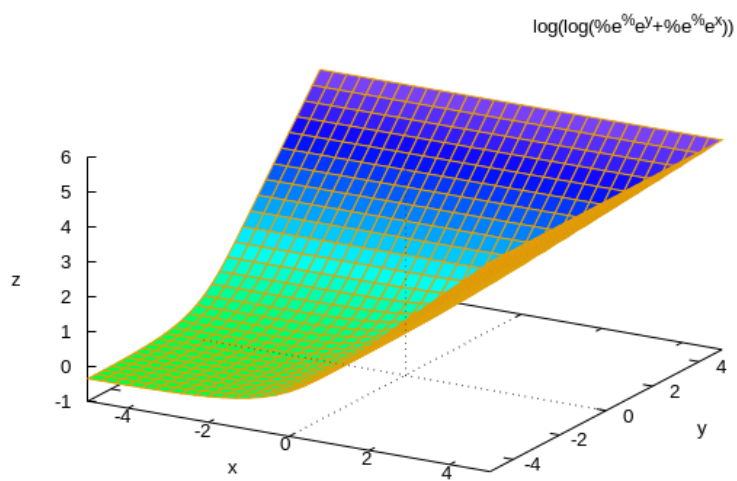
Let's take a look...

(% i71) wxplot2d ([down2(x,x),down2(x,x+1),down2(x,2*x),down2(x,3*x)], [x, -2, 2])\$



(% t71)

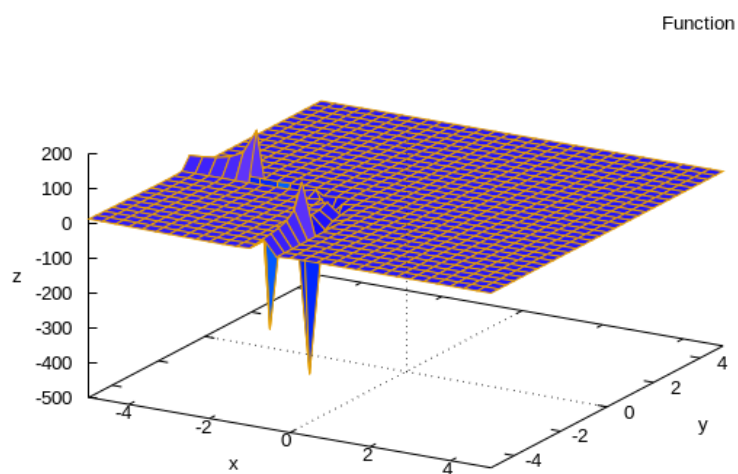
(% i72) wxplot3d (down2(x,y), [x, -5, 5], [y, -5, 5])\$



(% t72)

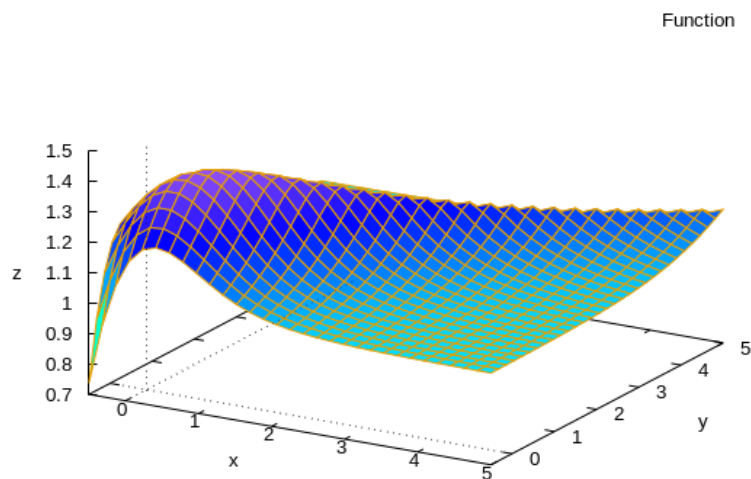
Comparing $\text{down}(x,y)$ with $\text{down2}(x,y)$:

(% i73) wxplot3d (down(x,y)/down2(x,y), [x, -5, 5], [y, -5, 5])\$



(% t73)

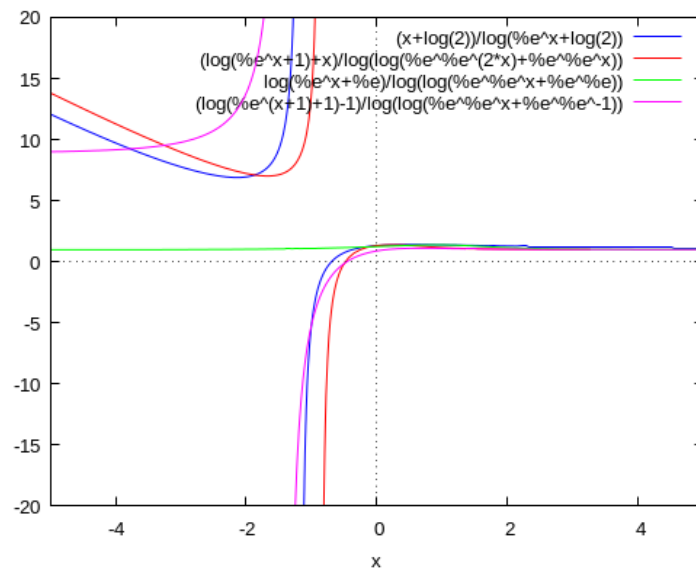
(% i74) wxplot3d (down(x,y)/down2(x,y), [x, -0.5, 5], [y, -0.5, 5])\$



(% t74)

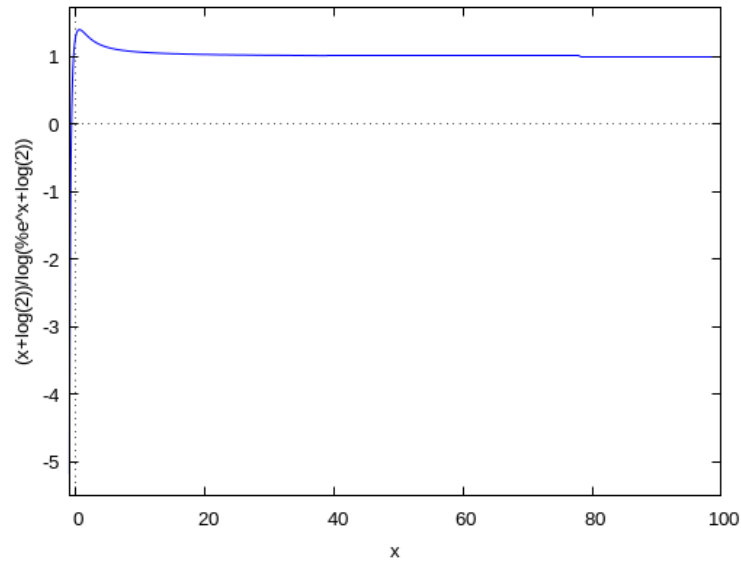
(% i75) wxplot2d ([down(x,x)/down2(x,x), down(x,2*x)/down2(x,2*x),down(x,1)/down2(x,1),down(x,-1)/down2(x,-1)], [x, -5, 5], [y, -20, 20])\$

plot2d : expression evaluate on non-numeric values somewhere in plotting range.



(% t75)

(% i76) wxplot2d (down(x,x)/down2(x,x), [x, -1, 100])\$



(% t76)

3 Up

Let's take a look in the "up" direction for an operation which is distributive over multiplication. Defining "star" (or "up"): $a * (b \cdot c) = (a * b) \cdot (a * c)$... Or: $\text{up}(a, b \cdot c) = \text{up}(a, b) \cdot \text{up}(a, c)$...

(% i77) str_eq : Str(a,b) = exp(log(a)*log(b));

$$\text{Str}(a, b) = e^{\log(a) \log(b)} \quad (\text{str_eq})$$

(% i78) up(a,b) := radcan(exp(log(a)*log(b)));

$$\text{up}(a, b) := \text{radcan}(\exp(\log(a) \log(b))) \quad (\% \text{ o78})$$

(% i79) up(a,b);

$$a^{\log(b)} \quad (\% \text{ o79})$$

(% i80) up(a,b*c);

$$a^{\log(c) + \log(b)} \quad (\% \text{ o80})$$


```
(% i81) up(a,b) * up(a,c);
```

$$a^{\log(c) + \log(b)} \quad (\% \text{ o81})$$

which is the same as the former. So, yes! Check with "down"-like operation:

```
(% i82) log( up(exp(a),exp(b)) );
```

$$ab \quad (\% \text{ o82})$$

yes! up(a,b) could also be expressed with b as the base:

```
(% i86) is( up(a,b) = b ^ log(a) );
is( up(a,b) = a ^ log(b) );
is( up(b,a) = a ^ log(b) );
is( up(a,b) = up(b,a) );
```

$$\text{false} \quad (\% \text{ o83})$$

$$\text{true} \quad (\% \text{ o84})$$

$$\text{true} \quad (\% \text{ o85})$$

$$\text{true} \quad (\% \text{ o86})$$

... although it seems Maxima has trouble (it enforces some type of alphabetization?) with that realization in the present form...? So one advantage of up(a,b), over "naive" exponentiation (a^b or b^a), can be seen immediately: namely commutativity... up(a,b)=up(b,a). Identity for up (or up_times):

```
(% i87) up(e,a);
```

$$a \quad (\% \text{ o87})$$

... the base e. Some special values:

```
(% i89) up(x,x);
x ^ log(x);
```

$$e^{\log(x)^2} \quad (\% \text{ o88})$$

$$x^{\log(x)} \quad (\% \text{ o89})$$

```
(% i92) diff(up(a,b),a);
        diff(up(a,b),b);
        diff(diff(up(a,b),a),b);
```

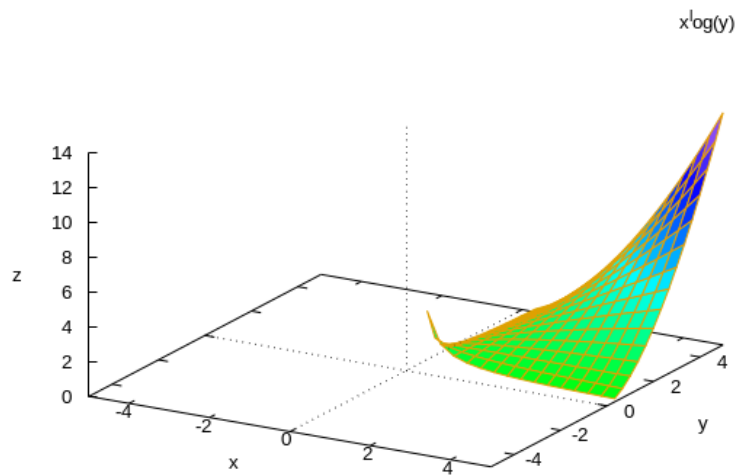
$$a^{\log(b)-1} \log(b) \quad (\% \text{ o90})$$

$$\frac{a^{\log(b)} \log(a)}{b} \quad (\% \text{ o91})$$

$$\frac{a^{\log(b)-1} \log(a) \log(b)}{b} + \frac{a^{\log(b)-1}}{b} \quad (\% \text{ o92})$$

Let's take a look:

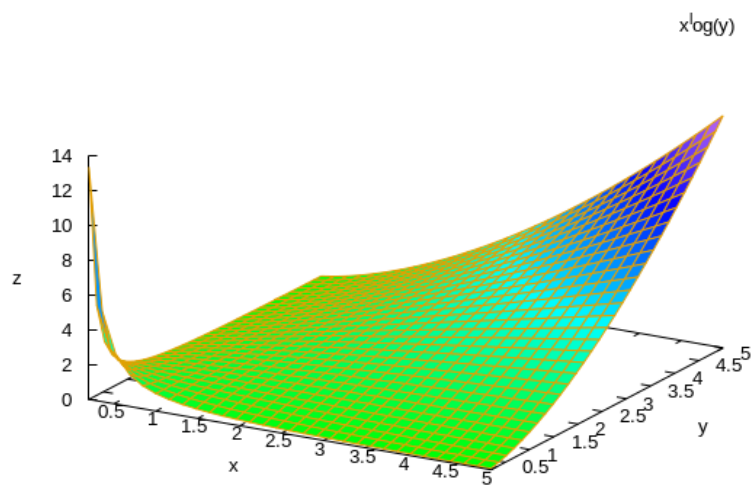
```
(% i93) wxplot3d (up(x,y), [x, -5, 5], [y, -5, 5])$
```



(% t93)

The singularity at the origin comes as no surprise. And for finite, real values, x and y should remain > 0 .

```
(% i94) wxplot3d (up(x,y), [x, 0.2, 5], [y, 0.2, 5])$
```



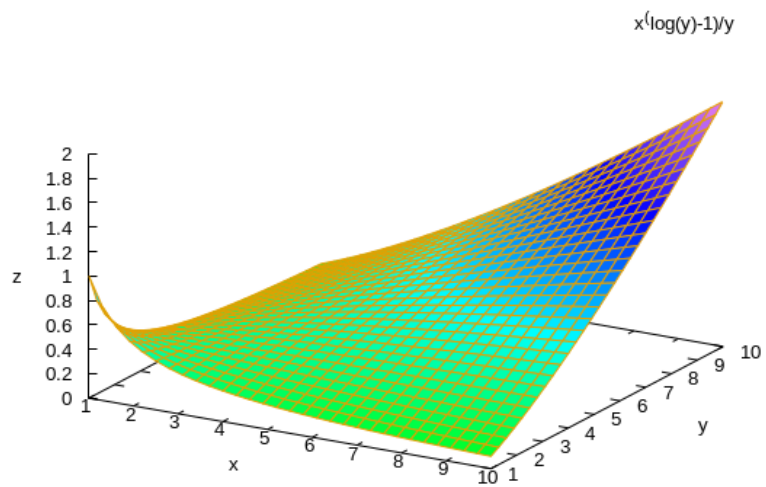
(% t94)

Compare with $x.y$:

```
(% i96) up(x,y)/(x*y);
wxplot3d (up(x,y)/(x*y), [x, 1, 10], [y, 1, 10])$
```

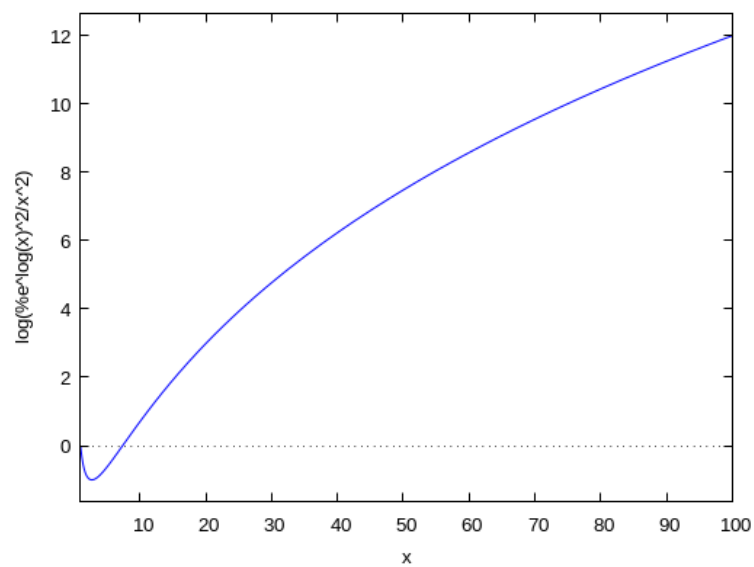
$$\frac{x^{\log(y)-1}}{y}$$

(% o95)



(% t96)

(% i97) wxplot2d (log(up(x,x)/(x*x)), [x, 1, 100])\$



(% t97)

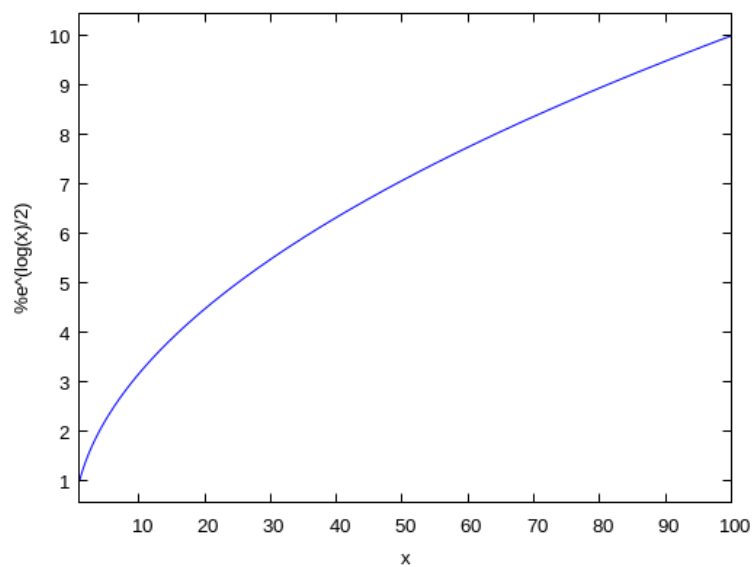
Using up_div for the "ratio"...

```
(% i99) exp(log(up(x,x))/log(x*x));
      radcan(%);
```

```
%e $\frac{\log(x)}{2}$                                      (% o98)
```

```
 $\sqrt{x}$                                              (% o99)
```

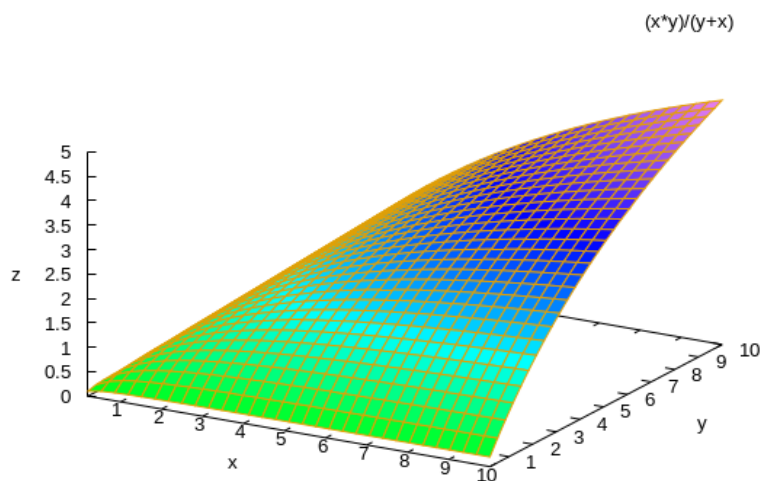
```
(% wxplot2d (exp(log(up(x,x))/log(x*x)), [x, 1, 100])$
i100)
```



```
(% t100)
```

And x,y with x+y :

```
(% wxplot3d (x*y/(x+y), [x, 0.2, 10], [y, 0.2, 10])$
i101)
```



(% t101)

3.1 Up differentiation

To avoid problems with alphabetical preferences...

```
(% up(x,h);
i103) up(a,h);
```

$$h^{\log(x)}$$

(% o102)

$$a^{\log(h)}$$

(% o103)

we'll use 'a' as the independent variable here. Working with up_/, up_+ as times, up_- as divide, and the new "up_+"-identity, h->1...

```
(% Uf(a,h) := f(a*h) / f(a) ;
i107) Uf(a,h);
Uf_Ua(a) := radcan( limit( exp( log(Uf(a,h)) / log(h) ) , h, 1, plus ) );
Uf_Ua(a);
```

$$Uf(a,h) := \frac{f(ah)}{f(a)}$$

(% o104)

$$\frac{f(ah)}{f(a)} \quad (\% \text{ o105})$$

$$\text{Uf_Ua}(a) := \text{radcan} \left(\text{limit} \left(\exp \left(\frac{\log(\text{Uf}(a, h))}{\log(h)} \right), h, 1, \text{plus} \right) \right) \quad (\% \text{ o106})$$

$$\lim_{\rightarrow h \setminus \setminus > 1+} \frac{f(ah)^{\frac{1}{\log(h)}}}{f(a)^{\frac{1}{\log(h)}}} \quad (\% \text{ o107})$$

```
(%      Uf_Ua(a) := ratsimp( limit( Uf(a,h)^ (1 / log(h)) , h, 1, plus));
i109)  Uf_Ua(a);
```

$$\text{Uf_Ua}(a) := \text{ratsimp} \left(\text{limit} \left(\text{Uf}(a, h)^{\frac{1}{\log(h)}}, h, 1, \text{plus} \right) \right) \quad (\% \text{ o108})$$

$$\lim_{\rightarrow h \setminus \setminus > 1+} \left(\frac{f(ah)}{f(a)} \right)^{\frac{1}{\log(h)}} \quad (\% \text{ o109})$$

```
(%      f(a) := n*a + c;
i112)  Uf(a,h);
      Uf_Ua(a);
```

$$f(a) := na + c \quad (\% \text{ o110})$$

$$\frac{ahn + c}{an + c} \quad (\% \text{ o111})$$

$${}_0e^{\frac{an}{an+c}} \quad (\% \text{ o112})$$

```
(%      assume(y>1);
i116)  f(a) := a^ y + c;
      Uf(a,h);
      Uf_Ua(a);
```

$$[y > 1] \quad (\% \text{ o113})$$

$$f(a) := a^y + c \quad (\% \text{ o114})$$

$$\frac{(ah)^y + c}{c + a^y} \quad (\% \text{ o115})$$

$$\%e^{\frac{a^y y}{c + a^y}} \quad (\% \text{ o116})$$

```
(%      f(a) := exp(b*a^ y + c);
i119)   Uf(a,h);
        Uf_ Ua(a);
```

$$f(a) := \exp(ba^y + c) \quad (\% \text{ o117})$$

$$\%e^{b(ah)^y - a^y b} \quad (\% \text{ o118})$$

"Is "

y

" an "

integer

"?"

no ;

$$\%e^{a^y by} \quad (\% \text{ o119})$$

```
(%      f(a) := b^ (a^ y + c);
i122)   Uf(a,h);
        Uf_ Ua(a);
```

$$f(a) := b^{a^y + c} \quad (\% \text{ o120})$$

$$b^{(ah)^y - a^y} \tag*{(\% o121)}$$

"Is "

$$y$$

" an "

integer

"?"

no;

"Is "

$$y - 1$$

" positive, negative or zero?"

positive;

$$\%e^{a^y \log(b)y} \tag*{(\% o122)}$$

(% f(a) := exp(b^ (a^ y + c));
i125) Uf(a,h);
Uf_ Ua(a);

$$f(a) := \exp\left(b^{a^y + c}\right) \tag*{(\% o123)}$$

$$\%e^{b^{(ah)^y + c} - b^{c + a^y}} \tag*{(\% o124)}$$

"Is "

$$c$$

" an "

integer

"?"

no;

$$\frac{1}{y} e^{a^y b^{c+a^y} \log(b)y} \quad (\% \text{ o125})$$

```
(%      f(a) := g(a) * p(a);
i128)   Uf(a,h);
        Uf_ Ua(a);
```

$$f(a) := g(a) p(a) \quad (\% \text{ o126})$$

$$\frac{g(ah) p(ah)}{g(a) p(a)} \quad (\% \text{ o127})$$

$$\lim_{h \rightarrow 1} \left(\frac{g(ah) p(ah)}{g(a) p(a)} \right)^{\frac{1}{\log(h)}} \quad (\% \text{ o128})$$

So the up-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: $U_Ua(e^a) = e^a$.

```
(%      f(a) := b^ exp(a^ y + c);
i131)   Uf(a,h);
        Uf_ Ua(a);
```

$$f(a) := b^{\exp(a^y + c)} \quad (\% \text{ o129})$$

$$b^{\frac{1}{y} e^{(ah)^y + c} - \frac{1}{y} e^{c+a^y}} \quad (\% \text{ o130})$$

"Is "

c

" an "

integer

"?"

no;

$$\%e^{a^y \log(b)y} \%e^{c+a^y} \quad (\% \text{ o131})$$

```
(%      f(a) := log(a);
i134)   Uf(a,h);
        Uf_ Ua(a);
```

$$f(a) := \log(a) \quad (\% \text{ o132})$$

$$\frac{\log(ah)}{\log(a)} \quad (\% \text{ o133})$$

$$\%e^{\frac{1}{\log(a)}} \quad (\% \text{ o134})$$

```
(%      kill(f);
i135)
```

$$\text{done} \quad (\% \text{ o135})$$

Alternative expressions for Uf_ Ua(a) (with a=exp(z)):

```
(%      a = exp(z);
i140)   exp('diff( log(f(exp(z))) ,z,1));
        exp(diff( log(f(exp(z))) ,z,1));
        exp( a * diff(f(a),a,1) / f(a) );
        exp('diff( log(f(a)) ,log(a),1));
```

$$a = \%e^z \quad (\% \text{ o136})$$

$$\%e^{\frac{d}{dz} \log(f(\%e^z))} \quad (\% \text{ o137})$$

$$\%e^{\frac{\frac{d}{da} f(\%e^z)}{f(\%e^z)}} \quad (\% \text{ o138})$$

$$\%e^{\frac{a \left(\frac{d}{da} f(a) \right)}{f(a)}} \quad (\% \text{ o139})$$

$$\%e^{\frac{d}{d \log(a)} \log(f(a))} \quad (\% \text{ o140})$$

This leads to an alternative expression for the "normal" derivative, d/dx , in terms of the down derivative:

```
(%      exp(x + 'Df_Dx(x) - f(x));
i144)   z = log(x); g(x) = log(f(x));
        exp('Dg_Dz(x));
```

$$\%e^{Df_Dx(x)-f(x)+x} \quad (\% \text{ o141})$$

$$z = \log(x) \quad (\% \text{ o142})$$

$$g(x) = \log(f(x)) \quad (\% \text{ o143})$$

$$\%e^{Dg_Dz(x)} \quad (\% \text{ o144})$$

4 Up2

Now to be distributive over "up": $\text{up2}(a, \text{up}(b,c)) = \text{up}(\text{up2}(a,b), \text{up2}(a,c)) \dots$

```
(%      up2(a,b) := radcan( exp( up(log(a),log(b)) ) );
i145)
```

$$\text{up2}(a, b) := \text{radcan}(\exp(\text{up}(\log(a), \log(b)))) \quad (\% \text{ o145})$$

```
(%      up(up2(a,b),up2(a,c));
i147)   up2(a,up(b,c));
```

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \quad (\% \text{ o146})$$

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \quad (\% \text{ o147})$$

Bingo! And obviously also commutative, unlike say, tetration ($\hat{y}(x)$ or $\hat{x}(y)$), which is neither. Identity for `up2_+`:

```
(%      up2(e^ e,a);
i148)

a                                     (% o148)
```

```
(%      diff(up2(a,b),a);
i149)


$$\frac{\log(a)^{\log(\log(b))-1} e^{\log(a)^{\log(\log(b))}} \log(\log(b))}{a}$$
                                     (% o149)
```

```
(%      up2(x,x);
i151)      exp(log(x)^ log(x));

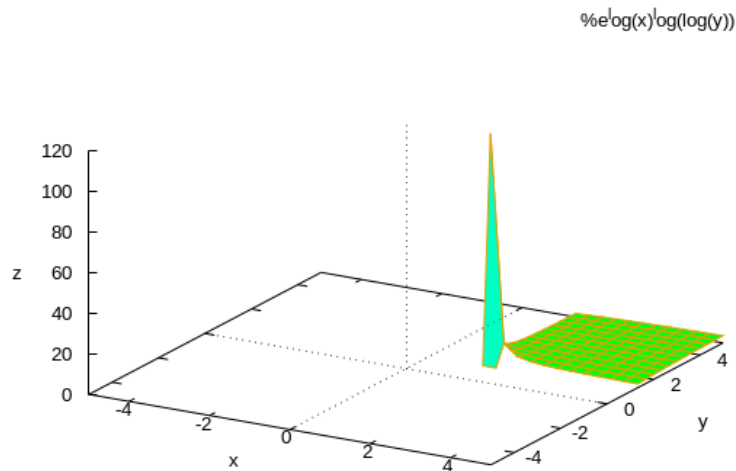

$$e^{e^{\log(\log(x))^2}}$$
                                     (% o150)
```

```

$$e^{\log(x)^{\log(x)}}$$
                                     (% o151)
```

Taking a look...

```
(%      wxplot3d (up2(x,y), [x, -5, 5], [y, -5, 5])$
i152)
```

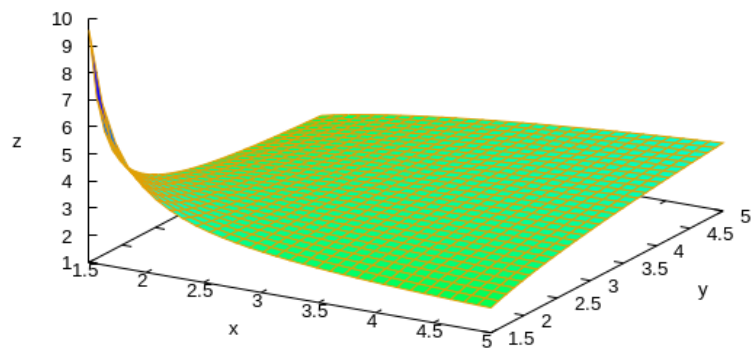


(% t152)

Problems now at x or $y = 1$.

(% wxplot3d (up2(x,y), [x, 1.5, 5], [y, 1.5, 5])\$
i153)

%e^{log(x)}log(log(y))

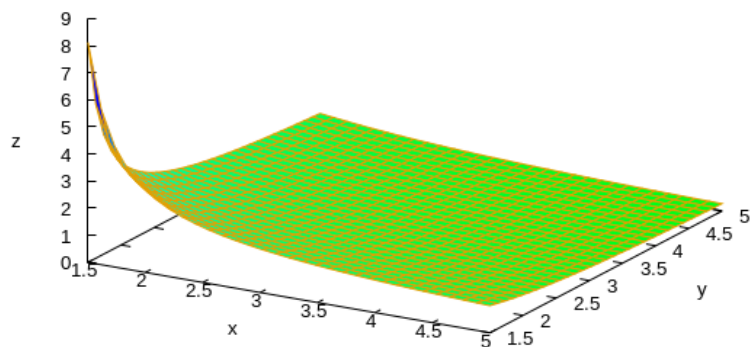


(% t153)

Comparing up2(x,y) with up(x,y) :

```
(% wxplot3d (up2(x,y)/up(x,y), [x, 1.5, 5], [y, 1.5, 5])$
i154)
```

$e^{\log(x)\log(\log(y))/x\log(y)}$

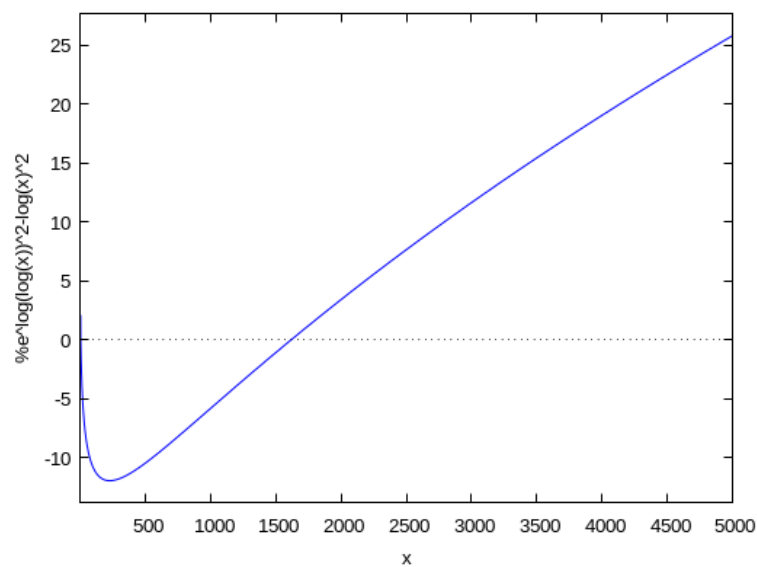


(% t154)

```
(% log(up2(x,x)/up(x,x));
i156) wxplot2d (log(up2(x,x)/up(x,x)), [x, 1.5, 5000])$
```

$e^{\log(\log(x))^2 - \log(x)^2}$

(% o155)



(% t156)

Using up2_div for the "ratio"...

```
(%      up_div(a,b) := radcan( exp(log(a) / log(b)) );
i158)  up2_div(a,b) := radcan( exp( up_div(log(a),log(b)) ) );
```

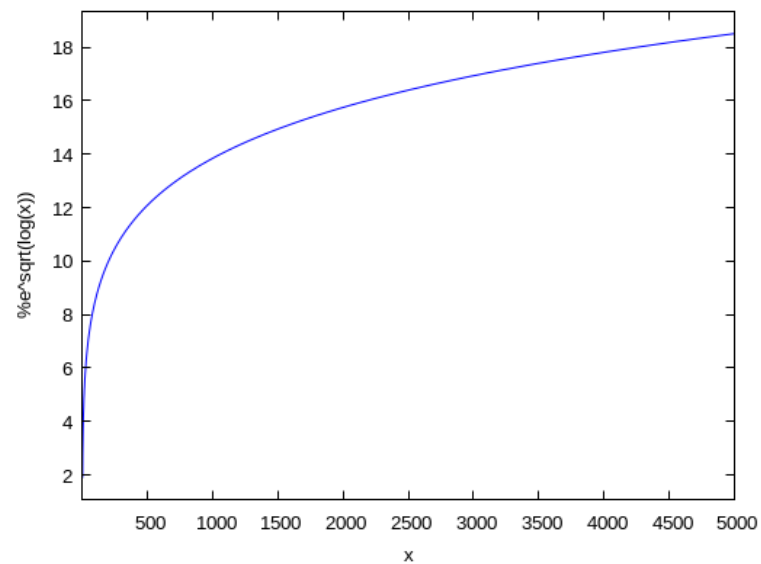
$$\text{up_div}(a, b) := \text{radcan} \left(\exp \left(\frac{\log(a)}{\log(b)} \right) \right) \quad (\% \text{ o157})$$

$$\text{up2_div}(a, b) := \text{radcan}(\exp(\text{up_div}(\log(a), \log(b)))) \quad (\% \text{ o158})$$

```
(%      up2_div(up2(x,x),up(x,x));
i159)
```

$${}_0e^{\sqrt{\log(x)}} \quad (\% \text{ o159})$$


```
(%  
i160) wxplot2d (up2_div(up2(x,x),up(x,x)), [x, 1.5, 5000])$
```

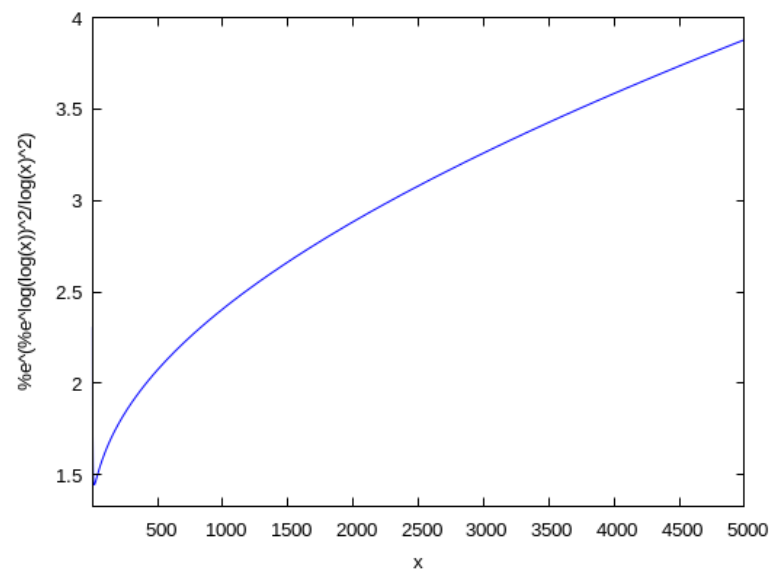


(% t160)

```
(%  
i162) up_div(up2(x,x),up(x,x));  
wxplot2d (up_div(up2(x,x),up(x,x)), [x, 3, 5000])$
```

$$\frac{{}_0e^{\frac{\log(\log(x))^2}{\log(x)^2}}}{\log(x)^2}$$

(% o161)



(% t162)