MA319 — 偏微分方程

Assignment 2

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习题 1.2/3

方程的通解为

$$u(x,t) = F(x-at) + G(x+at),$$

代入初值条件得

$$\varphi(x) = u|_{x-at=0} = F(0) + G(2x),$$

$$\psi(x) = u|_{x+at=0} = F(2x) + G(0).$$

故

$$F(x) = \psi(x/2) - G(0),$$

$$G(x) = \varphi(x/2) - F(0),$$

$$\varphi(0) = \psi(0) = F(0) + G(0).$$

$$u(x,t) = F(x-at) + G(x+at) = \psi\left(\frac{x-at}{2}\right) + \varphi\left(\frac{x+at}{2}\right) - \varphi(0).$$

习题 1.2/4

非齐次初值问题的解为

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau.$$

其中与 $\varphi(x)$ 和 $\psi(x)$ 有关的定义域范围都是 $[x-at, x+at]$.

(1)

区间 $[x_1, x_2]$ 的影响区域为

$$x_1 - at \leqslant x \leqslant x_2 + at,$$

不受影响的区域为

$$x \leqslant x_1 - at \quad \text{和} \quad x \geqslant x_2 + at,$$

对应的 $\varphi(x)$ 和 $\psi(x)$ 定义域范围是

$$x \leqslant x_1 \quad \text{和} \quad x \geqslant x_2.$$

故当 $\varphi(x)$ 和 $\psi(x)$ 在 $[x_1, x_2]$ 上变化时，以上定义域内函数取值不变，对应解也不变.

(2)

区间 $[x_1, x_2]$ 的决定区域为

$$x_1 + at \leqslant x \leqslant x_2 - at,$$

对应的 $\varphi(x)$ 和 $\psi(x)$ 定义域范围是

$$x_1 \leqslant x \leqslant x_2.$$

故 $[x_1, x_2]$ 上所给的初始条件唯一地确定该区间解的数值.

习题 1.2/5

方程的通解为

$$u(x,t) = F(x-at) + G(x+at),$$

代入初值条件得

$$\varphi(x) = u|_{t=0} = F(x) + G(x),$$

$$0 = u_t|_{t=0} = a[-F'(x) + G'(x)].$$

$$0 = u_x - ku_t \ |_{x=0} = F'(-at) + G'(at) - ka[-F'(-at) + G'(at)].$$

故当 $x \geqslant 0$ 时

$$F(x) - G(x) = C = F(0) - G(0),$$

$$F(x) = \frac{1}{2}\varphi(x) + \frac{C}{2} = \frac{1}{2}\varphi(x) + \frac{F(0) - G(0)}{2},$$

$$G(x) = \frac{1}{2}\varphi(x) - \frac{C}{2} = \frac{1}{2}\varphi(x) - \frac{F(0) - G(0)}{2}.$$

$$(ka+1)F'(-at) - (ka-1)G'(at) = 0,$$

$$(ka+1)F'(-x) - (ka-1)G'(x) = 0,$$

$$\int_0^x [(ka+1)F'(-\xi) - (ka-1)G'(\xi)]d\xi = 0,$$

$$-(ka+1)F(-x) - (ka-1)G(x) = C_1 = -(ka+1)F(0) - (ka-1)G(0),$$

$$F(-x) = \frac{(1-ka)G(x) - C_1}{1+ka} = \frac{1-ka}{1+ka}G(x) + F(0) + \frac{ka-1}{1+ka}G(0).$$

当 $x - at \geqslant 0$ 时

$$u(x,t) = F(x-at) + G(x+at) = \frac{\varphi(x-at) + \varphi(x+at)}{2}.$$

当 $x-at<0$ 时

$$\begin{aligned}
u(x,t) &= F(-(at-x))+G(x+at) \\
&= \frac{1-ka}{1+ka}G(at-x)+F(0)+\frac{ka-1}{1+ka}G(0)+G(x+at) \\
&= \frac{1-ka}{1+ka}\left[\frac{1}{2}\varphi(at-x)-\frac{F(0)-G(0)}{2}\right]+F(0)+\frac{ka-1}{1+ka}G(0)+\frac{1}{2}\varphi(x+at)-\frac{F(0)-G(0)}{2} \\
&= \frac{1-ka}{2(1+ka)}\varphi(at-x)+\frac{1}{2}\varphi(x+at)+\frac{[-(1-ka)+(1+ka)]F(0)+[-(1-ka)+(1+ka)]G(0)}{2(1+ka)} \\
&= \frac{1-ka}{2(1+ka)}\varphi(at-x)+\frac{1}{2}\varphi(x+at)+\frac{ka}{1+ka}\varphi(0).
\end{aligned}$$

故

$$u(x,t)=\frac{1}{2}\varphi(x+at)+\begin{cases}\frac{1}{2}\varphi(x-at), & x-at\geqslant 0\\ \frac{1-ka}{2(1+ka)}\varphi(at-x)+\frac{ka}{1+ka}\varphi(0), & x-at<0\end{cases}.$$

习题 1.2/6

方程的通解为

$$u(x,t)=F(x-t)+G(x+t),$$

代入初值条件得

$$\varphi_0(x)=u|_{t=0}=F(x)+G(x),$$

$$\varphi_1(x)=u_t|_{t=0}=a[-F'(x)+G'(x)].$$

$$\psi(x)=u|_{t=kx}=F((1-k)x)+G((1+k)x).$$

故当 $x\geqslant 0$ 时

$$F(x)-G(x)=-\int_{x_0}^{x}\psi(\alpha)d\alpha+C,$$

$$F(x)=\frac{1}{2}\varphi_0(x)-\frac{1}{2}\int_{x_0}^{x}\varphi_1(\alpha)d\alpha+\frac{C}{2},$$

$$G(x)=\frac{1}{2}\varphi_0(x)+\frac{1}{2}\int_{x_0}^{x}\varphi_1(\alpha)d\alpha-\frac{C}{2}.$$

当 $x-t\geqslant 0$ 时

$$u(x,t)=F(x-t)+G(x+t)=\frac{\varphi_0(x-t)+\varphi_0(x+t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}\varphi_1(\alpha)d\alpha.$$

当 $x-t\leqslant 0$ 时

$$u(x,x)=F(x-x)+G(x+x)=F(0)+G(2x)=\frac{\varphi_0(0)+\varphi_0(2x)}{2}+\frac{1}{2}\int_{0}^{2x}\varphi_1(\alpha)d\alpha,$$

$$G(x)=\frac{\varphi_0(0)+\varphi_0(x)}{2}+\frac{1}{2}\int_{0}^{x}\varphi_1(\alpha)d\alpha-F(0),$$

$$F((1-k)x)=\psi(x)-G((1+k)x)=\psi(x)-\frac{\varphi_0(0)+\varphi_0((1+k)x)}{2}-\frac{1}{2}\int_0^{(1+k)x}\varphi_1(\alpha)d\alpha+F(0),$$

$$F(x)=\psi\left(\frac{x}{1-k}\right)-\frac{1}{2}\varphi_0(0)-\frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}x\right)-\frac{1}{2}\int_0^{\frac{1+k}{1-k}x}\varphi_1(\alpha)d\alpha+F(0).$$

$$\begin{aligned}
u(x,t)&=F(x-t)+G(x+t)\\
&=\psi\left(\frac{x-t}{1-k}\right)-\frac{1}{2}\varphi_0(0)-\frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}(x-t)\right)-\frac{1}{2}\int_0^{\frac{1+k}{1-k}(x-t)}\varphi_1(\alpha)d\xi+F(0)\\
&\quad+\frac{\varphi_0(0)+\varphi_0(x+t)}{2}+\frac{1}{2}\int_0^{x+t}\varphi_1(\xi)d\xi-F(0)\\
&=\psi\left(\frac{x-t}{1-k}\right)-\frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}(x-t)\right)+\frac{1}{2}\varphi_0(x+t)+\frac{1}{2}\int_{\frac{1+k}{1-k}(x-t)}^{x+t}\varphi_1(\xi)d\xi.
\end{aligned}$$

故

$$u(x,t)=\begin{cases}\dfrac{\varphi_0(x-t)+\varphi_0(x+t)}{2}+\dfrac{1}{2}\displaystyle\int_{x-t}^{x+t}\varphi_1(\xi)d\xi, & 0\leqslant t\leqslant x\\ \psi\left(\dfrac{x-t}{1-k}\right)-\dfrac{1}{2}\varphi_0\left(\dfrac{1+k}{1-k}(x-t)\right)+\dfrac{1}{2}\varphi_0(x+t)+\dfrac{1}{2}\displaystyle\int_{\frac{1+k}{1-k}(x-t)}^{x+t}\varphi_1(\xi)d\xi, & x<t<kx\end{cases}.$$

习题　1.2/7

方程的通解为

$$u(x,t)=F(x-t)+G(x+t),$$

代入初值条件得

$$\varphi(x)=u|_{t=x}=F(0)+G(2x),$$

$$\psi(x)=u|_{t=f(x)}=F(x-f(x))+G(x+f(x)).$$

解得

$$G(x)=\varphi(x/2)-F(0),$$

$$F(x-f(x))=\psi(x)-G(x+f(x))=\psi(x)-\varphi\left(\frac{x+f(x)}{2}\right)+F(0).$$

设 $g(x)=x-f(x)$, 由于 $f'(x)\neq 1$, 易知 $g'(x)\neq 0$, 即 $g(x)$ 为连续单调函数, $g^{-1}(x)$ 存在.

$$F(x-f(x))=F(g(x))=\psi(x)-\varphi\left(x-\frac{g(x)}{2}\right)+F(0),$$

$$F(x)=\psi(g^{-1}(x))-\varphi\left(g^{-1}(x)-\frac{x}{2}\right)+F(0).$$

故

$$
\begin{aligned}
u(x,t) &= F(x-t)+G(x+t) \\
&= \psi(g^{-1}(x-t)) - \varphi\left(g^{-1}(x-t)-\frac{x-t}{2}\right) + F(0) + \varphi\left(\frac{x+t}{2}\right) - F(0) \\
&= \psi(g^{-1}(x-t)) + \varphi\left(g^{-1}(x-t)-\frac{x-t}{2}\right) + \varphi\left(\frac{x+t}{2}\right).
\end{aligned}
$$

习题　1.2/8

代入 Kirchhoff 公式得

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t}\sin\xi d\xi + \frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\tau\sin\xi d\xi d\tau \\
&= \frac{1}{2}[\cos(x-t)-\cos(x+t)] + \frac{1}{2}\int_0^t \tau[\cos(x-t+\tau)-\cos(x+t-\tau)]d\tau \\
&= \frac{1}{2}[\cos(x-t)-\cos(x+t)] + \frac{1}{2}[\cos(-x)-t\sin(-x)-\cos(t-x)] - \frac{1}{2}[\cos x - t\sin x - \cos(t+x)] \\
&= t\sin x.
\end{aligned}
$$

习题　1.2/9

代入 Kirchhoff 公式得

$$
\begin{aligned}
u(x,t) &= \frac{1}{2a}\int_{x-at}^{x+at}\frac{1}{1+\xi^2}d\xi + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\frac{\tau\xi}{(1+\xi^2)^2}d\xi d\tau \\
&= \frac{1}{2a}[\arctan(x+at)-\arctan(x-at)] + \frac{1}{4a}\int_0^t \tau\left[\frac{1}{1+[x-a(t-\tau)]^2} - \frac{1}{1+[x+a(t-\tau)]^2}\right]d\tau \\
&= \frac{1}{2a}[\arctan(x+at)-\arctan(x-at)] + \frac{1}{4a^3}\left[(at-x)[\arctan x - \arctan(x-at)] + \frac{1}{2}\ln\frac{1+x^2}{1+(x-at)^2}\right] \\
&\quad - \frac{1}{4a^3}\left[-(at+x)[\arctan x - \arctan(x+at)] + \frac{1}{2}\ln\frac{1+x^2}{1+(x+at)^2}\right] \\
&= \frac{1}{2a}[\arctan(x+at)-\arctan(x-at)] + \frac{1}{8a^3}\ln\frac{1+(x+at)^2}{1+(x-at)^2} \\
&\quad + \frac{1}{4a^3}[2at\arctan x + (x-at)\arctan(x-at) - (x+at)\arctan(x+at)] \\
&= \frac{t}{2a^2}\arctan x + \frac{x-at-2a^2}{4a^3}\arctan(x-at) - \frac{x+at-2a^2}{4a^3}\arctan(x+at) + \frac{1}{8a^3}\ln\frac{1+(x+at)^2}{1+(x-at)^2}.
\end{aligned}
$$