MA319 — 偏微分方程

Assignment 3

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习题 1.3/1

(1)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u|_{t=0} = \sin \dfrac{3\pi x}{l}, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = x(l-x) \quad (0 < x < l), \\ u(0,t) = u(l,t) = 0; \end{cases}$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k \sin\sqrt{\lambda}x = C_k \sin\frac{k\pi}{l}x, \quad k = 1, 2, \cdots.$$

方程的通解为

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \left(A_k \cos\sqrt{\lambda}at + B_k \sin\sqrt{\lambda}at \right) \sin\sqrt{\lambda}x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos\frac{k\pi a}{l}t + B_k \sin\frac{k\pi a}{l}t \right) \sin\frac{k\pi}{l}x. \end{aligned}$$

代入初值条件得

$$\varphi(x) = \sin\frac{3\pi x}{l} = \sum_{k=1}^{\infty} A_k \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x,$$

$$\psi(x) = x(l-x) = \sum_{k=1}^{\infty} B_k\sqrt{\lambda}a \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} \frac{B_k k\pi a}{l} \sin\frac{k\pi}{l}x.$$

解得

$$A_k = \begin{cases} 1, & k = 3 \\ 0, & k \neq 3 \end{cases},$$

$$B_k = \frac{2}{l\sqrt{\lambda}a}\int_0^l \psi(x)\sin\sqrt{\lambda}x dx = \frac{2}{k\pi a}\int_0^l x(l-x)\sin\frac{k\pi}{l}x dx = -\frac{2l^3(-2+2\cos k\pi + k\pi\sin k\pi)}{k^4\pi^4 a}.$$

化简得

$$B_k = \begin{cases} \dfrac{8l^3}{k^4\pi^4 a}, & k = 2n-1 \\ 0, & k = 2n \end{cases}, \quad n = 1, 2, \cdots.$$

故

$$u(x,t)=\cos\frac{3\pi a}{l}t\sin\frac{3\pi}{l}x+\sum_{n=1}^{\infty}\frac{8l^3}{(2n-1)^4\pi^4 a}\sin\frac{(2n-1)\pi a}{l}t\sin\frac{(2n-1)\pi}{l}x.$$

(ii)

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0\\ u(0,t)=0,\quad \dfrac{\partial u}{\partial x}(l,t)=0,\\ u(x,0)=\dfrac{h}{l}x,\\ \dfrac{\partial u}{\partial x}(x,0)=0;\end{cases}.$$

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0$, $X'(l)=0$ 得

$$C_1=0,\quad C_2\sqrt{\lambda}\cos\sqrt{\lambda}l=0,$$

$$\lambda=\lambda_k=\frac{(2k-1)^2\pi^2}{4l^2},\quad X_k(x)=C_k\sin\frac{(2k-1)\pi}{2l}x,\quad k=1,2,\cdots.$$

方程的通解为

$$\begin{aligned}u(x,t)&=\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x\\&=\sum_{k=1}^{\infty}\left(A_k\cos\frac{(2k-1)\pi a}{2l}t+B_k\sin\frac{(2k-1)\pi a}{2l}t\right)\sin\frac{(2k-1)\pi}{2l}x.\end{aligned}$$

代入初值条件得

$$\varphi(x)=\frac{h}{l}x=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{(2k-1)\pi}{2l}x,$$

$$\psi(x)=0=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_k(2k-1)\pi a}{2l}\sin\frac{(2k-1)\pi}{2l}x.$$

解得

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\sqrt{\lambda}xdx=\frac{2}{l}\int_0^l\frac{h}{l}x\sin\frac{(2k-1)\pi}{2l}xdx=-\frac{4h[2\cos k\pi+(2k-1)\pi\sin k\pi]}{(2k-1)^2\pi^2},$$

$$B_k=0.$$

化简得

$$A_k=(-1)^{k+1}\frac{8h}{(2k-1)^2\pi^2}.$$

故

$$u(x,t)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{8h}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi a}{2l}t\sin\frac{(2k-1)\pi}{2l}x.$$

习题 1.3/2

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,t) = 0, \quad u(l,t) = A\sin\omega t \\ u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0; \end{cases}$$

设方程的一个特解为

$$U(x,t) = X(x)\sin\omega t,$$

$$U_{tt} = -\omega^2 X(x)\sin\omega t,$$

$$U_{xx} = X''(x)\sin\omega t.$$

代入方程和边界条件得

$$X'' + \frac{\omega^2}{a^2}X = 0, \quad X(0) = 0, \quad X(l) = A.$$

$$X(x) = C_1\cos\frac{\omega}{a}x + C_2\sin\frac{\omega}{a}x,$$

$$C_1 = 0, \quad C_2\sin\frac{\omega}{a}l = A \Longrightarrow C_2 = \frac{A}{\sin\frac{\omega}{a}l},$$

$$U(x,t) = X(x)\sin\omega t = \frac{A}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x\sin\omega t.$$

此时

$$U(0,t) = 0, \quad U(l,t) = A\sin\omega t, \quad U(x,0) = 0, \quad \frac{\partial U}{\partial t}(x,0) = \frac{A\omega}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x.$$

方程的通解为

$$u(x,t) = U(x,t) + V(x,t).$$

且 $V(x,t)$ 满足如下初边值问题

$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2} = a^2 \dfrac{\partial^2 V}{\partial x^2}, \\ V(0,t) = 0, \quad V(l,t) = 0 \\ V(x,0) = 0, \quad \dfrac{\partial V}{\partial t}(x,0) = -\dfrac{A\omega}{\sin\frac{\omega}{a}l}\sin\dfrac{\omega}{a}x; \end{cases}$$

$V(x,t)$ 的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k\sin\sqrt{\lambda}x = C_k\sin\frac{k\pi}{l}x, \quad k = 1, 2, \cdots.$$

$V(x,t)$ 的通解为

$$\begin{aligned} V(x,t) &= \sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at + B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x \\ &= \sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi a}{l}t + B_k\sin\frac{k\pi a}{l}t\right)\sin\frac{k\pi}{l}x. \end{aligned}$$

代入初值条件得

$$\varphi(x)=0=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi}{l}x,$$

$$\psi(x)=-\frac{A\omega}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_kk\pi a}{l}\sin\frac{k\pi}{l}x.$$

解得

$$A_k=0,$$

$$\begin{aligned}B_k&=\frac{2}{l\sqrt{\lambda}a}\int_0^l\psi(x)\sin\sqrt{\lambda}xdx=\frac{2}{k\pi a}\int_0^l-\frac{A\omega}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x\sin\frac{k\pi}{l}xdx\\&=-\frac{2A\omega}{k\pi\sin\frac{\omega}{a}l}\cdot\frac{l\left(\omega l\cos\frac{\omega}{a}l\sin k\pi-k\pi a\sin\frac{\omega}{a}l\cos k\pi\right)}{k^2\pi^2a^2-\omega^2l^2}.\end{aligned}$$

化简得

$$B_k=(-1)^k\frac{2A\omega al}{\omega^2l^2-k^2\pi^2a^2}.$$

故

$$V(x,t)=\sum_{k=1}^{\infty}(-1)^k\frac{2A\omega al}{\omega^2l^2-k^2\pi^2a^2}\sin\frac{k\pi a}{l}t\sin\frac{k\pi}{l}x,$$

$$u(x,t)=U(x,t)+V(x,t)=\frac{A}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x\sin\omega t+\sum_{k=1}^{\infty}(-1)^k\frac{2A\omega al}{\omega^2l^2-k^2\pi^2a^2}\sin\frac{k\pi a}{l}t\sin\frac{k\pi}{l}x.$$

习题　1.3/3

(1)

$$\begin{cases}u_{tt}-a^2u_{xx}=0,\quad 0<x<l,\quad t>0,\\ u|_{x=0}=u_x|_{x=l}=0,\\ u|_{t=0}=\sin\dfrac{3}{2l}\pi x,\quad u_t|_{t=0}=\sin\dfrac{5}{2l}\pi x;\end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0,\ X'(l)=0$ 得

$$C_1=0,\quad C_2\sqrt{\lambda}\cos\sqrt{\lambda}l=0,$$

$$\lambda=\lambda_k=\frac{(2k-1)^2\pi^2}{4l^2},\quad X_k(x)=C_k\sin\frac{(2k-1)\pi}{2l}x,\quad k=1,2,\cdots.$$

方程的通解为

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at + B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x \\
&= \sum_{k=1}^{\infty}\left(A_k\cos\frac{(2k-1)\pi a}{2l}t + B_k\sin\frac{(2k-1)\pi a}{2l}t\right)\sin\frac{(2k-1)\pi}{2l}x.
\end{aligned}
$$

代入初值条件得

$$
\varphi(x)=\sin\frac{3}{2l}\pi x=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{(2k-1)\pi}{2l}x,
$$

$$
\psi(x)=\sin\frac{5}{2l}\pi x=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_k(2k-1)\pi a}{2l}\sin\frac{(2k-1)\pi}{2l}x.
$$

解得

$$
A_k=\begin{cases}1, & k=2\\ 0, & k\neq 2\end{cases},\quad B_k=\begin{cases}\dfrac{2l}{5\pi a}, & k=3\\ 0, & k\neq 3\end{cases}.
$$

故

$$
u(x,t)=\cos\frac{3\pi a}{2l}t\sin\frac{3\pi}{2l}x+\frac{2l}{5\pi a}\sin\frac{5\pi a}{2l}t\sin\frac{5\pi}{2l}x.
$$

(2)

$$
\begin{cases}
u_{tt}-a^2u_{xx}=0, & 0<x<l,\quad t>0,\\
u_x|_{x=0}=u_x|_{x=l}=0,\\
u|_{t=0}=x,\quad u_t|_{t=0}=0;
\end{cases}
$$

方程的特征值和对应的特征函数为

$$
X''(x)+\lambda X(x)=0,
$$

$$
X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,
$$

代入 $X'(0)=0$, $X'(l)=0$ 得

$$
C_2=0,\quad -C_1\sqrt{\lambda}\sin\sqrt{\lambda}l=0,
$$

$$
\lambda=\lambda_k=\frac{k^2\pi^2}{l^2},\quad X_k(x)=C_k\cos\frac{k\pi}{l}x,\quad k=0,1,2,\cdots.
$$

方程的通解为

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}A_0+\frac{1}{2}B_0t+\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\cos\sqrt{\lambda}x \\
&= \frac{1}{2}A_0+\frac{1}{2}B_0t+\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi a}{l}t+B_k\sin\frac{k\pi a}{l}t\right)\cos\frac{k\pi}{l}x.
\end{aligned}
$$

代入初值条件得

$$
\varphi(x)=x=\frac{1}{2}A_0+\sum_{k=1}^{\infty}A_k\cos\sqrt{\lambda}x=\frac{1}{2}A_0+\sum_{k=1}^{\infty}A_k\cos\frac{k\pi}{l}x,
$$

$$\psi(x)=0=\frac{1}{2}B_0+\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\cos\sqrt{\lambda}x=\frac{1}{2}B_0+\sum_{k=1}^{\infty}\frac{B_k k\pi a}{l}\cos\frac{k\pi}{l}x.$$

解得

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\cos\sqrt{\lambda}xdx=\frac{2}{l}\int_0^l x\cos\frac{k\pi}{l}xdx=\begin{cases}l, & k=0\\ \dfrac{2l(-1+\cos k\pi+k\pi\sin k\pi)}{k^2\pi^2}, & k\neq 0\end{cases},$$

$$B_k=0.$$

化简得

$$A_k=\begin{cases}l, & k=0\\ -\dfrac{4l}{k^2\pi^2}, & k=2n-1\ ,\quad n=1,2,\cdots.\\ 0, & k=2n\end{cases}$$

故

$$u(x,t)=\frac{l}{2}+\sum_{n=1}^{\infty}-\frac{4l}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi a}{l}t\cos\frac{(2n-1)\pi}{l}x.$$

习题　1.3/4

$$\begin{cases}u_{tt}-a^2u_{xx}=g, & 0<x<l, \quad t>0,\\ u|_{x=0}=u_x|_{x=l}=0,\\ u|_{t=0}=0, \quad u_t|_{t=0}=\sin\dfrac{\pi x}{2l};\end{cases}$$

利用叠加原理, 设

$$u(x,t)=U(x,t)+V(x,t).$$

先求

$$\begin{cases}U_{tt}-a^2U_{xx}=0, & 0<x<l, \quad t>0,\\ U|_{x=0}=U_x|_{x=l}=0,\\ U|_{t=0}=0, \quad U_t|_{t=0}=\sin\dfrac{\pi x}{2l};\end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0$, $X'(l)=0$ 得

$$C_1=0, \quad C_2\sqrt{\lambda}\cos\sqrt{\lambda}l=0,$$

$$\lambda=\lambda_k=\frac{(2k-1)^2\pi^2}{4l^2}, \quad X_k(x)=C_k\sin\frac{(2k-1)\pi}{2l}x, \quad k=1,2,\cdots.$$

$U(x,t)$ 的通解为

$$U(x,t)=\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x$$
$$=\sum_{k=1}^{\infty}\left(A_k\cos\frac{(2k-1)\pi a}{2l}t+B_k\sin\frac{(2k-1)\pi a}{2l}t\right)\sin\frac{(2k-1)\pi}{2l}x.$$

代入初值条件得

$$\varphi(x)=0=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{(2k-1)\pi}{2l}x,$$

$$\psi(x)=\sin\frac{\pi x}{2l}=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_k(2k-1)\pi a}{2l}\sin\frac{(2k-1)\pi}{2l}x.$$

解得

$$A_k=0,\quad B_k=\begin{cases}\frac{2l}{\pi a}, & k=1\\ 0, & k\neq 1\end{cases}.$$

故

$$U(x,t)=\frac{2l}{\pi a}\sin\frac{\pi a}{2l}t\sin\frac{\pi}{2l}x.$$

再求

$$\begin{cases}V_{tt}-a^2V_{xx}=g, \quad 0<x<l, \quad t>0,\\ V|_{x=0}=V_x|_{x=l}=0,\\ V|_{t=0}=V_t|_{t=0}=0;\end{cases}$$

令 $f(x,\tau)=g$, 代入齐次化原理公式得

$$B_k(\tau)=\frac{2}{l\sqrt{\lambda}a}\int_0^l f(x,\tau)\sin\sqrt{\lambda}xdx=\frac{4}{(2k-1)\pi a}\int_0^l g\sin\frac{(2k-1)\pi}{2l}xdx$$
$$=-\frac{8gl(\sin k\pi-1)}{(2k-1)^2\pi^2a}=\frac{8gl}{(2k-1)^2\pi^2a}.$$

故

$$V(x,t)=\sum_{k=1}^{\infty}\int_0^t B_k(\tau)\sin\sqrt{\lambda}a(t-\tau)d\tau\cdot\sin\sqrt{\lambda}x$$
$$=\sum_{k=1}^{\infty}\int_0^t\frac{8gl}{(2k-1)^2\pi^2a}\sin\frac{(2k-1)\pi a}{2l}(t-\tau)d\tau\cdot\sin\frac{(2k-1)\pi}{2l}x$$
$$=\sum_{k=1}^{\infty}\frac{8gl}{(2k-1)^2\pi^2a}\cdot\frac{4l\sin^2\frac{(2k-1)\pi a}{4l}t}{(2k-1)\pi a}\cdot\sin\frac{(2k-1)\pi}{2l}x$$
$$=\sum_{k=1}^{\infty}\frac{32gl^2}{(2k-1)^3\pi^3a^2}\sin^2\frac{(2k-1)\pi a}{4l}t\sin\frac{(2k-1)\pi}{2l}x.$$

$$u(x,t)=U(x,t)+V(x,t)=\frac{2l}{\pi a}\sin\frac{\pi a}{2l}t\sin\frac{\pi}{2l}x+\sum_{k=1}^{\infty}\frac{32gl^2}{(2k-1)^3\pi^3a^2}\sin^2\frac{(2k-1)\pi a}{4l}t\sin\frac{(2k-1)\pi}{2l}x.$$

习题 1.3/5

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + b \sinh x, \\
u|_{x=0} = u|_{x=l} = 0, \\
u|_{t=0} = \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0;
\end{cases}
$$

令 $f(x,\tau) = b\sinh x$, 代入齐次化原理公式得

$$
\begin{aligned}
B_k(\tau) &= \frac{2}{l\sqrt{\lambda}a}\int_0^l f(x,\tau)\sin\sqrt{\lambda}x dx = \frac{2}{k\pi a}\int_0^l b\sinh x \sin\frac{k\pi}{l}x dx \\
&= \frac{2bl(l\cosh l \sin k\pi - k\pi \sinh l \cos k\pi)}{k\pi a(k^2\pi^2 + l^2)} \\
&= (-1)^{k+1}\frac{2bl\sinh l}{a(k^2\pi^2+l^2)}.
\end{aligned}
$$

故

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty}\int_0^t B_k(\tau)\sin\sqrt{\lambda}a(t-\tau)d\tau \cdot \sin\sqrt{\lambda}x \\
&= \sum_{k=1}^{\infty}\int_0^t (-1)^{k+1}\frac{2bl\sinh l}{a(k^2\pi^2+l^2)}\sin\frac{k\pi}{l}a(t-\tau)d\tau\cdot\sin\frac{k\pi}{l}x \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2bl\sinh l}{a(k^2\pi^2+l^2)}\cdot\frac{2l\sin^2\frac{k\pi a}{2l}t}{k\pi a}\cdot\sin\frac{k\pi}{l}x \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2bl^2\sinh l}{k\pi a^2(k^2\pi^2+l^2)}\sin^2\frac{k\pi a}{2l}t\sin\frac{k\pi}{l}x.
\end{aligned}
$$

习题 1.3/6

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} + 2b\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}, \\
u|_{x=0} = u|_{x=l} = 0, \\
u|_{t=0} = \dfrac{h}{l}x, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0;
\end{cases}
$$

设 $u(x,t) = X(x)T(t)$, 代入得

$$
X(x)T''(t) + 2bX(x)T'(t) = a^2X''(x)T(t),
$$

$$
\frac{T''(t)+2bT'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda.
$$

则有两个常微分方程

$$
T''(t) + 2bT'(t) + \lambda a^2 T(t) = 0,
$$

$$
X''(x) + \lambda X(x) = 0.
$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k \sin\sqrt{\lambda}x = C_k \sin\frac{k\pi}{l}x, \quad k = 1, 2, \cdots .$$

代入关于 T 的常微分方程可得

$$T''(t) + 2bT'(t) + \frac{k^2\pi^2a^2}{l^2}T(t) = 0,$$

$$\Delta = 4b^2 - \frac{4k^2\pi^2a^2}{l^2},$$

设

$$\lambda' = \frac{k^2\pi^2a^2}{l^2} - b^2,$$

$$r_1 = -b + \sqrt{b^2 - \frac{k^2\pi^2a^2}{l^2}} = -b + \sqrt{-\lambda'}, \quad r_2 = -b - \sqrt{b^2 - \frac{k^2\pi^2a^2}{l^2}} = -b - \sqrt{-\lambda'}.$$

当 $\lambda' < 0$ 时

$$k < \frac{bl}{\pi a}, \quad T(t) = C_1e^{r_1t} + C_2e^{r_2t},$$

$$U_k(x,t) = T_k(t)X_k(x) = (A_ke^{r_1t} + B_ke^{r_2t})\sin\sqrt{\lambda}x,$$

$$\frac{\partial U_k}{\partial t}(x,t) = T_k'(t)X_k(x) = (A_kr_1e^{r_1t} + B_kr_2e^{r_2t})\sin\sqrt{\lambda}x.$$

当 $\lambda' = 0$ 时

$$k = \frac{bl}{\pi a}, \quad T(t) = e^{-bt}(C_1 + C_2t),$$

$$U_k(x,t) = T_k(t)X_k(x) = e^{-bt}(A_k + B_kt)\sin\sqrt{\lambda}x,$$

$$\frac{\partial U_k}{\partial t}(x,t) = T_k'(t)X_k(x) = e^{-bt}(-bA_k + B_k - bB_kt)\sin\sqrt{\lambda}x.$$

当 $\lambda' > 0$ 时

$$k > \frac{bl}{\pi a}, \quad T(t) = e^{-bt}(C_1\cos\sqrt{\lambda'}t + C_2\sin\sqrt{\lambda'}t),$$

$$U_k(x,t) = T_k(t)X_k(x) = e^{-bt}(A_k\cos\sqrt{\lambda'}t + B_k\sin\sqrt{\lambda'}t)\sin\sqrt{\lambda}x,$$

$$\frac{\partial U_k}{\partial t}(x,t) = T_k'(t)X_k(x) = e^{-bt}(-bA_k\cos\sqrt{\lambda'}t - bB_k\sin\sqrt{\lambda'}t - \sqrt{\lambda'}A_k\sin\sqrt{\lambda'}t + \sqrt{\lambda'}B_k\sin\sqrt{\lambda'}t)\sin\sqrt{\lambda}x.$$

设

$$n = \frac{bl}{\pi a}, \quad \underline{n} = \lceil n-1 \rceil, \quad \overline{n} = \lfloor n+1 \rfloor, \quad \delta(n) = \begin{cases} 1, & n \in Z \\ 0, & n \notin Z \end{cases}.$$

方程的通解为

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} T_k(t)X_k(x) \\ &= \left[\sum_{k=1}^{\underline{n}}(A_ke^{r_1t} + B_ke^{r_2t}) + \delta(n)e^{-bt}(A_n + B_nt) + \sum_{k=\overline{n}}^{\infty} e^{-bt}(A_k\cos\sqrt{\lambda'}t + B_k\sin\sqrt{\lambda'}t)\right]\sin\sqrt{\lambda}x. \end{aligned}$$

代入初值条件得

$$u(x,0)=\left[\sum_{k=1}^{\underline{n}}(A_k+B_k)+\delta(n)A_n+\sum_{k=\overline{n}}^{\infty}A_k\right]\sin\sqrt{\lambda}x=\frac{h}{l}x,$$

$$\frac{\partial u}{\partial t}(x,0)=\left[\sum_{k=1}^{\underline{n}}(A_kr_1+B_kr_2)+\delta(n)(-A_n+B_n)+\sum_{k=\overline{n}}^{\infty}(-bA_k+\sqrt{\lambda'}B_k)\right]\sin\sqrt{\lambda}x=0.$$

解得

$$A_k'=\begin{cases}A_k+B_k, & k<n\\ A_k, & k\geqslant n\end{cases}=\frac{2}{l}\int_0^l\frac{h}{l}x\sin\sqrt{\lambda}xdx=\frac{h}{l}\cdot-\frac{2(\sqrt{\lambda}l\cos\sqrt{\lambda}l-\sin\sqrt{\lambda}l)}{\lambda l}=(-1)^{k+1}\frac{2h}{k\pi},$$

$$B_k'=\begin{cases}A_kr_1+B_kr_2, & k<n\\ -A_k+B_k, & k=n\\ -bA_k+\sqrt{\lambda'}B_k, & k>n\end{cases}=0.$$

化简得

$$A_k=\begin{cases}\dfrac{r_2}{r_2-r_1}A_k', & k<n\\ A_k', & k\geqslant n\end{cases},\quad B_k=\begin{cases}-\dfrac{r_1}{r_2-r_1}A_k', & k<n\\ A_k', & k=n\\ \dfrac{b}{\sqrt{\lambda'}}A_k', & k>n\end{cases}.$$

故将 $r_1,r_2,\lambda,\lambda',n,\underline{n},\overline{n},\delta(n),A_k,B_k$ 代入可得 $u(x,t)$.

例题

(1)

$$\begin{cases}u_{tt}-a^2u_{xx}=A\sin\omega t,\\ u|_{x=0}=u_x|_{x=l}=0,\\ u|_{t=0}=1,\quad u_t|_{t=0}=x;\end{cases}$$

利用叠加原理, 设

$$u(x,t)=U(x,t)+V(x,t).$$

先求

$$\begin{cases}U_{tt}-a^2U_{xx}=0,\\ U|_{x=0}=U_x|_{x=l}=0,\\ U|_{t=0}=1,\quad U_t|_{t=0}=x;\end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0$, $X'(l)=0$ 得

$$C_1=0,\quad C_2\sqrt{\lambda}\cos\sqrt{\lambda}l=0,$$

$$\lambda=\lambda_k=\frac{(2k-1)^2\pi^2}{4l^2},\quad X_k(x)=C_k\sin\frac{(2k-1)\pi}{2l}x,\quad k=1,2,\cdots.$$

$U(x,t)$ 的通解为

$$\begin{aligned}U(x,t)&=\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x\\&=\sum_{k=1}^{\infty}\left(A_k\cos\frac{(2k-1)\pi a}{2l}t+B_k\sin\frac{(2k-1)\pi a}{2l}t\right)\sin\frac{(2k-1)\pi}{2l}x.\end{aligned}$$

代入初值条件得

$$\varphi(x)=1=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{(2k-1)\pi}{2l}x,$$

$$\psi(x)=x=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_k(2k-1)\pi a}{2l}\sin\frac{(2k-1)\pi}{2l}x.$$

解得

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\sqrt{\lambda}xdx=\frac{2}{l}\int_0^l\sin\frac{(2k-1)\pi}{2l}xdx=\frac{4(-1+\sin k\pi)}{(2k-1)\pi}=-\frac{4}{(2k-1)\pi},$$

$$B_k=\frac{2}{l\sqrt{\lambda}a}\int_0^l\psi(x)\sin\sqrt{\lambda}xdx=\frac{4}{(2k-1)\pi a}\int_0^l x\sin\frac{(2k-1)\pi}{2l}xdx=-\frac{4l^2[2\cos k\pi+(2k-1)\pi\sin k\pi]}{(2k-1)^3\pi^3a}.$$

化简得

$$B_k=(-1)^{k+1}\frac{8l^2}{(2k-1)^3\pi^3a}.$$

故

$$U(x,t)=\sum_{k=1}^{\infty}\left[-\frac{4}{(2k-1)\pi}\cos\frac{(2k-1)\pi a}{2l}t+(-1)^{k+1}\frac{8l^2}{(2k-1)^3\pi^3a}\sin\frac{(2k-1)\pi a}{2l}t\right]\sin\frac{(2k-1)\pi}{2l}x.$$

再求

$$\begin{cases}V_{tt}-a^2V_{xx}=A\sin\omega t,\\V|_{x=0}=V_x|_{x=l}=0,\\V|_{t=0}=V_t|_{t=0}=0;\end{cases}$$

令 $f(x,\tau)=A\sin\omega t$, 代入齐次化原理公式得

$$\begin{aligned}B_k(\tau)&=\frac{2}{l\sqrt{\lambda}a}\int_0^l f(x,\tau)\sin\sqrt{\lambda}xdx=\frac{4}{(2k-1)\pi a}\int_0^l A\sin\omega\tau\sin\frac{(2k-1)\pi}{2l}xdx\\&=-\frac{8A\sin\omega\tau\cdot l(\sin k\pi-1)}{(2k-1)^2\pi^2a}=\frac{8lA\sin\omega\tau}{(2k-1)^2\pi^2a}.\end{aligned}$$

故

$$
\begin{aligned}
V(x,t) &= \sum_{k=1}^{\infty}\int_0^t B_k(\tau)\sin\sqrt{\lambda}a(t-\tau)d\tau\cdot\sin\sqrt{\lambda}x\\
&= \sum_{k=1}^{\infty}\int_0^t \frac{8lA\sin\omega\tau}{(2k-1)^2\pi^2 a}\sin\frac{(2k-1)\pi a}{2l}(t-\tau)d\tau\cdot\sin\frac{(2k-1)\pi}{2l}x\\
&= \sum_{k=1}^{\infty}\frac{8lA}{(2k-1)^2\pi^2 a}\cdot\frac{-2l[2\omega l\sin\frac{(2k-1)\pi a}{2l}t+(2k-1)\pi a\sin\omega t]}{(2k-1)^2\pi^2a^2-4\omega^2l^2}\cdot\sin\frac{(2k-1)\pi}{2l}x\\
&= \sum_{k=1}^{\infty}-\frac{16l^2A}{(2k-1)^4\pi^4a^3-4(2k-1)^2\pi^2a\omega^2l^2}\left[2\omega l\sin\frac{(2k-1)\pi a}{2l}t+(2k-1)\pi a\sin\omega t\right]\sin\frac{(2k-1)\pi}{2l}x.
\end{aligned}
$$

$$u(x,t)=U(x,t)+V(x,t).$$

(2)

$$
\begin{cases}
u_{tt}-u_{xx}-4u=2\sin^2 x,\\
u_x|_{x=0}=u_x|_{x=\pi}=0,\\
u|_{t=0}=0,\quad u_t|_{t=0}=0;
\end{cases}
$$

将原方程化为

$$
\begin{cases}
t'=t-\tau\\
W_{tt}-W_{xx}-4W=0,\\
W_x|_{x=0}=W_x|_{x=\pi}=0,\\
W|_{t'=0}=0,\quad W_{t'}|_{t'=0}=2\sin^2 x;
\end{cases}
$$

设 $W(x,t)=X(x)T(t)$, 代入得

$$X(x)T''(t)-X''(x)T(t)-4X(x)T(t)=0,$$

$$\frac{T''(t)-4T(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda.$$

则有两个常微分方程

$$T''(t)+(\lambda-4)T(t)=0,$$

$$X''(x)+\lambda X(x)=0.$$

方程的特征值和对应的特征函数为

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X'(0)=0$, $X'(\pi)=0$ 得

$$C_2=0,\quad -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi=0,$$

$$\lambda=\lambda_k=k^2,\quad X_k(x)=C_k\cos\sqrt{\lambda}x=C_k\cos kx,\quad k=0,1,2,\cdots.$$

代入关于 T 的常微分方程可得

$$T''(t)+(k^2-4)T(t)=0,$$

设

$$\lambda'=k^2-4,$$

$$r_1=\sqrt{4-k^2}=\sqrt{-\lambda'},\quad r_2=-\sqrt{4-k^2}=-\sqrt{-\lambda'}.$$

当 $\lambda'<0$ 时

$$k<2,\quad T(t)=C_1e^{\sqrt{-\lambda'}t}+C_2e^{-\sqrt{-\lambda'}t},$$

$$W_k(x,t)=T_k(t)X_k(x)=(A_ke^{\sqrt{-\lambda'}t}+B_ke^{-\sqrt{-\lambda'}t})\cos\sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x,t)=T_k'(t)X_k(x)=\sqrt{-\lambda'}(A_ke^{\sqrt{-\lambda'}t}-B_ke^{-\sqrt{-\lambda'}t})\cos\sqrt{\lambda}x.$$

当 $\lambda'=0$ 时

$$k=2,\quad T(t)=C_1+C_2t,$$

$$W_k(x,t)=T_k(t)X_k(x)=(A_k+B_kt)\cos\sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x,t)=T_k'(t)X_k(x)=B_k\cos\sqrt{\lambda}x.$$

当 $\lambda'>0$ 时

$$k>2,\quad T(t)=C_1\cos\sqrt{\lambda'}t+C_2\sin\sqrt{\lambda'}t,$$

$$W_k(x,t)=T_k(t)X_k(x)=(A_k\cos\sqrt{\lambda'}t+B_k\sin\sqrt{\lambda'}t)\cos\sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x,t)=T_k'(t)X_k(x)=\sqrt{\lambda'}(-A_k sin\sqrt{\lambda'}t+B_k\cos\sqrt{\lambda'}t)\cos\sqrt{\lambda}x.$$

方程的通解为

$$\begin{aligned}W(x,t)&=\sum_{k=0}^{\infty}T_k(t)X_k(x)\\&=\left[\sum_{k=0}^{1}(A_ke^{\sqrt{-\lambda'}t}+B_ke^{-\sqrt{-\lambda'}t})+(A_2+B_2t)+\sum_{k=3}^{\infty}(A_k\cos\sqrt{\lambda'}t+B_k\sin\sqrt{\lambda'}t)\right]\cos\sqrt{\lambda}x.\end{aligned}$$

代入初值条件得

$$W(x,0)=\left[\sum_{k=0}^{1}(A_k+B_k)+A_2+\sum_{k=3}^{\infty}A_k\right]\cos\sqrt{\lambda}x=0,$$

$$\frac{\partial W}{\partial t}(x,0)=\left[\sum_{k=0}^{1}\sqrt{-\lambda'}(A_k-B_k)+B_2+\sum_{k=3}^{\infty}\sqrt{\lambda'}B_k\right]\cos\sqrt{\lambda}x=2\sin^2x=1-\cos 2x.$$

解得

$$A_k'=\begin{cases}A_k+B_k, & k<2\\ A_k, & k\geqslant 2\end{cases}=0,$$

$$B_k'=\begin{cases}\sqrt{-\lambda'}(A_k-B_k), & k<2\\ B_k, & k=2\\ \sqrt{\lambda'}B_k, & k>2\end{cases}=\begin{cases}1, & k=0\\ -1, & k=2\\ 0, & k\neq 0,2\end{cases}.$$

化简得

$$A_k=\begin{cases}\frac{1}{4}, & k=0\\ 0, & k\neq 0\end{cases},\quad B_k=\begin{cases}-\frac{1}{4}, & k=0\\ -1, & k=2\\ 0, & k\neq 0,2\end{cases}\cdot$$

故

$$W(x,t)=\frac{1}{4}(e^{2t}-e^{-2t})-t\cos 2x,$$

$$W(x,t;\tau)=\frac{1}{4}(e^{2(t-\tau)}-e^{-2(t-\tau)})-(t-\tau)\cos 2x,$$

$$u(x,t)=\int_0^t W(x,t;\tau)d\tau=\int_0^t \frac{1}{4}(e^{2(t-\tau)}-e^{-2(t-\tau)})-(t-\tau)\cos 2xd\tau=\frac{1}{4}\cosh 2t-\frac{1}{2}t^2\cos 2x-\frac{1}{4}.$$