MA319 — 偏微分方程

Assignment 7

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习题　2.2/3

如果有一长度为 l 的均匀细棒，其周围及两端 $x=0$，$x=l$ 均为绝热，初始温度分布为 $u(x,0)=f(x)$，问以后时刻的温度分布如何？且证明当 $f(x)$ 等于常数 u_0 时，恒有 $u(x,t)=u_0$.

题目可归结为求解如下的定解问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(l,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x,$$

代入 $X'(0)=0$，$X'(l)=0$ 得

$$C_2 = 0, \quad C_1\sqrt{\lambda}\sin\sqrt{\lambda}l = 0,$$

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k \cos\frac{k\pi}{l}x, \quad k = 0,1,2,\cdots.$$

方程的通解为

$$u(x,t) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k e^{-a^2\lambda t}\cos\sqrt{\lambda}x = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2}{l^2}a^2 t}\cos\frac{k\pi}{l}x.$$

代入初值条件得

$$u(x,0) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos\sqrt{\lambda}x = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos\frac{k\pi}{l}x = f(x).$$

解得

$$A_k = \frac{2}{l}\int_0^l f(x)\cos\sqrt{\lambda}x dx = \frac{2}{l}\int_0^l f(x)\cos\frac{k\pi}{l}x dx.$$

故

$$u(x,t) = \frac{1}{l}\int_0^l f(x)dx + \sum_{k=1}^{\infty}\frac{2}{l}\int_0^l f(x)\cos\frac{k\pi}{l}x dx \cdot e^{-\frac{k^2\pi^2}{l^2}a^2 t}\cos\frac{k\pi}{l}x.$$

当 $f(x)$ 等于常数 u_0 时

$$A_k = \begin{cases} 2u_0, & k=0, \\ \dfrac{u_0 l \sin k\pi}{k\pi} = 0, & k > 0. \end{cases}$$

故

$$u(x,t) = \frac{1}{2}A_0 = u_0.$$

习题 2.2/4

在区域 $t>0$, $0<x<l$ 中求解如下的定解问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \beta(u-u_0), \\ u(0,t) = u(1,t) = u_0, \\ u(x,0) = f(x), \end{cases}$$

其中 a, β, u_0 均为常数, $f(x)$ 为已知函数.

设

$$u(x,t) = u_0 + v(x,t)e^{-\beta t}.$$

则

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \beta(u-u_0) = \frac{\partial v}{\partial t}e^{-\beta t} - \beta v e^{-\beta t} - \frac{\partial^2 v}{\partial x^2}e^{-\beta t} + \beta v e^{-\beta t} = e^{-\beta t}\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right) = 0,$$

$$v(0,t)e^{-\beta t} = v(l,t)e^{-\beta t} = u(0,t) - u_0 = u(l,t) - u_0 = 0,$$

$$v(x,0) = u(x,0) - u_0 = f(x) - u_0.$$

故可以先求解如下定解问题:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2}, \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = f(x) - u_0. \end{cases}$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k \sin\sqrt{\lambda}x = C_k \sin\frac{k\pi}{l}x, \quad k = 1, 2, \cdots.$$

方程的通解为

$$v(x,t) = \sum_{k=1}^{\infty} A_k e^{-a^2\lambda t}\sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2}{l^2}a^2 t}\sin\frac{k\pi}{l}x.$$

代入初值条件得

$$v(x,0) = \sum_{k=1}^{\infty} A_k \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x = f(x) - u_0.$$

解得

$$A_k = \frac{2}{l}\int_0^l [f(x)-u_0]\sin\sqrt{\lambda}xdx = \frac{2}{l}\int_0^l [f(x)-u_0]\sin\frac{k\pi}{l}xdx = \frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx - \frac{2u_0(1-\cos k\pi)}{k\pi}.$$

化简得

$$A_k = \frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx + \frac{2u_0[(-1)^k-1]}{k\pi}.$$

故

$$v(x,t) = \sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx + \frac{2u_0[(-1)^k-1]}{k\pi}\right\}e^{-\frac{k^2\pi^2}{l^2}a^2t}\sin\frac{k\pi}{l}x,$$

$$u(x,t) = u_0 + e^{-\beta t}\sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx + \frac{2u_0[(-1)^k-1]}{k\pi}\right\}e^{-\frac{k^2\pi^2}{l^2}a^2t}\sin\frac{k\pi}{l}x.$$

习题 2.2/5

长度为 l 的均匀细杆的初始温度为 $0^\circ C$，端点 $x=0$ 保持常温 u_0，而在 $x=l$ 和侧面上，热量可以发散到周围的介质中去，介质的温度为 $^\circ C$，此时杆上的温度分布函数 $u(x,t)$ 满足下述定解问题:

$$\begin{cases}\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2} - b^2u, \\ u(0,t)=u_0, \quad \left.\left(\dfrac{\partial u}{\partial x}+Hu\right)\right|_{x=l}=0, \\ u(x,0)=0,\end{cases}$$

其中 a,b,H 均为常数，试求出 $u(x,t)$.

设

$$u(x,t) = v(x,t)e^{-b^2t} + f(x).$$

则

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} + b^2u = \frac{\partial v}{\partial t}e^{-b^2t} - b^2ve^{-b^2t} - a^2\frac{\partial^2 v}{\partial x^2}e^{-b^2t} - a^2f''(x) + b^2ve^{-b^2t} + b^2f(x) = 0,$$

$$v(0,t)e^{-b^2t} + f(0) = u_0, \quad \left.\left(\frac{\partial v}{\partial x}+Hv\right)\right|_{x=l} + f'(l) + Hf(l) = 0,$$

$$v(x,0) + f(x) = 0.$$

可找到 $f(x)$ 使以下成立

$$\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right)e^{-b^2t} = 0, \quad f''(x) - \frac{b^2}{a^2}f(x) = 0,$$

$$v(0,t)e^{-b^2t} = 0, \quad f(0) = u_0, \quad \left.\left(\frac{\partial v}{\partial x}+Hv\right)\right|_{x=l} = 0, \quad f'(l) + Hf(l) = 0.$$

故 $f(x)$ 的通解为

$$f(x) = C_1e^{\frac{b}{a}x} + C_2e^{-\frac{b}{a}x}.$$

代入初值条件得

$$f(0) = C_1 + C_2 = u_0,$$

$$f'(l) + Hf(l) = \frac{b}{a}\left(C_1 e^{\frac{bl}{a}} - C_2 e^{-\frac{bl}{a}}\right) + H\left(C_1 e^{\frac{bl}{a}} + C_2 e^{-\frac{bl}{a}}\right) = 0.$$

解得

习题　2.3/1

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习题　2.3/2