MA319 — 偏微分方程

Assignment 6

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习题 1.6/3

证明波动方程

$$u_{tt} = a^2(u_{xx} + u_{yy}) + f(x, y, t)$$

的自由项 f 在 $L^2(K)$ 意义下作微小改变时, 对应的柯西问题的解 u 在 $L^2(K)$ 意义下改变也是微小的, 其中 K 是由

$$(x - x_0)^2 + (y - y_0)^2 \leqslant (R - at)^2$$

所表示的锥体.

在 K 内任一截面 Ω_t 上成立能量不等式

$$E_0(\Omega_t) = \iint\limits_{\Omega_t} u^2 dxdy, \quad E_1(\Omega_t) = \iint\limits_{\Omega_t} \left[u_t^2 + a^2\left(u_x^2 + u_y^2\right)\right] dxdy.$$

对 $E_1(t)$ 有

$$\begin{aligned}
\frac{dE_1(\Omega_t)}{dt} &= \frac{d}{dt}\int_0^{R-at}\int_0^{2\pi r}\left[u_t^2 + a^2\left(u_x^2 + u_y^2\right)\right] dsdr \\
&= 2\int_0^{R-at}\int_0^{2\pi r}\left[u_t u_{tt} + a^2\left(u_x u_{xt} + u_y u_{yt}\right)\right] dsdr - a\int_{\Gamma_t}\left[u_t^2 + a^2\left(u_x^2 + u_y^2\right)\right] ds. \\
&= 2\int_0^{R-at}\int_0^{2\pi r} u_t\left[u_{tt} - a^2\left(u_x^2 + u_y^2\right)\right] dsdr \\
&\quad + 2\int_{\Gamma_t}\left\{a^2[u_x u_t \cos(n,x) + u_y u_t \sin(n,x)] - \frac{a}{2}\left[u_t^2 + a^2\left(u_x^2 + u_y^2\right)\right]\right\} ds. \\
&= 2\iint\limits_{\Omega_t} u_t f dxdy - a\int_{\Gamma_t}\left[\left(au_x - u_t\cos(n,x)\right)^2 + \left(au_y - u_t\cos(n,y)\right)^2\right] ds \\
&\leqslant 2\iint\limits_{\Omega_t} u_t f dxdy \leqslant \iint\limits_{\Omega_t} u_t^2 dxdy + \iint\limits_{\Omega_t} f^2 dxdy \leqslant E_1(\Omega_t) + \iint\limits_{\Omega_t} f^2 dxdy,
\end{aligned}$$

$$\frac{d}{dt}e^{-t}E_1(\Omega_t) = -e^{-t}E_1(\Omega_t) + e^{-t}\frac{dE_1(\Omega_t)}{dt} \leqslant e^{-t}\iint\limits_{\Omega_t} f^2 dxdy,$$

$$E_1(\Omega_t) \leqslant e^t\int_0^t e^{-\tau}\int_0^l f^2 dxd\tau + e^t E_1(\Omega_0) \leqslant e^t E_1(\Omega_0) + e^t\int_0^t\iint\limits_{\Omega_t} f^2 dxdyd\tau = \overline{E_1(\Omega_t)}.$$

对 $E_0(t)$ 有

$$\frac{dE_0(\Omega_t)}{dt}=2\iint\limits_{\Omega_t}uu_tdxdy-a\int_{\Gamma_t}u^2ds\leqslant 2\iint\limits_{\Omega_t}uu_tdxdy\leqslant\iint\limits_{\Omega_t}u^2dxdy+\iint\limits_{\Omega_t}u_t^2dxdy\leqslant E_0(\Omega_t)+E_1(\Omega_t),$$

$$\frac{d}{dt}e^{-t}E_0(\Omega_t)=-e^{-t}E_0(\Omega_t)+e^{-t}\frac{dE_0(\Omega_t)}{dt}\leqslant e^{-t}E_1(\Omega_t),$$

$$E_0(\Omega_t)\leqslant e^t\int_0^t e^{-\tau}E_1(\Omega_\tau)d\tau+e^tE_0(\Omega_0)\leqslant e^t\overline{E_1(\Omega_t)}(1-e^{-t})+e^tE_0(\Omega_0)=e^tE_0(\Omega_0)+(e^t-1)\overline{E_1(\Omega_t)},$$

$$\begin{aligned}E_0(\Omega_t)+E_1(\Omega_t)&\leqslant e^tE_0(\Omega_0)+(e^t-1)\overline{E_1(\Omega_t)}+E_1(\Omega_t)\leqslant e^t(E_0(\Omega_0)+\overline{E_1(\Omega_t)})\\&=e^tE_0(\Omega_0)+e^{2t}E_1(\Omega_0)+e^{2t}\int_0^t\iint\limits_{\Omega_t}f^2dxdyd\tau.\end{aligned}$$

在 $0\leqslant t\leqslant T$ 上

$$\begin{aligned}E_0(\Omega_t)+E_1(\Omega_t)&\leqslant e^TE_0(\Omega_0)+e^{2T}E_1(\Omega_0)+e^{2T}\int_0^T\iint\limits_{\Omega_t}f^2dxdyd\tau\\&\leqslant e^{2T}\left(E_0(\Omega_0)+E_1(\Omega_0)+\int_0^T\iint\limits_{\Omega_t}f^2dxdyd\tau\right).\end{aligned}$$

存在 C 使得

$$\sqrt{T(E_0(\Omega_t)+E_1(\Omega_t))}\leqslant\sqrt{Te^{2T}\left(E_0(\Omega_0)+E_1(\Omega_0)+\int_0^T\iint\limits_{\Omega_t}f^2dxdyd\tau\right)}\leqslant C\eta,$$

$$\sqrt{E_0(\Omega_0)+E_1(\Omega_0)+\int_0^T\iint\limits_{\Omega_t}f^2dxdyd\tau}\leqslant\eta.$$

任取 $\varepsilon>0$, 可以找到 $\eta=\dfrac{\varepsilon}{C}$, 使得

$$\|f_1-f_2\|_{L^2(K)}\leqslant\sqrt{E_0(\Omega_0)+E_1(\Omega_0)+\int_0^T\iint\limits_{\Omega_t}f^2dxdyd\tau}\leqslant\eta,$$

$$\|u_1-u_2\|_{L^2(K)}\leqslant\sqrt{T(E_0(\Omega_t)+E_1(\Omega_t))}\leqslant\varepsilon.$$

故自由项 f 在 $L^2(K)$ 意义下作微小改变时, 对应的柯西问题的解 u 在 $L^2(K)$ 意义下改变也是微小的.

习题 2.1/2

试直接推导扩散过程所满足的微分方程.

设 $N(x,y,t)$ 表示扩散物质的浓度, dm 表示在无穷小时段 dt 内沿法线方向 **n** 经过一个无穷小面积 dS 的

扩散物质的质量, $D(x,y,t)$ 为扩散系数, 其总取正值, 则

$$dm=-D(x,y,t)\frac{\partial N(x,y,t)}{\partial \mathbf{n}}dSdt.$$

故从 t_1 到 t_2 进入扩散面为 Γ 的区域 Ω 的质量为

$$\int_{t_1}^{t_2}\iint\limits_{\Gamma}-dm=\int_{t_1}^{t_2}\iint\limits_{\Gamma}D\frac{\partial N}{\partial t}dSdt=\int_{t_1}^{t_2}\iiint\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial N}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial N}{\partial z}\right)\right]dxdydzdt.$$

区域 Ω 内, 从 t_1 到 t_2 物质的增加量也可表示为

$$\iiint\limits_{\Omega}\left[N(x,y,z,t_2)-N(x,y,z,t_1)\right]dxdydz=\iiint\limits_{\Omega}\int_{t_1}^{t_2}\frac{\partial N}{\partial t}dtdxdydz=\int_{t_1}^{t_2}\iiint\limits_{\Omega}\frac{\partial N}{\partial t}dxdydzdt.$$

由于这两个质量相等, 且 t_1, t_2, Ω 的取值是任意的, 可得

$$\frac{\partial N}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial N}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial N}{\partial z}\right).$$

习题　2.2/1

用分离变量法求下列定解问题的解:

$$\begin{cases}\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}\quad(t>0,0<x<\pi),\\ u(0,t)=\dfrac{\partial u}{\partial x}(\pi,t)=0\quad(t>0),\\ u(x,0)=f(x)\quad(0<x<\pi).\end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0$, $X'(\pi)=0$ 得

$$C_1=0,\quad C_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi=0,$$

$$\lambda=\lambda_k=\frac{(2k-1)^2\pi^2}{4l^2}=\frac{(2k-1)^2}{4},\quad X_k(x)=C_k\sin\frac{(2k-1)}{2}x,\quad k=1,2,\cdots.$$

方程的通解为

$$u(x,t)=\sum_{k=1}^{\infty}A_ke^{-a^2\lambda t}\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_ke^{-\frac{(2k-1)^2}{4}a^2t}\sin\frac{2k-1}{2}x.$$

代入初值条件得

$$u(x,0)=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{2k-1}{2}x=f(x).$$

解得

$$A_k=\frac{2}{l}\int_0^l f(x)\sin\sqrt{\lambda}xdx=\frac{2}{\pi}\int_0^{\pi}f(x)\sin\frac{2k-1}{2}xdx.$$

故

$$u(x,t)=\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^{\pi}f(x)\sin\frac{2k-1}{2}xdx\cdot e^{-\frac{(2k-1)^2}{4}a^2t}\sin\frac{2k-1}{2}x.$$

习题 2.2/2

用分离变量法求解热传导方程的初边值问题:

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}\quad(t>0,0<x<1),\\ u(x,0)=\begin{cases}x, & 0<x\leqslant\dfrac{1}{2},\\ 1-x, & \dfrac{1}{2}<x<1,\end{cases}\\ u(0,t)=u(1,t)=0.\end{cases}$$

方程的特征值和对应的特征函数为

$$\lambda=\lambda_k=\frac{k^2\pi^2}{l^2}=k^2\pi^2,\quad X_k(x)=C_k\sin\sqrt{\lambda}x=C_k\sin k\pi x,\quad k=1,2,\cdots.$$

方程的通解为

$$u(x,t)=\sum_{k=1}^{\infty}A_ke^{-a^2\lambda t}\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_ke^{-k^2\pi^2t}\sin k\pi x.$$

代入初值条件得

$$u(x,0)=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin k\pi x=\begin{cases}x, & 0<x\leqslant\dfrac{1}{2},\\ 1-x, & \dfrac{1}{2}<x<1.\end{cases}=f(x).$$

解得

$$A_k=\frac{2}{l}\int_0^l f(x)\sin\sqrt{\lambda}xdx=2\int_0^1 f(x)\sin k\pi xdx=2\left(\frac{-k\pi\cos\frac{k\pi}{2}+2\sin\frac{k\pi}{2}+k\pi\cos\frac{k\pi}{2}+2\sin\frac{k\pi}{2}-2\sin k\pi}{2k^2\pi^2}\right).$$

化简得

$$A_k=\frac{4}{k^2\pi^2}\sin\frac{k\pi}{2}=\begin{cases}\dfrac{4(-1)^{(k-1)/2}}{k^2\pi^2}, & k=2n-1\\ 0, & k=2n\end{cases},\quad n=1,2,\cdots.$$

故

$$u(x,t)=\sum_{n=1}^{\infty}\frac{4(-1)^{n-1}}{(2n-1)^2\pi^2}e^{-(2n-1)^2\pi^2t}\sin(2n-1)\pi x.$$