MA319 — 偏微分方程

Assignment 2

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习题 1.2/3

利用传播波法, 求解波动方程的古尔萨问题:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u|_{x-at=0} = \varphi(x), \\ u|_{x+at=0} = \psi(x), \quad \varphi(0) = \psi(0). \end{cases}$$

方程的通解为

$$u(x,t) = F(x-at) + G(x+at),$$

代入初值条件得

$$\varphi(x) = u|_{x-at=0} = F(0) + G(2x),$$

$$\psi(x) = u|_{x+at=0} = F(2x) + G(0).$$

故

$$F(x) = \psi(x/2) - G(0),$$

$$G(x) = \varphi(x/2) - F(0),$$

$$\varphi(0) = \psi(0) = F(0) + G(0).$$

$$u(x,t) = F(x-at) + G(x+at) = \psi\left(\frac{x-at}{2}\right) + \varphi\left(\frac{x+at}{2}\right) - \varphi(0).$$

习题 1.2/4

对非齐次波动方程的初值问题 (2.5), (2.6), 证明: 当 $f(x,t)$ 不变时,

(1) 如果初始条件在 x 轴的区间 $[x_1, x_2]$ 发生变化, 那么对应的解在区间 $[x_1, x_2]$ 的影响区域以外不发生变化;

(2) 在 x 轴区间 $[x_1, x_2]$ 上所给的初始条件唯一地确定区间 $[x_1, x_2]$ 的决定区域中解的数值.

非齐次初值问题的解为

$$u(x,t) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)d\alpha + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau.$$

其中与 $\varphi(x)$ 和 $\psi(x)$ 有关的定义域范围都是 $[x-at, x+at]$.

(1)

区间 $[x_1, x_2]$ 的影响区域为

$$x_1 - at \leqslant x \leqslant x_2 + at,$$

不受影响的区域为

$$x \leqslant x_1 - at \quad \text{和} \quad x \geqslant x_2 + at,$$

对应的 $\varphi(x)$ 和 $\psi(x)$ 定义域范围是

$$x \leqslant x_1 \quad \text{和} \quad x \geqslant x_2.$$

故当 $\varphi(x)$ 和 $\psi(x)$ 在 $[x_1, x_2]$ 上变化时, 以上定义域内函数取值不变, 对应解也不变.

(2)

区间 $[x_1, x_2]$ 的决定区域为

$$x_1 + at \leqslant x \leqslant x_2 - at,$$

对应的 $\varphi(x)$ 和 $\psi(x)$ 定义域范围是

$$x_1 \leqslant x \leqslant x_2.$$

故 $[x_1, x_2]$ 上所给的初始条件唯一地确定该区间解的数值.

习题　1.2/5

求解

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & x > 0, t > 0, \\ u|_{t=0} = \varphi(x), & u_t|_{t=0} = 0, \\ u_x - k u_t|_{x=0} = 0, \end{cases}$$

其中 k 为正常数.

方程的通解为

$$u(x,t) = F(x-at) + G(x+at),$$

代入初值条件得

$$\varphi(x) = u|_{t=0} = F(x) + G(x),$$

$$0 = u_t|_{t=0} = a[-F'(x) + G'(x)].$$

$$0 = u_x - k u_t\ |_{x=0} = F'(-at) + G'(at) - ka[-F'(-at) + G'(at)].$$

故当 $x \geqslant 0$ 时

$$F(x) - G(x) = C = F(0) - G(0),$$

$$F(x) = \frac{1}{2}\varphi(x) + \frac{C}{2} = \frac{1}{2}\varphi(x) + \frac{F(0) - G(0)}{2},$$

$$G(x)=\frac{1}{2}\varphi(x)-\frac{C}{2}=\frac{1}{2}\varphi(x)-\frac{F(0)-G(0)}{2}.$$

$$(ka+1)F'(-at)-(ka-1)G'(at)=0,$$

$$(ka+1)F'(-x)-(ka-1)G'(x)=0,$$

$$\int_0^x[(ka+1)F'(-\xi)-(ka-1)G'(\xi)]d\xi=0,$$

$$-(ka+1)F(-x)-(ka-1)G(x)=C_1=-(ka+1)F(0)-(ka-1)G(0),$$

$$F(-x)=\frac{(1-ka)G(x)-C_1}{1+ka}=\frac{1-ka}{1+ka}G(x)+F(0)+\frac{ka-1}{1+ka}G(0).$$

当 $x-at\geqslant 0$ 时

$$u(x,t)=F(x-at)+G(x+at)=\frac{\varphi(x-at)+\varphi(x+at)}{2}.$$

当 $x-at<0$ 时

$$\begin{aligned}
u(x,t)&=F(-(at-x))+G(x+at)\\
&=\frac{1-ka}{1+ka}G(at-x)+F(0)+\frac{ka-1}{1+ka}G(0)+G(x+at)\\
&=\frac{1-ka}{1+ka}\left[\frac{1}{2}\varphi(at-x)-\frac{F(0)-G(0)}{2}\right]+F(0)+\frac{ka-1}{1+ka}G(0)+\frac{1}{2}\varphi(x+at)-\frac{F(0)-G(0)}{2}\\
&=\frac{1-ka}{2(1+ka)}\varphi(at-x)+\frac{1}{2}\varphi(x+at)+\frac{[-(1-ka)+(1+ka)]F(0)+[-(1-ka)+(1+ka)]G(0)}{2(1+ka)}\\
&=\frac{1-ka}{2(1+ka)}\varphi(at-x)+\frac{1}{2}\varphi(x+at)+\frac{ka}{1+ka}\varphi(0).
\end{aligned}$$

故

$$u(x,t)=\frac{1}{2}\varphi(x+at)+\begin{cases}\frac{1}{2}\varphi(x-at), & x-at\geqslant 0\\ \frac{1-ka}{2(1+ka)}\varphi(at-x)+\frac{ka}{1+ka}\varphi(0), & x-at<0\end{cases}.$$

习题　1.2/6

求解初边值问题

$$\begin{cases}u_{tt}-u_{xx}=0, \quad 0<t<kx, k>1,\\ u|_{t=0}=\varphi_0(x), \quad x\geqslant 0,\\ u_t|_{t=0}=\varphi_1(x), \quad x\geqslant 0,\\ u|_{t=kx}=\psi(x),\end{cases}$$

其中 $\varphi_0(0)=\psi(0)$.

方程的通解为

$$u(x,t)=F(x-t)+G(x+t),$$

代入初值条件得

$$\varphi_0(x)=u|_{t=0}=F(x)+G(x),$$

$$\varphi_1(x) = u_t|_{t=0} = a[-F'(x) + G'(x)].$$

$$\psi(x) = u|_{t=kx} = F((1-k)x) + G((1+k)x).$$

故当 $x \geqslant 0$ 时

$$F(x) - G(x) = -\int_{x_0}^{x} \psi(\alpha)d\alpha + C,$$

$$F(x) = \frac{1}{2}\varphi_0(x) - \frac{1}{2}\int_{x_0}^{x} \varphi_1(\alpha)d\alpha + \frac{C}{2},$$

$$G(x) = \frac{1}{2}\varphi_0(x) + \frac{1}{2}\int_{x_0}^{x} \varphi_1(\alpha)d\alpha - \frac{C}{2}.$$

当 $x - t \geqslant 0$ 时

$$u(x,t) = F(x-t) + G(x+t) = \frac{\varphi_0(x-t) + \varphi_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \varphi_1(\alpha)d\alpha.$$

当 $x - t \leqslant 0$ 时

$$u(x,x) = F(x-x) + G(x+x) = F(0) + G(2x) = \frac{\varphi_0(0) + \varphi_0(2x)}{2} + \frac{1}{2}\int_{0}^{2x} \varphi_1(\alpha)d\alpha,$$

$$G(x) = \frac{\varphi_0(0) + \varphi_0(x)}{2} + \frac{1}{2}\int_{0}^{x} \varphi_1(\alpha)d\alpha - F(0),$$

$$F((1-k)x) = \psi(x) - G((1+k)x) = \psi(x) - \frac{\varphi_0(0) + \varphi_0((1+k)x)}{2} - \frac{1}{2}\int_{0}^{(1+k)x} \varphi_1(\alpha)d\alpha + F(0),$$

$$F(x) = \psi\left(\frac{x}{1-k}\right) - \frac{1}{2}\varphi_0(0) - \frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}x\right) - \frac{1}{2}\int_{0}^{\frac{1+k}{1-k}x} \varphi_1(\alpha)d\alpha + F(0).$$

$$\begin{aligned}
u(x,t) &= F(x-t) + G(x+t) \\
&= \psi\left(\frac{x-t}{1-k}\right) - \frac{1}{2}\varphi_0(0) - \frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}(x-t)\right) - \frac{1}{2}\int_{0}^{\frac{1+k}{1-k}(x-t)} \varphi_1(\alpha)d\xi + F(0) \\
&\quad + \frac{\varphi_0(0) + \varphi_0(x+t)}{2} + \frac{1}{2}\int_{0}^{x+t} \varphi_1(\xi)d\xi - F(0) \\
&= \psi\left(\frac{x-t}{1-k}\right) - \frac{1}{2}\varphi_0\left(\frac{1+k}{1-k}(x-t)\right) + \frac{1}{2}\varphi_0(x+t) + \frac{1}{2}\int_{\frac{1+k}{1-k}(x-t)}^{x+t} \varphi_1(\xi)d\xi.
\end{aligned}$$

故

$$u(x,t) = \begin{cases} \dfrac{\varphi_0(x-t) + \varphi_0(x+t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} \varphi_1(\xi)d\xi, & 0 \leqslant t \leqslant x \\ \psi\left(\dfrac{x-t}{1-k}\right) - \dfrac{1}{2}\varphi_0\left(\dfrac{1+k}{1-k}(x-t)\right) + \dfrac{1}{2}\varphi_0(x+t) + \dfrac{1}{2}\displaystyle\int_{\frac{1+k}{1-k}(x-t)}^{x+t} \varphi_1(\xi)d\xi, & x < t < kx \end{cases}.$$

习题　1.2/7

求边值问题

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad f(t) < x < t, \\ u|_{x=t} = \varphi(t), \\ u|_{x=f(t)} = \psi(t) \end{cases}$$

的解, 其中 $\varphi(0) = \psi(x)$, $x = f(t)$ 为由原点出发的, 介于 $x = t$ 和 $x = -t$ 之间的光滑曲线, 且 $|f'(t)| \neq 1$ 对一切 t 成立.

方程的通解为

$$u(x,t) = F(x-t) + G(x+t),$$

代入初值条件得

$$\varphi(x) = u|_{t=x} = F(0) + G(2x),$$

$$\psi(x) = u|_{t=f(x)} = F(x - f(x)) + G(x + f(x)).$$

解得

$$G(x) = \varphi(x/2) - F(0),$$

$$F(x - f(x)) = \psi(x) - G(x + f(x)) = \psi(x) - \varphi\left(\frac{x + f(x)}{2}\right) + F(0).$$

设 $g(x) = x - f(x)$, 由于 $f'(x) \neq 1$, 易知 $g'(x) \neq 0$, 即 $g(x)$ 为连续单调函数, $g^{-1}(x)$ 存在.

$$F(x - f(x)) = F(g(x)) = \psi(x) - \varphi\left(x - \frac{g(x)}{2}\right) + F(0),$$

$$F(x) = \psi(g^{-1}(x)) - \varphi\left(g^{-1}(x) - \frac{x}{2}\right) + F(0).$$

故

$$\begin{aligned} u(x,t) &= F(x-t) + G(x+t) \\ &= \psi(g^{-1}(x-t)) - \varphi\left(g^{-1}(x-t) - \frac{x-t}{2}\right) + F(0) + \varphi\left(\frac{x+t}{2}\right) - F(0) \\ &= \psi(g^{-1}(x-t)) + \varphi\left(g^{-1}(x-t) - \frac{x-t}{2}\right) + \varphi\left(\frac{x+t}{2}\right). \end{aligned}$$

习题　1.2/8

求解波动方程的初值问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = t \sin x, \\ u|_{t=0} = 0, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \sin x. \end{cases}$$

代入 Kirchhoff 公式得

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t}\sin\xi d\xi + \frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\tau\sin\xi d\xi d\tau \\
&= \frac{1}{2}[\cos(x-t)-\cos(x+t)] + \frac{1}{2}\int_0^t \tau[\cos(x-t+\tau)-\cos(x+t-\tau)]d\tau \\
&= \frac{1}{2}[\cos(x-t)-\cos(x+t)] + \frac{1}{2}[\cos(-x)-t\sin(-x)-\cos(t-x)] - \frac{1}{2}[\cos x - t\sin x - \cos(t+x)] \\
&= t\sin x.
\end{aligned}
$$

习题　1.2/9

求解波动方程的初值问题

$$
\begin{cases}
u_{tt} = a^2 u_{xx} + \dfrac{tx}{(1+x^2)^2}, \\
u|_{t=0} = 0, \\
u_t|_{t=0} = \dfrac{1}{1+x^2}.
\end{cases}
$$

代入 Kirchhoff 公式得

$$
\begin{aligned}
u(x,t) &= \frac{1}{2a}\int_{x-at}^{x+at}\frac{1}{1+\xi^2}d\xi + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\frac{\tau\xi}{(1+\xi^2)^2}d\xi d\tau \\
&= \frac{1}{2a}[\arctan(x+at)-\arctan(x-at)] + \frac{1}{4a}\int_0^t \tau\left[\frac{1}{1+[x-a(t-\tau)]^2} - \frac{1}{1+[x+a(t-\tau)]^2}\right]d\tau \\
&= \frac{1}{2a}[\arctan(x+at)-\arctan(x-at)] + \frac{1}{4a^3}\left[(at-x)[\arctan x - \arctan(x-at)] + \frac{1}{2}\ln\frac{1+x^2}{1+(x-at)^2}\right] \\
&\quad - \frac{1}{4a^3}\left[-(at+x)[\arctan x - \arctan(x+at)] + \frac{1}{2}\ln\frac{1+x^2}{1+(x+at)^2}\right] \\
&= \frac{1}{2a}[\arctan(x+at)-\arctan(x-at)] + \frac{1}{8a^3}\ln\frac{1+(x+at)^2}{1+(x-at)^2} \\
&\quad + \frac{1}{4a^3}[2at\arctan x + (x-at)\arctan(x-at) - (x+at)\arctan(x+at)] \\
&= \frac{t}{2a^2}\arctan x + \frac{x-at-2a^2}{4a^3}\arctan(x-at) - \frac{x+at-2a^2}{4a^3}\arctan(x+at) + \frac{1}{8a^3}\ln\frac{1+(x+at)^2}{1+(x-at)^2}.
\end{aligned}
$$