MA319 — 偏微分方程

Assignment 6

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习题 1.6/3

证明波动方程

$$u_{tt}=a^2(u_{xx}+u_{yy})+f(x,y,t)$$

的自由项 f 在 $L^2(K)$ 意义下作微小改变时, 对应的柯西问题的解 u 在 $L^2(K)$ 意义下改变也是微小的, 其中 K 是由

$$(x-x_0)^2+(y-y_0)^2\leqslant(R-at)^2$$

所表示的锥体.

在 K 内任一截面 Ω_t 上成立能量不等式

$$E_0(\Omega_t)=\iint\limits_{\Omega_t}u^2dxdy,\quad E_1(\Omega_t)=\iint\limits_{\Omega_t}\left[u_t^2+a^2\left(u_x^2+u_y^2\right)\right]dxdy.$$

对 $E_1(t)$ 有

$$\begin{aligned}
\frac{dE_1(\Omega_t)}{dt}&=\frac{d}{dt}\int_0^{R-at}\int_0^{2\pi r}\left[u_t^2+a^2\left(u_x^2+u_y^2\right)\right]dsdr\\
&=2\int_0^{R-at}\int_0^{2\pi r}\left[u_tu_{tt}+a^2\left(u_xu_{xt}+u_yu_{yt}\right)\right]dsdr-a\int_{\Gamma_t}\left[u_t^2+a^2\left(u_x^2+u_y^2\right)\right]ds.\\
&=2\int_0^{R-at}\int_0^{2\pi r}u_t\left[u_{tt}-a^2\left(u_x^2+u_y^2\right)\right]dsdr\\
&\quad+2\int_{\Gamma_t}\left\{a^2[u_xu_t\cos(n,x)+u_yu_t\sin(n,x)]-\frac{a}{2}\left[u_t^2+a^2\left(u_x^2+u_y^2\right)\right]\right\}ds.\\
&=2\iint\limits_{\Omega_t}u_tfdxdy-a\int_{\Gamma_t}\left[\left(au_x-u_t\cos(n,x)\right)^2+\left(au_y-u_t\cos(n,y)\right)^2\right]ds\\
&\leqslant2\iint\limits_{\Omega_t}u_tfdxdy\leqslant\iint\limits_{\Omega_t}u_t^2dxdy+\iint\limits_{\Omega_t}f^2dxdy\leqslant E_1(\Omega_t)+\iint\limits_{\Omega_t}f^2dxdy,
\end{aligned}$$

$$\frac{d}{dt}e^{-t}E_1(\Omega_t)=-e^{-t}E_1(\Omega_t)+e^{-t}\frac{dE_1(\Omega_t)}{dt}\leqslant e^{-t}\iint\limits_{\Omega_t}f^2dxdy,$$

$$E_1(\Omega_t)\leqslant e^t\int_0^te^{-\tau}\int_0^lf^2dxd\tau+e^tE_1(\Omega_0)\leqslant e^tE_1(\Omega_0)+e^t\int_0^t\iint\limits_{\Omega_t}f^2dxdyd\tau=\overline{E_1(\Omega_t)}.$$

对 $E_0(t)$ 有

$$\frac{dE_0(\Omega_t)}{dt} = 2\iint\limits_{\Omega_t} uu_t dxdy - a\int_{\Gamma_t} u^2 ds \leqslant 2\iint\limits_{\Omega_t} uu_t dxdy \leqslant \iint\limits_{\Omega_t} u^2 dxdy + \iint\limits_{\Omega_t} u_t^2 dxdy \leqslant E_0(\Omega_t) + E_1(\Omega_t),$$

$$\frac{d}{dt} e^{-t} E_0(\Omega_t) = -e^{-t} E_0(\Omega_t) + e^{-t}\frac{dE_0(\Omega_t)}{dt} \leqslant e^{-t} E_1(\Omega_t),$$

$$E_0(\Omega_t) \leqslant e^t \int_0^t e^{-\tau} E_1(\Omega_\tau) d\tau + e^t E_0(\Omega_0) \leqslant e^t \overline{E_1(\Omega_t)}(1 - e^{-t}) + e^t E_0(\Omega_0) = e^t E_0(\Omega_0) + (e^t - 1)\overline{E_1(\Omega_t)},$$

$$\begin{aligned} E_0(\Omega_t) + E_1(\Omega_t) &\leqslant e^t E_0(\Omega_0) + (e^t - 1)\overline{E_1(\Omega_t)} + E_1(\Omega_t) \leqslant e^t (E_0(\Omega_0) + \overline{E_1(\Omega_t)}) \\ &= e^t E_0(\Omega_0) + e^{2t} E_1(\Omega_0) + e^{2t} \int_0^t \iint\limits_{\Omega_t} f^2 dxdyd\tau. \end{aligned}$$

在 $0 \leqslant t \leqslant T$ 上

$$\begin{aligned} E_0(\Omega_t) + E_1(\Omega_t) &\leqslant e^T E_0(\Omega_0) + e^{2T} E_1(\Omega_0) + e^{2T} \int_0^T \iint\limits_{\Omega_t} f^2 dxdyd\tau \\ &\leqslant e^{2T} \left(E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint\limits_{\Omega_t} f^2 dxdyd\tau \right). \end{aligned}$$

存在 C 使得

$$\sqrt{T(E_0(\Omega_t) + E_1(\Omega_t))} \leqslant \sqrt{Te^{2T} \left(E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint\limits_{\Omega_t} f^2 dxdyd\tau \right)} \leqslant C\eta,$$

$$\sqrt{E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint\limits_{\Omega_t} f^2 dxdyd\tau} \leqslant \eta.$$

任取 $\varepsilon > 0$, 可以找到 $\eta = \dfrac{\varepsilon}{C}$, 使得

$$\|f_1 - f_2\|_{L^2(K)} \leqslant \sqrt{E_0(\Omega_0) + E_1(\Omega_0) + \int_0^T \iint\limits_{\Omega_t} f^2 dxdyd\tau} \leqslant \eta,$$

$$\|u_1 - u_2\|_{L^2(K)} \leqslant \sqrt{T(E_0(\Omega_t) + E_1(\Omega_t))} \leqslant \varepsilon.$$

故自由项 f 在 $L^2(K)$ 意义下作微小改变时, 对应的柯西问题的解 u 在 $L^2(K)$ 意义下改变也是微小的.

习题　2.1/2

试直接推导扩散过程所满足的微分方程.

设 $N(x, y, t)$ 表示扩散物质的浓度, dm 表示在无穷小时段 dt 内沿法线方向 $\mathbf{n}$ 经过一个无穷小面积 dS 的

扩散物质的质量, $D(x,y,t)$ 为扩散系数, 其总取正值, 则

$$dm = -D(x,y,t)\frac{\partial N(x,y,t)}{\partial \mathbf{n}}dSdt.$$

故从 t_1 到 t_2 进入扩散面为 Γ 的区域 Ω 的质量为

$$\int_{t_1}^{t_2}\iint\limits_{\Gamma} -dm = \int_{t_1}^{t_2}\iint\limits_{\Gamma} D\frac{\partial N}{\partial t}dSdt = \int_{t_1}^{t_2}\iiint\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial N}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial N}{\partial z}\right)\right]dxdydzdt.$$

区域 Ω 内, 从 t_1 到 t_2 物质的增加量也可表示为

$$\iiint\limits_{\Omega}\left[N(x,y,z,t_2)-N(x,y,z,t_1)\right]dxdydz = \iiint\limits_{\Omega}\int_{t_1}^{t_2}\frac{\partial N}{\partial t}dtdxdydz = \int_{t_1}^{t_2}\iiint\limits_{\Omega}\frac{\partial N}{\partial t}dxdydzdt.$$

由于这两个质量相等, 且 t_1, t_2, Ω 的取值时任意的, 可得

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial N}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial N}{\partial z}\right).$$

习题　2.2/1

习题　2.2/2