

MA319 — 偏微分方程

Assignment 8

Instructor: 许德良

Author: 刘逸灏 (515370910207)

— SJTU (Fall 2019)

习题 2.3/5

求解热传导方程 (3.17) 的柯西问题, 已知

- (1) $u|_{t=0} = \sin x$,
- (2) 用延拓法求解半有界直线上的热传导方程 (3.17), 假设

$$\begin{cases} u(x, 0) = \varphi(x) & (0 < x < \infty), \\ u(0, t) = 0. \end{cases}$$

(i)

初值条件为

$$\varphi(x) = u|_{t=0} = \sin x.$$

故

$$\begin{aligned} u(x, t) &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi \\ &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(x + 2a\sqrt{t}\eta) e^{-\eta^2} d(x + 2a\sqrt{t}\eta) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sin(x + 2a\sqrt{t}\eta) e^{-\eta^2} d\eta \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [\sin x \cos(2a\sqrt{t}\eta) + \cos x \sin(2a\sqrt{t}\eta)] e^{-\eta^2} d\eta \\ &= \frac{\sin x}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos(2a\sqrt{t}\eta) e^{-\eta^2} d\eta \\ &= \frac{\sin x}{2\sqrt{\pi}} \int_{-\infty}^{\infty} [e^{-\eta^2 + i2a\sqrt{t}\eta} + e^{-\eta^2 - i2a\sqrt{t}\eta}] d\eta \\ &= \frac{\sin x}{2\sqrt{\pi}} e^{-a^2 t} \left[\int_{-\infty}^{\infty} e^{-(\eta - ia\sqrt{t})^2} d\eta + \int_{-\infty}^{\infty} e^{-(\eta + ia\sqrt{t})^2} d\eta \right] \\ &= \frac{\sin x}{\sqrt{\pi}} e^{-a^2 t} \int_{-\infty}^{\infty} e^{-\chi^2} d\chi \\ &= e^{-a^2 t} \sin x. \end{aligned}$$

(2)

使用奇延拓

$$u(x, 0) = \begin{cases} \varphi(x) & (0 < x < \infty), \\ 0 & (x = 0), \\ -\varphi(-x) & (-\infty < x < 0). \end{cases}$$

代入公式可得

$$\begin{aligned} u(x, t) &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} u(\xi, 0) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi \\ &= \frac{1}{2a\sqrt{\pi t}} \left(\int_0^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi + \int_{-\infty}^0 -\varphi(-\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi \right) \\ &= \frac{1}{2a\sqrt{\pi t}} \left(\int_0^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi + \int_0^{\infty} -\varphi(\xi) e^{-\frac{(x+\xi)^2}{4a^2 t}} d\xi \right) \\ &= \frac{1}{2a\sqrt{\pi t}} \int_0^{\infty} \varphi(\xi) e^{-\frac{x^2+\xi^2}{4a^2 t}} \left(e^{\frac{2x\xi}{4a^2 t}} - e^{-\frac{2x\xi}{4a^2 t}} \right) d\xi \\ &= \frac{1}{a\sqrt{\pi t}} \int_0^{\infty} \varphi(\xi) e^{-\frac{x^2+\xi^2}{4a^2 t}} \sinh \frac{x\xi}{2a^2 t} d\xi. \end{aligned}$$

习题 2.3/7

证明: 如果 $u_1(x, t)$, $u_2(x, t)$ 分别是下述两个定解问题的解:

$$\begin{cases} \frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2}, & \begin{cases} \frac{\partial u_2}{\partial t} = a^2 \frac{\partial^2 u_2}{\partial y^2}, \\ u_2|_{t=0} = \varphi_2(y). \end{cases} \\ u_1|_{t=0} = \varphi_1(x); \end{cases}$$

则 $u(x, y, t) = u_1(x, t)u_2(y, t)$ 是定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\ u|_{t=0} = \varphi_1(x)\varphi_2(y) \end{cases}$$

的解.

由 $u(x, y, t) = u_1(x, t)u_2(y, t)$ 可得

$$\frac{\partial u}{\partial t} = \frac{\partial u_1}{\partial t} u_2 + \frac{\partial u_2}{\partial t} u_1 = a^2 \left(\frac{\partial^2 u_1}{\partial x^2} u_2 + \frac{\partial^2 u_2}{\partial y^2} u_1 \right) = a^2 \left(\frac{\partial^2 u_1 u_2}{\partial x^2} + \frac{\partial^2 u_1 u_2}{\partial y^2} \right) = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

$$u|_{t=0} = u_1|_{t=0} \cdot u_2|_{t=0} = \varphi_1(x)\varphi_2(y).$$

故得证.

习题 2.3/8

导出下列热传导方程柯西问题的解的表达式:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\ u|_{t=0} = \sum_{i=0}^n \alpha_i(x) \beta_i(y). \end{cases}$$

由上题结论易知

$$\begin{cases} \frac{\partial u_i}{\partial t} = a^2 \left(\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} \right), \\ u_i|_{t=0} = \alpha_i(x) \beta_i(y). \end{cases}$$

的解 $u_i(x, y, t)$ 为

$$\begin{aligned} u_i(x, y, t) &= u_{i1}(x, t) u_{i2}(y, t) \\ &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \alpha_i(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi \cdot \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \beta_i(\eta) e^{-\frac{(y-\eta)^2}{4a^2 t}} d\eta \\ &= \frac{1}{4\pi a^2 t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_i(\xi) \beta_i(\eta) e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4a^2 t}} d\xi d\eta. \end{aligned}$$

根据叠加原理

$$u(x, y, t) = \sum_{i=0}^n u_i(x, y, t) = \sum_{i=0}^n \frac{1}{4\pi a^2 t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_i(\xi) \beta_i(\eta) e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4a^2 t}} d\xi d\eta.$$

习题 2.4/2

利用证明热传导方程极值原理的方法, 证明满足方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 的函数在有界闭区域上的最大值不会超过它在边界上的最大值.

设 $u(x, y)$ 在区域 Ω 内连续, 并且在区域内部满足方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, 且 Γ 为区域 Ω 的边界. 设 M 为 Ω 内 $u(x, y)$ 的最大值, m 为 Γ 上 $u(x, y)$ 的最大值. 使用反证法, 如果命题不真, 那么 $M > m$. 此时在 Ω 内一定存在着一点 (x^*, y^*) , 使函数 $u(x, y)$ 在该点取值 M . 作函数

$$V(x, y) = u(x, y) + \frac{M - m}{4R^2} [(x - x^*)^2 + (y - y^*)^2],$$

其中 $R^2 > (x - x^*)^2 + (y - y^*)^2$. 由于在 Γ 上

$$V(x, y) < m + \frac{M - m}{4} = \frac{M}{4} + \frac{3}{4}m = \theta M \quad (0 < \theta < 1),$$

而

$$V(x^*, y^*) = M,$$

因此, 函数 $V(x, y)$ 和 $u(x, y)$ 一样, 它不在 Γ 上取到最大值. 设 $V(x, y)$ 在 Ω 中的某一点 (x_1, y_1) 上取到

最大值, 则在此点应有 $\frac{\partial^2 V}{\partial x^2} \leq 0, \frac{\partial^2 V}{\partial y^2} \leq 0$, 因此在点 (x_1, y_1) 处

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \leq 0.$$

但由直接计算方程可得

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{M-m}{2R^2} + \frac{\partial^2 u}{\partial y^2} + \frac{M-m}{2R^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{M-m}{R^2} > 0,$$

这就得到矛盾. 这说明原先的假设是不正确的, 证毕.

习题 2.4/3

导出初边值问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x, t), \\ u|_{x=0} = \mu_1(x), \quad \left(\frac{\partial u}{\partial x} + hu \right) \Big|_{x=l} = \mu_2(t) \quad (h > 0) \\ u|_{t=0} = \varphi(x) \end{cases}$$

的解 $u(x, t)$ 在 $R_T : \{0 \leq t \leq T, 0 \leq x \leq l\}$ 中满足的估计.

$$u(x, t) \leq e^{\lambda T} \max \left(0, \max_{0 \leq x \leq l} \varphi(x), \max_{0 \leq t \leq T} \left(e^{-\lambda t} \mu_1(t), \frac{e^{-\lambda t} \mu_2(t)}{h} \right), \frac{1}{\lambda} \max_{R_T} (e^{-\lambda t} f) \right),$$

其中 $\lambda > 0$ 为任意正常数.