A Mathematical Perspective on the Vietnam War: Guerrilla Versus Conventional Combat

Thibault Busidan, Jonathan Kwan, George Zhang

Abstract

The VIETNAM model describes combat involving an army employing guerrilla warfare tactics against an army with conventional tactics. Performing linear stability analysis and finding both analytical and numerical solutions allow us to identify the initial conditions and parameters that predict each force's victory. We then apply the model to the Battle of Hue in the Vietnam War using historical statistics. We see that, while our findings are not entirely historically accurate, they illustrate the strength of a guerrilla force over a long duration of time and against better-equipped armies. These results highlight the difference between guerrilla warfare and conventional warfare.

1 Introduction

The Vietnam War from 1954 to 1975 was a military conflict between the communist North Vietnam with its Vietcong allies in the South and South Vietnam with its allies from the United States. [5] North Vietnam wished to unify the country under a communist regime while South Vietnam fought to retain relations with the West. These opposing ideologies ignited the spark that rapidly grew to a full-scale war. Afraid of letting another country fall into the grasp of communism, the U.S. committed more than 500,000 soldiers to Vietnam by 1969 [5]. However, these reinforcements appeared to be ineffective. Although both sides suffered heavy losses, the U.S. was forced to withdraw by 1973, and South Vietnam fell to the North just two years later. Not only did the Vietnam War have lasting social and political impacts, it also demonstrated the difficulty for larger, better equipped forces to defeat smaller guerrilla troops. The Vietcong were built around village guerrilla units [5] who hid and evaded U.S. forces instead of fighting in open battlefields.

The term “guerrilla” describes an unconventional war, derived from the term “little war” in Spanish and conceptualized by military theorist Carl von Clausewitz [1]. Many different types of guerrilla warfare exist. Despite the etymology behind it, guerrilla tactics are not exclusively used by smaller forces or in smaller conflicts. However, guerrilla tactics tend to be historically used by initially disadvantaged forces, such as tribes fighting against colonial powers or minorities revolting against ruling governments [1]. Thus, a central part of many applications of guerrilla warfare is ambushing the conventional force. We aim to view the Vietnam War through a mathematical lens by describing and analyzing a model for combat between guerrilla and conventional troops.

One battle that will be analyzed as an example is the Battle of Hue. As one of the battles in North Vietnam’s Tet Offensive, the Battle of Hue saw South Vietnam fighting to recapture the ancient city. Although the South Vietnamese and American forces won militarily, the destruction of the city and casualties of combatants and civilians alike made the Battle of Hue a common anecdote in discouraging American participation in the Vietnam War [2]. In approaching battles as important as this one, it would be worthwhile to analyze how many troops are necessary, given an enemy force’s strategy or efficiency, to secure victory and prepare for the loss of life that may entail.

Combat models are generally derived from the Lanchester Models for Combat or Attrition [6]. One example of a Lanchester model assumes that, for any force of combatants x:

$$\frac{dx}{dt} = -(OLR + CLR) + RR \tag{1}$$

Here, OLR represents the “Operational Loss Rate,” accounting for diseases, desertions, and other noncombat means by which troops are unable to fight. The “Combat Loss Rate” is represented by CLR and defined as the rate at which troops are incapacitated as a result of combat. Lanchester’s laws, namely the square law and linear law, are used to derive these based on the particular combat situation [4]. The last term, RR, represents the “Reinforcement Rate”, as more troops may enter a combat area over time.

This paper will overview the aptly named VIETNAM model, which models a combat situation between the Vietcong, a guerrilla force, and the U.S., a conventional force. The VIETNAM model uses elements from the CONCOM model, which reflects combat between two conventional forces, and the GUERCOM model, which reflects combat between two guerrilla forces. [6]

Let $x(t)$ denote the number of Vietcong troops at time t, and let $y(t)$ denote the number of U.S. troops at time t. The VIETNAM model assumes that:

$$\begin{aligned}\frac{dx}{dt} &= -axy - cx + e(t),\\ \frac{dy}{dt} &= -bx - dy + f(t),\end{aligned} \tag{2}$$

where $a, b, c, d \geq 0$ are constants and e, f are functions of t.

While the operational loss rate and reinforcement rates are similar between the two forces, they have notably different combat loss rates. For the conventional force, $y(t)$, the combat loss rate is proportional to x to reflect the number of Vietcong troops that are attempting to incapacitate U.S. troops. This is derived from Lanchester's square law, which tends to be applied to situations involving direct fire [4]. However, for the guerrilla force, $x(t)$, the combat loss rate is proportional to both x and y. Proportionality to y has the same reasoning for $x(t)$ as x does for $y(t)$. Introducing x as a factor considers how guerrilla tactics are dependent on the opposing force being unable to find and take aim at the guerrilla force. So, the more guerrilla troops there are, the easier it is to find them, given the amount of cover on the battlefield remains constant.

The combat effectiveness of the U.S. force against the hiding Vietcong is measured by a, and b is a constant measuring the same for the Vietcong against the U.S. force in the open. These parameters at which Vietcong or U.S. forces are incapacitated in combat, respectively. The effect of losses outside of combat on the Vietcong and U.S. forces are measured by c and d, respectively. The rates at which reinforcements for the Vietcong or U.S. arrive at the battlefield vary over time, being represented by $e(t)$ and $f(t)$ respectively.

In the sections below, we will be focusing on a simplified version of (2):

$$\frac{dx}{dt} = -axy, \tag{3a}$$

$$\frac{dy}{dt} = -bx, \tag{3b}$$

where constants $c = d = 0$ and functions $e(t) = f(t) = 0$. During a combat situation, the number of troops incapacitated for non-combat reasons will be negligible compared to the number incapacitated due to combat. The number of reinforcements can also be reduced to zero if we assume that the majority of troops are within firing range of an enemy combatant by the start of the battle. These assumptions will simplify and focus our analysis of the model on the differences between conventional and guerrilla strategies.

2 Thesis Statement

We aim to model the combat between a conventional army and a guerrilla army given various initial conditions and parameters. Specifically, we will apply our model to the Battle of Hue, one of the Vietnam War's most violent conflicts. To begin, we utilize linear stability analysis and find analytical and numerical solutions to fully understand the dynamics of our system. Then, we estimate the parameters and initial conditions for the Battle of Hue and draw conclusions from and regarding our model given the predicted trajectory of the battle.

3 Data and Methods

3.1 Data

Tung provides equations to quantify the conventional force's and guerrilla force's combat effectiveness, a and b [6]:

$$a = c_y \frac{A_g}{A_x} \tag{4}$$

$$b = c_x p_x \tag{5}$$

The combat effectiveness of the guerrilla force, b, is dependent on the guerrilla force's firing rate, c_x, and the probability that a guerrilla combatant's shot incapacitates a soldier, p_x. Tung approximates this probability to be approximately 0.1. The combat effectiveness of the conventional force, a, is similarly calculated, but the probability factor is more objective. The area where a single guerrilla combatant can be hit, A_g, is divided by the area in which the guerrilla force is evenly distributed, A_x, to result in the probability that a conventional combatant's shot incapacitates a guerrilla soldier. The firing rate of the conventional force, due to the advanced weaponry of the United States compared to the Vietcong's, is expected to be significantly higher than that of the guerrilla force. We will assume that $c_x = 1$ and

$c_y = 500$.

The aforementioned Battle of Hue saw sixteen battalions (approximately 12,800 troops) from South Vietnam and the U.S. versus ten battalions (approximately 8,000 troops) from North Vietnam and the Vietcong [2]. We will use these numbers as a heuristic for analysis, assuming a conventional force where $y_0 = 12800$ and a guerrilla force where $x_0 = 8000$. The area in which the battle took place is approximated to be that of Hue's citadel, which has a circumference of seven miles and therefore an area of approximately 108 million square feet [3]. Thus, we approximate $A_x = 1.08 * 10^8$. A large number of soldiers were incapacitated on both sides during this battle, measuring around 6000 troops from North Vietnam and the Vietcong and 3600 troops from South Vietnam and the U.S. [2].

3.2 Methods

Here, we describe three methods for obtaining solutions to our simplified system (3) described in the introduction in which neither army has operational losses nor reinforcements. Linear stability analysis, as is often applied to nonlinear systems of ODEs, is utilized first to gain an understanding of the phase space. Unfortunately, as we will see, linear stability analysis alone cannot predict the outcome of battles. Thus, we turn to exact analytical solutions and numerical solutions. In our simplified case, exact analytical solutions can be found following a procedure similar to problems 3g and 4d on homework 5. We obtain numerical solutions through the Runge-Kutta method mentioned in class and implemented through MATLAB's ODE45 function. We report no bifurcations in our system assuming positive constants a and b.

We begin by presenting a linear stability analysis of (3). To find the fixed points (x_*, y_*) of the system, we set the derivatives of (3a) and (3b) to 0 and solve the system of equations in x and y. (3a) and (3b) become

$$0 = -ax_*y_*, \tag{6a}$$

$$0 = -bx_*. \tag{6b}$$

If we consider (6b) first, then it is clear that $x_* = 0$. In that case, (6a) tells us that regardless of the y_* value, we will have a fixed point. Therefore, we have a line of fixed points at $(x_* = 0, y)$.

We then calculate the Jacobian matrix for our nonlinear system

$$J = \begin{pmatrix} -ay & -ax \\ -b & 0 \end{pmatrix}. \tag{7}$$

Evaluating the Jacobian at the fixed point $(0, y)$ yields

$$J|_{(0,y)} = \begin{pmatrix} -ay & 0 \\ -b & 0 \end{pmatrix}. \tag{8}$$

The eigenvalues of the Jacobian are $\lambda = 0, -ay$ with corresponding eigenvectors $(0, 1), (\frac{ay}{b}, 1)$. The eigenvector for $\lambda = 0$ points straight up and down towards the other fixed points. Thus, all of the fixed points along $x = 0$, assuming $y > 0$, are stable because the other eigenvalue, $\lambda = -ay$, is negative. Similarly, all of the fixed points along $x = 0$, assuming $y < 0$, are unstable. In this case, initial conditions where $y > 0$ will be attracted to the line of fixed points along $x = 0$, and initial conditions where $y < 0$ will be repelled from the line of fixed points.

We cannot identify whether trajectories intersect the x-axis first (meaning the guerrilla army wins) or the y-axis first (meaning the conventional army wins) from exclusively the linear stability analysis. Therefore, to identify exact analytic solutions, we divide (3b) by (3a) to get

$$\frac{dy}{dx} = \frac{-bx}{-axy}. \tag{9}$$

Solving this differential equation outputs the general solution

$$y = \pm\sqrt{\frac{2b}{a}x + C} \tag{10}$$

to the trajectories of our system (variable by a constant C). This parabolic trajectory intersects the origin when $C = 0$. Therefore, given $y > 0$, trajectories corresponding to $C = 0$ are attracted to the origin:

$$y = \sqrt{\frac{2b}{a}x}. \tag{11}$$

This trajectory is key in determining the effects of parameters and initial conditions on our model. First, we will determine what parameters are required for trajectories to go toward the origin given equal positive x and y initial conditions. We set $y = x = p_0 > 0$ in (11) and solve for a in terms of b:

$$a = \frac{2b}{p_0}. \tag{12}$$

Lastly, we compute the trajectories numerically in MATLAB using the ODE45 method. The error between the numerical and analytical solutions should be minimal given we select small time steps ($h = 0.05$). This is because ODE45, based on the Runge-Kutta method introduced in class, has an error of $O(h^4)$ globally. With the numerical trajectories on a vector field plotted using the command *quiver*, we obtained the phase plane. Looking at the phase plane, we can get a visual sense of the direction of a battle over time under varying initial conditions and parameters. We experiment with numerous initial conditions to see how the trajectories are affected by one army having or not having a numerical advantage. Using MATLAB we can also determine the global error in the numerical solution by calculating the maximum absolute value of the difference between the ODE45 and analytical solutions. To do so we must find the analytical solution corresponding to the numerical solution with parameters a, b and initial conditions x_0, y_0. Using (10) and solving for C yields

$$C = y_0^2 - \frac{2b}{a}x_0. \tag{13}$$

4 Results

A linear stability analysis of our system (3) revealed the existence of a line of fixed points along $x = 0$, where fixed points above $y = 0$ would be stable and fixed points below $y = 0$ would be unstable. However, knowing the location of the fixed points and their stability is not enough to identify the exact trajectories in the phase space, which we need to declare a winner for each battle. Finding numerical solutions and a vector field enables us to visualize and improve our understanding of these trajectories. Figure 1 shows what the phase plane and sample trajectories look like numerically when we used parameters $a = 1$, $b = 1$ and initial conditions $x_0 = 10, y_0 = 10, 8, 6, 4, 3, 2$.

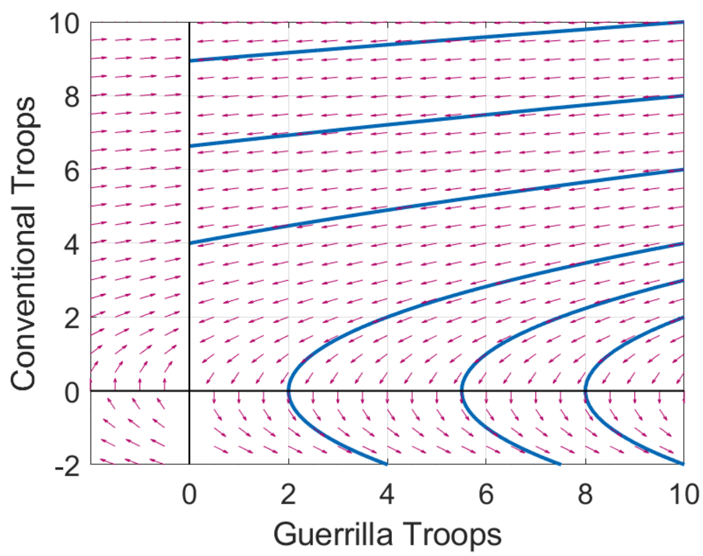

Figure 1: Phase plane of conventional vs. guerrilla forces given $a = 1, b = 1$ and initial conditions $x_0 = 10, y_0 = 10, 8, 6, 4, 3, 2$.

Now we can see that the trajectories are parabolic in nature, centered at $y = 0$, and open towards the right. The phase space also confirms the results of our linear stability analysis. The trajectories are attracted to the line $x = 0$ when above the x-axis and repelled away from $x = 0$ when below the x-axis. Notice that there exists one parabolic trajectory which intersects the origin. All initial conditions above will result in a win for the conventional army and any initial conditions below will result in a win for the guerrilla army.

Earlier in Section 3.2, we calculated the exact analytic trajectories in (10). Figure 2 displays three of the analytic trajectories ($y > 0$) with parameter values $a = 1, b = 1$ and

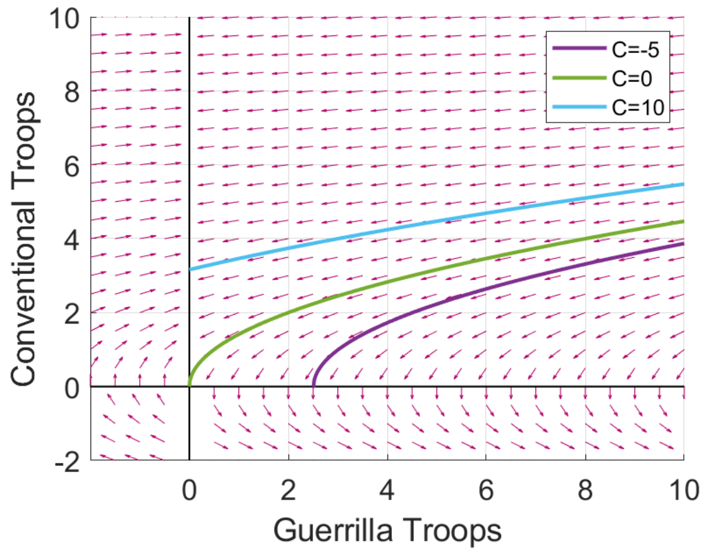

Figure 2: Analytic solutions of the conventional vs. guerrilla forces trajectories for $x > 0, y > 0$ given parameters $a = 1, b = 1$ and $C = -5, 0, 10$.

constant C values $-5, 0, 5$. As Figure 2 demonstrates, the analytic solution intersects the origin when $C = 0$. Any solutions with $C > 0$ will intersect the y-axis first (meaning the conventional army wins) and any solutions with $C < 0$ will intersect the x-axis first (meaning the guerrilla army wins). We can visually inspect the analytic trajectories to verify that they follow the vector field in Figure 2. We can also calculate C given parameter values and initial conditions using 13.

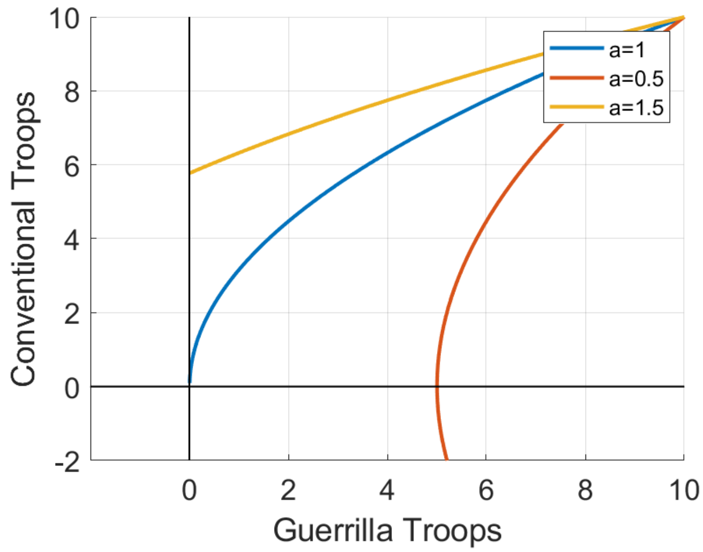

Figure 3: Phase plane of conventional vs. guerrilla forces with initial conditions $x_0 = 10, y_0 = 10$ and parameter values $b = 10$ and $a = 0.5, 1, 1.5$.

Looking more carefully at the parameters a and b given the analytic solution, we find that for an equal number of initial troops, condition (12) must be satisfied for the trajectory to reach the origin, resulting in a stalemate. Figure 3 demonstrates the phase plane using initial conditions $p_0 = x_0 = y_0 = 10$ with parameters $b = 10$ and $a = 0.5, 1, 1.5$. Given these initial conditions, we know that $a = \frac{1}{5}b$ is necessary for a stalemate. This fact is reflected in Figure 3 as the trajectory with $a = 1$ intersects the origin. With these initial conditions, the conventional force wins when $a > 1$, and the guerrilla force wins when $a < 1$. Condition (12) also tells us that a and b are not on the same scale. For a stalemate to occur given equal forces, a is inversely proportional to the initial condition p_0. With large starting armies, a will necessarily be much smaller than b for the strengths of both armies to be remotely comparable.

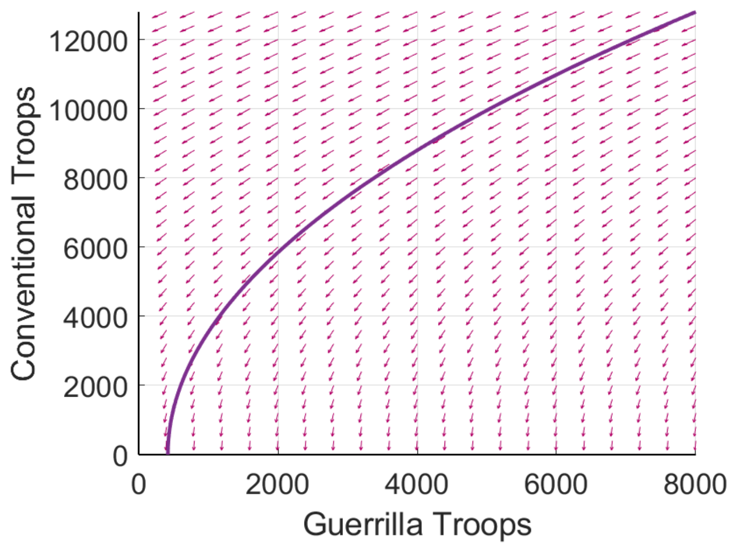

Figure 4: Phase plane of conventional vs. guerrilla forces with parameter values $a \approx 9.26 * 10^{-6}, b = 0.1$ and initial conditions $x_0 = 8000, y_0 = 12800$.

Equipped with a better understanding of the phase plane, we turn to inspect a realistic scenario at the Battle of Hue. As calculated in Section 3.1, the parameter values we estimated for that battle are $a \approx 9.26 * 10^{-6}, b = 0.1$ with initial conditions $x_0 = 8000, y_0 = 12800$. Figure 4 shows our prediction for the Battle of Hue over time on the phase plane. Our model predicts that after a long time, despite the superior firepower and troop count of the U.S., the Vietcong guerrilla troops will win this battle of attrition with very few remaining survivors.

For Figures 1, 2, 4, we found numerical solutions using MATLAB's ODE45 method. Numerical solutions inherently come with error, in this case, an $O(h^5)$ local error and an $O(h^4)$ global error. By using (13), we can find analytic solutions given specific parameters and initial conditions and compare them to our numerical solutions to find the approximate errors. We report a local error of approximately $1.82 * 10^{-12}$ and a global error of approximately 7.82 for Figure 4 from $t = 0$ to $t = 50$ with $h = 0.05$. The local error is below the expected order of $O(h^5)$, but the global error is much higher than one on the order of $O(h^4)$. Even with a higher than expected global error, both the local and global errors are relatively low enough such that numerical solutions may possibly be considered. However, we will rely upon analytical solutions in drawing our conclusions out of an abundance of caution.

5 Discussion

To verify the validity of our model, we look at the effects of varying initial conditions and parameters on the trajectory in the phase plane. The battle ends when the trajectory reaches either the x-axis, meaning a win for the guerrilla troops as the conventional army is wiped out, or the y-axis, meaning a win for the conventional troops as the guerrilla army is wiped out.

In Figure 1, we maintained the parameter values of $a = 1, b = 1$ and varied the initial condition y_0 between the values of $10, 8, 6, 4, 3, 2$ ($x_0 = 10$ was kept the same). When the number of starting conventional army soldiers is set to $y_0 = 10$, the conventional army won in a convincing fashion, with few casualties and the guerrilla troops completely wiped out. As the number of conventional army soldiers decreased, our trajectories logically reflected an increase in the number of conventional army casualties until $y_0 = 4$, where the tides of battle changed. For $y_0 = 4, 3, 2$, the superior number of guerrilla troops began to overwhelm the conventional army and the guerilla army wins convincingly with few casualties at $y_0 = 2$.

Similarly, we can examine the trajectories of battle under various parameter regimes. Figure 3 considered equivalent initial conditions of $x_0 = y_0 = 10$ and varies a between $0.5, 1, 1.5$ ($b = 10$ is kept constant). Since we know that a stalemate occurs when $a = 1$, we would expect a win for the guerrilla army if a, the combat effectiveness of the conventional army against the guerrilla army, is less than 1. Conversely, we would expect a win for the conventional army if $a > 1$. In fact, that is what we observe in Figure 3.

In a battle between two conventional armies, the number of troops remaining over time can be modeled by

$$\frac{dx}{dt} = -ay, \tag{14}$$

$$\frac{dy}{dt} = -bx \tag{15}$$

where a, b are positive constants [4]. Compared to our model, a and b hold the same general meaning; that is, increasing a will favor army y, and increasing b will favor army x. However, a is on the same scale as b which differs from our model. The condition for a stalemate between two conventional armies is described by the square law $ay^2 = bx^2$, where to stalemate an opponent twice your size, you must be four times as effective [6]. The condition for a stalemate for our model between one conventional army and one guerrilla army differs and is described by the parabolic law (11).

Our main situation of interest for this work is the Battle of Hue between the Vietcong guerrilla army and the U.S. conventional army. From our estimated parameters and initial troop level estimates, we obtain a predicted trajectory of the battle in Figure 4. Note that although we have been considering defeating the entirety of an opposing army as a victory, it is often the case that armies surrender or retreat before all personnel are incapacitated. Recall from Section 3.1 that in fact, by the end of the Battle of Hue, approximately 6000 out of 8000 Vietcong troops were incapacitated, and 3600 out of 12800 U.S. troops were incapacitated. Our model predicts that when there are 2000 Vietcong troops left in fighting conditions, there will only be about 5900 U.S. troops left standing, which is approximately $(12800 - 3600) - 5900 = 3300$ troops less than the actual count. Given the uncertain nature of our parameters, it is not such a surprise that our predictions do not closely match reality. But while the parameter estimates might not be the most accurate, it is clear that our model reflects the inability of a superiorly equipped U.S. conventional army to completely overwhelm a smaller but more agile Vietcong guerrilla army. Given the shape of the parabolic trajectory, our model affirms that the longer a battle lasts between these two armies, the more favored the guerrilla army becomes. This persistence of the Vietcong army is what eventually caused the U.S. to withdraw completely from Vietnam in 1973.

6 Conclusions

The VIETNAM model designed to model combat between a guerrilla army and a conventional army proves to be an accurate way to distinguish between the two tactics. The model's resultant trajectories, which are parabolas open to the right, illustrate the increasing strength of guerrilla combatants as their numbers dwindle. Analyzing our model has enabled us to derive equations that estimate the number of troops necessary to win given combat effectiveness and vice versa. These findings differed from models that assume both forces employ conventional strategies.

Our model, provides an approximation of how the forces involved in the Battle of Hue may have been affected throughout the conflict and could be affected if North Vietnam and the Vietcong continued to fight. Although the strategies employed at the Battle of Hue likely differed from a purely conventional force against a purely guerrilla force, the model we studied is meant to use the statistics from the battle as if this was the case. Using the Battle of Hue as an example further illustrates how military strength may not guarantee success against a guerrilla force as clearly as it does against another conventional force.

The simplified model analyzed in this paper excludes the operational loss rate and reinforcement rate, both of which will alter the dynamics of the system in noticeable ways. For instance, operational loss may decrease the casualties the guerrilla force face due to combat by reducing their numbers. Furthermore, more work could be done to give the model's combat effectiveness more accurate values or formulas. While the reasoning of differing weaponry or estimates of the area of the battlefield may have led to a working heuristic in our analysis of the Battle of Hue, there are many other factors in combat to consider.

Bibliography

[1] P. Allan and A. Stahel. *Tribal Guerrilla Warfare against a Colonial Power: Analyzing the War in Afghanistan.* Sage Publications, Inc., 1983.

[2] Crafted Knowledge. *Battle of Hue: American Casualties.* Gold Star Archive, 2022.

[3] Editors of Encyclopaedia Britannica. *Hue.* Encyclopedia Britannica, 2021.

[4] J. Lepingwell. *The Laws of Combat? Lanchester Reexamined.* The MIT Press, 1987.

[5] H. Spector. *Vietnam War.* Encyclopedia Britannica, 2022.

[6] K. Tung. *Topics in Mathematical Modeling.* Princeton University Press, 2016.