MA319 — 偏微分方程

Assignment 8

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习题 2.3/5

求解热传导方程 (3.17) 的柯西问题, 已知

(1) $u|_{t=0} = \sin x$,

(2) 用延拓法求解半有界直线上的热传导方程 (3.17), 假设

$$\begin{cases} u(x,0) = \varphi(x) \quad (0 < x < \infty), \\ u(0,t) = 0. \end{cases}$$

(i)

初值条件为

$$\varphi(x) = u|_{t=0} = \sin x.$$

故

$$\begin{aligned} u(x,t) &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} \, d\xi \\ &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(x + 2a\sqrt{t}\eta) e^{-\eta^2} \, d(x + 2a\sqrt{t}\eta) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sin\left(x + 2a\sqrt{t}\eta\right) e^{-\eta^2} \, d\eta \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\sin x \cos\left(2a\sqrt{t}\eta\right) + \cos x \sin\left(2a\sqrt{t}\eta\right)\right] e^{-\eta^2} \, d\eta \\ &= \frac{\sin x}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos\left(2a\sqrt{t}\eta\right) e^{-\eta^2} \, d\eta \\ &= \frac{\sin x}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \left[e^{-\eta^2 + i2a\sqrt{t}\eta} + e^{-\eta^2 - i2a\sqrt{t}\eta}\right] d\eta \\ &= \frac{\sin x}{2\sqrt{\pi}} e^{-a^2 t} \left[\int_{-\infty}^{\infty} e^{-(\eta - ia\sqrt{t})^2} \, d\eta + \int_{-\infty}^{\infty} e^{-(\eta + ia\sqrt{t})^2} \, d\eta\right] \\ &= \frac{\sin x}{\sqrt{\pi}} e^{-a^2 t} \int_{-\infty}^{\infty} e^{-\chi^2} \, d\chi \\ &= e^{-a^2 t} \sin x. \end{aligned}$$

(2)

使用奇延拓

$$u(x,0)=\begin{cases}\varphi(x) & (0<x<\infty),\\ 0 & (x=0),\\ -\varphi(-x) & (-\infty<x<0).\end{cases}$$

带入公式可得

$$\begin{aligned}u(x,t)&=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}u(\xi,0)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi\\&=\frac{1}{2a\sqrt{\pi t}}\left(\int_0^{\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi+\int_{-\infty}^{0}-\varphi(-\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi\right)\\&=\frac{1}{2a\sqrt{\pi t}}\left(\int_0^{\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi+\int_{0}^{\infty}-\varphi(\xi)e^{-\frac{(x+\xi)^2}{4a^2t}}d\xi\right)\\&=\frac{1}{2a\sqrt{\pi t}}\int_0^{\infty}\varphi(\xi)e^{-\frac{x^2+\xi^2}{4a^2t}}\left(e^{\frac{2x\xi}{4a^2t}}-e^{-\frac{2x\xi}{4a^2t}}\right)d\xi\\&=\frac{1}{a\sqrt{\pi t}}\int_0^{\infty}\varphi(\xi)e^{-\frac{x^2+\xi^2}{4a^2t}}\sinh\frac{x\xi}{2a^2t}d\xi.\end{aligned}$$

习题　2.3/7

证明: 如果 $u_1(x,t)$, $u_2(x,t)$ 分别是下述两个定解问题的解:

$$\begin{cases}\dfrac{\partial u_1}{\partial t}=a^2\dfrac{\partial^2 u_1}{\partial x^2},\\ u_1|_{t=0}=\varphi_1(x);\end{cases}\qquad\begin{cases}\dfrac{\partial u_2}{\partial t}=a^2\dfrac{\partial^2 u_2}{\partial y^2},\\ u_2|_{t=0}=\varphi_2(y).\end{cases}$$

则 $u(x,y,t)=u_1(x,t)u_2(y,t)$ 是定解问题

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right),\\ u|_{t=0}=\varphi_1(x)\varphi_2(y)\end{cases}$$

的解.

由 $u(x,y,t)=u_1(x,t)u_2(y,t)$ 可得

$$\frac{\partial u}{\partial t}=\frac{\partial u_1}{\partial t}u_2+\frac{\partial u_2}{\partial t}u_1=a^2\left(\frac{\partial^2 u_1}{\partial x^2}u_2+\frac{\partial^2 u_2}{\partial y^2}u_1\right)=a^2\left(\frac{\partial^2 u_1u_2}{\partial x^2}+\frac{\partial^2 u_1u_2}{\partial y^2}\right)=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$

$$u|_{t=0}=u_1|_{t=0}\cdot u_2|_{t=0}=\varphi_1(x)\varphi_2(y).$$

故得证.

习题 2.3/8

导出下列热传导方程柯西问题的解的表达式:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\
u|_{t=0} = \displaystyle\sum_{i=0}^{n} \alpha_i(x)\beta_i(y).
\end{cases}
$$

由上题结论易知

$$
\begin{cases}
\dfrac{\partial u_i}{\partial t} = a^2\left(\dfrac{\partial^2 u_i}{\partial x^2} + \dfrac{\partial^2 u_i}{\partial y^2}\right), \\
u_i|_{t=0} = \alpha_i(x)\beta_i(y).
\end{cases}
$$

的解 $u_i(x,y,t)$ 为

$$
\begin{aligned}
u_i(x,y,t) &= u_{i1}(x,t)y_{i2}(y,t) \\
&= \frac{1}{2a\sqrt{\pi}t}\int_{-\infty}^{\infty}\alpha_i(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi \cdot \frac{1}{2a\sqrt{\pi}t}\int_{-\infty}^{\infty}\beta_i(\eta)e^{-\frac{(y-\eta)^2}{4a^2t}}d\eta \\
&= \frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_i(\xi)\beta_i(\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}d\xi d\eta.
\end{aligned}
$$

根据叠加原理

$$
u(x,y,t) = \sum_{i=0}^{n} u_i(x,y,t) = \sum_{i=0}^{n}\frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_i(\xi)\beta_i(\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}d\xi d\eta.
$$

习题 2.4/2

利用证明热传导方程极值原理的方法, 证明满足方程 $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ 的函数在有界闭区域上的最大值不会超过它在边界上的最大值.

习题　2.4/3

导出初边值问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t), \\ u|_{x=0} = \mu_1(x), \quad \left(\dfrac{\partial u}{\partial x} + hu\right)\Big|_{x=l} = \mu_2(t) \quad (h>0) \\ u|_{t=0} = \varphi(x) \end{cases}$$

的解 $u(x,t)$ 在 $R_T : \{0 \leqslant t \leqslant T, 0 \leqslant x \leqslant l\}$ 中满足的估计.

$$u(x,t) \leqslant e^{\lambda T} \max\left(0, \max_{0 \leqslant x \leqslant l} \varphi(x), \max_{0 \leqslant t \leqslant T}\left(e^{-\lambda t}\mu_1(t), \frac{e^{-\lambda r}\mu_2(t)}{h}\right), \frac{1}{\lambda}\max_{R_T}(e^{-\lambda t} f)\right),$$

其中 $\lambda > 0$ 为任意正常数.