MA319 — 偏微分方程

Assignment 4

Instructor: 许德良

Author: 刘逸灏 (515370910207)

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习题 1.4/1

用泊松公式求解波动方程的柯西问题:

$$(1)\ \begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u|_{t=0} = 0, \quad u_t|_{t=0} = x^2 + yz; \end{cases}$$

$$(2)\ \begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u|_{t=0} = x^3 + y^2 z, \quad u_t|_{t=0} = 0; \end{cases}$$

(1)

设

$$\xi = x + at\sin\theta\cos\phi, \quad \eta = y + at\sin\theta\sin\phi, \quad \zeta = z + at\cos\theta.$$

方程的初值条件为

$$\varphi(\xi,\eta,\zeta) = 0,$$

$$\psi(\xi,\eta,\zeta) = \xi^2 + \eta\zeta = (x + at\sin\theta\cos\phi)^2 + (y + at\sin\theta\sin\phi)(z + at\cos\theta).$$

代入泊松公式得

$$\begin{aligned}
u(x,y,z,t) &= \frac{1}{4\pi a^2}\frac{\partial}{\partial t}\iint\limits_{S_{at}^M}\frac{\varphi(\xi,\eta,\zeta)}{t}dS + \frac{1}{4\pi a^2}\iint\limits_{S_{at}^M}\frac{\psi(\xi,\eta,\zeta)}{t}dS \\
&= \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} t\varphi(\xi,\eta,\zeta)\sin\theta d\theta d\phi + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\psi(\xi,\eta,\zeta)\sin\theta d\theta d\phi \\
&= \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}[(x + at\sin\theta\cos\phi)^2 + (y + at\sin\theta\sin\phi)(z + at\cos\theta)]\sin\theta d\theta d\phi \\
&= \frac{t}{4\pi}\int_0^{2\pi}\left[\frac{2}{3}a^2t^2\cos 2\phi + \frac{2}{3}a^2t^2 + \pi atx\cos\phi + \frac{1}{2}\pi atz\sin\phi + 2x^2 + 2yz\right]d\phi \\
&= \frac{t}{4\pi}\cdot\frac{4}{3}\pi\left[a^2t^2 + 3\left(x^2 + yz\right)\right] \\
&= \frac{1}{3}a^2t^3 + t\left(x^2 + yz\right).
\end{aligned}$$

(2)

设

$$\xi = x + at\sin\theta\cos\phi, \quad \eta = y + at\sin\theta\sin\phi, \quad \zeta = z + at\cos\theta.$$

方程的初值条件为

$$\varphi(\xi,\eta,\zeta)=\xi^2+\eta\zeta=(x+at\sin\theta\cos\phi)^3+(y+at\sin\theta\sin\phi)^2(z+at\cos\theta),$$

$$\psi(\xi,\eta,\zeta)=0.$$

代入泊松公式得

$$\begin{aligned}
u(x,y,z,t)&=\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\iint\limits_{S_{at}^M}\frac{\varphi(\xi,\eta,\zeta)}{t}dS+\frac{1}{4\pi a^2}\iint\limits_{S_{at}^M}\frac{\psi(\xi,\eta,\zeta)}{t}dS\\
&=\frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi}t\varphi(\xi,\eta,\zeta)\sin\theta d\theta d\phi+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\psi(\xi,\eta,\zeta)\sin\theta d\theta d\phi\\
&=\frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi}t[(x+at\sin\theta\cos\phi)^3+(y+at\sin\theta\sin\phi)^2(z+at\cos\theta)]\sin\theta d\theta d\phi\\
&=\frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}t\begin{bmatrix}\frac{3}{32}\pi a^3t^3\cos3\phi+\frac{3}{32}\pi at\cos\phi\left(3a^2t^2+16x^2\right)+2a^2t^2x+\\ \frac{2}{3}a^2t^2(3x-z)\cos2\phi+\frac{2}{3}a^2t^2z+\pi atyz\sin\phi+2x^3+2y^2z\end{bmatrix}d\phi\\
&=\frac{1}{4\pi}\frac{\partial}{\partial t}\frac{4}{3}\pi t\left[a^2t^2(3x+z)+3\left(x^3+y^2z\right)\right]\\
&=a^2t^2(3x+z)+x^3+y^2z.
\end{aligned}$$

习题 1.4/3

求解平面波动方程的柯西问题:

$$(1)\ \begin{cases}u_{tt}=a^2(u_{xx}+u_{yy}),\\ u|_{t=0}=x^2(x+y),\\ u_t|_{t=0}=0;\end{cases}$$

$$(2)\ \begin{cases}u_{tt}-3(u_{xx}+u_{yy})=x^3+y^3,\\ u|_{t=0}=0,\\ u_t|_{t=0}=x^2;\end{cases}$$

(1)

设

$$\xi=x+r\cos\theta,\quad \eta=y+r\sin\theta.$$

方程的初值条件为

$$\varphi(\xi,\eta)=\xi^2(\xi+\eta)=(x+r\cos\theta)^2(x+r\cos\theta+y+r\sin\theta),$$

$$\psi(\xi,\eta)=0.$$

代入泊松公式得

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{C_{at}^M}\frac{\varphi(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}+\frac{1}{2\pi a}\iint\limits_{C_{at}^M}\frac{\psi(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}\\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{\varphi(\xi,\eta)}{\sqrt{a^2t^2-r^2}}rd\theta dr+\frac{1}{2\pi a}\int_0^{at}\int_0^{2\pi}\frac{\psi(\xi,\eta)}{\sqrt{a^2t^2-r^2}}rd\theta dr\\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{(x+r\cos\theta)^2(x+r\cos\theta+y+r\sin\theta)}{\sqrt{a^2t^2-r^2}}rd\theta dr\\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\frac{\pi r\left[r^2(3x+y)+2x^2(x+y)\right]}{\sqrt{a^2t^2-r^2}}dr\\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\pi\left[\frac{2}{3}a^3t^3(3x+y)+2atx^2(x+y)\right]\\
&= a^2t^2(3x+y)+x^2(x+y).
\end{aligned}
$$

(2)

设

$$\xi=x+r\cos\theta,\quad \eta=y+r\sin\theta,\quad \tau=a(t-s).$$

方程的初值条件为

$$\varphi(\xi,\eta)=0,$$

$$\psi(\xi,\eta)=\xi^2=(x+r\cos\theta)^2,$$

$$f(\xi,\eta,s)=\xi^3+\eta^3=(x+r\cos\theta)^3+(x+r\sin\theta)^3$$

代入泊松公式得

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{C_{at}^M}\frac{\varphi(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}+\frac{1}{2\pi a}\iint\limits_{C_{at}^M}\frac{\psi(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}\\
&\quad+\frac{1}{2\pi a^2}\int_0^{at}\iint\limits_{C_{at}^M}\frac{f(\xi,\eta,t-\tau/a)d\xi d\eta d\tau}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}\\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{\varphi(\xi,\eta)}{\sqrt{a^2t^2-r^2}}rd\theta dr+\frac{1}{2\pi a}\int_0^{at}\int_0^{2\pi}\frac{\psi(\xi,\eta)}{\sqrt{a^2t^2-r^2}}rd\theta dr\\
&\quad+\frac{1}{2\pi a}\int_0^{t}\int_0^{a(t-s)}\int_0^{2\pi}\frac{f(\xi,\eta,s)}{\sqrt{a^2(t-s)^2-r^2}}rd\theta drds\\
&= \frac{1}{2\pi a}\int_0^{at}\int_0^{2\pi}\frac{(x+r\cos\theta)^2}{\sqrt{a^2t^2-r^2}}rd\theta dr+\frac{1}{2\pi a}\int_0^{t}\int_0^{a(t-s)}\int_0^{2\pi}\frac{(x+r\cos\theta)^3+(x+r\sin\theta)^3}{\sqrt{a^2(t-s)^2-r^2}}rd\theta drds\\
&= \frac{1}{2\pi a}\int_0^{at}\frac{\pi r(r^2+2x)}{\sqrt{a^2t^2-r^2}}dr+\frac{1}{2\pi a}\int_0^{t}\int_0^{a(t-s)}\frac{\pi r\left[3r^2(x+y)+2\left(x^3+y^3\right)\right]}{\sqrt{a^2(s-t)^2-r^2}}drds\\
&= \frac{1}{2\pi a}\cdot\frac{2}{3}\pi at\left(a^2t^2+3x^2\right)+\frac{1}{2\pi a}\int_0^{t}2\pi(x+y)a(t-s)\left[a^2(s-t)^2+x^2-xy+y^2\right]ds\\
&= \frac{1}{3}a^2t^3+tx^2+\frac{1}{2\pi a}\cdot\frac{1}{2}\pi at^2(x+y)\left[a^2t^2+2\left(x^2-xy+y^2\right)\right]\\
&= t^3+tx^2+\frac{1}{4}t^2(x+y)\left[3t^2+2\left(x^2-xy+y^2\right)\right].
\end{aligned}
$$

习题 1.4/5

求解下列柯西问题:

$$\begin{cases} u_{tt} = a^2(u_{xx}+u_{yy}) + c^2 u, \\ u|_{t=0} = \varphi(x,y), \\ u_t|_{t=0} = \psi(x,y). \end{cases}$$

设

$$v(x,y,z,t) = e^{\frac{cz}{a}} u(x,y,t).$$

则

$$v_{tt} - a^2(v_{xx}+v_{yy}+v_{zz}) = e^{\frac{cz}{a}} u_{tt} - a^2\left(e^{\frac{cz}{a}} u_{xx} + e^{\frac{cz}{a}} u_{yy} + \frac{c^2}{a^2} e^{\frac{cz}{a}} u\right) = e^{\frac{cz}{a}}\left[u_{tt} - a^2(u_{xx}+u_{yy}) + c^2 u\right] = 0,$$

$$v|_{t=0} = e^{\frac{cz}{a}} u|_{t=0} = e^{\frac{cz}{a}} \varphi(x,y),$$

$$v_t|_{t=0} = \frac{\partial}{\partial t} e^{\frac{cz}{a}} u\bigg|_{t=0} = e^{\frac{cz}{a}} u_t|_{t=0} = e^{\frac{cz}{a}} \psi(x,y).$$

故可以先求解以下柯西问题:

$$\begin{cases} v_{tt} = a^2(v_{xx}+v_{yy}+v_{zz}), \\ v|_{t=0} = e^{\frac{cz}{a}} \varphi(x,y), \\ v_t|_{t=0} = e^{\frac{cz}{a}} \psi(x,y). \end{cases}$$

设

$$\xi = x + at\sin\theta\cos\phi, \quad \eta = y + at\sin\theta\sin\phi, \quad \zeta = z + at\cos\theta.$$

方程的初值条件为

$$\varphi(\xi,\eta,\zeta) = e^{\frac{c\zeta}{a}} \varphi(\xi,\eta) = e^{\frac{c}{a}(z+at\cos\theta)} \varphi(x+at\sin\theta\cos\phi, y+at\sin\theta\sin\phi),$$

$$\psi(\xi,\eta,\zeta) = e^{\frac{c\zeta}{a}} \psi(\xi,\eta) = e^{\frac{c}{a}(z+at\cos\theta)} \psi(x+at\sin\theta\cos\phi, y+at\sin\theta\sin\phi).$$

代入泊松公式得

$$\begin{aligned} v(x,y,z,t) &= \frac{1}{4\pi a^2}\frac{\partial}{\partial t}\iint\limits_{S_{at}^M} \frac{\varphi(\xi,\eta,\zeta)}{t} dS + \frac{1}{4\pi a^2}\iint\limits_{S_{at}^M} \frac{\psi(\xi,\eta,\zeta)}{t} dS \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} t\varphi(\xi,\eta,\zeta)\sin\theta d\theta d\phi + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\psi(\xi,\eta,\zeta)\sin\theta d\theta d\phi \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} te^{\frac{c}{a}(z+at\cos\theta)}\varphi(\xi,\eta)\sin\theta d\theta d\phi + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{\frac{c}{a}(z+at\cos\theta)}\psi(\xi,\eta)\sin\theta d\theta d\phi \\ &= e^{\frac{cz}{a}}\left[\frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} te^{ct\cos\theta}\varphi(\xi,\eta)\sin\theta d\theta d\phi + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{ct\cos\theta}\psi(\xi,\eta)\sin\theta d\theta d\phi\right], \\ u(x,y,t) &= e^{-\frac{cz}{a}} v(x,y,z,t) \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} te^{ct\cos\theta}\varphi(\xi,\eta)\sin\theta d\theta d\phi + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{ct\cos\theta}\psi(\xi,\eta)\sin\theta d\theta d\phi \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} te^{ct\cos\theta}\varphi(x+at\sin\theta\cos\phi, y+at\sin\theta\sin\phi)\sin\theta d\theta d\phi \\
&\quad + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{ct\cos\theta}\psi(x+at\sin\theta\cos\phi, y+at\sin\theta\sin\phi)\sin\theta d\theta d\phi.
\end{aligned}
$$

习题 1.4/6

试用齐次化原理导出平面非齐次波动方程

$$u_{tt} = a^2(u_{xx}+u_{yy}) + f(x,y,t)$$

在齐次初始条件

$$\begin{cases} u|_{t=0} = 0, \\ u_t|_{t=0} = 0 \end{cases}$$

下的求解公式.

设以下齐次柯西问题的解为 $W = (W, x, y; \tau)$:

$$\begin{cases} W_{tt} = a^2(W_{xx}+W_{yy}), \\ W|_{t=\tau} = 0, \quad W_t|_{t=\tau} = f(x,y,\tau). \end{cases}$$

然后关于参数 τ 积分得

$$u(x,y,t) = \int_0^t W(x,y,t;\tau)d\tau.$$

然后验证 $u(x,y,t)$ 就是原柯西问题的解:

$$u_t = W(x,y,t;t) + \int_0^t W_t(x,y,t;\tau)d\tau = \int_0^t W_t(x,y,t;\tau)d\tau,$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0,$$

$$u_{tt} = W_t(x,y,t;t) + \int_0^t W_{tt}(x,y,t;\tau)d\tau = f(x,y,t) + \int_0^t W_{tt}(x,y,t;\tau)d\tau,$$

$$\int_0^t W_{tt}(x,y,t;\tau)d\tau = a^2\left(\int_0^t W_{xx}(x,y,t;\tau)d\tau + \int_0^t W_{yy}(x,y,t;\tau)d\tau\right) = a^2(u_{xx}+u_{yy}).$$

故

$$\begin{cases} u_{tt} = a^2(u_{xx}+u_{yy}) + f(x,y,t), \\ u|_{t=0} = 0, \quad t_t|_{t=0} = 0. \end{cases}$$

令 $t' = t-\tau$, 则 W 满足

$$\begin{cases} W_{tt} = a^2(W_{xx}+W_{yy}), \\ W|_{t'=0} = 0, \quad W_{t'}|_{t'=0} = f(x,y,\tau). \end{cases}$$

设

$$\xi = x + r\cos\theta, \quad \eta = y + r\sin\theta.$$

方程的初值条件为

$$\varphi(\xi, \eta) = 0,$$

$$\psi(\xi, \eta) = f(\xi, \eta, \tau) = f(x + r\cos\theta, y + r\sin\theta, \tau).$$

代入泊松公式得

$$\begin{aligned}
W(x, y, t; \tau) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{C_{at'}^M} \frac{\varphi(\xi, \eta)d\xi d\eta}{\sqrt{a^2t'^2 - (\xi - x)^2 - (\eta - y)^2}} + \frac{1}{2\pi a}\iint\limits_{C_{at'}^M} \frac{\psi(\xi, \eta)d\xi d\eta}{\sqrt{a^2t'^2 - (\xi - x)^2 - (\eta - y)^2}} \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at'}\int_0^{2\pi} \frac{\varphi(\xi, \eta)}{\sqrt{a^2t'^2 - r^2}} r d\theta dr + \frac{1}{2\pi a}\int_0^{at'}\int_0^{2\pi} \frac{\psi(\xi, \eta)}{\sqrt{a^2t'^2 - r^2}} r d\theta dr \\
&= \frac{1}{2\pi a}\int_0^{a(t-\tau)}\int_0^{2\pi} \frac{f(x + r\cos\theta, y + r\sin\theta, \tau)}{\sqrt{a^2(t - \tau)^2 - r^2}} r d\theta dr, \\
u(x, y, t) &= \int_0^t W(x, y, t; \tau)d\tau \\
&= \frac{1}{2\pi a}\int_0^t\int_0^{a(t-\tau)}\int_0^{2\pi} \frac{f(x + r\cos\theta, y + r\sin\theta, \tau)}{\sqrt{a^2(t - \tau)^2 - r^2}} r d\theta dr d\tau.
\end{aligned}$$

习题　1.4/8

解非齐次方程的柯西问题:

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz} + 2(y - t), \\ u|_{t=0} = 0, \quad u_t|_{t=0} = x^2 + yz. \end{cases}$$

设

$$\xi = x + t\sin\theta\cos\phi, \quad \eta = y + t\sin\theta\sin\phi, \quad \zeta = z + t\cos\theta.$$

方程的初值条件为

$$\varphi(\xi, \eta, \zeta) = 0,$$

$$\psi(\xi, \eta, \zeta) = \xi^2 + \eta\zeta = (x + t\sin\theta\cos\phi)^2 + (y + t\sin\theta\sin\phi)(z + t\cos\theta),$$

$$f(\xi, \eta, \zeta, t - r) = 2[\eta - (t - r)] = 2(y + t\sin\theta\sin\phi - t + r).$$

代入泊松公式得

$$\begin{aligned}
u(x, y, z, t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\iint\limits_{S_t^M} \frac{\varphi(\xi, \eta, \zeta)}{t} dS + \frac{1}{4\pi}\iint\limits_{S_t^M} \frac{\psi(\xi, \eta, \zeta)}{t} dS + \frac{1}{4\pi}\iiint\limits_{r \leqslant r} \frac{f(\xi, \eta, \zeta, t - r)}{r} dV \\
&= \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} t\varphi(\xi, \eta, \zeta)\sin\theta d\theta d\phi + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} \psi(\xi, \eta, \zeta)\sin\theta d\theta d\phi \\
&\quad + \frac{1}{4\pi}\int_0^t\int_0^{2\pi}\int_0^{\pi} f(\xi, \eta, \zeta, t - r) r\sin\theta d\theta d\phi dr \\
&= \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} [(x + t\sin\theta\cos\phi)^2 + (y + t\sin\theta\sin\phi)(z + t\cos\theta)]\sin\theta d\theta d\phi \\
&\quad + \frac{1}{4\pi}\int_0^t\int_0^{2\pi}\int_0^{\pi} 2(y + t\sin\theta\sin\phi - t + r) r\sin\theta d\theta d\phi dr \\
&= \frac{t}{4\pi}\int_0^{2\pi}\left[\frac{2}{3}t^2\cos 2\phi + \frac{2}{3}t^2 + \pi t x\cos\phi + \frac{1}{2}\pi t z\sin\phi + 2x^2 + 2yz\right] d\phi
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4\pi}\int_0^t\int_0^{2\pi} r[\pi t\sin\phi + 4(r-t+y)]d\phi dr \\
&= \frac{t}{4\pi}\cdot\frac{4}{3}\pi\left[t^2+3\left(x^2+yz\right)\right] + \frac{1}{4\pi}\int_0^t 8\pi r(r-t+y)dr \\
&= \frac{1}{3}t^3 + t\left(x^2+yz\right) + \frac{1}{4\pi}\cdot -\frac{4}{3}\pi t^2(t-3y) \\
&= t\left(ty+x^2+\mathrm{yz}\right).
\end{aligned}
$$