MA319 — 偏微分方程

Assignment 5

Instructor: 许德良

Author: 刘逸灏 (515370910207)

— SJTU (Fall 2019)

习题 1.5/2

> 试说明: 对一维波动方程, 即使初始资料具有紧支集, 当 $t \to +\infty$ 时其柯西问题的解没有衰减性.

若初始资料 φ, ψ 具有紧支集, 则存在一个常数 $\rho > 0$, 使 φ 和 ψ 在以原点为中心, ρ 为半径的区间 $[-\rho, \rho]$ 外恒为零, 而在区间内成立

$$|\varphi|, |\psi| \leqslant C.$$

代入达朗贝尔公式得

$$u(x, t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi.$$

由于在 $[-\rho, \rho]$ 外, $\psi(x) = 0$, $x - at$ 到 $x + at$ 的积分可以写为 $-\rho$ 到 ρ 的积分

$$\lim_{t \to +\infty} u(x, t) = \frac{C_1 + C_2}{2} + \frac{1}{2a} \int_{-\rho}^{\rho} \psi(\xi) d\xi = \frac{C_1 + C_2}{2} + C_3,$$

$$|C_1|, |C_2| \leqslant C, \quad C_3 \leqslant \frac{\rho C}{a}.$$

故当 $t \to +\infty$ 时其柯西问题的解趋于一个常数, 没有衰减性.

习题 1.5/3

> 设 u 为初始资料 φ 及 ψ 具有紧支集的二维波动方程的解. 试证明: 对任意固定的 $(x_0, y_0) \in \mathbf{R}^2$, 成立
>
> $$\lim_{t \to +\infty} u(x_0, y_0, t) = 0.$$

若初始资料 φ, ψ 具有紧支集, 则存在一个常数 $\rho > 0$, 使 φ 和 ψ 在以原点为中心, ρ 为半径的圆 C_ρ^O 外恒为零, 而在 C_ρ^O 内成立

$$|\varphi|, |\psi| \leqslant C.$$

代入泊松公式得

$$u(x, y, t) = \frac{1}{2\pi a} \frac{\partial}{\partial t} \iint_{C_{at}^M} \frac{\varphi(\xi, \eta) d\xi d\eta}{\sqrt{a^2 t^2 - (\xi - x)^2 - (\eta - y)^2}} + \frac{1}{2\pi a} \iint_{C_{at}^M} \frac{\psi(\xi, \eta) d\xi d\eta}{\sqrt{a^2 t^2 - (\xi - x)^2 - (\eta - y)^2}}$$

$$= \frac{1}{2\pi a} \frac{\partial}{\partial t} \int_0^{at} \int_0^{2\pi} \frac{\varphi(\xi, \eta)}{\sqrt{a^2 t^2 - r^2}} r d\theta dr + \frac{1}{2\pi a} \int_0^{at} \int_0^{2\pi} \frac{\psi(\xi, \eta)}{\sqrt{a^2 t^2 - r^2}} r d\theta dr.$$

由于在 C_ρ^O 外, $\psi(x) = 0$, 0 到 at 的积分可以写为 0 到 ρ 的积分. 当 $at > \rho$ 时

$$
\begin{aligned}
|u(x, y, t)| &= \left| \frac{1}{2\pi a} \frac{\partial}{\partial t} \int_0^\rho \int_0^{2\pi} \frac{\varphi(\xi, \eta)}{\sqrt{a^2 t^2 - r^2}} r d\theta dr + \frac{1}{2\pi a} \int_0^\rho \int_0^{2\pi} \frac{\psi(\xi, \eta)}{\sqrt{a^2 t^2 - r^2}} r d\theta dr \right| \\
&\leqslant \frac{1}{2\pi a} \left| \frac{\partial}{\partial t} \int_0^\rho \int_0^{2\pi} \frac{C}{\sqrt{a^2 t^2 - r^2}} r d\theta dr \right| + \frac{1}{2\pi a} \left| \int_0^\rho \int_0^{2\pi} \frac{C}{\sqrt{a^2 t^2 - r^2}} r d\theta dr \right| \\
&= \frac{1}{2\pi a} \left| \frac{\partial}{\partial t} \int_0^\rho \frac{2\pi C r}{\sqrt{a^2 t^2 - r^2}} dr \right| + \frac{1}{2\pi a} \left| \int_0^\rho \frac{2\pi C r}{\sqrt{a^2 t^2 - r^2}} dr \right| \\
&= \frac{1}{2\pi a} \left| \frac{\partial}{\partial t} 2\pi C \left(at - \sqrt{a^2 t^2 - \rho^2} \right) \right| + \frac{1}{2\pi a} \left| 2\pi C \left(at - \sqrt{a^2 t^2 - \rho^2} \right) \right| \\
&= \frac{C}{a} \left| a - \frac{a^2 t}{\sqrt{a^2 t^2 - \rho^2}} + \left(at - \sqrt{a^2 t^2 - \rho^2} \right) \right|.
\end{aligned}
$$

当 $t \to +\infty$ 时

$$
\lim_{t \to +\infty} \frac{at}{\sqrt{a^2 t^2 - \rho^2}} = 1, \quad \lim_{t \to +\infty} \frac{a^2 t}{\sqrt{a^2 t^2 - \rho^2}} = a,
$$

$$
\lim_{t \to +\infty} |u(x, y, t)| \leqslant \frac{C}{a} |a - a + 0| = 0.
$$

故取任意固定的 $(x_0, y_0) \in \mathbf{R}^2$, 成立

$$
\lim_{t \to +\infty} u(x_0, y_0, t) = 0.
$$

习题 1.6/1

对受摩擦力作用且具固定端点的有界弦振动, 满足方程

$$
u_{tt} = a^2 u_{xx} - c u_t,
$$

其中常数 $c > 0$, 证明其能量是减少的, 并由此证明方程

$$
u_{tt} = a^2 u_{xx} - c u_t + f
$$

的初边值问题解的唯一性以及关于初始条件及自由项的稳定性.

弦的总能量可写成

$$
E(t) = \int_0^l (u_t^2 + a^2 u_x^2) dx.
$$

能量变化率为

$$
\begin{aligned}
\frac{dE(t)}{dt} &= 2 \int_0^l \left(u_t u_{tt} + a^2 u_x u_{xt} \right) dx \\
&= 2 \int_0^l \left[u_t \left(a^2 u_{xx} - c u_t \right) + a^2 u_x u_{xt} \right] dx \\
&= 2 \int_0^l \left(-c u_t^2 + a^2 \frac{\partial}{\partial x} u_t u_x \right) dx \\
&= -2 \int_0^l c u_t^2 dx + 2 a^2 u_t u_x \Big|_0^l.
\end{aligned}
$$

由于端点是固定的

$$2a^2 u_t u_x \Big|_0^l = 0.$$

故

$$\frac{dE(t)}{dt} = -2\int_0^l c u_t^2 dx \leqslant 0.$$

要证明方程

$$u_{tt} = a^2 u_{xx} - c u_t + f$$

的初边值问题解的唯一性，只需证明零初始条件方程只有零解

$$\begin{cases} u_{tt} = a^2 u_{xx} - c u_t, \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \end{cases}$$

根据能量不等式得

$$E(t) \leqslant E(0) = \int_0^l \left(u_t(x,0)^2 + a^2 u_x(x,0)^2 \right) dx = 0,$$

$$u_t = u_x = 0 \Longrightarrow u = C.$$

根据初始条件易知 $u \equiv 0$，故初边值问题解的唯一性得证.

对于方程的稳定性，设

$$E_0(t) = \int_0^l u^2 dx, \quad E_1(t) = \int_0^l (u_t^2 + a^2 u_x^2) dx.$$

对 $E_1(t)$ 有

$$\frac{dE_1(t)}{dt} = 2\int_0^l \left[u_t \left(a^2 u_{xx} - c u_t + f \right) + a^2 u_x u_{xt} \right] dx = 2\int_0^l (-c u_t^2 + u_t f) dx \leqslant \int_0^l (u_t^2 + f^2) dx \leqslant E_1(t) + \int_0^l f^2 dx,$$

$$\frac{d}{dt} e^{-t} E_1(t) = -e^{-t} E_1(t) + e^{-t} \frac{dE_1(t)}{dt} \leqslant e^{-t} \int_0^l f^2 dx,$$

$$E_1(t) \leqslant e^t \int_0^t e^{-\tau} \int_0^l f^2 dx d\tau + e^t E_1(0) \leqslant e^t E_1(0) + e^t \int_0^t \int_0^l f^2 dx d\tau = \overline{E_1(t)}.$$

对 $E_0(t)$ 有

$$\frac{dE_0(t)}{dt} = 2\int_0^l u u_t dx \leqslant \int_0^l u^2 dx + \int_0^l u_t^2 dx \leqslant E_0(t) + E_1(t),$$

$$\frac{d}{dt} e^{-t} E_0(t) = -e^{-t} E_0(t) + e^{-t} \frac{dE_0(t)}{dt} \leqslant e^{-t} E_1(t),$$

$$E_0(t) \leqslant e^t \int_0^t e^{-\tau} E_1(\tau) d\tau + e^t E_0(0) \leqslant e^t \overline{E_1(t)}(1 - e^{-t}) + e^t E_0(0) = e^t E_0(0) + (e^t - 1)\overline{E_1(t)},$$

$$E_0(t) + E_1(t) \leqslant e^t E_0(0) + (e^t - 1)\overline{E_1(t)} + E_1(t) \leqslant e^t (E_0(0) + \overline{E_1(t)}) = e^t E_0(0) + e^{2t} E_1(0) + e^{2t} \int_0^t \int_0^l f^2 dx d\tau.$$

在 $0 \leqslant t \leqslant T$ 上

$$E_0(t) + E_1(t) \leqslant e^T E_0(0) + e^{2T} E_1(0) + e^{2T} \int_0^T \int_0^l f^2 dx d\tau \leqslant e^{2T} \left(E_0(0) + E_1(0) + \int_0^T \int_0^l f^2 dx d\tau \right).$$

存在 C 使得

$$\sqrt{T(E_0(t) + E_1(t))} \leqslant \sqrt{Te^{2T}\left(E_0(0) + E_1(0) + \int_0^T \int_0^l f^2 dx d\tau\right)} \leqslant C\eta,$$

$$\sqrt{E_0(0) + E_1(0) + \int_0^T \int_0^l f^2 dx d\tau} \leqslant \eta.$$

任取 $\varepsilon > 0$, 可以找到 $\eta = \dfrac{\varepsilon}{C}$, 使得

$$\|\varphi_1 - \varphi_2\|_{L^2(t)}, \|\varphi_{1x} - \varphi_{2x}\|_{L^2(t)}, \|\psi_1 - \psi_2\|_{L^2(t)}, \|f_1 - f_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{E_0(0) + E_1(0) + \int_0^T \int_0^l f^2 dx d\tau} \leqslant \eta,$$

$$\|u_1 - u_2\|_{L^2(t)}, \|u_{1x} - u_{2x}\|_{L^2(t)}, \|u_{1t} - u_{2t}\|_{L^2(t)}, \|u_1 - u_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{T(E_0(t) + E_1(t))} \leqslant \varepsilon.$$

故初始条件和自由项都是稳定的.

习题　1.6/2

> 证明函数 $f(x,t)$ 在 $G: 0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T$ 作微笑改变时, 方程
>
> $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) - q(x)u + f(x,t)$$
>
> (其中 $k(x) > 0$, $q(x) > 0$ 和 $f(x,t)$ 都是一些充分光滑的函数) 具固定端点边界条件的初边值问题的解在 G 内的改变也是很微小的.

设

$$E_0(t) = \int_0^l u^2 dx, \quad E_1(t) = \int_0^l \left[u_t^2 + k(x)u_x^2 + q(x)u^2\right] dx.$$

对 $E_1(t)$ 有

$$\begin{aligned}
\frac{dE_1(t)}{dt} &= 2\int_0^l (u_t u_{tt} + k(x)u_x u_{xt} + q(x)uu_t)dx \\
&= 2\int_0^l \{u_t[u_{tt} - (k(x)u_x)_x + q(x)u] + u_t(k(x)u_x)_x + k(x)u_x u_{xt}\}dx \\
&= 2\int_0^l \left(u_t f(x,t) + \frac{\partial}{\partial x}k(x)u_x u_t\right)dx \\
&= 2\int_0^l u_t f(x,t)dx + 2k(x)u_x u_t\Big|_0^l.
\end{aligned}$$

由于端点是固定的

$$2k(x)u_x u_t\Big|_0^l = 0.$$

故

$$\frac{dE_1(t)}{dt} = 2\int_0^l u_t f(x,t)dx \leqslant \int_0^l (u_t^2 + f^2)dx = E_1(t) + \int_0^l f^2 dx,$$

$$\frac{d}{dt}e^{-t}E_1(t) = -e^{-t}E_1(t) + e^{-t}\frac{dE_1(t)}{dt} \leqslant e^{-t}\int_0^l f^2 dx,$$

$$E_1(t) \leqslant e^t \int_0^t e^{-\tau} \int_0^l f^2 dx d\tau + e^t E_1(0) \leqslant e^t E_1(0) + e^t \int_0^t \int_0^l f^2 dx d\tau = \overline{E_1(t)}.$$

对 $E_0(t)$ 有

$$\frac{dE_0(t)}{dt} = 2\int_0^l u u_t dx \leqslant \int_0^l u^2 dx + \int_0^l u_t^2 dx \leqslant E_0(t) + E_1(t),$$

$$\frac{d}{dt}e^{-t}E_0(t) = -e^{-t}E_0(t) + e^{-t}\frac{dE_0(t)}{dt} \leqslant e^{-t}E_1(t),$$

$$E_0(t) \leqslant e^t \int_0^t e^{-\tau}E_1(\tau)d\tau + e^t E_0(0) \leqslant e^t\overline{E_1(t)}(1-e^{-t}) + e^t E_0(0) = e^t E_0(0) + (e^t-1)\overline{E_1(t)},$$

$$E_0(t)+E_1(t) \leqslant e^t E_0(0)+(e^t-1)\overline{E_1(t)}+E_1(t) \leqslant e^t(E_0(0)+\overline{E_1(t)}) = e^t E_0(0)+e^{2t}E_1(0)+e^{2t}\int_0^t \int_0^l f^2 dx d\tau.$$

在 $0 \leqslant t \leqslant T$ 上

$$E_0(t) + E_1(t) \leqslant e^T E_0(0) + e^{2T}E_1(0) + e^{2T}\int_0^T \int_0^l f^2 dx d\tau \leqslant e^{2T}\left(E_0(0)+E_1(0)+\int_0^T \int_0^l f^2 dx d\tau\right).$$

设 $u(x,t)$ 为齐次初边值问题的解, 有 $E_0(0) = E(0) = 0$, 则存在 C 使得

$$\sqrt{TE_0(t)} \leqslant \sqrt{T(E_0(t)+E_1(t))} \leqslant \sqrt{Te^{2T}\int_0^T \int_0^l f^2 dx d\tau} \leqslant C\eta, \quad \sqrt{\int_0^T \int_0^l f^2 dx d\tau} \leqslant \eta$$

任取 $\varepsilon > 0$, 可以找到 $\eta = \dfrac{\varepsilon}{C}$, 使得

$$\|f_1 - f_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{\int_0^T \int_0^l f^2 dx d\tau} \leqslant \eta,$$

$$\|u_1 - u_2\|_{L^2(t)}, \|u_1 - u_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{TE_0(t)} \leqslant \varepsilon.$$

故自由项 $f(x,t)$ 是稳定的.

习题 1.6/5

考虑波动方程的第三类边值问题

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy}) = 0, & t > 0, (x, y) \in \Omega, \\ u|_{t=0} = \varphi(x, y), & u_t|_{t=0} = \psi(x, y), \\ \left(\dfrac{\partial u}{\partial \mathbf{n}} + \sigma u\right)\Big|_{\Gamma} = 0, \end{cases}$$

其中 $\sigma > 0$ 是常数，Γ 为 Ω 的边界，$\mathbf{n}$ 为 Γ 上的单位外法向量. 对于上述问题的解, 定义能量积分

$$E(t) = \iint\limits_{\Omega} (u_t^2 + a^2(u_x^2 + u_y^2))dxdy + a^2 \int_{\Gamma} \sigma u^2 ds,$$

试证明 $E(t) \equiv$ 常数, 并由此证明上述定解问题解的唯一性.

$$\begin{aligned} \frac{dE(t)}{dt} &= 2\iint\limits_{\Omega} (u_t u_{tt} + a^2(u_t u_{xt} + u_t u_{yt}))dxdy + 2a^2 \int_{\Gamma} \sigma u u_t ds \\ &= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy})) + a^2(u_x u_{xt} + u_y u_{yt} + u_t u_{xx} + u_t u_{yy}) \right] dxdy + 2a^2 \int_{\Gamma} \sigma u u_t ds \\ &= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy})) + a^2 \left(\frac{\partial}{\partial x} u_x u_t + \frac{\partial}{\partial y} u_y u_t \right) \right] dxdy + 2a^2 \int_{\Gamma} \sigma u u_t ds \\ &= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy})) \right] dxdy + 2a^2 \int_{\Gamma} \frac{\partial u}{\partial \mathbf{n}} u_t ds + 2a^2 \int_{\Gamma} \sigma u u_t ds \\ &= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy})) \right] dxdy + 2a^2 \int_{\Gamma} u_t \left(\frac{\partial u}{\partial \mathbf{n}} + \sigma u \right) ds \\ &= 0. \end{aligned}$$

故 $E(t) \equiv$ 常数. 要证明方程解的唯一性, 只需证明满足方程的解只有零解

$$E(t) = E(0) = \iint\limits_{\Omega} (u_t(x, y, 0)^2 + a^2(u_x(x, y, 0)^2 + u_y(x, y, 0)^2))dxdy + a^2 \int_{\Gamma} \sigma u(x, y, 0)^2 ds = 0,$$

$$u_t = u_x = u_y = u = 0.$$

故得证.