On the Intra-Household Public Goods and Fertility Trade-Off

Arnab K. Basu, Tsenguunjav Byambasuren and Nancy H. Chau*

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Abstract

This paper explores an infinite-horizon dynamic model of intra-household public goods, fertility, and child mortality in which child survival is subject to the provision of household public goods. An overlapping generation model we developed explains the empirically observed relationship between fertility rate, household-level public goods, and infant mortality. In this framework, the household public goods provision reduces child mortality and enhances the earning capacity of surviving children. The model demonstrates how intra-household public goods provision is associated with fertility decisions through its impacts on child health and parents' earning potential in the long run. The model with endogenous household public good provision and intergenerational transfer suggests that households make trade-offs between intra-household public goods and fertility conditional on the survival probability of offspring, a complement of child mortality.

JEL Classification: D11, D64, H41, J13, O12
Keywords: household public goods, fertility, child mortality, intergenerational transfers

*Basu: Charles H. Dyson School of Applied Economics and Management, Cornell University, 441 Warren Hall, Ithaca, NY 14853 & Institute for the Study of Labor (IZA) (email: arnab.basu@cornell.edu); Byambasuren: Charles H. Dyson School of Applied Economics and Management, Cornell University, 438 Warren Hall, Ithaca, NY 14853 (email: tb497@cornell.edu); Chau: Charles H. Dyson School of Applied Economics and Management, Cornell University, 201A Warren Hall, Ithaca, NY 14853 & Institute for the Study of Labor (IZA) (email: hyc3@cornell.edu).

1 Introduction

Households and family members, typically the parents, make numerous household-level decisions such as the purchase of house or car, number of children, and education of their own and their children (see Bergstrom, 1997; Strauss et al., 2000; Behrman, 1997, for literature on household decision making). These include decisions toward both pure and impure public and private goods, and in this paper, we focus on household decisions about household public goods. Although the literature considers children as the prime example of household-level public goods (Weiss and Willis, 1985; Chiappori and Weiss, 2001; Rasul, 2000), we separate fertility decisions from the other decisions about household public goods and analyze the interaction or trade-off between the two. We also model how a household can endogenously decide on the provision of household public goods[1] given its impact on child health, which could determine parents' earning potentials in the future.

This paper formulates an overlapping generations model of intra-household public goods in which the association between household-level public goods and fertility decisions has been explored from the economic perspective, in particular, through a trade-off between the provision of household public goods and fertility. There are two potential channels through which intra-household public good and fertility are associated with each other. First, households decide not to have a child to avoid a potential mortality incidence of their child due to a lack of investment in intra-household public goods. Second, consistent with the allocation of time approach (Becker, 1960, 1965), households would have to postpone their fertility decisions to be able to afford a household public goods provision by working additional hours.

Several interesting results emerge from our analysis. Notably, first, the maximal discounted utility was linearly proportional to the weighted average of utilities gained over the adulthood and old age periods when fertility level, one of two choice variables in our model, was optimally chosen by two-period lived individuals maximizing the present discounted value of their utility. Second, we find that household investment in health-beneficial intra-household public goods increases an individual's utility over the old age period through more consumption afforded by an increased intergenerational transfer receipt due to higher survival likelihood of the child. Third, we also find that an individual's utility during adulthood increases in response to investment in household public goods when the investment is less than the certain upper bound because too

[1] Throughout the paper, we use the term *household public good* mainly with fuels used at home or indoor air quality in mind, which is a key component so far overlooked. Previous empirical literature also uses the term *local (or intra-household) public good* to describe indoor air quality, arguing that larger households benefit more from a reduction in indoor air pollution or transition to clean fuels (Chaudhuri and Pfaff, 2002; Kumar and Viswanathan, 2007). In this paper, we thus use the terms of household public goods, household-level public goods, and intra-household public goods, interchangeably. Relatedly, Pfaff et al. (2004) examine a household-production model to demonstrate household-level environmental Kuznets curve, in particular, household Engel curve for indoor air quality. But it is worth noting that the household public good in our setting should be considered in general, not limited to household's energy choice.

much investment in household public goods lowers the adulthood consumption. Finally, there are two empirical implications of the model. First, investment in intra-household public goods from which each member of the household benefits and fertility are negatively associated, or there is a trade-off between household public goods and fertility decisions. Second, lack of household-level public goods provision is positively correlated with child mortality. These main findings are consistent with our empirical results that we found using India's DHS (1992-2016) data[2].

The main contributions of this paper to the long-standing economic literature on household fertility decisions, which goes back to the economic theory of population growth originated by seminal work of Malthus, are twofold. First, we add intra-household public goods as potential determinants of the household fertility decisions. In the neoclassical theory of fertility (Becker, 1960; Becker and Lewis, 1973), income, education, work status, and child costs, which include opportunity costs, play key roles in the fertility decision. The empirical results of this theory are mixed. There exist several empirical studies that support Becker's theory (Leibowitz, 1974; Heckman and Walker, 1990; Nguyen-Dinh, 1997). For instance, using a panel data of 3,458 households in the United States obtained from the Consumer Expenditure Survey, Caudill and Mixon (1995) show that family income, both wife's and husband's education, a family residence which captures variation in the opportunity cost of children, and work status are vital contributing factors in the fertility decision. Similarly, Wang and Famoye (1997) test the fertility hypothesis of Becker and Lewis using data of about 5,500 households from the Michigan Panel Study of Income Dynamics (PSID) by estimating a generalized Poisson regression model and find that wife's education level and working status, household income, and race are important determinants explaining the household fertility decisions. On the other hand, Melkersson and Rooth (2000), an example of empirical studies that oppose the Becker's theory of fertility, could not find any significant effects of proposed determinants on the number of births based on fertility data from the Swedish Level of Living Survey.

Second, the present paper also touches upon and adds to economic studies on the relationship between child mortality and fertility (Olsen and Wolpin, 1983; Wolpin, 1984; Barro and Becker, 1989; Sah, 1991; Boldrin and Jones, 2002; Doepke, 2005). Our model is consistent with the result from the Barro-Becker model, that is, a fall in child mortality rate enhances fertility. In the Barro-Becker framework, the main explanation of the fertility response to child mortality is a quantity-quality tradeoff: child mortality increases the cost of having a surviving child, and thus parents prefer smaller family to invest more on each child's education and quality in general, resulting in a fertility decline (Becker and Lewis, 1973; Barro and Becker, 1989). In addition to being consistent,

[2] We motivate our theoretical framework numerically and empirically by analyzing all four waves of publicly available household-level data from India's Demographic Health Survey (DHS) over the past 25 years since 1992. Using this data, rich in both temporal and spatial dimensions, we investigate the fertility behavior of Indian women and examine the relationship between household public goods (in the examples of household energy choice, indoor smoking behavior, and vaccination of the child) and fertility decisions among nationally representative households in India.

we extend the literature by interacting or relating the child mortality-fertility relationship with household public goods, e.g., household fuel choice or indoor air pollution, indoor smoking behavior, and child vaccination. Our model fundamentally differs from the Barro-Becker framework: lack of investment in household public goods causes child mortality, and thus parents (i) invest more in intra-household public goods to improve infant survival and to ultimately maximize intergenerational transfers that they expect to receive when they become old and (ii) postpone fertility decisions during adulthood (i.e., fertility declines). Hence, similar to the Barro-Becker model, fertility declines in response to growing child mortality; however, as a distinction from the Barro-Becker framework, our overlapping generations model focuses on child mortality in a family that lacks in the intra-household public good provision, which is motivated by our brief empirical analysis.

Household energy choice, the most significant contributor to indoor air quality with more than one-third of the world population relying on solid fuels, is an important example of an intra-household public good, in addition to children, that has been often ignored in the literature. Hence, we relate our analysis, especially the application of our theory, to the energy economics literature of household fuel choice considering this literature as one of the subfields where our theoretical framework relates and can be applied. This paper makes numerous contributions to economic literature on energy choice at the household level. Notably, we present what is, to our knowledge, the first economic model of fuel use characterizing the fuel choice as a household public good in the economic literature on household fuel use. The energy choice at the household level or indoor air quality can be viewed as household-level public good due to its non-rival and non-excludable characteristics[3].

The theoretical foundation of household energy choice and factors affecting the fuel-switching behavior of households has been limited, while there is a growing literature of empirical studies that attempt to identify the determinants driving household fuel transition (see Muller and Yan (2018) for a comprehensive survey of literature on conceptual and theoretical frameworks of household fuel choice in developing countries.) Household choice of cooking fuel was initially determined by household income according to the "energy ladder" theory, heavily relying on traditional income effects in consumer model (Heltberg, 2005; Chambwera and Folmer, 2007; Kowsari and Zerriffi, 2011; Lay et al., 2013; Das et al., 2014). However, the energy ladder model has been criticized by the fact that households in developing countries consume multiple fuels at the same time instead of completely transitioning from one fuel type to another (Masera et al., 2000; Ngui et al., 2011; Kroon et al., 2013, 2014). As a result, the "fuel stacking" theory has been proposed, suggesting that households use a

[3] Every family members benefit or harm from the household fuel choice without any competition, and it is expensive or impossible to prevent anyone from breathing air inside the house, while previous studies in household fuel use literature considered a fuel used at home as a non-food consumption good. As a public good, family members can have different willingness to pay for clean air or clean fuel, but they would consume the same quantity of air inside the house. Also child mortality is one of the negative externalities of indoor air pollution. Access to clean energy, therefore, exhibits the properties of public goods.

combination of clean and polluting fuels with a certain fraction[4].

Following those earlier theoretical models, more sophisticated utility maximization models of urban and rural (or agricultural) households emerged to better explain fuel choice patterns, a component of household public goods in our generic model. First, Edwards and Langpap (2005) and Gupta and Köhlin (2006) proposed urban households models in which utility-maximizing consumers subject to budget constraint choose commercial and non-commercial fuels simultaneously conditional on fuel prices. Second, another wave of studies including Heltberg et al. (2000), Chen et al. (2006) and, Manning and Taylor (2014) introduced market failures of rural households in developing countries (incomplete markets for fuels, agricultural products, labor, and finance or credit) in the theoretical framework of fuel choice. Those non-separated farm household models link the fuel choice with agricultural production and address a set of rural market failures. In opposition to earlier studies, in our theoretical framework, households maximize discounted utility by choice of two indicators including, first, investment in household-level public goods (for example, access to clean cooking fuel) that benefits the survival likelihood as well as future earning potentials of each offspring, and, second, fertility decisions conditional on the likelihood of survival. We solve the household maximization problem for each family dynasty over the infinite horizon.

The present study also contributes to literature on optimal provision of household public goods and household public good-child mortality link. First, the findings from previous empirical studies suggest that household fuel choice is endogenous in a mortality-fuel choice relationship (Duflo et al., 2008; Schindler et al., 2017; Basu et al., 2020). Since household fuel choice or indoor air quality can be generalized as household public good, motivated by those existing results, we present overlapping generations model in this paper to better understand the driving mechanism of household decisions about investment in household-level public goods and its association with child mortality or health in general[5]. Although there is a rich literature on the relationship

[4]Several energy studies conducted at the household level in developing countries documented the sociodemographic determinants of household fuel choice, which are consistent with the "fuel stacking" theory, including education and gender of the household heads in urban India (Farsi et al., 2007), tastes and cultural preferences in Burkina Faso (Ouedraogo, 2006), urban India (Farsi et al., 2007) and Mozambique (Arthur et al., 2012), household size in India (Rao and Reddy, 2007; Basu et al., 2020), Kenya (Ngui et al., 2011) and urban and rural areas in a set of developing countries (Kroon et al., 2013), and education or awareness in reducing the demand for traditional fuels (Heltberg, 2004, 2005; Rao and Reddy, 2007; Farsi et al., 2007; Chambwera and Folmer, 2007; Njong and Johannes, 2011; Alem et al., 2016). Alem et al. (2016) also find that households' economic status and price of alternative energy sources are important socioeconomic factors that determine household fuel choice in urban Ethiopia using a random-effects multinomial logit regression based on three rounds of the Ethiopian Urban Socioeconomic Survey over the period 2000-2009. Kuo and Azam (2019) empirically examined the stacking of fuels for rural and urban households in India and investigated the determinants of household cooking fuel choice using panel data from India Human Development Survey. Relatedly, their results from a random-effects multinomial logit model suggest that women's bargaining power and price of liquid petroleum gas (LPG) are found as significant contributors to urban household's decisions about the adoption of clean fuels.

[5]Basu et al. (2020) provide an extensive review of existing epidemiological and economic studies that investigate the simple association between indoor air pollution (IAP) and infant mortality and estimate the

between ambient air pollution, an important environmental example of a public good, or outdoor environmental conditions and fertility rates in developed and developing countries, no research theoretically addresses the interactions between intra-household public goods (e.g., household fuel choice or indoor air quality) and fertility behaviors[6]. As household public good determines the likelihood of child survival, it is an important factor for individuals or households to internalize in their fertility decisions. Our model thus is the first theoretical framework in which households optimize their fertility decisions for a given level of investment in household public goods, as well as clean fuel investment in the energy economics literature.

Second, as another contribution to the theoretical literature on household public good and its association with child mortality as well as fertility, we provide the first household public good model with dynamic structure. A dynamic investment decision model is needed to capture the decision about investing in household-level public goods, while all existing models of, for example, household fuel choice, are static. Like our setting, Manning and Taylor (2014) might be seen as an exception in the theoretical literature of fuel use in that it too attempts to depart from a static feature of the model and examines the decision whether invest in modern cooking and heating appliances (e.g., gas stoves or LPG purchase). But unlike our setting, investment in a gas stove is taken as exogenously given and, as such, it does not take into account endogenous investment in clean fuel–the central theme of our model in this paper. With a focus on the effect of migration on household's gas stove-investment decisions, Manning and Taylor (2014) provides a simplified two-stage model where the investment decision is limited only to gas stove investment due to increased income of remittances, while we accommodate a model with infinite periods, in each period, two generations live together. It is questionable to consider that their dynamic model endogenizes the investment decision of whether or not to buy a gas stove since it does not explicitly accommodate migration decision response to any change in gas stove investment, another major instance of a household public good provision. The dynamic feature of our setting instead aims to capture the intergenerational motivation of parents' investment in the household public goods and child's survival likelihood across generations.

Our intra-household public goods model is also consistent with experimental evidence from existing studies examining the determinants of an individual's willingness to pay for health-beneficial consumer products. For example, Levine et al. (2018) review the drivers of investment in products with health benefits and find that

causal effect of IAP on child mortality.

[6]See Nieuwenhuijsen et al. (2014), Vizcaíno et al. (2016) and Carré et al. (2017) for systematic surveys of epidemiological studies on the effect of outdoor air pollution on human fertility as studies on public good-fertility link in the example of ambient air quality. From the epidemiological perspective, indoor or household air pollution reduces the human fertility and even causes infertility through increasing oxidative stress that potentially leads to polycystic ovarian disease, which is one of the significant factors of infertility (Kelly, 2003; Lodovici and Bigagli, 2011; Apte and Salvi, 2016). On the other hand, Kishore and Spears (2014) shows that having a son as a firstborn in an urban Indian household increases the probability of using clean cooking fuel by 2 percentage points, i.e., enhances a household investment in intra-household public goods.

sales offer combined with free trials and time payments dramatically increases the uptake of fuel-efficient cookstoves in both rural and urban regions of Uganda. According to this technology adoption literature, one of the key factors for investment in health-beneficial household public goods, such as clean fuels and improved stoves, was found to be the provision of additional information on its health or economic benefits (Madajewicz et al., 2007; Meredith et al., 2013; Beltramo et al., 2015b). Hence, we model investment in household public goods conditional on health (e.g., probability of child survival in our model) and future economic or financial opportunities corresponding to the health conditions (transfers received during elder period)[7]. Recent theoretical and empirical studies also show that personality trait plays a critical role in human and physical capital investments. For example, Heckman et al. (2006) examine the effects of cognitive and non-cognitive skills on various behavioral and economic outcomes. For non-cognitive skills that affect individuals' investment decisions, recent empirical studies suggest that locus of control predicts health-related investments (Chiteji, 2010; Cobb-Clark et al., 2014). In addition, Abay et al. (2017) investigate the implication of non-cognitive skills (locus of control) on farmers' technology adoption decisions in Ethiopia and show that farmers with an internal locus of control are more likely to adopt new technologies or intra-household public goods such as improved cookstoves or clean energy, no indoor smoking behavior, and child vaccination.

The rest of the paper proceeds as follows. Section 2 provides background on the intra-household public goods, mortality and fertility rates, and household earnings in India, and presents the trend analysis of fertility rate associated with the household-level public good and household earnings. Section 3 formulates the model within which to examine the determination of household public goods and the potential for variation in household-level public goods to pass through into infant mortality or likelihood of survival and fertility decisions. Section 4 evaluates the welfare implications of the household public goods model. Section 5 concludes with some summary remarks.

2 Background

In this section, we discuss the trends in India's household fuel choice, indoor smoking behavior and child vaccination as major instances of investment in the intra-household public goods in our framework, fertility rate in relation to these household public goods, and household wealth. We chart the co-movement in investment in the household public goods, fertility, and earnings over time because such an exercise enables us to articulate the reasons for government intervention, if any, to influence private investment in the household public goods. For example, households in poverty and more polluted homes

[7] Another factor that explains the willingness to pay for health-beneficial products is the neighborhood or peer effect (Luoto et al., 2012; Meredith et al., 2013; Miller and Mobarak, 2014; Beltramo et al., 2015a). For example, Beltramo et al. (2015a) suggest that a neighbor's purchase of efficient cookstoves increases the odds of favoring the improved stoves by 17-22 percentage points based on a randomized control trial (RCT) in rural Uganda. However, they find that peer effects have no implications on final purchase decisions.

typically have higher fertility and a higher burden of disease due to exposure to indoor air pollution, keeping all other factors at constant (Bruce et al., 2006).

As illustrated in Figure 1a, total fertility rate[8] for households using polluting fuel for cooking is significantly higher than that for those using clean cooking fuel in India. The total fertility rate (TFR) represents the average number of births per woman, defined as the ratio of number of births to the total number of ever-married women of reproductive age (15–49 years old). The estimate of TFR ranges from 2.5% to 4.2% and from 1.7% to 3.1% for households using dirty and clean fuels, respectively, and the fertility rates for these two different households are statistically different from each other in all years over the period 1992–2016.

Similarly, we examine the relationship between the TFR and whether anyone is smoking inside the house alongside with other household members as another example of household-level public goods. Figure 1b demonstrates that having no individual who smokes indoors, which can be seen as an investment in the household public good, is negatively associated with fertility decisions, consistent with above result for access to clean energy. We then also check if the argument is true for the provision of vaccination for children, and Figure 1c likewise presents a negative relationship between the tendency to spend on a child's vaccination and to have a child.

In addition, we see that the TFR has been continuously decreasing over time while investment in household public goods (or use of clean fuel and child vaccination) generally has been increasing in the country (Figure 2), confirming that the household-level public goods provision in the case of clean energy access (no access) and vaccination (no vaccination) are negatively (positively) correlated with fertility across time. Additionally, Figure 3 shows that investment in the intra-household public goods (or greater use of clean fuels at homes in the left panel and vaccination of child in the right panel) is associated with greater household wealth over most of the time during the period 1992–2016.

There exist many studies that support the positive correlation between the use of clean fuel or household air quality, an important but largely ignored component of household public goods, and infant mortality. However, few empirical studies investigate the causal effect of cooking fuel choice on infant mortality. For example, Basu et al. (2020) provide empirical evidence on indoor air pollution-child mortality relationship and suggest that the use of polluting fuel for cooking significantly increases the risk of under-five and neonatal mortality using India's Demographic Health Survey (DHS) data.

[8]The Demographic and Health Survey (DHS) datasets for the period 1992–93, 1998–99 (in fact, some survey was collected in 2000 during the second wave), 2005–06, and 2015–16 was used to calculate the TFR for the three years before the survey, excluding the month of an interview (1-36 months before the survey). Appendix A.1 briefly describes the DHS data and provides a detailed definition of indicators used in our background checks.

These observations from the background checks and our trend analysis of the association between intra-household public goods and fertility decisions using all four rounds of India's DHS datasets of nearly a million nationally representative households over the period 1992–2016, taken together, offer descriptive motivations for our theoretical model. First, the descriptive results suggest that lack of household public good provision is associated with a higher fertility rate. Second, it indicates that greater investment in intra-household public goods is associated with lower child mortality rates. Third, in general, the co-movement of investment in the household public good and household earnings indicates that they are positively correlated.

Additionally, we empirically find that the data shows two "clusters" – high investment in household public goods and low fertility, and low investment in household public goods and high fertility. We show this by using an unsupervised machine learning technique – clustering – on a nationally representative data. In particular, we perform the K-modes clustering[9] given that we use categorical or dummy variables for our analysis. The optimal number of clusters (i.e., equilibria) is determined by plotting a scree plot, which shows the within-groups sum of squares[10] for different numbers of clusters. Unlike clustering algorithms such as X-means clustering, which is a variation of K-means clustering, any of the information criteria such as Akaike's Information Criterion (AIC) and Bayesian Information Criterion (BIC) is applicable for K-modes clustering to determine the optimal number of clusters. Hence, we use a scree plot visualization to determine the optimal number of clusters such that minimizes the within-groups sum of squares. The result of this analysis suggests that $K = 2$ is the optimal number of clusters (Figure 4).

With this as the selected optimal number of clusters, Figure 5 visualizes results from the K-modes clustering analysis using simple graphs[11]. For example, Figure 5b shows that the relationship between fertility and household investment in clean energy use at home is separately located (or clustered) at two different locations on the hyperplane: the first cluster is at the top-left where there is a low investment in clean energy and high fertility, and the second cluster is at the bottom-right where there is a high investment in clean energy and low fertility. Figure 5c also presents the same association and clusters between fertility and the status of no indoor smoking, which is another example of intra-household public goods. For the association between fertility and investment in child vaccination, we have slightly different picture in which most of the data points are located at the top-right where there are high child vaccination and high fertility, and points at top-left and bottom-right are not separately clustered into two groups; however, the clustering looks reasonably the same as above.

[9] See Appendix A.2 for a more detailed discussion of the K-modes algorithm.

[10] The within-groups sum of squares measures the variation of individual values or scores from each group or cluster center (e.g., mode, in our case) and is generally calculated as $SSW = \sum_k \sum_i (Y_{ik} - \bar{Y}_k)^2$ where Y_{ik} is the individual score and $\bar{Y}_k$ is the center of cluster k.

[11] Figure 6 presents the visualizations of K-modes clustering results for various values of $K \in \{1, 5\}$, including $K = 2$, which is the optimal number of clusters suggested by the scree plot for our 5-dimensional set of categorical objects or data vectors.

The average mortality and fertility rates of the first cluster (colored with black in Figure 5) are respectively 4% and 77%. Additionally, the average rates of mortality and fertility of the second cluster (colored with red in Figure 5) are zero and 39%, respectively. In the first cluster in which fertility is high, 18%, 25%, and 88% of households respectively invest in clean fuels, no indoor smoking behavior, and child vaccination. For the second cluster in which fertility is low, the percentages of households investing in clean fuels, no indoor smoking behavior, and child vaccination are 82%, 86%, and 89%, respectively. This confirms that households with many children lack in household-level public goods provision, and parents make tradeoffs between household public goods and fertility decisions as we see that fraction of households that invest in all our considered household public goods monotonically increases as the number of children in the household get fewer.

3 The Model

In this section, we first present our overlapping generations model of the intra-household public goods, and then define the equilibrium concepts, discuss the interpretation of our model findings and provide potential policy implications based on the comparative statics.

3.1 Setup

Consider a model of overlapping generations, in which each individual live for two-periods (adulthood and old age). In particular, there are L number of family lines in each period, and each family at each period t has two members, including adult and old age. Additionally, the collection of family lines at a given period t is referred to as a generation, while the collection of families across generations within a family line is referred to as a dynasty. Note that time is discrete, $t = 0, 1, \ldots$. Upon reaching adulthood, an individual allocates his or her earnings y_t at time t on three items. First, the representative individual transfers σ fraction of his/her income to the parent, $t_t = \sigma y_t$. Second, the agent spends $v n_t$ amount of income to raise n_t number of children, where v is the cost per child. Third, the adult spends $k_t \in [0, \bar{k}]$ amount of income to purchase a household public good, which (i) increases the likelihood of survival of each of the n_t number of children (π_{t+1}) from π_0 to $\pi_0(1 + k_t)$[12], and (ii) increases the earning capacity of surviving children (y_{t+1}) from y_0 to $y_0(1 + k_t)$.

An individual derives utility in period t from consumption c_t according to a utility function, U_t. We assume that the individual discounts each future period with a constant

[12]Since π_0 is the likelihood of child survival, then its complement, $1 - \pi_0$, is the likelihood of child mortality.

discount factor, $0 < \beta < 1$. The discounted utility of an adult over two time periods given a discount factor, β, therefore is

$$U_t = \log(c_t - \bar{c}) + \beta \log(c_{t+1} - \bar{c}),\ \beta \in (0, 1), \tag{1}$$

where $\bar{c}$ is a level of subsistence consumption. Since $y_t = y_0(1 + k_{t-1})$, and the adult spends c_t, vn_t, k_t and σy_t respectively on consumption, child rearing, household public goods and transfer, the budget constraint of the adult is

$$y_0(1 + k_{t-1}) = c_t + vn_t + k_t + \sigma y_0(1 + k_{t-1}). \tag{2}$$

Since $\pi_{t+1} = \pi_0(1 + k_t)$ is the likelihood of survival of each of the n children and $y_{t+1} = y_0(1 + k_t)$, the budget constraint of the individual during old age is

$$\pi_{t+1} n_t \sigma y_{t+1} = c_{t+1}, \tag{3}$$

where the amount of transfers (only source of revenue for old age) that an old age will receive from each of his or her surviving child is $t_{t+1} = \sigma y_{t+1}$.

3.2 Equilibrium and Steady State

We solve the utility maximization problem of each adult surviving into adulthood at time t, by maximizing the proposed utility function subject to the two budget constraints, taking agent's earning potential determined by investment undertaken by the parents, k_{t-1}, as given. Substituting the two budget constraints expressed in (2) and (3) into the utility function in (1) by defining c_t and c_{t+1} as a function of other inputs yields

$$\max_{\{k_t, n_t\}_{t=0}^{\infty}} U_t = \log\left[y_0(1 - \sigma)(1 + k_{t-1}) - vn_t - k_t - \bar{c}\right] + \beta \log\left(\pi_{t+1} n_t \sigma y_{t+1} - \bar{c}\right), \tag{4}$$

where the first term is adult's utility from consumption during adulthood while the second term refers to agent's utility gained over old age period. Maximizing U_t by choosing fertility n_t, the first order condition is given by, with assumption that k_t is provided for the moment

$$\frac{v}{y_0(1 - \sigma)(1 + k_{t-1}) - vn_t - k_t - \bar{c}} = \frac{\beta \pi_{t+1} \sigma y_{t+1}}{\pi_{t+1} n_t \sigma y_{t+1} - \bar{c}},$$

and utility maximizing or optimal fertility level, n_t^*, is defined as

$$n_t^* = \frac{\beta}{(1 + \beta)v}\left[y_0(1 - \sigma)(1 + k_{t-1}) - k_t - \bar{c}\left(1 - \frac{v}{\beta \pi_{t+1} \sigma y_{t+1}}\right)\right]. \tag{5}$$

If we substitute the utility maximizing fertility level, n_t^*, back into the utility function in (4), we derive the maximal discounted utility as

$$U_t(k_t, k_{t-1}) = (1+\beta)\log\left[y_0(1-\sigma)(1+k_{t-1}) - k_t - \bar{c}\left(1+\frac{v}{\pi_{t+1}\sigma y_{t+1}}\right)\right] + \\ + \beta\log\left(\pi_{t+1}\sigma y_{t+1}\right) + \log\left(\frac{1}{1+\beta}\right) + \beta\log\left[\frac{\beta}{(1+\beta)v}\right],$$

which is linearly proportional to the weighted average of the log of

$$y_0(1-\sigma)(1+k_{t-1}) - k_t - \bar{c}\left(1+\frac{v}{\pi_{t+1}\sigma y_{t+1}}\right),$$

and the log of

$$\pi_{t+1}\sigma y_{t+1},$$

plus a constant, $\log(1/(1+\beta)) + \beta\log(\beta/((1+\beta)v))$. We denote this a maximum $U_t(k_t, k_{t-1})$. Upon choosing the optimal level of fertility, or internalizing the fertility decision, a consumer's utility function or the consumer's maximal attainable utility is defined as a function of investment in household public goods in the following form

$$U_t(k_t, k_{t-1}) = (1+\beta)\log\left[y_0(1-\sigma)(1+k_{t-1}) - k_t - \bar{c}\left(1+\frac{v}{\pi_0 y_0\sigma(1+k_t)^2}\right)\right] + \\ + 2\beta\log(1+k_t) + (1+\beta)\log\left(\frac{1}{1+\beta}\right) + \beta\log\left(\frac{\beta\pi_0 y_0\sigma}{v}\right). \tag{6}$$

Now if we maximize $U_t(k_t, k_{t-1})$ in (6) by choice of k_t, the first order condition is given by

$$\frac{(1+\beta)\left[1-\frac{2v\bar{c}}{\pi_0 y_0\sigma(1+k_t)^3}\right]}{y_0(1-\sigma)(1+k_{t-1}) - k_t - \bar{c}\left[1+\frac{v}{\pi_0 y_0\sigma(1+k_t)^2}\right]} = \frac{2\beta}{1+k_t},$$

and we derive the law of motion equation, which illustrates the dynamic relationship between k_t and k_{t-1}, as follows

$$k_{t-1} = g(k_t) = \alpha k_t - \frac{\hat{k}_0}{(1+k_t)^2} + \tilde{k}_l, \tag{7}$$

where $\alpha = (1+3\beta)/(2\beta(1-\sigma)y_0)$ and $\hat{k}_0 = v\bar{c}/(\beta(1-\sigma)\pi_0\sigma y_0^2)$ are parameters, and $\tilde{k}_l = \bar{c}/((1-\sigma)y_0) + (1+\beta)/(2\beta(1-\sigma)y_0) - 1$ is a constant. In Appendix A.3, we show that such a relationship always exists in greater detail. Note that $g(\cdot)$ is strictly increasing in k_t, with

$$\frac{\partial g(k_t)}{\partial k_t} = \alpha + \frac{2\hat{k}_0}{(1+k_t)^3} > 0,$$

where $\alpha > 0$ and $\hat{k}_0 \geq 0$. For all $k_{t-1} \in g(\mathbb{R}_+)$, there exist a unique point whose image is k_{t-1}. Since $k_{t-1} \in g(\mathbb{R}_+)$, there exists some k_t such that $g(k_t) = k_{t-1}$. For all $k'_t \neq k_t$, from the strict increasing property of $g(\cdot)$, we must have $g(k'_t) \neq g(k_t)$. So the inverse function $g^{-1}(\cdot)$ exists or it is well-defined. Accordingly, we can write investment in intra-household public goods at any time t given k_{t-1} as

$$k_t = g^{-1}[g(k_t)] = g^{-1}(k_{t-1}) = f(k_{t-1}), \tag{8}$$

where $g^{-1}(\cdot) = f(\cdot)$ is solution to the following cubic equation,

$$\alpha k_t^3 - (k_{t-1} - 2\alpha - \tilde{k}_l)k_t^2 - (2k_{t-1} - \alpha - 2\tilde{k}_l)k_t - (k_{t-1} + \hat{k}_0 - \tilde{k}_l) = 0. \tag{9}$$

Since $g(\cdot)$ is strictly increasing, we must have $g^{-1}(k'_{t-1}) > g^{-1}(k''_{t-1})$ for any $k'_{t-1} > k''_{t-1}$, so $g^{-1}(\cdot) = f(\cdot)$ is also strictly increasing. It follows directly that an investment equilibrium k^* is a rest point or fixed point to (8) with

$$k^* = f(k^*).$$

With this in mind, a simple application of Brouwer's fixed point theorem (Debreu, 1959) provides the following proposition:

Proposition 1. *If $f(\cdot)$ is continuous, an investment equilibrium exists.*

To this end, (7) implies that given any feasible k_{t-1}, $k_t = f(k_{t-1})$ is

$$\begin{cases} 0 < k_t \leq \bar{k}, & \text{if } \tilde{k}_l - \hat{k}_0 < k_{t-1}; \\ 0, & \text{if } 0 \leq k_{t-1} \leq \tilde{k}_l - \hat{k}_0, \end{cases}$$

where $\tilde{k}_l - \hat{k}_0 = \bar{c}/((1-\sigma)y_0) + (1+\beta)/(2\beta(1-\sigma)y_0) - v\bar{c}/(\beta(1-\sigma)\pi_0\sigma y_0^2) - 1$. This indicates that the k_t schedule has a kink at $\tilde{k}_l - \hat{k}_0$ due to a non-negativity constraint of k_t and the k_t schedule takes strictly positive values beyond the point of inheritance $k_{t-1} = \tilde{k}_l - \hat{k}_0$.

Our findings so far take investment decision of generation $t-1$ as given. Figure 7 depicts the law of motion for investment in intra-household public goods, k_t, defined in (8) under the baseline and two alternative scenarios. The horizontal and the vertical axis respectively denotes the degree of investment in household-level public goods at time $t-1$ and t. The 45° line is the set of all possible steady-state solutions corresponding to

(8). There are two scenarios where non-zero stable fixed point of this dynamic system resides at $\bar{k}$ or optimal investment in the household public good is equal to maximum possible investment level. Begin with the baseline scenario where dynamically stable steady level of investment in household public goods resides at $\bar{k}$ when $f(k_{t-1})$ crosses 45^{o} line at the point of $k_{t-1} = k_t = \bar{k}$ after it reaches the $k_t = \bar{k}$ as numerically illustrated in Figure 7a. Alternatively, depending on parameter values, function k_t intersects 45^{o} line for the second time exactly at $\bar{k}$. One such scenario or equilibrium is shown in Figure 7b. Notice that, in these first two scenarios, we have k_{t-1}, which is weakly less than $\bar{k}$, being able to generate $\bar{k}$ in period t. Although there exist these two scenarios where k_t intersects 45^{o} line for strictly positive values of k_{t-1}, it is also possible that the k_t schedule intersects the 45^{o} line only once, at the origin, and never intersects the 45^{o} for any strictly positive values of k_{t-1} that is less than $\bar{k}$ (Figure 7c).

From (8), the investment schedule $f(k_{t-1})$ for inheritance $k_{t-1} \geq \tilde{k}_l - \hat{k}_0$ is both strictly increasing and convex in k_{t-1}[13]. In baseline scenario, the k_t schedule intersects the 45^{o} line three times, or there are total of three fixed points or equilibria where $k_t = k_{t-1}$. Interior steady states are given by the intersection of the 45^{o} line and the k_t schedule. Since $k_t \geq 0$, $k_t = f(k_{t-1})$ starts from the origin where it intersects the 45^{o} line for the first time. Then k_t function intersects the 45^{o} line at $k^* \in (0, \bar{k})$ for the second time. The investment in the household public good is bounded by construction, $k_t \in [0, \bar{k}]$, therefore, function k_t crosses the 45^{o} line at $k^{**} = \bar{k}$ for the last time. It can be readily verified that the steady states or fixed points k^* and k^{**} exist, and k^* is dynamically unstable or locally repelling, with $|(\partial k_t/\partial k_{t-1})|_{k_{t-1}=k^*}| > 1$, while k^{**} is dynamically stable or locally attracting, with $|(\partial k_t/\partial k_{t-1})|_{k_{t-1}=k^{**}}| < 1$. In each of the three scenarios except for the last one, there are two long-run dynamically stable equilibria (i) at $\bar{k}$ which is strictly greater than the repelling fixed point k^*, and (ii) at the origin which is strictly less than k^*. In particular, if k_{t-1} is less than the unstable equilibrium, the dynamic system will converge to zero. We have thus:

Proposition 2. *In the overlapping generations model with intra-household public goods that are complementary to child mortality, there exist two dynamically stable steady states, including one with zero investment (trivial) and another with investment at maximum possible level (non-zero or non-trivial).*

The result of Proposition 2 strongly suggests that there are two "clusters", which is supported by the data in the previous section.

[13]To see this, note that $\partial g(k_t)/\partial k_t > 0$ and $\partial^2 g(k_t)/\partial k_t^2 = -6\hat{k}_0(1 + k_t)^{-4} < 0$, which indicates that $g(k_t)$ is strictly concave, and $f(k_{t-1}) = g^{-1}(k_{t-1})$ is strictly convex because an inverse function of strictly increasing and concave function is strictly convex.

3.3 Comparative Statics

Effects of Investment in Intra-Household Public Goods on Fertility: The household makes a trade-off between investment in household-level public goods and fertility decisions as the utility-maximizing fertility level defined in (5) decreases in response to increase in investment in household public goods, assuming that k_{t-1} is given,

$$\frac{\partial n_t^*}{\partial k_t} = -\frac{\beta}{(1+\beta)v}\left[1 + \frac{2v}{\beta \pi_0 \sigma y_0 (1+k_t)^3}\right] < 0, \tag{10}$$

since $\pi_{t+1} = \pi_0(1+k_t)$ and $y_{t+1} = y_0(1+k_t)$. This provides the following proposition:

Proposition 3. *Households make tradeoffs between investment in intra-household public goods and fertility decisions.*

Effects of Investment in Intra-Household Public Goods on Utility: Taking the total derivative of (6), by implicit function theorem, the change in maximum utility from a change in household investment in household public goods can be written as

$$\frac{dU_t(k_t, k_{t-1})}{dk_t} = \frac{(1+\beta)\left[\frac{2v\bar{c}}{\pi_0 y_0 \sigma (1+k_t)^3} - 1\right]}{y_0(1-\sigma)(1+k_{t-1}) - k_t - \bar{c}\left[1 + \frac{v}{\pi_0 y_0 \sigma (1+k_t)^2}\right]} + \frac{2\beta}{1+k_t} > 0, \tag{11}$$

for $k_t < (2v\bar{c}/\pi_0 y_0 \sigma)^{1/3} - 1$ since all other terms in (11) are positive. The comparative statics of the maximum $U_t(k_t, k_{t-1})$ with respect to k_t suggests several important policy implications. First, investment in intra-household public goods (or increase in k_t) would directly decrease an individual's utility during the adulthood by reducing his/her disposable income for consumption; however, it can have a positive impact on utility over this period when $k_t < (2v\bar{c}/\pi_0 y_0 \sigma)^{1/3} - 1$ or investment in intra-household public goods is bounded above by some upper bound. This indicates that too much investment in household-level public goods reduces adults' maximum possible utility by lowering the expenditure potential on consumption goods. Second, investment in health-benefiting household public goods increases an individual's utility over an old age period at all times or for all non-negative values of k_t through improving the child's likelihood of survival and resulting increase in intergenerational transfer income or returns to investment in household public goods in the earlier period, i.e.,

$$\frac{\partial \beta \log\left[\pi_0 y_0 \sigma (1+k_t)^2\right]}{\partial k_t} = \frac{2\beta}{1+k_t} > 0,$$

since $\beta > 0$ and $k_t \geq 0$. This result is intuitive, given our two-generations setting, because an individual's income and thus consumption during the old age period increases in response to an increase in k_t, and elders do not allocate their income resource to investment in intra-household public goods. Hence, there is no reason that an individual's utility over an old age period to decrease in response to positive investment in k_t.

4 Welfare Analysis

Each adult makes decisions to maximize his or her welfare, rather than that of the family dynasty. We call this a "holdup problem". Consider therefore the problem of an adult faced with the decision whether investing in intra-household public goods. The maximal discounted utility of such an adult, as defined in (6) above, is given by

$$U(k^*) = (1+\beta)\log\left[y_0(1-\sigma)(1+k^*) - k^* - \bar{c}\left(1 + \frac{v}{\pi_0 y_0 \sigma (1+k^*)^2}\right)\right] + \beta\log\left[\pi_0 y_0 \sigma (1+k^*)^2\right] + \left\{\log\left(\frac{1}{1+\beta}\right) + \beta\log\left[\frac{\beta}{(1+\beta)v}\right]\right\},$$

where $k_{t-1} = k_t = k^*$. We maximize the utility function with respect to k^*, and the first order condition is given by

$$\frac{(1+\beta)\left[1 - y_0(1-\sigma) - \left(\frac{2v\bar{c}}{\pi_0 y_0 \sigma}\right)\frac{1}{(1+k^*)^3}\right]}{y_0(1-\sigma)(1+k^*) - k^* - \bar{c} - \frac{v\bar{c}}{\pi_0 y_0 \sigma (1+k^*)^2}} = \frac{2\beta}{1+k^*}.$$

Rearranging the above expression and collecting the terms yield

$$(1+3\beta)\left[1 - y_0(1-\sigma)\right]k^{*3} + \left[2\beta\bar{c} + 7\beta + 3 - 3(1+3\beta)y_0(1-\sigma)\right]k^{*2} + \\ + \left[4\beta\bar{c} + 5\beta + 3 - 3(1+3\beta)y_0(1-\sigma)\right]k^* - \\ -(1+3\beta)y_0(1-\sigma) + 2\beta\bar{c} + \beta + 1 - \frac{2v\bar{c}}{\pi_0 y_0 \sigma} = 0.$$

Then the steady state welfare of an adult can be found solving the above equality for k^*. Once we find the steady state welfare of an adult, we maximize the steady state welfare of the household to define under what condition will such k^* exceed (or fall below or match) the steady state equilibrium found in the previous section. This will be further explored in the future work.

5 Conclusion

This paper presents a dynamic model, which interlinks investment in intra-household public goods with both child mortality or infant survival and fertility decisions. By developing a model in which investment in household-level public goods affects infant survival based on the setup of overlapping generations of economic agents, we show that there is a trade-off between intra-household public goods and fertility decisions. The overlapping generations structure enables us to formulate the intergenerational effects of household public goods (e.g., indoor air pollution, indoor smoking behavior, and child vaccination) on long-term earning potential of both parents and children and child mortality or inverse of child survival likelihood as well as the potential tradeoffs between intra-household public goods and fertility decisions. We empirically motivate the key features of our model by leveraging multiple waves of a nationally representative household survey data.

We contribute to at least two distinct economic literature, with several extensions in some subfields in various directions. First, this paper adds to economic literature on household public goods by exploring the interaction between decisions about household public goods other than children, which has been considered as the main household public goods in the current literature, and the number of children or fertility. We model the household public good as it affects child mortality or survival using standard tools from the neoclassical growth theory of the consumer. Considering that indoor air pollution as the key component of intra-household public goods that have been undertaken, we argue that the present study contributes to the economic subfield of energy and fuel use models. In addition to endogenizing the investment in household public goods, second, we extend the economic literature on the model of fertility and household fertility decisions by endogenizing the fertility choice and showing that household investment in intra-household public goods is negatively associated with fertility rate.

Upon optimally choosing an investment in intra-household public goods or maximizing an indirect utility function with respect to household-level public goods, we show that the dynamic relationship of investment in household public goods exists. Notably, we find that there are two dynamically stable equilibrium outcomes both at the boundaries (instead of interior solutions) and non-zero dynamically stable steady-state equilibrium is uniquely determined at the upper bound of household investment in intra-household public goods.

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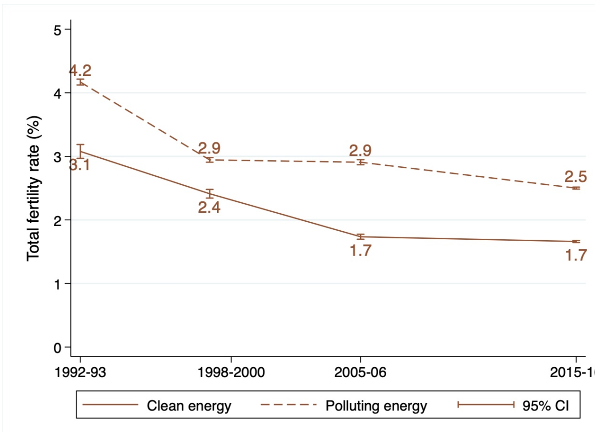
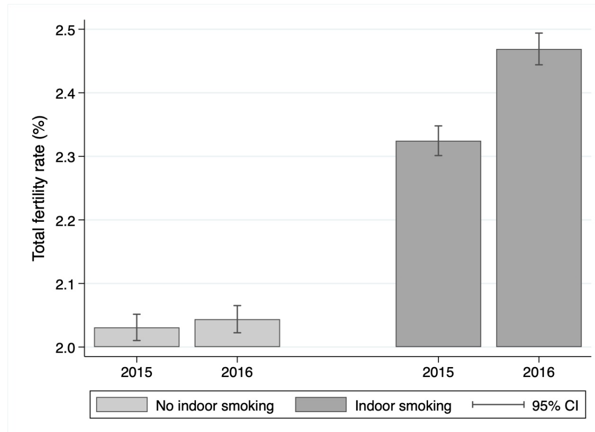
Figure 1: Trend of Total Fertility Rate associated with Household Public Goods

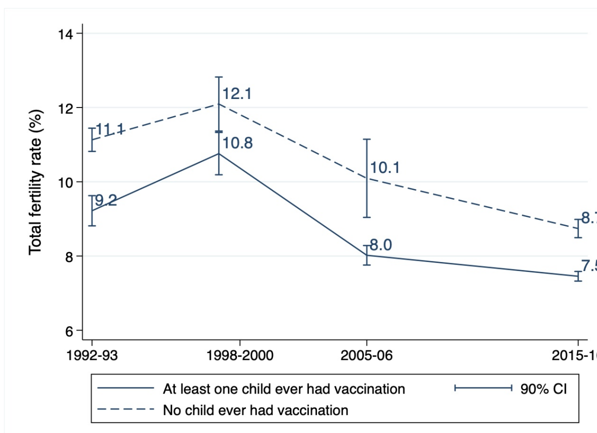
(a) Access to Clean Energy



(b) Indoor Smoking Behavior



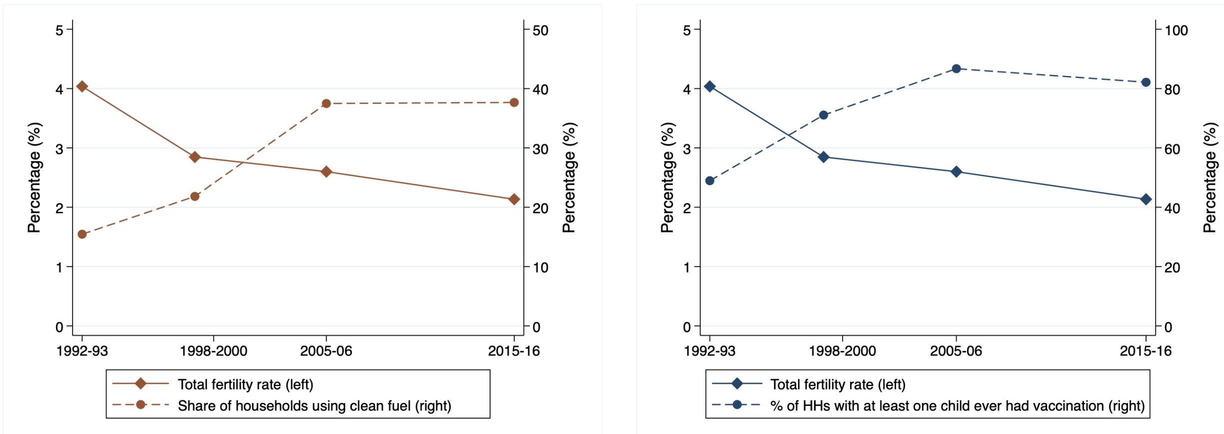
(c) Child Vaccination



Notes: Based on India's DHS datasets 1992–93, 1998–2000, 2005–06, and 2015–16. The figure presents the time trend of the total fertility rate (TFR) in India in association with intra-household public goods, including the types of fuels used for cooking (panel (a)), whether there is any household member smoking inside the house (panel (b)), and whether any one of the children is ever had vaccination (panel (c)). An indicator for household members' indoor smoking behavior is constructed based on a variable of the frequency of smoking by anyone of the family members in the house, which was only asked in the DHS-4 (2015–16). Despite this, we still can show the time trend of TFR in association with household members' indoor smoking behavior because a single wave of the DHS covers multiple years, 2015 and 2016 for DHS-4, for example. A binary variable of whether indoor smoking takes the value 1 if household members never smoke inside the house in household h, in village v, in district d of state s, in survey year t, and 0 otherwise. All four rounds of DHS collect data on vaccination of every under-five child of interviewed women, and we define a dummy variable indicating whether at least one of the children ever had a vaccination. The TFR for households at least one child ever treated with vaccination is different from that for those without any child ever received vaccination at the 95% level except for 1998–2000, where the two are significantly different at the 90% level.

Figure 2: Trends of Investment in Household Public Goods and Total Fertility Rates

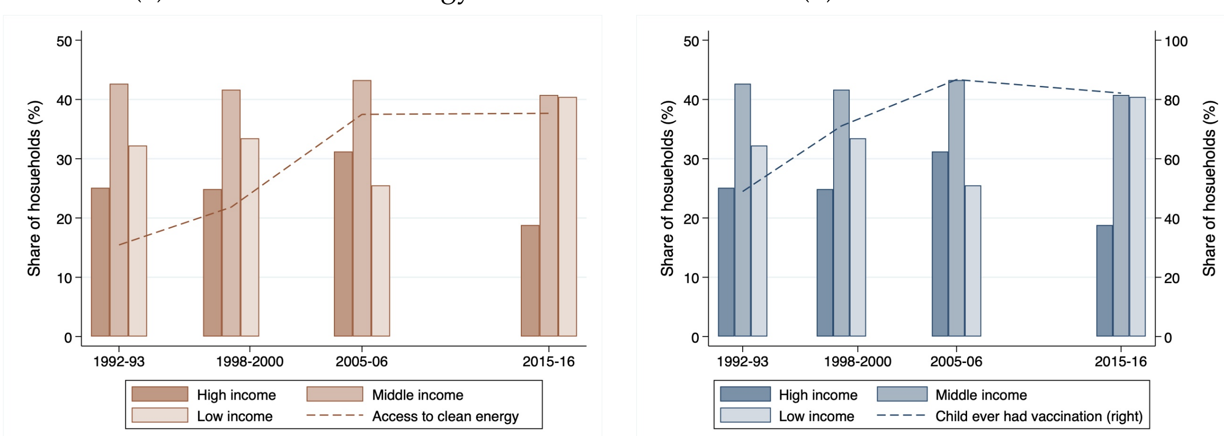
(a) Access to Clean Energy (b) Child Vaccination



Notes: The figure shows the co-movement of the tendency to invest in household public goods and total fertility rate over time, based on India's DHS data sets, in the examples of the use of clean fuels for cooking (left panel) and child vaccination (right panel). Panel (a) suggests that fertility has been continuously declining, while the share of households using clean fuels for cooking has been growing with level-off over the past ten years before DHS-4 (2015–16). Panel (b) similarly shows that child vaccination grows while fertility rate declines over the period before DHS-3 (2005–06), although share of households with at least one child ever had vaccination slightly drops during a decade between 2005–06 and 2015–16. We thus generally see that there is a negative relationship between fertility and investment in the household public goods. One may expect that intra-household public goods provision and fertility rates should be positively correlated, arguing that parents would prefer to have child and raise their children in a better or cleaner environment. However, it could be the opposite, i.e., the two are negatively associated because the decision to invest in household public goods makes households or individuals to work more instead of staying at home to afford clean fuels and vaccination and further leads to fertility drop.

Figure 3: Trends of Household Public Goods Provision and Household Earnings

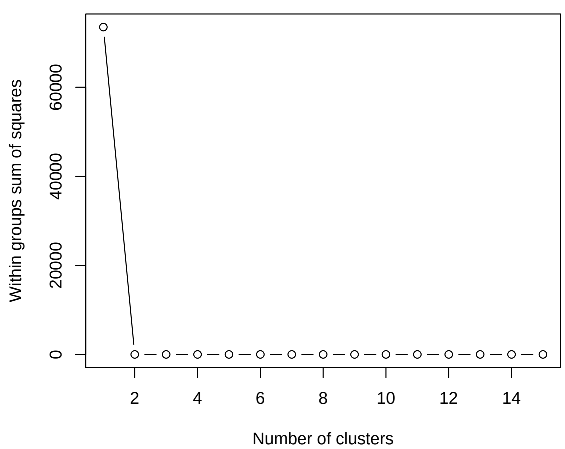
(a) Access to Clean Energy (b) Child Vaccination



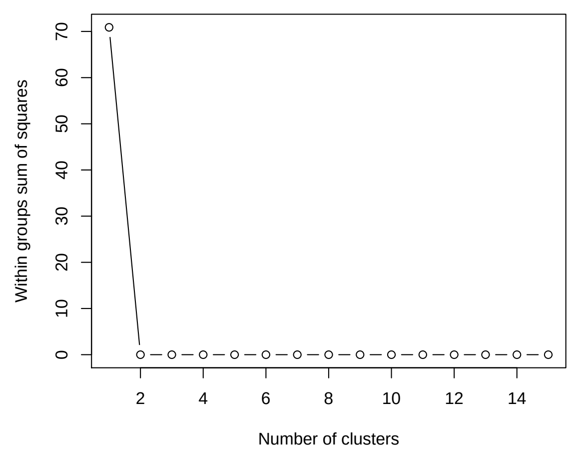
Notes: The figure shows the co-movement of household public goods provision and household wealth (described by the household wealth index) over time, based on India's DHS data sets. We observe that the share of middle-income households stabilizes at around 40% since the early 1990s. The share of high-income households has been growing over the considered period, while the share of low-income households has been declining since 1992–93, except for periods from 2005–06 to 2015–16. During a decade between 2005–06 and 2015–16, in contrast, the share of high-income households plummeted, whereas the share of low-income households shrinks down sharply. Hence, we see that a higher level of investment in the household public goods in the examples of household investment in clean fuel (left panel) and child vaccination (right panel) is associated with greater household earnings over the periods of first three rounds of DHS (from 1992–93 to 2005–06).

Figure 4: Scree Plot

(a) Full Sample



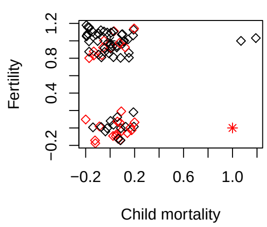
(b) Subsample



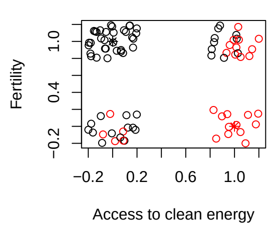
Notes: The figure depicts the scree plot to determine the optimal number of clusters. Panel (a) shows the scree plot based on the K-modes clustering performed on full sample of India's DHS-4 (2015–16) that consists of $N = 84,286$ ever-married women. Panel (b) similarly shows the scree plot of the K-modes clustering analysis using 0.1 percent of the full sample (i.e., $N = 84$ ever married women), which is used for the K-modes clustering visualized in Figure 5. The number of objects is five (i.e., $M = 5$), and fertility, child mortality, household fuel choice, indoor smoking behavior, and child vaccination status are included.

Figure 5: The K-Modes Clustering for Fertility, Child Mortality, and Household Public Goods ($K = 2$)

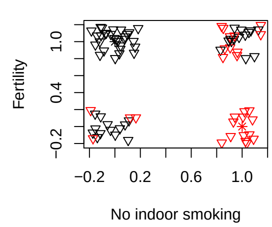
(a) Fertility vs. Child Mortality



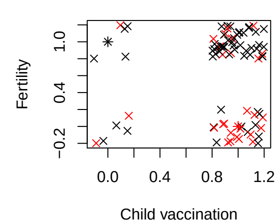
(b) Fertility vs. Access to Clean Energy



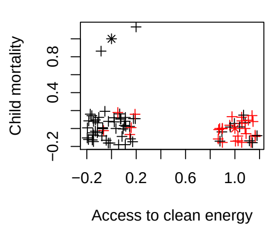
(c) Fertility vs. No Indoor Smoking



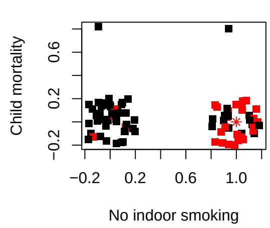
(d) Fertility vs. Child Vaccination



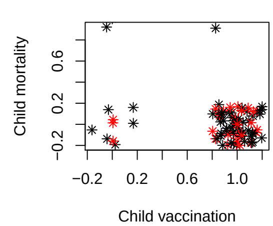
(e) Child Mortality vs. Access to Clean Energy



(f) Child Mortality vs. No Indoor Smoking



(g) Child Mortality vs. Child Vaccination



Notes: Based on a subsample of $N = 84$ (0.1 percent of the full sample) randomly drawn from India's DHS dataset 2015–16 for clearer illustration. Given that we have categorical data, the figure depicts the results from a 5-dimensional ($M = 5$) cluster analysis of fertility, child mortality, and considered three examples of intra-household public goods (including access to clean energy, indoor smoking behavior, and child vaccination) clustering using the K-modes algorithm ($K = 2$). The clusters are visualized with some jitter or random displacement for an illustration purpose.

Figure 6: The K-Modes Clustering for Fertility, Child Mortality, and Household Public Goods, $K \in \{1, 5\}$

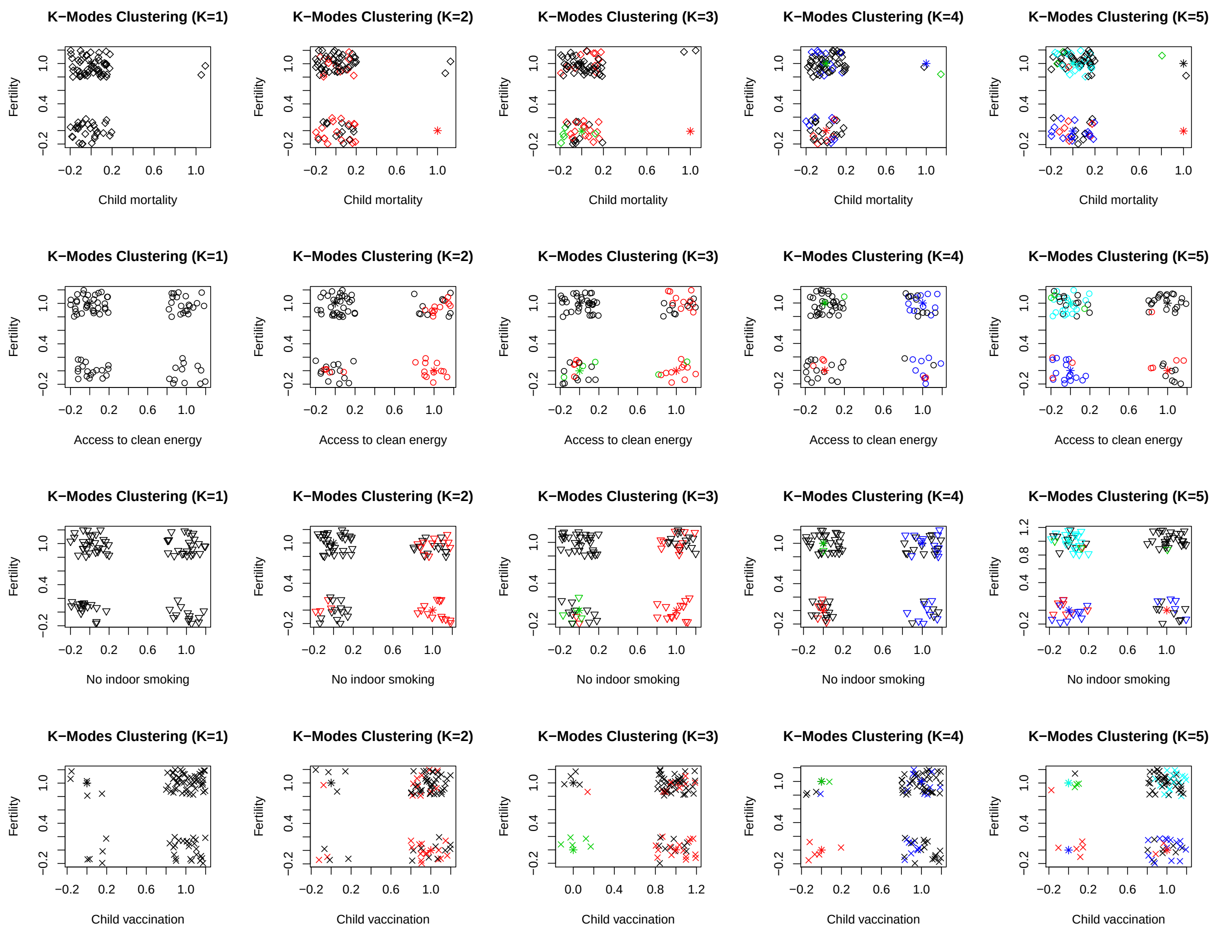
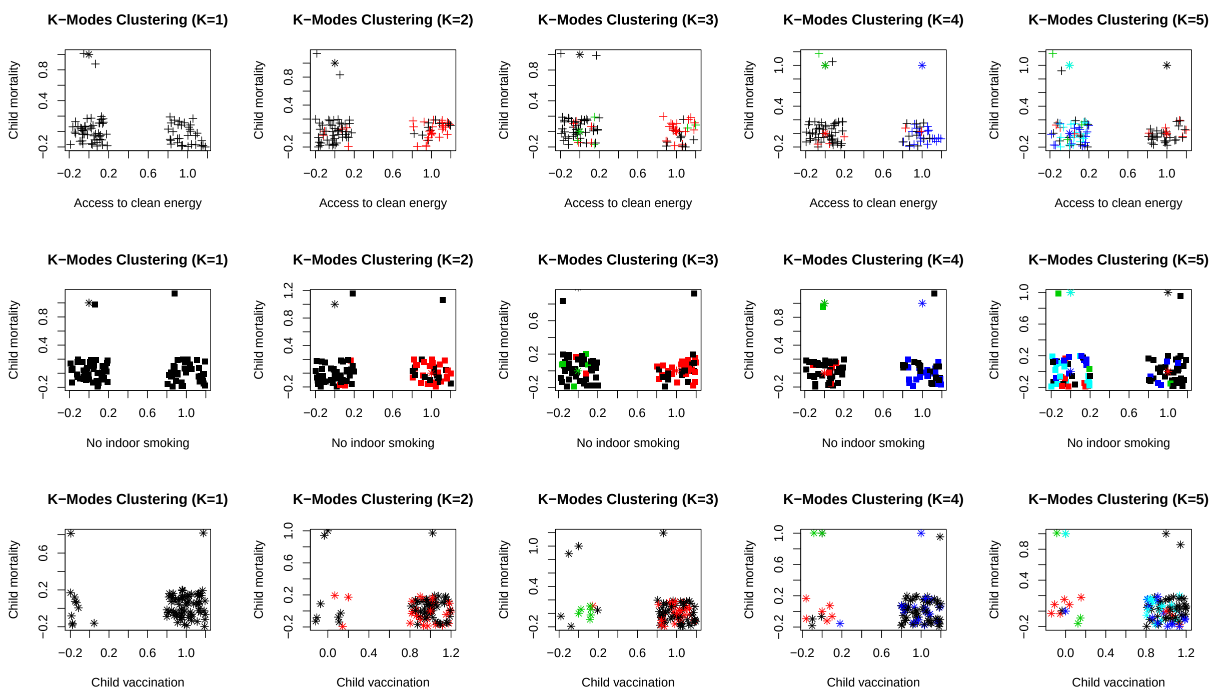


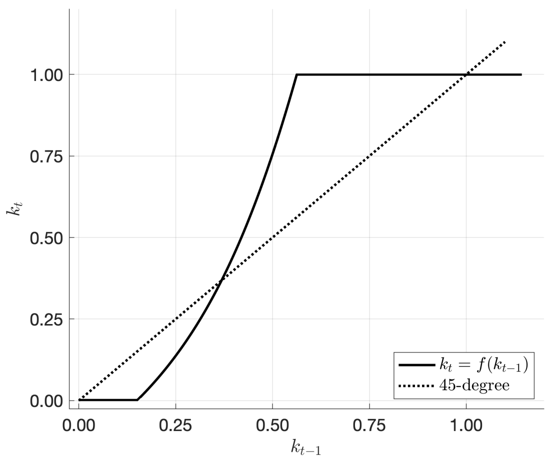
Figure 6: (Continued)



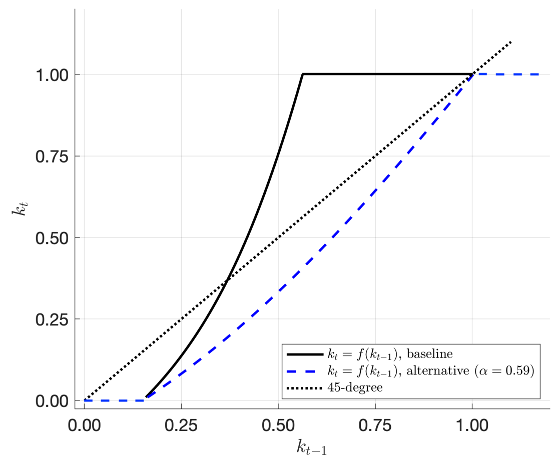
Notes: Based on a subsample of $N = 84$ (0.1 percent of the full sample) randomly drawn from India's DHS-4 (2015–16) dataset. The figure presents results from the K-modes clustering analysis for five objects of our interest ($M = 5$) and predetermined different numbers of clusters, $K \in \{1, 5\}$.

Figure 7: The Law of Motion for k_t

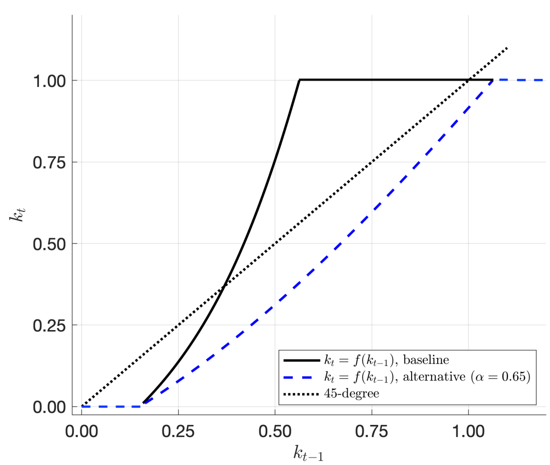
(a) Baseline:
k_t intersects 45° line after reaching $\bar{k}$



(b) Baseline vs. Alternative:
k_t intersects 45° line at $\bar{k}$



(c) Baseline vs. Alternative:
No Strictly Positive Stable Steady State



Notes: Based on expression in Equation (7), this figure plots the law of motion for investment in intra-household public goods, k_t, under the baseline (panel (a)) and two alternative scenarios (panels (b) and (c)). The baseline parameter calibration are $\alpha = 0.15$, $\bar{k} = 1$, $\hat{k}_0 = 0.35$, $\tilde{k}_l = 0.5$, and the point where k_t crosses the $k_t = 0$ line is thus $\tilde{k}_l - \hat{k}_0 = 0.15$.

A Appendix

A.1 Data

We use India's repeated cross-sectional data from the Demographic Health Survey (DHS) to empirically motivate the key features of our theoretical framework. All four waves of this qualitatively rich survey for the years 1992–93, 1998–2000, 2005–06, and 2015–16 have been utilized for the background checks and trend analysis. The full sample of four rounds of the DHS data contains 720,470 households with a total of 1,000,228 respondents or ever-married women under 49 years of age by the survey date from 38,436 villages/towns (or primary sampling units – PSUs) in all 36 states and union territories (UTs). The background checks based on trend analysis generally use a full sample, i.e., as many observations as possible. However, the K-modes clustering was performed based on a randomly drawn subsample, 0.1% of India's DHS-4 (2015–16) data, only in which indoor smoking behavior is recorded. We use a small portion of the data (84 observations) but still nationally representative for a clear illustration purpose.

Table A.1 provides detailed descriptions of the variables used in our analysis. It is important to note that the fertility measure is defined in two different ways. First, the fertility rate used in background checks and trend analysis is the total fertility rate (TFR). Second, we employ a dummy variable of fertility decisions for our clustering analysis, that is defined as whether a woman had a child over the period of 30 months before the date of the survey.

Table A.1: Variable Descriptions

Variable	Definition
(a) Instances of Household Public Goods, mother (j) × year of survey (t)	
Indoor air quality	Proxied by a dummy variable indicating whether household fuel choice or clean energy access. A total of ten different types of fuel are reported in India's DHS datasets as the main cooking fuels, and we reclassify these self-reported fuels used in the household into a dummy. 1 if a household in which the mother lives uses one of the clean fuels for cooking, 0 otherwise. Electricity, liquid petroleum gas (LPG) or natural gas, and biogas are counted as clean fuels, whereas kerosene, coal or lignite, charcoal, straw/shrubs/grass, wood, crop waste, and dung are considered as dirty fuels.
Indoor smoking behavior	A dummy variable indicating whether a household in which the mother lives has someone smoking inside the house. 1 if family members never smoked indoor, 0 otherwise.
Child vaccination	A dummy variable indicating whether a mother ever treated at least one of her under-five children with vaccination. 1 if a child ever had vaccination, 0 otherwise.
(b) Fertility Measures, mother (j) × year of survey (t)	
Fertility decision	A dummy variable of whether a woman had a child over the period of 30 months before the survey. 1 if a woman had a child within 30 months before the survey date, 0 otherwise.
Fertility rate, country (i) × year of survey (t)	The total fertility rate (TFR) – the sum of the age-specific fertility rate (or fertility rate for the seven five-year age groups from 15-19 to 45-49) for all eligible women in the household multiplied by five. When calculating the weighted estimates of the total fertility rates, we use a national women's sample weight to adjust the cluster sampling survey design.
(c) Child Mortality Measure, mother (j) × year of survey (t)	
Child mortality incidence	A dummy variable indicating whether a mother experienced mortality of a child born over the period 5 years (or 60 months) before the survey. 1 if a child died in the first 30 months of life, 0 otherwise. Twins are excluded.

Notes: The unit of observation is mother (j) × year of survey (t) unless otherwise specified.

A.2 *K*-Modes Clustering

To define the number of equilibria and explore the relationships between variables of interest, we use a simple machine learning technique that is an unsupervised clustering algorithm. Particularly, we use the K-modes clustering algorithm, and its paradigm is described here. The K-modes clustering algorithm is an extension of the K-means clustering algorithm (MacQueen et al., 1967) and was developed by Huang (1997, 1998) to cluster categorical data by dealing with modes instead of means.

The K-modes clustering method classifies the objects given by the data into K groups such that the distance from objects to the assigned cluster modes is minimized. The dissimilarity of two objects is determined by simple matching distance measure[14] for

[14]The simple matching distance also can be seen as the Hamming distance, which determines the dissimilarity between two string objects of equal length by counting the distinctive characteristics of the two

categorical objects, which counts the number of mismatches in all variables, and it is the first difference from the K-means clustering method that uses Euclidean distance function for continuous numeric values. Secondly, the K-modes method replaces mean of a cluster with the mode to compute the center of a cluster.

The steps of K-modes algorithm are then the same as those of K-means algorithm, except for the cost function it minimizes. Suppose $\boldsymbol{X} = [X_1 \cdots X_m \cdots X_M]_{(N\times M)}$ be a vector of categorical objects of interest or data vectors, where M is the number of objects or variables. The m-th and j-th objects can be written as $X_m = [x_{1m} \cdots x_{im} \cdots x_{Nm}]$ and $X_j = [x_{1j} \cdots x_{ij} \cdots x_{Nj}]$, where N is the total number of observations. It is considered that $X_m = X_j$ if and only if $x_{im} = x_{ij}$ for all $1 \le m, j \le M$ and $1 \le i \le N$, where m and j are indexes of objects and i is an index of observations.

The simple matching distance function of X_m and X_j represented above is defined as

$$d(X_m, X_j) = \sum_{i=1}^{N} \gamma(x_{im}, x_{ij}), \tag{B.1}$$

where

$$\gamma(x_{im}, x_{ij}) = \begin{cases} 0, & \text{if } x_{im} = x_{ij} \\ 1, & \text{if } x_{im} \neq x_{ij}. \end{cases} \tag{B.2}$$

The objective of clustering a vector of M categorical objects into K clusters is to find Y and S such that minimize the cost function $F(\cdot)$

$$F(S, Y) = \sum_{k=1}^{K} \sum_{m=1}^{M} s_{km} d(Y_k, X_m), \tag{B.3}$$

subject to

$$s_{km} \in \{0, 1\}, \quad \sum_{k=1}^{K} s_{km} = 1, \quad \text{and} \quad 0 < \sum_{m=1}^{M} s_{km} < M, \tag{B.4}$$

for all $1 \le k \le K$ and $1 \le m \le M$, where K is a predetermined number of clusters ($K \le M$), S is a $K \times M$ matrix of $s_{km} \in \{0, 1\}$, $Y = [Y_1 \cdots Y_k \cdots Y_K]$, and Y_k is center (e.g., mode) of the k-th cluster. In terms of optimization procedures, we first minimize the objective function $F(\cdot)$ with respect to S given Y, and then in the second step minimize $F(\cdot)$ with respect to Y by fixing the S at the level determined in the first step. We perform the K-modes clustering using `Rstudio` package `kmodes`.

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categorical objects.

A.3 Detailed Solution of the Model

In this section, we provide a solution to our model in greater detail. The consumer's utility maximization problem is defined as

$$\begin{aligned}
\max_{\{k_t, n_t\}_{t=0}^{\infty}} \quad & U_t = \log(c_t - \bar{c}) + \beta \log(c_{t+1} - \bar{c}) \\
\text{s.t.} \quad & y_0(1 + k_{t-1}) = c_t + v n_t + k_t + \sigma y_0(1 + k_{t-1}), \\
& \pi_{t+1} n_t \sigma y_{t+1} = c_{t+1}, \\
& 0 \le k_t \le \bar{k} \text{ and } c_t, n_t, k_{t-1} \ge 0 \text{ for all } t, \\
& y_0, \pi_0 \text{ given.}
\end{aligned}$$

To solve this maximization problem, we first find c_t and c_{t+1} from the two budget constraints and then plug them into the utility function. For the moment, we intentionally ignore non-negativity constraints and upper-bound constraint on investment in household public goods. Then we have the following unconstrained utility maximization problem:

$$\max_{\{k_t, n_t\}_{t=0}^{\infty}} U_t = \log\left[y_0(1-\sigma)(1+k_{t-1}) - v n_t - k_t - \bar{c}\right] + \beta \log\left(\pi_{t+1} n_t \sigma y_{t+1} - \bar{c}\right), \tag{B.5}$$

where the first term corresponds to utility during adulthood, while the utility during old age is captured by the second term. The first order condition with respect to fertility level, n_t, is given by

$$\frac{v}{y_0(1-\sigma)(1+k_{t-1}) - v n_t - k_t - \bar{c}} = \frac{\beta \pi_{t+1} \sigma y_{t+1}}{\pi_{t+1} n_t \sigma y_{t+1} - \bar{c}}.$$

After few manipulations, we can define the utility maximizing fertility level, n_t^*, as

$$n_t^* = \frac{\beta}{(1+\beta)v}\left[y_0(1-\sigma)(1+k_{t-1}) - k_t - \bar{c}\left(1 - \frac{v}{\beta \pi_{t+1} \sigma y_{t+1}}\right)\right].$$

To substitute the utility-maximizing fertility level, n_t^*, back into the utility function in (B.5), let us do this for adulthood and old age separately for notational economy and then we combine those two expressions together. First, utility during the adulthood is

$$\begin{aligned}
\log\left[y_0(1-\sigma)(1+k_{t-1}) - v n_t^* - k_t - \bar{c}\right] = \log\left(\frac{1}{1+\beta}\right) + \\
+ \log\left[y_0(1-\sigma)(1+k_{t-1}) - k_t - \bar{c}\left(1 + \frac{v}{\pi_{t+1}\sigma y_{t+1}}\right)\right].
\end{aligned}$$

Second, utility during the old age period is

$$\beta\log\left(\pi_{t+1}n_t^*\sigma y_{t+1}-\bar{c}\right)=\beta\log\left[\frac{\beta}{(1+\beta)v}\right]+\beta\log(\pi_{t+1}\sigma y_{t+1})+$$
$$+\beta\log\left[y_0(1-\sigma)(1+k_{t-1})-k_t-\bar{c}\left(1+\frac{v}{\pi_{t+1}\sigma y_{t+1}}\right)\right].$$

Combining these two expressions, we can write the maximal discounted utility as

$$U_t(k_t,k_{t-1})=(1+\beta)\log\left[y_0(1-\sigma)(1+k_{t-1})-k_t-\bar{c}\left(1+\frac{v}{\pi_{t+1}\sigma y_{t+1}}\right)\right]+$$
$$+\beta\log\left(\pi_{t+1}\sigma y_{t+1}\right)+\log\left(\frac{1}{1+\beta}\right)+\beta\log\left[\frac{\beta}{(1+\beta)v}\right].$$

Now we maximize this maximum $U_t(k_t,k_{t-1})$ by choice of k_t, and the first order condition (FOC) with respect to k_t is given by

$$\frac{(1+\beta)\left[1-\frac{2v\bar{c}}{\pi_0 y_0\sigma(1+k_t)^3}\right]}{y_0(1-\sigma)(1+k_{t-1})-k_t-\bar{c}\left[1+\frac{v}{\pi_0 y_0\sigma(1+k_t)^2}\right]}=\frac{2\beta}{1+k_t}.$$

By rearranging and collecting the terms, we obtain

$$2\beta y_0(1-\sigma)k_{t-1}=(1+3\beta)k_t-\left(\frac{2v\bar{c}}{\pi_0 y_0\sigma}\right)\frac{1}{(1+k_t)^2}-2\beta y_0(1-\sigma)+2\beta\bar{c}+\beta+1,$$

and if we divide the both sides of above expression by $2\beta y_0(1-\sigma)$ where we assume that $\sigma\neq 1$, we define the following law of motion equation which demonstrates the dynamic relationship between k_t and k_{t-1}

$$\begin{aligned}k_{t-1}&=\left[\frac{(1+3\beta)}{2\beta(1-\sigma)y_0}\right]k_t-\left[\frac{1}{\beta(1-\sigma)\pi_0\sigma}\right]\left(\frac{v}{y_0}\right)\left(\frac{\bar{c}}{y_0}\right)\frac{1}{(1+k_t)^2}+\\&\quad+\frac{1}{1-\sigma}\left(\frac{\bar{c}}{y_0}\right)+\left[\frac{1+\beta}{2\beta(1-\sigma)}\right]\frac{1}{y_0}-1,\\&=\alpha k_t-\frac{\hat{k}_0}{(1+k_t)^2}+\tilde{k}_l,\\&=g(k_t).\end{aligned}$$