

DISTRIBUTIVITY, EXTENDED

(Tommy Burch, on and off from 08.2024 to 06.2025) (wxMaxima does not export the cleanest of latex files, so the resulting pdf contains a number of quirks and errors)

1 At or Down

This all started back in the late '90s at the UofA in a physics lab prep session (with R. McCroskey and R. Haar, among others). The discussion hinged on finding an operation (i termed it 'at' with the symbol @) over which addition was distributive. The exp / log solution below i came up with followed later the same day... Note that, for multiplication and addition, the distributive property can be "revealed" through exponentials and their inverse, the natural logarithms:

(% i1) `add(a,b) := log(exp(a)*exp(b));`

$$\text{add}(a, b) := \log(\exp(a) \exp(b)) \quad (\% \text{ o1})$$

(% i3) `add(a,b);`
`add(c*a,c*b);`

$$b + a \quad (\% \text{ o2})$$

$$bc + ac \quad (\% \text{ o3})$$

(% i4) `ratsimp(%);`

$$(b + a) c \quad (\% \text{ o4})$$

Placing the multiplication between the exponentials in the logarithm, we arrive at the relation (addition) over which it is distributive. We can try the same with + between the exp's...

1.1 At _ +

Define the @ (or "down") relation such that addition is distributive over it: $a + (b @ c) = (a+b) @ (a+c)$...Or: $a + \text{down}(b,c) = \text{down}(a+b,a+c)$. Trying the following ansatz:

(% i5) at_eq : At(a,b) = log(exp(a) + exp(b));

$$\text{At}(a, b) = \log(e^b + e^a) \quad (\text{at_eq})$$

(% i6) down(a,b) := radcan(log(exp(a) + exp(b)));

$$\text{down}(a, b) := \text{radcan}(\log(\exp(a) + \exp(b))) \quad (\% \text{ o6})$$

Check:

(% i7) down(a+b,a+c);

$$\log(e^c + e^b) + a \quad (\% \text{ o7})$$

yes! (It is also an obviously commutative operation.) This should work with any base for the power / log's: e.g.,

(% i8) log10(x) := log(x) / log(10);

$$\log_{10}(x) := \frac{\log(x)}{\log(10)} \quad (\% \text{ o8})$$

(% i9) radcan(log10(10^ (a+b) + 10^ (a+c)));

$$\frac{\log(2^c 5^c + 2^b 5^b) + (\log(5) + \log(2)) a}{\log(5) + \log(2)} \quad (\% \text{ o9})$$

(% i10) logcontract(%);

$$\frac{\log(2^c 5^c + 2^b 5^b) + \log(10) a}{\log(10)} \quad (\% \text{ o10})$$

yep! The ambiguity probably speak volumes (see, e.g., M. Burgin, arXiv:1010.3287 and M. Czachor, arXiv:1412.8583 : there, the focus appears not to be in maintaining distributivity, unlike here, where it's the main focus; also, a specific choice for the "projective" function is usually made – like our choice of exp here – rather than exploring any implications of not having to choose; we should keep the same in mind by noting that the base can be changed). It may be worth exploring whether functions other than powers/logarithms can retain distributivity... at first glance, seemingly not. Some values with base e:

(% i11) down(0,0);

$$\log(2) \quad (\% \text{ o11})$$

(% i12) down(1,0);

$$\log(\%e + 1) \quad (\% \text{ o12})$$

(% i13) down(x,0);

$$\log(\%e^x + 1) \quad (\% \text{ o13})$$

(% i15) down(x,x);
down(x-log(2),x-log(2));

$$x + \log(2) \quad (\% \text{ o14})$$

$$x \quad (\% \text{ o15})$$

Associative:

(% i17) down(a,down(b,c));
down(down(a,b),c);

$$\log(\%e^c + \%e^b + \%e^a) \quad (\% \text{ o16})$$

$$\log(\%e^c + \%e^b + \%e^a) \quad (\% \text{ o17})$$

(% i20) diff(down(a,b),a);
diff(down(a,b),b);
diff(diff(down(a,b),b),a);

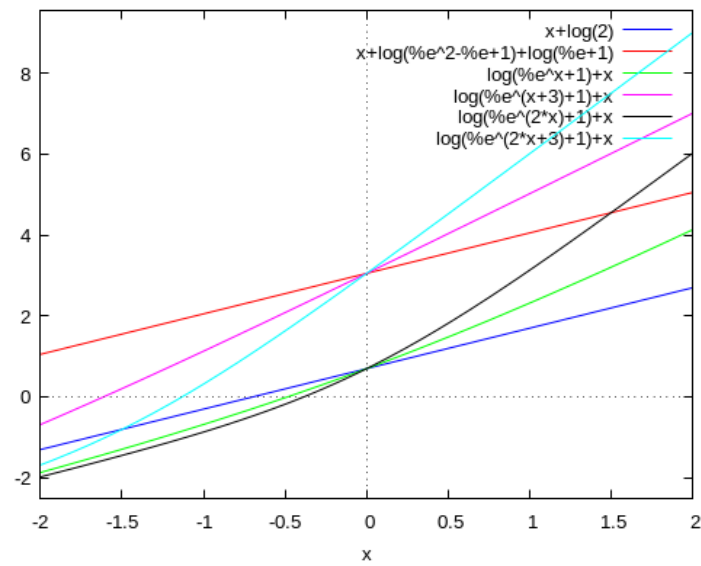
$$\frac{\%e^a}{\%e^b + \%e^a} \quad (\% \text{ o18})$$

$$\frac{\%e^b}{\%e^b + \%e^a} \quad (\% \text{ o19})$$

$$-\frac{\%e^{b+a}}{\left(\%e^b + \%e^a\right)^2} \quad (\% \text{ o20})$$

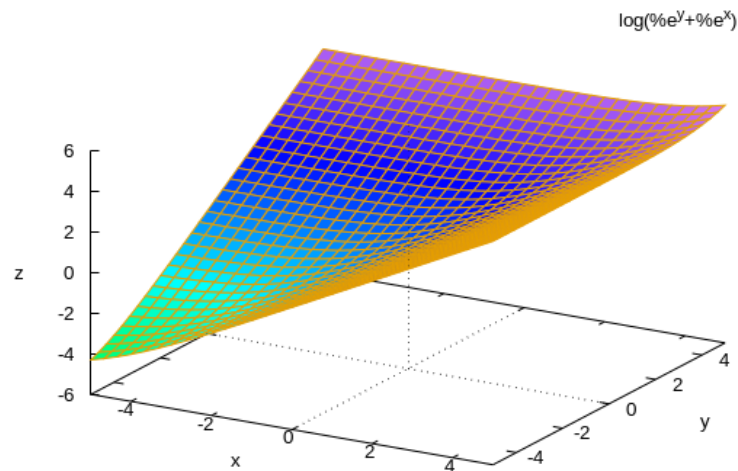
Let's have a look:

```
(% i21) wxplot2d ([down(x,x),down(x,x+3),down(x,2*x),down(x,2*x+3),down(x,3*x),down(x,3*x+3)],
[x, -2, 2])$
```



(% t21)

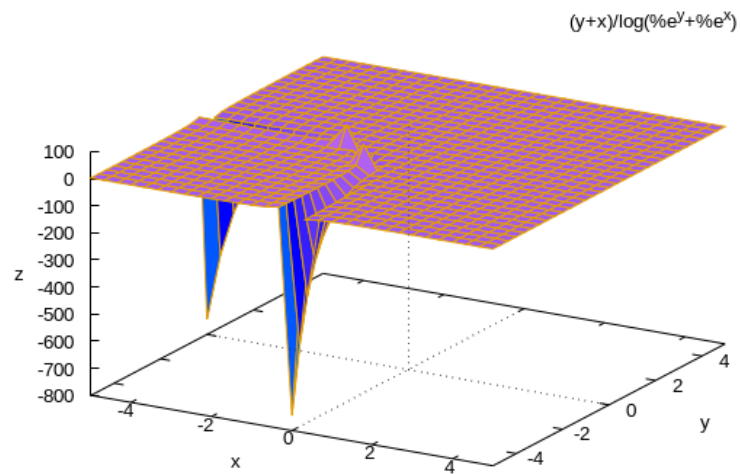
```
(% i22) wxplot3d (down(x,y), [x, -5, 5], [y, -5, 5])$
```



(% t22)

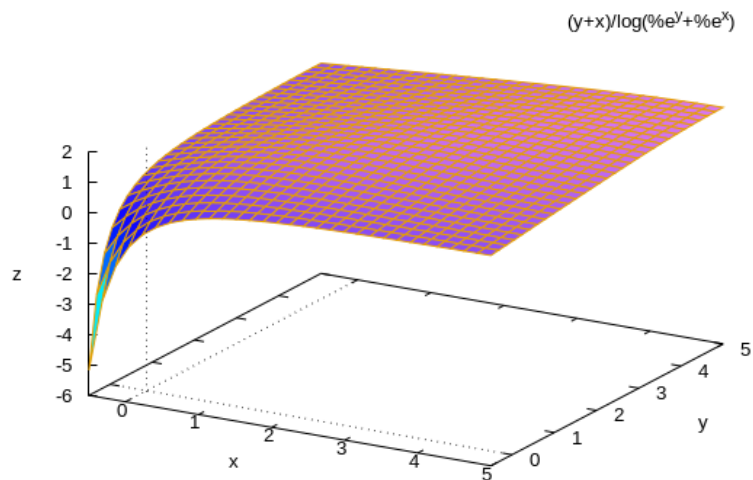
Comparing $x+y$ with $\text{down}(x,y)$:

(% i23) `wxplot3d ((x+y)/down(x,y), [x, -5, 5], [y, -5, 5])$`



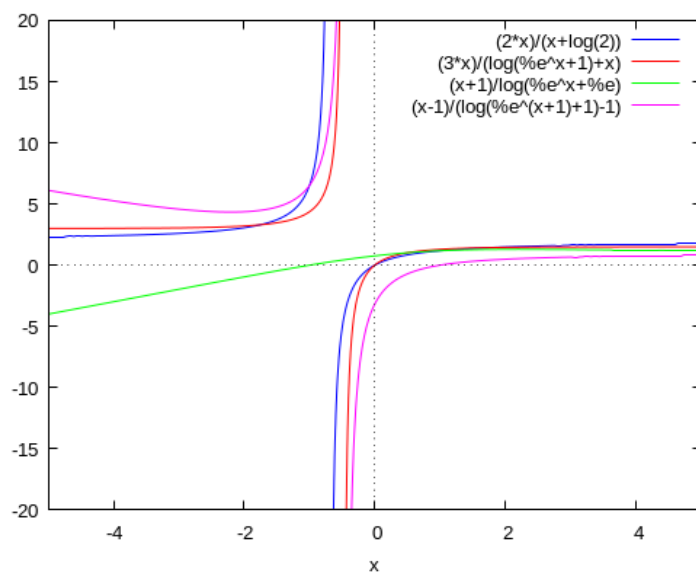
(% t23)

(% i24) wxplot3d ((x+y)/down(x,y), [x, -0.5, 5], [y, -0.5, 5])\$



(% t24)

(% i25) wxplot2d ([2*x/down(x,x), 3*x/down(x,2*x),(x+1)/down(x,1),(x-1)/down(x,-1)], [x, -5, 5], [y, -20, 20])\$



(% t25)

1.2 At₋,*,/

We can also consider a "down-like" operation with a minus instead a plus:

(% i26) `down_min(a,b) := radcan(log(exp(a) - exp(b)));`

$$\text{down_min}(a, b) := \text{radcan}(\log(\exp(a) - \exp(b))) \quad (\% \text{ o26})$$

Associativity:

(% i30) `down(down_min(a,b),c);`
`down_min(down(a,c),b);`
`down(down_min(a,b),down_min(c,d));`
`down(down_min(a,d),down_min(c,b));`

$$\log\left(\%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o27})$$

$$\log\left(\%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o28})$$

$$\log\left(-\%e^d + \%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o29})$$

$$\log\left(-\%e^d + \%e^c - \%e^b + \%e^a\right) \quad (\% \text{ o30})$$

Same distributive property:

(% i31) `down_min(a+b,a+c);`

$$\log\left(\%e^b - \%e^c\right) + a \quad (\% \text{ o31})$$

As we've already seen, with times, we achieve addition. With divide, one achieves subtraction:

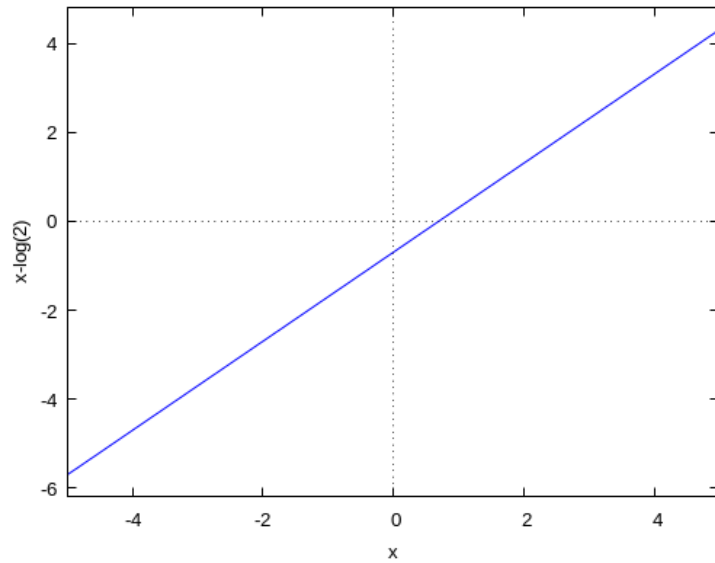
(% i33) `down_div(a,b) := radcan(log(exp(a) / exp(b)));`
`down_div(a,b);`

$$\text{down_div}(a, b) := \text{radcan}\left(\log\left(\frac{\exp(a)}{\exp(b)}\right)\right) \quad (\% \text{ o32})$$

$$a - b \quad (\% \text{ o33})$$

Using @_/ or down_div for the "ratio" of + and @...

```
(% i35) down_div(x+x,down(x,x));
wxplot2d (2*x - down(x,x), [x, -5, 5])$
x - log(2)                                     (% o34)
```



(% t35)

1.3 Down differentiation

Given down, down_min, and down_div, one is free to define a "down derivative". However, we must take care with the notion of infinitesimals: normally $h \rightarrow 0$ in the definition of the derivative, but here we need $h \rightarrow -\infty$ to approach the @_+ identity ($x @ -\infty = x$)...

```
(% i36) Df_Dx_eq : Df_Dx = limit( down_min( f(down(x,h)) , f(x) ) - h , h, -inf);
```

$$Df_Dx = \lim_{h \searrow -\infty} \log \left(\frac{e^{f(\log(e^x + e^h))}}{e^{f(x)}} \right) - h \quad (Df_Dx_eq)$$

```
(% i40) Df(x,h) := down_min( f(down(x,h)) , f(x) );
Df(x,h);
Df_Dx(x) := ratsimp( limit( Df(x,h) - h , h, minf) );
Df_Dx(x);
```

```
Df(x,h) := down_min( f(down(x,h)) , f(x))                                     (% o37)
```


$$\log \left(\%e^{\text{f}(\log (\%e^x + \%e^h))} - \%e^{\text{f}(x)} \right) \quad (\% \text{ o38})$$

$$\text{Df_Dx}(x) := \text{ratsimp}(\text{limit}(\text{Df}(x, h) - h, h, -\infty)) \quad (\% \text{ o39})$$

$$\lim_{\rightarrow h \setminus \setminus > -\infty} \log \left(\%e^{\text{f}(\log (\%e^x + \%e^h))} - \%e^{\text{f}(x)} \right) - h \quad (\% \text{ o40})$$

$$\begin{aligned} (\% \text{ i43}) \quad & \text{f(x)} := \text{A}^*\text{x}^b + \text{c}; \\ & \text{Df(x,h)}; \\ & \text{Df_Dx(x)}; \end{aligned}$$

$$\text{f}(x) := Ax^b + c \quad (\% \text{ o41})$$

$$\log \left(\%e^{A \log (\%e^x + \%e^h)^b} - \%e^{Ax^b} \right) + c \quad (\% \text{ o42})$$

$$(b-1) \log(x) + Ax^b - x + c + \log(b) + \log(A) \quad (\% \text{ o43})$$

$$\begin{aligned} (\% \text{ i46}) \quad & \text{f(x)} := \log(\text{A}^*\text{x}^b + \text{c}); \\ & \text{Df(x,h)}; \\ & \text{Df_Dx(x)}; \end{aligned}$$

$$\text{f}(x) := \log(Ax^b + c) \quad (\% \text{ o44})$$

$$\log \left(\log \left(\%e^x + \%e^h \right)^b - x^b \right) + \log(A) \quad (\% \text{ o45})$$

$$(b-1) \log(x) - x + \log(b) + \log(A) \quad (\% \text{ o46})$$

$$\begin{aligned} (\% \text{ i49}) \quad & \text{f(x)} := \exp(\text{A}^*\text{x}^b + \text{c}); \\ & \text{Df(x,h)}; \\ & \text{Df_Dx(x)}; \end{aligned}$$

$$\text{f}(x) := \exp(Ax^b + c) \quad (\% \text{ o47})$$

$$\log \left(e^{e^{A \log(e^x + e^h)^{b+c}}} - e^{e^{Ax^b+c}} \right) \quad (\% \text{ o48})$$

$$(b-1) \log(x) + e^{Ax^b+c} + Ax^b - x + c + \log(b) + \log(A) \quad (\% \text{ o49})$$

(% i52) f(x) := down(g(x) , p(x));
 Df(x,h);
 Df_Dx(x) ;
 f(x) := down(g(x) , p(x)) \quad (\% \text{ o50})

$$\log \left(e^{p(\log(e^x + e^h))} + e^{g(\log(e^x + e^h))} - e^{p(x)} - e^{g(x)} \right) \quad (\% \text{ o51})$$

$$\lim_{h \searrow -\infty} \log \left(e^{p(\log(e^x + e^h))} + e^{g(\log(e^x + e^h))} - e^{p(x)} - e^{g(x)} \right) - h \quad (\% \text{ o52})$$

So the down-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: $D_Dx(e^{\hat{x}}) = e^{\hat{x}}$.

(% i55) f(x) := b^ (A*x+c);
 Df(x,h);
 Df_Dx(x) ;
 f(x) := b^{Ax+c} \quad (\% \text{ o53})

$$\log \left(e^{b^{A \log(e^x + e^h)^{b+c}}} - e^{b^{Ax+c}} \right) \quad (\% \text{ o54})$$

$$(A \log(b) - 1)x + \log(b)c + b^{Ax+c} + \log(\log(b)) + \log(A) \quad (\% \text{ o55})$$

(% i58) f(x) := log(x);
 Df(x,h);
 Df_Dx(x) ;
 f(x) := log(x) \quad (\% \text{ o56})

$$\log \left(\log \left(e^x + e^h \right) - x \right) \quad (\% \text{ o57})$$

$$-x \quad (\% \text{ o58})$$

(% i59) kill(f);

done (\% o59)

(% i60) Eig_Df_eq : Df_Dx(x) = f(x);

$$\lim_{h \rightarrow -\infty} \log \left(e^{f(\log(e^x + e^h))} - e^{f(x)} \right) - h = f(x) \quad (\text{Eig_Df_eq})$$

As we can see from the previous solutions of Df_Dx, this DE is uniquely solved by f(x)=x.

(% i61) kill(f);

done (\% o61)

Alternative expressions for Df_Dx(x) (with x=log(z)):

(% i65) x = log(z);
log(diff(exp(f(log(z))) ,z,1));
log(diff(exp(f(log(z))) ,z,1));
f(x) - x + log(diff(f(x),x,1));

$$x = \log(z) \quad (\% \text{ o62})$$

$$\log \left(\frac{d}{dz} e^{f(\log(z))} \right) \quad (\% \text{ o63})$$

$$\log \left(e^{f(\log(z))} \left(\frac{d}{dz} f(\log(z)) \right) \right) \quad (\% \text{ o64})$$

$$\log \left(\frac{d}{dx} f(x) \right) + f(x) - x \quad (\% \text{ o65})$$

2 Down2

Heading further "down", define "three dots" (therefore), or down2, such that: a
 @ down2(b,c) = down2(a@b,a@c) ...Or: down(a,down2(b,c)) = down2(down(a,b),down(a,c))
 ...

(% i66) td_eq : Td(a,b) = log(down(exp(a),exp(b)));

$$\text{Td}(a, b) = \log \left(\log \left({}_0e^{{}_0e^b} + {}_0e^{{}_0e^a} \right) \right) \quad (\text{td_eq})$$

(% i67) down2(a,b) := radcan(log(down(exp(a),exp(b))));

$$\text{down2}(a, b) := \text{radcan} \left(\log \left(\text{down} \left(\exp(a), \exp(b) \right) \right) \right) \quad (\% \text{ o67})$$

(% i68) down2(down(a,b),down(a,c));

$$\log \left(\log \left({}_0e^{{}_0e^c} + {}_0e^{{}_0e^b} \right) + {}_0e^a \right) \quad (\% \text{ o68})$$

yes!down2-identity would have to be at $\infty + i\pi$.

(% i69) down2(x,x);

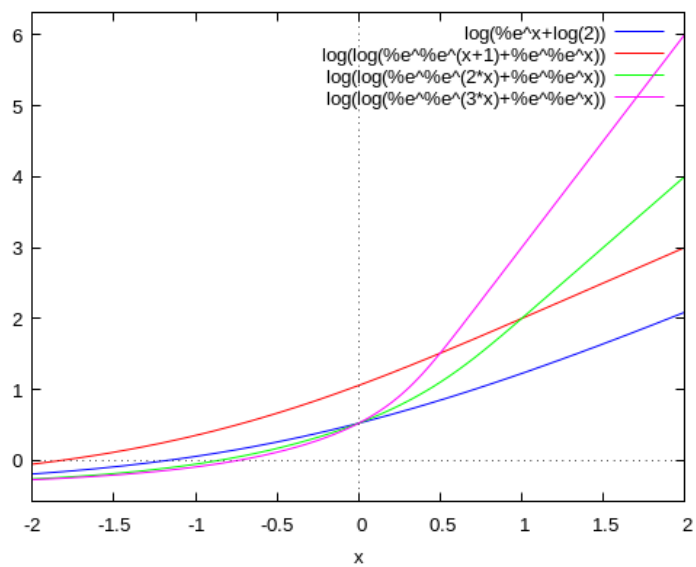
$$\log({}_0e^x + \log(2)) \quad (\% \text{ o69})$$

(% i70) diff(down2(a,b),a);

$$\frac{{}_0e^{{}_0e^a + a}}{\left({}_0e^{{}_0e^b} + {}_0e^{{}_0e^a} \right) \log \left({}_0e^{{}_0e^b} + {}_0e^{{}_0e^a} \right)} \quad (\% \text{ o70})$$

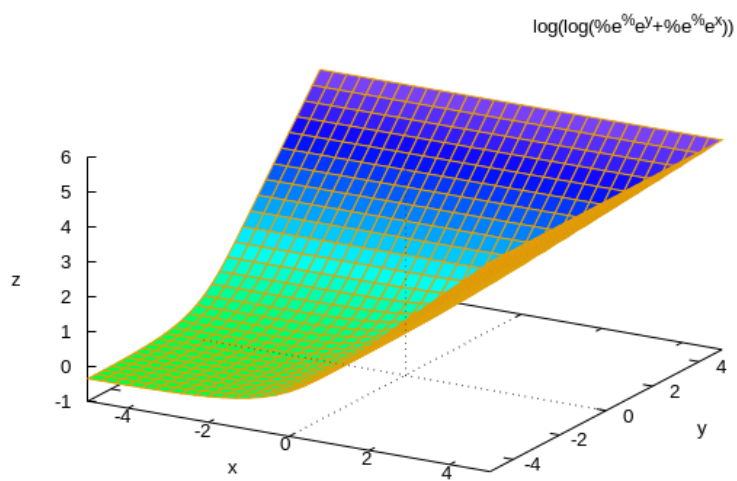
Let's take a look...

(% i71) wxplot2d ([down2(x,x),down2(x,x+1),down2(x,2*x),down2(x,3*x)], [x, -2, 2])\$



(% t71)

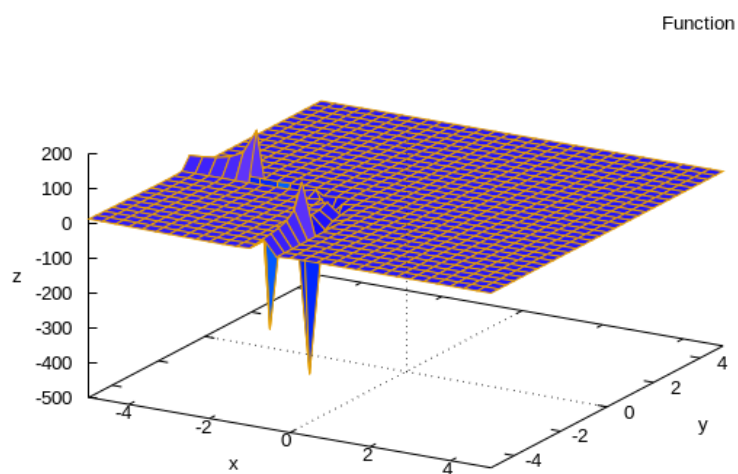
(% i72) wxplot3d (down2(x,y), [x, -5, 5], [y, -5, 5])\$



(% t72)

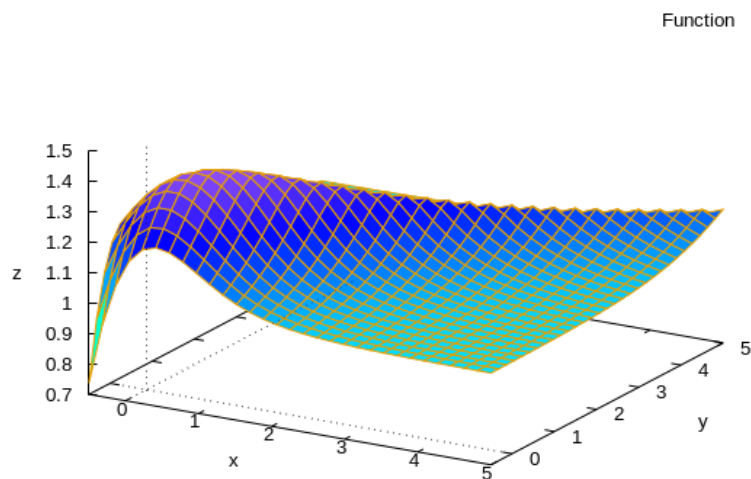
Comparing $\text{down}(x,y)$ with $\text{down2}(x,y)$:

(% i73) wxplot3d (down(x,y)/down2(x,y), [x, -5, 5], [y, -5, 5])\$



(% t73)

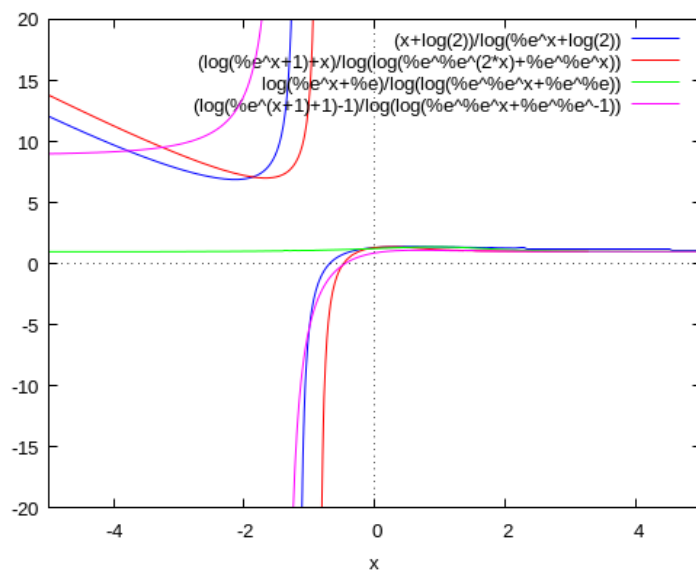
(% i74) wxplot3d (down(x,y)/down2(x,y), [x, -0.5, 5], [y, -0.5, 5])\$



(% t74)

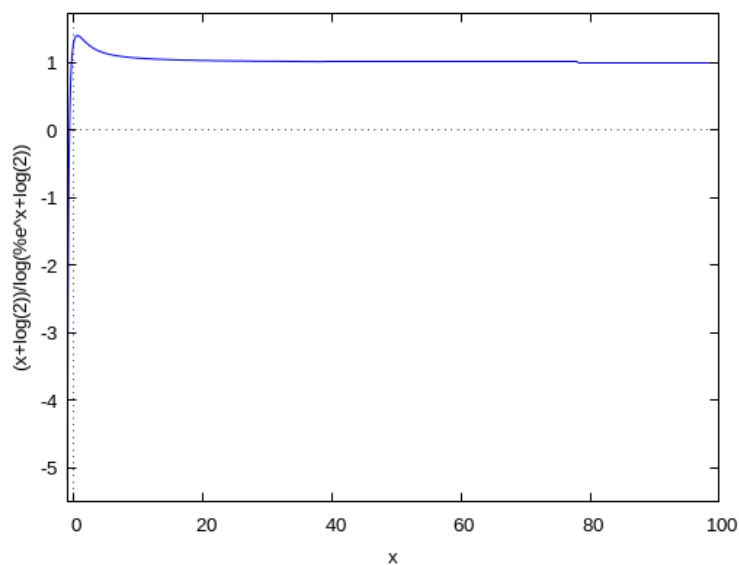
(% i75) wxplot2d ([down(x,x)/down2(x,x), down(x,2*x)/down2(x,2*x),down(x,1)/down2(x,1),down(x,-1)/down2(x,-1)], [x, -5, 5], [y, -20, 20])\$

plot2d : expression evaluate on non-numeric values somewhere in plotting range.



(% t75)

```
(% i76) wxplot2d (down(x,x)/down2(x,x), [x, -1, 100])$
```



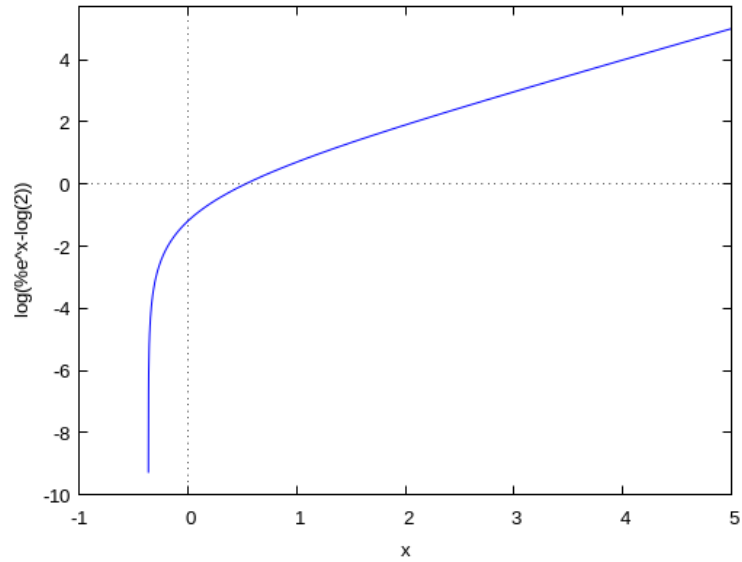
(% t76)

Using down2_div (or down_min) for the "ratio"...

```
(% i78) down_min(down(x,x),down2(x,x));
wxplot2d (down_min(down(x,x),down2(x,x)), [x, -1, 5])$
```

$\log(e^x - \log(2))$ plot2d : *expression evaluatest on non-numeric values somewhere in plotting range.*

(% o77)



(% t78)

3 Up

Let's take a look in the "up" direction for an operation which is distributive over multiplication. Defining "star" (or "up"): $a * (b \cdot c) = (a * b) \cdot (a * c)$...Or: $\text{up}(a, b \cdot c) = \text{up}(a, b) \cdot \text{up}(a, c)$...

(% i79) `str_eq : Str(a,b) = exp(log(a)*log(b));`

$$\text{Str}(a, b) = e^{\log(a) \log(b)} \quad (\text{str_eq})$$

(% i80) `up(a,b) := radcan(exp(log(a)*log(b)));`

$$\text{up}(a, b) := \text{radcan}(\exp(\log(a) \log(b))) \quad (\% \text{ o80})$$

(% i81) `up(a,b);`

$$a^{\log(b)} \quad (\% \text{ o81})$$

(% i82) `up(a,b*c);`

$$a^{\log(c) + \log(b)} \quad (\% \text{ o82})$$

```
(% i83) up(a,b) * up(a,c);
```

$$a^{\log(c)+\log(b)} \quad (\% \text{ o83})$$

which is the same as the former. So, yes! Check with "down"-like operation:

```
(% i84) log( up(exp(a),exp(b)) );
```

$$ab \quad (\% \text{ o84})$$

yes! up(a,b) could also be expressed with b as the base:

```
(% i88) is( up(a,b) = b^ log(a) );
         is( up(a,b) = a^ log(b) );
         is( up(b,a) = a^ log(b) );
         is( up(a,b) = up(b,a) );
```

$$\text{false} \quad (\% \text{ o85})$$

$$\text{true} \quad (\% \text{ o86})$$

$$\text{true} \quad (\% \text{ o87})$$

$$\text{true} \quad (\% \text{ o88})$$

... although it seems Maxima has trouble (it enforces some type of alphabetization?) with that realization in the present form...? So one advantage of up(a,b), over "naive" exponentiation (a^b or b^a), can be seen immediately: namely commutativity... up(a,b)=up(b,a). Identity for up (or up_times):

```
(% i89) up(e,a);
```

$$a \quad (\% \text{ o89})$$

... the base e. Some special values:

```
(% i91) up(x,x);
         x^ log(x);
```

$$e^{\log(x)^2} \quad (\% \text{ o90})$$

$$x^{\log(x)} \quad (\% \text{ o91})$$

```
(% i94) diff(up(a,b),a);
        diff(up(a,b),b);
        diff(diff(up(a,b),a),b);
```

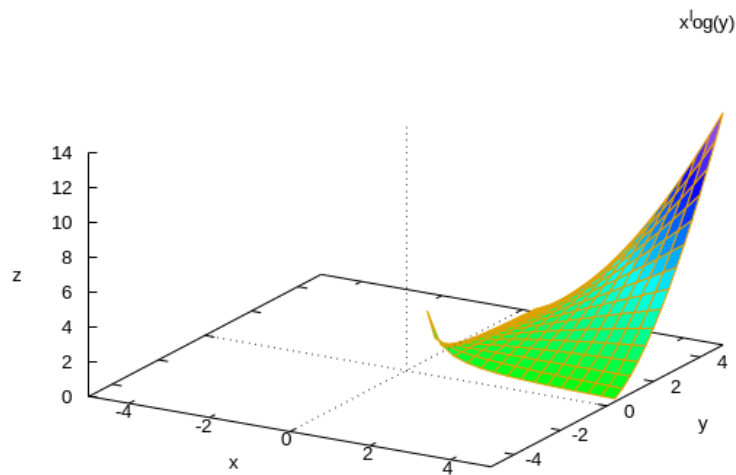
$$a^{\log(b)-1} \log(b) \quad (\% \text{ o92})$$

$$\frac{a^{\log(b)} \log(a)}{b} \quad (\% \text{ o93})$$

$$\frac{a^{\log(b)-1} \log(a) \log(b)}{b} + \frac{a^{\log(b)-1}}{b} \quad (\% \text{ o94})$$

Let's take a look:

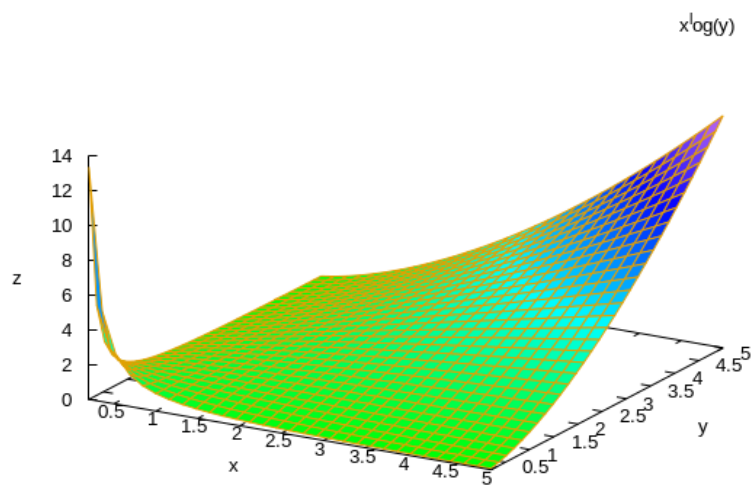
```
(% i95) wxplot3d (up(x,y), [x, -5, 5], [y, -5, 5])$
```



(% t95)

The singularity at the origin comes as no surprise. And for finite, real values, x and y should remain > 0 .

(% i96) wxplot3d (up(x,y), [x, 0.2, 5], [y, 0.2, 5])\$



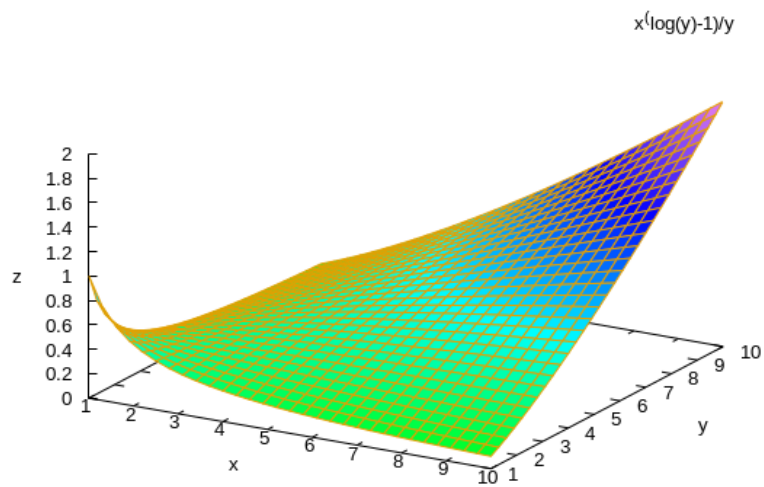
(% t96)

Compare with x.y :

(% i98) up(x,y)/(x*y);
wxplot3d (up(x,y)/(x*y), [x, 1, 10], [y, 1, 10])\$

$$\frac{x^{\log(y)-1}}{y}$$

(% o97)



(% t98)

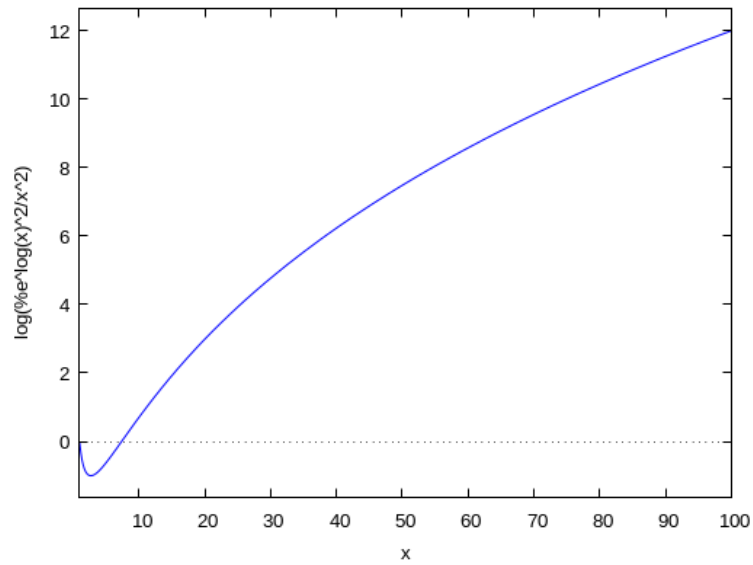
```
(%      up(x,x)/(x*x);
i101)  x^(log(x)-2);
        wxplot2d (log(up(x,x)/(x*x)), [x, 1, 100])$
```

$$\frac{e^{\log(x)^2}}{x^2}$$

(% o99)

$$x^{\log(x)-2}$$

(% o100)



(% t101)

Derivative of this ratio:

```
(%      factor(diff(x^ (log(x)-2),x,1));
i102)
```

$$2x^{\log(x)-3}(\log(x) - 1)$$

(% o102)

Minimum obviously at $x=e$:

```
(%      up(e,e);
i104)  find_root(diff(x^ (log(x)-2),x,1), x, 1, 10);
```

$$e$$

(% o103)

$$2.718281828459045$$

(% o104)

Also, $up(x,x)$ does eventually win over $x^2 \dots$ at $x=e^2$, where $up(x,x)$ is ...

```
(%      up(exp(2),exp(2));
i106)  float(%), numer;
```

$$e^4$$

(% o105)

54.59815003314423 (% o106)

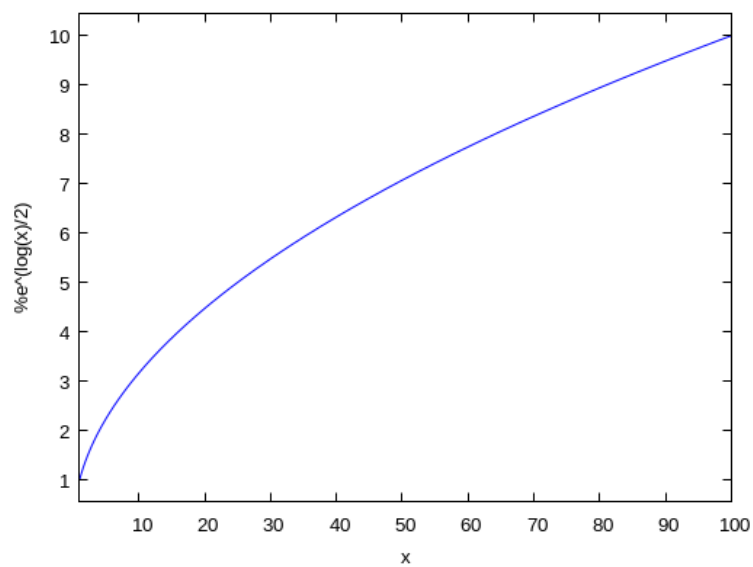
Using up_div for the "ratio"...

```
(%
i108)  exp(log(up(x,x))/log(x*x));
       radcan(%);
```

$e^{\frac{\log(x)}{2}}$ (% o107)

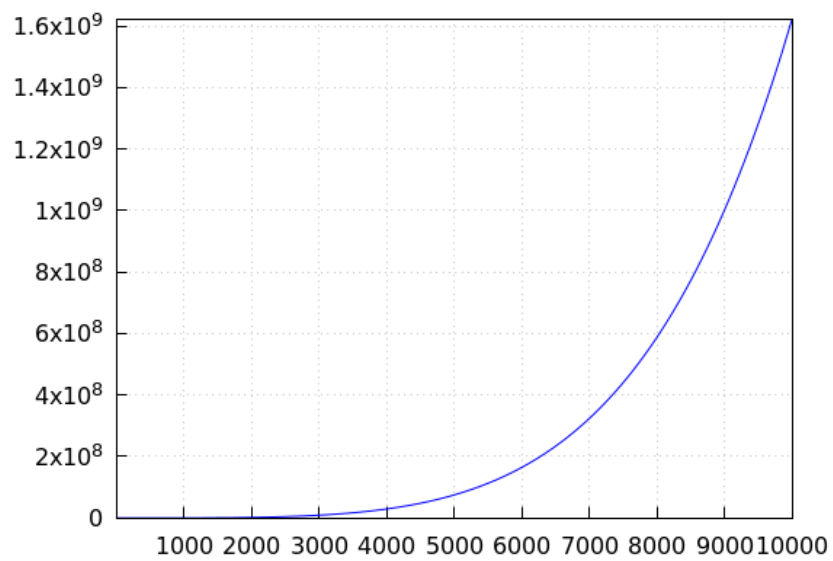
\sqrt{x} (% o108)

```
(%
i109)  wxplot2d (exp(log(up(x,x))/log(x*x)), [x, 1, 100])$
```



(% t109)

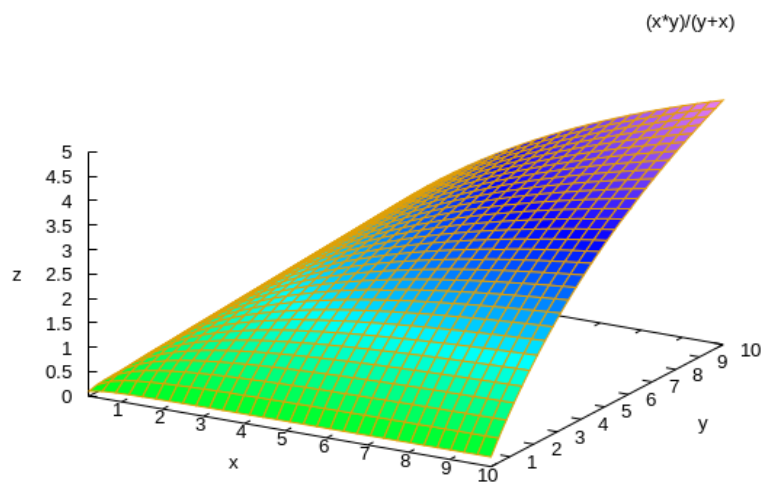
```
(%
i110)  wxdraw2d( grid = true, nticks = 1000,
          parametric( t^2 , up(t,t) ,
            t,1,100 ) )$
```



(% t110)

And x.y with x+y :


```
(% wxplot3d (x*y/(x+y), [x, 0.2, 10], [y, 0.2, 10])$
i111)
```



```
(% t111)
```

Possible relations to polylogarithms?:

```
(% integrate(log(t)/(t-1), t, 1, x);
i115) integrate(log(t)/(t+1), t, 1, x);
integrate(log(t)/t, t, 1, x);
integrate(log(y)/t, t, 1, x);
```

"Is "

$$x - 1$$

" positive, negative or zero?"

positive;

$$\log(1-x)\log(x) + \operatorname{li}_2(x) - \frac{\pi^2}{6}$$

```
(% o112)
```

"Is "

$$x - 1$$

" positive, negative or zero?"

positive;

$$\log (x) \log (x+1)+\operatorname{li}_2(-x)+\frac{\pi^2}{12} \quad (\% \text { o113})$$

"Is "

$$x-1$$

" positive, negative or zero?"

positive;

$$\frac{\log (x)^2}{2} \quad (\% \text { o114})$$

"Is "

$$x-1$$

" positive, negative or zero?"

positive;

$$\log (x) \log (y) \quad (\% \text { o115})$$

3.1 Up differentiation

To avoid problems with alphabetical preferences...

(% up(x,h);
i117) up(a,h);

$$h^{\log (x)} \quad (\% \text { o116})$$

$$a^{\log (h)} \quad (\% \text { o117})$$

we'll use 'a' as the independent variable here. Working with up_/, up_+ as times, up_- as divide, and the new "up_+"-identity, h->1...

```
(%      Uf(a,h) := f(a*h) / f(a) ;
i121)   Uf(a,h);
        Uf_Ua(a) := radcan( limit( exp( log(Uf(a,h)) / log(h) ) , h, 1, plus) );
        Uf_Ua(a);
```

$$Uf(a, h) := \frac{f(ah)}{f(a)} \quad (\% \text{ o118})$$

$$\frac{f(ah)}{f(a)} \quad (\% \text{ o119})$$

$$Uf_Ua(a) := \text{radcan} \left(\text{limit} \left(\exp \left(\frac{\log(Uf(a, h))}{\log(h)} \right), h, 1, \text{plus} \right) \right) \quad (\% \text{ o120})$$

$$\lim_{h \rightarrow 1} \frac{f(ah)^{\frac{1}{\log(h)}}}{f(a)^{\frac{1}{\log(h)}}} \quad (\% \text{ o121})$$

```
(%      Uf_Ua(a) := ratsimp( limit( Uf(a,h)^ (1 / log(h)) , h, 1, plus));
i123)   Uf_Ua(a);
```

$$Uf_Ua(a) := \text{ratsimp} \left(\text{limit} \left(Uf(a, h)^{\frac{1}{\log(h)}}, h, 1, \text{plus} \right) \right) \quad (\% \text{ o122})$$

$$\lim_{h \rightarrow 1} \left(\frac{f(ah)}{f(a)} \right)^{\frac{1}{\log(h)}} \quad (\% \text{ o123})$$

```
(%      f(a) := n*a + c;
i126)   Uf(a,h);
        Uf_Ua(a);
```

$$f(a) := na + c \quad (\% \text{ o124})$$

$$\frac{ahn + c}{an + c} \quad (\% \text{ o125})$$

$$\%_0 e^{\frac{an}{an+c}} \tag{\% o126}$$

```
(%      assume(y>1);
i130)  f(a) := a^ y + c;
        Uf(a,h);
        Uf_ Ua(a);
```

$$[y > 1] \tag{\% o127}$$

$$f(a) := a^y + c \tag{\% o128}$$

$$\frac{(ah)^y + c}{c + a^y} \tag{\% o129}$$

$$\%_0 e^{\frac{a^y y}{c+a^y}} \tag{\% o130}$$

```
(%      f(a) := exp(b*a^ y + c);
i133)  Uf(a,h);
        Uf_ Ua(a);
```

$$f(a) := \exp (ba^y + c) \tag{\% o131}$$

$$\%_0 e^{b(ah)^y - a^y b} \tag{\% o132}$$

"Is "

y

" an "

integer

"?"

no;

$$\%e^{a^yby} \qquad (\% \text{ o133})$$

```
(%      Uf_ Ua(a);
i134)
```

```
"Is "
```

$$y$$

```
" an "
```

```
integer
```

```
"?"
```

```
yes;
```

$$\%e^{a^yby} \qquad (\% \text{ o134})$$

```
(%      f(a) := b^ (a^ y + c);
i137)      Uf(a,h);
           Uf_ Ua(a);
```

$$f(a) := b^{a^y+c} \qquad (\% \text{ o135})$$

$$b^{(ah)^y-a^y} \qquad (\% \text{ o136})$$

```
"Is "
```

$$y$$

```
" an "
```

```
integer
```

```
"?"
```

```
no;
```

```
"Is "
```

$$y - 1$$

" positive, negative or zero?"

positive;

$$\%e^{a^y \log(b)y} \quad (\% \text{ o137})$$

(% Uf_Ua(a);
i138)

"Is "

y

" an "

integer

"?"

no;

"Is "

y - 1

" positive, negative or zero?"

negative;

$$\%e^{a^y \log(b)y} \quad (\% \text{ o138})$$

(% f(a) := exp(b^ (a^ y + c));
i141) Uf(a,h);
 Uf_Ua(a);

$$f(a) := \exp \left(b^{a^y+c} \right) \quad (\% \text{ o139})$$

$$\%e^{b^{(ah)^y+c}-b^{c+a^y}} \quad (\% \text{ o140})$$

```

"Is "

      c

" an "

      integer

"?"

      no;

      %e^{a^yb^{c+a^y}\log(b)y}                                (% o141)

(%      Uf_Ua(a);
i142)

"Is "

      c

" an "

      integer

"?"

      yes;

      %e^{a^yb^{c+a^y}\log(b)y}                                (% o142)

(%      f(a) := g(a) * p(a);
i145)      Uf(a,h);
           Uf_Ua(a);

      f(a) := g(a) p(a)                                         (% o143)

      g (ah) p (ah)
      -----
      g(a) p(a)                                                 (% o144)

```

$$\lim_{h \rightarrow 1} \left(\frac{g(ah)p(ah)}{g(a)p(a)} \right)^{\frac{1}{\log(h)}} \quad (\% \text{ o145})$$

So the up-derivative maintains not only the extended distributivity, but also the property that the exponential function keeps its special role: $U_Ua(\mathfrak{e}\hat{a}) = \mathfrak{e}\hat{a}$.

```
(%      f(a) := b^ exp(a^ y + c);
i148)   Uf(a,h);
        Uf_ Ua(a);
```

$$f(a) := b^{\exp(a^y + c)} \quad (\% \text{ o146})$$

$$b^{\%e^{(ah)^y + c} - \%e^{c + a^y}} \quad (\% \text{ o147})$$

```
"Is "
      c
" an "
      integer
"? "
      no;
```

$$\%e^{a^y \log(b)y} \%e^{c + a^y} \quad (\% \text{ o148})$$

```
(%      Uf_ Ua(a);
i149)
"Is "
      c
" an "
      integer
"? "
      yes;
```


$$\%e^{a^y \log(b)y} \%e^{c+a^y} \quad (\% \text{ o149})$$

```
(%      f(a) := log(a);
i152)   Uf(a,h);
        Uf_Ua(a);

f(a) := log(a) \quad (\% \text{ o150})
```

$$\frac{\log(ah)}{\log(a)} \quad (\% \text{ o151})$$

$$\%e^{\frac{1}{\log(a)}} \quad (\% \text{ o152})$$

```
(%      kill(f);
i153)

done \quad (\% \text{ o153})
```

Alternative expressions for Uf_Ua(a) (with a=exp(z)):

```
(%      a = exp(z);
i158)   exp('diff( log(f(exp(z))) ,z,1));
        exp(diff( log(f(exp(z))) ,z,1));
        exp( a * diff(f(a),a,1) / f(a) );
        exp('diff( log(f(a)) ,log(a),1));

a = \%e^z \quad (\% \text{ o154})
```

$$\%e^{\frac{d}{dz} \log(f(\%e^z))} \quad (\% \text{ o155})$$

$$\%e^{\frac{\frac{d}{da} f(\%e^z)}{f(\%e^z)}} \quad (\% \text{ o156})$$

$$\%e^{\frac{a \left(\frac{d}{da} f(a) \right)}{f(a)}} \quad (\% \text{ o157})$$

$$\%e^{\frac{d}{d \log(a)} \log(f(a))} \quad (\% \text{ o158})$$

This leads to an alternative expression for the "normal" derivative, d/dx , in terms of the down derivative:

```
(%      exp(x + 'Df_Dx(x) - f(x));
i162)   z = log(x); g(x) = log(f(x));
        exp('Dg_Dz(x));
```

$$\%e^{Df_Dx(x)-f(x)+x} \quad (\% \text{ o159})$$

$$z = \log(x) \quad (\% \text{ o160})$$

$$g(x) = \log(f(x)) \quad (\% \text{ o161})$$

$$\%e^{Dg_Dz(x)} \quad (\% \text{ o162})$$

4 Up2

Now to be distributive over "up": $\text{up2}(a, \text{up}(b,c)) = \text{up}(\text{up2}(a,b), \text{up2}(a,c)) \dots$

```
(%      up2(a,b) := radcan( exp( up(log(a),log(b)) ) );
i163)
```

$$\text{up2}(a, b) := \text{radcan}(\exp(\text{up}(\log(a), \log(b)))) \quad (\% \text{ o163})$$

```
(%      up(up2(a,b),up2(a,c));
i165)   up2(a,up(b,c));
```

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \quad (\% \text{ o164})$$

$$\%e^{\log(a)^{\log(\log(c))+\log(\log(b))}} \quad (\% \text{ o165})$$

Bingo! And obviously also commutative, unlike say, tetration (\hat{y}^x or \hat{x}^y), which is neither. Identity for up2 (or up2_times):

```
(%      up2(e^ e,a);
i166)
```

$$a \quad (\% \text{ o166})$$

... or base to the power of the base. So one can imagine that for up3 it would be e^{e^e} ... etc.

```
(%      diff(up2(a,b),a);
i167)
```

$$\frac{\log(a)^{\log(\log(b))-1} e^{\log(a)^{\log(\log(b))}} \log(\log(b))}{a} \quad (\% \text{ o167})$$

```
(%      up2(x,x);
i170)  exp(log(x)^ log(x));
      x^ (log(x)^ (log(x)-1));
```

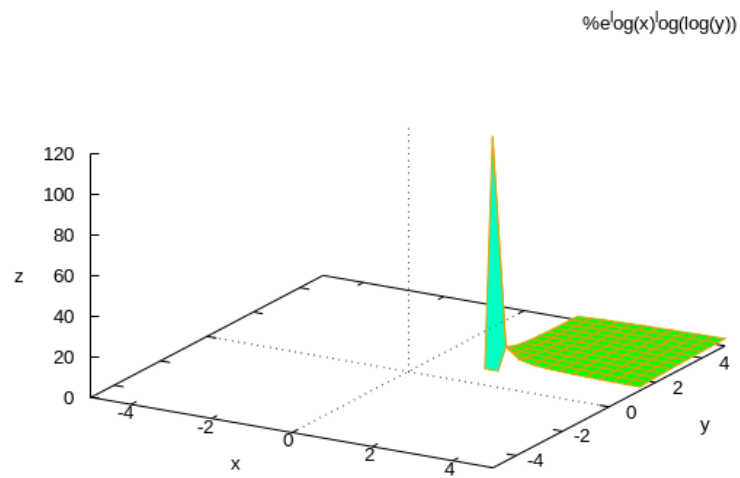
$$e^{e^{\log(\log(x))^2}} \quad (\% \text{ o168})$$

$$e^{\log(x)^{\log(x)}} \quad (\% \text{ o169})$$

$$x^{\log(x)^{\log(x)-1}} \quad (\% \text{ o170})$$

which is $\text{up}(x,x)[\log(x)(\log(x)-2)]$. Taking a look...

```
(% wxplot3d (up2(x,y), [x, -5, 5], [y, -5, 5])$
i171)
```

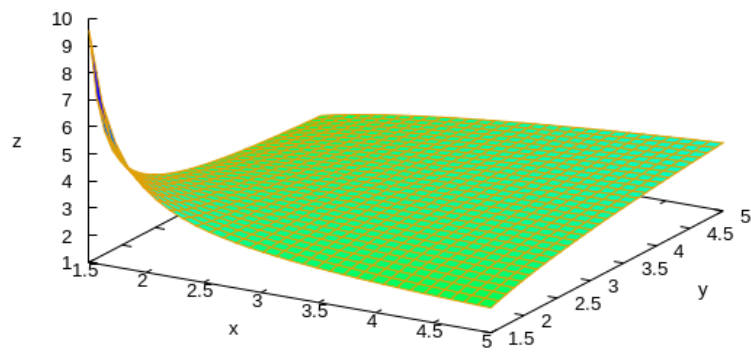


```
(% t171)
```

Problems now at x or $y = 1$.

```
(% wxplot3d (up2(x,y), [x, 1.5, 5], [y, 1.5, 5])$
i172)
```

```
%elog(x)log(log(y))
```

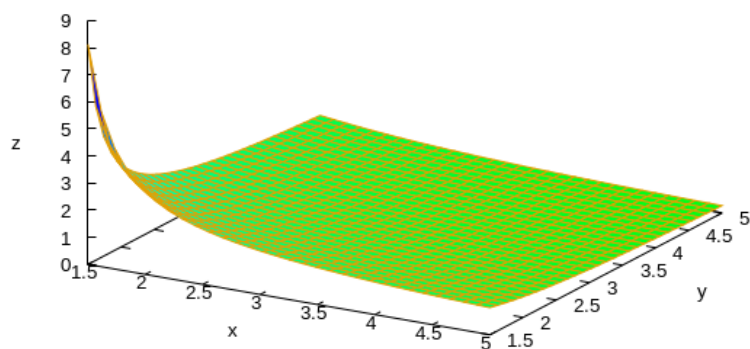


```
(% t172)
```

Comparing up2(x,y) with up(x,y) :

```
(% wxplot3d (up2(x,y)/up(x,y), [x, 1.5, 5], [y, 1.5, 5])$
i173)
```

```
%elog(x)log(log(y))/xlog(y)
```



```
(% t173)
```

```
(% up2(x,x)/up(x,x);
i177) log(up2(x,x)/up(x,x));
log(x)^log(log(x))-log(x)^2;
wxplot2d (log(up2(x,x)/up(x,x)), [x, 1.5, 5000])$
```

```
%e%elog(log(x))2-log(x)2
```

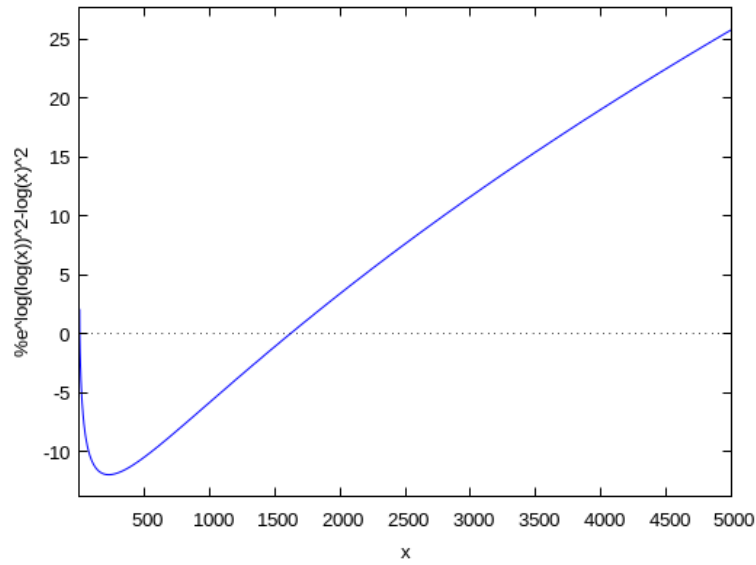
(% o174)

```
%elog(log(x))2-log(x)2
```

(% o175)

```
log(x)log(log(x))-log(x)2
```

(% o176)



(% t177)

```
(% factor(diff(exp(log(x)^ log(log(x))-log(x)^ 2),x,1));
i178)
```

$$\frac{2e^{\log(x)\log(\log(x))-\log(x)^2} \left(\log(x)^{\log(\log(x))} \log(\log(x)) - \log(x)^2 \right)}{x \log(x)} \quad (\% \text{ o178})$$

```
(% find_root('(%), x, 100, 300);
i179)
```

225.248329275192 (% o179)

Note also that up2(x,x) does eventually win over up(x,x)... at $x=e^{\hat{e}^2}$, where up2(x,x) is...

```
(% up2(exp(exp(2)),exp(exp(2)));
i181) float(%), numer;
```

e^{e^4} (% o180)

5.14843556263455710²³ (% o181)

Holy Avogadro, batman! Using up2_div for the "ratio"...

```
(%      up_div(a,b) := radcan( exp(log(a) / log(b)) );
i183)  up2_div(a,b) := radcan( exp( up_div(log(a),log(b)) ) );
```

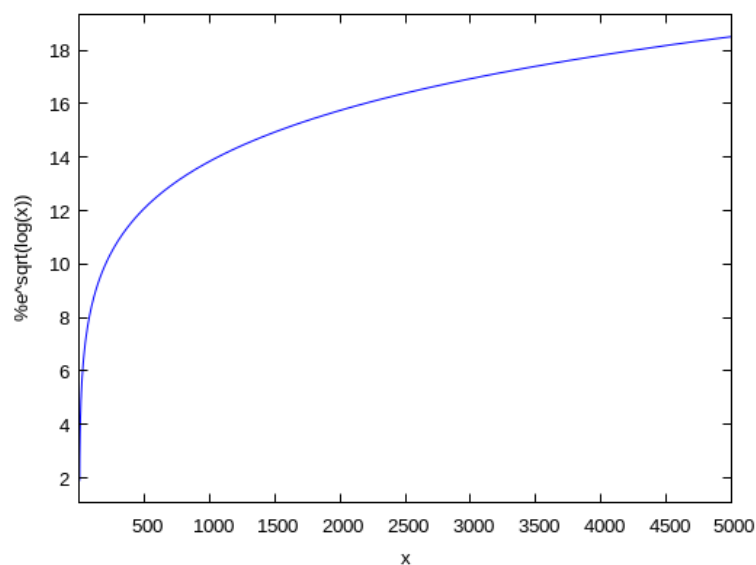
$$\text{up_div}(a, b) := \text{radcan} \left(\exp \left(\frac{\log(a)}{\log(b)} \right) \right) \quad (\% \text{ o182})$$

$$\text{up2_div}(a, b) := \text{radcan}(\exp(\text{up_div}(\log(a), \log(b)))) \quad (\% \text{ o183})$$

```
(%      up2_div(up2(x,x),up(x,x));
i184)
```

$$\%e^{\sqrt{\log(x)}} \quad (\% \text{ o184})$$

```
(%      wxplot2d (up2_div(up2(x,x),up(x,x)), [x, 1.5, 5000])$
i185)
```

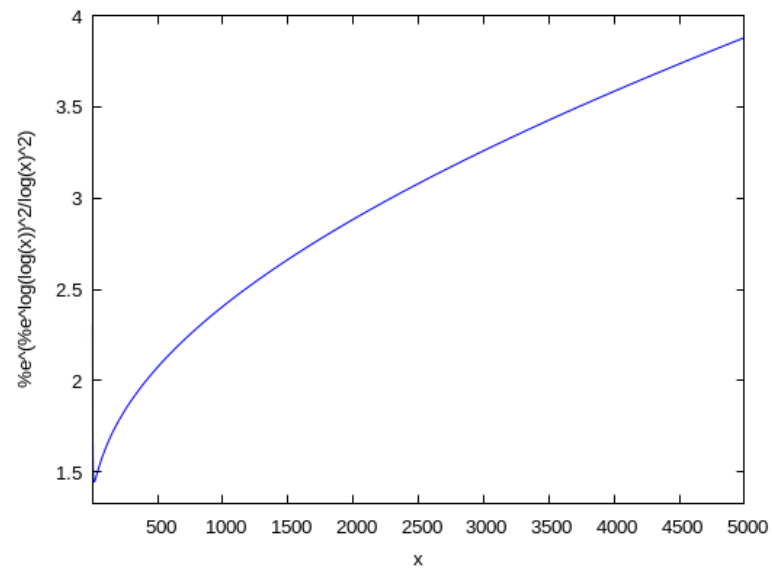


(% t185)

```
(%      up_div(up2(x,x),up(x,x));
i188)  exp(log(x)^(log(log(x))-2));
wxplot2d (up_div(up2(x,x),up(x,x)), [x, 3, 5000])$
```

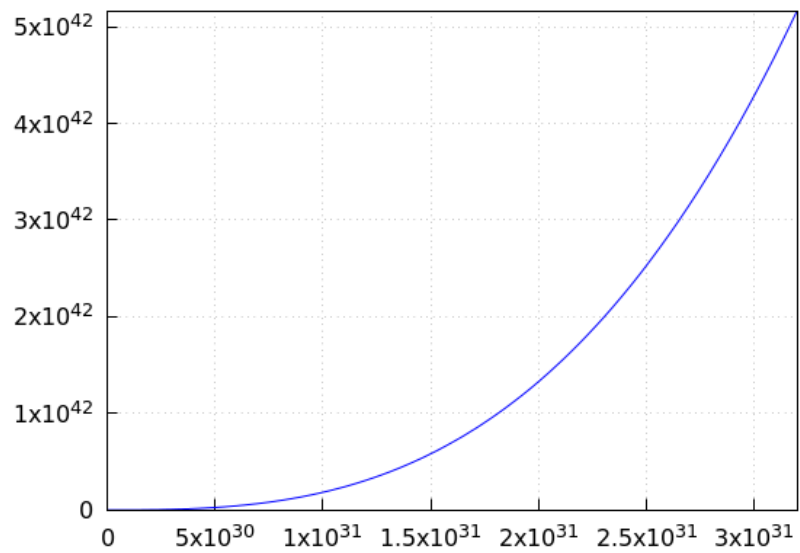
$$\%e^{\frac{\log(\log(x))^2}{\log(x)^2}} \quad (\% \text{ o186})$$


```
%e^log(x)^log(log(x))-2                                (% o187)
```



```
(% t188)
```

```
(% wxdraw2d( grid = true, nticks = 1000,
i189) parametric( up(t,t) , up2(t,t) ,
t,1.5,5000 ) )$
```



(% t189)