

MA319 — 偏微分方程

Assignment 10

Instructor: 许德良

Author: 刘逸灏 (515370910207)

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习题 3.1/6

用分离变量法求解由下述调和方程的第一边值问题所描述的矩形平板 ($0 \leq x \leq a$, $0 \leq y \leq b$) 上的稳定温度分布:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \\ u(0, y) = u(a, y) = 0, \\ u(x, 0) = \sin \frac{\pi x}{a}, \quad u(x, b) = 0. \end{cases}$$

设 $u(x, y) = X(x)Y(y)$, 代入得

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

则有两个常微分方程

$$X''(x) + \lambda X(x) = 0,$$

$$Y''(y) - \lambda Y(y) = 0.$$

关于 x 的方程的特征值和特征函数为

$$X(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x.$$

代入 $X(0) = 0$, $X(a) = 0$ 得

$$C_1 = 0, \quad C_2 \sin \sqrt{\lambda}a = 0,$$

$$\lambda = \lambda_k = \frac{k^2\pi^2}{a^2}, \quad X_k(x) = C_k \sin \sqrt{\lambda}x = C_k \sin \frac{k\pi}{a}x, \quad k = 1, 2, \dots.$$

代入关于 y 的常微分方程可得

$$Y''(y) - \frac{k^2\pi^2}{a^2}Y(y) = 0,$$

$$Y(y) = A_k e^{\frac{k\pi}{a}y} + B_k e^{-\frac{k\pi}{a}y}.$$

方程的通解为

$$u(x, y) = \sum_{k=1}^{\infty} X(x)Y(y) = \sum_{k=1}^{\infty} \left(A_k e^{\frac{k\pi}{a}y} + B_k e^{-\frac{k\pi}{a}y} \right) \sin \frac{k\pi}{a}x.$$

代入初值条件得

$$u(x, 0) = \sum_{k=1}^{\infty} (A_k + B_k) \sin \frac{k\pi}{a} x = \sin \frac{\pi x}{a},$$

$$u(x, b) = \sum_{k=1}^{\infty} \left(A_k e^{\frac{k\pi}{a} b} + B_k e^{-\frac{k\pi}{a} b} \right) \sin \frac{k\pi}{a} x = 0.$$

解得

$$A_k + B_k = \begin{cases} 1 & k = 1, \\ 0 & k > 1, \end{cases}$$

$$A_k e^{\frac{k\pi}{a} b} + B_k e^{-\frac{k\pi}{a} b} = 0.$$

化简得

$$A_k = \begin{cases} -\frac{e^{-\frac{\pi}{a} b}}{e^{\frac{\pi}{a} b} - e^{-\frac{\pi}{a} b}} & k = 1, \\ 0 & k > 1, \end{cases} \quad B_k = \begin{cases} \frac{e^{\frac{\pi}{a} b}}{e^{\frac{\pi}{a} b} - e^{-\frac{\pi}{a} b}} & k = 1, \\ 0 & k > 1. \end{cases}$$

故

$$u(x, y) = \frac{-e^{-\frac{\pi}{a}(b-y)} + e^{\frac{\pi}{a}(b-y)}}{e^{\frac{\pi}{a} b} - e^{-\frac{\pi}{a} b}} \sin \frac{\pi}{a} x = \frac{\sinh \frac{\pi}{a}(b-y)}{\sinh \frac{\pi}{a} b} \sin \frac{\pi}{a} x.$$

习题 3.1/7

在膜形扁壳渠道闸门的设计中, 为了考察闸门在水压力作用下的受力情况, 要在矩形区域 $0 \leq x \leq a$, $0 \leq y \leq b$ 上求解如下的非齐次调和方程的边值问题:

$$\begin{cases} \Delta u = py + q & (p < 0, q > 0 \text{ 常数}) \\ \frac{\partial u}{\partial x} \Big|_{x=0} = 0, & u|_{x=a} = 0, \\ u|_{y=0, y=b} = 0. \end{cases}$$

试求解之.

设

$$v(x, y) = u(x, y) + (x^2 - a^2)(fy + g).$$

则

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + 2(fy + g) + \frac{\partial^2 v}{\partial y^2} = \Delta u + 2(fy + g).$$

令 $f = -\frac{p}{2}$, $g = -\frac{q}{2}$ 可使 $\Delta v = 0$, 此时

$$v(x, y) = u(x, y) - \frac{1}{2}(x^2 - a^2)(py + q),$$

$$\frac{\partial v}{\partial x} \Big|_{x=0} = \left[\frac{\partial u}{\partial x} + 2x(fy + g) \right] \Big|_{x=0} = 0, \quad v|_{x=a} = [u + (x^2 - a^2)(fy + g)]|_{x=a} = 0,$$

$$v|_{y=0} = [u + (x^2 - a^2)(fy + g)]|_{y=0} = -\frac{q}{2}(x^2 - a^2), \quad v|_{y=b} = [u + (x^2 - a^2)(fy + g)]|_{y=b} = -\frac{pb + q}{2}(x^2 - a^2).$$

故可以先求解齐次调和方程的边值问题:

$$\begin{cases} \Delta v = 0 \\ \frac{\partial v}{\partial x} \Big|_{x=0} = 0, \quad v|_{x=a} = 0, \\ v|_{y=0} = -\frac{q}{2}(x^2 - a^2), \quad v|_{y=b} = -\frac{pb+q}{2}(x^2 - a^2). \end{cases}$$

设 $v(x, y) = X(x)Y(y)$, 代入得

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

则有两个常微分方程

$$X''(x) + \lambda X(x) = 0,$$

$$Y''(y) - \lambda Y(y) = 0.$$

关于 x 的方程的特征值和特征函数为

$$X(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x.$$

代入 $X'(0) = 0, X(a) = 0$ 得

$$C_2 = 0, \quad C_1 \cos \sqrt{\lambda}a = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2\pi^2}{4a^2}, \quad X_k(x) = C_k \cos \sqrt{\lambda}x = C_k \cos \frac{(2k-1)\pi}{2a}x, \quad k = 1, 2, \dots$$

代入关于 y 的常微分方程可得

$$Y''(y) - \frac{(2k-1)^2\pi^2}{4a^2}Y(y) = 0,$$

$$Y(y) = A_k e^{\frac{(2k-1)\pi}{2a}y} + B_k e^{-\frac{(2k-1)\pi}{2a}y}.$$

方程的通解为

$$v(x, y) = \sum_{k=1}^{\infty} X(x)Y(y) = \sum_{k=1}^{\infty} \left(A_k e^{\frac{(2k-1)\pi}{2a}y} + B_k e^{-\frac{(2k-1)\pi}{2a}y} \right) \cos \frac{(2k-1)\pi}{2a}x.$$

代入初值条件得

$$v(x, 0) = \sum_{k=1}^{\infty} (A_k + B_k) \cos \frac{(2k-1)\pi}{2a}x = -\frac{q}{2}(x^2 - a^2),$$

$$v(x, b) = \sum_{k=1}^{\infty} \left(A_k e^{\frac{(2k-1)\pi}{2a}b} + B_k e^{-\frac{(2k-1)\pi}{2a}b} \right) \cos \frac{(2k-1)\pi}{2a}x = -\frac{pb+q}{2}(x^2 - a^2).$$

解得

$$\frac{2}{a} \int_0^a (x^2 - a^2) \cos \frac{(2k-1)\pi}{2a}x dx = \frac{16a^2[2 \cos k\pi + (2k-1)\pi \sin k\pi]}{(2k-1)^3\pi^3} = (-1)^k \frac{32a^2}{(2k-1)^3\pi^3},$$

$$A_k + B_k = \frac{2}{a} \int_0^a -\frac{q}{2}(x^2 - a^2) \cos \frac{(2k-1)\pi}{2a}x dx = (-1)^{k+1} \frac{16qa^2}{(2k-1)^3\pi^3},$$

$$A_k e^{\frac{(2k-1)\pi}{2a}b} + B_k e^{-\frac{(2k-1)\pi}{2a}b} = \frac{2}{a} \int_0^a -\frac{pb+q}{2}(x^2-a^2) \cos \frac{(2k-1)\pi}{2a}x dx = (-1)^{k+1} \frac{16(pb+q)a^2}{(2k-1)^3\pi^3}.$$

化简得

$$A_k = (-1)^{k+1} \frac{(2k-1)^3\pi^3}{16(pb+q)a^2} \cdot \frac{pb+q - qe^{-\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b} - e^{-\frac{(2k-1)\pi}{2a}b}},$$

$$B_k = (-1)^{k+1} \frac{(2k-1)^3\pi^3}{16(pb+q)a^2} \cdot \frac{-(pb+q) + qe^{\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b} - e^{-\frac{(2k-1)\pi}{2a}b}}.$$

故

$$\begin{aligned} v(x, y) &= \sum_{k=1}^{\infty} \left((-1)^{k+1} \frac{(2k-1)^3\pi^3}{16(pb+q)a^2} \cdot \frac{pb+q - qe^{-\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b} - e^{-\frac{(2k-1)\pi}{2a}b}} e^{\frac{(2k-1)\pi}{2a}y} \right. \\ &\quad \left. + (-1)^{k+1} \frac{(2k-1)^3\pi^3}{16(pb+q)a^2} \cdot \frac{-(pb+q) + qe^{\frac{(2k-1)\pi}{2a}b}}{e^{\frac{(2k-1)\pi}{2a}b} - e^{-\frac{(2k-1)\pi}{2a}b}} e^{-\frac{(2k-1)\pi}{2a}y} \right) \cos \frac{(2k-1)\pi}{2a}x \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k-1)^3\pi^3}{16(pb+q)a^2} \cdot \frac{(pb+q) \sinh \frac{(2k-1)\pi}{2a}y + q \sinh \frac{(2k-1)\pi}{2a}(b-y)}{\sinh \frac{(2k-1)\pi}{2a}b} \cos \frac{(2k-1)\pi}{2a}x. \\ u(x, y) &= v(x, y) + \frac{1}{2}(x^2 - a^2)(py + q). \end{aligned}$$

习题 3.1/8

举例说明在二维调和方程的狄利克雷外问题中, 如对解 $u(x, y)$ 不加在无穷远点为有界的限制, 那么定解问题的解就不是唯一的.

设

$$u(x, y) = f(r),$$

根据 3.1/1 可知

$$u(x, y) = c_1 + c_2 \ln \frac{1}{r}.$$

此时只需令

$$u|_{r=1} = c_1,$$

由 c_2 的任意性即可知 u 有无数个解.

习题 2.2/6

半径为 a 的半圆形平板, 其表面绝热, 在板的圆周边界上保持常温 u_0 , 而在直径边界上保持常温 u_1 , 求圆板稳恒状态 (即与时间 t 无关的状态) 的温度分布.

根据题意可列出定解问题为

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & x^2 + y^2 \leq a^2, \quad 0 < y < a, \\ u(x, 0) = u_1, \\ u(\sqrt{a^2 - y^2}, y) = u_0. \end{cases}$$

设

$$u(x, y) = v(x, y) + u_1.$$

坐标变换为极坐标系可得

$$\begin{cases} \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0, & 0 \leq r \leq a, \quad 0 < \theta < \pi, \\ v(r, 0) = v(r, \pi) = 0, \\ v(a, \theta) = u_0 - u_1. \end{cases}$$

设 $v(r, \theta) = R(r)\Theta(\theta)$, 代入得

$$\begin{aligned} R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) &= 0, \\ -\frac{r^2R''(r) + rR'(r)}{R(r)} &= \frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda. \end{aligned}$$

则有两个常微分方程

$$\begin{aligned} r^2R''(r) + rR'(r) - \lambda R(r) &= 0, \\ \Theta''(\theta) + \lambda\Theta(\theta) &= 0. \end{aligned}$$

关于 θ 的方程的特征值和特征函数为

$$\Theta(\theta) = C_1 \cos \sqrt{\lambda}\theta + C_2 \sin \sqrt{\lambda}\theta.$$

代入 $\Theta(0) = 0, \Theta(\pi) = 0$ 得

$$\begin{aligned} C_1 &= 0, \quad C_2 \sin \sqrt{\lambda}\pi = 0, \\ \lambda = \lambda_k = k^2, \quad \Theta_k(\theta) &= C_k \sin \sqrt{\lambda}\theta = C_k \sin k\theta, \quad k = 1, 2, \dots \end{aligned}$$

代入关于 r 的常微分方程可得

$$\begin{aligned} r^2R''(r) + rR'(r) - k^2R(r) &= 0, \\ Y(y) &= A_k r^k + B_k r^{-k}. \end{aligned}$$

方程的通解为

$$v(r, \theta) = \sum_{k=1}^{\infty} R(r)\Theta(\theta) = \sum_{k=1}^{\infty} (A_k r^k + B_k r^{-k}) \sin k\theta.$$

由于

$$\lim_{r \rightarrow 0} r^{-k} \rightarrow \infty$$

可推出

$$B_k = 0.$$

代入初值条件得

$$v(a, \theta) = \sum_{k=1}^{\infty} A_k a^k \sin k\theta = u_0 - u_1.$$

解得

$$A_k = \frac{2(u_0 - u_1)}{a^k \pi} \int_0^\pi \sin k\theta d\theta = \frac{2(u_0 - u_1)}{a^k \pi} \cdot \frac{1 - \cos k\pi}{k} = \begin{cases} \frac{4(u_0 - u_1)}{a^k k \pi}, & k = 2n - 1, \\ 0, & k = 2n \end{cases}, \quad n = 1, 2, \dots.$$

故

$$v(k, \theta) = \sum_{n=1}^{\infty} \frac{4(u_0 - u_1)}{a^{2n-1}(2n-1)\pi} r^{2n-1} \sin(2n-1)\theta,$$
$$u(x, y) = u_1 + \sum_{n=1}^{\infty} \frac{4(u_0 - u_1)}{a^{2n-1}(2n-1)\pi} \sqrt{x^2 + y^2}^{2n-1} \sin \left[(2n-1) \arctan \frac{y}{x} \right].$$

其中

$$\arctan \frac{y}{x} \in [0, \pi].$$