MA319 — 偏微分方程

Assignment 8

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习题 2.3/5

求解热传导方程 (3.17) 的柯西问题, 已知

(1) $u|_{t=0} = \sin x$,

(2) 用延拓法求解半有界直线上的热传导方程 (3.17), 假设

$$\begin{cases} u(x,0) = \varphi(x) \quad (0 < x < \infty), \\ u(0,t) = 0. \end{cases}$$

(i)

初值条件为

$$\varphi(x) = u|_{t=0} = \sin x.$$

故

$$\begin{aligned}
u(x,t) &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} \, d\xi \\
&= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(x + 2a\sqrt{t}\eta) e^{-\eta^2} \, d(x + 2a\sqrt{t}\eta) \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sin\left(x + 2a\sqrt{t}\eta\right) e^{-\eta^2} \, d\eta \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\sin x \cos\left(2a\sqrt{t}\eta\right) + \cos x \sin\left(2a\sqrt{t}\eta\right)\right] e^{-\eta^2} \, d\eta \\
&= \frac{\sin x}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos\left(2a\sqrt{t}\eta\right) e^{-\eta^2} \, d\eta \\
&= \frac{\sin x}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \left[e^{-\eta^2 + i2a\sqrt{t}\eta} + e^{-\eta^2 - i2a\sqrt{t}\eta}\right] d\eta \\
&= \frac{\sin x}{2\sqrt{\pi}} e^{-a^2 t} \left[\int_{-\infty}^{\infty} e^{-(\eta - ia\sqrt{t})^2} \, d\eta + \int_{-\infty}^{\infty} e^{-(\eta + ia\sqrt{t})^2} \, d\eta\right] \\
&= \frac{\sin x}{\sqrt{\pi}} e^{-a^2 t} \int_{-\infty}^{\infty} e^{-\chi^2} \, d\chi \\
&= e^{-a^2 t} \sin x.
\end{aligned}$$

(2)

使用奇延拓

$$u(x,0)=\begin{cases}\varphi(x) & (0<x<\infty),\\ 0 & (x=0),\\ -\varphi(-x) & (-\infty<x<0).\end{cases}$$

代入公式可得

$$\begin{aligned}u(x,t)&=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}u(\xi,0)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi\\&=\frac{1}{2a\sqrt{\pi t}}\left(\int_0^{\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi+\int_{-\infty}^{0}-\varphi(-\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi\right)\\&=\frac{1}{2a\sqrt{\pi t}}\left(\int_0^{\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi+\int_{0}^{\infty}-\varphi(\xi)e^{-\frac{(x+\xi)^2}{4a^2t}}d\xi\right)\\&=\frac{1}{2a\sqrt{\pi t}}\int_0^{\infty}\varphi(\xi)e^{-\frac{x^2+\xi^2}{4a^2t}}\left(e^{\frac{2x\xi}{4a^2t}}-e^{-\frac{2x\xi}{4a^2t}}\right)d\xi\\&=\frac{1}{a\sqrt{\pi t}}\int_0^{\infty}\varphi(\xi)e^{-\frac{x^2+\xi^2}{4a^2t}}\sinh\frac{x\xi}{2a^2t}d\xi.\end{aligned}$$

习题 2.3/7

证明: 如果 $u_1(x,t)$, $u_2(x,t)$ 分别是下述两个定解问题的解:

$$\begin{cases}\dfrac{\partial u_1}{\partial t}=a^2\dfrac{\partial^2 u_1}{\partial x^2},\\ u_1|_{t=0}=\varphi_1(x);\end{cases}\qquad\begin{cases}\dfrac{\partial u_2}{\partial t}=a^2\dfrac{\partial^2 u_2}{\partial y^2},\\ u_2|_{t=0}=\varphi_2(y).\end{cases}$$

则 $u(x,y,t)=u_1(x,t)u_2(y,t)$ 是定解问题

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right),\\ u|_{t=0}=\varphi_1(x)\varphi_2(y)\end{cases}$$

的解.

由 $u(x,y,t)=u_1(x,t)u_2(y,t)$ 可得

$$\frac{\partial u}{\partial t}=\frac{\partial u_1}{\partial t}u_2+\frac{\partial u_2}{\partial t}u_1=a^2\left(\frac{\partial^2 u_1}{\partial x^2}u_2+\frac{\partial^2 u_2}{\partial y^2}u_1\right)=a^2\left(\frac{\partial^2 u_1u_2}{\partial x^2}+\frac{\partial^2 u_1u_2}{\partial y^2}\right)=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$

$$u|_{t=0}=u_1|_{t=0}\cdot u_2|_{t=0}=\varphi_1(x)\varphi_2(y).$$

故得证.

习题　2.3/8

导出下列热传导方程柯西问题的解的表达式:

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right),\\ u|_{t=0}=\displaystyle\sum_{i=0}^{n}\alpha_i(x)\beta_i(y).\end{cases}$$

由上题结论易知

$$\begin{cases}\dfrac{\partial u_i}{\partial t}=a^2\left(\dfrac{\partial^2 u_i}{\partial x^2}+\dfrac{\partial^2 u_i}{\partial y^2}\right),\\ u_i|_{t=0}=\alpha_i(x)\beta_i(y).\end{cases}$$

的解 $u_i(x,y,t)$ 为

$$\begin{aligned}u_i(x,y,t)&=u_{i_1}(x,t)u_{i_2}(y,t)\\&=\frac{1}{2a\sqrt{\pi}t}\int_{-\infty}^{\infty}\alpha_i(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi\cdot\frac{1}{2a\sqrt{\pi}t}\int_{-\infty}^{\infty}\beta_i(\eta)e^{-\frac{(y-\eta)^2}{4a^2t}}d\eta\\&=\frac{1}{4\pi a^2t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_i(\xi)\beta_i(\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}d\xi d\eta.\end{aligned}$$

根据叠加原理

$$u(x,y,t)=\sum_{i=0}^{n}u_i(x,y,t)=\sum_{i=0}^{n}\frac{1}{4\pi a^2t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_i(\xi)\beta_i(\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}d\xi d\eta.$$

习题　2.4/2

利用证明热传导方程极值原理的方法, 证明满足方程 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ 的函数在有界闭区域上的最大值不会超过它在边界上的最大值.

设 $u(x,y)$ 在区域 Ω 内连续, 并且在区域内部满足方程 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, 且 Γ 为区域 Ω 的边界. 设 M 为 Ω 内 $u(x,y)$ 的最大值, m 为 Γ 上 $u(x,y)$ 的最大值. 使用反证法, 如果命题不真, 那么 $M>m$. 此时在 Ω 内一定存在着一点 (x^*,y^*), 使函数 $u(x,y)$ 在该点取值 M. 作函数

$$V(x,y)=u(x,y)+\frac{M-m}{4R^2}[(x-x^*)^2+(y-y^*)^2],$$

其中 $R^2>(x-x^*)^2+(y-y^*)^2$. 由于在 Γ 上

$$V(x,y)<m+\frac{M-m}{4}=\frac{M}{4}+\frac{3}{4}m=\theta M\quad(0<\theta<1),$$

而

$$V(x^*,y^*)=M,$$

因此, 函数 $V(x,y)$ 和 $u(x,y)$ 一样, 它不在 Γ 上取到最大值. 设 $V(x,y)$ 在 Ω 中的某一点 (x_1,y_1) 上取到

最大值, 则在此点应有 $\dfrac{\partial^2 V}{\partial x^2} \leqslant 0$, $\dfrac{\partial^2 V}{\partial y^2} \leqslant 0$, 因此在点 (x_1, y_1) 处

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \leqslant 0.$$

但由直接计算方程可得

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{M-m}{2R^2} + \frac{\partial^2 u}{\partial y^2} + \frac{M-m}{2R^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{M-m}{R^2} > 0,$$

这就得到矛盾. 这说明原先的假设时不正确的, 证毕.

习题　2.4/3

导出初边值问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t), \\ u|_{x=0} = \mu_1(x), \quad \left(\dfrac{\partial u}{\partial x} + hu\right)\Big|_{x=l} = \mu_2(t) \quad (h>0) \\ u|_{t=0} = \varphi(x) \end{cases}$$

的解 $u(x,t)$ 在 $R_T : \{0 \leqslant t \leqslant T, 0 \leqslant x \leqslant l\}$ 中满足的估计.

$$u(x,t) \leqslant e^{\lambda T} \max\left(0, \max_{0\leqslant x\leqslant l} \varphi(x), \max_{0\leqslant t\leqslant T}\left(e^{-\lambda t}\mu_1(t), \frac{e^{-\lambda t}\mu_2(t)}{h}\right), \frac{1}{\lambda}\max_{R_T}(e^{-\lambda t} f)\right),$$

其中 $\lambda > 0$ 为任意正常数.

设

$$v(x,t) = u(x,t)e^{-\lambda t}.$$

则

$$u_t - a^2 u_{xx} - f(x,t) = v_t e^{\lambda t} + \lambda v e^{\lambda t} - a^2 v_{xx} e^{\lambda t} - f(x,t) = v_t - a^2 v_{xx} + \lambda v - f(x,t)e^{-\lambda t} = 0,$$

$$v|_{x=0} = (ue^{-\lambda t})|_{x=0} = \mu_1(x)e^{-\lambda t}, \quad \left(\frac{\partial v}{\partial x} + hv\right)\Big|_{x=l} = \left(\frac{\partial u}{\partial x}e^{-\lambda t} + hue^{-\lambda t}\right)\Big|_{x=l} = \mu_2(t)e^{-\lambda t},$$

$$v|_{t=0} = (ue^{-\lambda t})|_{t=0} = \varphi(x).$$

故可以先求以下初边值问题 $v(x,t)$ 在 R_T 中满足的估计:

$$\begin{cases} v_t - a^2 v_{xx} + \lambda v = f(x,t)e^{-\lambda t}, \\ v|_{x=0} = \mu_1(x)e^{-\lambda t}, \quad \left(\dfrac{\partial v}{\partial x} + hv\right)\Big|_{x=l} = \mu_2(t)e^{-\lambda t} \quad (h>0) \\ v|_{t=0} = \varphi(x) \end{cases}$$

若 v 在 (x^*, t^*) 取正的极大值, 则 $v(x^*, t^*) > 0$, 由极值原理知 v 的极大值只能在边界 $x = 0$, $x = l$ 或 $t = 0$ 上取到.

在 $(x^*, t^*) \in \{x = 0\}$ 或 $\{t = 0\}$ 的情形, 可知

$$v(x^*, t^*) \leqslant \max_{0 \leqslant t \leqslant T} e^{-\lambda t} \mu_1(t),$$

$$v(x^*, t^*) \leqslant \max_{0 \leqslant x \leqslant l} \varphi(x).$$

在 $(x^*, t^*) \in \{x = l\}$ 的情形, 由 $\dfrac{\partial v}{\partial x} \geqslant 0$ 可得

$$hv \leqslant e^{-\lambda t} \mu_2(t),$$

$$v(x^*, t^*) \leqslant \max_{0 \leqslant t \leqslant T} \frac{e^{-\lambda t} \mu_2(t)}{h}.$$

又由于在正的极大值点, $v_t \leqslant 0$, $v_{xx} \leqslant 0$, 可得

$$\lambda v \leqslant f(x, t) e^{-\lambda t},$$

$$v(x^*, t^*) \leqslant \frac{1}{\lambda} \max_{R_T} (e^{-\lambda} f).$$

综上可得

$$v(x, t) \leqslant \max \left(0, \max_{0 \leqslant x \leqslant l} \varphi(x), \max_{0 \leqslant t \leqslant T} \left(e^{-\lambda t} \mu_1(t), \frac{e^{-\lambda t} \mu_2(t)}{h} \right), \frac{1}{\lambda} \max_{R_T} (e^{-\lambda t} f) \right).$$

故

$$u(x, t) = e^{\lambda t} v(x, t) \leqslant e^{\lambda T} v(x, t) \leqslant e^{\lambda T} \max \left(0, \max_{0 \leqslant x \leqslant l} \varphi(x), \max_{0 \leqslant t \leqslant T} \left(e^{-\lambda t} \mu_1(t), \frac{e^{-\lambda t} \mu_2(t)}{h} \right), \frac{1}{\lambda} \max_{R_T} (e^{-\lambda t} f) \right).$$