MA319 — 偏微分方程

Assignment 7

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习题 2.2/3

如果有一长度为 l 的均匀细棒，其周围及两端 $x=0$，$x=l$ 均为绝热，初始温度分布为 $u(x,0)=f(x)$，问以后时刻的温度分布如何？且证明当 $f(x)$ 等于常数 u_0 时，恒有 $u(x,t)=u_0$.

题目可归结为求解如下的定解问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(l,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x,$$

代入 $X'(0)=0$，$X'(l)=0$ 得

$$C_2 = 0, \quad C_1\sqrt{\lambda}\sin\sqrt{\lambda}l = 0,$$

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k\cos\frac{k\pi}{l}x, \quad k = 0,1,2,\cdots.$$

方程的通解为

$$u(x,t) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k e^{-a^2\lambda t}\cos\sqrt{\lambda}x = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2}{l^2}a^2 t}\cos\frac{k\pi}{l}x.$$

代入初值条件得

$$u(x,0) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\cos\sqrt{\lambda}x = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\cos\frac{k\pi}{l}x = f(x).$$

解得

$$A_k = \frac{2}{l}\int_0^l f(x)\cos\sqrt{\lambda}x dx = \frac{2}{l}\int_0^l f(x)\cos\frac{k\pi}{l}x dx.$$

故

$$u(x,t) = \frac{1}{l}\int_0^l f(x)dx + \sum_{k=1}^{\infty}\frac{2}{l}\int_0^l f(x)\cos\frac{k\pi}{l}x dx \cdot e^{-\frac{k^2\pi^2}{l^2}a^2 t}\cos\frac{k\pi}{l}x.$$

当 $f(x)$ 等于常数 u_0 时

$$A_k = \begin{cases} 2u_0, & k = 0, \\ \dfrac{u_0 l \sin k\pi}{k\pi} = 0, & k > 0. \end{cases}$$

故

$$u(x,t) = \frac{1}{2}A_0 = u_0.$$

习题　2.2/4

在区域 $t > 0$, $0 < x < l$ 中求解如下的定解问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \beta(u - u_0), \\ u(0,t) = u(1,t) = u_0, \\ u(x,0) = f(x), \end{cases}$$

其中 a, β, u_0 均为常数, $f(x)$ 为已知函数.

设

$$u(x,t) = u_0 + v(x,t)e^{-\beta t}.$$

则

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \beta(u - u_0) = \frac{\partial v}{\partial t}e^{-\beta t} - \beta v e^{-\beta t} - \frac{\partial^2 v}{\partial x^2}e^{-\beta t} + \beta v e^{-\beta t} = e^{-\beta t}\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right) = 0,$$

$$v(0,t)e^{-\beta t} = v(l,t)e^{-\beta t} = u(0,t) - u_0 = u(l,t) - u_0 = 0,$$

$$v(x,0) = u(x,0) - u_0 = f(x) - u_0.$$

故可以先求解如下定解问题:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2}, \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = f(x) - u_0. \end{cases}$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k \sin\sqrt{\lambda}x = C_k \sin\frac{k\pi}{l}x, \quad k = 1, 2, \cdots.$$

方程的通解为

$$v(x,t) = \sum_{k=1}^{\infty} A_k e^{-a^2\lambda t}\sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2}{l^2}a^2 t}\sin\frac{k\pi}{l}x.$$

代入初值条件得

$$v(x,0) = \sum_{k=1}^{\infty} A_k \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x = f(x) - u_0.$$

解得

$$A_k=\frac{2}{l}\int_0^l[f(x)-u_0]\sin\sqrt{\lambda}xdx=\frac{2}{l}\int_0^l[f(x)-u_0]\sin\frac{k\pi}{l}xdx=\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx-\frac{2u_0(1-\cos k\pi)}{k\pi}.$$

化简得

$$A_k=\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx+\frac{2u_0[(-1)^k-1]}{k\pi}.$$

故

$$v(x,t)=\sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx+\frac{2u_0[(-1)^k-1]}{k\pi}\right\}e^{-\frac{k^2\pi^2}{l^2}a^2t}\sin\frac{k\pi}{l}x,$$

$$u(x,t)=u_0+e^{-\beta t}\sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx+\frac{2u_0[(-1)^k-1]}{k\pi}\right\}e^{-\frac{k^2\pi^2}{l^2}a^2t}\sin\frac{k\pi}{l}x.$$

习题 2.2/5

长度为 l 的均匀细杆的初始温度为 $0^\circ C$, 端点 $x=0$ 保持常温 u_0, 而在 $x=l$ 和侧面上, 热量可以发散到周围的介质中去, 介质的温度为 $^\circ C$, 此时杆上的温度分布函数 $u(x,t)$ 满足下述定解问题:

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}-b^2u,\\ u(0,t)=u_0,\quad \left(\dfrac{\partial u}{\partial x}+Hu\right)\Big|_{x=l}=0,\\ u(x,0)=0,\end{cases}$$

其中 a,b,H 均为常数, 试求出 $u(x,t)$.

设

$$u(x,t)=v(x,t)e^{-b^2t}+f(x).$$

则

$$\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}+b^2u=\frac{\partial v}{\partial t}e^{-b^2t}-b^2ve^{-b^2t}-a^2\frac{\partial^2 v}{\partial x^2}e^{-b^2t}-a^2f''(x)+b^2ve^{-b^2t}+b^2f(x)=0,$$

$$v(0,t)e^{-b^2t}+f(0)=u_0,\quad \left(\frac{\partial v}{\partial x}+Hv\right)\Big|_{x=l}+f'(l)+Hf(l)=0,$$

$$v(x,0)+f(x)=0.$$

可找到 $f(x)$ 使以下成立

$$\left(\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}\right)e^{-b^2t}=0,\quad f''(x)-\frac{b^2}{a^2}f(x)=0,$$

$$v(0,t)e^{-b^2t}=0,\quad f(0)=u_0,\quad \left(\frac{\partial v}{\partial x}+Hv\right)\Big|_{x=l}=0,\quad f'(l)+Hf(l)=0.$$

故 $f(x)$ 的通解为

$$f(x)=C_1e^{\frac{b}{a}x}+C_2e^{-\frac{b}{a}x}.$$

代入初值条件得

$$f(0)=C_1+C_2=u_0,$$

$$f'(l)+Hf(l)=\frac{b}{a}\left(C_1e^{\frac{bl}{a}}-C_2e^{-\frac{bl}{a}}\right)+H\left(C_1e^{\frac{bl}{a}}+C_2e^{-\frac{bl}{a}}\right)=0.$$

解得

$$C_1=\frac{u_0(b-aH)}{aHe^{\frac{2bl}{a}}+be^{\frac{2bl}{a}}-aH+b},\quad C_2=\frac{u_0(aH+b)e^{\frac{2bl}{a}}}{aHe^{\frac{2bl}{a}}+be^{\frac{2bl}{a}}-aH+b},$$

$$f(x)=\frac{u_0e^{-\frac{bx}{a}}\left[aH\left(e^{\frac{2bl}{a}}-e^{\frac{2bx}{a}}\right)+b\left(e^{\frac{2bl}{a}}+e^{\frac{2bx}{a}}\right)\right]}{aH\left(e^{\frac{2bl}{a}}-1\right)+b\left(e^{\frac{2bl}{a}}+1\right)}=\frac{u_0\left[aH\sinh\left(\frac{b(l-x)}{a}\right)+b\cosh\left(\frac{b(l-x)}{a}\right)\right]}{aH\sinh\left(\frac{bl}{a}\right)+b\cosh\left(\frac{bl}{a}\right)}.$$

故可以先求解如下定解问题:

$$\begin{cases}\dfrac{\partial v}{\partial t}=a^2\dfrac{\partial^2 v}{\partial x^2},\\ v(0,t)=0,\quad \left(\dfrac{\partial v}{\partial x}+Hv\right)\Big|_{x=l}=0,\\ v(x,0)=-f(x).\end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0$, $X'(l)+HX(l)=0$ 得

$$C_1=0,\quad C_2(\sqrt{\lambda}\cos\sqrt{\lambda}l+H\sin\sqrt{\lambda}l)=0,$$

$$v=\sqrt{\lambda}l,\quad \tan v=-\frac{\sqrt{\lambda}}{H}.$$

存在无数个正根 $v_k>0$, 满足

$$\left(k-\frac{1}{2}\right)\pi<v_k<k\pi,\quad \lambda_k=\left(\frac{v_k}{l}\right)^2,\quad X_k(x)=C_k\sin\frac{v_k}{l}x,\quad k=1,2,\cdots$$

方程的通解为

$$v(x,t)=\sum_{k=1}^{\infty}A_ke^{-a^2\lambda t}\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_ke^{-\frac{v_k^2}{l^2}a^2t}\sin\frac{v_k}{l}x.$$

代入初值条件得

$$v(x,0)=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{v_k}{l}x=-f(x).$$

由固有函数系 $\{X_k\}=\{\sin\sqrt{\lambda_k}x\}$ 的正交性可得

$$M_k=\int_0^l\sin^2\sqrt{\lambda_k}xdx=\frac{l}{2}+\frac{H}{2(H^2+\lambda_k)}=\frac{l}{2}+\frac{Hl^2}{2(H^2l^2+v_k^2)},$$

$$A_k=\frac{1}{M_k}\int_0^l-f(x)\sin\sqrt{\lambda_k}xdx=-\frac{1}{M_k}\int_0^lf(x)\sin\frac{v_k}{l}xdx.$$

故

$$v(x,t)=-\sum_{k=1}^{\infty}\frac{1}{M_k}\int_0^l f(\xi)\sin\frac{v_k}{l}\xi d\xi\cdot e^{-\frac{v_k^2}{l^2}a^2t}\sin\frac{v_k}{l}x,$$

$$u(x,t)=-e^{-b^2t}\sum_{k=1}^{\infty}\frac{1}{M_k}\int_0^l f(\xi)\sin\frac{v_k}{l}\xi d\xi\cdot e^{-\frac{v_k^2}{l^2}a^2t}\sin\frac{v_k}{l}x+\frac{u_0\left[aH\sinh\left(\frac{b(l-x)}{a}\right)+b\cosh\left(\frac{b(l-x)}{a}\right)\right]}{aH\sinh\left(\frac{bl}{a}\right)+b\cosh\left(\frac{bl}{a}\right)}.$$

习题 2.3/1

求下列函数的傅里叶变换:

(1) $e^{-\eta x^2}$ $(\eta>0)$;

(2) $e^{-a|x|}$ $(a>0)$.

(1)

$$g(\lambda)=\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}d\xi=\int_{-\infty}^{\infty}e^{-\eta\xi^2-i\lambda\xi}d\xi=\int_{-\infty}^{\infty}e^{-\eta\xi^2-i\lambda\xi+\frac{\lambda^2}{4\eta}-\frac{\lambda^2}{4\eta}}d\xi=e^{-\frac{\lambda^2}{4\eta}}\int_{-\infty}^{\infty}e^{-\eta\left(\xi+\frac{i\lambda}{2\eta}\right)^2}d\xi.$$

设

$$\int_{-\infty}^{\infty}e^{-\eta x^2}dx=\int_{-\infty}^{\infty}e^{-\eta y^2}dy=A,$$

$$A^2=\int_{-\infty}^{\infty}e^{-\eta x^2}dx\int_{-\infty}^{\infty}e^{-\eta y^2}dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\eta(x^2+y^2)^2}dxdy.$$

令 $r^2=x^2+y^2$

$$A^2=\int_0^{2\pi}\int_0^{\infty}e^{-\eta r^2}rdrd\theta=2\pi\int_0^{\infty}-\frac{1}{2\eta}e^{-\eta r^2}d(-\eta r^2)=\frac{\pi}{\eta}\int_{-\infty}^{0}e^s ds=\frac{\pi}{\eta},$$

$$g(\lambda)=e^{-\frac{\lambda^2}{4\eta}}\int_{-\infty}^{\infty}e^{-\eta\left(\xi+\frac{i\lambda}{2\eta}\right)^2}d\left(\xi+\frac{i\lambda}{2\eta}\right)=e^{-\frac{\lambda^2}{4\eta}}\int_{-\infty}^{\infty}e^{-\eta x^2}dx=Ae^{-\frac{\lambda^2}{4\eta}}=\sqrt{\frac{\pi}{\eta}}e^{-\frac{\lambda^2}{4\eta}}.$$

(2)

$$g(\lambda)=\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}d\xi=\int_{-\infty}^{\infty}e^{-a|\xi|-i\lambda\xi}d\xi=\int_{-\infty}^{\infty}e^{-a|\xi|}\cos\lambda\xi d\xi+\int_{-\infty}^{\infty}e^{-a|\xi|}i\sin\lambda\xi d\xi.$$

由于 $e^{-a|\xi|}$, $\cos\lambda\xi$ 是偶函数, $i\sin\lambda\xi$ 是奇函数,

$$\int_{-\infty}^{\infty}e^{-a|\xi|}\cos\lambda\xi d\xi=2\int_0^{\infty}e^{-a\xi}\cos\lambda\xi d\xi,\quad \int_{-\infty}^{\infty}e^{-a|\xi|}i\sin\lambda\xi d\xi=0,$$

$$g(\lambda)=2\int_0^{\infty}e^{-a\xi}\cos\lambda\xi d\xi=\frac{e^{-a\xi}(\lambda\sin(\lambda\xi)-a\cos(\lambda\xi))}{a^2+\lambda^2}\Bigg|_0^{\infty}=\frac{a}{a^2+\lambda^2}.$$

习题 2.3/2

证明: 当 $f(x)$ 在 $(-\infty,\infty)$ 上绝对可积时, $F[f]$ 为连续函数.

$$F[f](\lambda+h)-F[f](\lambda)=\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}d\xi-\int_{-\infty}^{\infty}f(\xi)e^{-i(\lambda+h)\xi}d\xi=\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}(1-e^{-ih\xi})d\xi,$$

$$|F[f](\lambda+h)-F[f](\lambda)|=\left|\int_{-\infty}^{\infty}f(\xi)e^{-i\lambda\xi}(1-e^{-ih\xi})d\xi\right|\leqslant\int_{-\infty}^{\infty}|f(\xi)|\left|e^{-i\lambda\xi}\right|\left|1-e^{-ih\xi}\right|d\xi.$$

易知当 $\xi\in\mathbf{R}$ 时,

$$\left|e^{-i\lambda\xi}\right|=\sqrt{\cos^2\lambda\xi+\sin^2\lambda\xi}=1,\quad \left|1-e^{-ih\xi}\right|=\sqrt{(1-\cos h\xi)^2+\sin^2 h\xi}=\sqrt{2}\cdot\sqrt{1-\cos h\xi},$$

$$|F[f](\lambda+h)-F[f](\lambda)|\leqslant\int_{-\infty}^{\infty}|f(\xi)|\sqrt{2}\cdot\sqrt{1-\cos h\xi}d\xi.$$

现只需证明, 任取 $\varepsilon>0$, 存在 $h>0$ 使得 $|F[f](\lambda+h)-F[f](\lambda)|\leqslant\varepsilon$. 将上式拆分为三个区间的积分

$$|F[f](\lambda+h)-F[f](\lambda)|\leqslant\sqrt{2}\left(\int_{-\infty}^{A}|f(\xi)|\sqrt{1-\cos h\xi}d\xi+\int_{A}^{B}|f(\xi)|\sqrt{1-\cos h\xi}d\xi+\int_{B}^{\infty}|f(\xi)|\sqrt{1-\cos h\xi}d\xi\right).$$

由于 $f(x)$ 在 $(-\infty,\infty)$ 上绝对可积, 可找到 $A<0$, $B>0$ 使得

$$\sqrt{2}\int_{-\infty}^{A}|f(\xi)|\sqrt{1-\cos h\xi}d\xi\leqslant\sqrt{2}\int_{-\infty}^{A}|f(\xi)|d\xi\leqslant\frac{\varepsilon}{3},$$

$$\sqrt{2}\int_{B}^{\infty}|f(\xi)|\sqrt{1-\cos h\xi}d\xi\leqslant\sqrt{2}\int_{B}^{\infty}|f(\xi)|d\xi\leqslant\frac{\varepsilon}{3}.$$

设

$$C=\max\{-A,B\},\quad M=\sup_{x\in[A,B]}|f(x)|,\quad L=B-A,$$

$$\sqrt{2}\int_{A}^{B}|f(\xi)|\sqrt{1-\cos h\xi}d\xi\leqslant\sqrt{2}(B-A)\sup_{x\in[A,B]}|f(x)|\sup_{x\in[A,B]}\sqrt{1-\cos hx}=\sqrt{2}ML\sqrt{1-\cos hC}.$$

令

$$h\leqslant\frac{1}{C}\arccos\left(1-\frac{\varepsilon^2}{18M^2L^2}\right),$$

则

$$\sqrt{2}\int_{A}^{B}|f(\xi)|\sqrt{1-\cos h\xi}d\xi\leqslant\frac{\varepsilon}{3},$$
$$|F[f](\lambda+h)-F[f](\lambda)|\leqslant\varepsilon.$$

故得证.