MA319 — 偏微分方程

Assignment 7

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在区域 $t > 0$, $0 < x < l$ 中求解如下的定解问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \beta(u - u_0), \\ u(0, t) = u(1, t) = u_0, \\ u(x, 0) = f(x), \end{cases}$$

其中 a, β, u_0 均为常数, $f(x)$ 为已知函数.

设

$$u(x, t) = u_0 + v(x, t)e^{-\beta t}.$$

则

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \beta(u - u_0) = \frac{\partial v}{\partial t}e^{-\beta t} - \beta v e^{-\beta t} - \frac{\partial^2 v}{\partial x^2}e^{-\beta t} + \beta v e^{-\beta t} = e^{-\beta t}\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right) = 0,$$

$$v(0, t)e^{-\beta t} = v(l, t)e^{-\beta t} = u(0, t) - u_0 = u(l, t) - u_0 = 0,$$

$$v(x, 0) = u(x, 0) - u_0 = f(x) - u_0.$$

故可以先求解如下定解问题:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2}, \\ v(0, t) = v(1, t) = 0, \\ v(x, 0) = f(x) - u_0. \end{cases}$$

方程的特征值和对应的特征函数为

$$\lambda = \lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = C_k \sin\sqrt{\lambda}x = C_k \sin\frac{k\pi}{l}x, \quad k = 1, 2, \cdots.$$

方程的通解为

$$v(x, t) = \sum_{k=1}^{\infty} A_k e^{-a^2\lambda t} \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k e^{-\frac{k^2\pi^2}{l^2}a^2 t} \sin\frac{k\pi}{l}x.$$

代入初值条件得

$$v(x, 0) = \sum_{k=1}^{\infty} A_k \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x = f(x) - u_0.$$

解得

$$A_k = \frac{2}{l}\int_0^l [f(x) - u_0] \sin\sqrt{\lambda} x dx = \frac{2}{l}\int_0^l [f(x) - u_0] \sin\frac{k\pi}{l} x dx = \frac{2}{l}\int_0^l f(x) \sin\frac{k\pi}{l} x dx - \frac{2u_0(1-\cos k\pi)}{k\pi}.$$

化简得

$$A_k = \frac{2}{l}\int_0^l f(x) \sin\frac{k\pi}{l} x dx + \frac{2u_0[(-1)^k - 1]}{k\pi}.$$

故

$$v(x,t) = \sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x) \sin\frac{k\pi}{l} x dx + \frac{2u_0[(-1)^k - 1]}{k\pi}\right\} e^{-\frac{k^2\pi^2}{l^2}a^2 t} \sin\frac{k\pi}{l} x,$$

$$u(x,t) = u_0 + e^{-\beta t}\sum_{k=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x) \sin\frac{k\pi}{l} x dx + \frac{2u_0[(-1)^k - 1]}{k\pi}\right\} e^{-\frac{k^2\pi^2}{l^2}a^2 t} \sin\frac{k\pi}{l} x.$$

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(1)

(2)

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