MA319 — 偏微分方程

Assignment 3

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习题 1.3/1

用分离变量法求下列问题的解:

$$(1)\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2},\\ u|_{t=0}=\sin\dfrac{3\pi x}{l},\quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=x(l-x)\quad (0<x<l),\\ u(0,t)=u(l,t)=0;\end{cases}$$

$$(2)\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0\\ u(0,t)=0,\quad \dfrac{\partial u}{\partial x}(l,t)=0,\\ u(x,0)=\dfrac{h}{l}x,\\ \dfrac{\partial u}{\partial x}(x,0)=0.\end{cases}.$$

(1)

方程的特征值和对应的特征函数为

$$\lambda=\lambda_k=\frac{k^2\pi^2}{l^2},\quad X_k(x)=C_k\sin\sqrt{\lambda}x=C_k\sin\frac{k\pi}{l}x,\quad k=1,2,\cdots.$$

方程的通解为

$$\begin{aligned}u(x,t)&=\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x\\&=\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi a}{l}t+B_k\sin\frac{k\pi a}{l}t\right)\sin\frac{k\pi}{l}x.\end{aligned}$$

代入初值条件得

$$\varphi(x)=\sin\frac{3\pi x}{l}=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi}{l}x,$$

$$\psi(x)=x(l-x)=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_kk\pi a}{l}\sin\frac{k\pi}{l}x.$$

解得

$$A_k=\begin{cases}1, & k=3\\ 0, & k\neq 3\end{cases},$$

$$B_k=\frac{2}{l\sqrt{\lambda}a}\int_0^l \psi(x)\sin\sqrt{\lambda}xdx=\frac{2}{k\pi a}\int_0^l x(l-x)\sin\frac{k\pi}{l}xdx=-\frac{2l^3(-2+2\cos k\pi+k\pi\sin k\pi)}{k^4\pi^4 a}.$$

化简得

$$B_k=\begin{cases}\dfrac{8l^3}{k^4\pi^4 a}, & k=2n-1\\ 0, & k=2n\end{cases},\quad n=1,2,\cdots.$$

故

$$u(x,t)=\cos\frac{3\pi a}{l}t\sin\frac{3\pi}{l}x+\sum_{n=1}^{\infty}\frac{8l^3}{(2n-1)^4\pi^4 a}\sin\frac{(2n-1)\pi a}{l}t\sin\frac{(2n-1)\pi}{l}x.$$

(ii)

方程的特征值和对应的特征函数为

$$X''(x)+\lambda X(x)=0,$$

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X(0)=0,\ X'(l)=0$ 得

$$C_1=0,\quad C_2\sqrt{\lambda}\cos\sqrt{\lambda}l=0,$$

$$\lambda=\lambda_k=\frac{(2k-1)^2\pi^2}{4l^2},\quad X_k(x)=C_k\sin\frac{(2k-1)\pi}{2l}x,\quad k=1,2,\cdots.$$

方程的通解为

$$\begin{aligned}u(x,t)&=\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x\\&=\sum_{k=1}^{\infty}\left(A_k\cos\frac{(2k-1)\pi a}{2l}t+B_k\sin\frac{(2k-1)\pi a}{2l}t\right)\sin\frac{(2k-1)\pi}{2l}x.\end{aligned}$$

代入初值条件得

$$\varphi(x)=\frac{h}{l}x=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{(2k-1)\pi}{2l}x,$$

$$\psi(x)=0=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_k(2k-1)\pi a}{2l}\sin\frac{(2k-1)\pi}{2l}x.$$

解得

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\sqrt{\lambda}xdx=\frac{2}{l}\int_0^l\frac{h}{l}x\sin\frac{(2k-1)\pi}{2l}xdx=-\frac{4h[2\cos k\pi+(2k-1)\pi\sin k\pi]}{(2k-1)^2\pi^2},$$

$$B_k=0.$$

化简得

$$A_k=(-1)^{k+1}\frac{8h}{(2k-1)^2\pi^2}.$$

故

$$u(x,t)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{8h}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi a}{2l}t\sin\frac{(2k-1)\pi}{2l}x.$$

习题 1.3/2

设弹簧一端固定，一端在外力作用下作周期运动，此时定解问题归结为

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2},\\ u(0,t)=0,\quad u(l,t)=A\sin\omega t\\ u(x,0)=\dfrac{\partial u}{\partial t}(x,0)=0.\end{cases}$$

求解此问题.

设方程的一个特解为

$$U(x,t)=X(x)\sin\omega t,$$

$$U_{tt}=-\omega^2X(x)\sin\omega t,$$

$$U_{xx}=X''(x)\sin\omega t.$$

代入方程和边界条件得

$$X''+\frac{\omega^2}{a^2}X=0,\quad X(0)=0,\quad X(l)=A.$$

$$X(x)=C_1\cos\frac{\omega}{a}x+C_2\sin\frac{\omega}{a}x,$$

$$C_1=0,\quad C_2\sin\frac{\omega}{a}l=A\Longrightarrow C_2=\frac{A}{\sin\frac{\omega}{a}l},$$

$$U(x,t)=X(x)\sin\omega t=\frac{A}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x\sin\omega t.$$

此时

$$U(0,t)=0,\quad U(l,t)=A\sin\omega t,\quad U(x,0)=0,\quad \frac{\partial U}{\partial t}(x,0)=\frac{A\omega}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x.$$

方程的通解为

$$u(x,t)=U(x,t)+V(x,t).$$

且 $V(x,t)$ 满足如下初边值问题

$$\begin{cases}\dfrac{\partial^2 V}{\partial t^2}=a^2\dfrac{\partial^2 V}{\partial x^2},\\ V(0,t)=0,\quad V(l,t)=0\\ V(x,0)=0,\quad \dfrac{\partial V}{\partial t}(x,0)=-\dfrac{A\omega}{\sin\frac{\omega}{a}l}\sin\dfrac{\omega}{a}x.\end{cases}$$

$V(x,t)$ 的特征值和对应的特征函数为

$$\lambda=\lambda_k=\frac{k^2\pi^2}{l^2},\quad X_k(x)=C_k\sin\sqrt{\lambda}x=C_k\sin\frac{k\pi}{l}x,\quad k=1,2,\cdots.$$

$V(x,t)$ 的通解为

$$V(x,t)=\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x$$
$$=\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi a}{l}t+B_k\sin\frac{k\pi a}{l}t\right)\sin\frac{k\pi}{l}x.$$

代入初值条件得

$$\varphi(x)=0=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi}{l}x,$$

$$\psi(x)=-\frac{A\omega}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x=\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\sin\sqrt{\lambda}x=\sum_{k=1}^{\infty}\frac{B_kk\pi a}{l}\sin\frac{k\pi}{l}x.$$

解得

$$A_k=0,$$

$$B_k=\frac{2}{l\sqrt{\lambda}a}\int_0^l\psi(x)\sin\sqrt{\lambda}xdx=\frac{2}{k\pi a}\int_0^l-\frac{A\omega}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x\sin\frac{k\pi}{l}xdx$$
$$=-\frac{2A\omega}{k\pi\sin\frac{\omega}{a}l}\cdot\frac{l\left(\omega l\cos\frac{\omega}{a}l\sin k\pi-k\pi a\sin\frac{\omega}{a}l\cos k\pi\right)}{k^2\pi^2a^2-\omega^2l^2}.$$

化简得

$$B_k=(-1)^k\frac{2A\omega al}{\omega^2l^2-k^2\pi^2a^2}.$$

故

$$V(x,t)=\sum_{k=1}^{\infty}(-1)^k\frac{2A\omega al}{\omega^2l^2-k^2\pi^2a^2}\sin\frac{k\pi a}{l}t\sin\frac{k\pi}{l}x,$$

$$u(x,t)=U(x,t)+V(x,t)=\frac{A}{\sin\frac{\omega}{a}l}\sin\frac{\omega}{a}x\sin\omega t+\sum_{k=1}^{\infty}(-1)^k\frac{2A\omega al}{\omega^2l^2-k^2\pi^2a^2}\sin\frac{k\pi a}{l}t\sin\frac{k\pi}{l}x.$$

习题　1.3/3

求弦震动方程

$$u_{tt}-a^2u_{xx}=0,\quad 0<x<l,\quad t>0$$

满足以下定解条件的解

(1) $u|_{x=0}=u_x|_{x=l}=0,$
$u|_{t=0}=\sin\frac{3}{2l}\pi x,\quad u_t|_{t=0}=\sin\frac{5}{2l}\pi x;$

(2) $u_x|_{x=0}=u_x|_{x=l}=0,$
$u|_{t=0}=x,\quad u_t|_{t=0}=0.$

(1)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad 0 < x < l, \quad t > 0, \\ u|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = \sin \dfrac{3}{2l}\pi x, \quad u_t|_{t=0} = \sin \dfrac{5}{2l}\pi x. \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X(0) = 0$, $X'(l) = 0$ 得

$$C_1 = 0, \quad C_2 \sqrt{\lambda} \cos \sqrt{\lambda} l = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2 \pi^2}{4l^2}, \quad X_k(x) = C_k \sin \frac{(2k-1)\pi}{2l} x, \quad k = 1, 2, \cdots.$$

方程的通解为

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} at + B_k \sin \sqrt{\lambda} at \right) \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{(2k-1)\pi a}{2l} t + B_k \sin \frac{(2k-1)\pi a}{2l} t \right) \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

代入初值条件得

$$\varphi(x) = \sin \frac{3}{2l}\pi x = \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} A_k \sin \frac{(2k-1)\pi}{2l} x,$$

$$\psi(x) = \sin \frac{5}{2l}\pi x = \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} \frac{B_k (2k-1)\pi a}{2l} \sin \frac{(2k-1)\pi}{2l} x.$$

解得

$$A_k = \begin{cases} 1, & k = 2 \\ 0, & k \neq 2 \end{cases}, \quad B_k = \begin{cases} \dfrac{2l}{5\pi a}, & k = 3 \\ 0, & k \neq 3 \end{cases}.$$

故

$$u(x,t) = \cos \frac{3\pi a}{2l} t \sin \frac{3\pi}{2l} x + \frac{2l}{5\pi a} \sin \frac{5\pi a}{2l} t \sin \frac{5\pi}{2l} x.$$

(2)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad 0 < x < l, \quad t > 0, \\ u_x|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = x, \quad u_t|_{t=0} = 0. \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X'(0)=0$, $X'(l)=0$ 得

$$C_2=0,\quad -C_1\sqrt{\lambda}\sin\sqrt{\lambda}l=0,$$

$$\lambda=\lambda_k=\frac{k^2\pi^2}{l^2},\quad X_k(x)=C_k\cos\frac{k\pi}{l}x,\quad k=0,1,2,\cdots.$$

方程的通解为

$$\begin{aligned}u(x,t)&=\frac{1}{2}A_0+\frac{1}{2}B_0t+\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda}at+B_k\sin\sqrt{\lambda}at\right)\cos\sqrt{\lambda}x\\&=\frac{1}{2}A_0+\frac{1}{2}B_0t+\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi a}{l}t+B_k\sin\frac{k\pi a}{l}t\right)\cos\frac{k\pi}{l}x.\end{aligned}$$

代入初值条件得

$$\varphi(x)=x=\frac{1}{2}A_0+\sum_{k=1}^{\infty}A_k\cos\sqrt{\lambda}x=\frac{1}{2}A_0+\sum_{k=1}^{\infty}A_k\cos\frac{k\pi}{l}x,$$

$$\psi(x)=0=\frac{1}{2}B_0+\sum_{k=1}^{\infty}B_k\sqrt{\lambda}a\cos\sqrt{\lambda}x=\frac{1}{2}B_0+\sum_{k=1}^{\infty}\frac{B_kk\pi a}{l}\cos\frac{k\pi}{l}x.$$

解得

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\cos\sqrt{\lambda}xdx=\frac{2}{l}\int_0^l x\cos\frac{k\pi}{l}xdx=\begin{cases}l, & k=0\\ \dfrac{2l(-1+\cos k\pi+k\pi\sin k\pi)}{k^2\pi^2}, & k\neq 0\end{cases},$$

$$B_k=0.$$

化简得

$$A_k=\begin{cases}l, & k=0\\ -\dfrac{4l}{k^2\pi^2}, & k=2n-1\\ 0, & k=2n\end{cases},\quad n=1,2,\cdots.$$

故

$$u(x,t)=\frac{l}{2}+\sum_{n=1}^{\infty}-\frac{4l}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi a}{l}t\cos\frac{(2n-1)\pi}{l}x.$$

习题 1.3/4

用分离变量法求解初边值问题:

$$\begin{cases}u_{tt}-a^2u_{xx}=g, \quad 0<x<l, \quad t>0,\\ u|_{x=0}=u_x|_{x=l}=0,\\ u|_{t=0}=0, \quad u_t|_{t=0}=\sin\dfrac{\pi x}{2l}.\end{cases}$$

其中 g 是常数.

利用叠加原理, 设

$$u(x,t)=U(x,t)+V(x,t).$$

先求

$$\begin{cases} U_{tt} - a^2 U_{xx} = 0, \quad 0 < x < l, \quad t > 0, \\ U|_{x=0} = U_x|_{x=l} = 0, \\ U|_{t=0} = 0, \quad U_t|_{t=0} = \sin\dfrac{\pi x}{2l}. \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x,$$

代入 $X(0) = 0$, $X'(l) = 0$ 得

$$C_1 = 0, \quad C_2\sqrt{\lambda}\cos\sqrt{\lambda}l = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2\pi^2}{4l^2}, \quad X_k(x) = C_k \sin\frac{(2k-1)\pi}{2l}x, \quad k = 1, 2, \cdots.$$

$U(x,t)$ 的通解为

$$\begin{aligned} U(x,t) &= \sum_{k=1}^{\infty}\left(A_k \cos\sqrt{\lambda}at + B_k \sin\sqrt{\lambda}at\right)\sin\sqrt{\lambda}x \\ &= \sum_{k=1}^{\infty}\left(A_k \cos\frac{(2k-1)\pi a}{2l}t + B_k \sin\frac{(2k-1)\pi a}{2l}t\right)\sin\frac{(2k-1)\pi}{2l}x. \end{aligned}$$

代入初值条件得

$$\varphi(x) = 0 = \sum_{k=1}^{\infty} A_k \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty} A_k \sin\frac{(2k-1)\pi}{2l}x,$$

$$\psi(x) = \sin\frac{\pi x}{2l} = \sum_{k=1}^{\infty} B_k\sqrt{\lambda}a \sin\sqrt{\lambda}x = \sum_{k=1}^{\infty}\frac{B_k(2k-1)\pi a}{2l}\sin\frac{(2k-1)\pi}{2l}x.$$

解得

$$A_k = 0, \quad B_k = \begin{cases} \dfrac{2l}{\pi a}, & k = 1 \\ 0, & k \neq 1 \end{cases}.$$

故

$$U(x,t) = \frac{2l}{\pi a}\sin\frac{\pi a}{2l}t \sin\frac{\pi}{2l}x.$$

再求

$$\begin{cases} V_{tt} - a^2 V_{xx} = g, \quad 0 < x < l, \quad t > 0, \\ V|_{x=0} = V_x|_{x=l} = 0, \\ V|_{t=0} = V_t|_{t=0} = 0. \end{cases}$$

令 $f(x,\tau) = g$, 代入齐次化原理公式得

$$\begin{aligned} B_k(\tau) &= \frac{2}{l\sqrt{\lambda}a}\int_0^l f(x,\tau)\sin\sqrt{\lambda}x dx = \frac{4}{(2k-1)\pi a}\int_0^l g \sin\frac{(2k-1)\pi}{2l}x dx \\ &= -\frac{8gl(\sin k\pi - 1)}{(2k-1)^2\pi^2 a} = \frac{8gl}{(2k-1)^2\pi^2 a}. \end{aligned}$$

故

$$
\begin{aligned}
V(x,t)&=\sum_{k=1}^{\infty}\int_0^t B_k(\tau)\sin\sqrt{\lambda}a(t-\tau)d\tau\cdot\sin\sqrt{\lambda}x\\
&=\sum_{k=1}^{\infty}\int_0^t \frac{8gl}{(2k-1)^2\pi^2 a}\sin\frac{(2k-1)\pi a}{2l}(t-\tau)d\tau\cdot\sin\frac{(2k-1)\pi}{2l}x\\
&=\sum_{k=1}^{\infty}\frac{8gl}{(2k-1)^2\pi^2 a}\cdot\frac{4l\sin^2\frac{(2k-1)\pi a}{4l}t}{(2k-1)\pi a}\cdot\sin\frac{(2k-1)\pi}{2l}x\\
&=\sum_{k=1}^{\infty}\frac{32gl^2}{(2k-1)^3\pi^3 a^2}\sin^2\frac{(2k-1)\pi a}{4l}t\sin\frac{(2k-1)\pi}{2l}x.
\end{aligned}
$$

$$
u(x,t)=U(x,t)+V(x,t)=\frac{2l}{\pi a}\sin\frac{\pi a}{2l}t\sin\frac{\pi}{2l}x+\sum_{k=1}^{\infty}\frac{32gl^2}{(2k-1)^3\pi^3 a^2}\sin^2\frac{(2k-1)\pi a}{4l}t\sin\frac{(2k-1)\pi}{2l}x.
$$

习题　1.3/5

用分离变量法求下面问题的解:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}+b\sinh x,\\
u|_{x=0}=u|_{x=l}=0,\\
u|_{t=0}=\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0.
\end{cases}
$$

令 $f(x,\tau)=b\sinh x$, 代入齐次化原理公式得

$$
\begin{aligned}
B_k(\tau)&=\frac{2}{l\sqrt{\lambda}a}\int_0^l f(x,\tau)\sin\sqrt{\lambda}xdx=\frac{2}{k\pi a}\int_0^l b\sinh x\sin\frac{k\pi}{l}xdx\\
&=\frac{2bl(l\cosh l\sin k\pi-k\pi\sinh l\cos k\pi)}{k\pi a(k^2\pi^2+l^2)}\\
&=(-1)^{k+1}\frac{2bl\sinh l}{a(k^2\pi^2+l^2)}.
\end{aligned}
$$

故

$$
\begin{aligned}
u(x,t)&=\sum_{k=1}^{\infty}\int_0^t B_k(\tau)\sin\sqrt{\lambda}a(t-\tau)d\tau\cdot\sin\sqrt{\lambda}x\\
&=\sum_{k=1}^{\infty}\int_0^t(-1)^{k+1}\frac{2bl\sinh l}{a(k^2\pi^2+l^2)}\sin\frac{k\pi}{l}a(t-\tau)d\tau\cdot\sin\frac{k\pi}{l}x\\
&=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2bl\sinh l}{a(k^2\pi^2+l^2)}\cdot\frac{2l\sin^2\frac{k\pi a}{2l}t}{k\pi a}\cdot\sin\frac{k\pi}{l}x\\
&=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2bl^2\sinh l}{k\pi a^2(k^2\pi^2+l^2)}\sin^2\frac{k\pi a}{2l}t\sin\frac{k\pi}{l}x.
\end{aligned}
$$

习题　1.3/6

用分离变量法求下面问题的解:

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}+2b\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2},\\ u|_{x=0}=u|_{x=l}=0,\\ u|_{t=0}=\dfrac{h}{l}x,\quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0.\end{cases}$$

设 $u(x,t)=X(x)T(t)$, 代入得

$$X(x)T''(t)+2bX(x)T'(t)=a^2X''(x)T(t),$$

$$\frac{T''(t)+2bT'(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda.$$

则有两个常微分方程

$$T''(t)+2bT'(t)+\lambda a^2T(t)=0,$$

$$X''(x)+\lambda X(x)=0.$$

方程的特征值和对应的特征函数为

$$\lambda=\lambda_k=\frac{k^2\pi^2}{l^2},\quad X_k(x)=C_k\sin\sqrt{\lambda}x=C_k\sin\frac{k\pi}{l}x,\quad k=1,2,\cdots.$$

代入关于 T 的常微分方程可得

$$T''(t)+2bT'(t)+\frac{k^2\pi^2a^2}{l^2}T(t)=0,$$

$$\Delta=4b^2-\frac{4k^2\pi^2a^2}{l^2},$$

设

$$\lambda'=\frac{k^2\pi^2a^2}{l^2}-b^2,$$

$$r_1=-b+\sqrt{b^2-\frac{k^2\pi^2a^2}{l^2}}=-b+\sqrt{-\lambda'},\quad r_2=-b-\sqrt{b^2-\frac{k^2\pi^2a^2}{l^2}}=-b-\sqrt{-\lambda'}.$$

当 $\lambda'<0$ 时

$$k<\frac{bl}{\pi a},\quad T(t)=C_1e^{r_1t}+C_2e^{r_2t},$$

$$U_k(x,t)=T_k(t)X_k(x)=(A_ke^{r_1t}+B_ke^{r_2t})\sin\sqrt{\lambda}x,$$

$$\frac{\partial U_k}{\partial t}(x,t)=T_k'(t)X_k(x)=(A_kr_1e^{r_1t}+B_kr_2e^{r_2t})\sin\sqrt{\lambda}x.$$

当 $\lambda'=0$ 时

$$k=\frac{bl}{\pi a},\quad T(t)=e^{-bt}(C_1+C_2t),$$

$$U_k(x,t)=T_k(t)X_k(x)=e^{-bt}(A_k+B_kt)\sin\sqrt{\lambda}x,$$

$$\frac{\partial U_k}{\partial t}(x,t)=T_k'(t)X_k(x)=e^{-bt}(-bA_k+B_k-bB_kt)\sin\sqrt{\lambda}x.$$

当 $\lambda'>0$ 时

$$k>\frac{bl}{\pi a},\quad T(t)=e^{-bt}(C_1\cos\sqrt{\lambda'}t+C_2\sin\sqrt{\lambda'}t),$$

$$U_k(x,t)=T_k(t)X_k(x)=e^{-bt}(A_k\cos\sqrt{\lambda'}t+B_k\sin\sqrt{\lambda'}t)\sin\sqrt{\lambda}x,$$

$$\frac{\partial U_k}{\partial t}(x,t)=T_k'(t)X_k(x)=e^{-bt}(-bA_k\cos\sqrt{\lambda'}t-bB_k\sin\sqrt{\lambda'}t-\sqrt{\lambda'}A_k\sin\sqrt{\lambda'}t+\sqrt{\lambda'}B_k\sin\sqrt{\lambda'}t)\sin\sqrt{\lambda}x.$$

设

$$n=\frac{bl}{\pi a},\quad \underline{n}=\lceil n-1\rceil,\quad \overline{n}=\lfloor n+1\rfloor,\quad \delta(n)=\begin{cases}1, & n\in Z\\ 0, & n\notin Z\end{cases}.$$

方程的通解为

$$\begin{aligned}u(x,t)&=\sum_{k=1}^{\infty}T_k(t)X_k(x)\\&=\left[\sum_{k=1}^{\underline{n}}(A_ke^{r_1t}+B_ke^{r_2t})+\delta(n)e^{-bt}(A_n+B_nt)+\sum_{k=\overline{n}}^{\infty}e^{-bt}(A_k\cos\sqrt{\lambda'}t+B_k\sin\sqrt{\lambda'}t)\right]\sin\sqrt{\lambda}x.\end{aligned}$$

代入初值条件得

$$u(x,0)=\left[\sum_{k=1}^{\underline{n}}(A_k+B_k)+\delta(n)A_n+\sum_{k=\overline{n}}^{\infty}A_k\right]\sin\sqrt{\lambda}x=\frac{h}{l}x,$$

$$\frac{\partial u}{\partial t}(x,0)=\left[\sum_{k=1}^{\underline{n}}(A_kr_1+B_kr_2)+\delta(n)(-A_n+B_n)+\sum_{k=\overline{n}}^{\infty}(-bA_k+\sqrt{\lambda'}B_k)\right]\sin\sqrt{\lambda}x=0.$$

解得

$$A_k'=\begin{cases}A_k+B_k, & k<n\\ A_k, & k\geqslant n\end{cases}=\frac{2}{l}\int_0^l\frac{h}{l}x\sin\sqrt{\lambda}xdx=\frac{h}{l}\cdot-\frac{2(\sqrt{\lambda}l\cos\sqrt{\lambda}l-\sin\sqrt{\lambda}l)}{\lambda l}=(-1)^{k+1}\frac{2h}{k\pi},$$

$$B_k'=\begin{cases}A_kr_1+B_kr_2, & k<n\\ -A_k+B_k, & k=n\\ -bA_k+\sqrt{\lambda'}B_k, & k>n\end{cases}=0.$$

化简得

$$A_k=\begin{cases}\dfrac{r_2}{r_2-r_1}A_k', & k<n\\ A_k', & k\geqslant n\end{cases},\quad B_k=\begin{cases}-\dfrac{r_1}{r_2-r_1}A_k', & k<n\\ A_k', & k=n\\ \dfrac{b}{\sqrt{\lambda'}}A_k', & k>n\end{cases}.$$

故将 $r_1,r_2,\lambda,\lambda',n,\underline{n},\overline{n},\delta(n),A_k,B_k$ 代入可得 $u(x,t)$.

例题

(1)

$$\begin{cases} u_{tt} - a^2 u_{xx} = A \sin \omega t, \\ u|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = 1, \quad u_t|_{t=0} = x. \end{cases}$$

利用叠加原理, 设

$$u(x,t) = U(x,t) + V(x,t).$$

先求

$$\begin{cases} U_{tt} - a^2 U_{xx} = 0, \\ U|_{x=0} = U_x|_{x=l} = 0, \\ U|_{t=0} = 1, \quad U_t|_{t=0} = x. \end{cases}$$

方程的特征值和对应的特征函数为

$$X''(x) + \lambda X(x) = 0,$$

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

代入 $X(0) = 0$, $X'(l) = 0$ 得

$$C_1 = 0, \quad C_2 \sqrt{\lambda} \cos \sqrt{\lambda} l = 0,$$

$$\lambda = \lambda_k = \frac{(2k-1)^2 \pi^2}{4l^2}, \quad X_k(x) = C_k \sin \frac{(2k-1)\pi}{2l} x, \quad k = 1, 2, \cdots.$$

$U(x,t)$ 的通解为

$$\begin{aligned} U(x,t) &= \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda} at + B_k \sin \sqrt{\lambda} at \right) \sin \sqrt{\lambda} x \\ &= \sum_{k=1}^{\infty} \left(A_k \cos \frac{(2k-1)\pi a}{2l} t + B_k \sin \frac{(2k-1)\pi a}{2l} t \right) \sin \frac{(2k-1)\pi}{2l} x. \end{aligned}$$

代入初值条件得

$$\varphi(x) = 1 = \sum_{k=1}^{\infty} A_k \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} A_k \sin \frac{(2k-1)\pi}{2l} x,$$

$$\psi(x) = x = \sum_{k=1}^{\infty} B_k \sqrt{\lambda} a \sin \sqrt{\lambda} x = \sum_{k=1}^{\infty} \frac{B_k (2k-1)\pi a}{2l} \sin \frac{(2k-1)\pi}{2l} x.$$

解得

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \sin \sqrt{\lambda} x dx = \frac{2}{l} \int_0^l \sin \frac{(2k-1)\pi}{2l} x dx = \frac{4(-1 + \sin k\pi)}{(2k-1)\pi} = -\frac{4}{(2k-1)\pi},$$

$$B_k = \frac{2}{l\sqrt{\lambda} a} \int_0^l \psi(x) \sin \sqrt{\lambda} x dx = \frac{4}{(2k-1)\pi a} \int_0^l x \sin \frac{(2k-1)\pi}{2l} x dx = -\frac{4l^2 [2\cos k\pi + (2k-1)\pi \sin k\pi]}{(2k-1)^3 \pi^3 a}.$$

化简得

$$B_k = (-1)^{k+1} \frac{8l^2}{(2k-1)^3 \pi^3 a}.$$

故

$$U(x,t)=\sum_{k=1}^{\infty}\left[-\frac{4}{(2k-1)\pi}\cos\frac{(2k-1)\pi a}{2l}t+(-1)^{k+1}\frac{8l^2}{(2k-1)^3\pi^3 a}\sin\frac{(2k-1)\pi a}{2l}t\right]\sin\frac{(2k-1)\pi}{2l}x.$$

再求

$$\begin{cases} V_{tt}-a^2V_{xx}=A\sin\omega t,\\ V|_{x=0}=V_x|_{x=l}=0,\\ V|_{t=0}=V_t|_{t=0}=0.\end{cases}$$

令 $f(x,\tau)=A\sin\omega t$, 代入齐次化原理公式得

$$\begin{aligned} B_k(\tau)&=\frac{2}{l\sqrt{\lambda}a}\int_0^l f(x,\tau)\sin\sqrt{\lambda}xdx=\frac{4}{(2k-1)\pi a}\int_0^l A\sin\omega\tau\sin\frac{(2k-1)\pi}{2l}xdx\\ &=-\frac{8A\sin\omega\tau\cdot l(\sin k\pi-1)}{(2k-1)^2\pi^2 a}=\frac{8lA\sin\omega\tau}{(2k-1)^2\pi^2 a}.\end{aligned}$$

故

$$\begin{aligned} V(x,t)&=\sum_{k=1}^{\infty}\int_0^t B_k(\tau)\sin\sqrt{\lambda}a(t-\tau)d\tau\cdot\sin\sqrt{\lambda}x\\ &=\sum_{k=1}^{\infty}\int_0^t\frac{8lA\sin\omega\tau}{(2k-1)^2\pi^2 a}\sin\frac{(2k-1)\pi a}{2l}(t-\tau)d\tau\cdot\sin\frac{(2k-1)\pi}{2l}x\\ &=\sum_{k=1}^{\infty}\frac{8lA}{(2k-1)^2\pi^2 a}\cdot\frac{-2l[2\omega l\sin\frac{(2k-1)\pi a}{2l}t+(2k-1)\pi a\sin\omega t]}{(2k-1)^2\pi^2a^2-4\omega^2l^2}\cdot\sin\frac{(2k-1)\pi}{2l}x\\ &=\sum_{k=1}^{\infty}-\frac{16l^2A}{(2k-1)^4\pi^4a^3-4(2k-1)^2\pi^2a\omega^2l^2}\left[2\omega l\sin\frac{(2k-1)\pi a}{2l}t+(2k-1)\pi a\sin\omega t\right]\sin\frac{(2k-1)\pi}{2l}x.\end{aligned}$$

$$u(x,t)=U(x,t)+V(x,t).$$

(2)

$$\begin{cases} u_{tt}-u_{xx}-4u=2\sin^2 x,\\ u_x|_{x=0}=u_x|_{x=\pi}=0,\\ u|_{t=0}=0,\quad u_t|_{t=0}=0.\end{cases}$$

将原方程化为

$$\begin{cases} t'=t-\tau\\ W_{tt}-W_{xx}-4W=0,\\ W_x|_{x=0}=W_x|_{x=\pi}=0,\\ W|_{t'=0}=0,\quad W_{t'}|_{t'=0}=2\sin^2 x.\end{cases}$$

设 $W(x,t)=X(x)T(t)$, 代入得

$$X(x)T''(t)-X''(x)T(t)-4X(x)T(t)=0,$$

$$\frac{T''(t)-4T(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda.$$

则有两个常微分方程

$$T''(t)+(\lambda-4)T(t)=0,$$

$$X''(x)+\lambda X(x)=0.$$

方程的特征值和对应的特征函数为

$$X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x,$$

代入 $X'(0)=0,\ X'(\pi)=0$ 得

$$C_2=0,\quad -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi=0,$$

$$\lambda=\lambda_k=k^2,\quad X_k(x)=C_k\cos\sqrt{\lambda}x=C_k\cos kx,\quad k=0,1,2,\cdots.$$

代入关于 T 的常微分方程可得

$$T''(t)+(k^2-4)T(t)=0,$$

设

$$\lambda'=k^2-4,$$

$$r_1=\sqrt{4-k^2}=\sqrt{-\lambda'},\quad r_2=-\sqrt{4-k^2}=-\sqrt{-\lambda'}.$$

当 $\lambda'<0$ 时

$$k<2,\quad T(t)=C_1e^{\sqrt{-\lambda'}t}+C_2e^{-\sqrt{-\lambda'}t},$$

$$W_k(x,t)=T_k(t)X_k(x)=(A_ke^{\sqrt{-\lambda'}t}+B_ke^{-\sqrt{-\lambda'}t})\cos\sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x,t)=T_k'(t)X_k(x)=\sqrt{-\lambda'}(A_ke^{\sqrt{-\lambda'}t}-B_ke^{-\sqrt{-\lambda'}t})\cos\sqrt{\lambda}x.$$

当 $\lambda'=0$ 时

$$k=2,\quad T(t)=C_1+C_2t,$$

$$W_k(x,t)=T_k(t)X_k(x)=(A_k+B_kt)\cos\sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x,t)=T_k'(t)X_k(x)=B_k\cos\sqrt{\lambda}x.$$

当 $\lambda'>0$ 时

$$k>2,\quad T(t)=C_1\cos\sqrt{\lambda'}t+C_2\sin\sqrt{\lambda'}t,$$

$$W_k(x,t)=T_k(t)X_k(x)=(A_k\cos\sqrt{\lambda'}t+B_k\sin\sqrt{\lambda'}t)\cos\sqrt{\lambda}x,$$

$$\frac{\partial W_k}{\partial t}(x,t)=T_k'(t)X_k(x)=\sqrt{\lambda'}(-A_k sin\sqrt{\lambda'}t+B_k\cos\sqrt{\lambda'}t)\cos\sqrt{\lambda}x.$$

方程的通解为

$$\begin{aligned}W(x,t)&=\sum_{k=0}^{\infty}T_k(t)X_k(x)\\&=\left[\sum_{k=0}^{1}(A_ke^{\sqrt{-\lambda'}t}+B_ke^{-\sqrt{-\lambda'}t})+(A_2+B_2t)+\sum_{k=3}^{\infty}(A_k\cos\sqrt{\lambda'}t+B_k\sin\sqrt{\lambda'}t)\right]\cos\sqrt{\lambda}x.\end{aligned}$$

代入初值条件得

$$W(x,0)=\left[\sum_{k=0}^{1}(A_k+B_k)+A_2+\sum_{k=3}^{\infty}A_k\right]\cos\sqrt{\lambda}x=0,$$

$$\frac{\partial W}{\partial t}(x,0)=\left[\sum_{k=0}^{1}\sqrt{-\lambda'}(A_k-B_k)+B_2+\sum_{k=3}^{\infty}\sqrt{\lambda'}B_k\right]\cos\sqrt{\lambda}x=2\sin^2 x=1-\cos 2x.$$

解得

$$A_k'=\begin{cases}A_k+B_k, & k<2\\ A_k, & k\geqslant 2\end{cases}=0,$$

$$B_k'=\begin{cases}\sqrt{-\lambda'}(A_k-B_k), & k<2\\ B_k, & k=2\\ \sqrt{\lambda'}B_k, & k>2\end{cases}=\begin{cases}1, & k=0\\ -1, & k=2\\ 0, & k\neq 0,2\end{cases}.$$

化简得

$$A_k=\begin{cases}\frac{1}{4}, & k=0\\ 0, & k\neq 0\end{cases},\quad B_k=\begin{cases}-\frac{1}{4}, & k=0\\ -1, & k=2\\ 0, & k\neq 0,2\end{cases}.$$

故

$$W(x,t)=\frac{1}{4}(e^{2t}-e^{-2t})-t\cos 2x,$$

$$W(x,t;\tau)=\frac{1}{4}(e^{2(t-\tau)}-e^{-2(t-\tau)})-(t-\tau)\cos 2x,$$

$$u(x,t)=\int_0^t W(x,t;\tau)d\tau=\int_0^t\frac{1}{4}(e^{2(t-\tau)}-e^{-2(t-\tau)})-(t-\tau)\cos 2xd\tau=\frac{1}{4}\cosh 2t-\frac{1}{2}t^2\cos 2x-\frac{1}{4}.$$