# MA319 — 偏微分方程

*Assignment 5*

Instructor: 许德良

Author: 刘逸灏 (515370910207)

— SJTU (Fall 2019)

## 习题　1.5/2

试说明: 对一维波动方程, 即使初始资料具有紧支集, 当 $t \to +\infty$ 时其柯西问题的解没有衰减性.

若初始资料 $\varphi, \psi$ 具有紧支集, 则存在一个常数 $\rho > 0$, 使 $\varphi$ 和 $\psi$ 在以原点为中心, $\rho$ 为半径的区间 $[-\rho, \rho]$ 外恒为零, 而在区间内成立

$$|\varphi|, |\psi| \leqslant C.$$

代入达朗贝尔公式得

$$u(x,t) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi.$$

由于在 $[-\rho, \rho]$ 外, $\psi(x) = 0$, $x - at$ 到 $x + at$ 的积分可以写为 $-\rho$ 到 $\rho$ 的积分

$$\lim_{t\to+\infty} u(x,t) = \frac{C_1 + C_2}{2} + \frac{1}{2a}\int_{-\rho}^{\rho} \psi(\xi)d\xi = \frac{C_1 + C_2}{2} + C_3,$$

$$|C_1|, |C_2| \leqslant C, \quad C_3 \leqslant \frac{\rho C}{a}.$$

故当 $t \to +\infty$ 时其柯西问题的解趋于一个常数, 没有衰减性.

## 习题　1.5/3

设 $u$ 为初始资料 $\varphi$ 及 $\psi$ 具有紧支集的二维波动方程的解. 试证明: 对任意固定的 $(x_0, y_0) \in \mathbf{R}^2$, 成立

$$\lim_{t\to+\infty} u(x_0, y_0, t) = 0.$$

若初始资料 $\varphi, \psi$ 具有紧支集, 则存在一个常数 $\rho > 0$, 使 $\varphi$ 和 $\psi$ 在以原点为中心, $\rho$ 为半径的圆 $C_\rho^O$ 外恒为零, 而在 $C_\rho^O$ 内成立

$$|\varphi|, |\psi| \leqslant C.$$

代入泊松公式得

$$\begin{aligned} u(x,y,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{C_{at}^M} \frac{\varphi(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2 - (\xi-x)^2 - (\eta-y)^2}} + \frac{1}{2\pi a}\iint\limits_{C_{at}^M} \frac{\psi(\xi,\eta)d\xi d\eta}{\sqrt{a^2t^2 - (\xi-x)^2 - (\eta-y)^2}} \\ &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi} \frac{\varphi(\xi,\eta)}{\sqrt{a^2t^2 - r^2}} r d\theta dr + \frac{1}{2\pi a}\int_0^{at}\int_0^{2\pi} \frac{\psi(\xi,\eta)}{\sqrt{a^2t^2 - r^2}} r d\theta dr. \end{aligned}$$

由于在 $C_\rho^O$ 外, $\psi(x)=0$, 0 到 $at$ 的积分可以写为 0 到 $\rho$ 的积分. 当 $at>\rho$ 时

$$
\begin{aligned}
|u(x,y,t)| &= \left|\frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^\rho\int_0^{2\pi}\frac{\varphi(\xi,\eta)}{\sqrt{a^2t^2-r^2}}rd\theta dr+\frac{1}{2\pi a}\int_0^\rho\int_0^{2\pi}\frac{\psi(\xi,\eta)}{\sqrt{a^2t^2-r^2}}rd\theta dr\right| \\
&\leqslant \frac{1}{2\pi a}\left|\frac{\partial}{\partial t}\int_0^\rho\int_0^{2\pi}\frac{C}{\sqrt{a^2t^2-r^2}}rd\theta dr\right|+\frac{1}{2\pi a}\left|\int_0^\rho\int_0^{2\pi}\frac{C}{\sqrt{a^2t^2-r^2}}rd\theta dr\right| \\
&= \frac{1}{2\pi a}\left|\frac{\partial}{\partial t}\int_0^\rho\frac{2\pi Cr}{\sqrt{a^2t^2-r^2}}dr\right|+\frac{1}{2\pi a}\left|\int_0^\rho\frac{2\pi Cr}{\sqrt{a^2t^2-r^2}}dr\right| \\
&= \frac{1}{2\pi a}\left|\frac{\partial}{\partial t}2\pi C\left(at-\sqrt{a^2t^2-\rho^2}\right)\right|+\frac{1}{2\pi a}\left|2\pi C\left(at-\sqrt{a^2t^2-\rho^2}\right)\right| \\
&= \frac{C}{a}\left|a-\frac{a^2t}{\sqrt{a^2t^2-\rho^2}}+\left(at-\sqrt{a^2t^2-\rho^2}\right)\right|.
\end{aligned}
$$

当 $t\to+\infty$ 时

$$
\lim_{t\to+\infty}\frac{at}{\sqrt{a^2t^2-\rho^2}}=1,\quad \lim_{t\to+\infty}\frac{a^2t}{\sqrt{a^2t^2-\rho^2}}=a,
$$

$$
\lim_{t\to+\infty}|u(x,y,t)|\leqslant\frac{C}{a}|a-a+0|=0.
$$

故取任意固定的 $(x_0,y_0)\in\mathbf{R}^2$, 成立

$$
\lim_{t\to+\infty}u(x_0,y_0,t)=0.
$$

## 习题 1.6/1

对受摩擦力作用且具固定端点的有界弦振动, 满足方程

$$
u_{tt}=a^2u_{xx}-cu_t,
$$

其中常数 $c>0$, 证明其能量是减少的, 并由此证明方程

$$
u_{tt}=a^2u_{xx}-cu_t+f
$$

的初边值问题解的唯一性以及关于初始条件及自由项的稳定性.

弦的总能量可写成

$$
E(t)=\int_0^l(u_t^2+a^2u_x^2)dx.
$$

能量变化率为

$$
\begin{aligned}
\frac{dE(t)}{dt} &= 2\int_0^l\left(u_tu_{tt}+a^2u_xu_{xt}\right)dx \\
&= 2\int_0^l\left[u_t\left(a^2u_{xx}-cu_t\right)+a^2u_xu_{xt}\right]dx \\
&= 2\int_0^l\left(-cu_t^2+a^2\frac{\partial}{\partial x}u_tu_x\right)dx \\
&= -2\int_0^l cu_t^2dx+2a^2u_tu_x\Big|_0^l.
\end{aligned}
$$

由于端点是固定的

$$2a^2u_tu_x\Big|_0^l=0.$$

故

$$\frac{dE(t)}{dt}=-2\int_0^l cu_t^2dx\leqslant 0.$$

要证明方程

$$u_{tt}=a^2u_{xx}-cu_t+f$$

的初边值问题解的唯一性, 只需证明零初始条件方程只有零解

$$\begin{cases}u_{tt}=a^2u_{xx}-cu_t,\\ u|_{x=0}=u|_{x=l}=0,\\ u|_{t=0}=0,\quad u_t|_{t=0}=0.\end{cases}$$

根据能量不等式得

$$E(t)\leqslant E(0)=\int_0^l\left(u_t(x,0)^2+a^2u_x(x,0)^2\right)dx=0,$$

$$u_t=u_x=0\Longrightarrow u=C.$$

根据初始条件易知 $u\equiv 0$, 故初边值问题解的唯一性得证.
对于方程的稳定性, 设

$$E_0(t)=\int_0^l u^2dx,\quad E_1(t)=\int_0^l(u_t^2+a^2u_x^2)dx.$$

对 $E_1(t)$ 有

$$\frac{dE_1(t)}{dt}=2\int_0^l\left[u_t\left(a^2u_{xx}-cu_t+f\right)+a^2u_xu_{xt}\right]dx=2\int_0^l(-cu_t^2+u_tf)dx\leqslant\int_0^l(u_t^2+f^2)dx\leqslant E_1(t)+\int_0^l f^2dx,$$

$$\frac{d}{dt}e^{-t}E_1(t)=-e^{-t}E_1(t)+e^{-t}\frac{dE_1(t)}{dt}\leqslant e^{-t}\int_0^l f^2dx,$$

$$E_1(t)\leqslant e^t\int_0^t e^{-\tau}\int_0^l f^2dxd\tau+e^tE_1(0)\leqslant e^tE_1(0)+e^t\int_0^t\int_0^l f^2dxd\tau=\overline{E_1(t)}.$$

对 $E_0(t)$ 有

$$\frac{dE_0(t)}{dt}=2\int_0^l uu_tdx\leqslant\int_0^l u^2dx+\int_0^l u_t^2dx\leqslant E_0(t)+E_1(t),$$

$$\frac{d}{dt}e^{-t}E_0(t)=-e^{-t}E_0(t)+e^{-t}\frac{dE_0(t)}{dt}\leqslant e^{-t}E_1(t),$$

$$E_0(t)\leqslant e^t\int_0^t e^{-\tau}E_1(\tau)d\tau+e^tE_0(0)\leqslant e^t\overline{E_1(t)}(1-e^{-t})+e^tE_0(0)=e^tE_0(0)+(e^t-1)\overline{E_1(t)},$$

$$E_0(t)+E_1(t)\leqslant e^tE_0(0)+(e^t-1)\overline{E_1(t)}+E_1(t)\leqslant e^t(E_0(0)+\overline{E_1(t)})=e^tE_0(0)+e^{2t}E_1(0)+e^{2t}\int_0^t\int_0^l f^2dxd\tau.$$

在 $0\leqslant t\leqslant T$ 上

$$E_0(t)+E_1(t)\leqslant e^TE_0(0)+e^{2T}E_1(0)+e^{2T}\int_0^T\int_0^l f^2dxd\tau\leqslant e^{2T}\left(E_0(0)+E_1(0)+\int_0^T\int_0^l f^2dxd\tau\right).$$

存在 $C$ 使得

$$\sqrt{T(E_0(t)+E_1(t))} \leqslant \sqrt{Te^{2T}\left(E_0(0)+E_1(0)+\int_0^T\int_0^l f^2dxd\tau\right)} \leqslant C\eta,$$

$$\sqrt{E_0(0)+E_1(0)+\int_0^T\int_0^l f^2dxd\tau} \leqslant \eta.$$

任取 $\varepsilon>0$, 可以找到 $\eta=\dfrac{\varepsilon}{C}$, 使得

$$\|\varphi_1-\varphi_2\|_{L^2(t)}, \|\varphi_{1x}-\varphi_{2x}\|_{L^2(t)}, \|\psi_1-\psi_2\|_{L^2(t)}, \|f_1-f_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{E_0(0)+E_1(0)+\int_0^T\int_0^l f^2dxd\tau} \leqslant \eta,$$

$$\|u_1-u_2\|_{L^2(t)}, \|u_{1x}-u_{2x}\|_{L^2(t)}, \|u_{1t}-u_{2t}\|_{L^2(t)}, \|u_1-u_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{T(E_0(t)+E_1(t))} \leqslant \varepsilon.$$

故初始条件和自由项都是稳定的.

## 习题　1.6/2

证明函数 $f(x,t)$ 在 $G: 0\leqslant x\leqslant l, 0\leqslant t\leqslant T$ 作微笑改变时, 方程

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)-q(x)u+f(x,t)$$

(其中 $k(x)>0$, $q(x)>0$ 和 $f(x,t)$ 都是一些充分光滑的函数) 具固定端点边界条件的初边值问题的解在 $G$ 内的改变也是很微小的.

设

$$E_0(t)=\int_0^l u^2dx, \quad E_1(t)=\int_0^l \left[u_t^2+k(x)u_x^2+q(x)u^2\right]dx.$$

对 $E_1(t)$ 有

$$\begin{aligned}
\frac{dE_1(t)}{dt} &= 2\int_0^l (u_tu_{tt}+k(x)u_xu_{xt}+q(x)uu_t)dx \\
&= 2\int_0^l \{u_t[u_{tt}-(k(x)u_x)_x+q(x)u]+u_t(k(x)u_x)_x+k(x)u_xu_{xt}\}dx \\
&= 2\int_0^l \left(u_tf(x,t)+\frac{\partial}{\partial x}k(x)u_xu_t\right)dx \\
&= 2\int_0^l u_tf(x,t)dx+2k(x)u_xu_t\Big|_0^l.
\end{aligned}$$

由于端点是固定的

$$2k(x)u_xu_t\Big|_0^l=0.$$

故

$$\frac{dE_1(t)}{dt}=2\int_0^l u_tf(x,t)dx \leqslant \int_0^l (u_t^2+f^2)dx = E_1(t)+\int_0^l f^2dx,$$

$$\frac{d}{dt}e^{-t}E_1(t) = -e^{-t}E_1(t) + e^{-t}\frac{dE_1(t)}{dt} \leqslant e^{-t}\int_0^l f^2 dx,$$

$$E_1(t) \leqslant e^t \int_0^t e^{-\tau} \int_0^l f^2 dx d\tau + e^t E_1(0) \leqslant e^t E_1(0) + e^t \int_0^t \int_0^l f^2 dx d\tau = \overline{E_1(t)}.$$

对 $E_0(t)$ 有

$$\frac{dE_0(t)}{dt} = 2\int_0^l u u_t dx \leqslant \int_0^l u^2 dx + \int_0^l u_t^2 dx \leqslant E_0(t) + E_1(t),$$

$$\frac{d}{dt}e^{-t}E_0(t) = -e^{-t}E_0(t) + e^{-t}\frac{dE_0(t)}{dt} \leqslant e^{-t}E_1(t),$$

$$E_0(t) \leqslant e^t \int_0^t e^{-\tau}E_1(\tau)d\tau + e^t E_0(0) \leqslant e^t\overline{E_1(t)}(1-e^{-t}) + e^t E_0(0) = e^t E_0(0) + (e^t - 1)\overline{E_1(t)},$$

$$E_0(t)+E_1(t) \leqslant e^t E_0(0)+(e^t-1)\overline{E_1(t)}+E_1(t) \leqslant e^t(E_0(0)+\overline{E_1(t)}) = e^t E_0(0)+e^{2t}E_1(0)+e^{2t}\int_0^t\int_0^l f^2 dx d\tau.$$

在 $0 \leqslant t \leqslant T$ 上

$$E_0(t) + E_1(t) \leqslant e^T E_0(0) + e^{2T}E_1(0) + e^{2T}\int_0^T\int_0^l f^2 dx d\tau \leqslant e^{2T}\left(E_0(0) + E_1(0) + \int_0^T\int_0^l f^2 dx d\tau\right).$$

设 $u(x,t)$ 为齐次初边值问题的解, 有 $E_0(0) = E(0) = 0$, 则存在 $C$ 使得

$$\sqrt{TE_0(t)} \leqslant \sqrt{T(E_0(t)+E_1(t))} \leqslant \sqrt{Te^{2T}\int_0^T\int_0^l f^2 dx d\tau} \leqslant C\eta, \quad \sqrt{\int_0^T\int_0^l f^2 dx d\tau} \leqslant \eta$$

任取 $\varepsilon > 0$, 可以找到 $\eta = \frac{\varepsilon}{C}$, 使得

$$\|f_1 - f_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{\int_0^T\int_0^l f^2 dx d\tau} \leqslant \eta,$$

$$\|u_1 - u_2\|_{L^2((0,T)\times t)} \leqslant \sqrt{TE_0(t)} \leqslant \varepsilon.$$

故自由项 $f(x,t)$ 是稳定的.

## 习题 1.6/5

考虑波动方程的第三类边值问题

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy}) = 0, \quad t > 0, (x, y) \in \Omega, \\ u|_{t=0} = \varphi(x, y), \quad u_t|_{t=0} = \psi(x, y), \\ \left( \dfrac{\partial u}{\partial \mathbf{n}} + \sigma u \right)\Big|_{\Gamma} = 0, \end{cases}$$

其中 $\sigma > 0$ 是常数, $\Gamma$ 为 $\Omega$ 的边界, $\mathbf{n}$ 为 $\Gamma$ 上的单位外法向量. 对于上述问题的解, 定义能量积分

$$E(t) = \iint\limits_{\Omega} (u_t^2 + a^2(u_x^2 + u_y^2))dxdy + a^2 \int_{\Gamma} \sigma u^2 ds,$$

试证明 $E(t) \equiv$ 常数, 并由此证明上述定解问题解的唯一性.

$$\begin{aligned}
\frac{dE(t)}{dt} &= 2\iint\limits_{\Omega} (u_t u_{tt} + a^2(u_t u_{xt} + u_t u_{yt}))dxdy + 2a^2 \int_{\Gamma} \sigma u u_t ds \\
&= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy})) + a^2(u_x u_{xt} + u_y u_{yt} + u_t u_{xx} + u_t u_{yy})\right] dxdy + 2a^2 \int_{\Gamma} \sigma u u_t ds \\
&= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy})) + a^2 \left(\frac{\partial}{\partial x} u_x u_t + \frac{\partial}{\partial y} u_y u_t\right)\right] dxdy + 2a^2 \int_{\Gamma} \sigma u u_t ds \\
&= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy}))\right] dxdy + 2a^2 \int_{\Gamma} \frac{\partial u}{\partial \mathbf{n}} u_t ds + 2a^2 \int_{\Gamma} \sigma u u_t ds \\
&= 2\iint\limits_{\Omega} \left[u_t(u_{tt} - a^2(u_{xx} + u_{yy}))\right] dxdy + 2a^2 \int_{\Gamma} u_t \left(\frac{\partial u}{\partial \mathbf{n}} + \sigma u\right) ds \\
&= 0.
\end{aligned}$$

故 $E(t) \equiv$ 常数. 要证明方程解的唯一性, 只需证明满足方程的解只有零解

$$E(t) = E(0) = \iint\limits_{\Omega} (u_t(x, y, 0)^2 + a^2(u_x(x, y, 0)^2 + u_y(x, y, 0)^2))dxdy + a^2 \int_{\Gamma} \sigma u(x, y, 0)^2 ds = 0,$$

$$u_t = u_x = u_y = u = 0.$$

故得证.