

MA319 — 偏微分方程

Assignment 3

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习题 1.3/1

(1)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u|_{t=0} = \sin \frac{3\pi x}{l}, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = x(l-x) \quad (0 < x < l), \\ u(0, t) = u(l, t) = 0; \end{cases}$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2 \pi^2}{l^2}, \quad X_k(x) = C_k \sin \sqrt{\lambda} x = C_k \sin \frac{k\pi}{l} x, \quad k = 1, 2, \dots$$

方程的通解为

$$\begin{aligned} u(x, t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} at + B_k \sin \sqrt{\lambda} at \right) \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi a}{l} t + B_k \sin \frac{k\pi a}{l} t \right) \sin \frac{k\pi}{l} x. \end{aligned}$$

代入初值条件得

$$\begin{aligned} \varphi(x) &= \sin \frac{3\pi x}{l} = \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi}{l} x, \\ \psi(x) &= x(l-x) = \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} \frac{B_k k\pi a}{l} \sin \frac{k\pi}{l} x. \end{aligned}$$

解得

$$A_k = \begin{cases} 1, & k = 3 \\ 0, & k \neq 3 \end{cases},$$

$$B_k = \frac{2}{l\sqrt{\lambda}a} \int_0^l \psi(x) \sin \sqrt{\lambda} x dx = \frac{2}{k\pi a} \int_0^l x(l-x) \sin \frac{k\pi}{l} x dx = -\frac{2l^3(-2 + 2\cos k\pi + k\pi \sin k\pi)}{k^4\pi^4 a}.$$

化简得

$$B_k = \begin{cases} \frac{8l^3}{k^4\pi^4 a}, & k = 2n-1 \\ 0, & k = 2n \end{cases}, \quad n = 1, 2, \dots$$

故

$$u(x, t) = \cos \frac{3\pi a}{l} t \sin \frac{3\pi}{l} x + \sum_{n=1}^{\infty} \frac{8l^3}{(2n-1)^4 \pi^4 a} \sin \frac{(2n-1)\pi a}{l} t \sin \frac{(2n-1)\pi}{l} x.$$

(ii)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \\ u(0, t) = 0, \quad \frac{\partial u}{\partial x}(l, t) = 0, \\ u(x, 0) = \frac{h}{l} x, \\ \frac{\partial u}{\partial x}(x, 0) = 0; \end{cases}.$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X(0) = 0, X'(l) = 0$ 得

$$C_1 = 0, \quad C_2 \sqrt{\lambda} \cos \sqrt{\lambda} l = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2 \pi^2}{4l^2}, \quad X_k(x) = C_k \sin \frac{(2k-1)\pi}{2l} x, \quad k = 1, 2, \dots$$

方程的通解为

$$\begin{aligned} u(x, t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} a t + B_k \sin \sqrt{\lambda} a t \right) \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{(2k-1)\pi a}{2l} t + B_k \sin \frac{(2k-1)\pi a}{2l} t \right) \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

代入初值条件得

$$\varphi(x) = \frac{h}{l} x = \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} A_k \sin \frac{(2k-1)\pi}{2l} x,$$

$$\psi(x) = 0 = \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} \frac{B_k (2k-1)\pi a}{2l} \sin \frac{(2k-1)\pi}{2l} x.$$

解得

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \sin \sqrt{\lambda} x dx = \frac{2}{l} \int_0^l \frac{h}{l} x \sin \frac{(2k-1)\pi}{2l} x dx = -\frac{4h[2 \cos k\pi + (2k-1)\pi \sin k\pi]}{(2k-1)^2 \pi^2},$$

$$B_k = 0.$$

化简得

$$A_k = (-1)^{k+1} \frac{8h}{(2k-1)^2 \pi^2}.$$

故

$$u(x, t) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{8h}{(2k-1)^2 \pi^2} \cos \frac{(2k-1)\pi a}{2l} t \sin \frac{(2k-1)\pi}{2l} x.$$

习题 1.3/2

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u(0, t) = 0, \quad u(l, t) = A \sin \omega t \\ u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0; \end{cases}$$

设方程的一个特解为

$$U(x, t) = X(x) \sin \omega t,$$

$$U_{tt} = -\omega^2 X(x) \sin \omega t,$$

$$U_{xx} = X''(x) \sin \omega t.$$

代入方程和边界条件得

$$X'' + \frac{\omega^2}{a^2} X = 0, \quad X(0) = 0, \quad X(l) = A.$$

$$X(x) = C_1 \cos \frac{\omega}{a} x + C_2 \sin \frac{\omega}{a} x,$$

$$C_1 = 0, \quad C_2 \sin \frac{\omega}{a} l = A \implies C_2 = \frac{A}{\sin \frac{\omega}{a} l},$$

$$U(x, t) = X(x) \sin \omega t = \frac{A}{\sin \frac{\omega}{a} l} \sin \frac{\omega}{a} x \sin \omega t.$$

此时

$$U(0, t) = 0, \quad U(l, t) = A \sin \omega t, \quad U(x, 0) = 0, \quad \frac{\partial U}{\partial t}(x, 0) = \frac{A\omega}{\sin \frac{\omega}{a} l} \sin \frac{\omega}{a} x.$$

方程的通解为

$$u(x, t) = U(x, t) + V(x, t).$$

且 $V(x, t)$ 满足如下初边值问题

$$\begin{cases} \frac{\partial^2 V}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial x^2}, \\ V(0, t) = 0, \quad V(l, t) = 0 \\ V(x, 0) = 0, \quad \frac{\partial V}{\partial t}(x, 0) = -\frac{A\omega}{\sin \frac{\omega}{a} l} \sin \frac{\omega}{a} x; \end{cases}$$

$V(x, t)$ 的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2 \pi^2}{l^2}, \quad X_k(x) = C_k \sin \sqrt{\lambda} x = C_k \sin \frac{k\pi}{l} x, \quad k = 1, 2, \dots$$

$V(x, t)$ 的通解为

$$\begin{aligned} V(x, t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} at + B_k \sin \sqrt{\lambda} at \right) \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi a}{l} t + B_k \sin \frac{k\pi a}{l} t \right) \sin \frac{k\pi}{l} x. \end{aligned}$$

代入初值条件得

$$\varphi(x) = 0 = \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi}{l} x,$$

$$\psi(x) = -\frac{A\omega}{\sin \frac{\omega}{a} l} \sin \frac{\omega}{a} x = \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} \frac{B_k k\pi a}{l} \sin \frac{k\pi}{l} x.$$

解得

$$A_k = 0,$$

$$B_k = \frac{2}{l\sqrt{\lambda}a} \int_0^l \psi(x) \sin \sqrt{\lambda} x dx = \frac{2}{k\pi a} \int_0^l -\frac{A\omega}{\sin \frac{\omega}{a} l} \sin \frac{\omega}{a} x \sin \frac{k\pi}{l} x dx$$

$$= -\frac{2A\omega}{k\pi \sin \frac{\omega}{a} l} \cdot \frac{l(\omega l \cos \frac{\omega}{a} l \sin k\pi - k\pi a \sin \frac{\omega}{a} l \cos k\pi)}{k^2\pi^2 a^2 - \omega^2 l^2}.$$

化简得

$$B_k = (-1)^k \frac{2A\omega a l}{\omega^2 l^2 - k^2\pi^2 a^2}.$$

故

$$V(x, t) = \sum_{k=1}^{\infty} (-1)^k \frac{2A\omega a l}{\omega^2 l^2 - k^2\pi^2 a^2} \sin \frac{k\pi a}{l} t \sin \frac{k\pi}{l} x,$$

$$u(x, t) = U(x, t) + V(x, t) = \frac{A}{\sin \frac{\omega}{a} l} \sin \frac{\omega}{a} x \sin \omega t + \sum_{k=1}^{\infty} (-1)^k \frac{2A\omega a l}{\omega^2 l^2 - k^2\pi^2 a^2} \sin \frac{k\pi a}{l} t \sin \frac{k\pi}{l} x.$$

习题 1.3/3

(1)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & 0 < x < l, \quad t > 0, \\ u|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = \sin \frac{3}{2l} \pi x, \quad u_t|_{t=0} = \sin \frac{5}{2l} \pi x; \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X(0) = 0, X'(l) = 0$ 得

$$C_1 = 0, \quad C_2 \sqrt{\lambda} \cos \sqrt{\lambda} l = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2 \pi^2}{4l^2}, \quad X_k(x) = C_k \sin \frac{(2k-1)\pi}{2l} x, \quad k = 1, 2, \dots$$

方程的通解为

$$\begin{aligned} u(x, t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} a t + B_k \sin \sqrt{\lambda} a t \right) \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{(2k-1)\pi a}{2l} t + B_k \sin \frac{(2k-1)\pi a}{2l} t \right) \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

代入初值条件得

$$\begin{aligned} \varphi(x) &= \sin \frac{3}{2l} \pi x = \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} A_k \sin \frac{(2k-1)\pi}{2l} x, \\ \psi(x) &= \sin \frac{5}{2l} \pi x = \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} \frac{B_k (2k-1)\pi a}{2l} \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

解得

$$A_k = \begin{cases} 1, & k=2 \\ 0, & k \neq 2 \end{cases}, \quad B_k = \begin{cases} \frac{2l}{5\pi a}, & k=3 \\ 0, & k \neq 3 \end{cases}.$$

故

$$u(x, t) = \cos \frac{3\pi a}{2l} t \sin \frac{3\pi}{2l} x + \frac{2l}{5\pi a} \sin \frac{5\pi a}{2l} t \sin \frac{5\pi}{2l} x.$$

(2)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & 0 < x < l, \quad t > 0, \\ u_x|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = x, \quad u_t|_{t=0} = 0; \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X'(0) = 0, X'(l) = 0$ 得

$$C_2 = 0, \quad -C_1 \sqrt{\lambda} \sin \sqrt{\lambda} l = 0,$$

$$\lambda = \lambda_k = \frac{k^2 \pi^2}{l^2}, \quad X_k(x) = C_k \cos \frac{k\pi}{l} x, \quad k = 0, 1, 2, \dots$$

方程的通解为

$$\begin{aligned} u(x, t) &= \frac{1}{2} A_0 + \frac{1}{2} B_0 t + \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} a t + B_k \sin \sqrt{\lambda} a t \right) \cos \sqrt{\lambda} x \\ &= \frac{1}{2} A_0 + \frac{1}{2} B_0 t + \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi a}{l} t + B_k \sin \frac{k\pi a}{l} t \right) \cos \frac{k\pi}{l} x. \end{aligned}$$

代入初值条件得

$$\varphi(x) = x = \frac{1}{2} A_0 + \sum_{k=1}^{\infty} A_k \cos \sqrt{\lambda} x = \frac{1}{2} A_0 + \sum_{k=1}^{\infty} A_k \cos \frac{k\pi}{l} x,$$

$$\psi(x) = 0 = \frac{1}{2}B_0 + \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \cos \sqrt{\lambda} x = \frac{1}{2}B_0 + \sum_{k=1}^{\infty} \frac{B_k k \pi a}{l} \cos \frac{k \pi}{l} x.$$

解得

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \cos \sqrt{\lambda} x dx = \frac{2}{l} \int_0^l x \cos \frac{k \pi}{l} x dx = \begin{cases} l, & k = 0 \\ \frac{2l(-1 + \cos k\pi + k\pi \sin k\pi)}{k^2 \pi^2}, & k \neq 0 \end{cases},$$

$$B_k = 0.$$

化简得

$$A_k = \begin{cases} l, & k = 0 \\ -\frac{4l}{k^2 \pi^2}, & k = 2n-1, \quad n = 1, 2, \dots \\ 0, & k = 2n \end{cases}.$$

故

$$u(x, t) = \frac{l}{2} + \sum_{n=1}^{\infty} -\frac{4l}{(2n-1)^2 \pi^2} \cos \frac{(2n-1)\pi a}{l} t \cos \frac{(2n-1)\pi}{l} x.$$

习题 1.3/4

$$\begin{cases} u_{tt} - a^2 u_{xx} = g, & 0 < x < l, \quad t > 0, \\ u|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = \sin \frac{\pi x}{2l}; \end{cases}$$

利用叠加原理, 设

$$u(x, t) = U(x, t) + V(x, t).$$

先求

$$\begin{cases} U_{tt} - a^2 U_{xx} = 0, & 0 < x < l, \quad t > 0, \\ U|_{x=0} = U_x|_{x=l} = 0, \\ U|_{t=0} = 0, \quad U_t|_{t=0} = \sin \frac{\pi x}{2l}; \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X(0) = 0, X'(l) = 0$ 得

$$C_1 = 0, \quad C_2 \sqrt{\lambda} \cos \sqrt{\lambda} l = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2 \pi^2}{4l^2}, \quad X_k(x) = C_k \sin \frac{(2k-1)\pi}{2l} x, \quad k = 1, 2, \dots$$

$U(x, t)$ 的通解为

$$\begin{aligned} U(x, t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} a t + B_k \sin \sqrt{\lambda} a t \right) \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{(2k-1)\pi a}{2l} t + B_k \sin \frac{(2k-1)\pi a}{2l} t \right) \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

代入初值条件得

$$\begin{aligned} \varphi(x) &= 0 = \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} A_k \sin \frac{(2k-1)\pi}{2l} x, \\ \psi(x) &= \sin \frac{\pi x}{2l} = \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} \frac{B_k (2k-1)\pi a}{2l} \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

解得

$$A_k = 0, \quad B_k = \begin{cases} \frac{2l}{\pi a}, & k = 1 \\ 0, & k \neq 1 \end{cases}.$$

故

$$U(x, t) = \frac{2l}{\pi a} \sin \frac{\pi a}{2l} t \sin \frac{\pi}{2l} x.$$

再求

$$\begin{cases} V_{tt} - a^2 V_{xx} = g, & 0 < x < l, \quad t > 0, \\ V|_{x=0} = V_x|_{x=l} = 0, \\ V|_{t=0} = V_t|_{t=0} = 0; \end{cases}$$

令 $f(x, \tau) = g$, 代入齐次化原理公式得

$$\begin{aligned} B_k(\tau) &= \frac{2}{l\sqrt{\lambda}a} \int_0^l f(x, \tau) \sin \sqrt{\lambda} x dx = \frac{4}{(2k-1)\pi a} \int_0^l g \sin \frac{(2k-1)\pi}{2l} x dx \\ &= -\frac{8gl(\sin k\pi - 1)}{(2k-1)^2\pi^2 a} = \frac{8gl}{(2k-1)^2\pi^2 a}. \end{aligned}$$

故

$$\begin{aligned} V(x, t) &= \sum_{k=1}^{\infty} \int_0^t B_k(\tau) \sin \sqrt{\lambda} a (t - \tau) d\tau \cdot \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \int_0^t \frac{8gl}{(2k-1)^2\pi^2 a} \sin \frac{(2k-1)\pi a}{2l} (t - \tau) d\tau \cdot \sin \frac{(2k-1)\pi}{2l} x \\ &= \sum_{k=1}^{\infty} \frac{8gl}{(2k-1)^2\pi^2 a} \cdot \frac{4l \sin^2 \frac{(2k-1)\pi a}{4l} t}{(2k-1)\pi a} \cdot \sin \frac{(2k-1)\pi}{2l} x \\ &= \sum_{k=1}^{\infty} \frac{32gl^2}{(2k-1)^3\pi^3 a^2} \sin^2 \frac{(2k-1)\pi a}{4l} t \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

$$u(x, t) = U(x, t) + V(x, t) = \frac{2l}{\pi a} \sin \frac{\pi a}{2l} t \sin \frac{\pi}{2l} x + \sum_{k=1}^{\infty} \frac{32gl^2}{(2k-1)^3\pi^3 a^2} \sin^2 \frac{(2k-1)\pi a}{4l} t \sin \frac{(2k-1)\pi}{2l} x.$$

习题 1.3/5

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b \sinh x, \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = \frac{\partial u}{\partial t} \Big|_{t=0} = 0; \end{cases}$$

令 $f(x, \tau) = b \sinh x$, 代入齐次化原理公式得

$$\begin{aligned} B_k(\tau) &= \frac{2}{l\sqrt{\lambda}a} \int_0^l f(x, \tau) \sin \sqrt{\lambda}x dx = \frac{2}{k\pi a} \int_0^l b \sinh x \sin \frac{k\pi}{l}x dx \\ &= \frac{2bl(l \cosh l \sin k\pi - k\pi \sinh l \cos k\pi)}{k\pi a(k^2\pi^2 + l^2)} \\ &= (-1)^{k+1} \frac{2bl \sinh l}{a(k^2\pi^2 + l^2)}. \end{aligned}$$

故

$$\begin{aligned} u(x, t) &= \sum_{k=1}^{\infty} \int_0^t B_k(\tau) \sin \sqrt{\lambda}a(t - \tau) d\tau \cdot \sin \sqrt{\lambda}x \\ &= \sum_{k=1}^{\infty} \int_0^t (-1)^{k+1} \frac{2bl \sinh l}{a(k^2\pi^2 + l^2)} \sin \frac{k\pi}{l}a(t - \tau) d\tau \cdot \sin \frac{k\pi}{l}x \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2bl \sinh l}{a(k^2\pi^2 + l^2)} \cdot \frac{2l \sin^2 \frac{k\pi a}{2l}t}{k\pi a} \cdot \sin \frac{k\pi}{l}x \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2bl^2 \sinh l}{k\pi a^2(k^2\pi^2 + l^2)} \sin^2 \frac{k\pi a}{2l}t \sin \frac{k\pi}{l}x. \end{aligned}$$

习题 1.3/6

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + 2b \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = \frac{h}{l}x, \quad \frac{\partial u}{\partial t} \Big|_{t=0} = 0; \end{cases}$$

设 $u(x, t) = X(x)T(t)$, 代入得

$$X(x)T''(t) + 2bX(x)T'(t) = a^2X''(x)T(t),$$

$$\frac{T''(t) + 2bT'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

则有两个常微分方程

$$T''(t) + 2bT'(t) + \lambda a^2T(t) = 0,$$

$$X''(x) + \lambda X(x) = 0.$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2 \pi^2}{l^2}, \quad X_k(x) = C_k \sin \sqrt{\lambda} x = C_k \sin \frac{k\pi}{l} x, \quad k = 1, 2, \dots$$

代入关于 T 的常微分方程可得

$$T''(t) + 2bT'(t) + \frac{k^2 \pi^2 a^2}{l^2} T(t) = 0,$$

$$\Delta = 4b^2 - \frac{4k^2 \pi^2 a^2}{l^2},$$

设

$$\lambda' = \frac{k^2 \pi^2 a^2}{l^2} - b^2,$$

$$r_1 = -b + \sqrt{b^2 - \frac{k^2 \pi^2 a^2}{l^2}} = -b + \sqrt{-\lambda'}, \quad r_2 = -b - \sqrt{b^2 - \frac{k^2 \pi^2 a^2}{l^2}} = -b - \sqrt{-\lambda'}.$$

当 $\lambda' < 0$ 时

$$k < \frac{bl}{\pi a}, \quad T(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t},$$

$$U_k(x, t) = T_k(t) X_k(x) = (A_k e^{r_1 t} + B_k e^{r_2 t}) \sin \sqrt{\lambda} x,$$

$$\frac{\partial U_k}{\partial t}(x, t) = T'_k(t) X_k(x) = (A_k r_1 e^{r_1 t} + B_k r_2 e^{r_2 t}) \sin \sqrt{\lambda} x.$$

当 $\lambda' = 0$ 时

$$k = \frac{bl}{\pi a}, \quad T(t) = e^{-bt} (C_1 + C_2 t),$$

$$U_k(x, t) = T_k(t) X_k(x) = e^{-bt} (A_k + B_k t) \sin \sqrt{\lambda} x,$$

$$\frac{\partial U_k}{\partial t}(x, t) = T'_k(t) X_k(x) = e^{-bt} (-bA_k + B_k - bB_k t) \sin \sqrt{\lambda} x.$$

当 $\lambda' > 0$ 时

$$k > \frac{bl}{\pi a}, \quad T(t) = e^{-bt} (C_1 \cos \sqrt{\lambda'} t + C_2 \sin \sqrt{\lambda'} t),$$

$$U_k(x, t) = T_k(t) X_k(x) = e^{-bt} (A_k \cos \sqrt{\lambda'} t + B_k \sin \sqrt{\lambda'} t) \sin \sqrt{\lambda} x,$$

$$\frac{\partial U_k}{\partial t}(x, t) = T'_k(t) X_k(x) = e^{-bt} (-bA_k \cos \sqrt{\lambda'} t - bB_k \sin \sqrt{\lambda'} t - \sqrt{\lambda'} A_k \sin \sqrt{\lambda'} t + \sqrt{\lambda'} B_k \cos \sqrt{\lambda'} t) \sin \sqrt{\lambda} x.$$

设

$$n = \frac{bl}{\pi a}, \quad \underline{n} = \lceil n - 1 \rceil, \quad \bar{n} = \lfloor n + 1 \rfloor, \quad \delta(n) = \begin{cases} 1, & n \in Z \\ 0, & n \notin Z \end{cases}.$$

方程的通解为

$$\begin{aligned} u(x, t) &= \sum_{k=1}^{\infty} T_k(t) X_k(x) \\ &= \left[\sum_{k=1}^{\underline{n}} (A_k e^{r_1 t} + B_k e^{r_2 t}) + \delta(n) e^{-bt} (A_n + B_n t) + \sum_{k=\bar{n}}^{\infty} e^{-bt} (A_k \cos \sqrt{\lambda'} t + B_k \sin \sqrt{\lambda'} t) \right] \sin \sqrt{\lambda} x. \end{aligned}$$

代入初值条件得

$$u(x, 0) = \left[\sum_{k=1}^n (A_k + B_k) + \delta(n)A_n + \sum_{k=\bar{n}}^{\infty} A_k \right] \sin \sqrt{\lambda}x = \frac{h}{l}x,$$

$$\frac{\partial u}{\partial t}(x, 0) = \left[\sum_{k=1}^n (A_k r_1 + B_k r_2) + \delta(n)(-A_n + B_n) + \sum_{k=\bar{n}}^{\infty} (-bA_k + \sqrt{\lambda'}B_k) \right] \sin \sqrt{\lambda}x = 0.$$

解得

$$A'_k = \begin{cases} A_k + B_k, & k < n \\ A_k, & k \geq n \end{cases} = \frac{2}{l} \int_0^l \frac{h}{l} x \sin \sqrt{\lambda}x dx = \frac{h}{l} \cdot \frac{2(\sqrt{\lambda}l \cos \sqrt{\lambda}l - \sin \sqrt{\lambda}l)}{\lambda l} = (-1)^{k+1} \frac{2h}{k\pi},$$

$$B'_k = \begin{cases} A_k r_1 + B_k r_2, & k < n \\ -A_k + B_k, & k = n \\ -bA_k + \sqrt{\lambda'}B_k, & k > n \end{cases} = 0.$$

化简得

$$A_k = \begin{cases} \frac{r_2}{r_2 - r_1} A'_k, & k < n \\ A'_k, & k \geq n \end{cases}, \quad B'_k = \begin{cases} -\frac{r_1}{r_2 - r_1} A'_k, & k < n \\ A'_k, & k = n \\ \frac{b}{\sqrt{\lambda'}} A'_k, & k > n \end{cases}.$$

故将 $r_1, r_2, \lambda, \lambda', n, \underline{n}, \bar{n}, \delta(n), A_k, B_k$ 代入可得 $u(x, t)$.

例题

(1)

$$\begin{cases} u_{tt} - a^2 u_{xx} = A \sin \omega t, \\ u|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = 1, \quad u_t|_{t=0} = x; \end{cases}$$

利用叠加原理, 设

$$u(x, t) = U(x, t) + V(x, t).$$

先求

$$\begin{cases} U_{tt} - a^2 U_{xx} = 0, \\ U|_{x=0} = U_x|_{x=l} = 0, \\ U|_{t=0} = 1, \quad U_t|_{t=0} = x; \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x,$$

代入 $X(0) = 0, X'(l) = 0$ 得

$$C_1 = 0, \quad C_2 \sqrt{\lambda} \cos \sqrt{\lambda}l = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2\pi^2}{4l^2}, \quad X_k(x) = C_k \sin \frac{(2k-1)\pi}{2l}x, \quad k = 1, 2, \dots$$

$U(x, t)$ 的通解为

$$\begin{aligned} U(x, t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda}at + B_k \sin \sqrt{\lambda}at \right) \sin \sqrt{\lambda}x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{(2k-1)\pi a}{2l}t + B_k \sin \frac{(2k-1)\pi a}{2l}t \right) \sin \frac{(2k-1)\pi}{2l}x. \end{aligned}$$

代入初值条件得

$$\begin{aligned} \varphi(x) = 1 &= \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k \sin \frac{(2k-1)\pi}{2l}x, \\ \psi(x) = x &= \sum_{k=1}^{\infty} B_k \sqrt{\lambda}a \sin \sqrt{\lambda}x = \sum_{k=1}^{\infty} \frac{B_k(2k-1)\pi a}{2l} \sin \frac{(2k-1)\pi}{2l}x. \end{aligned}$$

解得

$$\begin{aligned} A_k &= \frac{2}{l} \int_0^l \varphi(x) \sin \sqrt{\lambda}x dx = \frac{2}{l} \int_0^l \sin \frac{(2k-1)\pi}{2l}x dx = \frac{4(-1 + \sin k\pi)}{(2k-1)\pi} = -\frac{4}{(2k-1)\pi}, \\ B_k &= \frac{2}{l\sqrt{\lambda}a} \int_0^l \psi(x) \sin \sqrt{\lambda}x dx = \frac{4}{(2k-1)\pi a} \int_0^l x \sin \frac{(2k-1)\pi}{2l}x dx = -\frac{4l^2[2 \cos k\pi + (2k-1)\pi \sin k\pi]}{(2k-1)^3\pi^3 a}. \end{aligned}$$

化简得

$$B_k = (-1)^{k+1} \frac{8l^2}{(2k-1)^3\pi^3 a}.$$

故

$$U(x, t) = \sum_{k=1}^{\infty} \left[-\frac{4}{(2k-1)\pi} \cos \frac{(2k-1)\pi a}{2l}t + (-1)^{k+1} \frac{8l^2}{(2k-1)^3\pi^3 a} \sin \frac{(2k-1)\pi a}{2l}t \right] \sin \frac{(2k-1)\pi}{2l}x.$$

再求

$$\begin{cases} V_{tt} - a^2 V_{xx} = A \sin \omega t, \\ V|_{x=0} = V_x|_{x=l} = 0, \\ V|_{t=0} = V_t|_{t=0} = 0; \end{cases}$$

令 $f(x, \tau) = A \sin \omega \tau$, 代入齐次化原理公式得

$$\begin{aligned} B_k(\tau) &= \frac{2}{l\sqrt{\lambda}a} \int_0^l f(x, \tau) \sin \sqrt{\lambda}x dx = \frac{4}{(2k-1)\pi a} \int_0^l A \sin \omega \tau \sin \frac{(2k-1)\pi}{2l}x dx \\ &= -\frac{8A \sin \omega \tau \cdot l(\sin k\pi - 1)}{(2k-1)^2\pi^2 a} = \frac{8lA \sin \omega \tau}{(2k-1)^2\pi^2 a}. \end{aligned}$$

故

$$\begin{aligned}
 V(x, t) &= \sum_{k=1}^{\infty} \int_0^t B_k(\tau) \sin \sqrt{\lambda} a(t - \tau) d\tau \cdot \sin \sqrt{\lambda} x \\
 &= \sum_{k=1}^{\infty} \int_0^t \frac{8lA \sin \omega \tau}{(2k-1)^2 \pi^2 a} \sin \frac{(2k-1)\pi a}{2l}(t - \tau) d\tau \cdot \sin \frac{(2k-1)\pi}{2l} x \\
 &= \sum_{k=1}^{\infty} \frac{8lA}{(2k-1)^2 \pi^2 a} \cdot \frac{-2l[2\omega l \sin \frac{(2k-1)\pi a}{2l} t + (2k-1)\pi a \sin \omega t]}{(2k-1)^2 \pi^2 a^2 - 4\omega^2 l^2} \cdot \sin \frac{(2k-1)\pi}{2l} x \\
 &= \sum_{k=1}^{\infty} -\frac{16l^2 A}{(2k-1)^4 \pi^4 a^3 - 4(2k-1)^2 \pi^2 a \omega^2 l^2} \left[2\omega l \sin \frac{(2k-1)\pi a}{2l} t + (2k-1)\pi a \sin \omega t \right] \sin \frac{(2k-1)\pi}{2l} x.
 \end{aligned}$$

$$u(x, t) = U(x, t) + V(x, t).$$

(2)

$$\begin{cases} u_{tt} - u_{xx} - 4u = 2 \sin^2 x, \\ u_x|_{x=0} = u_x|_{x=\pi} = 0, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0; \end{cases}$$

将原方程化为

$$\begin{cases} W_{tt} - W_{xx} - 4W = 0, \\ W_x|_{x=0} = W_x|_{x=\pi} = 0, \\ W|_{t=0} = 0, \quad W_t|_{t=0} = 2 \sin^2 x; \end{cases}$$

设 $W(x, t) = X(x)T(t)$, 代入得

$$X(x)T''(t) - X''(x)T(t) - 4X(x)T(t) = 0,$$

$$\frac{T''(t) - 4T(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

则有两个常微分方程

$$T''(t) + (\lambda - 4)T(t) = 0,$$

$$X''(x) + \lambda X(x) = 0.$$

方程的特征值和对应的特征函数为

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X'(0) = 0, X'(\pi) = 0$ 得

$$C_2 = 0, \quad -C_1 \sqrt{\lambda} \sin \sqrt{\lambda} \pi = 0,$$

$$\lambda = \lambda_k = k^2, \quad X_k(x) = C_k \cos \sqrt{\lambda} x = C_k \cos kx, \quad k = 0, 1, 2, \dots$$

代入关于 T 的常微分方程可得

$$T''(t) + (k^2 - 4)T(t) = 0,$$

设

$$\lambda' = k^2 - 4,$$

$$r_1 = \sqrt{4 - k^2} = \sqrt{-\lambda'}, \quad r_2 = -\sqrt{4 - k^2} = -\sqrt{-\lambda'}.$$

当 $\lambda' < 0$ 时

$$k < 2, \quad T(t) = C_1 e^{\sqrt{-\lambda'}t} + C_2 e^{-\sqrt{-\lambda'}t},$$

$$W_k(x, t) = T_k(t)X_k(x) = (A_k e^{\sqrt{-\lambda'}t} + B_k e^{-\sqrt{-\lambda'}t}) \cos \sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x, t) = T'_k(t)X_k(x) = \sqrt{-\lambda'}(A_k e^{\sqrt{-\lambda'}t} - B_k e^{-\sqrt{-\lambda'}t}) \cos \sqrt{\lambda}x.$$

当 $\lambda' = 0$ 时

$$k = 2, \quad T(t) = C_1 + C_2 t,$$

$$W_k(x, t) = T_k(t)X_k(x) = (A_k + B_k t) \cos \sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x, t) = T'_k(t)X_k(x) = B_k \cos \sqrt{\lambda}x.$$

当 $\lambda' > 0$ 时

$$k > 2, \quad T(t) = C_1 \cos \sqrt{\lambda'}t + C_2 \sin \sqrt{\lambda'}t,$$

$$W_k(x, t) = T_k(t)X_k(x) = (A_k \cos \sqrt{\lambda'}t + B_k \sin \sqrt{\lambda'}t) \cos \sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x, t) = T'_k(t)X_k(x) = \sqrt{\lambda'}(-A_k \sin \sqrt{\lambda'}t + B_k \cos \sqrt{\lambda'}t) \cos \sqrt{\lambda}x.$$

方程的通解为

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{\infty} T_k(t)X_k(x) \\ &= \left[\sum_{k=0}^1 (A_k e^{r_1 t} + B_k e^{r_2 t}) + (A_2 + B_2 t) + \sum_{k=3}^{\infty} (A_k \cos \sqrt{\lambda'}t + B_k \sin \sqrt{\lambda'}t) \right] \sin \sqrt{\lambda}x. \end{aligned}$$

代入初值条件得

$$u(x, 0) = \left[\sum_{k=1}^n (A_k + B_k) + \delta(n)A_n + \sum_{k=\bar{n}}^{\infty} A_k \right] \sin \sqrt{\lambda}x = \frac{h}{l}x,$$

$$\frac{\partial u}{\partial t}(x, 0) = \left[\sum_{k=1}^n (A_k r_1 + B_k r_2) + \delta(n)(-A_n + B_n) + \sum_{k=\bar{n}}^{\infty} (-bA_k + \sqrt{\lambda'}B_k) \right] \sin \sqrt{\lambda}x = 0.$$

解得

$$A'_k = \begin{cases} A_k + B_k, & k < n \\ A_k, & k \geq n \end{cases} = \frac{2}{l} \int_0^l \frac{h}{l} x \sin \sqrt{\lambda}x dx = \frac{h}{l} \cdot \frac{2(\sqrt{\lambda}l \cos \sqrt{\lambda}l - \sin \sqrt{\lambda}l)}{\lambda l} = (-1)^{k+1} \frac{2h}{k\pi},$$

$$B'_k = \begin{cases} A_k r_1 + B_k r_2, & k < n \\ -A_k + B_k, & k = n \\ -bA_k + \sqrt{\lambda'}B_k, & k > n \end{cases} = 0.$$

化简得

$$A_k = \begin{cases} \frac{r_2}{r_2 - r_1} A'_k, & k < n \\ A'_k, & k \geq n \end{cases}, \quad B'_k = \begin{cases} -\frac{r_1}{r_2 - r_1} A'_k, & k < n \\ A'_k, & k = n \\ \frac{b}{\sqrt{\lambda'}} A'_k, & k > n \end{cases}.$$

故将 $r_1, r_2, \lambda, \lambda', n, \underline{n}, \bar{n}, \delta(n), A_k, B_k$ 代入可得 $u(x, t)$.