MA319 — 偏微分方程

Assignment 7

Instructor: **许德良**

Author: **刘逸灏** (515370910207)

— SJTU (Fall 2019)

习题 2.2/3

如果有一长度为 l 的均匀细棒，其周围及两端 $x=0$，$x=l$ 均为绝热，初始温度分布为 $u(x,0)=f(x)$，问以后时刻的温度分布如何？且证明当 $f(x)$ 等于常数 u_0 时，恒有 $u(x,t)=u_0$.

题目可归结为求解如下的定解问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(l,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x,$$

代入 $X'(0)=0$, $X'(l)=0$ 得

$$C_2 = 0, \quad C_1\sqrt{\lambda}\sin\sqrt{\lambda}l = 0,$$

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k\cos\frac{k\pi}{l}x, \quad k = 0,1,2,\cdots.$$

方程的通解为

$$u(x,t) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k e^{-a^2\lambda t}\cos\sqrt{\lambda}x = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2}{l^2}a^2 t}\cos\frac{k\pi}{l}x.$$

代入初值条件得

$$u(x,0) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\cos\sqrt{\lambda}x = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\cos\frac{k\pi}{l}x = f(x).$$

解得

$$A_k = \frac{2}{l}\int_0^l f(x)\cos\sqrt{\lambda}x dx = \frac{2}{l}\int_0^l f(x)\cos\frac{k\pi}{l}x dx.$$

故

$$u(x,t) = \frac{1}{l}\int_0^l f(x)dx + \sum_{k=1}^{\infty}\frac{2}{l}\int_0^l f(x)\cos\frac{k\pi}{l}x dx\cdot e^{-\frac{k^2\pi^2}{l^2}a^2 t}\cos\frac{k\pi}{l}x.$$

当 $f(x)$ 等于常数 u_0 时

$$A_k = \begin{cases} 2u_0, & k=0, \\ \dfrac{u_0 l \sin k\pi}{k\pi} = 0, & k>0. \end{cases}$$

故

$$u(x,t) = \frac{1}{2}A_0 = u_0.$$

习题　2.2/4

在区域 $t>0,\ 0<x<l$ 中求解如下的定解问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \beta(u-u_0), \\ u(0,t) = u(1,t) = u_0, \\ u(x,0) = f(x), \end{cases}$$

其中 $a,\ \beta,\ u_0$ 均为常数, $f(x)$ 为已知函数.

设

$$u(x,t) = u_0 + v(x,t)e^{-\beta t}.$$

则

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \beta(u-u_0) = \frac{\partial v}{\partial t}e^{-\beta t} - \beta v e^{-\beta t} - \frac{\partial^2 v}{\partial x^2}e^{-\beta t} + \beta v e^{-\beta t} = e^{-\beta t}\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right) = 0,$$

$$v(0,t)e^{-\beta t} = v(l,t)e^{-\beta t} = u(0,t) - u_0 = u(l,t) - u_0 = 0,$$

$$v(x,0) = u(x,0) - u_0 = f(x) - u_0.$$

故可以先求解如下定解问题:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2}, \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = f(x) - u_0. \end{cases}$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k \sin\sqrt{\lambda}x = C_k \sin\frac{k\pi}{l}x, \quad k=1,2,\cdots.$$

方程的通解为

$$v(x,t) = \sum_{k=1}^{\infty} A_k e^{-a^2\lambda t}\sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2}{l^2}a^2 t}\sin\frac{k\pi}{l}x.$$

代入初值条件得

$$v(x,0) = \sum_{k=1}^{\infty} A_k \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x = f(x) - u_0.$$

解得

$$A_k=\frac{2}{l}\int_0^l[f(x)-u_0]\sin\sqrt{\lambda}xdx=\frac{2}{l}\int_0^l[f(x)-u_0]\sin\frac{k\pi}{l}xdx=\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx-\frac{2u_0(1-\cos k\pi)}{k\pi}.$$

化简得

$$A_k=\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx+\frac{2u_0[(-1)^k-1]}{k\pi}.$$

故

$$v(x,t)=\sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx+\frac{2u_0[(-1)^k-1]}{k\pi}\right\}e^{-\frac{k^2\pi^2}{l^2}a^2t}\sin\frac{k\pi}{l}x,$$

$$u(x,t)=u_0+e^{-\beta t}\sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx+\frac{2u_0[(-1)^k-1]}{k\pi}\right\}e^{-\frac{k^2\pi^2}{l^2}a^2t}\sin\frac{k\pi}{l}x.$$

习题 2.2/5

长度为 l 的均匀细杆的初始温度为 $0°C$, 端点 $x=0$ 保持常温 u_0, 而在 $x=l$ 和侧面上, 热量可以发散到周围的介质中去, 介质的温度为 $°C$, 此时杆上的温度分布函数 $u(x,t)$ 满足下述定解问题:

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}-b^2u,\\ u(0,t)=u_0,\quad \left.\left(\dfrac{\partial u}{\partial x}+Hu\right)\right|_{x=l}=0,\\ u(x,0)=0,\end{cases}$$

其中 a,b,H 均为常数, 试求出 $u(x,t)$.

设

$$u(x,t)=v(x,t)e^{-b^2t}+f(x).$$

则

$$\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}+b^2u=\frac{\partial v}{\partial t}e^{-b^2t}-b^2ve^{-b^2t}-a^2\frac{\partial^2 v}{\partial x^2}e^{-b^2t}-a^2f''(x)+b^2ve^{-b^2t}+b^2f(x)=0,$$

$$v(0,t)e^{-b^2t}+f(0)=u_0,\quad \left.\left(\frac{\partial v}{\partial x}+Hv\right)\right|_{x=l}+f'(l)+Hf(l)=0,$$

$$v(x,0)+f(x)=0.$$

可找到 $f(x)$ 使以下成立

$$\left(\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}\right)e^{-b^2t}=0,\quad f''(x)-\frac{b^2}{a^2}f(x)=0,$$

$$v(0,t)e^{-b^2t}=0,\quad f(0)=u_0,\quad \left.\left(\frac{\partial v}{\partial x}+Hv\right)\right|_{x=l}=0,\quad f'(l)+Hf(l)=0.$$

故 $f(x)$ 的通解为

$$f(x)=C_1e^{\frac{b}{a}x}+C_2e^{-\frac{b}{a}x}.$$

代入初值条件得

$$f(0)=C_1+C_2=u_0,$$

$$f'(l)+Hf(l)=\frac{b}{a}\left(C_1e^{\frac{bl}{a}}-C_2e^{-\frac{bl}{a}}\right)+H\left(C_1e^{\frac{bl}{a}}+C_2e^{-\frac{bl}{a}}\right)=0.$$

解得

$$C_1=\frac{u_0(b-aH)}{aHe^{\frac{2bl}{a}}+be^{\frac{2bl}{a}}-aH+b},\quad C_2=\frac{u_0(aH+b)e^{\frac{2bl}{a}}}{aHe^{\frac{2bl}{a}}+be^{\frac{2bl}{a}}-aH+b},$$

$$f(x)=\frac{u_0e^{-\frac{bx}{a}}\left[aH\left(e^{\frac{2bl}{a}}-e^{\frac{2bx}{a}}\right)+b\left(e^{\frac{2bl}{a}}+e^{\frac{2bx}{a}}\right)\right]}{aH\left(e^{\frac{2bl}{a}}-1\right)+b\left(e^{\frac{2bl}{a}}+1\right)}=\frac{u_0\left[aH\sinh\left(\frac{b(l-x)}{a}\right)+b\cosh\left(\frac{b(l-x)}{a}\right)\right]}{aH\sinh\left(\frac{bl}{a}\right)+b\cosh\left(\frac{bl}{a}\right)}.$$

故可以先求解如下定解问题:

$$\begin{cases}\dfrac{\partial v}{\partial t}=a^2\dfrac{\partial^2 v}{\partial x^2},\\ v(0,t)=0,\quad \left.\left(\dfrac{\partial v}{\partial x}+Hv\right)\right|_{x=l}=0,\\ v(x,0)=-f(x).\end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0$, $X'(l)+HX(l)=0$ 得

$$C_1=0,\quad C_2(\sqrt{\lambda}\cos\sqrt{\lambda}l+H\sin\sqrt{\lambda}l)=0,$$

$$v=\sqrt{\lambda}l,\quad \tan v=-\frac{\sqrt{\lambda}}{H}.$$

存在无数个正根 $v_k>0$, 满足

$$\left(k-\frac{1}{2}\right)\pi<v_k<k\pi,\quad \lambda_k=\left(\frac{v_k}{l}\right)^2,\quad X_k(x)=C_k\sin\frac{v_k}{l}x,\quad k=1,2,\cdots$$

方程的通解为

$$v(x,t)=\sum_{k=1}^{\infty}A_ke^{-a^2\lambda t}\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_ke^{-\frac{v_k^2}{l^2}a^2t}\sin\frac{v_k}{l}x.$$

代入初值条件得

$$v(x,0)=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{v_k}{l}x=-f(x).$$

由固有函数系 $\{X_k\}=\{\sin\sqrt{\lambda_k}x\}$ 的正交性可得

$$M_k=\int_0^l\sin^2\sqrt{\lambda_k}xdx=\frac{l}{2}+\frac{H}{2(H^2+\lambda_k)}=\frac{l}{2}+\frac{Hl^2}{2(H^2l^2+v_k^2)},$$

$$A_k=\frac{1}{M_k}\int_0^l-f(x)\sin\sqrt{\lambda_k}xdx=-\frac{1}{M_k}\int_0^lf(x)\sin\frac{v_k}{l}xdx.$$

故

$$v(x,t) = -\sum_{k=1}^{\infty} \frac{1}{M_k} \int_0^l f(\xi) \sin \frac{v_k}{l} \xi d\xi \cdot e^{-\frac{v_k^2}{l^2} a^2 t} \sin \frac{v_k}{l} x,$$

$$u(x,t) = -e^{-b^2 t} \sum_{k=1}^{\infty} \frac{1}{M_k} \int_0^l f(\xi) \sin \frac{v_k}{l} \xi d\xi \cdot e^{-\frac{v_k^2}{l^2} a^2 t} \sin \frac{v_k}{l} x + \frac{u_0 \left[aH \sinh\left(\frac{b(l-x)}{a}\right) + b \cosh\left(\frac{b(l-x)}{a}\right) \right]}{aH \sinh\left(\frac{bl}{a}\right) + b \cosh\left(\frac{bl}{a}\right)}.$$

习题 2.3/1

求下列函数的傅里叶变换:

(1) $e^{-\eta x^2}$ $\quad(\eta > 0)$;

(2) $e^{-a|x|}$ $\quad(a > 0)$.

(1)

$$g(\lambda) = \int_{-\infty}^{\infty} f(\xi) e^{-i\lambda\xi} d\xi = \int_{-\infty}^{\infty} e^{-\eta\xi^2 - i\lambda\xi} d\xi = \int_{-\infty}^{\infty} e^{-\eta\xi^2 - i\lambda\xi + \frac{\lambda^2}{4\eta} - \frac{\lambda^2}{4\eta}} d\xi = e^{-\frac{\lambda^2}{4\eta}} \int_{-\infty}^{\infty} e^{-\eta\left(\xi + \frac{i\lambda}{2\eta}\right)^2} d\xi.$$

设

$$\int_{-\infty}^{\infty} e^{-\eta x^2} dx = \int_{-\infty}^{\infty} e^{-\eta y^2} dy = A,$$

$$A^2 = \int_{-\infty}^{\infty} e^{-\eta x^2} dx \int_{-\infty}^{\infty} e^{-\eta y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\eta (x^2+y^2)^2} dxdy.$$

令 $r^2 = x^2 + y^2$

$$A^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\eta r^2} r dr d\theta = 2\pi \int_0^{\infty} -\frac{1}{2\eta} e^{-\eta r^2} d(-\eta r^2) = \frac{\pi}{\eta} \int_{-\infty}^0 e^s ds = \frac{\pi}{\eta},$$

$$g(\lambda) = e^{-\frac{\lambda^2}{4\eta}} \int_{-\infty}^{\infty} e^{-\eta\left(\xi + \frac{i\lambda}{2\eta}\right)^2} d\left(\xi + \frac{i\lambda}{2\eta}\right) = e^{-\frac{\lambda^2}{4\eta}} \int_{-\infty}^{\infty} e^{-\eta x^2} dx = A e^{-\frac{\lambda^2}{4\eta}} = \sqrt{\frac{\pi}{\eta}} e^{-\frac{\lambda^2}{4\eta}}.$$

(2)

$$g(\lambda) = \int_{-\infty}^{\infty} f(\xi) e^{-i\lambda\xi} d\xi = \int_{-\infty}^{\infty} e^{-a|\xi| - i\lambda\xi} d\xi = \int_{-\infty}^{\infty} e^{-a|\xi|} \cos \lambda\xi d\xi + \int_{-\infty}^{\infty} e^{-a|\xi|} i \sin \lambda\xi d\xi.$$

由于 $e^{-a|\xi|}$, $\cos \lambda\xi$ 是偶函数, $i \sin \lambda\xi$ 是奇函数,

$$\int_{-\infty}^{\infty} e^{-a|\xi|} \cos \lambda\xi d\xi = 2 \int_0^{\infty} e^{-a\xi} \cos \lambda\xi d\xi, \quad \int_{-\infty}^{\infty} e^{-a|\xi|} i \sin \lambda\xi d\xi = 0,$$

$$g(\lambda) = 2 \int_0^{\infty} e^{-a\xi} \cos \lambda\xi d\xi = \frac{e^{-a\xi}(\lambda \sin(\lambda\xi) - a \cos(\lambda\xi))}{a^2 + \lambda^2} \Bigg|_0^{\infty} = \frac{a}{a^2 + \lambda^2}.$$

习题 2.3/2

证明: 当 $f(x)$ 在 $(-\infty,\infty)$ 上绝对可积时, $F[f]$ 为连续函数.

$$F[f](\lambda+h)-F[f](\lambda)=\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}d\xi-\int_{-\infty}^{\infty}f(\xi)e^{-i(\lambda+h)\xi}d\xi=\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}(1-e^{-ih\xi})d\xi,$$

$$|F[f](\lambda+h)-F[f](\lambda)|=\left|\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}(1-e^{-ih\xi})d\xi\right|\leqslant\int_{-\infty}^{\infty}|f(\xi)|\left|e^{-i\lambda\xi}\right|\left|1-e^{-ih\xi}\right|d\xi.$$

易知当 $\xi\in\mathbf{R}$ 时,

$$\left|e^{-i\lambda\xi}\right|=\sqrt{\cos^2\lambda\xi+\sin^2\lambda\xi}=1,\quad \left|1-e^{-ih\xi}\right|=\sqrt{(1-\cos h\xi)^2+\sin^2 h\xi}=\sqrt{2}\cdot\sqrt{1-\cos h\xi},$$

$$|F[f](\lambda+h)-F[f](\lambda)|\leqslant\int_{-\infty}^{\infty}|f(\xi)|\sqrt{2}\cdot\sqrt{1-\cos h\xi}d\xi\leqslant\sqrt{2}\int_{-\infty}^{\infty}|f(\xi)|d\xi.$$