MA319 — 偏微分方程

Assignment 2

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习题　1.2/3

方程的通解为

$$u(x,t) = F(x-at) + G(x+at),$$

代入初值条件得

$$\varphi(x) = u|_{x-at=0} = F(0) + G(2x),$$

$$\psi(x) = u|_{x+at=0} = F(2x) + G(0).$$

故

$$F(x) = \psi(x/2) - G(0),$$

$$G(x) = \varphi(x/2) - F(0),$$

$$\varphi(0) = \psi(0) = F(0) + G(0).$$

$$u(x,t) = F(x-at) + G(x+at) = \psi\left(\frac{x-at}{2}\right) + \varphi\left(\frac{x+at}{2}\right) - \varphi(0).$$

习题　1.2/4

非齐次初值问题的解为

$$u(x,t) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi d\tau.$$

其中与 $\varphi(x)$ 和 $\psi(x)$ 有关的定义域范围都是 $[x-at, x+at]$.

(1)

区间 $[x_1, x_2]$ 的影响区域为

$$x_1 - at \leqslant x \leqslant x_2 + at,$$

不受影响的区域为

$$x \leqslant x_1 - at \quad \text{和} \quad x \geqslant x_2 + at,$$

对应的 $\varphi(x)$ 和 $\psi(x)$ 定义域范围是

$$x \leqslant x_1 \quad \text{和} \quad x \geqslant x_2.$$

故当 $\varphi(x)$ 和 $\psi(x)$ 在 $[x_1, x_2]$ 上变化时, 以上定义域内函数取值不变, 对应解也不变.

(2)

区间 $[x_1, x_2]$ 的决定区域为

$$x_1 + at \leqslant x \leqslant x_2 - at,$$

对应的 $\varphi(x)$ 和 $\psi(x)$ 定义域范围是

$$x_1 \leqslant x \leqslant x_2.$$

故 $[x_1, x_2]$ 上所给的初始条件唯一地确定该区间解的数值.

习题 1.2/5

方程的通解为

$$u(x, t) = F(x - at) + G(x + at),$$

代入初值条件得

$$\varphi(x) = u|_{t=0} = F(x) + G(x),$$

$$0 = u_t|_{t=0} = a[-F'(x) + G'(x)].$$

$$0 = u_x - ku_t \mid_{x=0} = F'(-at) + G'(at) - ka[-F'(-at) + G'(at)].$$

故当 $x \geqslant 0$ 时

$$F(x) - G(x) = C = F(0) - G(0),$$

$$F(x) = \frac{1}{2}\varphi(x) + \frac{C}{2} = \frac{1}{2}\varphi(x) + \frac{F(0) - G(0)}{2},$$

$$G(x) = \frac{1}{2}\varphi(x) - \frac{C}{2} = \frac{1}{2}\varphi(x) - \frac{F(0) - G(0)}{2}.$$

$$(ka + 1)F'(-at) - (ka - 1)G'(at) = 0,$$

$$(ka + 1)F'(-x) - (ka - 1)G'(x) = 0,$$

$$\int_0^x [(ka + 1)F'(-\xi) - (ka - 1)G'(\xi)]d\xi = 0,$$

$$-(ka + 1)F(-x) - (ka - 1)G(x) = C_1 = -(ka + 1)F(0) - (ka - 1)G(0),$$

$$F(-x) = \frac{(1 - ka)G(x) - C_1}{1 + ka} = \frac{1 - ka}{1 + ka}G(x) + F(0) + \frac{ka - 1}{1 + ka}G(0).$$

当 $x - at \geqslant 0$ 时

$$u(x, t) = F(x - at) + G(x + at) = \frac{\varphi(x - at) + \varphi(x + at)}{2}.$$

当 $x - at < 0$ 时

$$u(x, t) = F(-(at - x)) + G(x + at)$$
$$= \frac{1 - ka}{1 + ka} G(at - x) + F(0) + \frac{ka - 1}{1 + ka} G(0) + G(x + at)$$
$$= \frac{1 - ka}{1 + ka} \left[\frac{1}{2} \varphi(at - x) - \frac{F(0) - G(0)}{2} \right] + F(0) + \frac{ka - 1}{1 + ka} G(0) + \frac{1}{2} \varphi(x + at) - \frac{F(0) - G(0)}{2}$$
$$= \frac{1 - ka}{2(1 + ka)} \varphi(at - x) + \frac{1}{2} \varphi(x + at) + \frac{[-(1 - ka) + (1 + ka)] F(0) + [-(1 - ka) + (1 + ka)] G(0)}{2(1 + ka)}$$
$$= \frac{1 - ka}{2(1 + ka)} \varphi(at - x) + \frac{1}{2} \varphi(x + at) + \frac{ka}{1 + ka} \varphi(0).$$

故

$$u(x, t) = \frac{1}{2} \varphi(x + at) + \begin{cases} \dfrac{1}{2} \varphi(x - at), & x - at \geqslant 0 \\ \dfrac{1 - ka}{2(1 + ka)} \varphi(at - x) + \dfrac{ka}{1 + ka} \varphi(0), & x - at < 0 \end{cases}.$$

习题 1.2/6

方程的通解为

$$u(x, t) = F(x - t) + G(x + t),$$

代入初值条件得

$$\varphi_0(x) = u|_{t=0} = F(x) + G(x),$$

$$\varphi_1(x) = u_t|_{t=0} = a[-F'(x) + G'(x)].$$

$$\psi(x) = u|_{t=kx} = F((1 - k)x) + G((1 + k)x).$$

故当 $x \geqslant 0$ 时

$$F(x) - G(x) = -\int_{x_0}^{x} \psi(\alpha) d\alpha + C,$$

$$F(x) = \frac{1}{2} \varphi_0(x) - \frac{1}{2} \int_{x_0}^{x} \varphi_1(\alpha) d\alpha + \frac{C}{2},$$

$$G(x) = \frac{1}{2} \varphi_0(x) + \frac{1}{2} \int_{x_0}^{x} \varphi_1(\alpha) d\alpha - \frac{C}{2}.$$

当 $x - t \geqslant 0$ 时

$$u(x, t) = F(x - t) + G(x + t) = \frac{\varphi_0(x - t) + \varphi_0(x + t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \varphi_1(\alpha) d\alpha.$$

当 $x - t \leqslant 0$ 时

$$u(x, x) = F(x - x) + G(x + x) = F(0) + G(2x) = \frac{\varphi_0(0) + \varphi_0(2x)}{2} + \frac{1}{2} \int_{0}^{2x} \varphi_1(\alpha) d\alpha,$$

$$G(x) = \frac{\varphi_0(0) + \varphi_0(x)}{2} + \frac{1}{2} \int_{0}^{x} \varphi_1(\alpha) d\alpha - F(0),$$

$$F((1-k)x) = \psi(x) - G((1+k)x) = \psi(x) - \frac{\varphi_0(0) + \varphi_0((1+k)x)}{2} - \frac{1}{2}\int_0^{(1+k)x} \varphi_1(\alpha)d\alpha + F(0),$$

$$F(x) = \psi\left(\frac{x}{1-k}\right) - \frac{1}{2}\varphi_0(0) - \frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}x\right) - \frac{1}{2}\int_0^{\frac{1+k}{1-k}x} \varphi_1(\alpha)d\alpha + F(0).$$

$$u(x,t) = F(x-t) + G(x+t)$$
$$= \psi\left(\frac{x-t}{1-k}\right) - \frac{1}{2}\varphi_0(0) - \frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}(x-t)\right) - \frac{1}{2}\int_0^{\frac{1+k}{1-k}(x-t)} \varphi_1(\alpha)d\alpha + F(0) +$$
$$\frac{\varphi_0(0) + \varphi_0(x+t)}{2} + \frac{1}{2}\int_0^{x+t} \varphi_1(\alpha)d\alpha - F(0)$$
$$= \psi\left(\frac{x-t}{1-k}\right) - \frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}(x-t)\right) + \frac{1}{2}\varphi_0(x+t) + \frac{1}{2}\int_{\frac{1+k}{1-k}(x-t)}^{x+t} \varphi_1(\alpha)d\alpha.$$

故

$$u(x,t) = \begin{cases} \dfrac{\varphi_0(x-t) + \varphi_0(x+t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} \varphi_1(\alpha)d\alpha, & 0 \leqslant t \leqslant x \\ \psi\left(\dfrac{x-t}{1-k}\right) - \dfrac{1}{2}\varphi_0\left(\dfrac{1+k}{1-k}(x-t)\right) + \dfrac{1}{2}\varphi_0(x+t) + \dfrac{1}{2}\displaystyle\int_{\frac{1+k}{1-k}(x-t)}^{x+t} \varphi_1(\alpha)d\alpha, & x < t < kx \end{cases}.$$