1 Definitions

1.1 Set

Set

Notation:
$X = \{x\}$
Qualitites:
- Is empty or consists of distinguishable subsets/elements $x \in X$.

1.2 Function

Function

Notation:
$f : X \to Y$
Components:
- Set Domain: X
- Set Codomain: Y

Qualitites:
- Mapping of X onto Y:
  A function f relates each $x \in X$ to exactly one element $f(x) \in Y$.

Classifications:
- injective
  For any $a, b \in X \quad a \neq b$ the result is different $f(a) \neq f(b)$.
- surjective
  For any $y \in Y$ there is a $x \in X$ such that $f(x) = y$.

1.3 Group

Group

Notation:
$\mathbb{G} \equiv (G, \circ)$
Components:
- Set: G
- Function: $\circ : G \times G \to G$

Qualitites:
- Associativity:
  For all $a, b, c \in G$: $a \circ (b \circ c) = (a \circ b) \circ c$
- Identity element:
  There exists an element $\mathbb{1}_\circ \in G$, for which $\mathbb{1}_\circ \circ a = a \circ \mathbb{1}_\circ = a$ for all $a \in G$.
- Inverse element:
  For each $a \in G$, there exists an element $b \in G$ such that $a \circ b = \mathbb{1}_\circ$.

Classifications:
- abelian
  For any $a, b \in G$, $a \circ b = b \circ a$.

Ideal

1.4 Ideal

Components:

- Group: $(G, \circ)$
- Set Ideal: $I_\circ \subset G$

Qualitites:

- $I_\circ$ "absorbs" element of G by application of $\circ$:
  $a \circ b \in I_\circ, \forall a \in G, b \in I_\circ$
  $b \circ a \in I_\circ, \forall a \in G, b \in I_\circ$

Algebraic Field

1.5 Algebraic Field

Notation:

$\mathbb{K} \equiv (K, +, \cdot)$

Components:

- Set: K
- Function Addition: $+ : K \times K \to K \qquad (\mathbb{1}_+ \equiv 0)$
- Function Multiplication: $\cdot : K \times K \to K \qquad (\mathbb{1}_\cdot \equiv 1)$

Qualitites:

- $(K, +)$ is a Group[abelian]
- $(K \setminus \mathbb{1}_+, \cdot)$ is a Group[abelian]
- Distributivity:
  $a \cdot (b + c) \equiv a \cdot b + a \cdot c$, which is equivalent to $\{\mathbb{1}_+\}$ is an Ideal $I_\cdot \subset K$.

Topological Space

1.6 Topological Space

Components:

- Set X:

Qualitites:

- Neighbourhood topology:
  1. Neighbourhoods of x: $N_x \equiv \{N \subset X | x \in N\}$.
  2. Any neighbourhood N_x contains a neighbourhood N'_x such that N_x is a neighbourhood of each point of N'_x.

Classifications:

- contractible
  The identity map on X is null-homotopic, i.e. it is homotopic to some constant map.

Hausdorff Space

1.7 Hausdorff Space

Components:

- Topological Space X:

Qualitites:

- Separation:
  Two distinct elements have some disjoint neighbourhoods.

1.8 Topological Manifold

Topological Manifold

Components:
- Topological Space X:
- Algebraic Field $\mathbb{K}$:

Qualitites:
- Each $x \in X$ has a neighbourhood which is a Topological Space[contractible].

Quantities:
- dimension
  Minimum number of functions $f : N_x \to \mathbb{K}$ required to uniquely describe x by its function values within N_x[1]. Such a minimal set of functions is a chart on N_x, its elements are called coordinate functions.

1.9 Differentiable Manifold

Differentiable Manifold

Components:
- Topological Manifold M:

Qualitites:
- The coordinate functions of any two intersecting neighbourhoods $N_x \bigcap N_y \neq \emptyset$ are C^k differentiable with respect to each other.
  This also implies the existence of a tangent space T_xM in each point $x \in M$: For an arbitrary function $f : M \to \mathbb{K}$, the derivatives along the coordinate functions gives a vector of the tangent space.

Quantities:
- differentiability class
  C^k: Up to k-th derivatives exist.

1.10 Smooth Manifold

Smooth Manifold

Components:
- Differentiable Manifold(differentiability class: C^∞) M:

1.11 Riemannian Manifold

Riemannian Manifold

Components:
- Smooth Manifold M:

[1] This has some similarities to the definition of an entropy: assuming the function's codomains consist only of $\{0, 1\}$, the dimension corresponds to the number of bits needed to encode all information about $x \in X$.