MA319 — 偏微分方程

Assignment 10

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— SJTU (Fall 2019)

习题 3.1/6

用分离变量法求解由下述调和方程的第一边值问题所描述的矩形平板 $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b)$ 上的稳定温度分布:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \\ u(0,y) = u(a,y) = 0, \\ u(x,0) = \sin \dfrac{\pi x}{a}, \quad u(x,b) = 0. \end{cases}$$

设 $u(x,y) = X(x)Y(y)$, 代入得

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

则有两个常微分方程

$$X''(x) + \lambda X(x) = 0,$$

$$Y''(y) - \lambda Y(y) = 0.$$

关于 x 的方程的特征值和特征函数为

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x.$$

代入 $X(0) = 0$, $X(a) = 0$ 得

$$C_1 = 0, \quad C_2 \sin \sqrt{\lambda} a = 0,$$

$$\lambda = \lambda_k = \frac{k^2 \pi^2}{a^2}, \quad X_k(x) = C_k \sin \sqrt{\lambda} x = C_k \sin \frac{k\pi}{a} x, \quad k = 1, 2, \cdots.$$

代入关于 y 的常微分方程可得

$$Y''(y) - \frac{k^2\pi^2}{a^2} Y(y) = 0,$$

$$Y(y) = A_k e^{\frac{k\pi}{a} y} + B_k e^{-\frac{k\pi}{a} y}.$$

方程的通解为

$$u(x,y) = \sum_{k=1}^{\infty} X(x)Y(y) = \sum_{k=1}^{\infty} \left(A_k e^{\frac{k\pi}{a} y} + B_k e^{-\frac{k\pi}{a} y} \right) \sin \frac{k\pi}{a} x.$$

代入初值条件得

$$u(x,0)=\sum_{k=1}^{\infty}(A_k+B_k)\sin\frac{k\pi}{a}x=\sin\frac{\pi x}{a},$$

$$u(x,b)=\sum_{k=1}^{\infty}\left(A_k e^{\frac{k\pi}{a}b}+B_k e^{-\frac{k\pi}{a}b}\right)\sin\frac{k\pi}{a}x=0.$$

解得

$$A_k+B_k=\begin{cases}1 & k=1,\\ 0 & k>1,\end{cases}$$

$$A_k e^{\frac{k\pi}{a}b}+B_k e^{-\frac{k\pi}{a}b}=0.$$

化简得

$$A_k=\begin{cases}-\dfrac{e^{-\frac{\pi}{a}b}}{e^{\frac{\pi}{a}b}-e^{-\frac{\pi}{a}b}} & k=1,\\ 0 & k>1,\end{cases}\qquad B_k=\begin{cases}\dfrac{e^{\frac{\pi}{a}b}}{e^{\frac{\pi}{a}b}-e^{-\frac{\pi}{a}b}} & k=1,\\ 0 & k>1.\end{cases}$$

故

$$u(x,y)=\frac{-e^{-\frac{\pi}{a}(b-y)}+e^{\frac{\pi}{a}(b-y)}}{e^{\frac{\pi}{a}b}-e^{-\frac{\pi}{a}b}}\sin\frac{\pi}{a}x=\frac{\sinh\frac{\pi}{a}(b-y)}{\sinh\frac{\pi}{a}b}\sin\frac{\pi}{a}x.$$

习题 3.1/7

在膜形扁壳渠道闸门的设计中，为了考察闸门在水压力作用下的受力情况，要在矩形区域 $0\leqslant x\leqslant a$，$0\leqslant y\leqslant b$ 上求解如下的非齐次调和方程的边值问题：

$$\begin{cases}\Delta u=py+q\quad(p<0,q>0\text{常数})\\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0}=0,\quad u|_{x=a}=0,\\ u|_{y=0,y=b}=0.\end{cases}$$

试求解之.

设

$$v(x,y)=u(x,y)+(x^2-a^2)(fy+g).$$

则

$$\Delta v=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}+2(fy+g)+\frac{\partial^2 v}{\partial y^2}=\Delta u+2(fy+g).$$

令 $f=-\dfrac{p}{2}$，$g=-\dfrac{q}{2}$ 可使 $\Delta v=0$，此时

$$v(x,y)=u(x,y)-\frac{1}{2}(x^2-a^2)(py+q),$$

$$\left.\frac{\partial v}{\partial x}\right|_{x=0}=\left.\left[\frac{\partial u}{\partial x}+2x(fy+g)\right]\right|_{x=0}=0,\quad v|_{x=a}=\left[u+(x^2-a^2)(fy+g)\right]|_{x=a}=0,$$

$$v|_{y=0}=\left[u+(x^2-a^2)(fy+g)\right]|_{y=0}=-\frac{q}{2}(x^2-a^2),\quad v|_{y=b}=\left[u+(x^2-a^2)(fy+g)\right]|_{y=b}=-\frac{pb+q}{2}(x^2-a^2).$$

故可以先求解齐次调和方程的边值问题:

$$\begin{cases} \Delta v = 0 \\ \left.\dfrac{\partial v}{\partial x}\right|_{x=0} = 0, \quad v|_{x=a} = 0, \\ v|_{y=0} = -\dfrac{q}{2}(x^2 - a^2), \quad v|_{y=b} = -\dfrac{pb+q}{2}(x^2 - a^2). \end{cases}$$

设 $v(x, y) = X(x)Y(y)$, 代入得

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

则有两个常微分方程

$$X''(x) + \lambda X(x) = 0,$$

$$Y''(y) - \lambda Y(y) = 0.$$

关于 x 的方程的特征值和特征函数为

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

代入 $X'(0) = 0$, $X(a) = 0$ 得

$$C_2 = 0, \quad C_1 \cos\sqrt{\lambda}a = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2\pi^2}{4a^2}, \quad X_k(x) = C_k \cos\sqrt{\lambda}x = C_k \cos\frac{(2k-1)\pi}{2a}x, \quad k = 1, 2, \cdots.$$

代入关于 y 的常微分方程可得

$$Y''(y) - \frac{(2k-1)^2\pi^2}{4a^2}Y(y) = 0,$$

$$Y(y) = A_k e^{\frac{(2k-1)\pi}{2a}y} + B_k e^{-\frac{(2k-1)\pi}{2a}y}.$$

方程的通解为

$$v(x, y) = \sum_{k=1}^{\infty} X(x)Y(y) = \sum_{k=1}^{\infty} \left(A_k e^{\frac{(2k-1)\pi}{2a}y} + B_k e^{-\frac{(2k-1)\pi}{2a}y}\right) \cos\frac{(2k-1)\pi}{2a}x.$$

代入初值条件得

$$v(x, 0) = \sum_{k=1}^{\infty} (A_k + B_k) \cos\frac{(2k-1)\pi}{2a}x = -\frac{q}{2}(x^2 - a^2),$$

$$v(x, b) = \sum_{k=1}^{\infty} \left(A_k e^{\frac{(2k-1)\pi}{2a}b} + B_k e^{-\frac{(2k-1)\pi}{2a}b}\right) \cos\frac{(2k-1)\pi}{2a}x = -\frac{pb+q}{2}(x^2 - a^2).$$

解得

$$\frac{2}{a}\int_0^a (x^2 - a^2) \cos\frac{(2k-1)\pi}{2a}x dx = \frac{16a^2[2\cos k\pi + (2k-1)\pi \sin k\pi]}{(2k-1)^3\pi^3} = (-1)^k \frac{32a^2}{(2k-1)^3\pi^3},$$

$$A_k + B_k = \frac{2}{a}\int_0^a -\frac{q}{2}(x^2 - a^2) \cos\frac{(2k-1)\pi}{2a}x dx = (-1)^{k+1} \frac{16qa^2}{(2k-1)^3\pi^3},$$

$$A_k e^{\frac{(2k-1)\pi}{2a}b} + B_k e^{-\frac{(2k-1)\pi}{2a}b} = \frac{2}{a}\int_0^a -\frac{pb+q}{2}(x^2-a^2)\cos\frac{(2k-1)\pi}{2a}xdx = (-1)^{k+1}\frac{16(pb+q)a^2}{(2k-1)^3\pi^3}.$$

化简得

$$A_k = (-1)^{k+1}\frac{(2k-1)^3\pi^3}{16(pb+q)a^2}\cdot\frac{pb+q-qe^{-\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b}-e^{-\frac{(2k-1)\pi}{2a}b}},$$

$$B_k = (-1)^{k+1}\frac{(2k-1)^3\pi^3}{16(pb+q)a^2}\cdot\frac{-(pb+q)+qe^{\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b}-e^{-\frac{(2k-1)\pi}{2a}b}}.$$

故

$$\begin{aligned}
v(x,y) &= \sum_{k=1}^{\infty}\left((-1)^{k+1}\frac{(2k-1)^3\pi^3}{16(pb+q)a^2}\cdot\frac{pb+q-qe^{-\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b}-e^{-\frac{(2k-1)\pi}{2a}b}}e^{\frac{(2k-1)\pi}{2a}y}\right.\\
&\quad\left.+(-1)^{k+1}\frac{(2k-1)^3\pi^3}{16(pb+q)a^2}\cdot\frac{-(pb+q)+qe^{\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b}-e^{-\frac{(2k-1)\pi}{2a}b}}e^{-\frac{(2k-1)\pi}{2a}y}\right)\cos\frac{(2k-1)\pi}{2a}x\\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2k-1)^3\pi^3}{16(pb+q)a^2}\cdot\frac{(pb+q)\sinh\frac{(2k-1)\pi}{2a}y+q\sinh\frac{(2k-1)\pi}{2a}(b-y)}{\sinh\frac{(2k-1)\pi}{2a}b}\cos\frac{(2k-1)\pi}{2a}x.\\
u(x,y) &= v(x,y)+\frac{1}{2}(x^2-a^2)(py+q).
\end{aligned}$$

习题　3.1/8

举例说明在二维调和方程的狄利克雷外问题中，如对解 $u(x,y)$ 不加在无穷远点为有界的限制，那么定解问题的解就不是唯一的.

设

$$u(x,y) = f(r),$$

根据 3.1/1 可知

$$u(x,y) = c_1 + c_2\ln\frac{1}{r}.$$

此时只需令

$$u|_{r=1} = c_1,$$

由 c_2 的任意性即可知 u 有无数个解.

习题　2.2/6

半径为 a 的半圆形平板，其表面绝热，在板的圆周边界上保持常温 u_0，而在直径边界上保持常温 u_1，求圆板稳恒状态（即与时间 t 无关的状态）的温度分布.

根据题意可列出定解问题为

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0, \quad x^2+y^2\leqslant a^2, \quad 0<y<a,\\ u(x,0)=u_1,\\ u(\sqrt{a^2-y^2},y)=u_0.\end{cases}$$

设

$$u(x,y)=v(x,y)+u_1.$$

坐标变换为极坐标系可得

$$\begin{cases}\dfrac{\partial^2 v}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial v}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2}=0, \quad 0\leqslant r\leqslant a, \quad 0<\theta<\pi,\\ v(r,0)=v(r,\pi)=0,\\ v(a,\theta)=u_0-u_1.\end{cases}$$

设 $v(r,\theta)=R(r)\Theta(\theta)$, 代入得

$$R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0,$$

$$-\frac{r^2R''(r)+rR'(r)}{R(r)}=\frac{\Theta''(\theta)}{\Theta(\theta)}=-\lambda.$$

则有两个常微分方程

$$r^2R''(r)+rR'(r)-\lambda R(r)=0,$$

$$\Theta''(\theta)+\lambda\Theta(\theta)=0.$$

关于 θ 的方程的特征值和特征函数为

$$\Theta(\theta)=C_1\cos\sqrt{\lambda}\theta+C_2\sin\sqrt{\lambda}\theta.$$

代入 $\Theta(0)=0$, $\Theta(\pi)=0$ 得

$$C_1=0, \quad C_2\sin\sqrt{\lambda}\pi=0,$$

$$\lambda=\lambda_k=k^2, \quad \Theta_k(\theta)=C_k\sin\sqrt{\lambda}\theta=C_k\sin k\theta, \quad k=1,2,\cdots.$$

代入关于 r 的常微分方程可得

$$r^2R''(r)+rR'(r)-k^2R(r)=0,$$

$$R(r)=A_kr^k+B_kr^{-k}.$$

方程的通解为

$$v(r,\theta)=\sum_{k=1}^{\infty}R(r)\Theta(\theta)=\sum_{k=1}^{\infty}\left(A_kr^k+B_kr^{-k}\right)\sin k\theta.$$

由于

$$\lim_{r\to 0}r^{-k}\to\infty$$

可推出

$$B_k = 0.$$

代入初值条件得

$$v(a,\theta) = \sum_{k=1}^{\infty} A_k a^k \sin k\theta = u_0 - u_1.$$

解得

$$A_k = \frac{2(u_0-u_1)}{a^k\pi}\int_0^{\pi}\sin k\theta d\theta = \frac{2(u_0-u_1)}{a^k\pi}\cdot\frac{1-\cos k\pi}{k} = \begin{cases} \dfrac{4(u_0-u_1)}{a^k k\pi}, & k=2n-1 \\ 0, & k=2n \end{cases}, \quad n=1,2,\cdots.$$

故

$$v(k,\theta) = \sum_{n=1}^{\infty}\frac{4(u_0-u_1)}{a^{2n-1}(2n-1)\pi}r^{2n-1}\sin(2n-1)\theta,$$

$$u(x,y) = u_1 + \sum_{n=1}^{\infty}\frac{4(u_0-u_1)}{a^{2n-1}(2n-1)\pi}\sqrt{x^2+y^2}^{2n-1}\sin\left[(2n-1)\arctan\frac{y}{x}\right].$$

其中

$$\arctan\frac{y}{x}\in[0,\pi].$$