

MA319 — 偏微分方程

Assignment 1

Instructor: 许德良

Author: 刘逸颢 (515370910207)

— SJTU (Fall 2019)

习题 1.1/6

对于 $F(\xi) = F(x - at)$

$$\xi = x - at, \quad \frac{\partial \xi}{\partial t} = -a, \quad \frac{\partial \xi}{\partial x} = 1,$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} = -aF'(\xi),$$

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial t}[-aF'(\xi)] = \frac{\partial}{\partial \xi}[-aF'(\xi)] \cdot \frac{\partial \xi}{\partial t} = a^2 F''(\xi),$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = F'(\xi),$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}[F'(\xi)] = \frac{\partial}{\partial \xi}[F'(\xi)] \cdot \frac{\partial \xi}{\partial x} = F''(\xi).$$

故

$$\frac{\partial^2 F}{\partial t^2} - a^2 \frac{\partial^2 F}{\partial x^2} = 0.$$

对于 $G(\xi) = G(x + at)$

$$\xi = x + at, \quad \frac{\partial \xi}{\partial t} = a, \quad \frac{\partial \xi}{\partial x} = 1,$$

$$\frac{\partial G}{\partial t} = \frac{\partial G}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} = aG'(\xi),$$

$$\frac{\partial^2 G}{\partial t^2} = \frac{\partial}{\partial t}[aG'(\xi)] = \frac{\partial}{\partial \xi}[aG'(\xi)] \cdot \frac{\partial \xi}{\partial t} = a^2 G''(\xi),$$

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = G'(\xi),$$

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}[G'(\xi)] = \frac{\partial}{\partial \xi}[G'(\xi)] \cdot \frac{\partial \xi}{\partial x} = G''(\xi).$$

故

$$\frac{\partial^2 G}{\partial t^2} - a^2 \frac{\partial^2 G}{\partial x^2} = 0.$$

习题 1.1/7

$$u(x, y, t) = (t^2 - x^2 - y^2)^{1/2}, \quad t^2 - x^2 - y^2 > 0.$$

一阶偏导数为

$$\frac{\partial u}{\partial t} = \frac{1}{2}(t^2 - x^2 - y^2)^{-1/2} \cdot 2t = \frac{t}{u},$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}(t^2 - x^2 - y^2)^{-1/2} \cdot -2x = -\frac{x}{u},$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}(t^2 - x^2 - y^2)^{-1/2} \cdot -2y = -\frac{y}{u}.$$

二阶偏导数为

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \frac{t}{u} = \frac{\partial t}{\partial t} \cdot \frac{1}{u} + \frac{\partial}{\partial t} \frac{1}{u} \cdot t = \frac{1}{u} - \frac{3t^2}{u^3},$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x} \frac{x}{u} = -\frac{\partial x}{\partial x} \cdot \frac{1}{u} - \frac{\partial}{\partial x} \frac{1}{u} \cdot t = -\frac{1}{u} - \frac{3x^2}{u^3},$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y} \frac{y}{u} = -\frac{\partial y}{\partial y} \cdot \frac{1}{u} - \frac{\partial}{\partial y} \frac{1}{u} \cdot t = -\frac{1}{u} - \frac{3y^2}{u^3}.$$

故

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \frac{3}{u} + \frac{-3t^2 + 3x^2 + 3y^2}{u^3} = \frac{3}{u} + \frac{-3u^2}{u^3} = 0,$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

习题 1.2/1

$$\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x} \left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x} \right],$$

$$(h-x)^2 \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x} \left[(h-x)^2 \frac{\partial u}{\partial x} \right].$$

设 $u(x, t) = v(x, t)/(h-x)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{h-x} \frac{\partial^2 v}{\partial t^2}, \quad (h-x)^2 \frac{\partial^2 u}{\partial t^2} = (h-x) \frac{\partial^2 v}{\partial t^2}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \cdot \frac{1}{h-x} + \frac{\partial}{\partial x} \left(\frac{1}{h-x} \right) \cdot v = \frac{1}{h-x} \frac{\partial v}{\partial x} + \frac{v}{(h-x)^2},$$

$$\frac{\partial}{\partial x} \left[(h-x)^2 \frac{\partial u}{\partial x} \right] = \frac{\partial}{\partial x} \left[(h-x) \frac{\partial v}{\partial x} + v \right] = \frac{\partial^2 v}{\partial x^2} \cdot (h-x) + \frac{\partial}{\partial x} (h-x) \cdot \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} = (h-x) \frac{\partial^2 v}{\partial x^2}.$$

故

$$\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} = 0.$$

其通解可以写为

$$v(x, t) = F(x-at) + G(x+at),$$

$$u(x, t) = \frac{v(x, t)}{h-x} = \frac{F(x-at) + G(x+at)}{h-x}.$$

且满足初值条件

$$t=0: \quad v = (h-x)\varphi(x), \quad \frac{\partial v}{\partial t} = (h-x)\psi(x).$$

代入达朗贝尔公式得

$$v(x, t) = \frac{(h-x+at)\varphi(x-at) + (h-x-at)\varphi(x+at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} (h-\alpha)\psi(\alpha)d\alpha,$$
$$u(x, t) = \frac{(h-x+at)\varphi(x-at) + (h-x-at)\varphi(x+at)}{2(h-x)} + \frac{1}{2a(h-x)} \int_{x-at}^{x+at} (h-\alpha)\psi(\alpha)d\alpha.$$

习题 1.2/2

齐次波动方程初值问题的解仅由右传播波组成时

$$G(x+at) = C_1,$$

$$G(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a} \int_{x_0}^x \psi(\alpha)d\alpha - \frac{C}{2a} = C_1.$$

故初始条件 $\varphi(x)$, $\psi(x)$ 需要满足

$$\varphi(x) + \frac{1}{a} \int_{x_0}^x \psi(\alpha)d\alpha = C_2.$$

其中 C , C_1 , C_2 为常数.

例题

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 0, & |x| < 1, \quad |y| < 1 \\ u|_{y=x^2} = \varphi(x) \\ u_y|_{y=x^2} = \psi(x) \end{cases}.$$

方程的通解为

$$u(x, y) = F(x) + G(y) + C,$$

代入初值条件得

$$\varphi(x) = u|_{y=x^2} = F(x) + G(x^2) + C,$$

$$\psi(x) = u_y|_{y=x^2} = G'(x^2).$$

故

$$G'(x) = \psi(\sqrt{|x|}),$$

$$G(x) = \int_{x_0}^x \psi(\sqrt{|\xi|})d\xi + C_1,$$

$$F(x) = \varphi(x) - G(x^2) - C = \varphi(x) - \int_{x_0}^{x^2} \psi(\sqrt{|\xi|})d\xi - C_1 - C,$$

$$u(x, y) = F(x) + G(y) + C = \varphi(x) + \int_{x^2}^y \psi(\sqrt{|\xi|})d\xi.$$