MA319 — 偏微分方程

Assignment 9

Instructor: 许德良

Author: 刘逸灏 (515370910207)

— SJTU (Fall 2019)

习题　2.5/1

证明下列热传导方程初边值问题

$$\begin{cases} u_t - a^2 u_{xx} = 0, \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = \varphi(x) \end{cases}$$

的解当 $t \to +\infty$ 时指数地衰减为零, 其中 $\varphi \in C^2$, 且 $\varphi(0) = \varphi(l) = 0$.

方程的解是

$$u(x,t) = \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2 a^2}{l^2}t} \sin\frac{k\pi}{l}x, \quad A_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x dx.$$

其中

$$\left|\sin\frac{k\pi}{l}x\right| \leqslant 1,$$

$$|A_k| \leqslant \frac{2}{l}\cdot l\cdot \max_{0\leqslant x\leqslant l}\left|\varphi(x)\sin\frac{k\pi}{l}x\right| \leqslant 2\max_{0\leqslant x\leqslant l}|\varphi(x)| = C$$

由 $\varphi \in C^2$ 且 $\varphi(0) = \varphi(l) = 0$ 可知 C 是一个确定的正常数.

$$|u(x,t)| \leqslant \sum_{k=1}^{\infty} Ce^{-\frac{k^2\pi^2a^2}{l^2}t} = C\left(1 + \sum_{k=2}^{\infty} e^{-\frac{(k^2-1)\pi^2a^2}{l^2}t}\right)e^{-\frac{\pi^2a^2}{l^2}t}.$$

当 $t \to +\infty$ 时

$$\sum_{k=2}^{\infty} e^{-\frac{(k^2-1)\pi^2a^2}{l^2}t} \leqslant \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2a^2}{l^2}t} \leqslant \int_0^{\infty} e^{-\frac{k^2\pi^2a^2}{l^2}t}dk = \frac{\sqrt{\pi}}{2}\sqrt{\frac{l^2}{\pi^2a^2t}} \leqslant C_1.$$

设 $C_2 = C(1+C_1)$ 即可得

$$|u(x,t)| \leqslant C_2 e^{-\frac{\pi^2a^2}{l^2}t}.$$

习题　2.5/2

证明: 当 $\varphi(x,y)$ 为 $\mathbf{R}^2$ 上的有界连续函数, 且 $\varphi \in L^1(\mathbf{R}^2)$ 时, 二维热传导方程柯西问题的解, 当 $t \to +\infty$ 时, 以 t^{-1} 衰减率趋于零.

方程的解是

$$u(x,y,t)=\frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi,\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}d\xi d\eta.$$

由 $\varphi(x,y)$ 为 $\mathbf{R}^2$ 上的有界连续函数, 且 $\varphi\in L^1(\mathbf{R}^2)$ 可知

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(\xi,\eta)|d\xi d\eta\leqslant C.$$

当 $t\to+\infty$ 时

$$\left|e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\right|\leqslant 1,$$

$$|u(x,y,t)|\leqslant\frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\varphi(\xi,\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\right|d\xi d\eta\leqslant\frac{C}{4\pi a^2 t}.$$

设 $C_1=\dfrac{C}{4\pi a^2}$ 即可得

$$|u(x,y,t)|\leqslant C_1t^{-1}.$$

习题 2.5/3

证明: 当 $\varphi(x,y,z)$ 为 $\mathbf{R}^3$ 上的有界连续函数, 且 $\varphi\in L^1(\mathbf{R}^3)$ 时, 三维热传导方程柯西问题的解, 当 $t\to+\infty$ 时, 以 $t^{-\frac{3}{2}}$ 衰减率趋于零.

方程的解是

$$u(x,y,z,t)=\frac{1}{8a^3\sqrt{\pi t}^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi,\eta,\zeta)e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4a^2t}}d\xi d\eta d\zeta.$$

由 $\varphi(x,y,z)$ 为 $\mathbf{R}^2$ 上的有界连续函数, 且 $\varphi\in L^1(\mathbf{R}^3)$ 可知

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(\xi,\eta,\zeta)|d\xi d\eta d\zeta\leqslant C.$$

当 $t\to+\infty$ 时

$$\left|e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4a^2t}}\right|\leqslant 1,$$

$$|u(x,y,t)|\leqslant\frac{1}{8a^3\sqrt{\pi t}^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\varphi(\xi,\eta,\zeta)e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4a^2t}}\right|d\xi d\eta d\zeta\leqslant\frac{C}{8a^3\sqrt{\pi t}^3}.$$

设 $C_1=\dfrac{C}{8a^3\sqrt{\pi}^3}$ 即可得

$$|u(x,y,t)|\leqslant C_1t^{-\frac{3}{2}}.$$

习题 3.1/1

设 $u(x_1, \cdots, x_n) = f(r)$(其中 $r = \sqrt{x_1^2 + \cdots + x_n^2}$) 是 n 维调和函数 (即满足方程 $\dfrac{\partial^2 u}{\partial x_1^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2} = 0$), 试证明

$$f(r) = c_1 + \frac{c_2}{r^{n-2}} \quad (n \neq 2),$$

$$f(r) = c_1 + c_2 \ln \frac{1}{r} \quad (n = 2),$$

其中 c_1, c_2 为任意常数.

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\sqrt{x_1^2 + \cdots + x_n^2} = \frac{1}{2\sqrt{x_1^2 + \cdots + x_n^2}} \cdot 2x_i = \frac{x_i}{r},$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial x_i} = f'(r)\frac{x_i}{r},$$

$$\frac{\partial^2 u}{\partial x_i^2} = f'(r)\frac{1}{r}\frac{\partial x_i}{\partial x_i} + x_i\frac{\partial}{\partial r}\left[f'(r)\frac{1}{r}\right]\frac{\partial r}{\partial x_i} = f'(r)\frac{1}{r} + \left[f''(r)\frac{x_i}{r} - f'(r)\frac{x_i}{r^2}\right]\frac{x_i}{r} = f''(r)\frac{x_i^2}{r^2} + \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)f'(r),$$

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = f''(r) + \frac{n-1}{r}f'(r) = 0,$$

$$\frac{f''(r)}{f'(r)} = \frac{\frac{df'(r)}{dr}}{f'(r)} = -\frac{n-1}{r},$$

$$\frac{df'(r)}{f'(r)} = -\frac{n-1}{r}dr.$$

两边积分得

$$\int \frac{1}{f'(r)}df'(r) = -\int \frac{n-1}{r}dr,$$

$$\ln f'(r) = -(n-1)\ln r + c_0,$$

$$f'(r) = c_0 e^{-(n-1)\ln r} = \frac{c_0}{r^{n-1}}.$$

再次积分得

$$f(r) = \begin{cases} c_1 + \dfrac{c_2}{r^{n-2}} & (n \neq 2), \\ c_1 + c_2 \ln \dfrac{1}{r} & (n = 2). \end{cases}$$

习题 3.1/3

证明: 拉普拉斯算子在柱坐标 (r, θ, z) 下可以写为

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x},$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta,\quad \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta,$$

$$\frac{\partial\theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{y}{r^2}=-\frac{\sin\theta}{r},\quad \frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{x}{r^2}=\frac{\cos\theta}{r}.$$

则

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r},$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}.$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\frac{\partial r}{\partial x}-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial\theta}{\partial x}\\&=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial u}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}-\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}+\frac{\partial u}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}\\&=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-\frac{\partial^2 u}{\partial r\partial\theta}\frac{2\sin\theta\cos\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}+\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}&=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)\frac{\partial r}{\partial y}-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial\theta}{\partial y}\\&=\frac{\partial^2 u}{\partial r^2}\sin^2\theta+\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}-\frac{\partial u}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}+\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}-\frac{\partial u}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}\\&=\frac{\partial^2 u}{\partial r^2}\sin^2\theta+\frac{\partial^2 u}{\partial r\partial\theta}\frac{2\sin\theta\cos\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}+\frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}-\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}.\end{aligned}$$

故

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial\theta^2}\frac{1}{r^2}+\frac{\partial u}{\partial r}\frac{1}{r}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}.$$

习题 3.1/5

证明用极坐标表示的下列函数都满足调和方程:

(1) $\ln r$ 和 θ;
(2) $r^n\cos n\theta$ 和 $r^n\sin n\theta$;
(3) $r\ln r\cos\theta-r\theta\sin\theta$ 和 $r\ln r\sin\theta+r\theta\cos\theta$.

由上题易知拉普拉斯算子在极坐标系 (r,θ) 下可以写为

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

(1)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ln r=\frac{1}{r}\frac{\partial}{\partial r}(1)=0,$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\theta=\frac{1}{r^2}\frac{\partial}{\partial\theta}(1)=0.$$

(2)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) r^n\cos n\theta=\frac{1}{r}\frac{\partial}{\partial r}nr^n\cos n\theta-\frac{1}{r^2}\frac{\partial}{\partial\theta}nr^n\sin n\theta=n^2r^{n-2}\cos n\theta-n^2r^{n-2}\cos n\theta=0,$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) r^n\sin n\theta=\frac{1}{r}\frac{\partial}{\partial r}nr^n\sin n\theta+\frac{1}{r^2}\frac{\partial}{\partial\theta}nr^n\cos n\theta=n^2r^{n-2}\sin n\theta-n^2r^{n-2}\sin n\theta=0.$$

(3)

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(r\ln r\cos\theta-r\theta\sin\theta)\\
=&\frac{1}{r}\frac{\partial}{\partial r}r(\cos\theta+\cos\theta\ln r-\theta\sin\theta)+\frac{1}{r^2}\frac{\partial}{\partial\theta}(-r\theta\cos\theta-r\sin\theta-r\ln r\sin\theta)\\
=&\frac{2\cos\theta+\cos\theta\ln r-\theta\sin\theta}{2}-\frac{2\cos\theta+\cos\theta\ln r-\theta\sin\theta}{2}\\
=&0,
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(r\ln r\cos\theta-r\theta\sin\theta)\\
=&\frac{1}{r}\frac{\partial}{\partial r}r(\sin\theta+\sin\theta\ln r+\theta\cos\theta)+\frac{1}{r^2}\frac{\partial}{\partial\theta}(-r\theta\sin\theta+r\cos\theta+r\ln r\cos\theta)\\
=&\frac{2\sin\theta+\sin\theta\ln r+\theta\cos\theta}{2}-\frac{2\sin\theta+\sin\theta\ln r+\theta\cos\theta}{2}\\
=&0.
\end{aligned}$$