

MA319 — 偏微分方程

Assignment 6

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习题 1.6/3

证明波动方程

$$u_{tt} = a^2(u_{xx} + u_{yy}) + f(x, y, t)$$

的自由项 f 在 $L^2(K)$ 意义下作微小改变时, 对应的柯西问题的解 u 在 $L^2(K)$ 意义下改变也是微小的, 其中 K 是由

$$(x - x_0)^2 + (y - y_0)^2 \leq (R - at)^2$$

所表示的锥体.

在 K 内任一截面 Ω_t 上成立能量不等式

$$E_0(\Omega_t) = \iint_{\Omega_t} u^2 dx dy, \quad E_1(\Omega_t) = \iint_{\Omega_t} [u_t^2 + a^2(u_x^2 + u_y^2)] dx dy.$$

对 $E_1(t)$ 有

$$\begin{aligned} \frac{dE_1(\Omega_t)}{dt} &= \frac{d}{dt} \int_0^{R-at} \int_0^{2\pi r} [u_t^2 + a^2(u_x^2 + u_y^2)] ds dr \\ &= 2 \int_0^{R-at} \int_0^{2\pi r} [u_t u_{tt} + a^2(u_x u_{xt} + u_y u_{yt})] ds dr - a \int_{\Gamma_t} [u_t^2 + a^2(u_x^2 + u_y^2)] ds. \\ &= 2 \int_0^{R-at} \int_0^{2\pi r} u_t [u_{tt} - a^2(u_x^2 + u_y^2)] ds dr \\ &\quad + 2 \int_{\Gamma_t} \left\{ a^2[u_x u_t \cos(n, x) + u_y u_t \sin(n, x)] - \frac{a}{2} [u_t^2 + a^2(u_x^2 + u_y^2)] \right\} ds. \\ &= 2 \iint_{\Omega_t} u_t f dx dy - a \int_{\Gamma_t} [(au_x - u_t \cos(n, x))^2 + (au_y - u_t \cos(n, y))^2] ds \\ &\leq 2 \iint_{\Omega_t} u_t f dx dy \leq \iint_{\Omega_t} u_t^2 dx dy + \iint_{\Omega_t} f^2 dx dy \leq E_1(\Omega_t) + \iint_{\Omega_t} f^2 dx dy, \\ \frac{d}{dt} e^{-t} E_1(\Omega_t) &= -e^{-t} E_1(\Omega_t) + e^{-t} \frac{dE_1(\Omega_t)}{dt} \leq e^{-t} \iint_{\Omega_t} f^2 dx dy, \\ E_1(\Omega_t) &\leq e^t \int_0^t e^{-\tau} \iint_{\Omega_t} f^2 dx dy d\tau + e^t E_1(\Omega_0) \leq e^t E_1(\Omega_0) + e^t \int_0^t \iint_{\Omega_t} f^2 dx dy d\tau = \overline{E_1(\Omega_t)}. \end{aligned}$$

对 $E_0(t)$ 有

$$\frac{dE_0(\Omega_t)}{dt} = 2 \iint_{\Omega_t} uu_t dx dy - a \int_{\Gamma_t} u^2 ds \leq 2 \iint_{\Omega_t} uu_t dx dy \leq \iint_{\Omega_t} u^2 dx dy + \iint_{\Omega_t} u_t^2 dx dy \leq E_0(\Omega_t) + E_1(\Omega_t),$$

$$\frac{d}{dt} e^{-t} E_0(\Omega_t) = -e^{-t} E_0(\Omega_t) + e^{-t} \frac{dE_0(\Omega_t)}{dt} \leq e^{-t} E_1(\Omega_t),$$

$$E_0(\Omega_t) \leq e^t \int_0^t e^{-\tau} E_1(\Omega_\tau) d\tau + e^t E_0(\Omega_0) \leq e^t \overline{E_1(\Omega_t)} (1 - e^{-t}) + e^t E_0(\Omega_0) = e^t E_0(\Omega_0) + (e^t - 1) \overline{E_1(\Omega_t)},$$

$$\begin{aligned} E_0(\Omega_t) + E_1(\Omega_t) &\leq e^t E_0(\Omega_0) + (e^t - 1) \overline{E_1(\Omega_t)} + E_1(\Omega_t) \leq e^t (E_0(\Omega_0) + \overline{E_1(\Omega_t)}) \\ &= e^t E_0(\Omega_0) + e^{2t} E_1(\Omega_0) + e^{2t} \int_0^t \iint_{\Omega_t} f^2 dx dy d\tau. \end{aligned}$$

在 $0 \leq t \leq T$ 上

$$\begin{aligned} E_0(\Omega_t) + E_1(\Omega_t) &\leq e^T E_0(\Omega_0) + e^{2T} E_1(\Omega_0) + e^{2T} \int_0^T \iint_{\Omega_t} f^2 dx dy d\tau \\ &\leq e^{2T} \left(E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint_{\Omega_t} f^2 dx dy d\tau \right). \end{aligned}$$

存在 C 使得

$$\begin{aligned} \sqrt{T(E_0(\Omega_t) + E_1(\Omega_t))} &\leq \sqrt{Te^{2T} \left(E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint_{\Omega_t} f^2 dx dy d\tau \right)} \leq C\eta, \\ \sqrt{E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint_{\Omega_t} f^2 dx dy d\tau} &\leq \eta. \end{aligned}$$

任取 $\varepsilon > 0$, 可以找到 $\eta = \frac{\varepsilon}{C}$, 使得

$$\begin{aligned} \|f_1 - f_2\|_{L^2(K)} &\leq \sqrt{E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint_{\Omega_t} f^2 dx dy d\tau} \leq \eta, \\ \|u_1 - u_2\|_{L^2(K)} &\leq \sqrt{T(E_0(\Omega_t) + E_1(\Omega_t))} \leq \varepsilon. \end{aligned}$$

故自由项 f 在 $L^2(K)$ 意义下作微小改变时, 对应的柯西问题的解 u 在 $L^2(K)$ 意义下改变也是微小的.

习题 2.1/2

试直接推导扩散过程所满足的微分方程.

设 $N(x, y, t)$ 表示扩散物质的浓度, dm 表示在无穷小时段 dt 内沿法线方向 \mathbf{n} 经过一个无穷小面积 dS 的

扩散物质的质量, $D(x, y, t)$ 为扩散系数, 其总取正值, 则

$$dm = -D(x, y, t) \frac{\partial N(x, y, t)}{\partial \mathbf{n}} dS dt.$$

故从 t_1 到 t_2 进入扩散面为 Γ 的区域 Ω 的质量为

$$\int_{t_1}^{t_2} \iint_{\Gamma} -dm = \int_{t_1}^{t_2} \iint_{\Gamma} D \frac{\partial N}{\partial t} dS dt = \int_{t_1}^{t_2} \iiint_{\Omega} \left[\frac{\partial}{\partial x} \left(D \frac{\partial N}{\partial x} \right) + \frac{\partial}{\partial y} \left(D \frac{\partial N}{\partial y} \right) + \frac{\partial}{\partial z} \left(D \frac{\partial N}{\partial z} \right) \right] dx dy dz dt.$$

区域 Ω 内, 从 t_1 到 t_2 物质的增加量也可表示为

$$\iiint_{\Omega} [N(x, y, z, t_2) - N(x, y, z, t_1)] dx dy dz = \iiint_{\Omega} \int_{t_1}^{t_2} \frac{\partial N}{\partial t} dt dx dy dz = \int_{t_1}^{t_2} \iiint_{\Omega} \frac{\partial N}{\partial t} dx dy dz dt.$$

由于这两个质量相等, 且 t_1, t_2, Ω 的取值是任意的, 可得

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial N}{\partial x} \right) + \frac{\partial}{\partial y} \left(D \frac{\partial N}{\partial y} \right) + \frac{\partial}{\partial z} \left(D \frac{\partial N}{\partial z} \right).$$

习题 2.2/1

用分离变量法求下列定解问题的解:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} & (t > 0, 0 < x < \pi), \\ u(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0 & (t > 0), \\ u(x, 0) = f(x) & (0 < x < \pi). \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X(0) = 0, X'(\pi) = 0$ 得

$$C_1 = 0, \quad C_2 \sqrt{\lambda} \cos \sqrt{\lambda} \pi = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2}{4}, \quad X_k(x) = C_k \sin \frac{(2k-1)}{2} x, \quad k = 1, 2, \dots$$

习题 2.2/2