Distributivity, Extended

(Tommy Burch, on and off from 08.2024 to 05.2025)

1 At or Down

This all started back in the late '90s at the UofA in a physics lab prep session (with R. McCroskey and R. Haar, among others). The discussion hinged on finding an operation (i termed it 'at' with the symbol @) over which addition was distributive. The exp / log solution below i came up with followed later the same day... Note that, for multiplcation and addition, the distributive property can be "revealed" through exponentials and their inverse, the natural logarithms:

(% i1) add(a,b) := log(exp(a)*exp(b));

$$\mathrm{add}\left(a , b\right) := \log\left(\exp(a)\exp(b)\right) \tag{\% o1}$$

(% i3) add(a,b);
add(c*a,c*b);

$$b + a \tag{\% o2}$$

$$bc + ac \tag{\% o3}$$

(% i4) ratsimp(%);

$$\left(b + a\right)c \tag{\% o4}$$

Placing the multiplication between the exponentials in the logarithm, we arrive at the relation (addition) over which it is disributive. We can try the same with + between the exp's...

1.1 At_+

Define the @ (or "down") relation such that addition is distributive over it: a + (b @ c) = (a+b) @ (a+c)...Or: a + down(b,c) = down(a+b,a+c). Trying the following ansatz:

(% i5) at_eq : At(a,b) = log(exp(a) + exp(b));

$$\mathrm{At}\left(a , b\right) = \log\left(\%e^{b} + \%e^{a}\right) \tag{at\_ eq}$$

(% i6) down(a,b) := radcan(log(exp(a) + exp(b)));

$$\mathrm{down}\,(a\,,b) := \mathrm{radcan}\,(\log\,(\exp(a) + \exp(b))) \qquad (\%\ o6)$$

Check:

(% i7) down(a+b,a+c);

$$\log\left(\%e^{c} + \%e^{b}\right) + a \qquad (\%\ o7)$$

yes! (It is also an obviously commutative operation.)This should work with any base for the power / log's: e.g.,

(% i8) log10(x) := log(x) / log(10);

$$\mathrm{log10}(x) := \frac{\log\,(x)}{\log\,(10)} \qquad (\%\ o8)$$

(% i9) radcan(log10(10ˆ (a+b) + 10ˆ (a+c)));

$$\frac{\log\left(2^{c}5^{c} + 2^{b}5^{b}\right) + (\log\,(5) + \log\,(2))\,a}{\log\,(5) + \log\,(2)} \qquad (\%\ o9)$$

(% i10) logcontract(%);

$$\frac{\log\left(2^{c}5^{c} + 2^{b}5^{b}\right) + \log\,(10)a}{\log\,(10)} \qquad (\%\ o10)$$

yep! The ambiguity probably speak volumes (see, e.g., M. Burgin, arXiv:1010.3287 : there, the focus appears not to be in maintaining distributivity, unlike here, where it's the main focus). It may be worth exploring whether functions other than powers/logarithms can retain distributivity... at first glance, seemingly not.Some values with base e:

(% i11) down(0,0);

$$\log\,(2) \qquad (\%\ o11)$$

(% i12) down(1,0);

$$\log\,(\%e + 1) \qquad (\%\ o12)$$

(% i13) down(x,0);

$$\log\,(\%e^{x} + 1) \qquad (\%\ o13)$$

(% i15) down(x,x);
down(x-log(2),x-log(2));

$$x + \log(2) \tag{\% o14}$$

$$x \tag{\% o15}$$

Associative:

(% i17) down(a,down(b,c));
down(down(a,b),c);

$$\log\left(\%e^{c} + \%e^{b} + \%e^{a}\right) \tag{\% o16}$$

$$\log\left(\%e^{c} + \%e^{b} + \%e^{a}\right) \tag{\% o17}$$

(% i20) diff(down(a,b),a);
diff(down(a,b),b);
diff(diff(down(a,b),b),a);

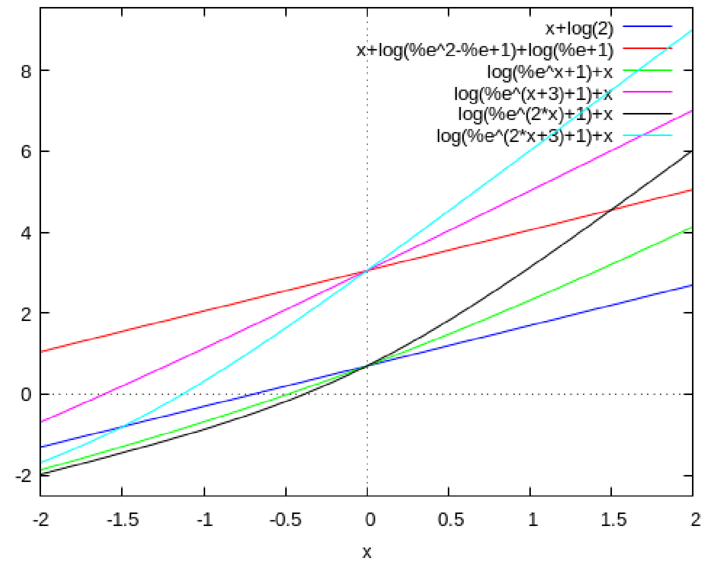
$$\frac{\%e^{a}}{\%e^{b} + \%e^{a}} \tag{\% o18}$$

$$\frac{\%e^{b}}{\%e^{b} + \%e^{a}} \tag{\% o19}$$

$$-\frac{\%e^{b+a}}{\left(\%e^{b} + \%e^{a}\right)^{2}} \tag{\% o20}$$

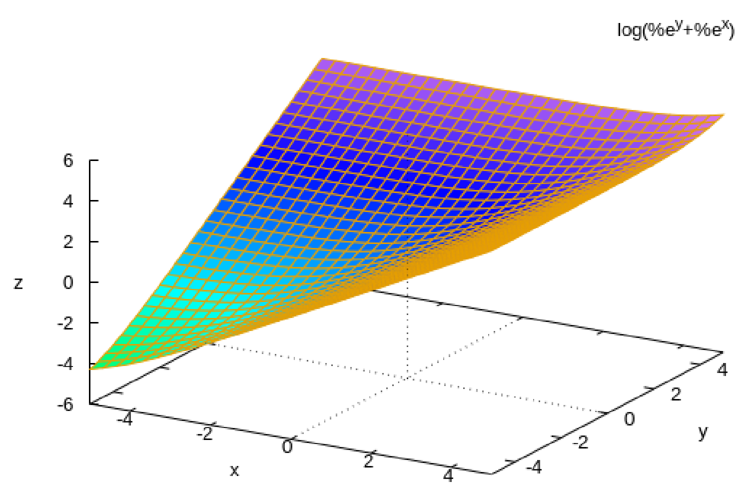
Let's have a look:

(% i21) wxplot2d ([down(x,x),down(x,x+3),down(x,2*x),down(x,2*x+3),down(x,3*x),down(x,3*x+3)], [x, -2, 2])$



(% t21)

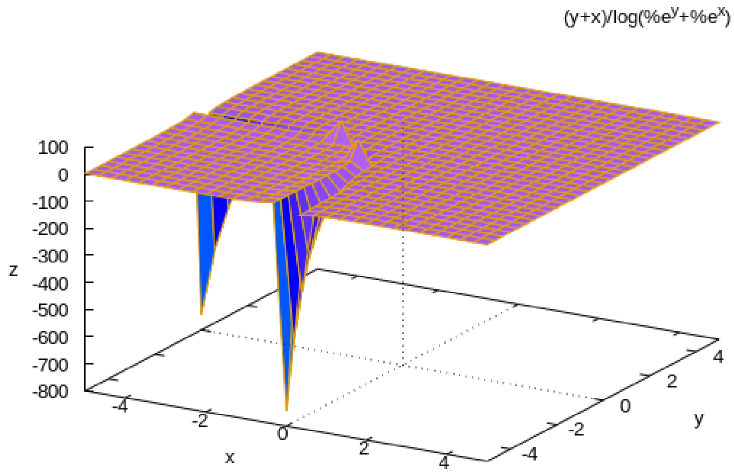
(% i22) wxplot3d (down(x,y), [x, -5, 5], [y, -5, 5])$



(% t22)

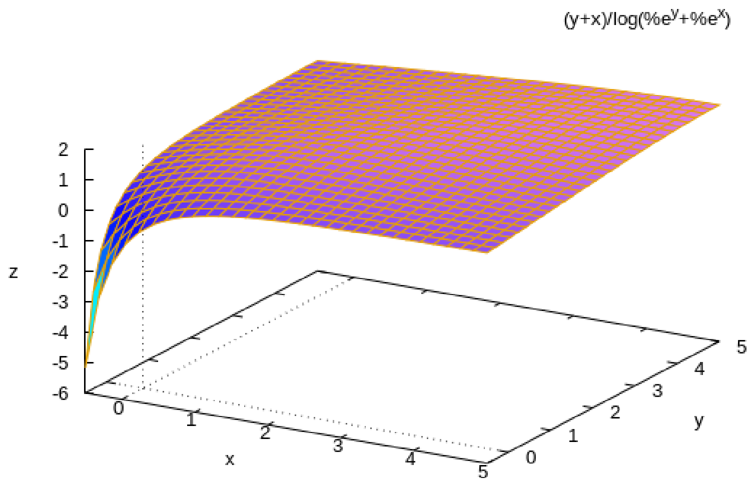
Comparing x+y with down(x,y) :

(% i23) wxplot3d ((x+y)/down(x,y), [x, -5, 5], [y, -5, 5])$



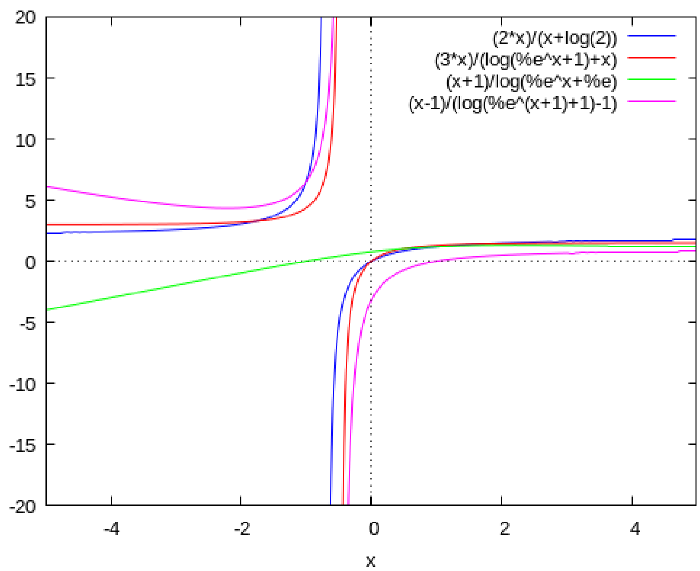
(% t23)

(% i24) wxplot3d ((x+y)/down(x,y), [x, -0.5, 5], [y, -0.5, 5])$



(% t24)

(% i25) wxplot2d ([2*x/down(x,x), 3*x/down(x,2*x),(x+1)/down(x,1),(x-1)/down(x,-1)], [x, -5, 5], [y, -20, 20])$



(% t25)

1.2 At_-,*,/

We can also consider a "down-like" operation with a minus instead a plus:

(% i26) down_min(a,b) := radcan(log(exp(a) - exp(b)));

$$\mathrm{down\_min}\left(a\,,b\right):=\mathrm{radcan}\left(\log\left(\exp(a)-\exp(b)\right)\right) \tag{\% o26}$$

Associativity:

(% i30) down(down_min(a,b),c);
down_min(down(a,c),b);
down(down_min(a,b),down_min(c,d));
down(down_min(a,d),down_min(c,b));

$$\log\left(\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o27}$$

$$\log\left(\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o28}$$

$$\log\left(-\%e^{d}+\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o29}$$

$$\log\left(-\%e^{d}+\%e^{c}-\%e^{b}+\%e^{a}\right) \tag{\% o30}$$

Same distributive property:

(% i31) down_min(a+b,a+c);

$$\log\left(\%e^{b}-\%e^{c}\right)+a \tag{\% o31}$$

As we've already seen, with times, we achieve addition. With divide, one achieves subtraction:

(% i33) down_div(a,b) := radcan(log(exp(a) / exp(b)));
down_div(a,b);

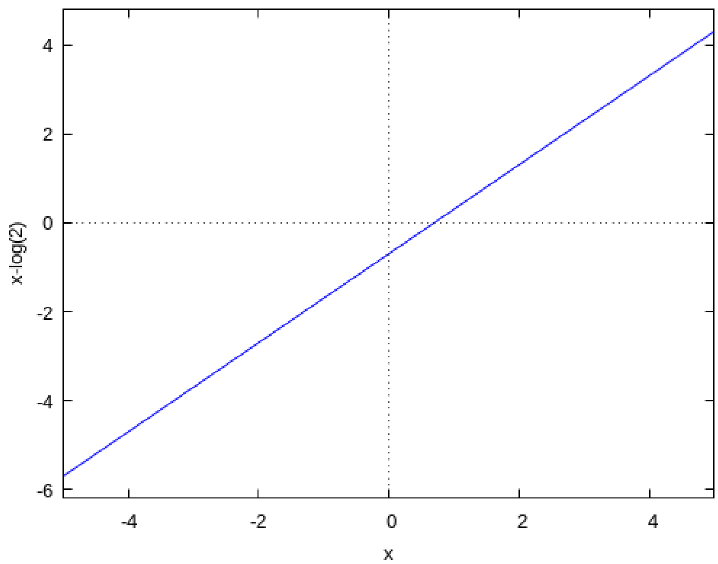
$$\mathrm{down\_div}\left(a\,,b\right):=\mathrm{radcan}\left(\log\left(\frac{\exp(a)}{\exp(b)}\right)\right) \tag{\% o32}$$

$$a-b \tag{\% o33}$$

Using @_/ or down_div for the "ratio" of + and @...

(% i35) down_div(x+x,down(x,x));
wxplot2d (2*x - down(x,x), [x, -5, 5])$

$$x - \log(2) \quad (\% o34)$$



(% t35)

1.3 Down differentiation

Given down, down_min, and down_div, one is free to define a "down derivative". However, we must take care with the notion of infinitesimals: normally h–>0 in the definition of the derivative, but here we need h–>-∞ to approach the @_+ identity (x @ -∞ = x)...

(% i36) Df_Dx_eq : Df_Dx = limit(down_min(f(down(x,h)) , f(x)) - h , h, -∞);

$$\mathrm{Df\_Dx} = \lim_{\to h \,\backslash\text{-}\backslash> -\infty} \log\left(\%e^{\mathrm{f}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)} - \%e^{\mathrm{f}(x)}\right) - h \quad (\mathrm{Df\_\ Dx\_\ eq})$$

(% i40) Df(x,h) := down_min(f(down(x,h)) , f(x));
Df(x,h);
Df_Dx(x) := ratsimp(limit(Df(x,h) - h , h, minf));
Df_Dx(x);

$$\mathrm{Df}(x, h) := \mathrm{down\_min}(\mathrm{f}(\mathrm{down}(x, h)), \mathrm{f}(x)) \quad (\% o37)$$

$$\log\left(\%e^{\mathrm{f}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{f}(x)}\right) \tag{\% o38}$$

$$\mathrm{Df\_Dx}(x):=\mathrm{ratsimp}\left(\mathrm{limit}\left(\mathrm{Df}\left(x\,,h\right)-h\,,h\,,\text{-}\infty\right)\right) \tag{\% o39}$$

$$\lim_{\to h\,\backslash\text{-}\backslash>\text{-}\infty}\log\left(\%e^{\mathrm{f}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{f}(x)}\right)-h \tag{\% o40}$$

(% i43) f(x) := A*x^ b + c;
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=A x^{b}+c \tag{\% o41}$$

$$\log\left(\%e^{A\log\left(\%e^{x}+\%e^{h}\right)^{b}}-\%e^{A x^{b}}\right)+c \tag{\% o42}$$

$$(b-1)\log(x)+A x^{b}-x+c+\log(b)+\log(A) \tag{\% o43}$$

(% i46) f(x) := log(A*x^ b + c);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=\log\left(A x^{b}+c\right) \tag{\% o44}$$

$$\log\left(\log\left(\%e^{x}+\%e^{h}\right)^{b}-x^{b}\right)+\log(A) \tag{\% o45}$$

$$(b-1)\log(x)-x+\log(b)+\log(A) \tag{\% o46}$$

(% i49) f(x) := exp(A*x^ b + c);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=\exp\left(A x^{b}+c\right) \tag{\% o47}$$

$$\log\left(\%e^{\%e^{A\log\left(\%e^{x}+\%e^{h}\right)^{b}+c}}-\%e^{\%e^{Ax^{b}+c}}\right) \tag{\% o48}$$

$$(b-1)\log(x)+\%e^{Ax^{b}+c}+Ax^{b}-x+c+\log(b)+\log(A) \tag{\% o49}$$

(% i52) f(x) := down(g(x) , p(x));
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=\mathrm{down}\left(\mathrm{g}(x),\mathrm{p}(x)\right) \tag{\% o50}$$

$$\log\left(\%e^{\mathrm{p}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}+\%e^{\mathrm{g}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{p}(x)}-\%e^{\mathrm{g}(x)}\right) \tag{\% o51}$$

$$\lim_{\rightarrow h \backslash\text{-}\backslash>\text{-}\infty}\log\left(\%e^{\mathrm{p}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}+\%e^{\mathrm{g}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{p}(x)}-\%e^{\mathrm{g}(x)}\right)-h \tag{\% o52}$$

So the down-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: D_Dx (e̊x̂) = e̊x̂.

(% i55) f(x) := b^ (A*x+c);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=b^{Ax+c} \tag{\% o53}$$

$$\log\left(\%e^{b^{A\log\left(\%e^{x}+\%e^{h}\right)+c}}-\%e^{b^{Ax+c}}\right) \tag{\% o54}$$

$$(A\log(b)-1)\,x+\log(b)c+b^{Ax+c}+\log(\log(b))+\log(A) \tag{\% o55}$$

(% i58) f(x) := log(x);
Df(x,h);
Df_Dx(x) ;

$$\mathrm{f}(x):=\log(x) \tag{\% o56}$$

$$\log\left(\log\left(\%e^{x}+\%e^{h}\right)-x\right) \tag{\% o57}$$

$$-x \tag{\% o58}$$

(% i59) kill(f);

$$\mathrm{done} \tag{\% o59}$$

(% i60) Eig_Df_eq : Df_Dx(x) = f(x);

$$\lim_{\to h \backslash\text{-}\backslash\text{>}\text{-}\infty} \log\left(\%e^{\mathrm{f}\left(\log\left(\%e^{x}+\%e^{h}\right)\right)}-\%e^{\mathrm{f}(x)}\right)-h=\mathrm{f}(x) \tag{Eig\_ Df\_ eq}$$

As we can see from the previous solutions of Df_Dx, this DE is uniquely solved by f(x)=eˆx.

(% i61) kill(f);

$$\mathrm{done} \tag{\% o61}$$

Alternative expressions for Df_Dx(x) (with x=log(z)):

(% i65) x = log(z);
log('diff(exp(f(log(z))) ,z,1));
log(diff(exp(f(log(z))) ,z,1));
f(x) - x + log(diff(f(x),x,1));

$$x=\log(z) \tag{\% o62}$$

$$\log\left(\frac{d}{dz}\%e^{\mathrm{f}(\log(z))}\right) \tag{\% o63}$$

$$\log\left(\%e^{\mathrm{f}(\log(z))}\left(\frac{d}{dz}\,\mathrm{f}(\log(z))\right)\right) \tag{\% o64}$$

$$\log\left(\frac{d}{dx}\,\mathrm{f}(x)\right)+\mathrm{f}(x)-x \tag{\% o65}$$

2 Down2

Heading further "down", define "three dots" (therefore), or down2, such that: a @ down2(b,c) = down2(a@b,a@c) ...Or: down(a,down2(b,c)) = down2(down(a,b),down(a,c)) ...

(% i66) td_eq : Td(a,b) = log(down(exp(a),exp(b)));

$$\mathrm{Td}\left(a\,,b\right)=\log\left(\log\left(\%e^{\%e^{b}}+\%e^{\%e^{a}}\right)\right) \qquad \text{(td\_ eq)}$$

(% i67) down2(a,b) := radcan(log(down(exp(a),exp(b))));

$$\mathrm{down2}\left(a\,,b\right):=\mathrm{radcan}\left(\log\left(\mathrm{down}\left(\exp(a)\,,\exp(b)\right)\right)\right) \qquad (\%\ o67)$$

(% i68) down2(down(a,b),down(a,c));

$$\log\left(\log\left(\%e^{\%e^{c}}+\%e^{\%e^{b}}\right)+\%e^{a}\right) \qquad (\%\ o68)$$

yes!down2-identity would have to be at ∞+iπ.

(% i69) down2(x,x);

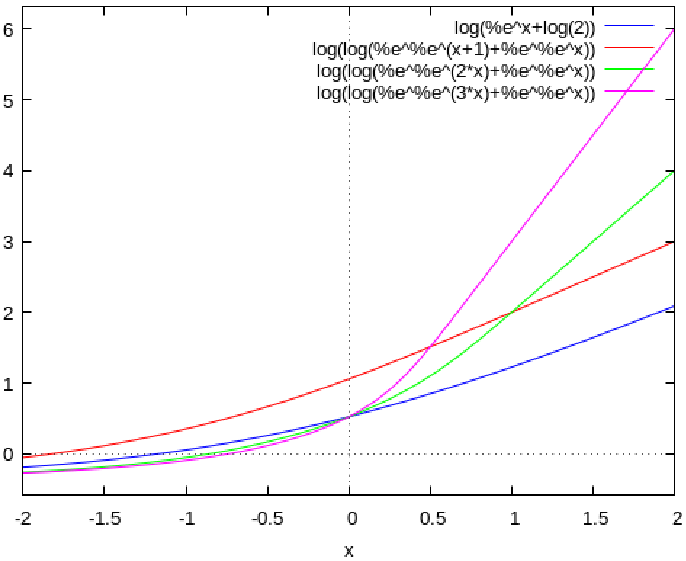
$$\log\left(\%e^{x}+\log\left(2\right)\right) \qquad (\%\ o69)$$

(% i70) diff(down2(a,b),a);

$$\frac{\%e^{\%e^{a}+a}}{\left(\%e^{\%e^{b}}+\%e^{\%e^{a}}\right)\log\left(\%e^{\%e^{b}}+\%e^{\%e^{a}}\right)} \qquad (\%\ o70)$$

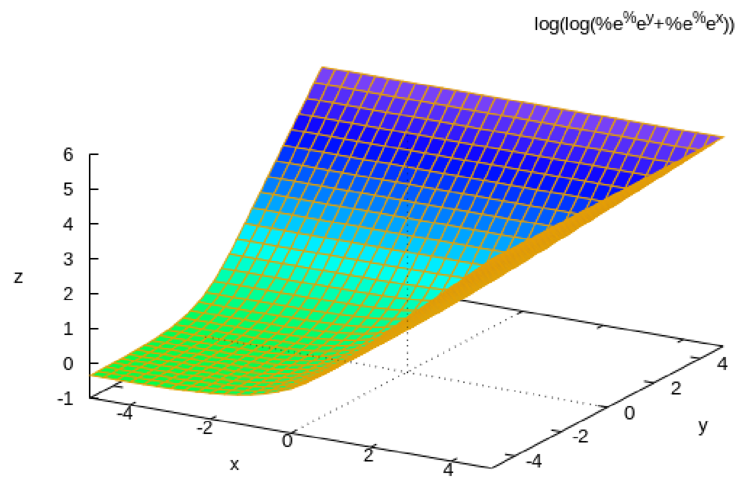
Let's take a look...

(% i71) wxplot2d ([down2(x,x),down2(x,x+1),down2(x,2*x),down2(x,3*x)], [x, -2, 2])$



(% t71)

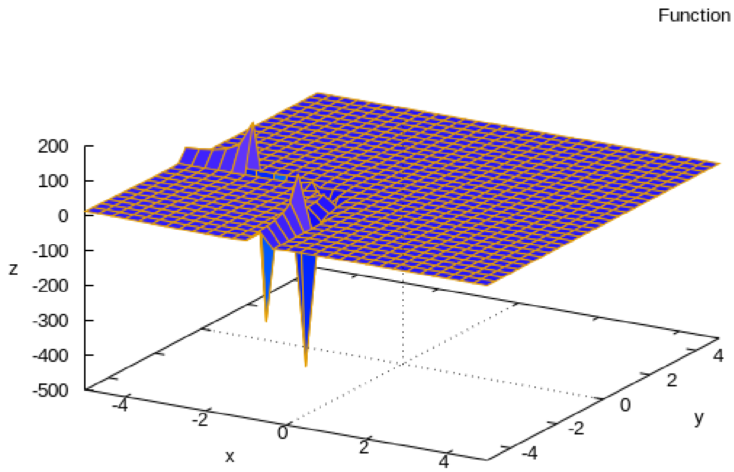
(% i72) wxplot3d (down2(x,y), [x, -5, 5], [y, -5, 5])$



(% t72)

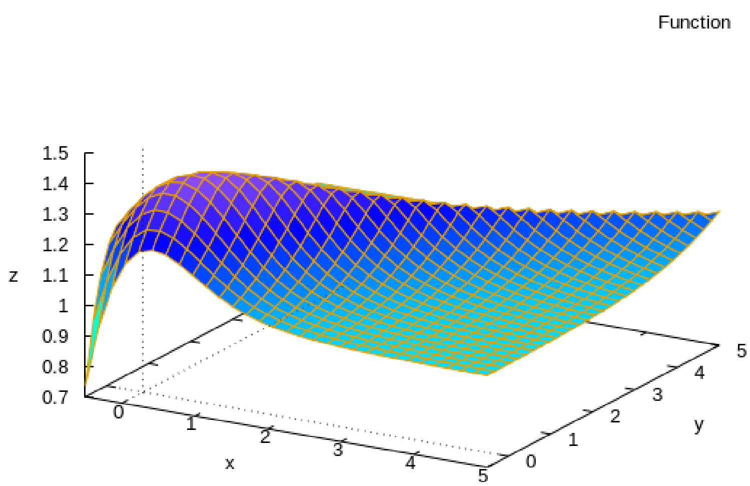
Comparing down(x,y) with down2(x,y) :

(% i73) wxplot3d (down(x,y)/down2(x,y), [x, -5, 5], [y, -5, 5])$



(% t73)

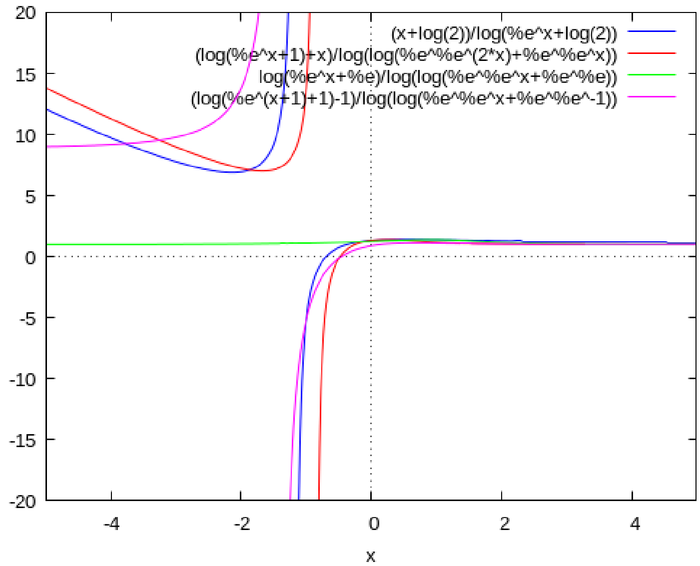
(% i74) wxplot3d (down(x,y)/down2(x,y), [x, -0.5, 5], [y, -0.5, 5])$



(% t74)

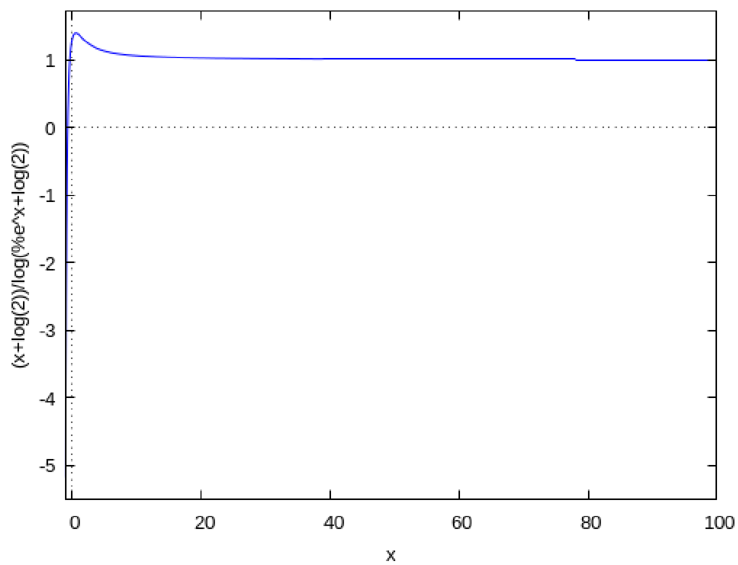
(% i75) wxplot2d ([down(x,x)/down2(x,x), down(x,2*x)/down2(x,2*x),down(x,1)/down2(x,1),down(x,-1)/down2(x,-1)], [x, -5, 5], [y, -20, 20])$

$plot2d : expression evaluates to non-numeric value somewhere in plotting range.$



(% t75)

(%i76) wxplot2d (down(x,x)/down2(x,x), [x, -1, 100])$



(%t76)

3 Up

Let's take a look in the "up" direction for an operation which is distrubutive over multiplication. Defining "star" (or "up"): a * (b . c) = (a * b) . (a * c) ...Or: up(a,b.c) = up(a,b) . up(a,c) ...

(%i77) str_eq : Str(a,b) = exp(log(a)*log(b));

$$\mathrm{Str}\,(a\,,b) = \%e^{\log(a)\,\log(b)} \tag{str\_ eq}$$

(%i78) up(a,b) := radcan(exp(log(a)*log(b)));

$$\mathrm{up}\,(a\,,b) := \mathrm{radcan}\,(\exp\,(\log\,(a)\,\log\,(b))) \tag{\% o78}$$

(%i79) up(a,b);

$$a^{\log(b)} \tag{\% o79}$$

(%i80) up(a,b*c);

$$a^{\log(c)+\log(b)} \tag{\% o80}$$

(% i81) up(a,b) * up(a,c);

$$a^{\log(c)+\log(b)} \quad (\% o81)$$

which is the same as the former. So, yes!Check with "down"-like operation:

(% i82) log(up(exp(a),exp(b)));

$$ab \quad (\% o82)$$

yes!up(a,b) could also be expressed with b as the base:

(% i86) is(up(a,b) = b^ log(a));
is(up(a,b) = a^ log(b));
is(up(b,a) = a^ log(b));
is(up(a,b) = up(b,a));

false (% o83)

true (% o84)

true (% o85)

true (% o86)

... although it seems Maxima has trouble (it enforces some type of alphabetization?) with that realization in the present form...?So one advantage of up(a,b), over "naive" exponentiation (a$\hat{b}$ or b$\hat{a}$), can be seen immediately: namely commutativity... up(a,b)=up(b,a). Identity for up (or up_times):

(% i87) up(%e,a);

$$a \quad (\% o87)$$

... the base %e. Some special values:

(% i89) up(x,x);
x^ log(x);

$$\%e^{\log(x)^2} \quad (\% o88)$$

$$x^{\log(x)} \tag{\% o89}$$

(% i92) diff(up(a,b),a);
diff(up(a,b),b);
diff(diff(up(a,b),a),b);

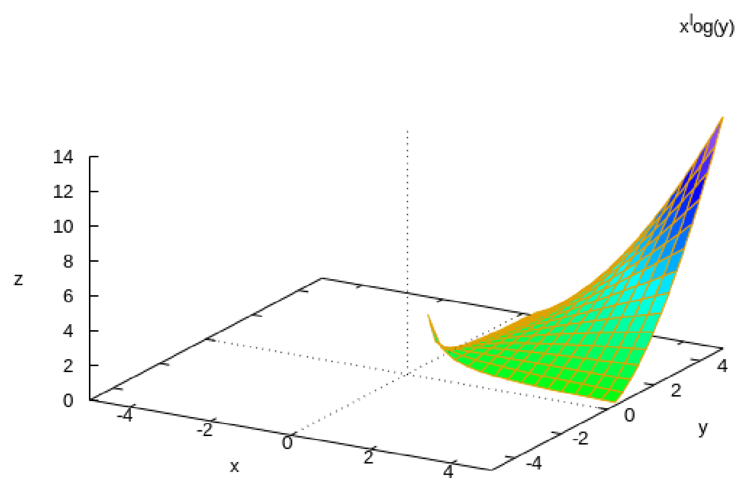
$$a^{\log(b)-1}\log(b) \tag{\% o90}$$

$$\frac{a^{\log(b)}\log(a)}{b} \tag{\% o91}$$

$$\frac{a^{\log(b)-1}\log(a)\log(b)}{b}+\frac{a^{\log(b)-1}}{b} \tag{\% o92}$$

Let's take a look:

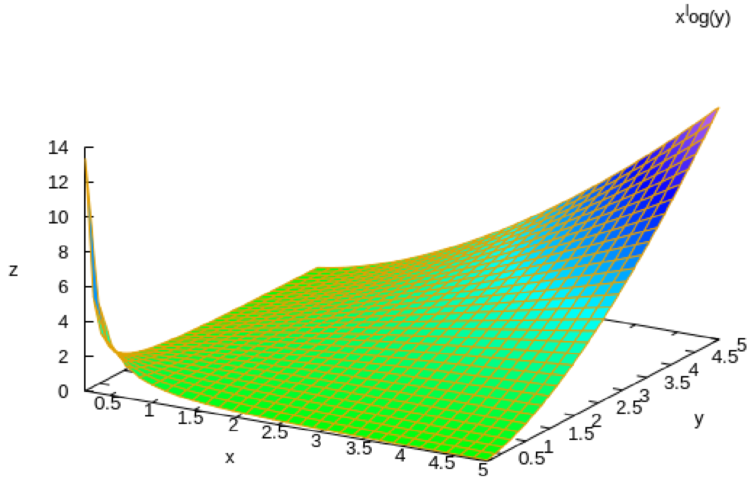
(% i93) wxplot3d (up(x,y), [x, -5, 5], [y, -5, 5])$



(% t93)

The singularity at the origin comes as no surprise. And for finite, real values, x and y should remain > 0.

(% i94) wxplot3d (up(x,y), [x, 0.2, 5], [y, 0.2, 5])$

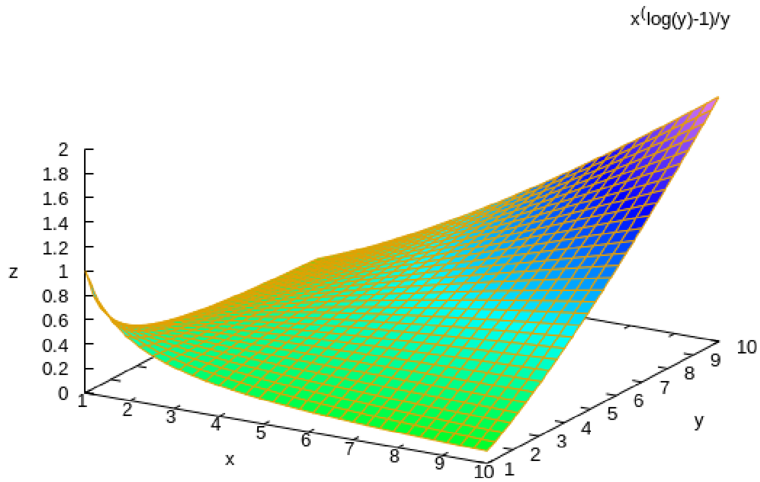


(% t94)

Compare with x.y :

(% i96) up(x,y)/(x*y);
wxplot3d (up(x,y)/(x*y), [x, 1, 10], [y, 1, 10])$

$$\frac{x^{\log(y)-1}}{y} \tag{\% o95}$$

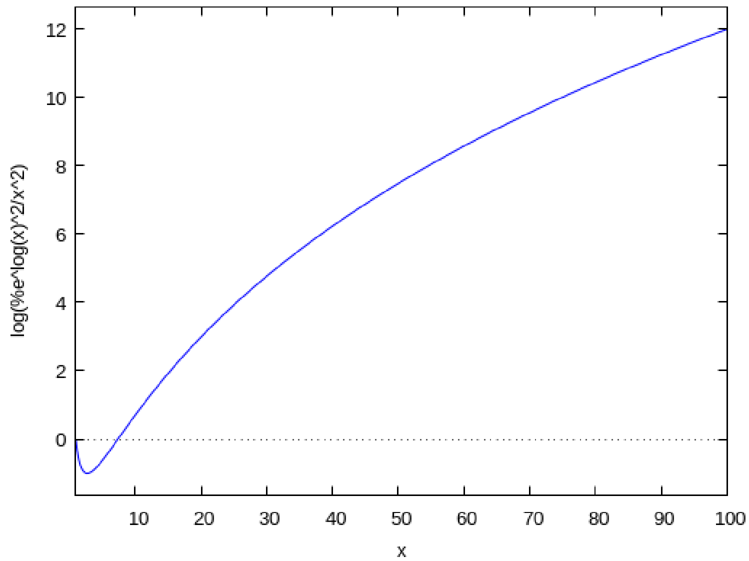


(% t96)

(% i99) up(x,x)/(x*x);
x^ (log(x)-2);
wxplot2d (log(up(x,x)/(x*x)), [x, 1, 100])$

$$\frac{\%e^{\log(x)^2}}{x^2} \quad (\%\ o97)$$

$$x^{\log(x)-2} \quad (\%\ o98)$$



(% t99)

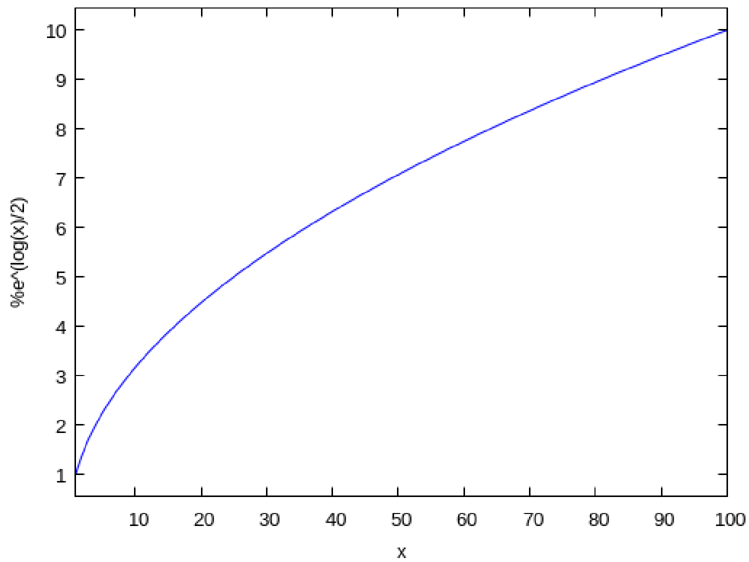
So up(x,x) does eventually win over x^2... at x=e^2. Using up_div for the "ratio"...

(% i101) exp(log(up(x,x))/log(x*x));
radcan(%);

$$\%e^{\frac{\log(x)}{2}} \quad (\%\ o100)$$

$$\sqrt{x} \quad (\%\ o101)$$

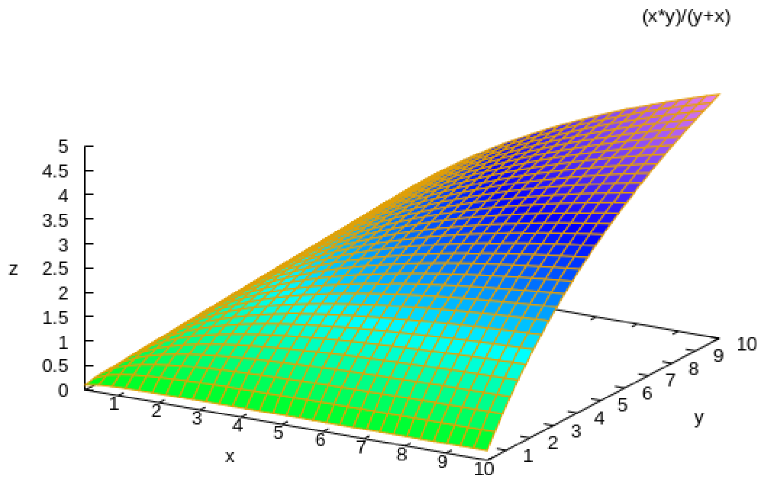
(%i102) wxplot2d (exp(log(up(x,x))/log(x*x)), [x, 1, 100])$



(%t102)

And x.y with x+y :

(% i103) wxplot3d (x*y/(x+y), [x, 0.2, 10], [y, 0.2, 10])$



(% t103)

3.1 Up differentiation

To avoid problems with alphabetical preferences...

(% i105) up(x,h);
up(a,h);

$$h^{\log(x)} \quad \text{(\% o104)}$$

$$a^{\log(h)} \quad \text{(\% o105)}$$

we'll use 'a' as the independent variable here.Working with up_/, up_+ as times, up_- as divide, and the new "up_+"-identity, h–>1...

(% i109) Uf(a,h) := f(a*h) / f(a) ;
Uf(a,h);
Uf_Ua(a) := radcan(limit(exp(log(Uf(a,h)) / log(h)) , h, 1, plus));
Uf_Ua(a);

$$\mathrm{Uf}(a, h) := \frac{\mathrm{f}(ah)}{\mathrm{f}(a)} \quad \text{(\% o106)}$$

$$\frac{\mathrm{f}(ah)}{\mathrm{f}(a)} \quad (\%\ o107)$$

$$\mathrm{Uf\_Ua}(a) := \mathrm{radcan}\left(\mathrm{limit}\left(\exp\left(\frac{\log(\mathrm{Uf}(a, h))}{\log(h)}\right), h, 1, \mathrm{plus}\right)\right) \quad (\%\ o108)$$

$$\lim_{\to h \text{ \textbackslash-\textbackslash> } 1+} \frac{\mathrm{f}(ah)^{\frac{1}{\log(h)}}}{\mathrm{f}(a)^{\frac{1}{\log(h)}}} \quad (\%\ o109)$$

(% i111) Uf_Ua(a) := ratsimp(limit(Uf(a,h)ˆ (1 / log(h)) , h, 1, plus));
Uf_Ua(a);

$$\mathrm{Uf\_Ua}(a) := \mathrm{ratsimp}\left(\mathrm{limit}\left(\mathrm{Uf}(a, h)^{\frac{1}{\log(h)}}, h, 1, \mathrm{plus}\right)\right) \quad (\%\ o110)$$

$$\lim_{\to h \text{ \textbackslash-\textbackslash> } 1+} \left(\frac{\mathrm{f}(ah)}{\mathrm{f}(a)}\right)^{\frac{1}{\log(h)}} \quad (\%\ o111)$$

(% i114) f(a) := n*a + c;
Uf(a,h);
Uf_Ua(a);

$$\mathrm{f}(a) := na + c \quad (\%\ o112)$$

$$\frac{ahn + c}{an + c} \quad (\%\ o113)$$

$$\%e^{\frac{an}{an+c}} \quad (\%\ o114)$$

(% i118) assume(y>1);
f(a) := aˆ y + c;
Uf(a,h);
Uf_Ua(a);

$$[y > 1] \quad (\%\ o115)$$

$$\mathrm{f}(a) := a^y + c \tag{\% o116}$$

$$\frac{(ah)^y + c}{c + a^y} \tag{\% o117}$$

$$\%e^{\frac{a^y y}{c+a^y}} \tag{\% o118}$$

(% i121)
```
f(a) := exp(b*a^ y + c);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := \exp\left(b a^y + c\right) \tag{\% o119}$$

$$\%e^{b(ah)^y - a^y b} \tag{\% o120}$$

"Is "

y

" an "

integer

"?"

no;

$$\%e^{a^y b y} \tag{\% o121}$$

(% i124)
```
f(a) := b^ (a^ y + c);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := b^{a^y + c} \tag{\% o122}$$

$$b^{(ah)^y - a^y} \tag{\% o123}$$

"Is "

$$y$$

" an "

integer

"?"

$$no;$$

"Is "

$$y - 1$$

" positive, negative or zero?"

$$positive;$$

$$\%e^{a^y \log(b) y} \tag{\% o124}$$

(% i127)

```
f(a) := exp(b^ (a^ y + c));
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := \exp\left(b^{a^y + c}\right) \tag{\% o125}$$

$$\%e^{b^{(ah)^y + c} - b^{c + a^y}} \tag{\% o126}$$

"Is "

$$c$$

" an "

integer

"?"

$no;$

$$\%e^{a^y b^{c+a^y}\log(b) y} \qquad (\%\ o127)$$

(% i130) f(a) := g(a) * p(a);
Uf(a,h);
Uf_Ua(a);

$$\mathrm{f}(a) := \mathrm{g}(a)\,\mathrm{p}(a) \qquad (\%\ o128)$$

$$\frac{\mathrm{g}\,(a h)\,\mathrm{p}\,(a h)}{\mathrm{g}(a)\,\mathrm{p}(a)} \qquad (\%\ o129)$$

$$\lim_{\to h \text{ \textbackslash-\textbackslash> } 1+} \left(\frac{\mathrm{g}\,(a h)\,\mathrm{p}\,(a h)}{\mathrm{g}(a)\,\mathrm{p}(a)}\right)^{\frac{1}{\log(h)}} \qquad (\%\ o130)$$

So the up-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: U_Ua (ⅇâ) = ⅇâ.

(% i133) f(a) := b^ exp(a^ y + c);
Uf(a,h);
Uf_Ua(a);

$$\mathrm{f}(a) := b^{\exp(a^y + c)} \qquad (\%\ o131)$$

$$b^{\%e^{(a h)^y + c} - \%e^{c + a^y}} \qquad (\%\ o132)$$

"Is "

c

" an "

integer

"?"

$no;$

$$\%e^{a^y \log(b) y \%e^{c+a^y}} \tag{\% o133}$$

(% i136)
```
f(a) := log(a);
Uf(a,h);
Uf_Ua(a);
```

$$\mathrm{f}(a) := \log(a) \tag{\% o134}$$

$$\frac{\log(ah)}{\log(a)} \tag{\% o135}$$

$$\%e^{\frac{1}{\log(a)}} \tag{\% o136}$$

(% i137)
```
kill(f);
```

$$\text{done} \tag{\% o137}$$

Alternative expressions for Uf_Ua(a) (with a=exp(z)):

(% i142)
```
a = exp(z);
exp('diff( log(f(exp(z))) ,z,1));
exp(diff( log(f(exp(z))) ,z,1));
exp( a * diff(f(a),a,1) / f(a) );
exp('diff( log(f(a)) ,log(a),1));
```

$$a = \%e^{z} \tag{\% o138}$$

$$\%e^{\frac{d}{dz} \log(\mathrm{f}(\%e^{z}))} \tag{\% o139}$$

$$\%e^{\frac{\frac{d}{dz} \mathrm{f}(\%e^{z})}{\mathrm{f}(\%e^{z})}} \tag{\% o140}$$

$$\%e^{\frac{a \left(\frac{d}{da} \mathrm{f}(a)\right)}{\mathrm{f}(a)}} \tag{\% o141}$$

$$\%e^{\frac{d}{d\log(a)}\log(\mathrm{f}(a))} \tag{\% o142}$$

This leads to an alternative expression for the "normal" derivative, d/dx, in terms of the down derivative:

(% i146)
```
exp(x + 'Df_Dx(x) - f(x));
z = log(x); g(x) = log(f(x));
exp('Dg_Dz(x));
```

$$\%e^{\mathrm{Df\_Dx}(x)-\mathrm{f}(x)+x} \tag{\% o143}$$

$$z = \log(x) \tag{\% o144}$$

$$\mathrm{g}(x) = \log(\mathrm{f}(x)) \tag{\% o145}$$

$$\%e^{\mathrm{Dg\_Dz}(x)} \tag{\% o146}$$

4 Up2

Now to be distributive over "up": up2(a , up(b,c)) = up(up2(a,b) , up2(a,c)) ...

(% i147)
```
up2(a,b) := radcan( exp( up(log(a),log(b)) ) );
```

$$\mathrm{up2}(a, b) := \mathrm{radcan}(\exp(\mathrm{up}(\log(a), \log(b)))) \tag{\% o147}$$

(% i149)
```
up(up2(a,b),up2(a,c));
up2(a,up(b,c));
```

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \tag{\% o148}$$

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \tag{\% o149}$$

Bingo! And obviously also commutative, unlike say, tetration ($\hat{y}$(x) or $\hat{x}$(y)), which is neither. Identity for up2 (or up2_times):

(% i150) up2(ⅇ^ ⅇ,a);

$$a \tag{\% o150}$$

... or base to the power of base. So one can imagine that for up3 it would be $e^{(e^e)}$... etc.

(% i151) diff(up2(a,b),a);

$$\frac{\log(a)^{\log(\log(b))-1}\%e^{\log(a)^{\log(\log(b))}}\log(\log(b))}{a} \tag{\% o151}$$

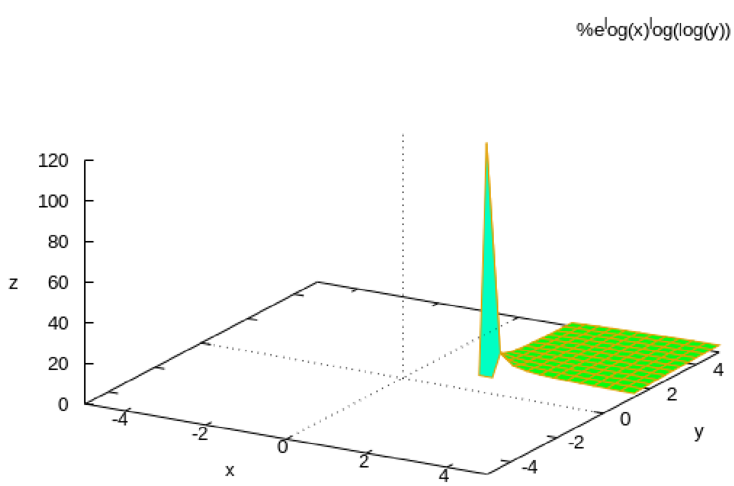
(% i153) up2(x,x);
exp(log(x)^ log(x));

$$\%e^{\%e^{\log(\log(x))^2}} \tag{\% o152}$$

$$\%e^{\log(x)^{\log(x)}} \tag{\% o153}$$

Taking a look...

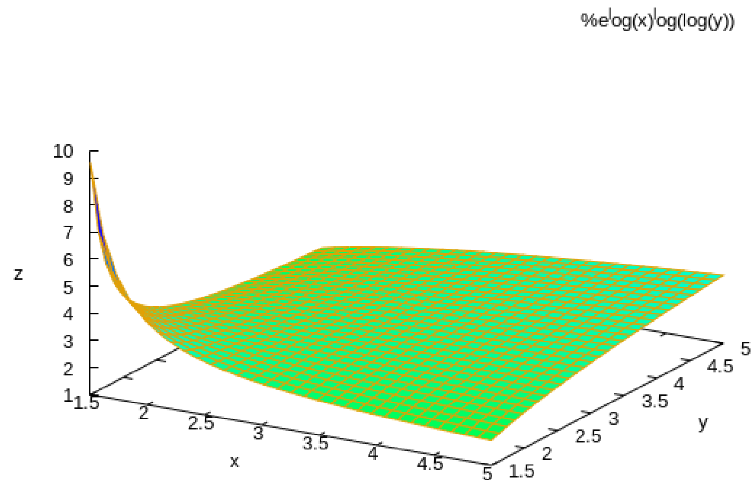
(% i154) wxplot3d (up2(x,y), [x, -5, 5], [y, -5, 5])$



(% t154)

Problems now at x or y = 1.

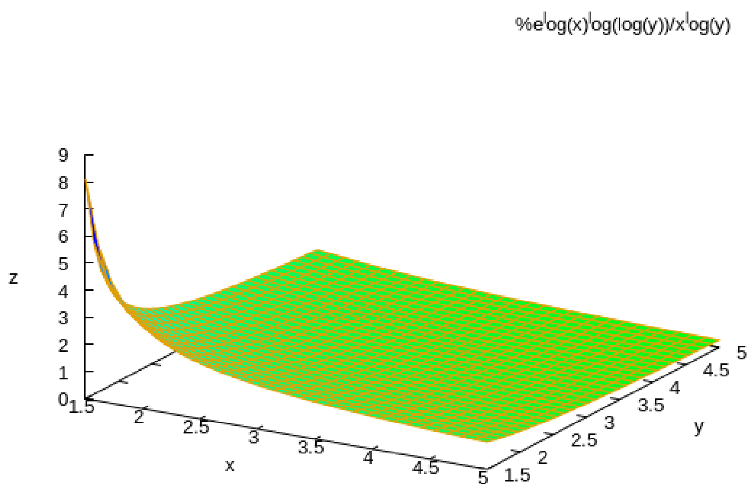
(% i155) wxplot3d (up2(x,y), [x, 1.5, 5], [y, 1.5, 5])$



(% t155)

Comparing up2(x,y) with up(x,y) :

(%i156) wxplot3d (up2(x,y)/up(x,y), [x, 1.5, 5], [y, 1.5, 5])$



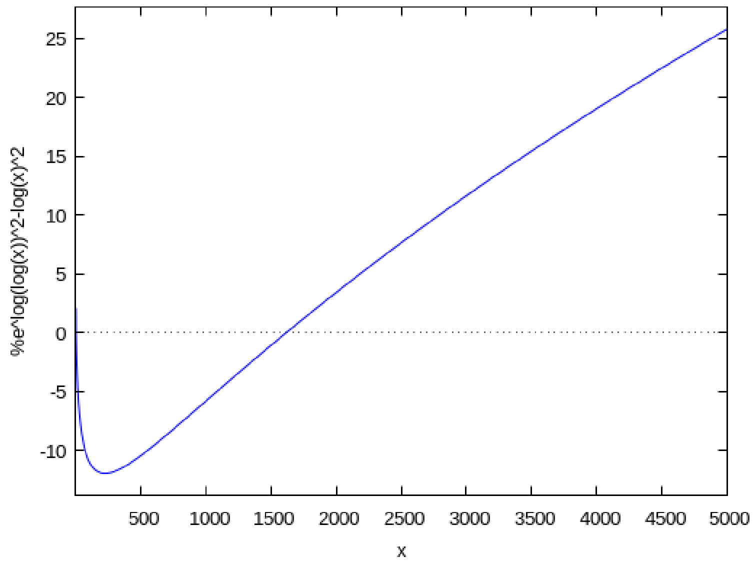
(%t156)

(%i160) up2(x,x)/up(x,x);
log(up2(x,x)/up(x,x));
log(x)^ log(log(x))-log(x)^ 2;
wxplot2d (log(up2(x,x)/up(x,x)), [x, 1.5, 5000])$

$$\%e^{\%e^{\log\left(\log\left(x\right)\right)^2}-\log\left(x\right)^2} \quad (\%o157)$$

$$\%e^{\log\left(\log\left(x\right)\right)^2}-\log\left(x\right)^2 \quad (\%o158)$$

$$\log\left(x\right)^{\log\left(\log\left(x\right)\right)}-\log\left(x\right)^2 \quad (\%o159)$$



(% t160)

So up2(x,x) does eventually win over up(x,x)... at x=e$\hat{(}$e^2). Using up2_div for the "ratio"...

(% i162) up_div(a,b) := radcan(exp(log(a) / log(b)));
up2_div(a,b) := radcan(exp(up_div(log(a),log(b))));

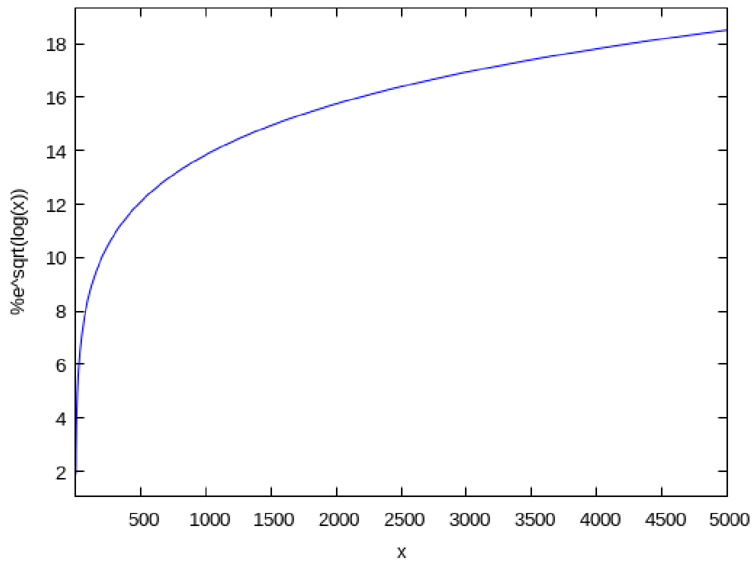
$$\mathrm{up\_div}\left(a\,,b\right):=\mathrm{radcan}\left(\exp\left(\frac{\log\left(a\right)}{\log\left(b\right)}\right)\right) \tag{\% o161}$$

$$\mathrm{up2\_div}\left(a\,,b\right):=\mathrm{radcan}\left(\exp\left(\mathrm{up\_div}\left(\log\left(a\right),\log\left(b\right)\right)\right)\right) \tag{\% o162}$$

(% i163) up2_div(up2(x,x),up(x,x));

$$\%e^{\sqrt{\log\left(x\right)}} \tag{\% o163}$$

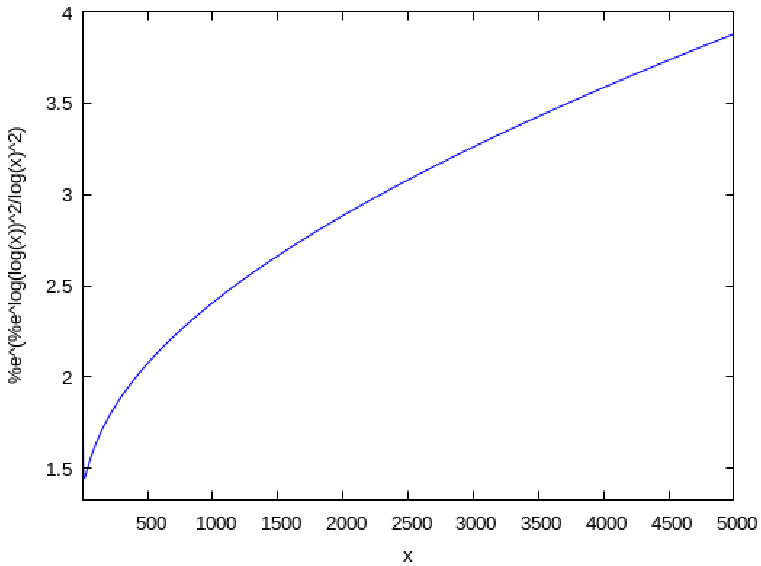
(% i164) wxplot2d (up2_div(up2(x,x),up(x,x)), [x, 1.5, 5000])$



(% t164)

(% i166) up_div(up2(x,x),up(x,x));
wxplot2d (up_div(up2(x,x),up(x,x)), [x, 3, 5000])$

$$\%e^{\frac{\%e^{\log(\log(x))^2}}{\log(x)^2}} \quad (\% \text{ o165})$$



%e^(%e^log(log(log(x)))^2/log(x)^2)
x
4
3.5
3
2.5
2
1.5
500
1000
1500
2000
2500
3000
3500
4000
4500
5000

(%t166)