MA319 — 偏微分方程

Assignment 11

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习题 3.3/2

证明格林函数的对称性: $G(M_1, M_2) = G(M_2, M_1)$.

设区域 Ω 的边界为 Γ, $K_1 = B(M_1, r)$, Γ_1 为 K_1 的边界, $K_2 = B(M_2, r)$, Γ_2 为 K_2 的边界, $u = G(M, M_2)$, $v = G(M, M_2)$, 根据格林函数的性质 1, 2 可知

$$\Delta u = \Delta v = 0, \quad M \neq M_1, \quad M \neq M_2,$$

$$u = v = 0, \quad M \in \Gamma.$$

根据格林第二公式可得

$$\iiint\limits_{\Omega - K_1 - K_2} (u\Delta v - v\Delta u) d\Omega = \iint\limits_{\Gamma + \Gamma_1 + \Gamma_2} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}} \right) dS.$$

在 $\Omega - K_1 - K_2$ 中满足 $\Delta u = \Delta v = 0$, 在 Γ 中满足 $u = v = 0$, 故

$$\iint\limits_{\Gamma_1 + \Gamma_2} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}} \right) dS = \iint\limits_{\Gamma_1 + \Gamma_2} \left[G(M, M_1)\frac{\partial G(M, M_2)}{\partial \mathbf{n}} - G(M, M_2)\frac{\partial G(M, M_1)}{\partial \mathbf{n}} \right] dS = 0.$$

$$\left| \iint\limits_{\Gamma_1} G(M, M_1)\frac{\partial G(M, M_2)}{\partial \mathbf{n}} dS \right| \leqslant \sup_{\Gamma_1} |G(M, M_1)| \sup_{\Gamma_1} \left| \frac{\partial G(M, M_2)}{\partial \mathbf{n}} \right| \iint\limits_{\Gamma_1} dS.$$

当 $r \to 0$ 时, 根据格林函数的性质 3 可知 $G(M, M_1) < \dfrac{1}{4\pi r}$, 且由 $G(M, M_2)$ 在 Γ_1 上调和可知 $\dfrac{\partial G(M, M_2)}{\partial \mathbf{n}}$ 有界, 故

$$\lim_{r\to 0} \left| \iint\limits_{\Gamma_1} G(M, M_1)\frac{\partial G(M, M_2)}{\partial \mathbf{n}} dS \right| \leqslant \lim_{r\to 0} \left(C \cdot \frac{1}{4\pi r} \cdot 4\pi r^2 \right) = \lim_{r\to 0} Cr = 0.$$

$$\iint\limits_{\Gamma_1} G(M, M_2)\frac{\partial G(M, M_1)}{\partial \mathbf{n}} dS = \iint\limits_{\Gamma_1} G(M, M_2) \left[\frac{\partial}{\partial \mathbf{n}} \frac{1}{4\pi r_{M_1 M}} - \frac{\partial g(M, M_1)}{\partial \mathbf{n}} \right] dS.$$

当 $r \to 0$ 时, 由 $g(M, M_1)$ 在 Γ_1 上调和可知 $\dfrac{\partial g(M, M_1)}{\partial \mathbf{n}}$ 有界, 故

$$\lim_{r\to 0} \left| \iint \frac{\partial g(M, M_1)}{\partial \mathbf{n}} dS \right| \leqslant \lim_{r\to 0} \left(\sup_{\Gamma_1} \left| \frac{\partial G(M, M_2)}{\partial \mathbf{n}} \right| \iint\limits_{\Gamma_1} dS \right) = \lim_{r\to 0} (C \cdot 4\pi r^2) = 0,$$

$$\iint\limits_{\Gamma_1} \frac{\partial}{\partial \mathbf{n}} \frac{1}{4\pi r_{M_1 M}} dS = \frac{1}{4\pi r^2} \cdot 4\pi r^2 = 1,$$

$$\lim_{r\to 0} \iint\limits_{\Gamma_1} G(M, M_2) \frac{\partial G(M, M_1)}{\partial \mathbf{n}} dS = G(M_1, M_2) \lim_{r\to 0} \iint\limits_{\Gamma_1} \frac{\partial G(M, M_1)}{\partial \mathbf{n}} dS = G(M_1, M_2).$$

综上并根据对称性可得,

$$\iint\limits_{\Gamma_1} \left[G(M, M_1) \frac{\partial G(M, M_2)}{\partial \mathbf{n}} - G(M, M_2) \frac{\partial G(M, M_1)}{\partial \mathbf{n}} \right] dS = -G(M_1, M_2),$$

$$\iint\limits_{\Gamma_2} \left[G(M, M_1) \frac{\partial G(M, M_2)}{\partial \mathbf{n}} - G(M, M_2) \frac{\partial G(M, M_1)}{\partial \mathbf{n}} \right] dS = G(M_2, M_1).$$

故

$$G(M_1, M_2) = G(M_2, M_1).$$

习题　3.3/5

求半圆区域上狄利克雷问题的格林函数.

圆的格林函数为

$$G(M, M_0) = \frac{1}{2\pi} \left(\ln \frac{1}{r_{M_0 M}} - \ln \frac{R}{\rho_0} \frac{1}{r_{M_1 M}} \right) = -\frac{1}{4\pi} \ln R^2 \frac{\rho_0^2 + \rho^2 - 2\rho_0 \rho \cos(\theta - \theta_0)}{R^4 + \rho_0 \rho^2 - 2R^2 \rho_1 \rho \cos(\theta - \theta_0)},$$

其中 R 为圆的半径, $\rho = r_{OM}$, $\rho_0 = r_{OM_1}$, θ 为 OM 的幅角, θ_0 为 OM_0 的幅角.
现只需取镜像点 M_0', 使得 $\rho_0' = \rho_0$, $\theta_0' = -\theta_0$, 即可得半圆的格林函数为

$$G(M, M_0) - G(M, M_0') = \frac{1}{4\pi} \left[\ln \frac{\rho_0^2 + \rho^2 - 2\rho_0 \rho \cos(\theta + \theta_0)}{R^4 + \rho_0 \rho^2 - 2R^2 \rho_1 \rho \cos(\theta + \theta_0)} - \ln \frac{\rho_0^2 + \rho^2 - 2\rho_0 \rho \cos(\theta - \theta_0)}{R^4 + \rho_0 \rho^2 - 2R^2 \rho_1 \rho \cos(\theta - \theta_0)} \right].$$

习题　3.3/9

试求一函数 u, 使其在半径为 a 的圆内部是调和的, 而且在圆周 C 上取下列的值:

(1) $u|_C = A \cos\varphi$,
(2) $u|_C = A + B \sin\varphi$,

其中 A, B 都是常数.

(1)

根据习题 3.1/5 可知 $Cr\cos\varphi$ 为调和函数, 代入边界条件可得

$$u = \frac{A}{a} r \cos\varphi.$$

(1)

根据习题 3.1/5 可知 $C_1 + C_2 r\sin\varphi$ 为调和函数, 代入边界条件可得

$$u = A + \frac{B}{a} r\sin\varphi.$$

习题　3.3/10

试用静电源像法导出二维调和方程在半平面上的狄利克雷问题:

$$\Delta u = u_{xx} + u_{yy} = 0, \quad y > 0,$$

$$u|_{y=0} = f(x)$$

的解.

平面的格林函数为

$$G(M, M_0) = \frac{1}{2\pi}\ln\frac{1}{r_{M_0M}} = \frac{1}{4\pi}\ln\frac{1}{(x-x_0)^2 + (y-y_0^2)},$$

其中 M 的坐标为 (x, y), M_0 的坐标为 (x_0, y_0).
现只需取镜像点 M_0', 使得 $x_0' = x_0$, $y_0' = -y_0$, 即可得半平面的格林函数为

$$G(M, M_0) - G(M, M_0') = \frac{1}{4\pi}\ln\frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2}.$$

对于半平面 $y > 0$ 来讲, 直线 $y = 0$ 的外法线方向是与 y 轴相反的方向, 即 $\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial y}$. 此外, 对于半平面的情形, 只要对调和函数 $u(x, y)$ 加上在无穷远处的条件:

$$u(M) = O\left(\ln\frac{1}{r_{OM}}\right), \quad \frac{\partial u}{\partial \mathbf{n}} = O\left(\frac{1}{r_{OM}}\right) \quad (r_{OM} \to \infty),$$

则仍可证明该公式成立:

$$u(M_0) = -\frac{1}{2\pi}\int\limits_{\Gamma}\left[u(M)\frac{\partial}{\partial \mathbf{n}}\left(\ln\frac{1}{r_{M_0M}}\right) - \ln\frac{1}{r_{M_0M}}\frac{\partial u(M)}{\partial \mathbf{n}}\right] dS_M.$$

故狄利克雷方程的求解式也成立

$$\begin{aligned}
u(x_0, y_0) &= -\int\limits_{\Gamma} f(x)\frac{\partial G(M, M_0)}{\partial \mathbf{n}} dS_M \\
&= \int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial y}\,\frac{1}{4\pi}\ln\frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2}\bigg|_{y=0} dx \\
&= \frac{y_0}{\pi}\int_{-\infty}^{\infty} f(x)\,\frac{(x-x_0^2) - y^2 + y_0^2}{[(x-x_0)^2 + (y+y_0)^2][(x-x_0)^2 + (y-y_0)^2]}\bigg|_{y=0} dx \\
&= \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{(x-x_0)^2 + y_0^2} dx.
\end{aligned}$$

习题 3.3/14

> 证明处处满足平均值公式的连续函数一定是调和函数.

设连续函数 u 在 Ω 内处处满足平均值公式, 设 $K = B(M_0, r) \subseteq \Omega$, 其边界为 Γ, 则在 K 内可以有狄利克雷问题

$$\Delta v = 0, \quad v|_\Gamma = u|_\Gamma.$$

易知 v 有唯一解, 且 v 是 K 内的调和函数, 故 $u - v$ 在 K 内处处满足平均值公式, 也成立极值原理. 由于 $(u - v)|_\Gamma = 0$, $u - v$ 在 K 内的最大值和最小值都为 0, 故 $u = v$. 由 v 是调和函数和 M_0 的任意性可知 u 是调和函数.

例题

(1)

> 求区域 $\Omega = \{x^2 + y^2 \leqslant 1, \quad x \geqslant 0, \quad y \geqslant 0\}$ 的格林函数.

由习题 3.3/5 可知上半单位圆的格林函数为

$$G(M, M_0) = \frac{1}{4\pi}\left[\ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta + \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta + \theta_0)} - \ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta - \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta - \theta_0)}\right],$$

其中 $\rho = r_{OM}$, $\rho_0 = r_{OM_1}$, θ 为 OM 的幅角, θ_0 为 OM_0 的幅角.
现只需取镜像点 M_0', 使得 $\rho_0' = \rho_0$, $\theta_0' = \pi - \theta_0$, 即可得右上四分之一单位圆的格林函数为

$$\begin{aligned}
G(M, M_0) - G(M, M_0') &= \frac{1}{4\pi}\left[\ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta + \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta + \theta_0)} - \ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta - \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta - \theta_0)}\right] \\
&\quad - \frac{1}{4\pi}\left[\ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta + \pi - \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta + \pi - \theta_0)} - \ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta - \pi + \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta - \pi + \theta_0)}\right] \\
&= \frac{1}{4\pi}\left[\ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta + \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta + \theta_0)} - \ln \frac{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta - \theta_0)}{1 + \rho_0\rho^2 - 2\rho_1\rho\cos(\theta - \theta_0)}\right. \\
&\quad \left. - \ln \frac{\rho_0^2 + \rho^2 + 2\rho_0\rho\cos(\theta - \theta_0)}{1 + \rho_0\rho^2 + 2\rho_1\rho\cos(\theta - \theta_0)} + \ln \frac{\rho_0^2 + \rho^2 + 2\rho_0\rho\cos(\theta + \theta_0)}{1 + \rho_0\rho^2 + 2\rho_1\rho\cos(\theta + \theta_0)}\right].
\end{aligned}$$

(2)

> $$\begin{cases} \Delta_2 u = 0, \quad 0 \leqslant r \leqslant R, \quad 0 < \theta \leqslant 2\pi \\ u|_{r=R} = \cos^2\theta + 1. \end{cases}$$

$$\cos^2\theta + 1 = \frac{1}{2}\cos 2\theta + \frac{3}{2}.$$

根据习题 3.1/5 可知 $C_1 + C_2 r^2 \cos 2\theta$ 为调和函数, 代入边界条件可得

$$u = \frac{3}{2} + \frac{1}{2R^2} r^2 \cos 2\theta.$$