MA319 — 偏微分方程

Assignment 1

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习题 1.1/6

若 $F(\xi)$, $G(\xi)$ 均为其变元的二次连续可导函数, 验证 $F(x-at)$, $G(x+at)$ 均满足弦振动方程.

对于 $F(\xi)=F(x-at)$

$$\xi = x - at, \quad \frac{\partial \xi}{\partial t} = -a, \quad \frac{\partial \xi}{\partial x} = 1,$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} = -aF'(\xi),$$

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial t}[-aF'(\xi)] = \frac{\partial}{\partial \xi}[-aF'(\xi)] \cdot \frac{\partial \xi}{\partial t} = a^2 F''(\xi),$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = F'(\xi),$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}[F'(\xi)] = \frac{\partial}{\partial \xi}[F'(\xi)] \cdot \frac{\partial \xi}{\partial x} = F''(\xi).$$

故

$$\frac{\partial^2 F}{\partial t^2} - a^2 \frac{\partial^2 F}{\partial x^2} = 0.$$

对于 $G(\xi)=G(x+at)$

$$\xi = x + at, \quad \frac{\partial \xi}{\partial t} = a, \quad \frac{\partial \xi}{\partial x} = 1,$$

$$\frac{\partial G}{\partial t} = \frac{\partial G}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} = aG'(\xi),$$

$$\frac{\partial^2 G}{\partial t^2} = \frac{\partial}{\partial t}[-aG'(\xi)] = \frac{\partial}{\partial \xi}[aG'(\xi)] \cdot \frac{\partial \xi}{\partial t} = a^2 G''(\xi),$$

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = G'(\xi),$$

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}[G'(\xi)] = \frac{\partial}{\partial \xi}[G'(\xi)] \cdot \frac{\partial \xi}{\partial x} = G''(\xi).$$

故

$$\frac{\partial^2 G}{\partial t^2} - a^2 \frac{\partial^2 G}{\partial x^2} = 0.$$

习题 1.1/7

验证

$$u(x,y,t)=\frac{1}{\sqrt{t^2-x^2-y^2}}$$

在锥 $t^2-x^2-y^2>0$ 中满足波动方程

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

一阶偏导数为

$$\begin{aligned}
\frac{\partial u}{\partial t}&=\frac{1}{2}(t^2-x^2-y^2)^{-1/2}\cdot 2t=\frac{t}{u},\\
\frac{\partial u}{\partial x}&=\frac{1}{2}(t^2-x^2-y^2)^{-1/2}\cdot -2x=-\frac{x}{u},\\
\frac{\partial u}{\partial y}&=\frac{1}{2}(t^2-x^2-y^2)^{-1/2}\cdot -2y=-\frac{y}{u}.
\end{aligned}$$

二阶偏导数为

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}&=\frac{\partial}{\partial t}\frac{t}{u}=\frac{\partial t}{\partial t}\cdot\frac{1}{u}+\frac{\partial}{\partial t}\frac{1}{u}\cdot t=\frac{1}{u}-\frac{3t^2}{u^3},\\
\frac{\partial^2 u}{\partial x^2}&=-\frac{\partial}{\partial x}\frac{x}{u}=-\frac{\partial x}{\partial x}\cdot\frac{1}{u}-\frac{\partial}{\partial x}\frac{1}{u}\cdot t=-\frac{1}{u}-\frac{3x^2}{u^3},\\
\frac{\partial^2 u}{\partial y^2}&=-\frac{\partial}{\partial y}\frac{y}{u}=-\frac{\partial y}{\partial y}\cdot\frac{1}{u}-\frac{\partial}{\partial y}\frac{1}{u}\cdot t=-\frac{1}{u}-\frac{3y^2}{u^3}.
\end{aligned}$$

故

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=\frac{3}{u}+\frac{-3t^2+3x^2+3y^2}{u^3}=\frac{3}{u}+\frac{-3u^2}{u^3}=0,$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

习题 1.2/1

证明方程

$$\frac{\partial}{\partial x}\left[\left(1-\frac{x}{h}\right)^2\frac{\partial u}{\partial x}\right]=\frac{1}{a^2}\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2}\quad (h>0,\text{常数})$$

的通解可以写成

$$u=\frac{F(x-at)+G(x+at)}{h-x},$$

其中 F, G 为任意的具有二阶连续导数的单变量函数, 并由此求它满足初始条件:

$$t=0:\quad u=\varphi(x),\frac{\partial u}{\partial t}=\psi(x)$$

的初值问题的解.

$$\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial}{\partial x}\left[\left(1-\frac{x}{h}\right)^2\frac{\partial u}{\partial x}\right],$$

$$(h-x)^2\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial}{\partial x}\left[(h-x)^2\frac{\partial u}{\partial x}\right].$$

设 $u(x,t)=v(x,t)/(h-x)$

$$\frac{\partial^2 u}{\partial t^2}=\frac{1}{h-x}\frac{\partial^2 v}{\partial t^2},\quad (h-x)^2\frac{\partial^2 u}{\partial t^2}=(h-x)\frac{\partial^2 v}{\partial t^2}.$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}\cdot\frac{1}{h-x}+\frac{\partial}{\partial x}\left(\frac{1}{h-x}\right)\cdot v=\frac{1}{h-x}\frac{\partial v}{\partial x}+\frac{v}{(h-x)^2},$$

$$\frac{\partial}{\partial x}\left[(h-x)^2\frac{\partial u}{\partial x}\right]=\frac{\partial}{\partial x}\left[(h-x)\frac{\partial v}{\partial x}+v\right]=\frac{\partial^2 v}{\partial x^2}\cdot(h-x)+\frac{\partial}{\partial x}(h-x)\cdot\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}=(h-x)\frac{\partial^2 v}{\partial x^2}.$$

故

$$\frac{\partial^2 v}{\partial t^2}-a^2\frac{\partial^2 v}{\partial x^2}=0.$$

其通解可以写为

$$v(x,t)=F(x-at)+G(x+at),$$

$$u(x,t)=\frac{v(x,t)}{h-x}=\frac{F(x-at)+G(x+at)}{h-x}.$$

且满足初值条件

$$t=0:\quad v=(h-x)\varphi(x),\quad \frac{\partial v}{\partial t}=(h-x)\psi(x).$$

代入达朗贝尔公式得

$$v(x,t)=\frac{(h-x+at)\varphi(x-at)+(h-x-at)\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}(h-\alpha)\psi(\alpha)d\alpha,$$

$$u(x,t)=\frac{(h-x+at)\varphi(x-at)+(h-x-at)\varphi(x+at)}{2(h-x)}+\frac{1}{2a(h-x)}\int_{x-at}^{x+at}(h-\alpha)\psi(\alpha)d\alpha.$$

习题 1.2/2

问初始条件 $\varphi(x)$ 和 $\psi(x)$ 满足怎样的条件时，齐次波动方程初值问题的解仅由右传播波组成?

齐次波动方程初值问题的解仅由右传播波组成时

$$G(x+at)=C_1,$$

$$G(x)=\frac{1}{2}\varphi(x)+\frac{1}{2a}\int_{x_0}^{x}\psi(\alpha)d\alpha-\frac{C}{2a}=C_1.$$

故初始条件 $\varphi(x)$, $\psi(x)$ 需要满足

$$\varphi(x)+\frac{1}{a}\int_{x_0}^{x}\psi(\alpha)d\alpha=C_2.$$

其中 C, C_1, C_2 为常数.

例题

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 0, \quad |x| < 1, \quad |y| < 1 \\ u|_{y=x^2} = \varphi(x) \\ u_y|_{y=x^2} = \psi(x) \end{cases}.$$

方程的通解为

$$u(x, y) = F(x) + G(y) + C,$$

代入初值条件得

$$\varphi(x) = u|_{y=x^2} = F(x) + G(x^2) + C,$$

$$\psi(x) = u_y|_{y=x^2} = G'(x^2).$$

故

$$G'(x) = \psi(\sqrt{|x|}),$$

$$G(x) = \int_{x_0}^{x} \psi(\sqrt{|\xi|}) d\xi + C_1,$$

$$F(x) = \varphi(x) - G(x^2) - C = \varphi(x) - \int_{x_0}^{x^2} \psi(\sqrt{|\xi|}) d\xi - C_1 - C,$$

$$u(x, y) = F(x) + G(y) + C = \varphi(x) + \int_{x^2}^{y} \psi(\sqrt{|\xi|}) d\xi.$$